

# Feed Forward Neural Networks and Building our own Neural Network code

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## Plan for week 3

- Building our own Feed-forward Neural Network and playing around with our own Feed-forward Neural Network and introduction to TensorFlow. Start convolutional Neural Networks (CNN).

Reading suggestions: [Aurelien Geron's chapters 10-11](#). For a more in depth discussion on neural networks we recommend Goodfellow et al chapters 6 and 7. For CNNs, see Goodfellow et al chapter 9. chapter 11 and 12 on practicalities and applications. We strongly recommend reading these chapters from Goodfellow et on Deep Learning, that is chapters 6-12. Bishop's chapter 5 on Neural Networks is an additional good read.

## Videos on Neural Networks

- [Video on Neural Networks](#)
- [Video on the back propagation algorithm](#)

## Neural networks

Artificial neural networks are computational systems that can learn to perform tasks by considering examples, generally without being programmed with any task-specific rules. It is supposed to mimic a biological system, wherein neurons interact by sending signals in the form of mathematical functions between layers. All layers can contain an arbitrary number of neurons, and each connection is represented by a weight variable.

## Artificial neurons

The field of artificial neural networks has a long history of development, and is closely connected with the advancement of computer science and computers in general. A model of artificial neurons was first developed by McCulloch and Pitts in 1943 to study signal processing in the brain and has later been refined by others. The general idea is to mimic neural networks in the human brain, which is composed of billions of neurons that communicate with each other by sending electrical signals. Each neuron accumulates its incoming signals, which must exceed an activation threshold to yield an output. If the threshold is not overcome, the neuron remains inactive, i.e. has zero output.

This behaviour has inspired a simple mathematical model for an artificial neuron.

$$y = f\left(\sum_{i=1}^n w_i x_i\right) = f(u) \quad (1)$$

Here, the output  $y$  of the neuron is the value of its activation function, which have as input a weighted sum of signals  $x_i, \dots, x_n$  received by  $n$  other neurons.

Conceptually, it is helpful to divide neural networks into four categories:

1. general purpose neural networks for supervised learning,
2. neural networks designed specifically for image processing, the most prominent example of this class being Convolutional Neural Networks (CNNs),
3. neural networks for sequential data such as Recurrent Neural Networks (RNNs), and
4. neural networks for unsupervised learning such as Deep Boltzmann Machines.

In natural science, DNNs and CNNs have already found numerous applications. In statistical physics, they have been applied to detect phase transitions in 2D Ising and Potts models, lattice gauge theories, and different phases of polymers, or solving the Navier-Stokes equation in weather forecasting. Deep learning has also found interesting applications in quantum physics. Various quantum phase transitions can be detected and studied using DNNs and CNNs, topological phases, and even non-equilibrium many-body localization. Representing quantum states as DNNs quantum state tomography are among some of the impressive achievements to reveal the potential of DNNs to facilitate the study of quantum systems.

In quantum information theory, it has been shown that one can perform gate decompositions with the help of neural.

The applications are not limited to the natural sciences. There is a plethora of applications in essentially all disciplines, from the humanities to life science and medicine.

## Neural network types

An artificial neural network (ANN), is a computational model that consists of layers of connected neurons, or nodes or units. We will refer to these interchangeably as units or nodes, and sometimes as neurons.

It is supposed to mimic a biological nervous system by letting each neuron interact with other neurons by sending signals in the form of mathematical functions between layers. A wide variety of different ANNs have been developed, but most of them consist of an input layer, an output layer and eventual layers in-between, called *hidden layers*. All layers can contain an arbitrary number of nodes, and each connection between two nodes is associated with a weight variable.

Neural networks (also called neural nets) are neural-inspired nonlinear models for supervised learning. As we will see, neural nets can be viewed as natural, more powerful extensions of supervised learning methods such as linear and logistic regression and soft-max methods we discussed earlier.

## Feed-forward neural networks

The feed-forward neural network (FFNN) was the first and simplest type of ANNs that were devised. In this network, the information moves in only one direction: forward through the layers.

Nodes are represented by circles, while the arrows display the connections between the nodes, including the direction of information flow. Additionally, each arrow corresponds to a weight variable (figure to come). We observe that each node in a layer is connected to *all* nodes in the subsequent layer, making this a so-called *fully-connected* FFNN.

## Convolutional Neural Network

A different variant of FFNNs are *convolutional neural networks* (CNNs), which have a connectivity pattern inspired by the animal visual cortex. Individual neurons in the visual cortex only respond to stimuli from small sub-regions of the visual field, called a receptive field. This makes the neurons well-suited to exploit the strong spatially local correlation present in natural images. The response of each neuron can be approximated mathematically as a convolution operation. (figure to come)

Convolutional neural networks emulate the behaviour of neurons in the visual cortex by enforcing a *local* connectivity pattern between nodes of adjacent layers: Each node in a convolutional layer is connected only to a subset of the nodes in the previous layer, in contrast to the fully-connected FFNN. Often, CNNs consist of several convolutional layers that learn local features of the input, with a fully-connected layer at the end, which gathers all the local data and produces the outputs. They have wide applications in image and video recognition.

## Recurrent neural networks

So far we have only mentioned ANNs where information flows in one direction: forward. *Recurrent neural networks* on the other hand, have connections between nodes that form directed *cycles*. This creates a form of internal memory which are able to capture information on what has been calculated before; the output is dependent on the previous computations. Recurrent NNs make use of sequential information by performing the same task for every element in a sequence, where each element depends on previous elements. An example of such information is sentences, making recurrent NNs especially well-suited for handwriting and speech recognition.

## Other types of networks

There are many other kinds of ANNs that have been developed. One type that is specifically designed for interpolation in multidimensional space is the radial basis function (RBF) network. RBFs are typically made up of three layers: an input layer, a hidden layer with non-linear radial symmetric activation functions and a linear output layer ("linear" here means that each node in the output layer has a linear activation function). The layers are normally fully-connected and there are no cycles, thus RBFs can be viewed as a type of fully-connected FFNN. They are however usually treated as a separate type of NN due the unusual activation functions.

## Multilayer perceptrons

One uses often so-called fully-connected feed-forward neural networks with three or more layers (an input layer, one or more hidden layers and an output layer) consisting of neurons that have non-linear activation functions.

Such networks are often called *multilayer perceptrons* (MLPs).

## Why multilayer perceptrons?

According to the *Universal approximation theorem*, a feed-forward neural network with just a single hidden layer containing a finite number of neurons can approximate a continuous multidimensional function to arbitrary accuracy, assuming the activation function for the hidden layer is a **non-constant, bounded and monotonically-increasing continuous function**.

Note that the requirements on the activation function only applies to the hidden layer, the output nodes are always assumed to be linear, so as to not restrict the range of output values.

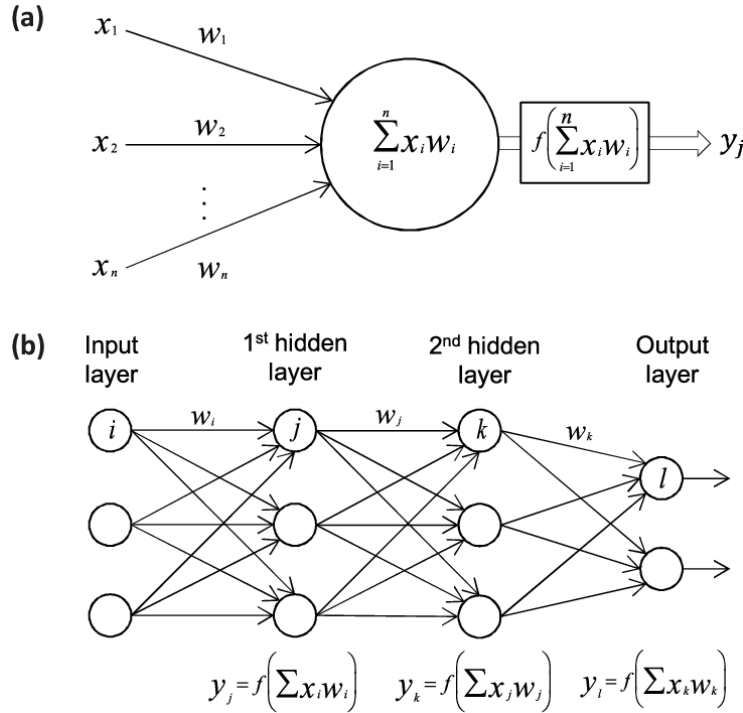


Figure 1: In a) we show a single perceptron model while in b) we display a network with two hidden layers, an input layer and an output layer.

## Illustration of a single perceptron model and a multi-perceptron model

### Examples of XOR, OR and AND gates

Let us first try to fit various gates using standard linear regression. The gates we are thinking of are the classical XOR, OR and AND gates, well-known elements in computer science. The tables here show how we can set up the inputs  $x_1$  and  $x_2$  in order to yield a specific target  $y_i$ .

```

"""
Simple code that tests XOR, OR and AND gates with linear regression
"""

import numpy as np
# Design matrix
X = np.array([ [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]], dtype=np.float64)
print(f"The X.TX matrix:{X.T @ X}")
Xinv = np.linalg.pinv(X.T @ X)
print(f"The invers of X.TX matrix:{Xinv}")

# The XOR gate
yXOR = np.array([ 0, 1, 1, 0])

```

```

ThetaXOR = Xinv @ X.T @ yXOR
print(f"The values of theta for the XOR gate:{ThetaXOR}")
print(f"The linear regression prediction for the XOR gate:{X @ ThetaXOR}")

# The OR gate
yOR = np.array([ 0, 1, 1, 1])
ThetaOR = Xinv @ X.T @ yOR
print(f"The values of theta for the OR gate:{ThetaOR}")
print(f"The linear regression prediction for the OR gate:{X @ ThetaOR}")

# The OR gate
yAND = np.array([ 0, 0, 0, 1])
ThetaAND = Xinv @ X.T @ yAND
print(f"The values of theta for the AND gate:{ThetaAND}")
print(f"The linear regression prediction for the AND gate:{X @ ThetaAND}")

```

What is happening here?

## Does Logistic Regression do a better Job?

```

"""
Simple code that tests XOR and OR gates with linear regression
and logistic regression
"""

import matplotlib.pyplot as plt
from sklearn.linear_model import LogisticRegression
import numpy as np

# Design matrix
X = np.array([ [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]], dtype=np.float64)
print(f"The X.TX matrix:{X.T @ X}")
Xinv = np.linalg.pinv(X.T @ X)
print(f"The invers of X.TX matrix:{Xinv}")

# The XOR gate
yXOR = np.array([ 0, 1, 1, 0])
ThetaXOR = Xinv @ X.T @ yXOR
print(f"The values of theta for the XOR gate:{ThetaXOR}")
print(f"The linear regression prediction for the XOR gate:{X @ ThetaXOR}")

# The OR gate
yOR = np.array([ 0, 1, 1, 1])
ThetaOR = Xinv @ X.T @ yOR
print(f"The values of theta for the OR gate:{ThetaOR}")
print(f"The linear regression prediction for the OR gate:{X @ ThetaOR}")

# The OR gate
yAND = np.array([ 0, 0, 0, 1])
ThetaAND = Xinv @ X.T @ yAND
print(f"The values of theta for the AND gate:{ThetaAND}")
print(f"The linear regression prediction for the AND gate:{X @ ThetaAND}")

# Now we change to logistic regression

```

```

# Logistic Regression
logreg = LogisticRegression()
logreg.fit(X, yOR)
print("Test set accuracy with Logistic Regression for OR gate: {:.2f}".format(logreg.score(X,yOR)))

logreg.fit(X, yXOR)
print("Test set accuracy with Logistic Regression for XOR gate: {:.2f}".format(logreg.score(X,yXOR)))

logreg.fit(X, yAND)
print("Test set accuracy with Logistic Regression for AND gate: {:.2f}".format(logreg.score(X,yAND)))

```

Not exactly impressive, but somewhat better.

