Griffiths Electrodynamics: Problem 5.14

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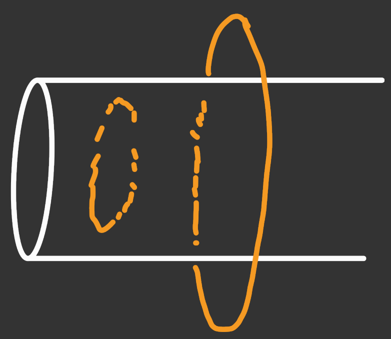
Part A

Just like Gauss' law, we create two *Ampereian* surfaces (or rings) , one inside and one outside.



Both of which are subject to Ampere's Law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

For the inside ring, $I_{\text{enc}} = 0$, thus,

$$\mathbf{B} = 0, \text{ for } s < a.$$

For the outside ring,

$$\begin{aligned} \oint \vec{B} \cdot dl &= \mu_0 I_{enc} \\ (B)(2\pi s) &= \mu_0 I \\ B &= \frac{\mu_0 I}{2\pi s} \end{aligned}$$

Performing the right-hand rule with our thumb, pointing in the direction of the current, we can see $\mathbf{B}$ curls in the direction of $\hat{\phi}$, so,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}, \text{ for } s > a.$$

Part B

We are given that, for some proportionality constant k,

$$J = ks.$$

So for an Amperian surface $d\mathbf{a}$ that is perpendicular to $\mathbf{J}$, our I is,

$$\begin{aligned} I &= \int \mathbf{J} \cdot d\mathbf{a} \\ &= \int (ks)(2\pi s)ds \end{aligned}$$

and we run ds from 0 to a,

$$\begin{aligned} I &= \int_0^a (ks)(2\pi s)ds \\ &= \frac{2\pi k a^3}{3}. \end{aligned}$$

Remember, we were given I, so we can solve for k,

$$k = \frac{3I}{2\pi a^3}.$$

Great. Now for a general expression of I_{enc}, that runs from 0 to some point s' from the axis of the wire,

$$\begin{aligned} I_{\text{enc}} &= \int_0^{s'} \mathbf{J} \cdot d\mathbf{a} \\ &= \int_0^{s'} (ks)(2\pi s)ds \\ &= \frac{2\pi k s^3}{3} \text{ (Dropping the s' for s.)} \\ &= I\frac{s^3}{a^3}. \end{aligned}$$

Finally, we can use this I_{enc} to solve Ampere's Law. For a ring inside the wire,

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$
$$(B)(2\pi s) = \mu_0 I \frac{s^3}{a^3}$$
$$\mathbf{B} = \frac{\mu_0 I s^2}{2\pi a^3}\,\hat{\phi}, \text{ for } s < a.$$

For a ring outside the wire,

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$
$$(B)(2\pi s) = \mu_0 I \frac{a^3}{a^3}$$
$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}, \quad \text{for } s > \alpha.$$