

1 **Geometric models reveal the hidden structure of**
2 **conceptual knowledge**

3 Paxton C. Fitzpatrick and Jeremy R. Manning^{*}

Dartmouth College

*Corresponding author: jeremy.r.manning@dartmouth.edu

4 **Abstract**

5 We develop a mathematical framework, based on natural language processing models, for
6 tracking and characterizing the acquisition of conceptual knowledge. Our approach embeds each
7 concept in a high dimensional representation space, where nearby coordinates reflect similar or
8 related concepts. We tested our approach using behavioral data collected from a group of
9 college students. In the experiment, we asked the participants to answer sets of quiz questions
10 interleaved between watching two course videos from the Khan Academy platform. We applied
11 our framework to the videos' transcripts, and to text of the quiz questions, to quantify the
12 content of each moment of video and each quiz question. We used these embeddings, along with
13 participants' quiz responses, to track how the learners' knowledge changed after watching each
14 video. Our findings show how a limited set of quiz questions may be used to construct rich and
15 meaningful representations of what each learner knows, and how their knowledge changes over
16 time as they learn.

17 **Keywords:** education, learning, knowledge, concepts, natural language processing

¹⁸ **Introduction**

¹⁹ How do we acquire conceptual knowledge? Memorizing course lectures or textbook chapters by
²⁰ rote can lead to the superficial *appearance* of understanding the underlying content, but achieving
²¹ true conceptual understanding seems to require something deeper and richer. Does conceptual
²² understanding entail connecting newly acquired information to the scaffolding of one's existing
²³ knowledge or experience [1, 5, 7, 8, 19]? Or weaving a lecture's atomic elements (e.g., its compo-
²⁴ nent words) into a structured network that describes how those individual elements are related?
²⁵ Conceptual understanding could also involve building a mental model that transcends the mean-
²⁶ ings of those individual atomic elements by reflecting the deeper meaning underlying the gestalt
²⁷ whole [12, 14, 18].

²⁸ The difference between “understanding” and “memorizing,” as framed by the researchers in
²⁹ education, cognitive psychology, and cognitive neuroscience [9–11, 14, 18] has profound analogs
³⁰ in the fields of natural language processing and natural language understanding. For example,
³¹ considering the raw contents of a document (e.g., its constituent symbols, letters, and words) might
³² provide some information about what the document is about, just as memorizing a passage might
³³ be used to answer simple questions about the passage [e.g., whether it might contain words related
³⁴ to furniture versus physics; 2, 3, 13]. However, modern natural language processing models [e.g.,
³⁵ 4, 6, 17] also attempt to capture the deeper meaning *underlying* those atomic elements. These
³⁶ models consider not only the co-occurrences of those elements within and across documents, but
³⁷ also patterns in how those elements appear across different scales (e.g., sentences, paragraphs,
³⁸ chapters, etc.), the temporal and grammatical properties of the elements, and other high-level
³⁹ characteristics of how they are used [15, 16]. According to these models, the deep conceptual
⁴⁰ meaning of a document may be captured by a feature vector in a high-dimensional representation
⁴¹ space, where nearby vectors reflect conceptually related documents. A model that succeeds at
⁴² capturing an analog of “understanding” is able to assign nearby feature vectors to two conceptually
⁴³ related documents, *even when the words contained in those documents have very little overlap.*

⁴⁴ What form might the representation of the sum total of a person's knowledge take? First,

45 we might require a means of systematically describing or representing the nearly infinite set of
46 possible things a person could know. Second, we might want to account for potential associations
47 between different concepts. For example, the concepts of “fish” and “water” might be associated in
48 the sense that fish live in water. Third, knowledge may have a critical dependency structure, such
49 that knowing about a particular concept might require first knowing about a set of other concepts.
50 For example, understanding the concept of a fish swimming in water first requires understanding
51 what fish and water *are*. Fourth, as we learn, our “current state of knowledge” should change
52 accordingly. Learning new concepts should both update our characterizations of “what is known”
53 and should also unlock any now-satisfied dependencies of that newly learned concept so that they
54 are “tagged” as available for future learning.

55 Here we develop a framework for modelling how knowledge is acquired during learning. The
56 central idea is to use text embedding models to define the coordinate systems of two maps: (a) a
57 *knowledge map* that describes the extent to which each concept is currently known and (b) a *learning*
58 *map* that describes the extent to which each concept could be learned. Each location on these maps
59 represents a single concept, and the geometries are defined such that related concepts are located
60 nearby in space. We use this framework to analyzing and interpreting behavioral data collected
61 from an experiment that has participants watch and answer conceptual questions about a series of
62 recorded course lectures.

