Classical Mechanics: Problem 7.13

Colton Kawamura

https://coltonkawamura.github.io/coltonkawamura/

Last updated: August 31, 2023

Part A

Just as suggested in the book, let's start with equation 7.41 in the text, which relates the potential energy to the non-constrain forces $\mathbf{F}$,

$$\mathbf{F} = -\nabla U(\mathbf{r}, t). \tag{1}$$

And the total forces on particle 1 is,

$$\mathbf{F}^1_{\text{tot}} = \mathbf{F}^1_{\text{cstr}} + \mathbf{F}^1.$$

Substituting equation 1 above,

$$\mathbf{F}^1_{\text{tot}} = \mathbf{F}^1_{\text{cstr}} - \nabla U^1.$$

We can apply the same logic to particle two. The net force on particle two is,

$$\mathbf{F}^2_{\text{tot}} = \mathbf{F}^2_{\text{cstr}} - \nabla U^2.$$

Now that we have the forces defined, we can keep moving along with the text as the problems requests. Looking at equation 7.43 in the text, we define the "wrong path" to be

$$\mathbf{R}(t) = \mathbf{r}(t) + \epsilon(t),$$

where $\mathbf{r}(t)$ is the "right path" and $\epsilon(t)$ is the infinitesimal vector pointing from a point on the right path to the corresponding point on the wrong path. For our system of two part ices,

$$\mathbf{R}_1(t) = \mathbf{r}_1(t) + \epsilon_1(t)$$
$$\mathbf{R}_2(t) = \mathbf{r}_2(t) + \epsilon_2(t).$$

We make the assumption (as in the text) that any two corresponding points on the right/wrong path lie in the same surface. Thus, $\epsilon(t)$ is also constrained to the same surface. Thus, $\epsilon(t) = 0$ at t_1 and t_2.

Now we define the action integral,

$$S = \int_{t_1}^{t_2} \mathcal{L}\,(\mathbf{R}, \dot{\mathbf{R}}, t)\, dt,$$

which describes how a physical system changes over time. For our system of *two* particles, it becomes,

$$S = \int_{t_1}^{t_2} \mathcal{L}\,(\mathbf{R_1}, \mathbf{R_2}, \dot{\mathbf{R_1}}, \dot{\mathbf{R_2}}, t)\, dt$$

Our intent, following the text, is the prove the difference in the action integrals

$$\delta S = S - S_o$$

is zero to the first order in the distance ϵ between the paths. Since the Lagrangian is defined as $\mathcal{L} = T - U$, for our two particles,

$$\delta\mathcal{S} = \mathcal{L}(\mathbf{R}_1, \mathbf{R}_2, \dot{\mathbf{R}}_1, \dot{\mathbf{R}}_2, t) - \mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_1, t). \tag{2}$$

If we substitute $\mathbf{R}(t) = \mathbf{r}(t) + \epsilon(t)$ and

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = T - U = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}, t),$$

this becomes

$$\begin{aligned}\delta\mathcal{L} &= \tfrac{1}{2}m\left[(\dot{\mathbf{r}}_1 + \dot{\epsilon_1})^2 + -\dot{\mathbf{r}}_1^2\right] + \tfrac{1}{2}m\left[(\dot{\mathbf{r}}_2 + \dot{\epsilon_2})^2 + -\dot{\mathbf{r}}_2^2\right] - \left[U(\mathbf{r}_1 + \epsilon_1, \mathbf{r}_2 + \epsilon_2, t) - U(\mathbf{r}_1, \mathbf{r}_2, t)\right] \\ &= m\,\dot{\mathbf{r}}_1 \cdot \dot{\epsilon_1} + m\,\dot{\mathbf{r}}_2 \cdot \dot{\epsilon_2} - \epsilon_1 \cdot \nabla U \,+ \epsilon_2 \cdot \nabla U \,+\, O(\epsilon_1^2) +\, O(\epsilon_2^2),\end{aligned}$$

where $O(\epsilon^2)$ denotes terms involving squares and higher powers of ϵ and $\dot{\epsilon}$. Now returning to equation 2, it now becomes,

$$\begin{aligned}\delta\mathcal{S} &= \int_{t_1}^{t_2} \delta\mathcal{L}\, dt \\ &= \int_{t_1}^{t_2} [m\,\dot{\mathbf{r}}_1 \cdot \dot{\epsilon_1} + m\,\dot{\mathbf{r}}_2 \cdot \dot{\epsilon_2} - \epsilon_1 \cdot \nabla U \,+ \epsilon_2 \cdot \nabla U \,+\, O(\epsilon_1^2) +\, O(\epsilon_2^2)]dt.\end{aligned}$$

The first term in the last line can be integrated by parts by just moving the time derivative from one factor to the other and changing the sign. The difference ϵ is zero at the two endpoints as stated earlier, so the endpoint contribution is zero, and we get,

$$\delta S = -\int_{t_1}^{t_2} \epsilon_1 \cdot [m\ddot{\mathbf{r}}_1 + \nabla U]dt - \int_{t_1}^{t_2} \epsilon_2 \cdot [m\ddot{\mathbf{r}}_2 + \nabla U]dt.$$

The $\mathbf{r}$ terms are the "right path" and satisfies Newton's second law. Therefore, the term $m\ddot{\mathbf{r}}$ is just the total force on the particle, $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F}$. Meanwhile,

∇U is the negative force on the particle. Thus,

$$\begin{aligned}
\delta S = &-\int_{t_1}^{t_2} \epsilon_1 \cdot [\mathbf{F}^1_{\text{cstr}} + \mathbf{F}^1 - \mathbf{F}^1]dt - \int_{t_1}^{t_2} \epsilon_2 \cdot [\mathbf{F}^2_{\text{cstr}} + \mathbf{F}^2 - \mathbf{F}^2]dt. \\
= &-\int_{t_1}^{t_2} \epsilon_1 \cdot \mathbf{F}^1_{\text{csir}}\, dt - \int_{t_1}^{t_2} \epsilon_2 \cdot \mathbf{F}^2_{\text{csir}}\, dt
\end{aligned}$$

But the constraint force is normal to the surface in which our particles move, while ϵ lies on the surface. Therefore, $\epsilon \cdot \mathbf{F}_{\text{const}} = 0$. Thus,

$$\delta \mathcal{S} = 0.$$

Thus, the action integral is stationary at the right path. For two particles, we have a total of six degrees of freedom, and the action integral has terms $q_1, q_2, q_3, q_4, q_3, q_4, q_5, q_6$, with an action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(q_1, q_2, q_3, q_4, q_3, q_4, q_5, q_6, \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5, \dot{q}_6, t\right) dt,$$

and this integral is stationary for any variation of those variables and must be satisfied by with the two Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\,.$$