Table ronde weak lensing calibration

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1 Abstract

Observed ellipticity of one individual galaxy with properties $\vec{P}$ (Eq.10 in Pujol+2018)

$$\vec{e}^{obs} = R(\vec{P})\vec{g} + \vec{a}(\vec{P}) + f(\vec{e}^I)$$

Response (Eq.2 in Pujol+2018)

$$R_{\alpha\beta} = \frac{\partial e_\alpha^{obs}}{\partial g_\beta} \approx \frac{e_\alpha^{obs,+} - e_\alpha^{obs,-}}{2\Delta g_\beta}$$

Taylor expansion of the exact expression from theory (for $|g| \le 1$)

$$e = \frac{e^I + g}{1 + g^* e^I} = e^I + g - g^*(e^I)^2 + O(g^2), \quad \text{if } |g^* e^I| < 1 \text{ as } \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \text{ for } |x| < 1$$

$$\vec{e} = \vec{e}^I + \vec{g}(1 + |\vec{e}^I|^2) - 2(\vec{e}^I \cdot \vec{g})\vec{e}^I + O(g^2)$$

$$\vec{e} = \vec{e}^I + \begin{pmatrix} 1 - (e_1^I)^2 + (e_2^I)^2 & -2e_1^I e_2^I \\ -2e_1^I e_2^I & 1 + (e_1^I)^2 - (e_2^I)^2 \end{pmatrix} \vec{g} = \vec{e}^I + A(\vec{e}^I)\vec{g}$$

Observed ellipticity of one individual galaxy with the next order term

$$\vec{e}^{obs} = R(\vec{P})A(\vec{e}^I)\vec{g} + \vec{a}(\vec{P}) + f(\vec{e}^I)$$

$$\frac{\partial e_\alpha^{obs}}{\partial g_\beta} = \left[R(\vec{P})A(\vec{e}^I) \right]_{\alpha\beta} = \widetilde{R}(\vec{P}, \vec{e}^I)_{\alpha\beta}$$

2 Update 2019

2.1 Predictions

Taylor expansion around $g = 0$

$$e_\alpha^{\text{true}} = e_\alpha^I + A_{\alpha\beta}g_\beta + B_{\alpha\beta\gamma}g_\beta g_\gamma + O(g^3)$$

The mean property $\langle e^{\text{true}} \rangle = g$ holds whatever the degree of the expansion (see proof below).

$$\begin{aligned} A_{11} &= 1 - (e_1^I)^2 + (e_2^I)^2 \\ A_{12} = A_{21} &= -2e_1^I e_2^I \\ A_{22} &= 1 - (e_1^I)^2 + (e_2^I)^2 \end{aligned}$$

$$\begin{aligned} B_{111} &= (e_1^I)^3 - 3e_1^I(e_2^I)^2 - e_1^I \\ B_{112} = B_{121} &= 3(e_1^I)^2 e_2^I - (e_2^I)^3 \\ B_{122} &= -(e_1^I)^3 + 3e_1^I(e_2^I)^2 - e_1^I \\ B_{211} &= -(e_2^I)^3 + 3(e_1^I)^2 e_2^I - e_2^I \\ B_{212} = B_{221} &= 3e_1^I(e_2^I)^2 - (e_1^I)^3 \\ B_{222} &= (e_2^I)^3 - 3(e_1^I)^2 e_2^I - e_2^I \end{aligned}$$

2.2 Observations

Observed ellipticity defined as the true ellipticity with a multiplicative response and an additive bias, both functions of the galaxy physical properties $\vec{P}$

$$\begin{aligned} e_\alpha^{\text{obs}} &= R_{\alpha\beta}e_\beta^{\text{true}} + a_\alpha + S \\ &= R_{\alpha\beta}e_\beta^I + R_{\alpha\beta}A_{\beta\gamma}g_\gamma + R_{\alpha\beta}B_{\beta\gamma\delta}g_\gamma g_\delta + a_\alpha + S \\ &= \tilde{R}_{\alpha\beta}g_\beta + \frac{1}{2}\tilde{Q}_{\alpha\beta\gamma}g_\beta g_\gamma + a_\alpha + f(e^I) \end{aligned}$$

with $\tilde{R}_{\alpha\beta} = R_{\alpha i}A_{i\beta}$ (4 independent values) and $\tilde{Q}_{\alpha\beta\gamma} = 2R_{\alpha i}B_{i\beta\gamma}$ (6 independent values)

