FYS5419 lecture, January 31, 2024

$$\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

ONB : $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sigma_z |0\rangle = +1 \boxed{|0\rangle}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$\sigma_z |1\rangle = -1 \boxed{|1\rangle}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_x |0\rangle \neq \lambda_i |0\rangle$$

$$\sigma_x |0\rangle = |1\rangle$$

$$|\psi_0\rangle = \alpha_0 |0\rangle + \beta_0 |1\rangle$$

$$|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$$

$$H |\psi_i\rangle = E_i |\psi_i\rangle$$

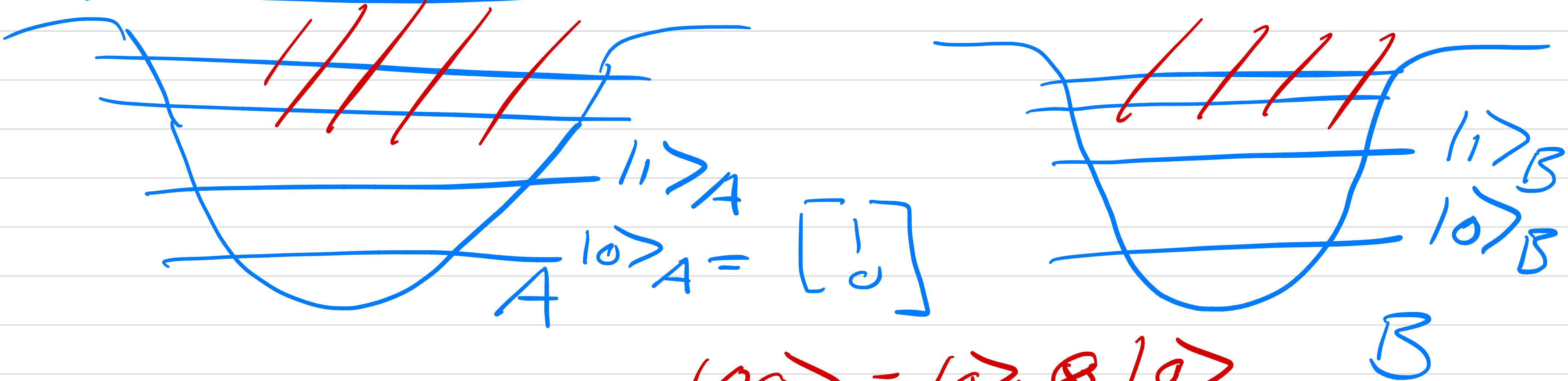
$$H = H_0 + \lambda H_I \qquad \lambda \in [0, 1]$$

$$|\psi\rangle = U |i\rangle$$

$$|\psi_0\rangle = \alpha_0 \overset{|0\rangle}{|00\rangle} + \alpha_1 \overset{|1\rangle}{|01\rangle} + \alpha_2 \overset{|2\rangle}{|10\rangle} + \alpha_3 \overset{|3\rangle}{|11\rangle}$$

$$\rho_0 = \sum_i \alpha_i |i\rangle\langle i|$$

<u>Entanglement</u>



$$|0\rangle_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B$$
$$= |0\rangle$$

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

introduce Bell states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$$|\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$|\psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

Exercise: show that these states are orthogonal to each other.

$|\phi^+\rangle$ and make measurement on qubit ($|0\rangle$ or $|1\rangle$) in A

$$M_0 = |0\rangle_A\langle 0|_A \otimes I_B$$

$$M_1 = |1\rangle_A\langle 1|_A \otimes I_B$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad n \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= \alpha|0\rangle$$

Probability of $|0\rangle$

$$P_{\phi^+}(0) = {}_{AB}\langle \phi^+ | M_0 | \phi^+ \rangle_{AB}$$

$$= 1/2$$

$$P_{\phi^+}(1) = {}_{AB}\langle \phi^+ | M_1 | \phi^+ \rangle_{AB} = 1/2$$

Probability link:

$$p(x, y) = p(x)\,p(y)$$

i.i.d = independent and identically distributed.

