Classical Mechanics: Problem 7.2

Colton Kawamura
https://coltonkawamura.github.io/coltonkawamura/

Last updated: July 29, 2023

Part A

We'll leverage the Biot-Savart law for this problem,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{}}{{}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l'}\times\hat{}}{{}^2}. \tag{1}$$

The direction of of current of the straight segments point directly away and to the point P. The cross product of $d\mathbf{I}$ and $\hat{}$ is zero and no contribution of magnetic field occurs.

For the outer curved section, the $d\mathbf{l'}$ is always perpendicular to $\hat{}$, so the cross product becomes dl that points into the into the page.

$$\mathbf{B}_{\text{outer arc}}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{b^2} \quad \text{(into the page)}.$$

$\int dl'$ is the length of the arc,

$$\begin{aligned}\mathbf{B}_{\text{outer arc}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \frac{2\pi b}{4}\frac{1}{b^2} \\ &= \frac{I\mu_0}{8}\frac{1}{b} \quad \text{(into the page)}\end{aligned}$$

For the inner curved section, the current flows in the opposite direction as the outer section, and the cross product points *out* of the page.

$$\begin{aligned}\mathbf{B}_{\text{inner arc}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \int \frac{dl'}{a^2} \\ &= \frac{\mu_0}{4\pi} I \frac{2\pi a}{4}\frac{1}{a^2} \\ &= \frac{I\mu_0}{8}\frac{1}{a} \quad \text{(out of the page)}.\end{aligned}$$

Putting these all together, we choose the direction as "positive" to out of the page,

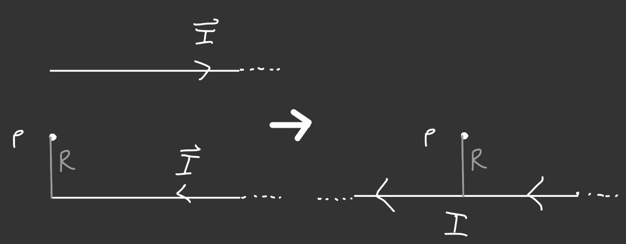
$$\begin{aligned}\sum \mathbf{B} &= \mathbf{B}_{\text{straight segment, top}} + \mathbf{B}_{\text{outer arc}} + \mathbf{B}_{\text{inner arc}} + \mathbf{B}_{\text{straight segment, side}} \\ &= 0 + -\frac{I\mu_0}{8}\frac{1}{b} + \frac{I\mu_0}{8}\frac{1}{a} + 0 \\ &= \frac{I\mu_0}{8}\left(\frac{1}{a} - \frac{1}{b}\right) \text{(out of the page)}\end{aligned}$$

Part B

We can break this system into a superposition of one half-circle with radius R and two straight line segments. First, focusing on the contribution of the half-circle with a total line segment length of $2\pi R$

$$\begin{aligned}\mathbf{B}_{\text{half-circle}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \int \frac{dl'}{R^2} \\ &= \frac{I\mu_0}{4}\frac{1}{R} \quad \text{(into the page).}\end{aligned}$$

For the two straight segments, the lines are equal distances R away from the point, and both have magnetic field contribution that point into the page, *and* they both start/stop at the same plane, we can consider these to be one long infinite wire.



We know from the text that the $\mathbf{B}$ for an infinite wire is,

$$\mathbf{B}_{\text{infinite wire}}(\mathbf{r}) = \frac{\mu_0 I}{2\pi R} \quad \text{(into the page)}$$

Choosing "positive" to be out of the page, the total magnetic field at point P is,

$$\begin{aligned}\sum \mathbf{B} &= -\frac{I\mu_0}{4}\frac{1}{R} + -\frac{\mu_0 I}{2\pi R} \\ &= -\frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right) \text{(out of the page)}\end{aligned}$$