Classical Mechanics: Problem 7.1

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Part A

The Lagrangian is defined as

$$\mathcal{L} = T - U \tag{1}$$

so we need to determine T and U for a projectile. We know that $T = \frac{1}{2}m\dot{r}^2$ where $\dot{r} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. For potential energy with a height z, $U = mgz$. Equation 1 becomes

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{2}$$

Part B

The three Lagrangian equations are

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}. \tag{3}$$

Because the only acceleration on the particle is gravity in the $-z$ direction, we expect the x and y terms to be zero for the Lagrange Equations. Let's be explicit. Looking at the x term left-hand side,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz\right)\\ &= 0.\end{aligned}$$

The right hand side,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-mgz\right)\\ &= \frac{d}{dt}\frac{1}{2}m(2\dot{x})\\ &= m\ddot{x}\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\\ 0 &= m\ddot{x}.\end{aligned}$$

Thus $\ddot{x}=0$, as expected. Performing the same flow for the y dimension,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} &= \frac{d}{dt}\frac{\partial}{\partial y}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-mgz\right)\\ &= 0.\end{aligned}$$

The right hand side,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-mgz\right)\\ &= \frac{d}{dt}\frac{1}{2}m(2\dot{y})\\ &= m\ddot{y}\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\\ 0 &= m\ddot{y}.\end{aligned}$$

Thus $\ddot{y}=0$, as expected. Finally, for the z dimension,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-mgz\right)\\ &= -mg.\end{aligned}$$

And the right hand side,

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{z}}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-mgz\right) \\
&= \frac{d}{dt}\frac{1}{2}m(2\dot{z}) \\
&= m\ddot{z}.
\end{aligned}
$$

And the Lagrangian equation becomes,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\
-mg &= m\ddot{z} \\
-g &= \ddot{z}
\end{aligned}
$$

As expected!