## Adding Neural Networks

```

# and now neural networks with Scikit-Learn and the XOR

from sklearn.neural_network import MLPClassifier
from sklearn.datasets import make_classification
X, yXOR = make_classification(n_samples=100, random_state=1)
FFNN = MLPClassifier(random_state=1, max_iter=300).fit(X, yXOR)
FFNN.predict_proba(X)
print(f"Test set accuracy with Feed Forward Neural Network for XOR gate:{FFNN.score(X, yXOR)}")

```

## Mathematical model

The output  $y$  is produced via the activation function  $f$

$$y = f\left(\sum_{i=1}^n w_i x_i + b_i\right) = f(z),$$

This function receives  $x_i$  as inputs. Here the activation  $z = (\sum_{i=1}^n w_i x_i + b_i)$ . In an FFNN of such neurons, the *inputs*  $x_i$  are the *outputs* of the neurons in the preceding layer. Furthermore, an MLP is fully-connected, which means that each neuron receives a weighted sum of the outputs of *all* neurons in the previous layer.

## Mathematical model

First, for each node  $i$  in the first hidden layer, we calculate a weighted sum  $z_i^1$  of the input coordinates  $x_j$ ,

$$z_i^1 = \sum_{j=1}^M w_{ij}^1 x_j + b_i^1 \quad (2)$$

Here  $b_i$  is the so-called bias which is normally needed in case of zero activation weights or inputs. How to fix the biases and the weights will be discussed below. The value of  $z_i^1$  is the argument to the activation function  $f_i$  of each node  $i$ . The variable  $M$  stands for all possible inputs to a given node  $i$  in the first layer. We define the output  $y_i^1$  of all neurons in layer 1 as

$$y_i^1 = f(z_i^1) = f\left(\sum_{j=1}^M w_{ij}^1 x_j + b_i^1\right) \quad (3)$$

where we assume that all nodes in the same layer have identical activation functions, hence the notation  $f$ . In general, we could assume in the more general case that different layers have different activation functions. In this case we would identify these functions with a superscript  $l$  for the  $l$ -th layer,

$$y_i^l = f^l(u_i^l) = f^l\left(\sum_{j=1}^{N_{l-1}} w_{ij}^l y_j^{l-1} + b_i^l\right) \quad (4)$$

where  $N_l$  is the number of nodes in layer  $l$ . When the output of all the nodes in the first hidden layer are computed, the values of the subsequent layer can be calculated and so forth until the output is obtained.

## Mathematical model

The output of neuron  $i$  in layer 2 is thus,

$$y_i^2 = f^2\left(\sum_{j=1}^N w_{ij}^2 y_j^1 + b_i^2\right) \quad (5)$$

$$= f^2\left[\sum_{j=1}^N w_{ij}^2 f^1\left(\sum_{k=1}^M w_{jk}^1 x_k + b_j^1\right) + b_i^2\right] \quad (6)$$

where we have substituted  $y_k^1$  with the inputs  $x_k$ . Finally, the ANN output reads

$$y_i^3 = f^3\left(\sum_{j=1}^N w_{ij}^3 y_j^2 + b_i^3\right) \quad (7)$$

$$= f^3\left[\sum_j w_{ij}^3 f^2\left(\sum_k w_{jk}^2 f^1\left(\sum_m w_{km}^1 x_m + b_k^1\right) + b_j^2\right) + b_i^3\right] \quad (8)$$

## Mathematical model

We can generalize this expression to an MLP with  $l$  hidden layers. The complete functional form is,

$$y_i^{l+1} = f^{l+1}\left[\sum_{j=1}^{N_l} w_{ij}^l f^l\left(\sum_{k=1}^{N_{l-1}} w_{jk}^{l-1}\left(\dots f^1\left(\sum_{n=1}^{N_0} w_{mn}^1 x_n + b_m^1\right)\dots\right) + b_k^2\right) + b_i^3\right] \quad (9)$$



which illustrates a basic property of MLPs: The only independent variables are the input values  $x_n$ .

## Mathematical model

This confirms that an MLP, despite its quite convoluted mathematical form, is nothing more than an analytic function, specifically a mapping of real-valued vectors  $\hat{x} \in \mathbb{R}^n \rightarrow \hat{y} \in \mathbb{R}^m$ .

Furthermore, the flexibility and universality of an MLP can be illustrated by realizing that the expression is essentially a nested sum of scaled activation functions of the form

$$f(x) = c_1 f(c_2 x + c_3) + c_4 \quad (10)$$

where the parameters  $c_i$  are weights and biases. By adjusting these parameters, the activation functions can be shifted up and down or left and right, change slope or be rescaled which is the key to the flexibility of a neural network.

**Matrix-vector notation.** We can introduce a more convenient notation for the activations in an A NN.

Additionally, we can represent the biases and activations as layer-wise column vectors  $\hat{b}_l$  and  $\hat{y}_l$ , so that the  $i$ -th element of each vector is the bias  $b_i^l$  and activation  $y_i^l$  of node  $i$  in layer  $l$  respectively.

We have that  $W_l$  is an  $N_{l-1} \times N_l$  matrix, while  $\hat{b}_l$  and  $\hat{y}_l$  are  $N_l \times 1$  column vectors. With this notation, the sum becomes a matrix-vector multiplication, and we can write the equation for the activations of hidden layer 2 (assuming three nodes for simplicity) as

$$\hat{y}_2 = f_2(W_2 \hat{y}_1 + \hat{b}_2) = f_2 \left( \begin{bmatrix} w_{11}^2 & w_{12}^2 & w_{13}^2 \\ w_{21}^2 & w_{22}^2 & w_{23}^2 \\ w_{31}^2 & w_{32}^2 & w_{33}^2 \end{bmatrix} \cdot \begin{bmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \end{bmatrix} + \begin{bmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{bmatrix} \right). \quad (11)$$

**Matrix-vector notation and activation.** The activation of node  $i$  in layer 2 is

$$y_i^2 = f_2(w_{i1}^2 y_1^1 + w_{i2}^2 y_2^1 + w_{i3}^2 y_3^1 + b_i^2) = f_2 \left( \sum_{j=1}^3 w_{ij}^2 y_j^1 + b_i^2 \right). \quad (12)$$

This is not just a convenient and compact notation, but also a useful and intuitive way to think about MLPs: The output is calculated by a series of matrix-vector multiplications and vector additions that are used as input to the activation functions. For each operation  $W_l \hat{y}_{l-1}$  we move forward one layer.

**Activation functions.** A property that characterizes a neural network, other than its connectivity, is the choice of activation function(s). As described in, the following restrictions are imposed on an activation function for a FFNN to fulfill the universal approximation theorem

- Non-constant
- Bounded
- Monotonically-increasing
- Continuous

**Activation functions, Logistic and Hyperbolic ones.** The second requirement excludes all linear functions. Furthermore, in a MLP with only linear activation functions, each layer simply performs a linear transformation of its inputs.

Regardless of the number of layers, the output of the NN will be nothing but a linear function of the inputs. Thus we need to introduce some kind of non-linearity to the NN to be able to fit non-linear functions Typical examples are the logistic *Sigmoid*

$$f(x) = \frac{1}{1 + e^{-x}},$$

and the *hyperbolic tangent* function

$$f(x) = \tanh(x)$$

**Relevance.** The *sigmoid* function are more biologically plausible because the output of inactive neurons are zero. Such activation function are called *one-sided*. However, it has been shown that the hyperbolic tangent performs better than the sigmoid for training MLPs. has become the most popular for *deep neural networks*

*"""The sigmoid function (or the logistic curve) is a function that takes any real number, z, and outputs a number (0,1). It is useful in neural networks for assigning weights on a relative scale. The value z is the weighted sum of parameters involved in the learning algorithm."""*

```
import numpy
import matplotlib.pyplot as plt
import math as mt

z = numpy.arange(-5, 5, .1)
sigma_fn = numpy.vectorize(lambda z: 1/(1+numpy.exp(-z)))
sigma = sigma_fn(z)

fig = plt.figure()
ax = fig.add_subplot(111)
ax.plot(z, sigma)
ax.set_ylim([-0.1, 1.1])
```

```

ax.set_xlim([-5,5])
ax.grid(True)
ax.set_xlabel('z')
ax.set_title('sigmoid function')

plt.show()

"""Step Function"""
z = numpy.arange(-5, 5, .02)
step_fn = numpy.vectorize(lambda z: 1.0 if z >= 0.0 else 0.0)
step = step_fn(z)

fig = plt.figure()
ax = fig.add_subplot(111)
ax.plot(z, step)
ax.set_ylim([-0.5, 1.5])
ax.set_xlim([-5,5])
ax.grid(True)
ax.set_xlabel('z')
ax.set_title('step function')

plt.show()

"""Sine Function"""
z = numpy.arange(-2*mt.pi, 2*mt.pi, 0.1)
t = numpy.sin(z)

fig = plt.figure()
ax = fig.add_subplot(111)
ax.plot(z, t)
ax.set_ylim([-1.0, 1.0])
ax.set_xlim([-2*mt.pi, 2*mt.pi])
ax.grid(True)
ax.set_xlabel('z')
ax.set_title('sine function')

plt.show()

"""Plots a graph of the squashing function used by a rectified linear
unit"""
z = numpy.arange(-2, 2, .1)
zero = numpy.zeros(len(z))
y = numpy.max([zero, z], axis=0)

fig = plt.figure()
ax = fig.add_subplot(111)
ax.plot(z, y)
ax.set_ylim([-2.0, 2.0])
ax.set_xlim([-2.0, 2.0])
ax.grid(True)
ax.set_xlabel('z')
ax.set_title('Rectified linear unit')

plt.show()

```

## The multilayer perceptron (MLP)

The multilayer perceptron is a very popular, and easy to implement approach, to deep learning. It consists of

1. A neural network with one or more layers of nodes between the input and the output nodes.
2. The multilayer network structure, or architecture, or topology, consists of an input layer, one or more hidden layers, and one output layer.
3. The input nodes pass values to the first hidden layer, its nodes pass the information on to the second and so on till we reach the output layer.

As a convention it is normal to call a network with one layer of input units, one layer of hidden units and one layer of output units as a two-layer network. A network with two layers of hidden units is called a three-layer network etc etc.

For an MLP network there is no direct connection between the output nodes/neurons/units and the input nodes/neurons/units. Hereafter we will call the various entities of a layer for nodes. There are also no connections within a single layer.

The number of input nodes does not need to equal the number of output nodes. This applies also to the hidden layers. Each layer may have its own number of nodes and activation functions.

The hidden layers have their name from the fact that they are not linked to observables and as we will see below when we define the so-called activation  $\hat{z}$ , we can think of this as a basis expansion of the original inputs  $\hat{x}$ . The difference however between neural networks and say linear regression is that now these basis functions (which will correspond to the weights in the network) are learned from data. This results in an important difference between neural networks and deep learning approaches on one side and methods like logistic regression or linear regression and their modifications on the other side.

## From one to many layers, the universal approximation theorem

A neural network with only one layer, what we called the simple perceptron, is best suited if we have a standard binary model with clear (linear) boundaries between the outcomes. As such it could equally well be replaced by standard linear regression or logistic regression. Networks with one or more hidden layers approximate systems with more complex boundaries.