63 Our primary research goal is to advance our understanding of what it means to acquire deep
64 real-world conceptual knowledge. Traditional laboratory approaches to studying learning and
65 memory (e.g., list learning studies) often draw little distinction between memorization and under-
66 standing. Instead, these studies typically focus on whether information is effectively encoded or
67 retrieved, rather than whether the information is *understood*. Approaches to studying conceptual
68 learning, such as category learning experiments, can start to investigate the distinction between
69 memorization and understanding, often by training participants to distinguish arbitrary or ran-
70 dom features in otherwise meaningless categorized stimuli. However the objective of real-world
71 training, or learning from life experiences more generally, is often to develop new knowledge
72 that may be applied in *useful* ways in the future. In this sense, the gap between modern learning

73 theories and modern pedagogical approaches and classroom learning strategies is enormous: most
74 of our theories about *how* people learn are inspired by experimental paradigms and models that
75 have only peripheral relevance to the kinds of learning that students and teachers actually seek.
76 To help bridge this gap, our study uses course materials from real online courses to inform, fit, and
77 test models of real-world conceptual learning.

78 **Results**

79 **Discussion**

80 **Materials and methods**

81 **Participants**

82 We enrolled a total of 50 Dartmouth undergraduate students in our study. Participants received
83 course credit for enrolling. We asked each participant to fill out a demographic survey that included
84 questions about their age, gender, native spoken language, ethnicity, race, hearing, color vision,
85 sleep, coffee consumption, level of alertness, and several aspects of their educational background
86 and prior coursework.

87 Participants' ages ranged from 18 to 22 years (mean: 19.52 years; standard deviation: 1.09
88 years). A total of 15 participants reported their gender as male and 35 participants reported their
89 gender as female. A total of 49 participants reported their native language as "English" and 1
90 reported having another native language. A total of 47 participants reported their ethnicity as
91 "Not Hispanic or Latino" and three reported their ethnicity as "Hispanic or Latino." Participants
92 reported their races as White (32 participants), Asian (14 participants), Black or African American
93 (5 participants), American Indian or Alaska Native (1 participant), and Native Hawaiian or Other
94 Pacific Islander (1 participant). (Note that some participants selected multiple racial categories.)

95 A total of 49 participants reporting having normal hearing and 1 participant reported having

96 some hearing impairment. A total of 49 participants reported having normal color vision and 1
97 participant reported being color blind. Participants reported having had, on the night prior to
98 testing, 2 – 4 hours of sleep (1 participant), 4 – 6 hours of sleep (9 participants), 6 – 8 hours of sleep
99 (35 participants), or 8+ hours of sleep (5 participants). They reported having consumed, on the
100 same day and leading up to their testing session, 0 cups of coffee (38 participants), 1 cup of coffee
101 (10 participants), 3 cups of coffee (1 participant), or 4+ cups of coffee (1 participant).

102 No participants reported that their focus was currently impaired (e.g., by drugs or alcohol).
103 Participants reported their current level of alertness, and we converted their responses to numerical
104 scores as follows: “very sluggish” (-2), “a little sluggish” (-1), “neutral” (0), “fairly alert” (1), and
105 “very alert” (2). Across all participants, a range of alertness levels were reported (range: -2 – 1;
106 mean: -0.10; standard deviation: 0.84).

107 Participants reported their undergraduate major(s) as Social Sciences (28 participants), Natural
108 sciences (16), Professional (e.g., pre-med or pre-law; 8 participants), Mathematics and engineering
109 (7 participants), Humanities (4 participants), or Undecided (3 participants). Note that some par-
110 ticipants selected multiple categories for their undergraduate major. We also asked participants
111 about the courses they had taken. In total, 46 participants reported having taken at least one Khan
112 academy course in the past or being familiar with the Khan academy, and 4 reported not having
113 taken any Khan academy courses. Of the participants who reported having watched at least one
114 Khan academy course, 1 participant declined to report the number of courses they had watched;
115 7 participants reported having watched 1–2 courses; 11 reported having watched 3–5 courses; 8
116 reported having watched 5–10 courses; and 19 reported having watched 10 or more courses. We
117 also asked participants about the specific courses they had watched, categorized under different
118 subject areas. In the “Mathematics” area participants reported having watched videos on AP
119 Calculus AB (21 participants), Precalculus (17 participants), Algebra 2 (14 participants), AP Cal-
120 culus BC (12 participants), Trigonometry (11 participants), Algebra 1 (10 participants), Geometry
121 (8 participants), Pre-algebra (7 participants), Multivariable Calculus (5 participants), Differential
122 Equations (5 participants, Statistics and Probability (4 participants), AP Statistics (2 participants),
123 Linear Algebra (2 participants), Early Math (1 participant), Arithmetic (1 participant), and other

