Modified Gravity

A list of modified gravity for reference use

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Well, since I moved all my documents and softwares to a organization called CosmologyTaskForce on github, I should use Cosmology-TaskForce as the author. However, there is no members but myself untill now. That's why I just wrote my name here.

Abstract

This is an article for myself. I listed some basic knowledge about modified gravity.

f(R) Gravity

Keypoint: Theories that changes R to f(R) in Hillbert-Einstein action.

$\mathcal{L} = \sqrt{-g}\ R$ -> $\mathcal{L} = \sqrt{-g}\ f(R)$

A standard action of f(R) gravity is

$$S = \frac{1}{2\kappa}\int d^4 x \sqrt{-g}\ f(R) + \int d^4 x\, \mathcal{L}_M(g_{\mu\nu}, \Psi_M) \tag{1}$$

Thus the field equation can be calculated using the variation principle.

Field Equation

Variation of action

$$\begin{aligned} \delta \mathrm{S} &\equiv \frac{d\, S[\lambda]}{d\lambda} |_{\lambda=0} \\ &= \int d^4 x\, (\delta\mathcal{L}_G + \delta\mathcal{L}_m) \end{aligned} \tag{2}$$

Calculate the terms seperately. Useful expressions are lised here. $\delta \mathrm{g} = g\, g^{\mathrm{ab}}\, \delta \mathrm{g}_{\mathrm{ab}}$, $\delta \mathrm{g}^{\mathrm{ab}} = -g^{\mathrm{ab}}\, g^{\mathrm{bd}}\, \delta \mathrm{g}_{\mathrm{cd}}$. Some definations are also listed here. $F[R] \equiv \frac{d f[R]}{d R}$.

$$\begin{aligned} \delta\mathcal{L}_G &= \frac{1}{2\kappa^2}\delta\left(\sqrt{-g[\lambda]}\ f[R[\lambda]]\right) \\ &= \frac{1}{2\kappa^2}\frac{d\sqrt{-g[\lambda]}}{d\lambda}|_{\lambda=0} f[R] + \frac{1}{2\kappa^2}\sqrt{-g}\ \frac{d\, f[R]}{d\lambda}|_{\lambda=0} \\ &= \left(f[R]\left(\frac{-1}{2\sqrt{-g}}\right)\delta \mathrm{g} + \sqrt{-g}\ F[R]\,\delta \mathrm{R}\right)\frac{1}{2\kappa^2} \\ &= \left(\frac{1}{2} f[R]\, g^{\mathrm{ab}}\, \delta \mathrm{g}_{\mathrm{ab}} + \sqrt{-g}\ F[R]\left(\delta \mathrm{g}^{\mathrm{ab}}\right) R_{\mathrm{ab}} + \sqrt{-g}\ F[R]\, g^{\mathrm{ab}}\, \delta \mathrm{R}_{\mathrm{ab}}\right)\frac{1}{2\kappa^2} \\ &\equiv (\mathrm{M1} + \mathrm{M2} + \mathrm{M3})\frac{1}{2\kappa^2} \end{aligned} \tag{3}$$

Calculate M1, M2 and M3.

$$\mathrm{M1} = \frac{1}{2} f[R]\, g^{\mathrm{ab}}\, \delta \mathrm{g}_{\mathrm{ab}} \tag{4}$$

$$\begin{aligned} \mathrm{M2} &= \sqrt{-g}\; F[R]\,\delta g^{ab}\,R_{ab} \\ &= \sqrt{-g}\; F[R]\,R_{ab}\left(-g^{ac}\,g^{bd}\,\delta g_{cd}\right) \\ &= -\sqrt{-g}\; F[R]\,R^{ab}\,\delta g_{ab} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{M3} &= \sqrt{-g}\; F[R]\,g^{ab}\,\delta R_{ab} \\ &= \sqrt{-g}\; F[R]\,\nabla^a\left(\nabla^b\,\delta g_{ab} - g^{bc}\,\nabla_a\,\delta g_{bc}\right) \\ &= \sqrt{-g}\; F[R]\,\nabla^a\,v_a \\ &= \sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;v_a\,\nabla^a\,F[R] \\ &= \sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;\left(\nabla^b\,\delta g_{ab} - g^{bc}\,\nabla_a\,\delta g_{bc}\right)\nabla^a\,F[R] \\ &= \sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;\left(\nabla^b\,\delta g_{ab}\right)\nabla^a\,F[R] + \sqrt{-g}\;g^{bc}\,\nabla_a\,\delta g_{bc}\,\nabla^a\,F[R] \\ &= \sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;\nabla^b\,(\delta g_{ab}\,\nabla^a\,F[R]) + \\ &\qquad \sqrt{-g}\;\nabla^b\,\nabla^a\,F[R]\,\delta g_{ab} + \sqrt{-g}\;\nabla_e\left(g^{ba}\,\delta g_{ba}\,\nabla^e\,F[R]\right) - \sqrt{-g}\;\nabla_e\left(g^{ab}\,\nabla^e\,F\right)\delta g_{ab} \\ &= \sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;\nabla^b\,(\delta g_{ab}\,\nabla^a\,F[R]) + \sqrt{-g}\;\nabla^b\,\nabla^a\,F[R]\,\delta g_{ab} + \\ &\qquad \sqrt{-g}\;\nabla_e\left(g^{ba}\,\delta g_{ba}\,\nabla^e\,F[R]\right) - \sqrt{-g}\;g^{ab}\,\nabla_e\,(\nabla^e\,F)\,\delta g_{ab} - \sqrt{-g}\;\nabla_e\left(g^{ab}\right)\nabla^e\,\mathrm{F}\delta g_{ab} \end{aligned} \tag{6}$$