As stated earlier, an important theorem in studies of neural networks, restated without proof here, is the [universal approximation theorem](#).

It states that a feed-forward network with a single hidden layer containing a finite number of neurons can approximate continuous functions on compact subsets of real functions. The theorem thus states that simple neural networks can represent a wide variety of interesting functions when given appropriate parameters. It is the multilayer feedforward architecture itself which gives neural networks the potential of being universal approximators.

## Deriving the back propagation code for a multilayer perceptron model

As we have seen now in a feed forward network, we can express the final output of our network in terms of basic matrix-vector multiplications. The unknown quantities are our weights  $w_{ij}$  and we need to find an algorithm for changing them so that our errors are as small as possible. This leads us to the famous [back propagation algorithm](#).

The questions we want to ask are how do changes in the biases and the weights in our network change the cost function and how can we use the final output to modify the weights?

To derive these equations let us start with a plain regression problem and define our cost function as

$$\mathcal{C}(\hat{W}) = \frac{1}{2} \sum_{i=1}^n (y_i - t_i)^2,$$

where the  $t_i$ s are our  $n$  targets (the values we want to reproduce), while the outputs of the network after having propagated all inputs  $\hat{x}$  are given by  $y_i$ . Below we will demonstrate how the basic equations arising from the back propagation algorithm can be modified in order to study classification problems with  $K$  classes.

### Definitions

With our definition of the targets  $\hat{t}$ , the outputs of the network  $\hat{y}$  and the inputs  $\hat{x}$  we define now the activation  $z_j^l$  of node/neuron/unit  $j$  of the  $l$ -th layer as a function of the bias, the weights which add up from the previous layer  $l-1$  and the forward passes/outputs  $\hat{a}^{l-1}$  from the previous layer as

$$z_j^l = \sum_{i=1}^{M_{l-1}} w_{ij}^l a_i^{l-1} + b_j^l,$$

where  $b_k^l$  are the biases from layer  $l$ . Here  $M_{l-1}$  represents the total number of nodes/neurons/units of layer  $l-1$ . The figure here illustrates this equation. We can rewrite this in a more compact form as the matrix-vector products we discussed earlier,

$$\hat{z}^l = \left(\hat{W}^l\right)^T \hat{a}^{l-1} + \hat{b}^l.$$

With the activation values  $\hat{z}^l$  we can in turn define the output of layer  $l$  as  $\hat{a}^l = f(\hat{z}^l)$  where  $f$  is our activation function. In the examples here we will use the sigmoid function discussed in our logistic regression lectures. We will also use the same activation function  $f$  for all layers and their nodes. It means we have

$$a_j^l = f(z_j^l) = \frac{1}{1 + \exp(-(z_j^l))}.$$

## Derivatives and the chain rule

From the definition of the activation  $z_j^l$  we have

$$\frac{\partial z_j^l}{\partial w_{ij}^l} = a_i^{l-1},$$

and

$$\frac{\partial z_j^l}{\partial a_i^{l-1}} = w_{ji}^l.$$

With our definition of the activation function we have that (note that this function depends only on  $z_j^l$ )

$$\frac{\partial a_j^l}{\partial z_j^l} = a_j^l(1 - a_j^l) = f(z_j^l)(1 - f(z_j^l)).$$

## Derivative of the cost function

With these definitions we can now compute the derivative of the cost function in terms of the weights.

Let us specialize to the output layer  $l = L$ . Our cost function is

$$\mathcal{C}(\hat{W}^L) = \frac{1}{2} \sum_{i=1}^n (y_i - t_i)^2 = \frac{1}{2} \sum_{i=1}^n (a_i^L - t_i)^2,$$

The derivative of this function with respect to the weights is

$$\frac{\partial \mathcal{C}(\hat{W}^L)}{\partial w_{jk}^L} = (a_j^L - t_j) \frac{\partial a_j^L}{\partial w_{jk}^L},$$

The last partial derivative can easily be computed and reads (by applying the chain rule)

$$\frac{\partial a_j^L}{\partial w_{jk}^L} = \frac{\partial a_j^L}{\partial z_j^L} \frac{\partial z_j^L}{\partial w_{jk}^L} = a_j^L(1 - a_j^L) a_k^{L-1},$$

## Bringing it together, first back propagation equation

We have thus

$$\frac{\partial \mathcal{C}(\hat{W}^L)}{\partial w_{jk}^L} = (a_j^L - t_j) a_j^L(1 - a_j^L) a_k^{L-1},$$

Defining

$$\delta_j^L = a_j^L(1 - a_j^L)(a_j^L - t_j) = f'(z_j^L) \frac{\partial \mathcal{C}}{\partial(a_j^L)},$$

and using the Hadamard product of two vectors we can write this as

$$\hat{\delta}^L = f'(\hat{z}^L) \circ \frac{\partial \mathcal{C}}{\partial(\hat{a}^L)}.$$

This is an important expression. The second term on the right handside measures how fast the cost function is changing as a function of the  $j$ th output activation. If, for example, the cost function doesn't depend much on a particular output node  $j$ , then  $\delta_j^L$  will be small, which is what we would expect. The first term on the right, measures how fast the activation function  $f$  is changing at a given activation value  $z_j^L$ .

Notice that everything in the above equations is easily computed. In particular, we compute  $z_j^L$  while computing the behaviour of the network, and it is only a small additional overhead to compute  $f'(z_j^L)$ . The exact form of the derivative with respect to the output depends on the form of the cost function. However, provided the cost function is known there should be little trouble in calculating

$$\frac{\partial \mathcal{C}}{\partial(a_j^L)}$$

With the definition of  $\delta_j^L$  we have a more compact definition of the derivative of the cost function in terms of the weights, namely

$$\frac{\partial \mathcal{C}(\hat{W}^L)}{\partial w_{jk}^L} = \delta_j^L a_k^{L-1}.$$

## Derivatives in terms of $z_j^L$

It is also easy to see that our previous equation can be written as

$$\delta_j^L = \frac{\partial \mathcal{C}}{\partial z_j^L} = \frac{\partial \mathcal{C}}{\partial a_j^L} \frac{\partial a_j^L}{\partial z_j^L},$$

which can also be interpreted as the partial derivative of the cost function with respect to the biases  $b_j^L$ , namely

$$\delta_j^L = \frac{\partial \mathcal{C}}{\partial b_j^L} \frac{\partial b_j^L}{\partial z_j^L} = \frac{\partial \mathcal{C}}{\partial b_j^L},$$

That is, the error  $\delta_j^L$  is exactly equal to the rate of change of the cost function as a function of the bias.

## Bringing it together

We have now three equations that are essential for the computations of the derivatives of the cost function at the output layer. These equations are needed to start the algorithm and they are

**The starting equations.**

$$\frac{\partial \mathcal{C}(\hat{W}^L)}{\partial w_{jk}^L} = \delta_j^L a_k^{L-1}, \quad (13)$$

and

$$\delta_j^L = f'(z_j^L) \frac{\partial \mathcal{C}}{\partial (a_j^L)}, \quad (14)$$

and

$$\delta_j^L = \frac{\partial \mathcal{C}}{\partial b_j^L}, \quad (15)$$

An interesting consequence of the above equations is that when the activation  $a_k^{L-1}$  is small, the gradient term, that is the derivative of the cost function with respect to the weights, will also tend to be small. We say then that the weight learns slowly, meaning that it changes slowly when we minimize the weights via say gradient descent. In this case we say the system learns slowly.

Another interesting feature is that is when the activation function, represented by the sigmoid function here, is rather flat when we move towards its end values 0 and 1 (see the above Python codes). In these cases, the derivatives of the activation function will also be close to zero, meaning again that the gradients will be small and the network learns slowly again.

We need a fourth equation and we are set. We are going to propagate backwards in order to determine the weights and biases. In order to do so we need to represent the error in the layer before the final one  $L - 1$  in terms of the errors in the final output layer.

## Final back propagating equation

We have that (replacing  $L$  with a general layer  $l$ )

$$\delta_j^l = \frac{\partial \mathcal{C}}{\partial z_j^l}.$$

We want to express this in terms of the equations for layer  $l + 1$ . Using the chain rule and summing over all  $k$  entries we have

$$\delta_j^l = \sum_k \frac{\partial \mathcal{C}}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} = \sum_k \delta_k^{l+1} \frac{\partial z_k^{l+1}}{\partial z_j^l},$$



and recalling that

$$z_j^{l+1} = \sum_{i=1}^{M_l} w_{ij}^{l+1} a_i^l + b_j^{l+1},$$

with  $M_l$  being the number of nodes in layer  $l$ , we obtain

$$\delta_j^l = \sum_k \delta_k^{l+1} w_{kj}^{l+1} f'(z_j^l),$$

This is our final equation.

We are now ready to set up the algorithm for back propagation and learning the weights and biases.

### Setting up the Back propagation algorithm

The four equations provide us with a way of computing the gradient of the cost function. Let us write this out in the form of an algorithm.

First, we set up the input data  $\hat{x}$  and the activations  $\hat{z}_1$  of the input layer and compute the activation function and the pertinent outputs  $\hat{a}^1$ .

Secondly, we perform then the feed forward till we reach the output layer and compute all  $\hat{z}_l$  of the input layer and compute the activation function and the pertinent outputs  $\hat{a}^l$  for  $l = 2, 3, \dots, L$ .

Thereafter we compute the output error  $\delta^L$  by computing all

$$\delta_j^L = f'(z_j^L) \frac{\partial \mathcal{C}}{\partial (a_j^L)}.$$

Then we compute the back propagate error for each  $l = L - 1, L - 2, \dots, 2$  as

$$\delta_j^l = \sum_k \delta_k^{l+1} w_{kj}^{l+1} f'(z_j^l).$$

Finally, we update the weights and the biases using gradient descent for each  $l = L - 1, L - 2, \dots, 2$  and update the weights and biases according to the rules

$$w_{jk}^l \leftarrow w_{jk}^l - \eta \delta_j^l a_k^{l-1},$$

$$b_j^l \leftarrow b_j^l - \eta \frac{\partial \mathcal{C}}{\partial b_j^l} = b_j^l - \eta \delta_j^l,$$

The parameter  $\eta$  is the learning parameter discussed in connection with the gradient descent methods. Here it is convenient to use stochastic gradient descent (see the examples below) with mini-batches with an outer loop that steps through multiple epochs of training.

## Setting up the Back propagation algorithm

We have the following equations:

First, we set up the input data  $\mathbf{x}$  and the activations  $\mathbf{z}_1$  of the input layer and compute the activation function and the pertinent outputs  $\mathbf{a}^1$ .

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## Setting up a Multi-layer perceptron model for classification

We are now going to develop an example based on the MNIST data base. This is a classification problem and we need to use our cross-entropy function we discussed in connection with logistic regression. The cross-entropy defines our cost function for the classification problems with neural networks.

In binary classification with two classes  $(0, 1)$  we define the logistic/sigmoid function as the probability that a particular input is in class 0 or 1. This is possible because the logistic function takes any input from the real numbers and inputs a number between 0 and 1, and can therefore be interpreted as a probability. It also has other nice properties, such as a derivative that is simple to calculate.

For an input  $\mathbf{a}$  from the hidden layer, the probability that the input  $\mathbf{x}$  is in class 0 or 1 is just. We let  $\theta$  represent the unknown weights and biases to be adjusted by our equations). The variable  $x$  represents our activation values  $z$ . We have

$$P(y = 0 \mid \mathbf{x}, \theta) = \frac{1}{1 + \exp(-\mathbf{x})},$$

and

$$P(y = 1 \mid \mathbf{x}, \theta) = 1 - P(y = 0 \mid \mathbf{x}, \theta),$$

where  $y \in \{0, 1\}$  and  $\theta$  represents the weights and biases of our network.

