CompSci Program, FEB21

Diffusion equation in 1-Dim

- explicit scheme
- implicit scheme

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

Scaled equation:

$x \in [0, 1]$ $\quad t \in [0, t_n]$

Boundary conditions:

$$u(0,t) = u(1,t) = 0$$

initial conditions:

$$u(x,0) = g(x)$$

Discretize

$x \Rightarrow x_i = x_0 + i\Delta x \quad i = 0, 1, \ldots m$

$t \Rightarrow t_j = t_0 + j\Delta t \quad j = 0, 1, \ldots n$

$$\Delta x = \frac{1-0}{m} = \frac{1}{m}$$

$$\Delta t = \frac{t_n - t_0}{n} = t_n/n$$

$$u(x,t) \Rightarrow u(x_i, t_j) = u_{ij}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + O(\Delta t)$$

Euler - Forward $\sim O(\Delta t)$

$$\frac{\partial u}{\partial t} \simeq \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t}$$

$$= \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} + u_{i-1,j} - u_{i,j} \cdot 2}{(\Delta x^2)} + O(\Delta x^2)$$

$$\simeq \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{(\Delta x)^2}$$

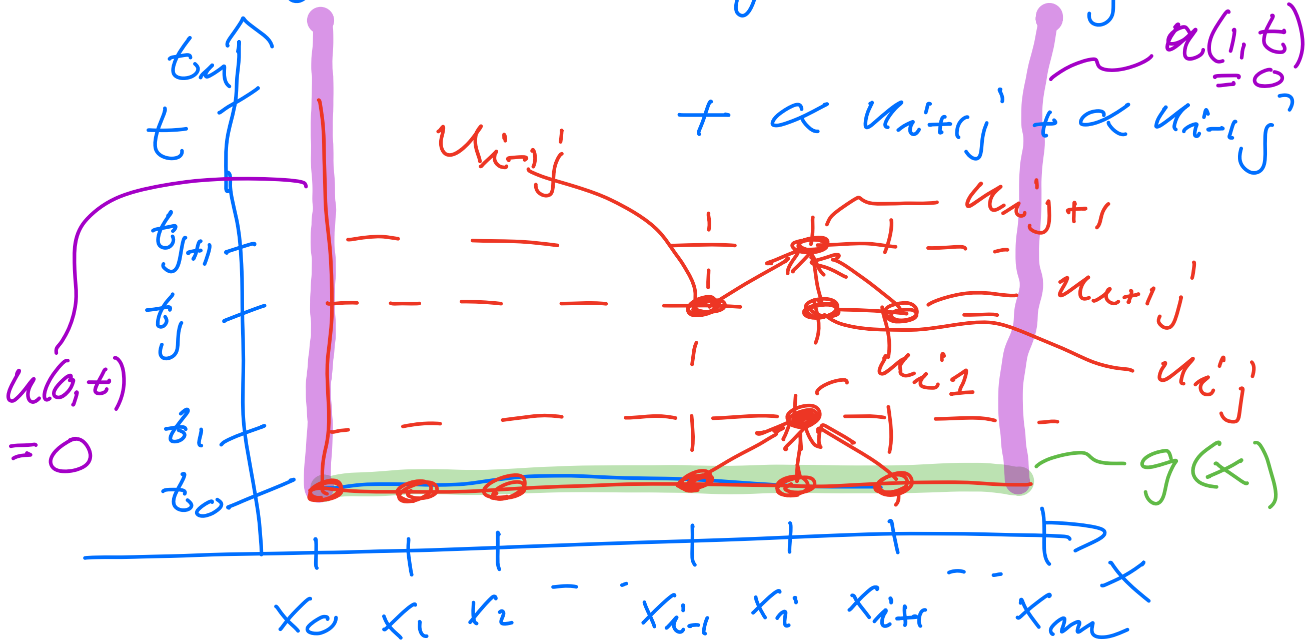
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \longrightarrow$$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{\Delta x^2}$$

$$\alpha = \Delta t$$

$\overline{}$
Δx^2

$$u_{i,j+1} = u_{i,j} - \alpha\, 2 \cdot u_{i,j} + \alpha\, u_{i+1,j} + \alpha\, u_{i-1,j}$$



$t_j = t_1 \qquad j = 1$

$$u_{i1} = \underbrace{u_{i0}}_{= g_i = g(x_i)} - 2\alpha \underbrace{u_{i0}}_{g_i} + \alpha \overbrace{u_{i+1\,0}}^{g_{i+1}} + \alpha \underbrace{u_{i-1\,0}}_{g_{i-1}}$$

but $i = 0$ and $i = m$
then use boundary
conditions.