The integration of some terms will be zero. They are

$$\sqrt{-g}\;\nabla^a\,(F[R]\,v_a)$$

(for this is just similar to a term classical Hilbert Einstein action which is cancel by a additional term or by restating the Hamilton principle.)

$$-\sqrt{-g}\;\nabla^b\,(\delta g_{ab}\,\nabla^a\,F[R])$$

(for at the boundary δg_{ab} becomes zero.)

$$\sqrt{-g}\;\nabla_e\left(g^{ba}\,\delta g_{ba}\,\nabla^e\,F[R]\right)$$

(for at boundary δg_{ab} is zero.)

$$-\sqrt{-g}\;\nabla_e\left(g^{ab}\right)\nabla^e\,\mathrm{F}\delta g_{ab}$$

(for this is the trandition we made.)

The variation of the action of matter is

$$\delta S_m = \int d^4\,x\left(\frac{1}{2}\,T^{ab}\,\sqrt{-g}\right) \tag{7}$$

Finally the integration of these terms is

$$\begin{aligned} \delta S &= \int d^4\,x\,(\delta\mathcal{L}_G) + \delta S_m \\ &= \int d^4\,x\Big(\sqrt{-g}\;\nabla^a\,(F[R]\,v_a) - \sqrt{-g}\;\nabla^b\,(\delta g_{ab}\,\nabla^a\,F[R]) + \sqrt{-g}\;\nabla^b\,\nabla^a\,F[R]\,\delta g_{ab} + \\ &\qquad \sqrt{-g}\;\nabla_e\left(g^{ba}\,\delta g_{ba}\,\nabla^e\,F[R]\right) - \sqrt{-g}\;g^{ab}\,\nabla_e\,(\nabla^e\,F)\,\delta g_{ab} - \sqrt{-g}\;\nabla_e\left(g^{ab}\right)\nabla^e\,\mathrm{F}\delta g_{ab}\Big) + \int d^4\,x\left(\frac{1}{2}\,T^{ab}\,\sqrt{-g}\right) \end{aligned} \tag{8}$$

Drop the zero terms and we find out the field equaiton.

$$F[R]\,R_{ab} - \frac{1}{2}\,f[R]\,g_{ab} - \nabla_a\,\nabla_b\,F[R] + g_{ab}\,\Box F[R] = \kappa^2\,T_{ab} \tag{9}$$

We can find the trace of this equation because this is not a good equation to use.

Most of the time the trace is more simple than the field equation since it is a scalar equation. And the trace of this field equaiton is

$$g^{ab}\,F[R]\,R_{ab}-\frac{1}{2}\,f[R]\,g_{ab}\,g^{ab}-g^{ab}\,\nabla_a\,\nabla_b\,F[R]+g^{ab}\,g_{ab}\,\Box F[R]=g^{ab}\,\kappa^2\,T_{ab}$$
$$F[R]\,R-2\,f[R]+3\,\Box F[R]=\kappa^2\,T \tag{10}$$

Next is to consider the cosmological application.

FRW metric is

$$\mathrm{ds}^2=-\mathrm{dt}^2+(a[t])^2\,d\boldsymbol{x}^2 \tag{11}$$

To calculate the Ricci tensor.

```
Begin["MyContext`"];
Quiet[<< COSPER`];
x = {t, x1, x2, x3};
(met = {{-1, 0, 0, 0}, {0, a[t] * a[t], 0, 0}, {0, 0, a[t] * a[t], 0},
      {0, 0, 0, a[t] * a[t]}}) // MatrixForm;
Ricci[met, x]
SCurvature[met, x]
End[];
```

```
Ricci[
  {{-1, 0, 0, 0}, {0, a[t]^2, 0, 0}, {0, 0, a[t]^2, 0}, {0, 0, 0, a[t]^2}}, {t, x1, x2, x3}]
```

```
SCurvature[
  {{-1, 0, 0, 0}, {0, a[t]^2, 0, 0}, {0, 0, a[t]^2, 0}, {0, 0, 0, a[t]^2}}, {t, x1, x2, x3}]
```

Then 00 component

$$R_{00}=-3\left(\dot{H}+H^2\right) \tag{12}$$

$$R_{11}=R_{22}=R_{33}=2\,a^2\,H^2+a^2\left(\dot{H}+H^2\right)=3\,a^2\,H^2+a^2\,\dot{H} \tag{13}$$

$$R=6\left(6\,H^2+\dot{H}\right) \tag{14}$$

The energy momentum tensor is

$$T^{\mu}{}_{\nu}=\begin{pmatrix}-\rho_M & 0 & 0 & 0\\ 0 & P_M & 0 & 0\\ 0 & 0 & P_M & 0\\ 0 & 0 & 0 & P_M\end{pmatrix} \tag{15}$$

Trace of energy momentum tensor

$$T=g^{\nu}{}_{\mu}\,T^{\mu}{}_{\nu}=g_{\mu\lambda}\,g^{\lambda\nu}\,T^{\mu}{}_{\nu} \tag{16}$$

The 00 component of energy momentum tensor

$$T_{00}=g_{0\,\lambda}\,T^{\lambda}{}_{0}=g_{00}\,T^{0}{}_{0}=\rho_M \tag{17}$$

The ii component of energy momentum tensor

$$T_{11} = g_{1\lambda} T^{\lambda}{}_{1} = g_{11} T^{1}{}_{1} = a^2 P_M \tag{18}$$

