

February 13-17,2023: Quantum Computing, Quantum Machine Learning and Quantum Information Theories

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Entanglement and Entropy

1. Review of density matrices and measurements from previous lecture
 2. Schmidt decomposition and entanglement
 3. Entropy and density matrices
- **Reading recommendation:** Scherrer, Mathematics of Quantum Computations, chapter 4
 - See also [whiteboard notes](#)
 - Note that the whiteboard notes will be added to this document later this week.

In order to study entanglement and why it is so important for quantum computing, we need to introduce some basic measures and useful quantities. For these endeavors, we will use our two-qubit system from the second lecture in order to introduce, through examples, density matrices and entropy. These two quantities, together with technicalities like the Schmidt decomposition define important quantities in analyzing quantum computing examples.

The Schmidt decomposition is again a linear decomposition which allows us to express a vector in terms of tensor product of two inner product spaces. In quantum information theory and quantum computing it is widely used as a way to define and describe entanglement.

First entanglement encounter

See [whiteboard notes](#)

Schmidt decomposition.

Two-qubit system and definition of density matrices. This system can be thought of as composed of two subsystems A and B . Each subsystem has computational basis states

$$|0\rangle_{A,B} = [1 \ 0]^T \quad |1\rangle_{A,B} = [0 \ 1]^T.$$

The subsystems could represent single particles or composite many-particle systems of a given symmetry. This leads to the many-body computational basis states

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B = [1 \ 0 \ 0 \ 0]^T,$$

and

$$|10\rangle = |1\rangle_A \otimes |0\rangle_B = [0 \ 1 \ 0 \ 0]^T,$$

and

$$|01\rangle = |0\rangle_A \otimes |1\rangle_B = [0 \ 0 \ 1 \ 0]^T,$$

and finally

$$|11\rangle = |1\rangle_A \otimes |1\rangle_B = [0 \ 0 \ 0 \ 1]^T.$$

These computational basis states define also the eigenstates of the non-interacting Hamiltonian

$$H_0|00\rangle = \epsilon_{00}|00\rangle,$$

$$H_0|10\rangle = \epsilon_{10}|10\rangle,$$

$$H_0|01\rangle = \epsilon_{01}|01\rangle,$$

and

$$H_0|11\rangle = \epsilon_{11}|11\rangle.$$

The interacting part of the Hamiltonian H_I is given by the tensor product of two σ_x and σ_z matrices, respectively, that is

$$H_I = H_x \sigma_x \otimes \sigma_x + H_z \sigma_z \otimes \sigma_z,$$

where H_x and H_z are interaction strength parameters. Our final Hamiltonian matrix is given by

$$\mathbf{H} = \begin{bmatrix} \epsilon_{00} + H_z & 0 & 0 & H_x \\ 0 & \epsilon_{10} - H_z & H_x & 0 \\ 0 & H_x & \epsilon_{01} + H_z & 0 \\ H_x & 0 & 0 & \epsilon_{11} - H_z \end{bmatrix}.$$

The four eigenstates of the above Hamiltonian matrix can in turn be used to define density matrices. As an example, the density matrix of the first eigenstate (lowest energy E_0) Ψ_0 is

$$\rho_0 = (\alpha_{00}|00\rangle\langle 00| + \alpha_{10}|10\rangle\langle 10| + \alpha_{01}|01\rangle\langle 01| + \alpha_{11}|11\rangle\langle 11|),$$

where the coefficients α_{ij} are the eigenvector coefficients resulting from the solution of the above eigenvalue problem.

We can then in turn define the density matrix for the subsets A or B as

$$\rho_A = \text{Tr}_B(\rho_0) = \langle 0|\rho_0|0\rangle_B + \langle 1|\rho_0|1\rangle_B,$$

or

$$\rho_B = \text{Tr}_A(\rho_{\Psi_0}) = \langle 0|\rho_0|0\rangle_A + \langle 1|\rho_0|1\rangle_A.$$

