Table ronde weak lensing calibration

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1 Abstract

Observed ellipticity of one individual galaxy with properties $\vec{P}$ (Eq.10 in Pujol+2018)

$$\vec{e}^{obs} = R(\vec{P})\vec{g} + \vec{a}(\vec{P}) + f(\vec{e}^I)$$

Response (Eq.2 in Pujol+2018)

$$R_{\alpha\beta} = \frac{\partial e^{obs}_\alpha}{\partial g_\beta} \approx \frac{e^{obs,+}_\alpha - e^{obs,-}_\alpha}{2\Delta g_\beta}$$

Taylor expansion of the exact expression from theory (for $|g| \leq 1$)

$$e = \frac{e^I + g}{1 + g^* e^I} = e^I + g - g^*(e^I)^2 + O(g^2), \quad \text{if } |g^* e^I| < 1 \text{ as } \frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k \text{ for } |x| < 1$$

$$\vec{e} = \vec{e}^I + \vec{g}(1 + |\vec{e}^I|^2) - 2(\vec{e}^I \cdot \vec{g})\vec{e}^I + O(g^2)$$

$$\vec{e} = \vec{e}^I + \begin{pmatrix} 1 - (e^I_1)^2 + (e^I_2)^2 & -2e^I_1 e^I_2 \\ -2e^I_1 e^I_2 & 1 + (e^I_1)^2 - (e^I_2)^2 \end{pmatrix} \vec{g} = \vec{e}^I + A(\vec{e}^I)\vec{g}$$

Observed ellipticity of one individual galaxy with the next order term

$$\vec{e}^{obs} = R(\vec{P})A(\vec{e}^I)\vec{g} + \vec{a}(\vec{P}) + f(\vec{e}^I)$$

$$\frac{\partial e^{obs}_\alpha}{\partial g_\beta} = \Big[R(\vec{P})A(\vec{e}^I)\Big]_{\alpha\beta} = \widetilde{R}(\vec{P}, \vec{e}^I)_{\alpha\beta}$$

2 Update 2019

2.1 Predictions

Taylor expansion around $g = 0$

$$e^{\text{true}}_\alpha = e^I_\alpha + A_{\alpha\beta} g_\beta + B_{\alpha\beta\gamma} g_\beta g_\gamma + O(g^3)$$

The mean property $\langle e^{\text{true}} \rangle = g$ holds whatever the degree of the expansion (see proof below).

$$\begin{aligned} A_{11} &= 1 - (e^I_1)^2 + (e^I_2)^2 \\ A_{12} = A_{21} &= -2e^I_1 e^I_2 \\ A_{22} &= 1 - (e^I_1)^2 + (e^I_2)^2 \end{aligned}$$

$$\begin{aligned} B_{111} &= (e^I_1)^3 - 3e^I_1(e^I_2)^2 - e^I_1 \\ B_{112} = B_{121} &= 3(e^I_1)^2 e^I_2 - (e^I_2)^3 \\ B_{122} &= -(e^I_1)^3 + 3e^I_1(e^I_2)^2 - e^I_1 \\ B_{211} &= -(e^I_2)^3 + 3(e^I_1)^2 e^I_2 - e^I_2 \\ B_{212} = B_{221} &= 3e^I_1(e^I_2)^2 - (e^I_1)^3 \\ B_{222} &= (e^I_2)^3 - 3(e^I_1)^2 e^I_2 - e^I_2 \end{aligned}$$

2.2 Observations

Observed ellipticity defined as the true ellipticity with a multiplicative response and an additive bias, both functions of the galaxy physical properties $\vec{P}$

$$\begin{aligned} e^{\text{obs}}_\alpha &= R_{\alpha\beta} e^{\text{true}}_\beta + a_\alpha + S \\ &= R_{\alpha\beta} e^I_\beta + R_{\alpha\beta} A_{\beta\gamma} g_\gamma + R_{\alpha\beta} B_{\beta\gamma\delta} g_\gamma g_\delta + a_\alpha + S \\ &= \tilde{R}_{\alpha\beta} g_\beta + \frac{1}{2}\tilde{Q}_{\alpha\beta\gamma} g_\beta g_\gamma + a_\alpha + f(e^I_\alpha) \end{aligned}$$

with $\tilde{R}_{\alpha\beta} = R_{\alpha i} A_{i\beta}$ (4 independent values) and $\tilde{Q}_{\alpha\beta\gamma} = 2 R_{\alpha i} B_{i\beta\gamma}$ (6 independent values)

