Lecture FYS5419, March 20, 2024

continuous function $f(t)$

$$\int_a^b |f(t)|^2 dt \leq M < \infty$$

$$f(t) = \sum_{n=1}^{N} A_n \sin(2\pi n \cdot t + \phi_n)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi n t) + b_n \sin(2\pi n t)]$$

$f(t+T) = f(t)$ periodic,

$$\cos t = (e^{it} + e^{-it})/2$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t}$$

$\to$ complex

$$c_0 = a_0/2$$

$$c_n^* = c_{-n}$$

$$f(t) = 2\mathrm{Re}\left\{ \sum_{n=0}^{N} c_n e^{i2\pi n t} \right\}$$

How do we find c_n?

Pick c_k multiply both sides by $e^{-2\pi i k t}$

$$e^{-2\pi i k t} f(t) = \dots + e^{-2\pi i k t} \times c_k e^{2\pi i k t} + \dots ,$$

$$c_k = e^{-2\pi i k t} f(t) - \sum_{\substack{n=-N \\ \{n \neq k\}}}^{N} c_n e^{2\pi i (n-k) \times t}$$

assuming period $\underline{1}$

$$\int_0^1 e^{2\pi i(n-k)t}\,dt = \frac{1}{2\pi i(n-k)}\Big[1-1\Big]_0 = $$

if $n \neq k$

$$c_k = \int_0^1 e^{-2\pi i k t} f(t)\,dt$$

if $f(t)$ is a real function

$c_n^* = c_{-n}$ (t, dt, $f(t)$ are real)

$$c_n^* = \int_0^1 e^{2\pi i n t} f(t)\,dt$$

$c_n \to \hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\,dt$

$\hat{f}(0) = \int_c^1 f(t)\,dt$, average value.

integrate from a to a+1

$\frac{d}{da}\left[\int_a^{a+1} e^{-2\pi i n t} f(t)\,dt\right]$

See how it varies as function of a.

$$= e^{-2\pi i n(a+1)} f(a+1)$$

$$- e^{-2\pi i n a} f(a)$$

$$f(a+1) = f(a)$$

$$= e^{-2\pi i n a} e^{-2\pi i n} f(a)$$

$$- e^{-2\pi i n a} f(a) = 0$$

$$\left[e^{i 2\pi i n} = 1 \right]$$

if f is even, so is $\hat{f}$

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\,dt$$

assume we have a period of length T

$$g(t) = f(T\cdot t) \qquad (2\pi\cdot t)$$

$$s = T\cdot t$$

$$g(t) = f(s)$$

$$f(s) = g(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n s}{T}}$$

$$\hat{g}(n) = \frac{1}{T}\int_0^T e^{-2\pi i n s/T} g(s)\,ds$$

Example

Square well pulse

$$f(t) = \begin{cases} +1 & 0 \le t \le 1/2 \\ -1 & 1/2 < t \le 1 \end{cases}$$

(period)

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\,dt$$

$$= \frac{1}{\pi i n}\left[1 - e^{-\pi i n}\right]$$

$$\sum_{n \neq 0} \frac{1}{\pi i n} \left[1 - e^{-\pi i n} \right] e^{2\pi i n t}$$

$$1 - e^{\pi i n} = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

$$f(t) = \sum_{n \text{ odd}} \frac{2}{\pi i n} e^{2\pi i n t}$$

Discrete Fourier transform

$$|x\rangle = X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

x_js are complex numbers

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{2\pi i (jk)/n}$$

$$|y\rangle = y = \frac{1}{\sqrt{n}} \begin{bmatrix} \overset{y_0}{\sum_{j=0}^{n-1} x_j \cdot e^{2\pi i \cdot j \cdot 0/n}} \\ \sum_{j=0}^{n-1} x_j \cdot e^{2\pi i \cdot j \cdot 1/n} \\ \vdots \\ \sum_{j=0}^{n-1} x_j \cdot e^{2\pi i \cdot j \cdot (n-1)/n} \end{bmatrix}$$

$k=0$

Example $x = \{1, 2\}$

x_0 x_1

$k=0$: $y_0 = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 3/\sqrt{2}$

$k=1$ $y_1 = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} e^{i\pi}$

$= \frac{1}{\sqrt{2}} - 2/\sqrt{2} = -\frac{1}{\sqrt{2}}$

$y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = U \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$u^T u = \mathbb{1}$$