## Electrodynamics: Problem 5.9

## Colton Kawamura

https://coltonkawamura.github.io/coltonkawamura/

Last updated: July 29, 2023

## Part A

We'll leverage the Biot-Savart law for this problem,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{r}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{r}}{r^2}.$$
 (1)

The direction of of current of the straight segments point directly away and to the point P. The cross product of  $d\mathbf{I}$  and  $\hat{r}$  is zero and no contribution of magnetic field occurs.

For the outer curved section, the  $d\mathbf{l}'$  is always perpendicular to  $\hat{r}$ , so the cross product becomes dl that points into the into the page.

$$\mathbf{B}_{\text{outer arc}}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{b^2}$$
 (into the page).

 $\int dl'$  is the length of the arc,

$$\begin{split} \mathbf{B}_{\text{outer arc}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \frac{2\pi b}{4} \frac{1}{b^2} \\ &= \frac{I \mu_0}{8} \frac{1}{b} \text{ (into the page)} \end{split}$$

For the inner curved section, the current flows in the opposite direction as the outer section, and the cross product points *out* of the page.

$$\mathbf{B}_{\text{inner arc}}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{a^2}$$

$$= \frac{\mu_0}{4\pi} I \frac{2\pi a}{4} \frac{1}{a^2}$$

$$= \frac{I\mu_0}{8} \frac{1}{a} \text{ (out of the page)}.$$

Putting these all together, we choose the direction as "positive" to out of the page,

$$\sum \mathbf{B} = \mathbf{B}_{\text{straight segment, top}} + \mathbf{B}_{\text{outer arc}} + \mathbf{B}_{\text{inner arc}} + \mathbf{B}_{\text{straight segment, side}}$$

$$= 0 + -\frac{I\mu_0}{8} \frac{1}{b} + \frac{I\mu_0}{8} \frac{1}{a} + 0$$

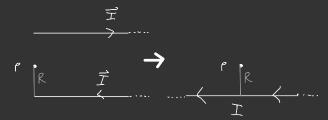
$$= \frac{I\mu_0}{8} \left(\frac{1}{a} - \frac{1}{b}\right) \text{ (out of the page)}$$

## Part B

We can break this system into a superposition of one half-circle with radius R and two straight line segments. First, focusing on the contribution of the half-circle with a total line segment length of  $2\pi R$ 

$$\begin{aligned} \mathbf{B}_{\text{half-circle}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \int \frac{dl'}{R^2} \\ &= \frac{I\mu_0}{4} \frac{1}{R} \text{ (into the page)}. \end{aligned}$$

For the two straight segments, the lines are equal distances R away from the point, and both have magnetic field contribution that point into the page, and they both start/stop at the same plane, we can consider these to be one long infinite wire.



We know from the text that the **B** for an infinite wire is,

$$\mathbf{B}_{\text{infinite wire}}(\mathbf{r}) = \frac{\mu_0 I}{2\pi R}$$
 (into the page)

Choosing "positive" to be out of the page, the total magnetic field at point P is,

$$\sum \mathbf{B} = -\frac{I\mu_0}{4} \frac{1}{R} + -\frac{\mu_0 I}{2\pi R}$$
$$= -\frac{\mu_0 I}{4R} \left(1 + \frac{2}{\pi}\right) \text{(out of the page)}$$