Electrodynamics: Problem 5.10

Colton Kawamura

https://coltonkawamura.github.io/coltonkawamura/

Last updated: July 29, 2023

Part A

From the Biot-Savart law, direction of the magnetic field from the infinite line current is out of the page on the top portion.

We look at each contribution of magnetic force $\mathbf{F}_{\text{mag}}$ for each segment of the square loop.

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})$$

Where $\mathbf{B}$ is out of the page for this region and has a magnitude of,

$$B = \frac{\mu_0 I}{2\pi \mathscr{r}}$$

Where $\mathscr{r}$ is the distance from the infinite wire to the point of measurement.

The two sides of the loop perpendicular to the infinite line current contribute no forces due to their cross product $(d\mathbf{l} \times \mathbf{B})$

The segment that is closest to the infinite wire segment,

$$\begin{aligned} \mathbf{F}_{\text{mag}} &= I \int \left(d\mathbf{l} \times \frac{\mu_0 I}{2\pi s} \right) \\ &= \frac{a\mu_0 I^2}{2\pi s} \text{ up the page.} \end{aligned} \tag{1}$$

For the side of the square further away,

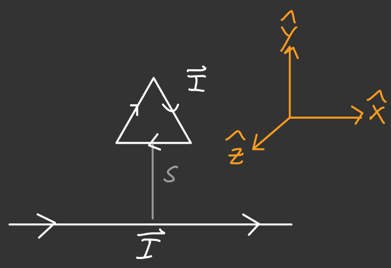
$$\begin{aligned} \mathbf{F}_{\text{mag}} &= I \int \left(d\mathbf{l} \times \frac{\mu_0 I}{2\pi (s+a)} \right) \\ &= \frac{a\mu_0 I^2}{2\pi (s+a)} \text{ down the page.} \end{aligned}$$

Combining all forces and choosing "up the page" to be positive,

$$\begin{aligned}\sum \mathbf{F}_{\text{mag}} &= \frac{a\mu_0 I^2}{2\pi s} + -\frac{a\mu_0 I^2}{2\pi(s+a)} \\ &= \frac{a\mu_0 I^2}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right) \quad \text{(up the page)}.\end{aligned}$$

Part B

Since the force is going to have more than one component, we're have to get a little more technical for this one and define a coordinate axis below.



From Part A, we know that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi r}\hat{z}.$$

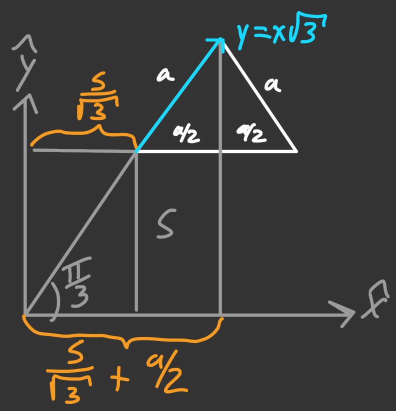
Looking at the two top parts of the triangular current loop, $d\mathbf{l} = dx\hat{x} + dy\hat{y}$ So so our magnetic force is,

$$\begin{aligned}\mathbf{F}_{\text{mag}} &= I\int (d\mathbf{l} \times \mathbf{B}) \\ &= \frac{\mu_0 I^2}{2\pi}\int (-\frac{dx}{y}\,\hat{y} + dy\,\hat{\mathbf{x}}).\end{aligned}$$

Before we move any further, let's try to get out of some calculations. If we perform the right rule on the two top sides, we can see that the left side points down and to the right ($\mathbf{F}_{\text{mag}} = (...)\hat{x} - (...)\hat{y}$). The right hand side points down and to the left ($\mathbf{F}_{\text{mag}} = -(...)\hat{x} - (...)\hat{y}$). So the x-components cancel and we're left with.

$$\mathbf{F}_{\text{mag}} = \frac{\mu_0 I^2}{2\pi}\int (-\frac{dx}{y}\,\hat{y}).$$

All that's left is to determine the limits of integration. After some geometry,



You can see that we integrate in respect to $y = x\sqrt{3}$ from $s/\sqrt{3}$ to $s/\sqrt{3}+a/2$. Our integral for one line segment becomes

$$\begin{aligned}\mathbf{F}_{\text{mag}} &= \frac{\mu_0 I^2}{2\pi} \int_{s/\sqrt{3}}^{s/\sqrt{3}+a/2} \left(-\frac{dx}{x\sqrt{3}}\,\hat{y}\right) \\ &= -\frac{\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(\frac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right)\hat{y} \end{aligned} \tag{2}$$

The *total* line contributions is is twice of equation 2 because of the other top side and the lower line segment we found in equation 1,

$$\sum \mathbf{F}_{\text{mag}} = \frac{a\mu_0 I^2}{2\pi s}\hat{z} - \frac{\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(\frac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right)\hat{y}.$$