Some useful techniques (in FRW space and F only envoles R):

$$\begin{aligned} \Box F &= \partial_a \partial^a F + \Gamma^{a}_{ac} \partial^c F \\ &= -\ddot{F} - 3 H \dot{F} \end{aligned} \tag{19}$$

$$\begin{aligned} \nabla_a \nabla_b F &= \partial_a \partial_b F - \Gamma^{c}_{ab} \partial_c F \\ &= \ddot{F} \end{aligned} \tag{20}$$

Put it into the two field equations

$$\begin{aligned} & 3 \Box F + F R - 2 f = \kappa^2 T \\ \Longrightarrow & 3 \Box F + F R - 2 f = \kappa^2 (-\rho_M + 3 P_M) \\ \Longrightarrow & 3 \left(-\ddot{F} - 3 \dot{F} H\right) + F R - 2 f = \kappa^2 (-\rho_M + 3 P_M) \\ \Longrightarrow & -3 \ddot{F} - 9 \dot{F} H + F R - 2 f = \kappa^2 (-\rho_M + 3 P_M) \end{aligned} \tag{21}$$

$$\begin{aligned} & F R_{00} - \frac{1}{2} f g_{00} - \nabla_0 \nabla_0 F + g_{00} \Box F = \kappa^2 T_{00} \\ \Longrightarrow & F \left(-3 \left(\dot{H} + H^2\right)\right) + \frac{1}{2} f - \nabla_0 \nabla_0 F - \Box F = \kappa^2 \rho_M \\ \Longrightarrow & -3 F \dot{H} - 3 F H^2 + \frac{1}{2} f - \ddot{F} - \Box F = \kappa^2 \rho_M \\ \Longrightarrow & -3 F \dot{H} - 3 F H^2 + \frac{1}{2} f - \ddot{F} + \ddot{F} + 3 H \dot{F} = \kappa^2 \rho_M \\ \Longrightarrow & -3 F \dot{H} - 3 F H^2 + \frac{1}{2} f + 3 H \dot{F} = \kappa^2 \rho_M \end{aligned} \tag{22}$$

$$\begin{aligned} & F R_{11} - \frac{1}{2} f g_{11} - \nabla_1 \nabla_1 F + g_{11} \Box F = \kappa^2 T_{11} \\ \Longrightarrow & F\left(3 a^2 H^2 + a^2 \dot{H}\right) - \frac{1}{2} f a^2 - \nabla_1 \nabla_1 F + a^2 \Box F = \kappa^2 a^2 P_M \\ \Longrightarrow & 3 F a^2 H^2 + a^2 F \dot{H} - \frac{1}{2} f a^2 - \nabla_1 \nabla_1 F + a^2 \Box F = \kappa^2 a^2 P_M \end{aligned} \tag{23}$$

Combine field equation (22) and its trace (21) above and we can find out two independent equations, because the trace of the equation is actually get by taking the trace of the field equation.

Eqn22 * 4 - Eqn21

$$-2 F \dot{H} = \ddot{F} - H \dot{F} + \kappa^2 (\rho_M + P_M) \tag{24}$$

Rewrite Eqn22 using the relation $R = 6 \left(6 H^2 + \dot{H}\right)$,

$$3 F H^2 = \frac{1}{2} (F R - f) - 3 H \dot{F} + \kappa^2 \rho_m \tag{25}$$

Most of the time we use equation (24) and equation (25) as the complete equation set to solve problem because this set is quite simple.

In Jordan frame, the conservation equation holds as the ordinary form which is derived from Bianchi identity.

$$T^{\mu\nu}{}_{;\mu} = G^{\mu\nu}{}_{;\mu} = 0 \tag{26}$$

This divergence can be calculated directly from a explicity $T^{\mu\nu}$. However it is simpler to switch to $T^{\mu}{}_{\nu;\mu}$ since we already defined $T^{\mu}{}_{\nu}$.

$$T^{\mu}{}_{\nu;\mu} = \partial_{\mu} T^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\lambda\mu} T^{\lambda}{}_{\nu} - \Gamma^{\lambda}{}_{\nu\mu} T^{\mu}{}_{\lambda} \tag{27}$$

$$\begin{aligned} T^{\mu}{}_{0;\mu} &= \partial_{\mu} T^{\mu}{}_{0} + \Gamma^{\mu}{}_{\lambda\mu} T^{\lambda}{}_{0} - \Gamma^{\lambda}{}_{0\mu} T^{\mu}{}_{\lambda} \\ &= \partial_{\mu} T^{\mu}{}_{0} + \Gamma^{\mu}{}_{\lambda\mu} T^{\lambda}{}_{0} - \Gamma^{\lambda}{}_{0\mu} T^{\mu}{}_{\lambda} \\ &= \partial_{0} T^{0}{}_{0} + \Gamma^{\mu}{}_{0\mu} T^{0}{}_{0} - \Gamma^{\lambda}{}_{0\mu} T^{\mu}{}_{\lambda} \\ &= -\dot{\rho}_M - 3\,\Gamma^{1}{}_{01}\,\rho_M - 3\,\Gamma^{1}{}_{01}\,T^{1}{}_{1} \\ &= -\dot{\rho}_M - 3\,H\,\rho_M - 3\,H\,P_M \end{aligned} \tag{28}$$

$$\begin{aligned} T^{\mu}{}_{i;\mu} &= \partial_{\mu} T^{\mu}{}_{i} + \Gamma^{\mu}{}_{\lambda\mu} T^{\lambda}{}_{i} - \Gamma^{\lambda}{}_{i\mu} T^{\mu}{}_{\lambda} \\ &= \partial_{i} T^{i}{}_{i} + \Gamma^{\mu}{}_{i\mu} T^{i}{}_{i} - \Gamma^{\lambda}{}_{i\mu} T^{\mu}{}_{\lambda} \end{aligned} \tag{29}$$

Attention: i is not summed.

