Griffiths Electrodynamics: Problem 5.18

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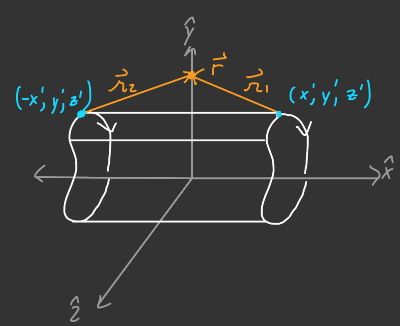
Part A

Consider the following figure of an infinite solenoid of a strange shape with the axis centered on the x-axis.



Qualitatively, for every point at (x', y', z') on the solenoid that contributes an infinitesimal amount to the electric field $d\mathbf{B}_1$ at some point $\mathbf{r}$, there exists another symmetrically placed point at $(-x', y', z')$, that produces $d\mathbf{B}_2$ at point $\mathbf{r}$ with with equal and opposite contributions in $\hat{y}$ and $\hat{z}$, leaving only $\hat{x}$.

Quantitatively, we define dl' from the loop in the right side from the previous figure as

$$dl' = dy'\,\hat{\mathbf{y}} + dz'\,\hat{\mathbf{z}}$$

because it only exists in the yz coordinate plane. Vector $\vec{\mathscr{r}}_1$ is the separation vector between the point we are evaluating the magnetic field, in our case at $(0, y, 0)$ the source of the field at some point (x', y', z').

$$\vec{\mathscr{r}}_1 = -x'\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} - z'\,\hat{\mathbf{z}}$$

For the contribution from some point at $(-x', y', z')$,

$$\vec{\mathscr{r}}_2 = +x'\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} - z'\,\hat{\mathbf{z}}.$$

Now looking at the Biot-Savart law,

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \int \frac{dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \\
&= \frac{\mu_0}{4\pi} I \int \frac{dl' \times \vec{\mathscr{r}}}{\mathscr{r}^3},
\end{aligned}$$

we focus on the $dl' \times \vec{\mathscr{r}}$ term. For the contribution at some point (x', y', z'),

$$\begin{aligned}
dl'_1 \times \vec{\mathscr{r}}_1 &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & dy' & dz' \\ -x' & (y-y') & -z' \end{vmatrix} \\
&= [-dyz' - dz\,(y-y')]\,\hat{x} + [dz(-x) - 0]\hat{y} + [0 - dy(-x')]\,\hat{z} \\
&= [-dyz' - dz\,(y-y')]\,\hat{x} + [-dzx']\,\hat{y} + [dyx']\,\hat{z}. \qquad (1)
\end{aligned}$$

But there's a another contribution from the point at $(-x', y', z')$,

$$\begin{aligned}
dl'_2 \times \vec{\mathscr{r}}_2 &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & dy' & dz' \\ +x' & (y-y') & -z' \end{vmatrix} \\
&= [-dyz' - dz\,(y-y')]\,\hat{x} + [dz(+x) - 0]\hat{y} + [0 - dy(+x')]\,\hat{z} \\
&= [-dyz' - dz\,(y-y')]\,\hat{x} + [-dzx']\,\hat{y} + [-dyx']\,\hat{z}. \qquad (2)
\end{aligned}$$

When we add these contributions $dl'_1 \times \vec{\mathscr{r}}_1 + dl'_2 \times \vec{\mathscr{r}}_2$ together for the total field, notice that the $\hat{y}$ and $\hat{z}$ components of equations 1 and 2 are equal and opposite, leaving behind only a component only in the $\hat{x}$ direction! So, no matter what the shape of the solenoid is, as long as the shape is constant along the axis, Ampere's law for that system is

$$\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{x}, & \text{inside solenoid,} \\ 0, & \text{outside the solenoid.} \end{cases}$$

Part B

When the radius of the doughnut is so large that a segment can be considered essentially straight, the difference between s to the inner radius and the outer radius is approximately constant. Thus,

$$N/2\pi s = n,$$

where n is the number of turns per unit length. Thus the electric field outside a toroid becomes,

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0 N I}{2\pi s}\hat{\phi} \\ &\approx \mu_0 n I \hat{\phi}.\end{aligned}$$