Classical Mechanics: Problem 7.8

Colton Kawamura
https://coltonkawamura.github.io/coltonkawamura/

Last updated: August 14, 2023

Part A

From the given information, the kinetic energy for particle 1 with mass $m_1 = m$ is

$$T_1 = \frac{1}{2}m\dot{x}_1^2.$$

And the kinetic energy for particle 2 with mass $m_2 = m$ is

$$T_2 = \frac{1}{2}m\dot{x}_2^2.$$

The potential energy for a displacement of $x = (x_1 - x_2 - l)$ is

$$U = \frac{1}{2}k\left(x_1 - x_2 - l\right)^2.$$

Together, these produce a Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k\left(x_1 - x_2 - l\right)^2 \tag{1}$$

Part B

To get equation 1 in terms of X and x, we leverage that we have two equations, $X = \frac{1}{2}(x_1 + x_2)$ and $x = (x_1 - x_2 - l)$, with two unknowns (x_1 and x_2). First, let's solve for x_1.

$$X = \frac{1}{2}\left(x_1 + x_2\right)$$
$$x_1 = 2X - x_2$$

Let's put this into $x = (x_1 - x_2 - l)$ to solve for x_2,

$$
\begin{aligned}
x &= (2X - x_2) - x_2 - l \\
x_2 &= -\frac{(x + l - 2X)}{2} \\
&= X - \frac{x}{2} - \frac{l}{2}
\end{aligned}
$$

Great! Now take the derivative of this so we can input into equation 1.

$$
\dot{x}_2 = \dot{X} - \frac{\dot{x}}{2} - 0
$$

All that is left is using our expression for x_2 to solve for an expression for x_1 that is only in terms of X and x. Starting with our expression for x_1

$$
\begin{aligned}
x_1 &= 2X - \left(X - \frac{x}{2} - \frac{1}{2} \right) \\
&= X + \frac{x}{2} + \frac{1}{2}
\end{aligned}
$$

and taking the derivative,

$$
\dot{x}_1 = \dot{X} + \frac{\dot{x}}{2} + 0.
$$

Finally, making the substitutions for x, $\dot{x}_1$, and $\dot{x}_2$ into equation 1,

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k \left(x_1 - x_2 - l \right)^2 \\
&= \frac{1}{2} m \left(\dot{X} + \frac{1}{2} \dot{x} \right)^2 + \frac{1}{2} m \left(\dot{X} - \frac{1}{2} \dot{x} \right)^2 - \frac{1}{2} k x^2 \\
&= m \dot{X}^2 + \frac{1}{4} m \dot{x}^2 - \frac{1}{2} k x^2 \\
&= \frac{1}{4} m [4 \dot{X}^2 + \dot{x}^2] - \frac{1}{2} k x^2
\end{aligned}
$$

Great. Now we can start working on the Lagrange equations. For X,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \\
\frac{\partial}{\partial X}\left[\frac{1}{4}m\left(4\dot{X}^2+\dot{x}^2\right)-\frac{1}{2}kx^2\right] &= \frac{\partial}{\partial t}\frac{\partial}{\partial \dot{X}}\left[\frac{1}{4}m\left(4\dot{X}^2+\dot{x}^2\right)\cdot\frac{1}{2}k\dot{x}^2\right] \\
0 &= \frac{\partial}{\partial t}\left[\frac{1}{4}m(8\dot{X}+0)-0\right] \\
&= \frac{\partial}{\partial t}[2m\dot{X}] \\
&= 2m\ddot{X}
\end{aligned}
$$

Thus,

$$
\ddot{X} = 0. \tag{2}
$$

For x,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\
\frac{\partial}{\partial x}\left[\frac{1}{4}m\left(4\dot{x}^2+\dot{x}^2\right)-\frac{1}{2}kx^2\right] &= \frac{\partial}{\partial t}\frac{\partial}{\partial \dot{x}}\left[\frac{1}{4}m\left(4\dot{x}^2+\dot{x}^2\right)-\frac{1}{2}kx^2\right] \\
-\frac{2}{2}kx &= \frac{\partial}{\partial t}\left[\frac{1}{4}m(0+2\dot{x})-0\right] \\
-kx &= \frac{\partial}{\partial t}\left[\frac{1}{2}mx\right] \\
&= \frac{1}{2}m\ddot{x}.
\end{aligned}
$$

Thus,

$$
\ddot{x} = -\left(\frac{2k}{m}\right)x.
$$

Part C

The general solution by integration to $\ddot{X} = 0$ for some initial velocity v_0 and center of mass position X_0,

$$
\begin{aligned}
\ddot{X} &= 0 \\
\dot{X} &= v_0 \\
X(t) &= v_0 t + x_0.
\end{aligned}
$$

The general solution to $\ddot{x} = -\left(\frac{2k}{m}\right)x$ is something we've seen lots of time before. The fact that our potential energy was $\frac{1}{2}kx^2$ should have hinted that this is a oscillator of the form $\ddot{x} = -\omega^2 x$, which has the general solution

$$x(t) = A\cos\left(\omega t + \phi\right)$$

where A is the amplitude, $\omega = \sqrt{\frac{2k}{m}}$, and ϕ is the phase.

What this means is that the two connected particles will oscillator relative to each other, but the center of mass $X(t)$ moves as a regular free particle!