# Lecture FYS5419, January 24, 2024

# Spectral decomposition & Measurements

Assume an ONB basis

$$|i\rangle \in \{|0\rangle, |1\rangle, \dots |n-1\rangle\}$$

eigenbasis of

$$A|i\rangle = \lambda_i |i\rangle$$

$$|\psi_a\rangle = \sum_{i=0}^{n-1} c_{ai} |i\rangle$$

$$c_{ai} = \langle i|\psi_a\rangle$$

$$\sum_{i=0}^{n-1} |c_{ai}|^2 = \underline{1} \quad \text{norm condition}$$

$$A|\psi_a\rangle = \sum_{i=0}^{n-1} c_{ai} A|i\rangle$$

$$= \sum_{i=0}^{n-1} c_{ai} \lambda_i |i\rangle$$

$$P_{\psi_a} = |\psi_a\rangle\langle\psi_a|$$

$$P_j = |j\rangle\langle j|$$

$$P_j|\psi_a\rangle = |j\rangle\langle j| \sum_{i=0}^{n-1} c_{ai}|i\rangle$$

$$= \sum_{i=0}^{n-1} c_{ai} |j\rangle \underbrace{\langle j|i\rangle}_{\delta_{ij}}$$

$$= c_{aj} |j\rangle$$

we substitute

$$P_j |\psi_a\rangle = c_{aj} |j\rangle$$

$$A|\psi_a\rangle = \sum_{i=0}^{n-1} c_{ai} \lambda_i |i\rangle$$

$$= \sum_{i=0}^{n-1} \lambda_i P_i |\psi_a\rangle$$

From this we deduce

$$A = \sum_{i=0}^{n-1} \lambda_i P_i$$

## Measurements

$$A|\psi_a\rangle = \lambda_a |\psi_a\rangle$$

$$\{ P_0, P_1 \ldots P_{m-1} \}$$

$|0\rangle\langle 0|$

$$m \leq n$$

$$|i\rangle \in \{ |0\rangle, |1\rangle, \ldots |n-1\rangle \}$$

$$\sum_{i=0}^{n-1} P_i = \mathbb{1}$$

Example

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \wedge \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sum_{i=0}^{1} |i\rangle\langle i| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathrm{Prob}(\lambda_i) = \langle \psi_a | P_i^{+} P_i | \psi_a \rangle$$

$$= | P_i | \psi_a \rangle |^2$$

$$P_i^{+} = P_i \wedge P_i^2 = P_i \,,\; P_i^{+} P_i = P_i$$

$$\mathrm{Prob}(\lambda_i) = \langle \psi_a | P_i | \psi_a \rangle$$

The total probability

$$\sum_{i=0}^{n-1} \mathrm{Prob}(\lambda_i) = \underline{1}$$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_0 = P_0^{\dagger}$$

$$P_0^2 = P_0$$

Post measurement normalized pure quantum state (after the outcome $\lambda_i$)

$$\frac{P_i |\psi_a\rangle}{\sqrt{\langle \psi_a | P_i | \psi_a \rangle}}$$

Example

$$|\psi_a\rangle = \alpha_x |x\rangle + \alpha_y |y\rangle$$

$$|\psi_b\rangle = \beta_x |x\rangle + \beta_y |y\rangle$$

$$P_x = |x\rangle\langle x| \;\wedge\; P_y = |y\rangle\langle y|$$

$$\text{Prob}(\psi_a) = p \quad (\text{lower case})$$

$$\text{Prob}(\psi_b) = 1-p$$

$$\text{Prob}(\lambda_i | \psi_a) = \langle \psi_a | P_x | \psi_a \rangle$$

$$= |\alpha_x|^2$$

$$\text{Prob}(\lambda_i | \psi_b) = \langle \psi_b | P_x | \psi_b \rangle = |\beta_x|^2$$

## Example II

$|0\rangle \wedge |1\rangle$

$P_0 = |0\rangle\langle 0| \wedge P_1 = |1\rangle\langle 1|$

$P_0^\dagger P_0 = P_0^2 = P_0 \wedge P_1^\dagger P_1 = P_1$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\text{Prob}(|\psi(0)\rangle)$

$= \langle \psi | P_0^\dagger P_0 | \psi \rangle$

$= (\alpha^* \langle 0| + \beta^* \langle 1|) |0\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle)$

$= |\alpha|^2$

# Post measurement state

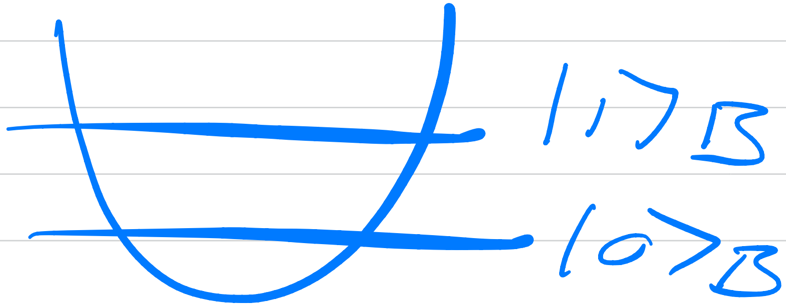
$$|\psi'(0)\rangle = \frac{P_0|\psi\rangle}{\sqrt{\langle\psi|P_0|\psi\rangle}}$$

$$= \frac{\alpha}{|\alpha|}|0\rangle$$

$$\mathrm{Prob}(\psi(1)) = |\beta|^2$$

$$|\psi'(1)\rangle = \frac{\beta}{|\beta|}|1\rangle$$


A




B

$$|0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|0\rangle_{A,B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$