Thus the conservation equation with ν=0 finally reads

$$\dot{\rho}_M + 3\,H(\rho_M + P_M) = 0 \tag{30}$$

Perturbation:

By light green and light blue I mean equations. By light purple I mean input and output.

Disformally Coupled Fields

Background: Field Eqns and Conservation Eqns

Conventions and Definations

Conservation Equation of Matter

Conservation Equation of Scalar Field

Calculate the energy momentum tensor of scalar fields.

$$X = -\frac{1}{2}\,\phi_{,\mu}\,\phi^{,\mu} \tag{47}$$

$$\frac{\delta \mathcal{L}_{\phi}(X,\phi)}{\delta g_{\mu\nu}} = \mathcal{L}_{\phi,X}\,\frac{\partial X}{\partial g_{\mu\nu}} \tag{48}$$

$$\frac{\partial X}{\partial g_{\mu\nu}} = -\frac{1}{2}\,\frac{\partial\,(g^{\sigma\tau}\,\phi_{,\sigma}\,\phi_{,\tau})}{\partial g_{\mu\nu}} = \frac{1}{2}\,\phi^{,\mu}\,\phi^{,\nu} \tag{49}$$

$$
\begin{aligned}
T^{\mu\nu}_{(\phi)} &\equiv \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\ \mathcal{L}_\phi\right)}{\delta g_{\mu\nu}} \\
&= \frac{2}{\sqrt{-g}} \left(\frac{\delta \sqrt{-g}}{\delta g_{\mu\nu}} \mathcal{L}_\phi + \sqrt{-g}\ \frac{\delta \mathcal{L}_\phi}{\delta g_{\mu\nu}} \right) \\
&= \frac{2}{\sqrt{-g}} \left(\frac{1}{2} \sqrt{-g}\ g^{\mu\nu} \mathcal{L}_\phi + \sqrt{-g} \left(\mathcal{L}_{\phi,X} \frac{\partial X}{\partial g_{\mu\nu}} + \mathcal{L}_{\phi,\phi} \frac{\partial \phi}{\partial g_{\mu\nu}} \right) \right) \\
&= g^{\mu\nu} \mathcal{L}_\phi + \phi^{\mu} \phi^{,\nu} \mathcal{L}_{\phi,X}
\end{aligned}
$$

Divergence of this scalar field energy momentum tensor

$$
\begin{aligned}
\nabla_\nu T^{\nu}_{\lambda(\phi)} &= \nabla_\nu \left(g_{\mu\lambda} T^{\mu\nu}_{(\phi)} \right) \\
&= g_{\mu\lambda} \nabla_\nu T^{\mu\nu}_{(\phi)} \\
&= g_{\mu\lambda} \nabla_\nu (g^{\mu\nu} \mathcal{L}_\phi + \phi^{\mu} \phi^{,\nu} \mathcal{L}_{\phi,X}) \\
&= g_{\mu\lambda} (g^{\mu\nu} \nabla_\nu \mathcal{L}_\phi + \nabla_\nu (\phi^{\mu} \phi^{,\nu}) \mathcal{L}_{\phi,X} + \phi^{\mu} \phi^{,\nu} \nabla_\nu \mathcal{L}_{\phi,X}) \\
&= g_{\mu\lambda} (g^{\mu\nu} (\mathcal{L}_{\phi,X} \nabla_\nu X + \mathcal{L}_{\phi,\phi} \phi_{,\nu}) + \nabla_\nu (\phi^{\mu} \phi^{,\nu}) \mathcal{L}_{\phi,X} + \phi^{\mu} \phi^{,\nu} (\mathcal{L}_{\phi,XX} \nabla_\nu X + \mathcal{L}_{\phi,X\phi} \phi_{,\nu}))
\end{aligned}
\tag{51}
$$

Use the following relation

$$
\nabla_\nu X = -\frac{1}{2} \nabla_\nu \left(g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) = -\frac{1}{2} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) = -\frac{1}{2} g_{\alpha\beta} \nabla_\nu \left(\phi^{,\alpha} \phi^{,\beta} \right)
\tag{52}
$$

$$
\nabla_\nu T^{\nu}_{\lambda(\phi)} = g_{\mu\lambda} \left(-g^{\mu\nu} \mathcal{L}_{\phi,X} \frac{1}{2} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + g^{\mu\nu} \mathcal{L}_{\phi,\phi} \phi_{,\nu} + \nabla_\nu (\phi^{\mu} \phi^{,\nu}) \mathcal{L}_{\phi,X} - \frac{1}{2} \mathcal{L}_{\phi,XX} \phi^{\mu} \phi^{,\nu} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + \mathcal{L}_{\phi,X\phi} \phi_{,\nu} \phi^{\mu} \phi^{,\nu} \right)
\tag{53}
$$

From action, we have a multicomponent field equation

$$
\nabla_\nu T^{\nu}_{\lambda(\phi)} + \nabla_\nu T^{\nu}_{\lambda(m)} = 0
\tag{54}
$$

We find

$$
\phi^{,\lambda} \nabla_\nu T^{\nu}_{\lambda(\phi)} = \mathrm{Q} \phi_{,\lambda} \phi^{,\lambda}
\tag{55}
$$

