Lecture Wednesday June 23

$$\boxed{m \frac{d^2x}{dt^2} = -k\,x(t)}$$

initial time $t_0 = 0$

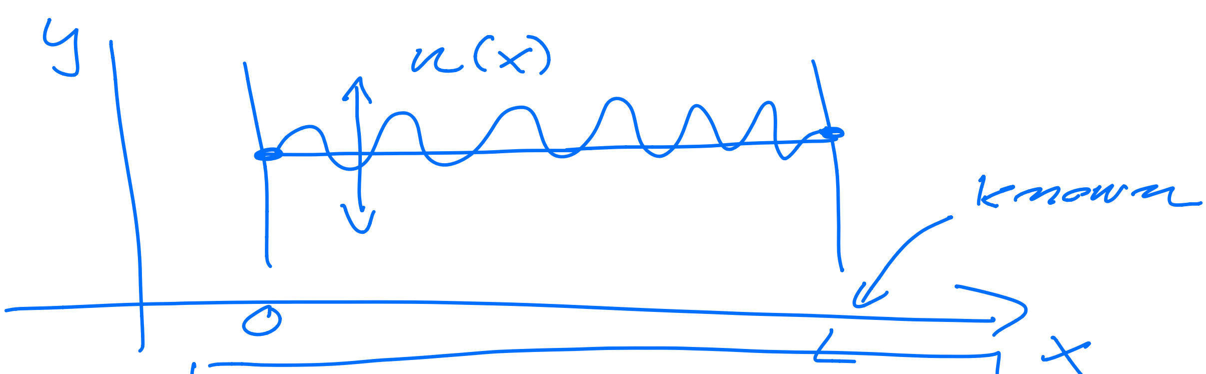
initial value problem

$x(t_0) = x_0$

$v(t_0) = v_0$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

– Boundary value problem



$$L\, d^2u(x) \quad = -F\,u(x)$$

dx^2 ↑known

$x \in [0, L]$

$u(x)$ is unknown.

— Scaling of equations

Dim less length

$$\xi = \frac{x}{L}$$

$$\xi \in [0, 1]$$

$$\frac{d^2u}{dx^2} = \frac{1}{L^2}\frac{d^2u}{d\xi^2}$$

$u(x)$

$$\boxed{\frac{d^2u}{d\xi^2} = -\frac{FL^2}{\gamma}u(\xi)}$$

$$= -\lambda u(\xi)$$

u has dimension length

what is the dimension of $\frac{FL^2}{\gamma}$?

FL^2/γ

$$\frac{d}{d\xi}\frac{du}{d\xi} = -\underline{\lambda}\, \underline{u}$$

Dim length

Dim length

Dim less

λ - Dim less

we went from

$$\gamma \frac{d^2u}{dx^2} = -F\, u(x)$$

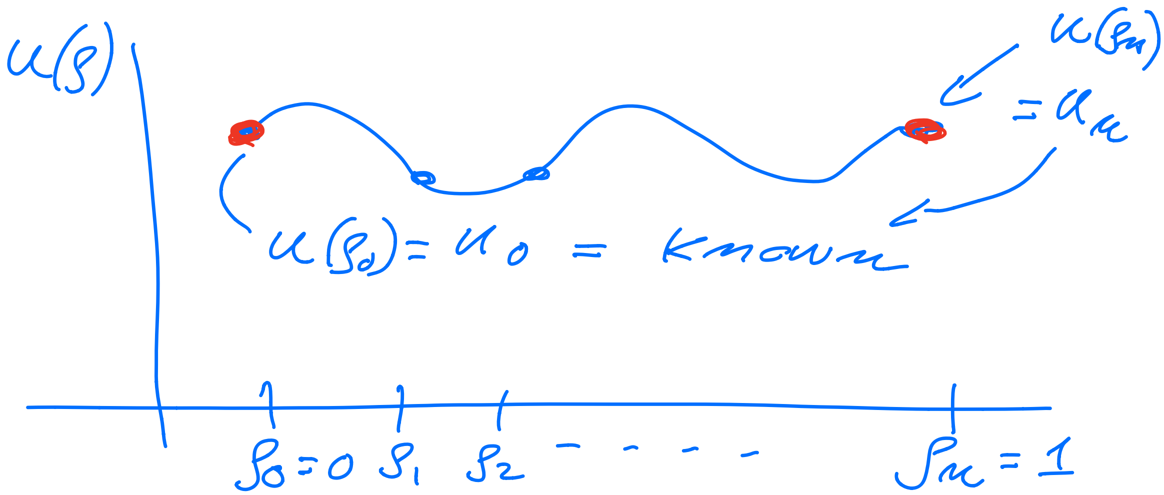
$$\downarrow$$

$$\frac{d^2u}{d\xi^2} = -\lambda u$$

$$\xi \in [0, 1]$$

$$u(0) = u(1) = 0$$

Discretize ξ



$$\underline{\rho_i = \rho_0 + i \cdot \Delta\rho} \qquad i = 0, 1, 2, \ldots n$$

