Classical Mechanics: Problem 7.5

Colton Kawamura
https://coltonkawamura.github.io/coltonkawamura/

Last updated: August 2, 2023

What this question is really asking for is to show that

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi}$$

starting with $df = \nabla f \cdot d\mathbf{r}$. We start by expanding $d\mathbf{r}$.

$$d\mathbf{r} = d\,r\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

So that,

$$\begin{aligned} df &= \nabla f \cdot \left(d\,r\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}\right) \\ &= \nabla f \cdot d\,r\,\hat{\mathbf{r}} + \nabla f \cdot r\,d\phi\,\hat{\boldsymbol{\phi}} \\ &= (\nabla f)_r\,d\,r\; + (\nabla f)_\phi\,r\,d\phi \end{aligned} \tag{1}$$

But we also know that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi. \tag{2}$$

Comparing equation 1 and equation 2, we can see that,

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi}$$