Simplify **LHS**

$$
\begin{aligned}
\textbf{LHS} &= \phi_{,\mu} \left(-g^{\mu\nu} \mathcal{L}_{\phi,X} \frac{1}{2} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + g^{\mu\nu} \mathcal{L}_{\phi,\phi} \phi_{,\nu} + \nabla_\nu (\phi^{\mu} \phi^{,\nu}) \mathcal{L}_{\phi,X} - \frac{1}{2} \mathcal{L}_{\phi,XX} \phi^{\mu} \phi^{,\nu} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + \mathcal{L}_{\phi,X\phi} \phi_{,\nu} \phi^{\mu} \phi^{,\nu} \right) \\
&= -\frac{1}{2} \phi_{,\mu} g^{\mu\nu} \mathcal{L}_{\phi,X} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + \phi_{,\mu} g^{\mu\nu} \phi_{,\nu} \mathcal{L}_{\phi,\phi} + \\
&\quad \phi_{,\mu} \mathcal{L}_{\phi,X} \nabla_\nu (\phi^{\mu} \phi^{,\nu}) - \frac{1}{2} \phi_{,\mu} \mathcal{L}_{\phi,XX} \phi^{\mu} \phi^{,\nu} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + \phi_{,\mu} \mathcal{L}_{\phi,X\phi} \phi_{,\nu} \phi^{\mu} \phi^{,\nu} \\
&= -\frac{1}{2} \phi_{,\mu} g^{\mu\nu} \mathcal{L}_{\phi,X} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) - 2\,\mathrm{X} \mathcal{L}_{\phi,\phi} + \phi_{,\mu} \mathcal{L}_{\phi,X} \nabla_\nu (\phi^{\mu} \phi^{,\nu}) + \mathrm{X} \mathcal{L}_{\phi,XX} \phi^{,\nu} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + 4\,\mathrm{XX} \mathcal{L}_{\phi,X\phi} \\
&= -\frac{1}{2} \phi_{,\mu} g^{\mu\nu} \mathcal{L}_{\phi,X} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + \phi_{,\mu} \mathcal{L}_{\phi,X} \nabla_\nu (\phi^{\mu} \phi^{,\nu}) + \mathrm{X} \mathcal{L}_{\phi,XX} \phi^{,\nu} g^{\alpha\beta} \nabla_\nu (\phi_{,\alpha} \phi_{,\beta}) + 2\,\mathrm{X} \mathcal{V}
\end{aligned}
\tag{56}
$$

Simplify **RHS**

$$
\textbf{RHS} = \left(\nabla_\nu \left(T^{\mu\nu} \frac{D}{C} \phi_{,\mu} \right) - T \frac{C'}{2\,C} - T^{\mu\nu} \frac{D'}{2\,C} \phi_{,\mu} \phi_{,\nu} \right) \phi_{,\lambda} \phi^{,\lambda}
$$

$$= -2X\left(\nabla_\nu\left(T^{\mu\nu}\frac{D}{C}\phi_\mu\right) - T\frac{C'}{2C} - T^{\mu\nu}\frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)$$
$$= -2X\left(\nabla_\nu\left(T^{\mu\nu}\frac{D}{C}\phi_\mu\right) - T^{\mu\nu}g_{\mu\nu}\frac{C'}{2C} - T^{\mu\nu}\frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)$$
$$= -2X\left(\nabla_\nu\left(T^{\mu\nu}\frac{D}{C}\phi_\mu\right) - T^{\mu\nu}\left(g_{\mu\nu}\frac{C'}{2C} + \frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)\right)$$
$$= -2X\left(\nabla_\nu(T^{\mu\nu})\frac{D}{C}\phi_\mu + T^{\mu\nu}\nabla_\nu\left(\frac{D}{C}\right)\phi_\mu + T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}\left(g_{\mu\nu}\frac{C'}{2C} + \frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)\right)$$
$$= -2X\left(\nabla_\nu(T^{\mu\nu})\frac{D}{C}\phi_\mu + T^{\mu\nu}\left(\frac{D'}{C} - \frac{C'D}{C^2}\right)\phi_{,\nu}\phi_\mu + T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}\left(g_{\mu\nu}\frac{C'}{2C} + \frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)\right)$$
$$= -2X\left(\nabla_\nu(T^{\mu\nu})\frac{D}{C}\phi_\mu + T^{\mu\nu}\left(-\frac{C'D}{C^2}\right)\phi_{,\nu}\phi_\mu + T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}\left(g_{\mu\nu}\frac{C'}{2C} - \frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)\right)$$
$$= -2X\left(\nabla_\nu(T^{\mu\nu})\frac{D}{C}\phi_\mu + T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}\left(g_{\mu\nu}\frac{C'}{2C} + \frac{C'D}{C^2}\phi_{,\nu}\phi_\mu - \frac{D'}{2C}\phi_\mu\phi_{,\nu}\right)\right)$$
$$= -2X\left(\nabla_\nu(T^{\mu\nu})\frac{D}{C}\phi_\mu + T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right)$$
$$= -2X\left(g^{\tau\mu}\nabla_\nu(T^\nu_\tau)\frac{D}{C}\phi_\mu\right) - 2X\left(T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right)$$
$$= -2X\left(g^{\tau\mu}(-Q\phi_{,\tau})\frac{D}{C}\phi_\mu\right) - 2X\left(T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right)$$
$$= 2X\left(\frac{D}{C}\textbf{RHS}\right) - 2X\left(T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right)$$

This gives us the result of **RHS**

$$\textbf{RHS} = -\frac{2XC}{C-2XD}\left(T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right) \tag{58}$$

