Classical Mechanics: Problem 7.7

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Part A

For N particles,

$$F_{Nx} = m_N \ddot{x}_N$$
$$F_{Ny} = m_N \ddot{y}_N$$
$$F_{Nz} = m_N \ddot{z}_N$$

Part B

In order to create the Lagrange equations, we need to determine kinetic energy K and potential energy U for each particle. Kinetic energy for a particle is

$$K = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 + ... + \frac{1}{2}m_N\dot{r}_N^2$$

We don't know what the potential is, but that shouldn't stop us from turning U in terms of force F as the problem stated. So our Langragian is,

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 + ... + \frac{1}{2}m_N\dot{r}_N^2 - U(r_1, r_2, ..., r_N)\end{aligned}$$

Our Langrange equation of motions are

$$\frac{\partial \mathcal{L}}{\partial r_N} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r_N}}$$

Looking at particle N, the right-hand side,

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r_N}} &= \frac{d}{dt}\left[\frac{1}{2}m_N(2\dot{r}_N)\right] \\
&= m\ddot{r}_N. \qquad (1)
\end{aligned}
$$

Now the left-hand side, the only term in $\mathcal{L}$ that varies with r_1 is potential energy. Since we know that

$$
\frac{\partial \mathcal{L}}{\partial r_N} = -\nabla U(r_N).
$$

Now,

$$
\begin{aligned}
-\nabla U(r_N) &= m\ddot{r}_N \\
-\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z} &= m_N(\hat{x} + \hat{y} + \hat{z}) \qquad (2)
\end{aligned}
$$

But we know that

$$
-\nabla U(r_N) = F(r_N)
$$

so by inspection, equation 2 becomes,

$$
\begin{aligned}
F_{1N} &= m_N\ddot{x}_N \\
F_{1N} &= m_N\ddot{y}_N \\
F_{1N} &= m_N\ddot{z}_N.
\end{aligned}
$$