Griffiths Electrodynamics: Problem 5.23

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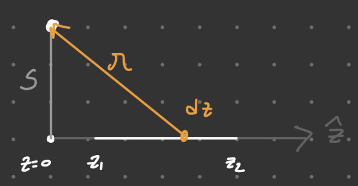
Part A

The magnetic potential of a line current lying on the z-axis from z_1 to z_2 is given by equation 5.66 of the text,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{I\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz.$$

For simplicity, we'll place the point of evaluating the magnetic potential right above $z = 0$.



From the diagram above we can see that our $\mathscr{r} = \sqrt{s^2 + z^2}$, where z is the distance from $z = 0$ to the dl that is contributing to the magnetic field. Now,

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}} \\
&= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left[\ln\left(z + \sqrt{z^2+s^2}\right)\right]_{z_1}^{z_2} \\
&= \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2 + \sqrt{(z_2)^2+s^2}}{z_1 + \sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}.
\end{aligned}$$

Part B

Now we need to check that our answer is consistent with equation 5.37 in the text. To do this, we need to leverage $\mathbf{B} = \nabla \times \mathbf{A}$ to solve for the magnetic field. Since we're dealing with polar coordinates around the line segment and our $\mathbf{A}$ only has a $\hat{\mathbf{z}}$ component, our curl reduces to $-\frac{\partial A}{\partial s}\,\hat{\phi}$,

$$\begin{aligned}
\mathbf{B} &= \mathbf{V} \times \mathbf{A} \\
&= -\frac{\partial A}{\partial s}\hat{\phi} \\
&= \frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2 + \sqrt{(z_2)^2 + s^2}}\frac{s}{\sqrt{(z_2)^2 + s^2}} - \frac{1}{z_1 + \sqrt{(z_1)^2 + s^2}}\frac{s}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\phi} \\
&= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2 - \sqrt{(z_2)^2 + s^2}}{(z_2)^2 - [(z_2)^2 + s^2]}\frac{1}{\sqrt{(z_2)^2 + s^2}} - \frac{z_1 - \sqrt{(z_1)^2 + s^2}}{z_1^2 - [(z_1)^2 + s^2]}\frac{1}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\phi}
\end{aligned}$$

Now we cancel out of the z_2^2 in the denominators and pull out the $-1/s^2$,

$$\begin{aligned}
&= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2 + s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2 + s^2}} + 1\right]\hat{\phi} \\
&= \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2 + s^2}} - \frac{z_1}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\phi}. \qquad (1)
\end{aligned}$$

Remember, we are trying to get this in the form of equation 5.37 in the text, which has $\sin\theta$, where θ is the angle between s and the two endpoints of the two endpoints of the segment In our case,

$$\sin\theta_1 = \frac{z_1}{\sqrt{(z_1)^2 + s^2}} \text{ and } \sin\theta_2 = \frac{z_2}{\sqrt{(z_2)^2 + s^2}},$$

so, equation 1 becomes,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\widehat{\phi}.$$