Finally combine the results of **LHS** and **RHS**

$$\begin{aligned}
&-\frac{1}{2}\phi_\mu g^{\mu\nu}\mathcal{L}_{\phi X}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha}\phi_\beta) + \phi_\mu\mathcal{L}_{\phi X}\nabla_\nu(\phi^\mu\phi^{,\nu}) + X\mathcal{L}_{\phi,XX}\phi^{,\nu}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha}\phi_\beta) + 2X\mathcal{V} = \\
&\quad -\frac{2XC}{C-2XD}\left(T^{\mu\nu}\frac{D}{C}\nabla_\nu(\phi_\mu) - T^{\mu\nu}Q_{\mu\nu}\right) \\
&\Longrightarrow -\frac{1}{2}\phi_\mu g^{\mu\nu}\mathcal{L}_{\phi X}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha}\phi_\beta) + \phi_\mu\mathcal{L}_{\phi X}\nabla_\nu(\phi^\mu\phi^{,\nu}) + X\mathcal{L}_{\phi,XX}\phi^{,\nu}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha}\phi_\beta) \\
&\qquad +2X\mathcal{V} + \frac{2XD}{C-2XD}T^{\mu\nu}\nabla_\nu(\phi_\mu) - \frac{2XC}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0 \\
&\Longrightarrow -\phi^{,\nu}\mathcal{L}_{\phi X}\nabla_\nu(\phi_{,\alpha})\phi^{,\alpha} + \phi_\mu\mathcal{L}_{\phi X}\nabla_\nu(\phi_{,\sigma}g^{\sigma\mu}g^{\tau\nu}\phi_{,\tau}) + 2X\mathcal{L}_{\phi,XX}\phi^{,\nu}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha})\phi_\beta \\
&\qquad + \frac{2XD}{C-2XD}T^{\mu\nu}\nabla_\nu(\phi_\mu) + 2X\mathcal{V} - \frac{2XC}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0 \\
&\Longrightarrow -\phi^{,\nu}\mathcal{L}_{\phi X}\nabla_\nu(\phi_{,\alpha})\phi^{,\alpha} + \phi^{,\sigma}\mathcal{L}_{\phi X}\nabla_\nu(\phi_{,\sigma})\phi^{\nu} + \phi_\mu\mathcal{L}_{\phi X}\phi^\mu(\nabla_\nu(g^{\tau\nu}\phi_{,\tau})) + \\
&\qquad 2X\mathcal{L}_{\phi,XX}\phi^{,\nu}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha})\phi_\beta + \frac{2XD}{C-2XD}T^{\mu\nu}\nabla_\nu(\phi_\mu) + 2X\mathcal{V} - \frac{2XC}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0 \\
&\Longrightarrow -2X\mathcal{L}_{\phi X}(g^{\tau\nu}\nabla_\nu(\phi_{,\tau})) + 2X\mathcal{L}_{\phi,XX}\phi^{,\nu}g^{\alpha\beta}\nabla_\nu(\phi_{,\alpha})\phi_\beta + \frac{2XD}{C-2XD}T^{\mu\nu}\nabla_\nu(\phi_\mu) + 2X\mathcal{V} - \frac{2XC}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0 \\
&\Longrightarrow \left(-\mathcal{L}_{\phi X}g^{\mu\nu} + \mathcal{L}_{\phi,XX}\phi^{,\nu}\phi^\mu + \frac{D}{C-2XD}T^{\mu\nu}\right)\nabla_\nu(\phi_\mu) + \mathcal{V} - \frac{C}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0 \\
&\Longrightarrow \mathcal{M}^{\mu\nu}\nabla_\nu\nabla_\mu\phi + \mathcal{V} - \frac{C}{C-2XD}T^{\mu\nu}Q_{\mu\nu} = 0
\end{aligned} \tag{59}$$

Application to Cosmology

We use metric with a sign of +2. Choose the simplest model,

$$\mathcal{L}_\phi = X + V \tag{60}$$

We have to know how to calculate covariant derivative, so the Christoffel has to be calculated. Now we just assume the universe has FRW metric, which is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a[t]^2 & 0 & 0 \\ 0 & 0 & a[t]^2 & 0 \\ 0 & 0 & 0 & a[t]^2 \end{pmatrix} \tag{61}$$

The Christoffel is ($\Gamma^\mu_{\nu\lambda}$, μ as the first index, which means $\Gamma^0_{01} = \frac{a'[t]}{a[t]}$)

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \mathrm{a[t]\,a'[t]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \mathrm{a[t]\,a'[t]} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{a[t]\,a'[t]} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{a'[t]}}{\mathrm{a[t]}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

Here we calculate some terms first, remember we only choose the case that $\dot{\phi} \neq 0$

$$T^0_{0(m)} = -\rho, \;\; T^{00}{}_{(m)} = \rho, \;\; T^i_{j(m)} = 0 \tag{62}$$

$$\mathcal{L}_{\phi,X} = 1 \tag{63}$$

$$\mathcal{L}_{\phi,\phi} = V' \tag{64}$$

$$\mathcal{L}_{\phi,\mathrm{XX}} = \mathcal{L}_{\phi,\mathrm{X}\phi} = 0 \tag{65}$$

$$X = -\frac{1}{2}\dot{\phi}^2\, g^{00} = \frac{1}{2}\dot{\phi}^2 \tag{66}$$

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\alpha \phi_\beta = \partial_\alpha \phi_\beta - \Gamma^\lambda_{\alpha\beta}\, \phi_\lambda \tag{67}$$