## Defining the cost function

Our cost function is given as (see the Logistic regression lectures)

$$\mathcal{C}(\theta) = -\ln P(\mathcal{D} \mid \theta) = -\sum_{i=1}^n y_i \ln[P(y_i = 0)] + (1 - y_i) \ln[1 - P(y_i = 0)] = \sum_{i=1}^n \mathcal{L}_i(\theta).$$

This last equality means that we can interpret our *cost* function as a sum over the *loss* function for each point in the dataset  $\mathcal{L}_i(\theta)$ . The negative sign is just so that we can think about our algorithm as minimizing a positive number, rather than maximizing a negative number.

In *multiclass* classification it is common to treat each integer label as a so called *one-hot* vector:

$$y = 5 \rightarrow \mathbf{y} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0), \text{ and}$$

$$y = 1 \rightarrow \mathbf{y} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0),$$

i.e. a binary bit string of length  $C$ , where  $C = 10$  is the number of classes in the MNIST dataset (numbers from 0 to 9)..

If  $\mathbf{x}_i$  is the  $i$ -th input (image),  $y_{ic}$  refers to the  $c$ -th component of the  $i$ -th output vector  $\mathbf{y}_i$ . The probability of  $\mathbf{x}_i$  being in class  $c$  will be given by the softmax function:

$$P(y_{ic} = 1 \mid \mathbf{x}_i, \theta) = \frac{\exp((\mathbf{a}_i^{hidden})^T \mathbf{w}_c)}{\sum_{c'=0}^{C-1} \exp((\mathbf{a}_i^{hidden})^T \mathbf{w}_{c'})},$$

which reduces to the logistic function in the binary case. The likelihood of this  $C$ -class classifier is now given as:

$$P(\mathcal{D} \mid \theta) = \prod_{i=1}^n \prod_{c=0}^{C-1} [P(y_{ic} = 1)]^{y_{ic}}.$$

Again we take the negative log-likelihood to define our cost function:

$$\mathcal{C}(\theta) = -\log P(\mathcal{D} \mid \theta).$$

See the logistic regression lectures for a full definition of the cost function.

The back propagation equations need now only a small change, namely the definition of a new cost function. We are thus ready to use the same equations as before!

### Example: binary classification problem

As an example of the above, relevant for project 2 as well, let us consider a binary class. As discussed in our logistic regression lectures, we defined a cost function in terms of the parameters  $\beta$  as

$$\mathcal{C}(\beta) = - \sum_{i=1}^n (y_i \log p(y_i|x_i, \beta) + (1 - y_i) \log 1 - p(y_i|x_i, \beta)),$$

where we had defined the logistic (sigmoid) function

$$p(y_i = 1|x_i, \beta) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)},$$

and

$$p(y_i = 0|x_i, \beta) = 1 - p(y_i = 1|x_i, \beta).$$

The parameters  $\beta$  were defined using a minimization method like gradient descent or Newton-Raphson's method.

Now we replace  $x_i$  with the activation  $z_i^l$  for a given layer  $l$  and the outputs as  $y_i = a_i^l = f(z_i^l)$ , with  $z_i^l$  now being a function of the weights  $w_{ij}^l$  and biases  $b_i^l$ . We have then

$$a_i^l = y_i = \frac{\exp(z_i^l)}{1 + \exp(z_i^l)},$$

with

$$z_i^l = \sum_j w_{ij}^l a_j^{l-1} + b_i^l,$$

where the superscript  $l - 1$  indicates that these are the outputs from layer  $l - 1$ . Our cost function at the final layer  $l = L$  is now

$$\mathcal{C}(\mathbf{W}) = - \sum_{i=1}^n (t_i \log a_i^L + (1 - t_i) \log (1 - a_i^L)),$$

where we have defined the targets  $t_i$ . The derivatives of the cost function with respect to the output  $a_i^L$  are then easily calculated and we get

$$\frac{\partial \mathcal{C}(\mathbf{W})}{\partial a_i^L} = \frac{a_i^L - t_i}{a_i^L (1 - a_i^L)}.$$

In case we use another activation function than the logistic one, we need to evaluate other derivatives.

### The Softmax function

In case we employ the more general case given by the Softmax equation, we need to evaluate the derivative of the activation function with respect to the activation  $z_i^l$ , that is we need

$$\frac{\partial f(z_i^l)}{\partial w_{jk}^l} = \frac{\partial f(z_i^l)}{\partial z_j^l} \frac{\partial z_j^l}{\partial w_{jk}^l} = \frac{\partial f(z_i^l)}{\partial z_j^l} a_k^{l-1}.$$

For the Softmax function we have

$$f(z_i^l) = \frac{\exp(z_i^l)}{\sum_{m=1}^K \exp(z_m^l)}.$$

Its derivative with respect to  $z_j^l$  gives

$$\frac{\partial f(z_i^l)}{\partial z_j^l} = f(z_i^l) (\delta_{ij} - f(z_j^l)),$$

which in case of the simply binary model reduces to having  $i = j$ .

## Developing a code for doing neural networks with back propagation

One can identify a set of key steps when using neural networks to solve supervised learning problems:

1. Collect and pre-process data
2. Define model and architecture
3. Choose cost function and optimizer
4. Train the model
5. Evaluate model performance on test data
6. Adjust hyperparameters (if necessary, network architecture)

## Collect and pre-process data

Here we will be using the MNIST dataset, which is readily available through the **scikit-learn** package. You may also find it for example [here](#). The *MNIST* (Modified National Institute of Standards and Technology) database is a large database of handwritten digits that is commonly used for training various image processing systems. The MNIST dataset consists of 70 000 images of size  $28 \times 28$  pixels, each labeled from 0 to 9. The scikit-learn dataset we will use consists of a selection of 1797 images of size  $8 \times 8$  collected and processed from this database.

To feed data into a feed-forward neural network we need to represent the inputs as a design/feature matrix  $X = (n_{inputs}, n_{features})$ . Each row represents an *input*, in this case a handwritten digit, and each column represents a *feature*, in this case a pixel. The correct answers, also known as *labels* or *targets* are represented as a 1D array of integers  $Y = (n_{inputs}) = (5, 3, 1, 8, \dots)$ .

As an example, say we want to build a neural network using supervised learning to predict Body-Mass Index (BMI) from measurements of height (in m) and weight (in kg). If we have measurements of 5 people the design/feature matrix could be for example:

$$X = \begin{bmatrix} 1.85 & 81 \\ 1.71 & 65 \\ 1.95 & 103 \\ 1.55 & 42 \\ 1.63 & 56 \end{bmatrix},$$

and the targets would be:

$$Y = (23.7, 22.2, 27.1, 17.5, 21.1)$$

Since each input image is a 2D matrix, we need to flatten the image (i.e. "unravel" the 2D matrix into a 1D array) to turn the data into a design/feature matrix. This means we lose all spatial information in the image, such as locality and translational invariance. More complicated architectures such as Convolutional Neural Networks can take advantage of such information, and are most commonly applied when analyzing images.

```
# import necessary packages
import numpy as np
import matplotlib.pyplot as plt
from sklearn import datasets

# ensure the same random numbers appear every time
np.random.seed(0)

# display images in notebook
%matplotlib inline
plt.rcParams['figure.figsize'] = (12,12)

# download MNIST dataset
digits = datasets.load_digits()

# define inputs and labels
inputs = digits.images
labels = digits.target

print("inputs = (n_inputs, pixel_width, pixel_height) = " + str(inputs.shape))
print("labels = (n_inputs) = " + str(labels.shape))

# flatten the image
# the value -1 means dimension is inferred from the remaining dimensions: 8x8 = 64
n_inputs = len(inputs)
inputs = inputs.reshape(n_inputs, -1)
print("X = (n_inputs, n_features) = " + str(inputs.shape))

# choose some random images to display
indices = np.arange(n_inputs)
random_indices = np.random.choice(indices, size=5)

for i, image in enumerate(digits.images[random_indices]):
    plt.subplot(1, 5, i+1)
    plt.axis('off')
```

```
plt.imshow(image, cmap=plt.cm.gray_r, interpolation='nearest')
plt.title("Label: %d" % digits.target[random_indices[i]])
plt.show()
```

## Train and test datasets

Performing analysis before partitioning the dataset is a major error, that can lead to incorrect conclusions.

We will reserve 80% of our dataset for training and 20% for testing.

It is important that the train and test datasets are drawn randomly from our dataset, to ensure no bias in the sampling. Say you are taking measurements of weather data to predict the weather in the coming 5 days. You don't want to train your model on measurements taken from the hours 00.00 to 12.00, and then test it on data collected from 12.00 to 24.00.

```
from sklearn.model_selection import train_test_split

# one-liner from scikit-learn library
train_size = 0.8
test_size = 1 - train_size
X_train, X_test, Y_train, Y_test = train_test_split(inputs, labels, train_size=train_size,
                                                    test_size=test_size)

# equivalently in numpy
def train_test_split_numpy(inputs, labels, train_size, test_size):
    n_inputs = len(inputs)
    inputs_shuffled = inputs.copy()
    labels_shuffled = labels.copy()

    np.random.shuffle(inputs_shuffled)
    np.random.shuffle(labels_shuffled)

    train_end = int(n_inputs*train_size)
    X_train, X_test = inputs_shuffled[:train_end], inputs_shuffled[train_end:]
    Y_train, Y_test = labels_shuffled[:train_end], labels_shuffled[train_end:]

    return X_train, X_test, Y_train, Y_test

#X_train, X_test, Y_train, Y_test = train_test_split_numpy(inputs, labels, train_size, test_size)

print("Number of training images: " + str(len(X_train)))
print("Number of test images: " + str(len(X_test)))
```

## Define model and architecture

Our simple feed-forward neural network will consist of an *input* layer, a single *hidden* layer and an *output* layer. The activation  $y$  of each neuron is a weighted sum of inputs, passed through an activation function. In case of the simple perceptron model we have

$$z = \sum_{i=1}^n w_i a_i,$$

$$y = f(z),$$

where  $f$  is the activation function,  $a_i$  represents input from neuron  $i$  in the preceding layer and  $w_i$  is the weight to input  $i$ . The activation of the neurons in the input layer is just the features (e.g. a pixel value).

The simplest activation function for a neuron is the *Heaviside* function:

$$f(z) = \begin{cases} 1, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

A feed-forward neural network with this activation is known as a *perceptron*. For a binary classifier (i.e. two classes, 0 or 1, dog or not-dog) we can also use this in our output layer. This activation can be generalized to  $k$  classes (using e.g. the *one-against-all* strategy), and we call these architectures *multiclass perceptrons*.

However, it is now common to use the terms Single Layer Perceptron (SLP) (1 hidden layer) and Multilayer Perceptron (MLP) (2 or more hidden layers) to refer to feed-forward neural networks with any activation function.

Typical choices for activation functions include the sigmoid function, hyperbolic tangent, and Rectified Linear Unit (ReLU). We will be using the sigmoid function  $\sigma(x)$ :

$$f(x) = \sigma(x) = \frac{1}{1 + e^{-x}},$$

which is inspired by probability theory (see logistic regression) and was most commonly used until about 2011. See the discussion below concerning other activation functions.

## Layers

- Input

Since each input image has  $8 \times 8 = 64$  pixels or features, we have an input layer of 64 neurons.

- Hidden layer

We will use 50 neurons in the hidden layer receiving input from the neurons in the input layer. Since each neuron in the hidden layer is connected to the 64 inputs we have  $64 \times 50 = 3200$  weights to the hidden layer.