$i = 1$

$$u_{1\,j+1} = u_{1j} - 2\alpha u_{1j} + \alpha(u_{2j} + \overbrace{u_{0j}}^{\text{boundary}})$$

$i=2$

$$u_{2j+1} = u_{2j} - 2\alpha u_{2j} + \alpha(u_{3j} + u_{1j})$$

$\vdots$

$i = m-1$

$$u_{m-1\,j+1} = u_{m-1\,j} - 2\alpha u_{m-1\,j} + \alpha(u_{mj} + u_{m-2\,j})$$

$$U_{j+1} = \begin{bmatrix} u_{1\,j+1} \\ u_{2\,j+1} \\ \vdots \\ u_{m-1\,j+1} \end{bmatrix} \quad \text{unknown}$$

$$U_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{m-1\,j} \end{bmatrix} \quad \text{known}$$

$$U_{j+1} = A \quad U_j$$

$$A = \begin{bmatrix} 1-2\alpha & \alpha & & 0 \\ \alpha & 1-2\alpha & \ddots & \\ & \ddots & \ddots & \alpha \\ 0 & & \alpha & 1-2\alpha \end{bmatrix}$$

$$= \mathbb{I} - \alpha B$$

$$B = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & -1 & \\ & \ddots & \ddots & \ddots \\ 0 & & -1 & 2 \quad 1 \end{bmatrix}$$ Tridiagonal

$$u_1 = A u_0$$

$$u_j = A^j u_0$$

Drawback $\dfrac{\Delta t}{(\Delta x)^2} \leq \dfrac{1}{2}$

$$\Delta x = 10^{-2} \implies$$

$$\Delta t \leq \frac{1}{2}\left(10^{-2}\right)^2$$

<u>implicit scheme</u>

$$u(x \pm \Delta x) = u(x) \pm \Delta x\, u'(x) + \frac{\Delta x^2}{2!} u''(x) + \dots$$

$$\underline{\partial u} \simeq u_{i,j} - u_{i,j-1}$$

∂t $\qquad \dfrac{\qquad}{\Delta t}$

$$u_{ij} = u_{ij-1} + \alpha\left(u_{i+1j} + u_{i-1j} - 2\alpha\, u_{ij}\right)$$

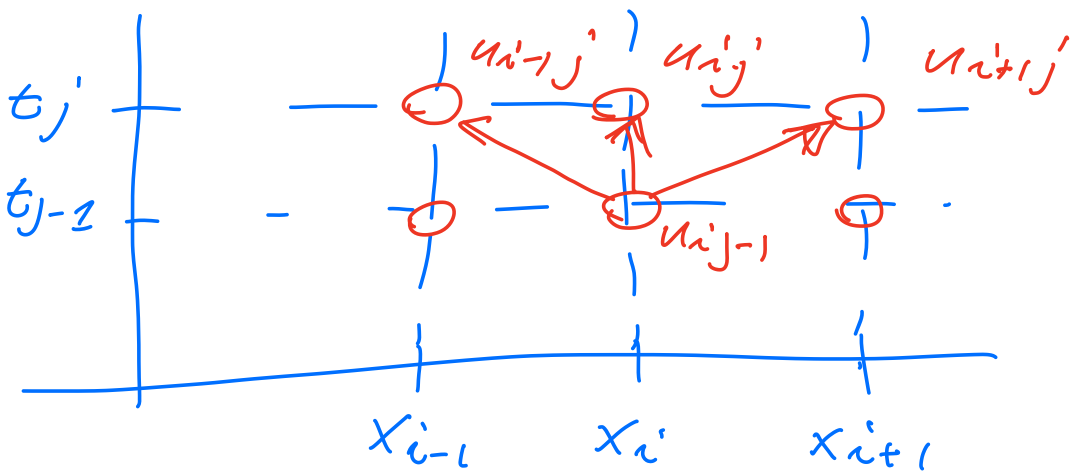
$$\alpha = \frac{\Delta t}{(\Delta x)^2}$$

unknown $\qquad\qquad\qquad\qquad$ known

$$u_{ij}(1+2\alpha) - \alpha(u_{i+1j} + u_{i-1j}) = u_{ij-1}$$

$j = 1$

$$u_{i1}(1+2\alpha) - \alpha(u_{i+1\,1} + u_{i-1\,1}) = \underline{\underline{u_{i0}}}$$



$$u_{j-1} = \begin{bmatrix} u_{1j-1} \\ u_{2j-1} \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ u_{m-1\,j-1} \end{bmatrix}$$

$$u_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{m-1\,j} \end{bmatrix}$$

$$\tilde{A}\, u_j = u_{j-1}$$

$$\tilde{A} = \mathbb{1} + \alpha B$$

$$B = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ 0 & & -1 & \ddots & \\ & & & \ddots\,-1 & 2 \quad -1 \end{bmatrix}$$

Solved by using the Thomas algorithm ($4m$ FLOPS)

stable for all Δx and Δt combinations.

Error in time $O(\Delta t)$

$-1 \quad - \quad x \qquad O(\Delta x^2)$