Then we subsitute them into the conservation equations, before that,

$$\mathcal{M}^{\mu\nu} = -\mathcal{L}_{\phi,X}\, g^{\mu\nu} + \mathcal{L}_{\phi,XX}\, \phi^{,\nu}\, \phi^{,\mu} + \frac{D}{C-2XD} T^{\mu\nu}$$
$$= -g^{\mu\nu} + \frac{D}{C-2XD} T^{\mu\nu} \tag{68}$$

$$\mathcal{V} = 2X\,\mathcal{L}_{\phi,X\phi} - \mathcal{L}_{\phi,\phi} = -V' \tag{69}$$

$$Q_{00} = -\frac{C'}{2C} + \left(\frac{C'D}{C^2} - \frac{D'}{2C}\right)\dot{\phi}^2 \tag{70}$$

Put them in

$$\mathcal{M}^{\mu\nu}\, \nabla_\nu \nabla_\mu \phi + \mathcal{V} - \frac{C}{C-2XD} T^{\mu\nu} Q_{\mu\nu} = 0$$
$$\Longrightarrow \left(\frac{D}{C-2XD}\rho\right) \nabla_0 \phi_{,0} - g^{\mu\nu}\, \nabla_\mu \nabla_\nu \phi - V' - \frac{C}{C-2XD}\rho\left(-\frac{C'}{2C} + \frac{C'D}{C^2}\dot{\phi}^2 - \frac{D'}{2C}\dot{\phi}^2\right) = 0$$
$$\Longrightarrow \left(\frac{D}{C-2XD}\rho\right)\left(\ddot{\phi} - \Gamma^0_{00}\,\phi_{,0}\right) - g^{\mu\nu}\left(\partial_\mu \phi_{,\nu} - \Gamma^\lambda_{\mu\nu}\,\phi_{,\lambda}\right) - V' - \frac{C}{C-2XD}\rho\left(-\frac{C'}{2C} + \frac{C'D}{C^2}\dot{\phi}^2 - \frac{D'}{2C}\dot{\phi}^2\right) = 0$$
$$\Longrightarrow \left(\frac{C-2XD+D\rho}{C-2XD}\right)\ddot{\phi} - g^{ij}\,\Gamma^0_{ij}\,\dot{\phi} - V' - \frac{C}{C-2XD}\rho\left(-\frac{C'}{2C} + \frac{C'D}{C^2}\dot{\phi}^2 - \frac{D'}{2C}\dot{\phi}^2\right) = 0$$
$$\Longrightarrow \ddot{\phi} + \frac{C-2XD}{C-2XD+D\rho} 3H\dot{\phi} - \frac{C-2XD}{C-2XD+D\rho} V' + \frac{\rho}{C-2XD+D\rho}\frac{C'}{2} - \frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C} - \frac{D'}{2}\right)\dot{\phi}^2 = 0 \tag{71}$$

To make it more comfortable, format it as the normal scalar field equation

$$\ddot{\phi} = -\frac{C-2XD}{C-2XD+D\rho} V' - \frac{\rho}{C-2XD+D\rho}\frac{C'}{2} + \frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C} - \frac{D'}{2}\right)\dot{\phi}^2$$
$$\Longrightarrow \ddot{\phi} + 3H\dot{\phi} - V' =$$
$$3H\dot{\phi} - \frac{C-2XD}{C-2XD+D\rho} 3H\dot{\phi} - V' + \frac{C-2XD}{C-2XD+D\rho} V' - \frac{\rho}{C-2XD+D\rho}\frac{C'}{2} + \frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C} - \frac{D'}{2}\right)\dot{\phi}^2$$
$$\Longrightarrow \ddot{\phi} + 3H\dot{\phi} - V' = \frac{3H\dot{\phi}(C-2XD+D\rho)}{C-2XD+D\rho} - \frac{C-2XD}{C-2XD+D\rho} 3H\dot{\phi} - \frac{V'(C-2XD+D\rho)}{C-2XD+D\rho} +$$
$$\frac{(C-2XD)V'}{C-2XD+D\rho} - \frac{\rho}{C-2XD+D\rho}\frac{C'}{2} + \frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C} - \frac{D'}{2}\right)\dot{\phi}^2$$
$$\Longrightarrow \ddot{\phi} + 3H\dot{\phi} - V' = \frac{3H\dot{\phi}D\rho}{C-2XD+D\rho} - \frac{V'D\rho}{C-2XD+D\rho} - \frac{\rho}{C-2XD+D\rho}\frac{C'}{2} + \frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C} - \frac{D'}{2}\right)\dot{\phi}^2$$
$$\Longrightarrow \ddot{\phi} + 3H\dot{\phi} - V' = \left(3H\dot{\phi}D - V'D - \frac{C'}{2} + \dot{\phi}^2\left(\frac{C'D}{C} - \frac{D'}{2}\right)\right)\frac{\rho}{C-2XD+D\rho} \tag{72}$$

For a quintessence model, $-\mathcal{L}_\phi = \frac{1}{2}\phi_\mu\,\phi^\mu + V$ gives the result

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{73}$$

So it is better to use the form $\mathcal{L}_\phi = X - V$ here.