The density matrices for these subsets can be used to compute the so-called von Neumann entropy, which is one of the possible measures of entanglement. A pure state has entropy equal zero while entangled state have an entropy larger than zero. The von-Neumann entropy is defined as

$$S(A, B) = -\text{Tr}(\rho_{A,B} \log_2(\rho_{A,B})).$$

The example here shows the above von Neumann entropy based on the density matrix for the lowest many-body state. We see clearly a jump in the entropy around the point where we have a level crossing. At interaction strength $\lambda = 0$ we have many-body states purely defined by their computational basis states. As we switch on the interaction strength, we obtain an increased degree of mixing and the entropy increases till we reach the level crossing point where we see an additional and sudden increase in entropy. Similar behaviors are observed for the other states. The most important result from this example is that entanglement is driven by the Hamiltonian itself and the strength of the interaction matrix elements and the non-interacting energies.

```
%matplotlib inline
from matplotlib import pyplot as plt
import numpy as np
from scipy.linalg import logm, expm
def log2M(a): # base 2 matrix logarithm
    return logm(a)/np.log(2.0)

dim = 4
Hamiltonian = np.zeros((dim,dim))
#number of lambda values
n = 40
lmbd = np.linspace(0.0,1.0,n)
Hx = 2.0
Hz = 3.0
# Non-diagonal part as sigma_x tensor product with sigma_x
sx = np.matrix([[0,1],[1,0]])
sx2 = Hx*np.kron(sx, sx)
# Diagonal part as sigma_z tensor product with sigma_z
sz = np.matrix([[1,0],[0,-1]])
```

```

sz2 = Hz*np.kron(sz, sz)
noninteracting = [0.0, 2.5, 6.5, 7.0]
D = np.diag(noninteracting)
Eigenvalue = np.zeros((dim,n))
Entropy = np.zeros(n)

for i in range(n):
    Hamiltonian = lmbd[i]*(sx2+sz2)+D
    # diagonalize and obtain eigenvalues, not necessarily sorted
    EigValues, EigVectors = np.linalg.eig(Hamiltonian)
    # sort eigenvectors and eigenvalues
    permute = EigValues.argsort()
    EigValues = EigValues[permute]
    EigVectors = EigVectors[:,permute]
    # Compute density matrix for selected system state, here ground state
    DensityMatrix = np.zeros((dim,dim))
    DensityMatrix = np.outer(EigVectors[:,0],EigVectors[:,0])
    # Project down on substates and find density matrix for subsystem
    d = np.matrix([[1,0],[0,1]])
    v1 = [1.0,0.0]
    proj1 = np.kron(v1,d)
    x1 = proj1 @ DensityMatrix @ proj1.T
    v2 = [0.0,1.0]
    proj2 = np.kron(v2,d)
    x2 = proj2 @ DensityMatrix @ proj2.T
    # Total density matrix for subsystem
    total = x1+x2
    # von Neumann Entropy for subsystem
    Entropy[i] = -np.matrix.trace(total @ log2M(total))
    # Plotting eigenvalues and entropy as functions of interaction strengths
    Eigenvalue[0,i] = EigValues[0]
    Eigenvalue[1,i] = EigValues[1]
    Eigenvalue[2,i] = EigValues[2]
    Eigenvalue[3,i] = EigValues[3]
plt.plot(lmbd, Eigenvalue[0,:], 'b-', lmbd, Eigenvalue[1,:], 'g-',)
plt.plot(lmbd, Eigenvalue[2,:], 'r-', lmbd, Eigenvalue[3,:], 'y-',)
plt.xlabel('$\lambda$')
plt.ylabel('Eigenvalues')
plt.show()
plt.plot(lmbd, Entropy)
plt.xlabel('$\lambda$')
plt.ylabel('Entropy')
plt.show

```

Entropies and density matrices

Shannon information entropy. We start our discussions with the classical information entropy, or just Shannon entropy, before we move over to a quantum mechanical way to define the entropy based on the density matrices discussed earlier.

We define a set of random variables $X = \{x_0, x_1, \dots, x_{n-1}\}$ with probability for an outcome $x \in X$ given by $p_X(x)$, the information entropy is defined as

$$S = - \sum_{x \in X} p_X(x) \log_2 p_X(x).$$

Von Neumann entropy.

$$S = -\text{Tr}[\rho \log_2 \rho].$$

Lecture next week.

1. More on entanglement and entropies
2. Einstein-Podolsky-Rosen paradox
3. Bell's inequalities