- Output

If we were building a binary classifier, it would be sufficient with a single neuron in the output layer, which could output 0 or 1 according to the Heaviside function. This would be an example of a *hard* classifier, meaning it outputs the class of the input directly. However, if we are dealing with noisy data it is often beneficial to use a *soft* classifier, which outputs the probability of being in class 0 or 1.

For a soft binary classifier, we could use a single neuron and interpret the output as either being the probability of being in class 0 or the probability of



being in class 1. Alternatively we could use 2 neurons, and interpret each neuron as the probability of being in each class.

Since we are doing multiclass classification, with 10 categories, it is natural to use 10 neurons in the output layer. We number the neurons  $j = 0, 1, \dots, 9$ . The activation of each output neuron  $j$  will be according to the *softmax* function:

$$P(\text{class } j \mid \text{input } \mathbf{a}) = \frac{\exp(\mathbf{a}^T \mathbf{w}_j)}{\sum_{c=0}^9 \exp(\mathbf{a}^T \mathbf{w}_c)},$$

i.e. each neuron  $j$  outputs the probability of being in class  $j$  given an input from the hidden layer  $\mathbf{a}$ , with  $\mathbf{w}_j$  the weights of neuron  $j$  to the inputs. The denominator is a normalization factor to ensure the outputs (probabilities) sum up to 1. The exponent is just the weighted sum of inputs as before:

$$z_j = \sum_{i=1}^n w_{ij} a_i + b_j.$$

Since each neuron in the output layer is connected to the 50 inputs from the hidden layer we have  $50 \times 10 = 500$  weights to the output layer.

## Weights and biases

Typically weights are initialized with small values distributed around zero, drawn from a uniform or normal distribution. Setting all weights to zero means all neurons give the same output, making the network useless.

Adding a bias value to the weighted sum of inputs allows the neural network to represent a greater range of values. Without it, any input with the value 0 will be mapped to zero (before being passed through the activation). The bias unit has an output of 1, and a weight to each neuron  $j$ ,  $b_j$ :

$$z_j = \sum_{i=1}^n w_{ij} a_i + b_j.$$

The bias weights  $\mathbf{b}$  are often initialized to zero, but a small value like 0.01 ensures all neurons have some output which can be backpropagated in the first training cycle.

```
# building our neural network

n_inputs, n_features = X_train.shape
n_hidden_neurons = 50
n_categories = 10

# we make the weights normally distributed using numpy.random.randn

# weights and bias in the hidden layer
hidden_weights = np.random.randn(n_features, n_hidden_neurons)
hidden_bias = np.zeros(n_hidden_neurons) + 0.01

# weights and bias in the output layer
output_weights = np.random.randn(n_hidden_neurons, n_categories)
output_bias = np.zeros(n_categories) + 0.01
```

## Feed-forward pass

Denote  $F$  the number of features,  $H$  the number of hidden neurons and  $C$  the number of categories. For each input image we calculate a weighted sum of input features (pixel values) to each neuron  $j$  in the hidden layer  $l$ :

$$z_j^l = \sum_{i=1}^F w_{ij}^l x_i + b_j^l,$$

this is then passed through our activation function

$$a_j^l = f(z_j^l).$$

We calculate a weighted sum of inputs (activations in the hidden layer) to each neuron  $j$  in the output layer:

$$z_j^L = \sum_{i=1}^H w_{ij}^L a_i^l + b_j^L.$$

Finally we calculate the output of neuron  $j$  in the output layer using the softmax function:

$$a_j^L = \frac{\exp(z_j^L)}{\sum_{c=0}^{C-1} \exp(z_c^L)}.$$

## Matrix multiplications

Since our data has the dimensions  $X = (n_{inputs}, n_{features})$  and our weights to the hidden layer have the dimensions  $W_{hidden} = (n_{features}, n_{hidden})$ , we can easily feed the network all our training data in one go by taking the matrix product

$$XW^h = (n_{inputs}, n_{hidden}),$$

and obtain a matrix that holds the weighted sum of inputs to the hidden layer for each input image and each hidden neuron. We also add the bias to obtain a matrix of weighted sums to the hidden layer  $Z^h$ :

$$\mathbf{z}^l = \mathbf{X}\mathbf{W}^l + \mathbf{b}^l,$$

meaning the same bias (1D array with size equal number of hidden neurons) is added to each input image. This is then passed through the activation:

$$\mathbf{a}^l = f(\mathbf{z}^l).$$

This is fed to the output layer:

$$\mathbf{z}^L = \mathbf{a}^L \mathbf{W}^L + \mathbf{b}^L.$$

Finally we receive our output values for each image and each category by passing it through the softmax function:

$$\text{output} = \text{softmax}(\mathbf{z}^L) = (n_{\text{inputs}}, n_{\text{categories}}).$$

```
# setup the feed-forward pass, subscript h = hidden layer

def sigmoid(x):
    return 1/(1 + np.exp(-x))

def feed_forward(X):
    # weighted sum of inputs to the hidden layer
    z_h = np.matmul(X, hidden_weights) + hidden_bias
    # activation in the hidden layer
    a_h = sigmoid(z_h)

    # weighted sum of inputs to the output layer
    z_o = np.matmul(a_h, output_weights) + output_bias
    # softmax output
    # axis 0 holds each input and axis 1 the probabilities of each category
    exp_term = np.exp(z_o)
    probabilities = exp_term / np.sum(exp_term, axis=1, keepdims=True)

    return probabilities

probabilities = feed_forward(X_train)
print("probabilities = (n_inputs, n_categories) = " + str(probabilities.shape))
print("probability that image 0 is in category 0,1,2,...,9 = \n" + str(probabilities[0]))
print("probabilities sum up to: " + str(probabilities[0].sum()))
print()

# we obtain a prediction by taking the class with the highest likelihood
def predict(X):
    probabilities = feed_forward(X)
    return np.argmax(probabilities, axis=1)

predictions = predict(X_train)
print("predictions = (n_inputs) = " + str(predictions.shape))
print("prediction for image 0: " + str(predictions[0]))
print("correct label for image 0: " + str(Y_train[0]))
```

## Choose cost function and optimizer

To measure how well our neural network is doing we need to introduce a cost function. We will call the function that gives the error of a single sample output the *loss* function, and the function that gives the total error of our network across all samples the *cost* function. A typical choice for multiclass classification is the *cross-entropy* loss, also known as the negative log likelihood.

In *multiclass* classification it is common to treat each integer label as a so called *one-hot* vector:

$$y = 5 \quad \rightarrow \quad \mathbf{y} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0),$$

$$y = 1 \quad \rightarrow \quad \mathbf{y} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0),$$

i.e. a binary bit string of length  $C$ , where  $C = 10$  is the number of classes in the MNIST dataset.

Let  $y_{ic}$  denote the  $c$ -th component of the  $i$ -th one-hot vector. We define the cost function  $\mathcal{C}$  as a sum over the cross-entropy loss for each point  $\mathbf{x}_i$  in the dataset.

In the one-hot representation only one of the terms in the loss function is non-zero, namely the probability of the correct category  $c'$  (i.e. the category  $c'$  such that  $y_{ic'} = 1$ ). This means that the cross entropy loss only punishes you for how wrong you got the correct label. The probability of category  $c$  is given by the softmax function. The vector  $\boldsymbol{\theta}$  represents the parameters of our network, i.e. all the weights and biases.

## Optimizing the cost function

The network is trained by finding the weights and biases that minimize the cost function. One of the most widely used classes of methods is *gradient descent* and its generalizations. The idea behind gradient descent is simply to adjust the weights in the direction where the gradient of the cost function is large and negative. This ensures we flow toward a *local* minimum of the cost function. Each parameter  $\theta$  is iteratively adjusted according to the rule

$$\theta_{i+1} = \theta_i - \eta \nabla \mathcal{C}(\theta_i),$$

where  $\eta$  is known as the *learning rate*, which controls how big a step we take towards the minimum. This update can be repeated for any number of iterations, or until we are satisfied with the result.

A simple and effective improvement is a variant called *Batch Gradient Descent*. Instead of calculating the gradient on the whole dataset, we calculate an approximation of the gradient on a subset of the data called a *minibatch*. If there are  $N$  data points and we have a minibatch size of  $M$ , the total number of batches is  $N/M$ . We denote each minibatch  $B_k$ , with  $k = 1, 2, \dots, N/M$ . The gradient then becomes:

$$\nabla \mathcal{C}(\theta) = \frac{1}{N} \sum_{i=1}^N \nabla \mathcal{L}_i(\theta) \quad \rightarrow \quad \frac{1}{M} \sum_{i \in B_k} \nabla \mathcal{L}_i(\theta),$$

i.e. instead of averaging the loss over the entire dataset, we average over a minibatch.

This has two important benefits:

1. Introducing stochasticity decreases the chance that the algorithm becomes stuck in a local minima.
2. It significantly speeds up the calculation, since we do not have to use the entire dataset to calculate the gradient.

The various optimization methods, with codes and algorithms, are discussed in our lectures on [Gradient descent approaches](#).

## Regularization

It is common to add an extra term to the cost function, proportional to the size of the weights. This is equivalent to constraining the size of the weights, so that they do not grow out of control. Constraining the size of the weights means that the weights cannot grow arbitrarily large to fit the training data, and in this way reduces *overfitting*.

We will measure the size of the weights using the so called *L2-norm*, meaning our cost function becomes:

$$\mathcal{C}(\theta) = \frac{1}{N} \sum_{i=1}^N \mathcal{L}_i(\theta) \quad \rightarrow \quad \frac{1}{N} \sum_{i=1}^N \mathcal{L}_i(\theta) + \lambda \|\mathbf{w}\|_2^2 = \frac{1}{N} \sum_{i=1}^N \mathcal{L}(\theta) + \lambda \sum_{ij} w_{ij}^2,$$

i.e. we sum up all the weights squared. The factor  $\lambda$  is known as a regularization parameter.

In order to train the model, we need to calculate the derivative of the cost function with respect to every bias and weight in the network. In total our network has  $(64+1) \times 50 = 3250$  weights in the hidden layer and  $(50+1) \times 10 = 510$  weights to the output layer (+1 for the bias), and the gradient must be calculated for every parameter. We use the *backpropagation* algorithm discussed above. This is a clever use of the chain rule that allows us to calculate the gradient efficiently.

## Matrix multiplication

To more efficiently train our network these equations are implemented using matrix operations. The error in the output layer is calculated simply as, with  $\mathbf{t}$  being our targets,

$$\delta_L = \mathbf{t} - \mathbf{y} = (n_{inputs}, n_{categories}).$$

The gradient for the output weights is calculated as

$$\nabla W_L = \mathbf{a}^T \delta_L = (n_{hidden}, n_{categories}),$$

where  $\mathbf{a} = (n_{inputs}, n_{hidden})$ . This simply means that we are summing up the gradients for each input. Since we are going backwards we have to transpose the activation matrix.

The gradient with respect to the output bias is then

$$\nabla \mathbf{b}_L = \sum_{i=1}^{n_{inputs}} \delta_L = (n_{categories}).$$

The error in the hidden layer is

$$\Delta_h = \delta_L W_L^T \circ f'(z_h) = \delta_L W_L^T \circ a_h \circ (1 - a_h) = (n_{inputs}, n_{hidden}),$$

where  $f'(a_h)$  is the derivative of the activation in the hidden layer. The matrix products mean that we are summing up the products for each neuron in the output layer. The symbol  $\circ$  denotes the *Hadamard product*, meaning element-wise multiplication.