$$u_i = u(\rho_i)$$

$\Delta\rho = $ step size

$$\Delta\rho = \frac{\rho_n - \rho_0}{n} = \frac{1-0}{n}$$

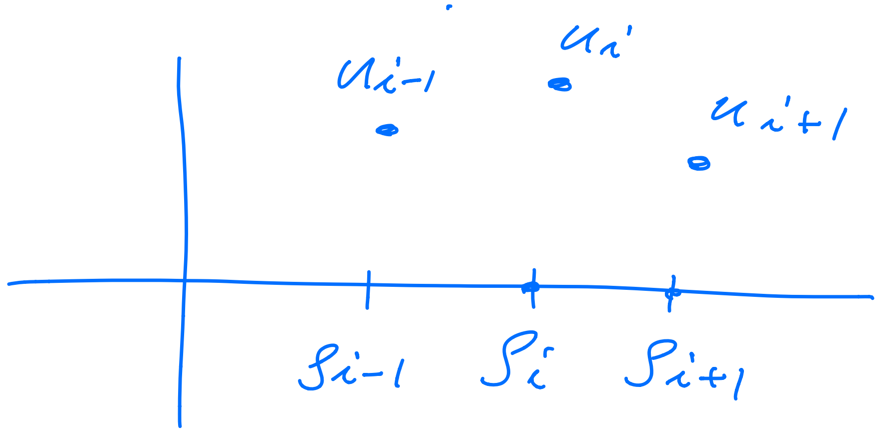
$$= \frac{1}{n}$$

$$u_{i+1} = u(\rho_i + \Delta\rho)$$

$$u_{i-1} = u(\rho_i - \Delta\rho)$$

$$\frac{d\,u}{d\rho^2} = -\lambda u(\rho) = -\lambda u_i$$

$\approx$?



$$\frac{d^2u}{d\rho^2} = \frac{u(\rho_i+\Delta\rho)+u(\rho_i-\Delta\rho)-2u(\rho_i)}{(\Delta\rho)^2}$$

$$+ \underline{O(\Delta\rho^2)}$$

$$\frac{d^2u}{d\rho^2} \simeq \frac{u_{i+1}+u_{i-1}-2u_i}{\Delta\rho^2}$$

$$\boxed{\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta\rho^2} = -\lambda u_i}$$

Taylor expansion

$$u(\rho_i+\Delta\rho) = u(\rho_i) + \Delta\rho\, u' + \frac{\Delta\rho^2}{2!}u'' + \frac{\Delta\rho^3}{3!}u''' + O(\Delta\rho^4)$$

$$u(\rho_i-\Delta\rho) = u(\rho_i) - \Delta\rho\, u' + \frac{\Delta\rho^2}{2!}u'' - \frac{\Delta\rho^3}{3!}u''' + O(\Delta\rho^4)$$

$$u(\rho_i+\Delta\rho) + u(\rho_i-\Delta\rho) = 2u(\rho_i) + \Delta\rho^2 u'' + O(\Delta\rho^4)$$

$$u'' = \frac{u(\rho_i+\Delta\rho) + u(\rho_i-\Delta\rho) - 2u(\rho_i)}{\Delta\rho^2} + O(\Delta\rho^2)$$

Final step

$$\boxed{u_{i+1} + u_{i-1} - 2u_i}$$

$$\frac{\qquad}{\Delta \rho^2} = -\lambda u_1$$

$$i = 0, 1, 2, \ldots n \qquad u_0 = u_n = 0$$

$i = 1$

$$\frac{u_2 + \overset{=0}{u_0} - 2u_1}{\Delta \rho^2} = -\lambda u_1$$

$i = 2$

$$\frac{u_3 + u_1 - 2u_2}{\Delta \rho^2} = -\lambda u_2$$

$i = 3$

$$\frac{u_4 + u_2 - 2u_3}{\Delta \rho^2} = -\lambda u_3$$

$\vdots$

$i = n-1$

$$\frac{\overset{=0}{u_n} + u_{n-2} - 2u_{n-1}}{\Delta \rho^2} = -\lambda u_{n-1}$$

$$\left[\; u_1 \;\right]$$

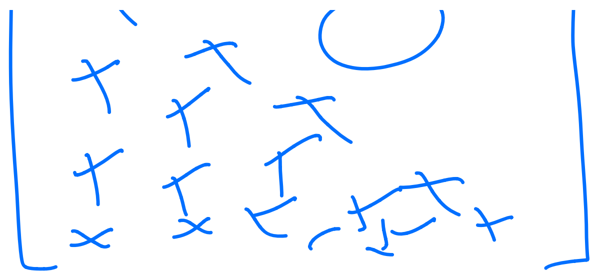
$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{m-1} \end{bmatrix}$$

$$A u = \lambda \cdot u$$

$$A = \begin{bmatrix} d & a & & & & & \\ a & d & a & & & 0 & \\ & a & d & a & & & \\ & & a & d & a & & \\ & 0 & & a & \ddots & \ddots & \\ & & & & \ddots & \ddots & a \\ & & & & & a & d \end{bmatrix}$$

$$d = +2/\Delta\rho^2$$

$$a = -1/\Delta\rho^2$$

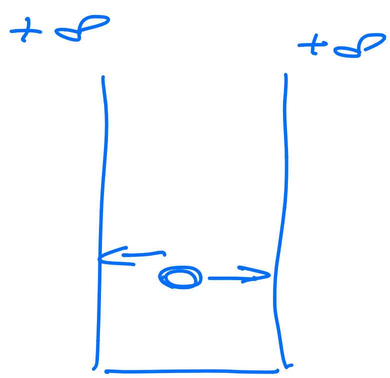


TRIDIAGONAL. (TÖPLITZ)

$$\gamma \frac{d^2u}{dx^2} = -F\,u(x)$$

infinite walls (QM)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = E\,u(x)$$



$$\frac{d^2u}{d\xi^2} = -\lambda u$$