¹²⁴ videos not listed in our survey (6 participants). In the “Science and engineering” area participants
¹²⁵ reported having watched videos on Chemistry, AP Chemistry, or Organic Chemistry (21 participants);
¹²⁶ Physics, AP Physics I, or AP Physics II (15 participants); Biology, AP Biology; or High
¹²⁷ school Biology (15 participants); Health and Medicine (1 participant); or other videos not listed in
¹²⁸ our survey (20 participants). We also asked participants if they had specifically seen the videos
¹²⁹ used in our experiment. When we asked about the *Four Fundamental Forces* video, 45 participants
¹³⁰ reported not having watched it before, 1 participant reported that they were not sure if they had
¹³¹ watched it before, and 4 participants declined to respond. When we asked about the *Birth of*
¹³² *Stars* video, 46 participants reported not having watched it before and 4 participants declined to
¹³³ respond. When we asked participants about non-Khan academy online courses, they reported
¹³⁴ having watched or taken courses on Mathematics (15 participants), Science and engineering (11
¹³⁵ participants), Test preparation (9 participants), Economics and finance (3 participants), Arts and
¹³⁶ humanities (2 participants), Computing (2 participants), and other categories not listed in our
¹³⁷ survey (18 participants). Finally, we asked participants about in-person courses they had taken in
¹³⁸ different subject areas. They reported taking courses in Mathematics (39 participants), Science and
¹³⁹ engineering (38 participants), Arts and humanities (35 participants), Test preparation (27 participants),
¹⁴⁰ Economics and finance (26 participants), Computing (15 participants), College and careers
¹⁴¹ (7 participants), or other courses not listed in our survey (6 participants).

¹⁴² Experiment

¹⁴³ We hand-selected two roughly 10-minute course videos from the Khan Academy platform: *The*
¹⁴⁴ *Four Fundamental Forces* (an introduction to gravity, electromagnetism, the weak nuclear force, and
¹⁴⁵ the strong nuclear force; duration: 10 minutes and 29 seconds) and *Birth of Stars* (an introduction
¹⁴⁶ to how stars are formed; duration: 7 minutes and 57 seconds). We hand-wrote 39 multiple
¹⁴⁷ choice questions: 15 about the conceptual content of *The Four Fundamental Forces*, another 15 about
¹⁴⁸ the conceptual content of *Birth of Stars*, and 9 other questions that tested for general conceptual
¹⁴⁹ knowledge about basic physics (covering material that was not presented in either video). The full
¹⁵⁰ set of questions may be found in Table S1.

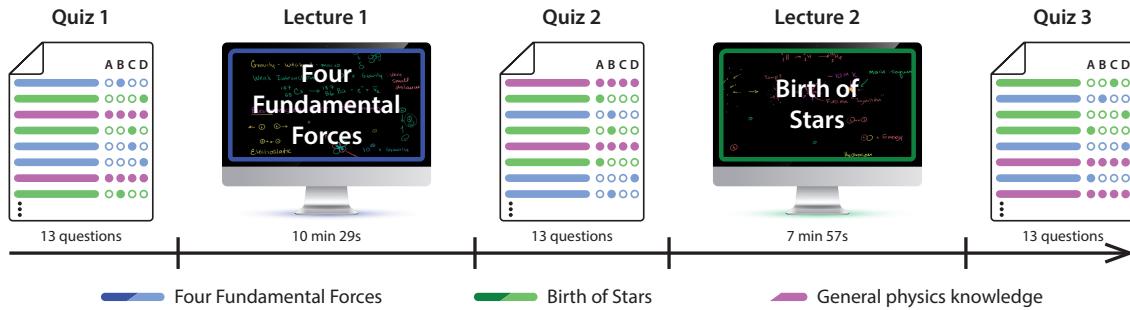


Figure 1: Experimental paradigm. Participants alternate between answering 13-question multiple choice quizzes and watching two Khan academy videos. Each quiz contains a mix of 5 questions about lecture 1, 5 questions about lecture 2, and 3 general physics knowledge questions. The specific questions reflected on each quiz, and the orders of each quiz's questions, were randomized across participants.

151 Participants began the main experiment by answering a battery of 13 randomly selected ques-
 152 tions (chosen from the full set of 39). Then they watched the *The Four Fundamental Forces* video.
 153 Next, they answered a second set of 13 questions (chosen at random from the remaining 20 ques-
 154 tions). Fourth, participants watch the *Birth of Stars* video, and finally they answered the remaining
 155 13 questions. Our experimental procedure is diagrammed in Figure 1. We used the experiment to
 156 develop and test our computational framework for estimating knowledge and learning maps.