This again gives us the gradients in the hidden layer:

$$\nabla W_h = X^T \delta_h = (n_{features}, n_{hidden}),$$

$$\nabla b_h = \sum_{i=1}^{n_{inputs}} \delta_h = (n_{hidden}).$$

```
# to categorical turns our integer vector into a onehot representation
from sklearn.metrics import accuracy_score

# one-hot in numpy
def to_categorical_numpy(integer_vector):
    n_inputs = len(integer_vector)
    n_categories = np.max(integer_vector) + 1
    onehot_vector = np.zeros((n_inputs, n_categories))
    onehot_vector[range(n_inputs), integer_vector] = 1

    return onehot_vector

#Y_train_onehot, Y_test_onehot = to_categorical(Y_train), to_categorical(Y_test)
Y_train_onehot, Y_test_onehot = to_categorical_numpy(Y_train), to_categorical_numpy(Y_test)

def feed_forward_train(X):
    # weighted sum of inputs to the hidden layer
    z_h = np.matmul(X, hidden_weights) + hidden_bias
    # activation in the hidden layer
    a_h = sigmoid(z_h)

    # weighted sum of inputs to the output layer
    z_o = np.matmul(a_h, output_weights) + output_bias
    # softmax output
    # axis 0 holds each input and axis 1 the probabilities of each category
    exp_term = np.exp(z_o)
    probabilities = exp_term / np.sum(exp_term, axis=1, keepdims=True)

    # for backpropagation need activations in hidden and output layers
    return a_h, probabilities

def backpropagation(X, Y):
    a_h, probabilities = feed_forward_train(X)

    # error in the output layer
    error_output = probabilities - Y
    # error in the hidden layer
    error_hidden = np.matmul(error_output, output_weights.T) * a_h * (1 - a_h)

    # gradients for the output layer
    output_weights_gradient = np.matmul(a_h.T, error_output)
    output_bias_gradient = np.sum(error_output, axis=0)

    # gradient for the hidden layer
    hidden_weights_gradient = np.matmul(X.T, error_hidden)
    hidden_bias_gradient = np.sum(error_hidden, axis=0)
```

```

        return output_weights_gradient, output_bias_gradient, hidden_weights_gradient, hidden_bias_gradient

print("Old accuracy on training data: " + str(accuracy_score(predict(X_train), Y_train)))

eta = 0.01
lmbd = 0.01
for i in range(1000):
    # calculate gradients
    dWo, dBo, dWh, dBh = backpropagation(X_train, Y_train_onehot)

    # regularization term gradients
    dWo += lmbd * output_weights
    dWh += lmbd * hidden_weights

    # update weights and biases
    output_weights -= eta * dWo
    output_bias -= eta * dBo
    hidden_weights -= eta * dWh
    hidden_bias -= eta * dBh

print("New accuracy on training data: " + str(accuracy_score(predict(X_train), Y_train)))

```

## Improving performance

As we can see the network does not seem to be learning at all. It seems to be just guessing the label for each image. In order to obtain a network that does something useful, we will have to do a bit more work.

The choice of *hyperparameters* such as learning rate and regularization parameter is hugely influential for the performance of the network. Typically a *grid-search* is performed, wherein we test different hyperparameters separated by orders of magnitude. For example we could test the learning rates  $\eta = 10^{-6}, 10^{-5}, \dots, 10^{-1}$  with different regularization parameters  $\lambda = 10^{-6}, \dots, 10^{-0}$ .

Next, we haven't implemented minibatching yet, which introduces stochasticity and is thought to act as an important regularizer on the weights. We call a feed-forward + backward pass with a minibatch an *iteration*, and a full training period going through the entire dataset ( $n/M$  batches) an *epoch*.

If this does not improve network performance, you may want to consider altering the network architecture, adding more neurons or hidden layers. Andrew Ng goes through some of these considerations in this [video](#). You can find a summary of the video [here](#).

## Full object-oriented implementation

It is very natural to think of the network as an object, with specific instances of the network being realizations of this object with different hyperparameters. An implementation using Python classes provides a clean structure and interface, and the full implementation of our neural network is given below.

```

class NeuralNetwork:
    def __init__(
        self,

```

```

        X_data,
        Y_data,
        n_hidden_neurons=50,
        n_categories=10,
        epochs=10,
        batch_size=100,
        eta=0.1,
        lmbd=0.0):

    self.X_data_full = X_data
    self.Y_data_full = Y_data

    self.n_inputs = X_data.shape[0]
    self.n_features = X_data.shape[1]
    self.n_hidden_neurons = n_hidden_neurons
    self.n_categories = n_categories

    self.epochs = epochs
    self.batch_size = batch_size
    self.iterations = self.n_inputs // self.batch_size
    self.eta = eta
    self.lmbd = lmbd

    self.create_biases_and_weights()

def create_biases_and_weights(self):
    self.hidden_weights = np.random.randn(self.n_features, self.n_hidden_neurons)
    self.hidden_bias = np.zeros(self.n_hidden_neurons) + 0.01

    self.output_weights = np.random.randn(self.n_hidden_neurons, self.n_categories)
    self.output_bias = np.zeros(self.n_categories) + 0.01

def feed_forward(self):
    # feed-forward for training
    self.z_h = np.matmul(self.X_data, self.hidden_weights) + self.hidden_bias
    self.a_h = sigmoid(self.z_h)

    self.z_o = np.matmul(self.a_h, self.output_weights) + self.output_bias

    exp_term = np.exp(self.z_o)
    self.probabilities = exp_term / np.sum(exp_term, axis=1, keepdims=True)

def feed_forward_out(self, X):
    # feed-forward for output
    z_h = np.matmul(X, self.hidden_weights) + self.hidden_bias
    a_h = sigmoid(z_h)

    z_o = np.matmul(a_h, self.output_weights) + self.output_bias

    exp_term = np.exp(z_o)
    probabilities = exp_term / np.sum(exp_term, axis=1, keepdims=True)
    return probabilities

def backpropagation(self):
    error_output = self.probabilities - self.Y_data
    error_hidden = np.matmul(error_output, self.output_weights.T) * self.a_h * (1 - self.a_h)

    self.output_weights_gradient = np.matmul(self.a_h.T, error_output)
    self.output_bias_gradient = np.sum(error_output, axis=0)

    self.hidden_weights_gradient = np.matmul(self.X_data.T, error_hidden)

```



```

self.hidden_bias_gradient = np.sum(error_hidden, axis=0)

if self.lmbd > 0.0:
    self.output_weights_gradient += self.lmbd * self.output_weights
    self.hidden_weights_gradient += self.lmbd * self.hidden_weights

self.output_weights -= self.eta * self.output_weights_gradient
self.output_bias -= self.eta * self.output_bias_gradient
self.hidden_weights -= self.eta * self.hidden_weights_gradient
self.hidden_bias -= self.eta * self.hidden_bias_gradient

def predict(self, X):
    probabilities = self.feed_forward_out(X)
    return np.argmax(probabilities, axis=1)

def predict_probabilities(self, X):
    probabilities = self.feed_forward_out(X)
    return probabilities

def train(self):
    data_indices = np.arange(self.n_inputs)

    for i in range(self.epochs):
        for j in range(self.iterations):
            # pick datapoints with replacement
            chosen_datapoints = np.random.choice(
                data_indices, size=self.batch_size, replace=False
            )

            # minibatch training data
            self.X_data = self.X_data_full[chosen_datapoints]
            self.Y_data = self.Y_data_full[chosen_datapoints]

            self.feed_forward()
            self.backpropagation()

```

## Evaluate model performance on test data

To measure the performance of our network we evaluate how well it does on data it has never seen before, i.e. the test data. We measure the performance of the network using the *accuracy* score. The accuracy is as you would expect just the number of images correctly labeled divided by the total number of images. A perfect classifier will have an accuracy score of 1.

$$\text{Accuracy} = \frac{\sum_{i=1}^n I(\tilde{y}_i = y_i)}{n},$$

where  $I$  is the indicator function, 1 if  $\tilde{y}_i = y_i$  and 0 otherwise.

```

epochs = 100
batch_size = 100

dnn = NeuralNetwork(X_train, Y_train_onehot, eta=eta, lmbd=lmbd, epochs=epochs, batch_size=batch_size,
                    n_hidden_neurons=n_hidden_neurons, n_categories=n_categories)
dnn.train()
test_predict = dnn.predict(X_test)

# accuracy score from scikit library

```

```

print("Accuracy score on test set: ", accuracy_score(Y_test, test_predict))

# equivalent in numpy
def accuracy_score_numpy(Y_test, Y_pred):
    return np.sum(Y_test == Y_pred) / len(Y_test)

#print("Accuracy score on test set: ", accuracy_score_numpy(Y_test, test_predict))

```

## Adjust hyperparameters

We now perform a grid search to find the optimal hyperparameters for the network. Note that we are only using 1 layer with 50 neurons, and human performance is estimated to be around 98% (2% error rate).

```

eta_vals = np.logspace(-5, 1, 7)
lmbd_vals = np.logspace(-5, 1, 7)
# store the models for later use
DNN_numpy = np.zeros((len(eta_vals), len(lmbd_vals)), dtype=object)

# grid search
for i, eta in enumerate(eta_vals):
    for j, lmbd in enumerate(lmbd_vals):
        dnn = NeuralNetwork(X_train, Y_train_onehot, eta=eta, lmbd=lmbd, epochs=epochs, batch_size=batch_size,
                             n_hidden_neurons=n_hidden_neurons, n_categories=n_categories)
        dnn.train()

        DNN_numpy[i][j] = dnn

        test_predict = dnn.predict(X_test)

        print("Learning rate = ", eta)
        print("Lambda = ", lmbd)
        print("Accuracy score on test set: ", accuracy_score(Y_test, test_predict))
        print()

```

## Visualization

```

# visual representation of grid search
# uses seaborn heatmap, you can also do this with matplotlib imshow
import seaborn as sns

sns.set()

train_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))
test_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))

for i in range(len(eta_vals)):
    for j in range(len(lmbd_vals)):
        dnn = DNN_numpy[i][j]

        train_pred = dnn.predict(X_train)
        test_pred = dnn.predict(X_test)

        train_accuracy[i][j] = accuracy_score(Y_train, train_pred)
        test_accuracy[i][j] = accuracy_score(Y_test, test_pred)

fig, ax = plt.subplots(figsize = (10, 10))

```

```

sns.heatmap(train_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Training Accuracy")
ax.set_ylabel(" $\eta$ ")
ax.set_xlabel(" $\lambda$ ")
plt.show()

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(test_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Test Accuracy")
ax.set_ylabel(" $\eta$ ")
ax.set_xlabel(" $\lambda$ ")
plt.show()

```

## scikit-learn implementation

**scikit-learn** focuses more on traditional machine learning methods, such as regression, clustering, decision trees, etc. As such, it has only two types of neural networks: Multi Layer Perceptron outputting continuous values, *MLPRegressor*, and Multi Layer Perceptron outputting labels, *MLPClassifier*. We will see how simple it is to use these classes.