First derivatives (4 equations):

$$\frac{\partial e_\alpha^{\mathrm{obs}}}{\partial g_\beta} = \tilde{R}_{\alpha\beta} + \frac{1}{2}\left(\tilde{Q}_{\alpha\beta\gamma}g_\gamma + \tilde{Q}_{\alpha\beta\gamma}g_\beta\right)$$

Second derivatives (6 equations):

$$\frac{\partial^2 e_\alpha^{\mathrm{obs}}}{\partial g_\beta \partial g_\gamma} = \frac{1}{2}\left(\tilde{Q}_{\alpha\beta\gamma} + \tilde{Q}_{\alpha\gamma\beta}\right)$$

Finite differences for first derivatives, with 2 points in the (g_1, g_2) plane:

$$\frac{\partial e_\alpha^{\mathrm{obs}}}{\partial g_\beta} \approx \frac{e_\alpha^{\mathrm{obs}}(+\Delta g_\beta) - e_\alpha^{\mathrm{obs}}(-\Delta g_\beta)}{2\Delta g_\beta}$$

Finite differences for second derivatives, with 5 points in the (g_1, g_2) plane:

$$\frac{\partial^2 e_\alpha^{\mathrm{obs}}}{\partial g_\beta \partial g_\gamma} \approx \frac{e_\alpha^{\mathrm{obs}}(+\Delta g_\beta) + e_\alpha^{\mathrm{obs}}(-\Delta g_\beta) + e_\alpha^{\mathrm{obs}}(+\Delta g_\gamma) + e_\alpha^{\mathrm{obs}}(-\Delta g_\gamma) - 4e_\alpha^{\mathrm{obs}}(0)}{4\Delta g_\beta \Delta g_\gamma}$$

3   Proof of the mean property

This is the proof of that the mean of the ellipticity estimator e^{true} is g. The mean is taken over intrinsic ellipticities e^I with probability density $p(e^I)$ assumed to be isotropic.

Proof with exact expression

$$
\begin{aligned}
\langle e^{\mathrm{true}} \rangle &= \int_{D(\vec{0},1)} e^{\mathrm{true}}(e^I, g)p(e^I)de^I && D(\vec{0},1) \text{ is the disk of center } \vec{0} \text{ and radius 1 (hereafter } C(0,1) \text{ the circle)} \\
&= \int_0^1 \left(\int_0^{2\pi} e^{\mathrm{true}}(ye^{2i\phi}, g)d\phi \right) p(y)y\,dy && e^I = ye^{2i\phi},\ p(e^I) \text{ must be polar symmetric} \\
&= \int_0^1 \left(\int_0^{2\pi} \frac{ye^{2i\phi} + g}{1 + g^*ye^{2i\phi}}d\phi \right) p(y)y\,dy \\
&= \int_0^1 \left(-i \oint_{C(0,1)} \frac{yu + g}{u(1 + g^*yu)}du \right) p(y)y\,dy && u = e^{2i\phi},\ \frac{du}{d\phi} = 2iu,\ \frac{d\phi}{du} = \frac{-i}{2u} \text{ but as } \phi \in [0, 2\pi), \text{ there are two circles} \\
&= 2\pi g \int_0^1 p(y)y\,dy && \text{by residue theorem, with poles } u = 0 \text{ (inside) and } u = -1/(g^*y) \text{ (outside)} \\
&= g && \text{imposed by normalization of } p
\end{aligned}
$$

Proof with Taylor expansion

$$
\begin{aligned}
\langle e^{\mathrm{true}} \rangle &= \int_0^1 \left(-i \oint_{C(0,1)} \frac{yu + g}{u(1 + g^*yu)}du \right) p(y)y\,dy \\
&= \int_0^1 \left(-i \oint_{C(0,1)} y + \frac{g}{u} - g^*y^2u + \sum_{k=2}^\infty f_k u^k du \right) p(y)y\,dy && \text{with } f_k \text{ some factors} \\
&= 2\pi g \int_0^1 p(y)y\,dy && \text{as } \oint_{C(0,1)} z^m dz = \begin{cases} 2\pi i & m = -1 \\ 0 & \text{otherwise} \end{cases} \\
&= g
\end{aligned}
$$