157 Analysis

158 Constructing text embeddings of multiple videos and questions

159 Estimating held-out conceptual knowledge

160 Creating knowledge and learning map visualizations

161 References

- 162 [1] Blaye, A., Bernard-Peyron, V., Paour, J.-L., and Bonthoux, F. (2006). Category flexibility in chil-
 163 dren: distinguishing response flexibility from conceptual flexibility; the protracted development
 164 of taxonomic representations. *European Journal of Developmental Psychology*, 3(2):163–188.
- 165 [2] Blei, D. M. and Lafferty, J. D. (2006). Dynamic topic models. In *Proceedings of the International*

- 166 Conference on Machine Learning, pages 113–120, New York, NY. Association for Computing
167 Machinery.
- 168 [3] Blei, D. M., Ng, A. Y., and Jordan, M. I. (2003). Latent dirichlet allocation. *Journal of Machine*
169 *Learning Research*, 3:993–1022.
- 170 [4] Brown, T. B., Mann, B., Ryder, N., Subbiah, M., Kaplan, J., Dhariwal, P., Neelakantan, A.,
171 Shyam, P., Sastry, G., Askell, A., Agarwal, S., Herbert-Voss, A., Krueger, G., Henighan, T., Child,
172 R., Ramesh, A., Ziegler, D. M., Wu, J., Winter, C., Hesse, C., Chen, M., Sigler, E., Litwin, M.,
173 Gray, S., Chess, B., Clark, J., Berner, C., McCandlish, S., Radford, A., Sutskever, I., and Amodei,
174 D. (2020). Language models are few-shot learners. *arXiv*, 2005.14165.
- 175 [5] Caramazza, A. and Mahon, B. Z. (2003). The organization of conceptual knowledge: the
176 evidence from category-specific semantic deficits. *Trends in Cognitive Sciences*, 7(8):354–361.
- 177 [6] Cer, D., Yang, Y., Kong, S. Y., Hua, N., Limtiaco, N., John, R. S., Constant, N., Guajardo-
178 Cespedes, M., Yuan, S., Tar, C., Sung, Y.-H., Strope, B., and Kurzweil, R. (2018). Universal
179 sentence encoder. *arXiv*, 1803.11175.
- 180 [7] Constantinescu, A. O., O'Reilly, J. X., and Behrens, T. E. J. (2016). Organizing conceptual
181 knowledge in humans with a gridlike code. *Science*, 352(6292):1464–1468.
- 182 [8] Deacon, D., Grose-Fifer, J., Yang, C. M., Stanick, V., Hewitt, S., and Dynowska, A. (2004).
183 Evidence for a new conceptualization of semantic representation in the left and right cerebral
184 hemispheres. *Cortex*, 40(3):467–478.
- 185 [9] Gallagher, J. J. (2000). Teaching for understanding and application of science knowledge. *School*
186 *Science and Mathematics*, 100(6):310–318.
- 187 [10] Hall, R. and Greeno, J. (2008). *21st century education: A reference handbook*, chapter Conceptual
188 learning, pages 212–221. Sage Publications.
- 189 [11] Katona, G. (1940). *Organizing and memorizing: studies in the psychology of learning and teaching*.
190 Columbia University Press.

- ¹⁹¹ [12] Kintsch (1970). *Learning, memory, and conceptual processes*. Wiley.
- ¹⁹² [13] Landauer, T. K. and Dumais, S. T. (1997). A solution to Plato's problem: the latent semantic
¹⁹³ analysis theory of acquisition, induction, and representation of knowledge. *Psychological Review*,
¹⁹⁴ 104:211–240.
- ¹⁹⁵ [14] Macellan, E. (2005). Conceptual learning: the priority for higher education. *British Journal of
196 Educational Studies*, 53(2):129–147.
- ¹⁹⁷ [15] Manning, J. R. (2020). Context reinstatement. In Kahana, M. J. and Wagner, A. D., editors,
¹⁹⁸ *Handbook of Human Memory*. Oxford University Press.
- ¹⁹⁹ [16] Manning, J. R. (2021). Episodic memory: mental time travel or a quantum “memory wave”
²⁰⁰ function? *Psychological Review*, 128(4):711–725.
- ²⁰¹ [17] Mikolov, T., Chen, K., Corrado, G., and Dean, J. (2013). Efficient estimation of word represen-
²⁰² tations in vector space. *arXiv*, 1301.3781.
- ²⁰³ [18] Scott, P., Asoko, H., and Leach, J. (2007). *Handbook of research on science education*, chapter
²⁰⁴ Student conceptions and conceptual learning in science. Routledge.
- ²⁰⁵ [19] Simon, M. A., Tzur, R., Heinz, K., and Kinzel, M. (2004). Explicating a mechanism for
²⁰⁶ conceptual learning: elaborating the construct of reflective abstraction. *Journal for Research in
207 Mathematics Education*, 35(5):305–329.