**scikit-learn** implements a few improvements from our neural network, such as early stopping, a varying learning rate, different optimization methods, etc. We would therefore expect a better performance overall.

```

from sklearn.neural_network import MLPClassifier
# store models for later use
DNN_scikit = np.zeros((len(eta_vals), len(lmbd_vals)), dtype=object)

for i, eta in enumerate(eta_vals):
    for j, lmbd in enumerate(lmbd_vals):
        dnn = MLPClassifier(hidden_layer_sizes=(n_hidden_neurons), activation='logistic',
                           alpha=lmbd, learning_rate_init=eta, max_iter=epochs)
        dnn.fit(X_train, Y_train)

        DNN_scikit[i][j] = dnn

    print("Learning rate = ", eta)
    print("Lambda = ", lmbd)
    print("Accuracy score on test set: ", dnn.score(X_test, Y_test))
    print()

```

## Visualization

```

# optional
# visual representation of grid search
# uses seaborn heatmap, could probably do this in matplotlib
import seaborn as sns

sns.set()

train_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))
test_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))

for i in range(len(eta_vals)):
    for j in range(len(lmbd_vals)):
        dnn = DNN_scikit[i][j]

```

```

train_pred = dnn.predict(X_train)
test_pred = dnn.predict(X_test)

train_accuracy[i][j] = accuracy_score(Y_train, train_pred)
test_accuracy[i][j] = accuracy_score(Y_test, test_pred)

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(train_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Training Accuracy")
ax.set_ylabel("$\eta$")
ax.set_xlabel("$\lambda$")
plt.show()

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(test_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Test Accuracy")
ax.set_ylabel("$\eta$")
ax.set_xlabel("$\lambda$")
plt.show()

```

## Testing our code for the XOR, OR and AND gates

Last week we discussed three different types of gates, the so-called XOR, the OR and the AND gates. Their inputs and outputs can be summarized using the following tables, first for the OR gate with inputs  $x_1$  and  $x_2$  and outputs  $y$ :

$x_1$	$x_2$	$y$
0	0	0
0	1	1
1	0	1
1	1	1

## The AND and XOR Gates

The AND gate is defined as

$x_1$	$x_2$	$y$
0	0	0
0	1	0
1	0	0
1	1	1

And finally we have the XOR gate

$x_1$	$x_2$	$y$
0	0	0
0	1	1
1	0	1
1	1	0

## Representing the Data Sets

Our design matrix is defined by the input values  $x_1$  and  $x_2$ . Since we have four possible outputs, our design matrix reads

$$\mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

while the vector of outputs is  $\mathbf{y}^T = [0, 1, 1, 0]$  for the XOR gate,  $\mathbf{y}^T = [0, 0, 0, 1]$  for the AND gate and  $\mathbf{y}^T = [0, 1, 1, 1]$  for the OR gate.

## Setting up the Neural Network

We define first our design matrix and the various output vectors for the different gates.

```
"""
Simple code that tests XOR, OR and AND gates with linear regression
"""

# import necessary packages
import numpy as np
import matplotlib.pyplot as plt
from sklearn import datasets

def sigmoid(x):
    return 1/(1 + np.exp(-x))

def feed_forward(X):
    # weighted sum of inputs to the hidden layer
    z_h = np.matmul(X, hidden_weights) + hidden_bias
    # activation in the hidden layer
    a_h = sigmoid(z_h)

    # weighted sum of inputs to the output layer
    z_o = np.matmul(a_h, output_weights) + output_bias
    # softmax output
    # axis 0 holds each input and axis 1 the probabilities of each category
    probabilities = sigmoid(z_o)
    return probabilities

# we obtain a prediction by taking the class with the highest likelihood
def predict(X):
    probabilities = feed_forward(X)
    return np.argmax(probabilities, axis=1)

# ensure the same random numbers appear every time
np.random.seed(0)

# Design matrix
X = np.array([ [0, 0], [0, 1], [1, 0], [1, 1]], dtype=np.float64)

# The XOR gate
yXOR = np.array([0, 1, 1, 0])
# The OR gate
yOR = np.array([0, 1, 1, 1])
# The AND gate
```

```

yAND = np.array([ 0, 0, 0, 1])

# Defining the neural network
n_inputs, n_features = X.shape
n_hidden_neurons = 2
n_categories = 2
n_features = 2

# we make the weights normally distributed using numpy.random.randn

# weights and bias in the hidden layer
hidden_weights = np.random.randn(n_features, n_hidden_neurons)
hidden_bias = np.zeros(n_hidden_neurons) + 0.01

# weights and bias in the output layer
output_weights = np.random.randn(n_hidden_neurons, n_categories)
output_bias = np.zeros(n_categories) + 0.01

probabilities = feed_forward(X)
print(probabilities)

predictions = predict(X)
print(predictions)

```

Not an impressive result, but this was our first forward pass with randomly assigned weights. Let us now add the full network with the back-propagation algorithm discussed above.

## The Code using Scikit-Learn

```

# import necessary packages
import numpy as np
import matplotlib.pyplot as plt
from sklearn.neural_network import MLPClassifier
from sklearn.metrics import accuracy_score
import seaborn as sns

# ensure the same random numbers appear every time
np.random.seed(0)

# Design matrix
X = np.array([ [0, 0], [0, 1], [1, 0], [1, 1]], dtype=np.float64)

# The XOR gate
yXOR = np.array([ 0, 1, 1, 0])
# The OR gate
yOR = np.array([ 0, 1, 1, 1])
# The AND gate
yAND = np.array([ 0, 0, 0, 1])

# Defining the neural network
n_inputs, n_features = X.shape
n_hidden_neurons = 2
n_categories = 2
n_features = 2

eta_vals = np.logspace(-5, 1, 7)
lmbd_vals = np.logspace(-5, 1, 7)

```

```

# store models for later use
DNN_scikit = np.zeros((len(eta_vals), len(lmbd_vals)), dtype=object)
epochs = 100

for i, eta in enumerate(eta_vals):
    for j, lmbd in enumerate(lmbd_vals):
        dnn = MLPClassifier(hidden_layer_sizes=(n_hidden_neurons), activation='logistic',
                           alpha=lmbd, learning_rate_init=eta, max_iter=epochs)
        dnn.fit(X, yXOR)
        DNN_scikit[i][j] = dnn
        print("Learning rate = ", eta)
        print("Lambda = ", lmbd)
        print("Accuracy score on data set: ", dnn.score(X, yXOR))
        print()

sns.set()
test_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))
for i in range(len(eta_vals)):
    for j in range(len(lmbd_vals)):
        dnn = DNN_scikit[i][j]
        test_pred = dnn.predict(X)
        test_accuracy[i][j] = accuracy_score(yXOR, test_pred)

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(test_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Test Accuracy")
ax.set_ylabel("$\eta$")
ax.set_xlabel("$\lambda$")
plt.show()

```

## Building neural networks in Tensorflow and Keras

Now we want to build on the experience gained from our neural network implementation in NumPy and scikit-learn and use it to construct a neural network in Tensorflow. Once we have constructed a neural network in NumPy and Tensorflow, building one in Keras is really quite trivial, though the performance may suffer.

In our previous example we used only one hidden layer, and in this we will use two. From this it should be quite clear how to build one using an arbitrary number of hidden layers, using data structures such as Python lists or NumPy arrays.

## Tensorflow

Tensorflow is an open source library machine learning library developed by the Google Brain team for internal use. It was released under the Apache 2.0 open source license in November 9, 2015.

Tensorflow is a computational framework that allows you to construct machine learning models at different levels of abstraction, from high-level, object-oriented APIs like Keras, down to the C++ kernels that Tensorflow is built upon. The higher levels of abstraction are simpler to use, but less flexible, and our choice of implementation should reflect the problems we are trying to solve.

Tensorflow uses so-called graphs to represent your computation in terms of the dependencies between individual operations, such that you first build a Tensorflow *graph* to represent your model, and then create a Tensorflow *session* to run the graph.

In this guide we will analyze the same data as we did in our NumPy and scikit-learn tutorial, gathered from the MNIST database of images. We will give an introduction to the lower level Python Application Program Interfaces (APIs), and see how we use them to build our graph. Then we will build (effectively) the same graph in Keras, to see just how simple solving a machine learning problem can be.

To install tensorflow on Unix/Linux systems, use pip as

```
pip3 install tensorflow
```

and/or if you use **anaconda**, just write (or install from the graphical user interface) (current release of CPU-only TensorFlow)

```
conda create -n tf tensorflow
conda activate tf
```

To install the current release of GPU TensorFlow

```
conda create -n tf-gpu tensorflow-gpu
conda activate tf-gpu
```

## Using Keras

Keras is a high level [neural network](#) that supports Tensorflow, CTNK and Theano as backends. If you have Anaconda installed you may run the following command

```
conda install keras
```

You can look up the [instructions here](#) for more information.  
We will to a large extent use **keras** in this course.

## Collect and pre-process data

Let us look again at the MINST data set.

```
# import necessary packages
import numpy as np
import matplotlib.pyplot as plt
import tensorflow as tf
from sklearn import datasets

# ensure the same random numbers appear every time
np.random.seed(0)

# display images in notebook
%matplotlib inline
```



```

plt.rcParams['figure.figsize'] = (12,12)

# download MNIST dataset
digits = datasets.load_digits()

# define inputs and labels
inputs = digits.images
labels = digits.target

print("inputs = (n_inputs, pixel_width, pixel_height) = " + str(inputs.shape))
print("labels = (n_inputs) = " + str(labels.shape))

# flatten the image
# the value -1 means dimension is inferred from the remaining dimensions: 8x8 = 64
n_inputs = len(inputs)
inputs = inputs.reshape(n_inputs, -1)
print("X = (n_inputs, n_features) = " + str(inputs.shape))

# choose some random images to display
indices = np.arange(n_inputs)
random_indices = np.random.choice(indices, size=5)

for i, image in enumerate(digits.images[random_indices]):
    plt.subplot(1, 5, i+1)
    plt.axis('off')
    plt.imshow(image, cmap=plt.cm.gray_r, interpolation='nearest')
    plt.title("Label: %d" % digits.target[random_indices[i]])
plt.show()

from tensorflow.keras.layers import Input
from tensorflow.keras.models import Sequential #This allows appending layers to existing models
from tensorflow.keras.layers import Dense #This allows defining the characteristics of layers
from tensorflow.keras import optimizers #This allows using whichever optimiser we want
from tensorflow.keras import regularizers #This allows using whichever regularizer we want
from tensorflow.keras.utils import to_categorical #This allows using categorical cross entropy loss

from sklearn.model_selection import train_test_split

# one-hot representation of labels
labels = to_categorical(labels)

# split into train and test data
train_size = 0.8
test_size = 1 - train_size
X_train, X_test, Y_train, Y_test = train_test_split(inputs, labels, train_size=train_size,
                                                    test_size=test_size)

epochs = 100
batch_size = 100
n_neurons_layer1 = 100
n_neurons_layer2 = 50
n_categories = 10
eta_vals = np.logspace(-5, 1, 7)
lmbd_vals = np.logspace(-5, 1, 7)
def create_neural_network_keras(n_neurons_layer1, n_neurons_layer2, n_categories, eta, lmbd):
    model = Sequential()
    model.add(Dense(n_neurons_layer1, activation='sigmoid', kernel_regularizer=regularizers.l2(lmbd)))
    model.add(Dense(n_neurons_layer2, activation='sigmoid', kernel_regularizer=regularizers.l2(lmbd)))

```

```

        model.add(Dense(n_categories, activation='softmax'))

        sgd = optimizers.SGD(lr=eta)
        model.compile(loss='categorical_crossentropy', optimizer=sgd, metrics=['accuracy'])

        return model

DNN_keras = np.zeros((len(eta_vals), len(lmbd_vals)), dtype=object)

for i, eta in enumerate(eta_vals):
    for j, lmbd in enumerate(lmbd_vals):
        DNN = create_neural_network_keras(n_neurons_layer1, n_neurons_layer2, n_categories,
                                          eta=eta, lmbd=lmbd)
        DNN.fit(X_train, Y_train, epochs=epochs, batch_size=batch_size, verbose=0)
        scores = DNN.evaluate(X_test, Y_test)

        DNN_keras[i][j] = DNN

        print("Learning rate = ", eta)
        print("Lambda = ", lmbd)
        print("Test accuracy: %.3f" % scores[1])
        print()

# optional
# visual representation of grid search
# uses seaborn heatmap, could probably do this in matplotlib
import seaborn as sns

sns.set()

train_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))
test_accuracy = np.zeros((len(eta_vals), len(lmbd_vals)))

for i in range(len(eta_vals)):
    for j in range(len(lmbd_vals)):
        DNN = DNN_keras[i][j]

        train_accuracy[i][j] = DNN.evaluate(X_train, Y_train)[1]
        test_accuracy[i][j] = DNN.evaluate(X_test, Y_test)[1]

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(train_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Training Accuracy")
ax.set_ylabel("$\eta$")
ax.set_xlabel("$\lambda$")
plt.show()

fig, ax = plt.subplots(figsize = (10, 10))
sns.heatmap(test_accuracy, annot=True, ax=ax, cmap="viridis")
ax.set_title("Test Accuracy")
ax.set_ylabel("$\eta$")
ax.set_xlabel("$\lambda$")
plt.show()

```

## The Mathematics of Neural Networks

1. Activation functions and vanishing gradients
2. Brief summary of gradient methods

3. Approximation theorems, in particular the *universal approximation theorem* for neural networks by Cybenko and Hornik

I strongly recommend Michael Nielsen's intuitive approach to the neural networks and the universal approximation theorem, see the slides at <http://neuralnetworksanddeeplearning.com/chap4.html>.

