Classical Mechanics: Problem 7.14

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Part A

To find the Lagrangian, we just need to determine expressions for kinetic and potential energy. Let's knock out the easy one first, potential energy. The only potential force acting on our yo-yo is gravity, so,

$$U = mgx$$

because we are using the x as our coordinate system. Next, let's look at kinetic energy, we were given that,

$$T = \frac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2,$$

and together with a given expression for I and knowing that for a cylinder, $\omega = \dot{x}/R$,

$$\begin{aligned} T &= \frac{1}{2}\,m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}\,mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{4}m\dot{x}^2 \\ &= \frac{1}{2}m\dot{x}^2. \end{aligned}$$

We combine these two for our Lagrangian,

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{3}{4}m\dot{x}^2 + mgx. \end{aligned}$$

Part B

Next we find the Langrange equation of motion,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right).$$

Applying our answer from Part A,

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{3}{4}m\dot{x}^2 + mgx\right) &= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{3}{4}m\dot{x}^2 + mgx\right)\right] \\
mg &= \frac{d}{dt}\left(\frac{3}{2}m\dot{x}\right) \\
\ddot{x} &= \frac{2g}{3}.
\end{aligned}$$