Entropy

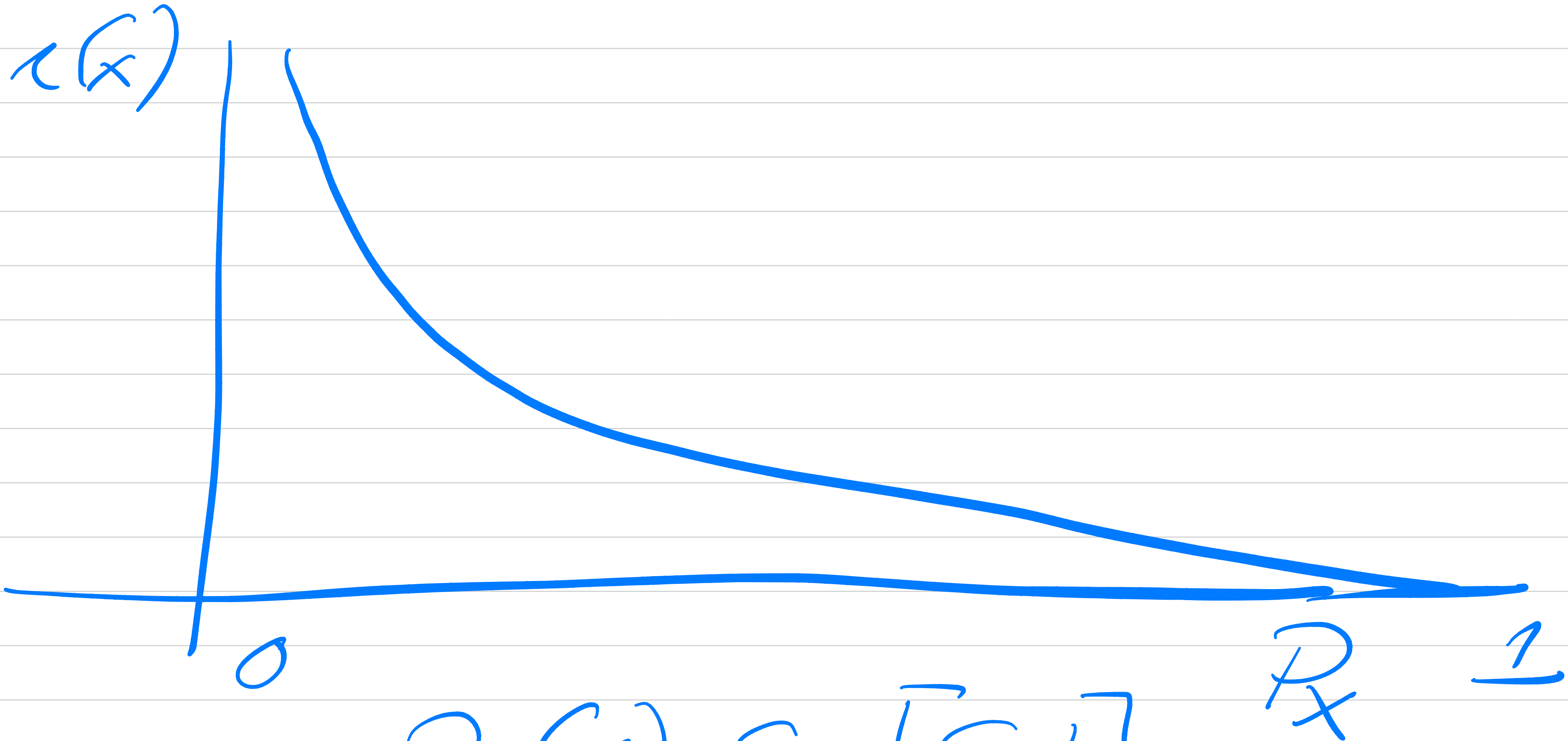
$$S(X) = - \sum_{x} P_X(x) \log_2 P_X(x)$$

considering a random variable $x \in X$ with probability $P_X(x)$

Shannon entropy.

measure of unlikeliness

$$I(x) = -\log_2 P_X(x)$$



$P_x(x) \in [0,1]$

(non-negative)