First derivatives (4 equations):

$$\frac{\partial e^{obs}_{\alpha}}{\partial g_{\beta}} = \tilde{R}_{\alpha\beta} + \frac{1}{2}\left(\tilde{Q}_{\alpha\beta\gamma} g_{\gamma} + \tilde{Q}_{\alpha\beta\gamma} g_{\beta}\right)$$

Second derivatives (6 equations):

$$\frac{\partial^2 e^{obs}_{\alpha}}{\partial g_{\beta} \partial g_{\gamma}} = \frac{1}{2}\left(\tilde{Q}_{\alpha\beta\gamma} + \tilde{Q}_{\alpha\gamma\beta}\right) = P_{\alpha\beta\gamma}$$

Finite differences for second derivatives, with 5 points in the (g_1, g_2) plane:

$$P_{\alpha\beta\gamma} \approx \frac{e^{\rm obs}_{\alpha}(+\Delta g_{\beta}) + e^{\rm obs}_{\alpha}(-\Delta g_{\beta}) + e^{\rm obs}_{\alpha}(+\Delta g_{\gamma}) + e^{\rm obs}_{\alpha}(-\Delta g_{\gamma}) - 4 e^{\rm obs}_{\alpha}(0)}{4 \Delta g_{\beta} \Delta g_{\gamma}}$$

3 Proof of the mean property

This is the proof of that the mean of the ellipticity estimator $e^{\rm true}$ is zero. The mean is taken over intrinsic ellipticities e^I with probability density $p(e^I)$.

Proof with exact expression

$$\begin{aligned}
\langle e^{\rm true} \rangle &= \int_{D(\vec{0},1)} e^{\rm true}(e^I, g) p(e^I) de^I && D(\vec{0},1) \text{ is the disk of center } \vec{0} \text{ and radius 1 (hereafter } C(0,1) \text{ the circle)} \\
&= \int_0^1 \left(\int_0^{2\pi} e^{\rm true}(y e^{2i\phi}, g) d\phi \right) p(y) y dy && e^I = y e^{2i\phi},\ p(e^I) \text{ must be polar symmetric} \\
&= \int_0^1 \left(\int_0^{2\pi} \frac{y e^{2i\phi} + g}{1 + g^* y e^{2i\phi}} d\phi \right) p(y) y dy && \\
&= \int_0^1 \left(-i \oint_{C(0,1)} \frac{yu + g}{u(1 + g^* yu)} du \right) p(y) y dy && u = e^{2i\phi},\ \frac{du}{d\phi} = 2iu,\ \frac{d\phi}{du} = \frac{-i}{2u} \text{ but as } \phi \in [0, 2\pi) \text{, there are two circles} \\
&= 2\pi g \int_0^1 p(y) y dy && \text{by residue theorem, with poles } u = 0 \text{ (inside) and } u = -1/(g^* y) \text{ (outside)} \\
&= g && \text{imposed by normalization of } p
\end{aligned}$$

Proof with Taylor expansion

$$\begin{aligned}
\langle e^{\rm true} \rangle &= \int_0^1 \left(-i \oint_{C(0,1)} \frac{yu + g}{u(1 + g^* yu)} du \right) p(y) y dy && \\
&= \int_0^1 \left(-i \oint_{C(0,1)} y + \frac{g}{u} - g^* y^2 u + \sum_{k=2}^{\infty} f_k u^k du \right) p(y) y dy && \text{with } f_k \text{ some factors} \\
&= 2\pi g \int_0^1 p(y) y dy && \text{as } \oint_{C(0,1)} z^m dz = \begin{cases} 2\pi i & m = -1 \\ 0 & \text{otherwise} \end{cases} \\
&= g &&
\end{aligned}$$