ONLY PROBLEM V'! If we have a different sign in front of V', or if we have $\mathcal{L}_\phi = X - V$.

$$\ddot{\phi} + 3H\dot{\phi} + V' = \frac{3H\dot{\phi}(C-2XD+D\rho)}{C-2XD+D\rho} - \frac{C-2XD}{C-2XD+D\rho} 3H\dot{\phi} +$$

$$
\begin{aligned}
&\frac{V'(C-2XD+D\rho)}{C-2XD+D\rho}-\frac{(C-2XD)V'}{C-2XD+D\rho}-\frac{\rho}{C-2XD+D\rho}\frac{C'}{2}+\frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C}-\frac{D'}{2}\right)\dot{\phi}^2\\
&\Longrightarrow \ddot{\phi}+3H\dot{\phi}+V'=\frac{3H\dot{\phi}D\rho}{C-2XD+D\rho}+\frac{V'D\rho}{C-2XD+D\rho}-\frac{\rho}{C-2XD+D\rho}\frac{C'}{2}+\frac{\rho}{C-2XD+D\rho}\left(\frac{C'D}{C}-\frac{D'}{2}\right)\dot{\phi}^2\\
&\Longrightarrow \ddot{\phi}+3H\dot{\phi}+V'=\left(3H\dot{\phi}D+V'D-\frac{C'}{2}+\left(\frac{C'D}{C}-\frac{D'}{2}\right)\dot{\phi}^2\right)\frac{\rho}{C-2XD+D\rho}\\
&\Longrightarrow \ddot{\phi}+3H\dot{\phi}+V'=\left(\left(3H\dot{\phi}+V'+\frac{C'}{C}\dot{\phi}^2\right)2D-C'-D'\dot{\phi}^2\right)\frac{\rho}{2(C-2XD+D\rho)}\\
&\Longrightarrow \ddot{\phi}+3H\dot{\phi}+V'=-Q_0\,\rho
\end{aligned}
$$

In which

$$
Q_0=-\frac{\left(3H\dot{\phi}+V'+\frac{C'}{C}\dot{\phi}^2\right)2D+C'+D'\dot{\phi}^2}{2(C-2XD+D\rho)} \tag{75}
$$

Perturbation

Perturbation of metric and Γ can be calculated with the program CosPer. Try it yourself.

Perturbed energy tensor of scalar field.

Energy momentum tensor of a scalar field with $\mathcal{L}_\phi = X + V$, in which $X = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu}$.

$$
\begin{aligned}
T^{\mu\nu}{}_{(\phi)} &= \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\ \mathcal{L}_\phi\right)}{\delta g_{\mu\nu}} = g^{\mu\nu}\mathcal{L}_\phi + \phi^{,\mu}\phi^{,\nu}\mathcal{L}_{\phi X}\\
&= g^{\mu\nu}(X+V)+\phi^{,\mu}\phi^{,\nu}\\
T^{00}{}_{(\phi)} &= -\left(\frac{1}{2}\dot{\phi}^2+V\right)+\dot{\phi}^2=\frac{1}{2}\dot{\phi}^2-V\\
T^{\mathrm{ij}}{}_{(\phi)} &= \frac{1}{a^2}\left(\frac{1}{2}\dot{\phi}^2+V\right)
\end{aligned} \tag{76}
$$

With one lower index

$$
T^{\mu}_{\nu(\phi)} = g_{\nu\lambda}T^{\mu\lambda}_{(\phi)} = g_{\nu\lambda}\left(g^{\mu\lambda}\mathcal{L}_\phi+\phi^{,\mu}\phi^{,\lambda}\mathcal{L}_{\phi X}\right)=\delta^\mu_\nu\mathcal{L}_\phi+g_{\nu\lambda}\phi^{,\mu}\phi^{,\lambda}\mathcal{L}_{\phi X}=\delta^\mu_\nu\mathcal{L}_\phi+g_{\nu\lambda}\phi^{,\mu}\phi^{,\lambda}=\delta^\mu_\nu\mathcal{L}_\phi+g^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} \tag{77}
$$

Perturbation of this scalar field EM tensor

$$
\begin{aligned}
\delta T^\mu_\nu &= \delta^\mu_\nu\,\delta\mathcal{L}_\phi+\left(\delta g^{\mu\lambda}\right)\phi_{,\lambda}\phi_{,\nu}+g^{\mu\lambda}\,\delta(\phi_{,\lambda}\phi_{,\nu})\\
&=\delta^\mu_\nu\,\delta\left(-\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}+V\right)+\left(\delta g^{\mu\lambda}\right)\phi_{,\lambda}\phi_{,\nu}+g^{\mu\lambda}\,\delta(\phi_{,\lambda}\phi_{,\nu})\\
&=\delta^\mu_\nu\left(-\frac{1}{2}\delta\left(g^{\alpha\beta}\right)\phi_{,\alpha}\phi_{,\beta}-\frac{1}{2}g^{\alpha\beta}\,\delta\left(\phi_{,\alpha}\phi_{,\beta}\right)+\delta V\right)+\left(\delta g^{\mu\lambda}\right)\phi_{,\lambda}\phi_{,\nu}+g^{\mu\lambda}\,\delta(\phi_{,\lambda}\phi_{,\nu})
\end{aligned} \tag{78}
$$

0-0 component

$$
\begin{aligned}
\delta T^0_0 &= -\frac{1}{2}\delta\left(g^{\alpha\beta}\right)\phi_{,\alpha}\phi_{,\beta}-\frac{1}{2}g^{\alpha\beta}\,\delta\left(\phi_{,\alpha}\phi_{,\beta}\right)+\delta V+\left(\delta g^{0\lambda}\right)\phi_{,\lambda}\phi_{,0}+g^{0\lambda}\,\delta(\phi_{,\lambda}\phi_{,0})\\
&=-\frac{1}{2}\delta\left(g^{00}\right)\phi_{,0}\phi_{,0}-\frac{1}{2}g^{\alpha\beta}(\delta\left(\phi_{,\alpha}\right)\phi_{,\beta}+\phi_{,\alpha}\,\delta\left(\phi_{,\beta}\right))+V'\,\delta\phi+\left(\delta g^{00}\right)\phi_{,0}\phi_{,0}+g^{0\lambda}(\delta(\phi_{,\lambda})\phi_{,0}+\phi_{,\lambda}\,\delta(\phi_{,0}))
\end{aligned} \tag{79}
$$