$p(x_1, x_2) = p(x_1)\,p(x_2)$

$I(x_1\, x_2) = -\log_2 p(x_1) \overset{?}{+} \log_2 p(x_2)$

$= i(x_1) + i(x_2)$

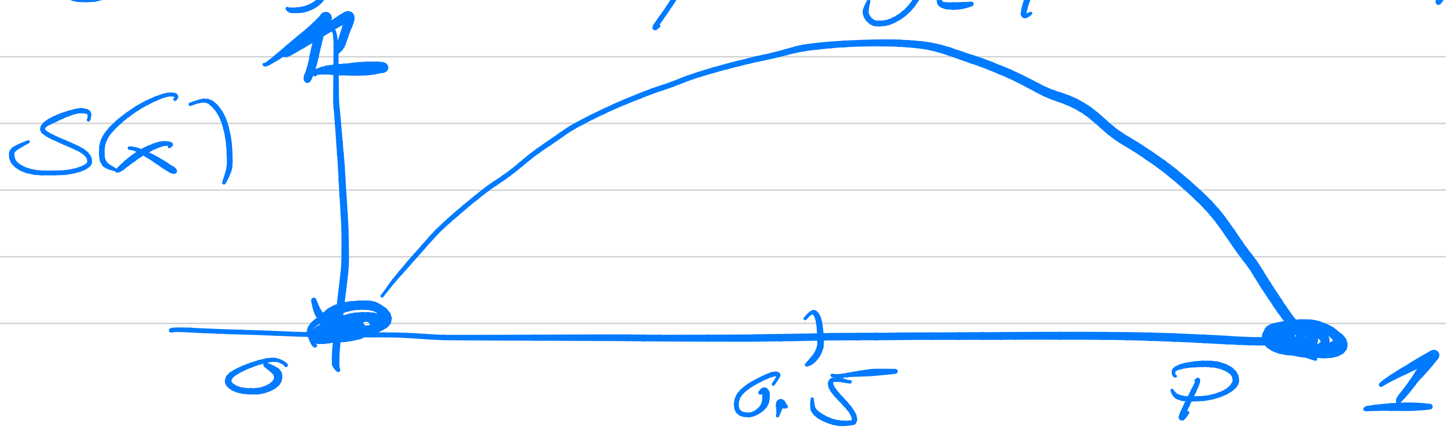
Example: binary system

$x \in [0, 1]$

$P_x(0) = p \wedge P_x(1) = 1 - p$

$$\lim_{\varepsilon \to 0} \varepsilon \log_2 \varepsilon = 0$$

$$S(x) = -p \log_2 p - (1-p) \log_2 (1-p)$$



$$S(X) = -\sum_x P_X(x) \log_2 P_X(x)$$

Shannon entropy.

Quantum mechanical equivalent is

von Neumann entropy

$$S(A) = -\mathrm{Tr}\left[\rho_A \log_2(\rho_A)\right]$$

$$\rho_A = \sum_j \lambda_j |\psi_j\rangle_A \langle\psi_j|_A$$

$S(x)$ when $p(x) = 1$ or $p(x) = 0$ should be zero.

λ_j are the eigenvalues (probabilities) and $|\psi_j\rangle_A$ are the corresponding ONBs.

ρ_A is semi-positive definite and it is always diagonalizable.

$$\rho_A = U^\dagger \rho_A' U =$$

$$U^\dagger D_A U$$

$$D_A = \begin{bmatrix} \lambda_0 & & \\ & \lambda_1 & 0 \\ 0 & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}$$

$$u^{+}u = uu^{+} = \mathbb{1}$$

$$\underbrace{u^{+}\rho_A}\ \underbrace{\log \rho_A\ u}$$

$$uu^{+}$$

$$u^{+}\log\rho_A u = \log D_A$$

$$= D_A \log D_A$$

$$\log D_A = \begin{bmatrix} \log\lambda_0 & & 0 \\ 0 & \ddots & \\ & & \log\lambda_{n-1} \end{bmatrix}$$

$$\mathrm{Tr}\, D_A = \sum_{i=0}^{n-1} \lambda_i$$

$$\Rightarrow \mathrm{Tr}\left[D_A \log D_A\right]$$

$$= \sum_{i=0}^{n-1} \lambda_i \log \lambda_i$$

$$= \sum_{i=0}^{n-1} p_i \log p_i$$

$$p_i = \lambda_i$$

$$\mathrm{Tr}[\rho_A] \longrightarrow S_A$$