## Fine-tuning neural network hyperparameters

The flexibility of neural networks is also one of their main drawbacks: there are many hyperparameters to tweak. Not only can you use any imaginable network topology (how neurons/nodes are interconnected), but even in a simple FFNN you can change the number of layers, the number of neurons per layer, the type of activation function to use in each layer, the weight initialization logic, the stochastic gradient optimized and much more. How do you know what combination of hyperparameters is the best for your task?

- You can use grid search with cross-validation to find the right hyperparameters.

However, since there are many hyperparameters to tune, and since training a neural network on a large dataset takes a lot of time, you will only be able to explore a tiny part of the hyperparameter space.

- You can use randomized search.
- Or use tools like [Oscar](#), which implements more complex algorithms to help you find a good set of hyperparameters quickly.

## Hidden layers

For many problems you can start with just one or two hidden layers and it will work just fine. For the MNIST data set you can easily get a high accuracy using just one hidden layer with a few hundred neurons. You can reach for this data set above 98% accuracy using two hidden layers with the same total amount of neurons, in roughly the same amount of training time.

For more complex problems, you can gradually ramp up the number of hidden layers, until you start overfitting the training set. Very complex tasks, such as large image classification or speech recognition, typically require networks with dozens of layers and they need a huge amount of training data. However, you will rarely have to train such networks from scratch: it is much more common to reuse parts of a pretrained state-of-the-art network that performs a similar task.

## Which activation function should I use?

The Back propagation algorithm we derived above works by going from the output layer to the input layer, propagating the error gradient on the way. Once the algorithm has computed the gradient of the cost function with regards to each parameter in the network, it uses these gradients to update each parameter with a Gradient Descent (GD) step.

Unfortunately for us, the gradients often get smaller and smaller as the algorithm progresses down to the first hidden layers. As a result, the GD update leaves the lower layer connection weights virtually unchanged, and training never converges to a good solution. This is known in the literature as **the vanishing gradients problem**.

In other cases, the opposite can happen, namely the the gradients can grow bigger and bigger. The result is that many of the layers get large updates of the weights the algorithm diverges. This is the **exploding gradients problem**, which is mostly encountered in recurrent neural networks. More generally, deep neural networks suffer from unstable gradients, different layers may learn at widely different speeds

## Is the Logistic activation function (Sigmoid) our choice?

Although this unfortunate behavior has been empirically observed for quite a while (it was one of the reasons why deep neural networks were mostly abandoned for a long time), it is only around 2010 that significant progress was made in understanding it.

A paper titled [Understanding the Difficulty of Training Deep Feedforward Neural Networks](#) by Xavier Glorot and Yoshua Bengio found that the problems with the popular logistic sigmoid activation function and the weight initialization technique that was most popular at the time, namely random initialization using a normal distribution with a mean of 0 and a standard deviation of 1.

They showed that with this activation function and this initialization scheme, the variance of the outputs of each layer is much greater than the variance of its inputs. Going forward in the network, the variance keeps increasing after each layer until the activation function saturates at the top layers. This is actually made worse by the fact that the logistic function has a mean of 0.5, not 0 (the hyperbolic tangent function has a mean of 0 and behaves slightly better than the logistic function in deep networks).

## The derivative of the Logistic function

Looking at the logistic activation function, when inputs become large (negative or positive), the function saturates at 0 or 1, with a derivative extremely close to 0. Thus when backpropagation kicks in, it has virtually no gradient to propagate back through the network, and what little gradient exists keeps getting diluted as backpropagation progresses down through the top layers, so there is really nothing left for the lower layers.

In their paper, Glorot and Bengio propose a way to significantly alleviate this problem. We need the signal to flow properly in both directions: in the forward direction when making predictions, and in the reverse direction when backpropagating gradients. We don't want the signal to die out, nor do we want it to explode and saturate. For the signal to flow properly, the authors argue that we need the variance of the outputs of each layer to be equal to the variance of its inputs, and we also need the gradients to have equal variance before and after flowing through a layer in the reverse direction.

One of the insights in the 2010 paper by Glorot and Bengio was that the vanishing/exploding gradients problems were in part due to a poor choice of activation function. Until then most people had assumed that if Nature had chosen to use roughly sigmoid activation functions in biological neurons, they must be an excellent choice. But it turns out that other activation functions behave much better in deep neural networks, in particular the ReLU activation function, mostly because it does not saturate for positive values (and also because it is quite fast to compute).

## The ReLU function family

The ReLU activation function suffers from a problem known as the dying ReLUs: during training, some neurons effectively die, meaning they stop outputting anything other than 0.

In some cases, you may find that half of your network's neurons are dead, especially if you used a large learning rate. During training, if a neuron's weights get updated such that the weighted sum of the neuron's inputs is negative, it will start outputting 0. When this happens, the neuron is unlikely to come back to life since the gradient of the ReLU function is 0 when its input is negative.

To solve this problem, nowadays practitioners use a variant of the ReLU function, such as the leaky ReLU discussed above or the so-called exponential linear unit (ELU) function

$$ELU(z) = \begin{cases} \alpha (\exp(z) - 1) & z < 0, \\ z & z \geq 0. \end{cases}$$

## Which activation function should we use?

In general it seems that the ELU activation function is better than the leaky ReLU function (and its variants), which is better than ReLU. ReLU performs better than tanh which in turn performs better than the logistic function.

If runtime performance is an issue, then you may opt for the leaky ReLU function over the ELU function. If you don't want to tweak yet another hyperparameter, you may just use the default  $\alpha$  of 0.01 for the leaky ReLU, and 1 for ELU. If you have spare time and computing power, you can use cross-validation or bootstrap to evaluate other activation functions.

## More on activation functions, output layers

In most cases you can use the ReLU activation function in the hidden layers (or one of its variants).

It is a bit faster to compute than other activation functions, and the gradient descent optimization does in general not get stuck.

### For the output layer:

- For classification the softmax activation function is generally a good choice for classification tasks (when the classes are mutually exclusive).
- For regression tasks, you can simply use no activation function at all.

## Batch Normalization

Batch Normalization aims to address the vanishing/exploding gradients problems, and more generally the problem that the distribution of each layer's inputs changes during training, as the parameters of the previous layers change.

The technique consists of adding an operation in the model just before the activation function of each layer, simply zero-centering and normalizing the inputs, then scaling and shifting the result using two new parameters per layer (one for scaling, the other for shifting). In other words, this operation lets the model learn the optimal scale and mean of the inputs for each layer. In order to zero-center and normalize the inputs, the algorithm needs to estimate the inputs' mean and standard deviation. It does so by evaluating the mean and standard deviation of the inputs over the current mini-batch, from this the name batch normalization.

## Dropout

It is a fairly simple algorithm: at every training step, every neuron (including the input neurons but excluding the output neurons) has a probability  $p$  of being temporarily dropped out, meaning it will be entirely ignored during this training step, but it may be active during the next step.

The hyperparameter  $p$  is called the dropout rate, and it is typically set to 50%. After training, the neurons are not dropped anymore. It is viewed as one of the most popular regularization techniques.

## Gradient Clipping

A popular technique to lessen the exploding gradients problem is to simply clip the gradients during backpropagation so that they never exceed some threshold (this is mostly useful for recurrent neural networks).

This technique is called Gradient Clipping.

In general however, Batch Normalization is preferred.

## A very nice website on Neural Networks

You may find this [website](#) very useful. Thx a million to Ghadi for sharing.

## A top-down perspective on Neural networks

The first thing we would like to do is divide the data into two or three parts. A training set, a validation or dev (development) set, and a test set. The test set is the data on which we want to make predictions. The dev set is a subset of the training data we use to check how well we are doing out-of-sample, after training the model on the training dataset. We use the validation error as a proxy for the test error in order to make tweaks to our model. It is crucial that we do not use any of the test data to train the algorithm. This is a cardinal sin in ML. Then:

- Estimate optimal error rate
- Minimize underfitting (bias) on training data set.
- Make sure you are not overfitting.

If the validation and test sets are drawn from the same distributions, then a good performance on the validation set should lead to similarly good performance on the test set.

However, sometimes the training data and test data differ in subtle ways because, for example, they are collected using slightly different methods, or because it is cheaper to collect data in one way versus another. In this case, there can be a mismatch between the training and test data. This can lead to the neural network overfitting these small differences between the test and training sets, and a poor performance on the test set despite having a good performance on the validation set. To rectify this, Andrew Ng suggests making two validation or dev sets, one constructed from the training data and one constructed from the test data. The difference between the performance of the algorithm on these two validation sets quantifies the train-test mismatch. This can serve as another important diagnostic when using DNNs for supervised learning.

## Limitations of supervised learning with deep networks

Like all statistical methods, supervised learning using neural networks has important limitations. This is especially important when one seeks to apply these methods, especially to physics problems. Like all tools, DNNs are not a universal solution. Often, the same or better performance on a task can be achieved by using a few hand-engineered features (or even a collection of random features).

Here we list some of the important limitations of supervised neural network based models.

- **Need labeled data.** All supervised learning methods, DNNs for supervised learning require labeled data. Often, labeled data is harder to acquire than unlabeled data (e.g. one must pay for human experts to label images).

- **Supervised neural networks are extremely data intensive.** DNNs are data hungry. They perform best when data is plentiful. This is doubly so for supervised methods where the data must also be labeled. The utility of DNNs is extremely limited if data is hard to acquire or the datasets are small (hundreds to a few thousand samples). In this case, the performance of other methods that utilize hand-engineered features can exceed that of DNNs.
- **Homogeneous data.** Almost all DNNs deal with homogeneous data of one type. It is very hard to design architectures that mix and match data types (i.e. some continuous variables, some discrete variables, some time series). In applications beyond images, video, and language, this is often what is required. In contrast, ensemble models like random forests or gradient-boosted trees have no difficulty handling mixed data types.
- **Many problems are not about prediction.** In natural science we are often interested in learning something about the underlying distribution that generates the data. In this case, it is often difficult to cast these ideas in a supervised learning setting. While the problems are related, it is possible to make good predictions with a *wrong* model. The model might or might not be useful for understanding the underlying science.

Some of these remarks are particular to DNNs, others are shared by all supervised learning methods. This motivates the use of unsupervised methods which in part circumvent these problems.