Griffiths Electrodynamics: Problem 5.12

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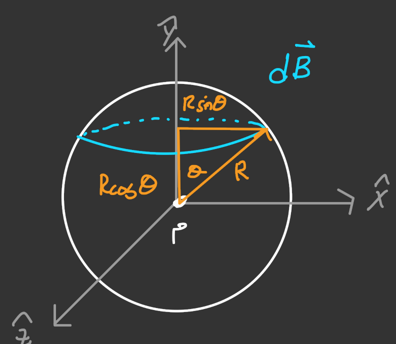
Part A

In example 5.6 of the text, we found that the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I horizontal components cancel, and the vertical components combine to give,

$$\mathbf{B}(\mathbf{z}) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z} \tag{1}$$

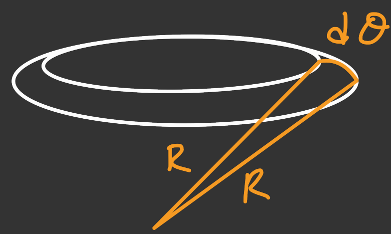
What we can do for this problem is divide the sphere into ring segments according to the diagram below.



Here, our axis has changed so $z \rightarrow y$. Each ring will contribute some $d\mathbf{B}$ as we span θ from 0 to pi. Now we transform equation 1 for some small current (dl) of each ring,

$$\mathbf{B} = \int_0^\pi \frac{\mu_0 dI}{2} \frac{(R\sin\theta)^2}{((R\sin\theta)^2 + (R\cos\theta)^2)^{3/2}} \hat{y}$$

We're varying θ, so we need to get dl in terms of θ. We can think of the width of each ring as $Rd\theta$.



So the small surface current in each "ribbon" on the ring is $K = dI/(Rd\theta)$. We now have an expression for dI,

$$dI = KRd\theta.$$

Making this substitution,

$$\mathbf{B} = \int_0^\pi \frac{\mu_0 KR}{2} \frac{(R\sin\theta)^2}{((R\sin\theta)^2 + (R\cos\theta)^2)^{3/2}} d\theta \ \hat{y}$$

We're not done yet! K is a function of surface charge σ and velocity v, the latter depends on θ. So,

$$\begin{aligned} K &= \sigma v \\ &= \left(\frac{Q}{4\pi R^2}\right)(\omega R \sin\theta) \end{aligned}$$

So now,

$$\begin{aligned} \mathbf{B} &= \int_0^\pi \frac{\mu_0 R}{2}\left(\frac{Q}{4\pi R^2}\right)(\omega R\sin\theta)\frac{(R\sin\theta)^2}{((R\sin\theta)^2 + (R\cos\theta)^2)^{3/2}} d\theta \ \hat{y} \\ &= \frac{\omega\mu_0 R}{2}\frac{Q}{4\pi R^2}\int_0^\pi \frac{R\sin\theta(R\sin\theta)^2}{((R\sin\theta)^2 + (R\cos\theta)^2)^{3/2}} d\theta \ \hat{y} \end{aligned}$$

However, since,

$$\frac{(R\sin\theta)^2}{[(R\sin\theta)^2 + (R\cos\theta)^2]^{3/2}} = \frac{\sin^2\theta}{R},$$

we have,

$$\begin{aligned} \mathbf{B} &= \frac{\omega\mu_0 R^2}{2}\frac{Q}{4\pi R^2}\int_0^\pi \frac{\sin^3\theta}{R} d\theta \ \hat{y} \\ &= \frac{\mu_0 Q\omega}{8\pi R}\left(\frac{4}{3}\right)\hat{y} \\ &= \frac{\mu_0 Q\omega}{6\pi R}\hat{y}. \end{aligned}$$

This supports our intuition if we use the right-hand rule to "curl" dI with a point at the center all around the sphere; all components should cancel leaving a $\hat{y}$ component!