Spectral Sequence, My Homological Saw (Lecture Notes of Spectral Sequences)

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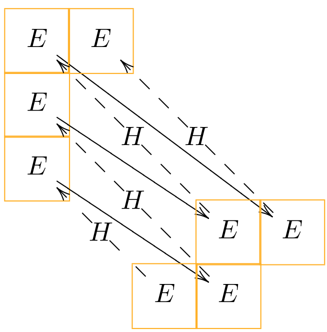
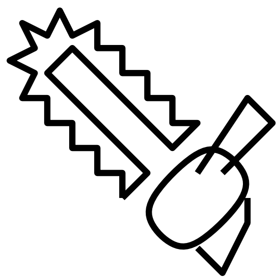
Contents





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0 Introduction

0.1. Quotation **Spectral sequence is the thing which really controls when we guess something should be controlled by one.**

0.2. Introduction **Spectral sequence** is a **powerful tool** widely used in different branches of modern mathematics. It plays a crucial role in **homological algebra**, **algebraic topology**, **algebraic geometry**, etc. Nowadays, it should be mastered by any master students in the relative areas. This lecture note is devoted to present its foundation and applications.

0.3. Construction In this part, the construction of spectral sequences will be discussed, including **filtered complexes** 1.8, **double complexes** 2.2, and **exact couples** 3.5. **Correct** and **self-contained** proofs will be presented.

First examples are given after the construction, including **simplicial cohomology** of a CW complex 1.12; the **Mayer–Vietoris spectral sequence** of topological spaces 2.11; how spectral sequences imply diagram chasing propositions 2.7.

0.4. Topology In this part, the **Leray–Serre spectral sequences** 4.6 are discussed in detail, with classic applications **Gysin sequences** 4.10, Thom isomorphisms 4.16.

Next, more contruction of spectral sequences are reviewed, **Eilenberg–Moore spectral sequences** 5.3, **Cartan–Leray spectral sequences** 5.6, **Atiyah–Hirzebruch spectral sequences** 5.8, and **Adams spectral sequences** 5.17.

0.5. Algebra In this part, we will discuss the **Künneth spectral sequences** 6.9 which is a corollary of its generalization 6.6. The tool is the **hyper-resolution** 6.2. We get properties of **Auslander–Reiten transpose** 6.12 using spectral sequence as an application.

Next, we will discuss the **Grothendieck spectral sequences** 7.2 which are very important. we will discuss **group (co)homology**, and the **Hochschild spectral sequence** 7.14. We will also define **Hochschild (co)homology** 7.20.

0.6. Geometry In this part, we will compute the cohomology of **projective spaces** 8.6, **Grassmannians** and **flag varieties** 8.7. Taking advantage of the computation, we will give an introduction to **Chern classes** 8.14.

Lastly, we will review more geometry, including **sheaf cohomology**, and **Hodge theory**. We will meet **Leray spectral sequence** 9.5 again, **Čech cohomology** 9.8, **spectral sequences for staratification** 9.19, and see **Frölicher spectral sequence** 9.24 and **Deligne degeneration theorem** 9.26.

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Comments and criticisms are welcome!

1 Construction (I)

Quick Definition

1.1. Given a complex (C, d), we define its **cohomology** $H(C) = \ker d/\operatorname{im} d$. It looks like

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & C^{i-1} \arrow[rr, "d"] \arrow[dr] & & \operatorname{im} d \arrow[rr, "d"] \arrow[dr] & & C^{i} \arrow[rr, "d"] \arrow[dr] & & \operatorname{im} d \arrow[rr, "d"] \arrow[dr] & & C^{i+1} \arrow[r, "d"] & \cdots \\
& & \operatorname{cok} d \arrow[ur] \arrow[rr, bend right] & & \ker d \arrow[ur] \arrow[dr] & & \operatorname{cok} d \arrow[ur] \arrow[rr, bend right] & & \ker d \arrow[ur] \arrow[dr] & & \\
& H(C) \arrow[ur] & & & & H(C) \arrow[ur] & & & & H(C) &
\end{tikzcd}$$

where the notation takes the usual cochain complex convention.

> We have $C = (C^i)_{i\in\mathbb{Z}}$, with $d_i : C^i \to C^{i+1}$, and $H^i(C)$ is a subquotient of C^i, i.e. $\ker d_i/\operatorname{im} d_{i+1}$.

Note that there is no natural map between $H(C)$ and C or among $H(C)$.

1.2. A **spectral sequence** is a series of complexes $(E_r)_{r=r_0}^{\infty}$ with $H(E_*) = E_{*+1}$. Note that the differential of E_r is "given", rather than induced. Usually, a spectral sequence is double graded in the following convention.

> We have $E_r = (\bigoplus_{n=p+q} E_r^{pq})_{n\in\mathbb{Z}}$ with $d : E_r \to E_r$ of double degree $(r, -r+1)$.

$$\begin{tikzcd}
E_1^{03} \arrow[r] & E_1^{13} \arrow[r] & E_1^{23} \arrow[r] & E_1^{33} \\
E_1^{02} \arrow[r] & E_1^{12} \arrow[r] & E_1^{22} \arrow[r] & E_1^{32} \\
E_1^{01} \arrow[r] & E_1^{11} \arrow[r] & E_1^{21} \arrow[r] & E_1^{31} \\
E_1^{00} \arrow[r] & E_1^{10} \arrow[r] & E_1^{20} \arrow[r] & E_1^{30}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2^{03} \arrow[drr] & E_2^{13} \arrow[drr] & E_2^{23} & E_2^{33} \\
E_2^{02} \arrow[drr] & E_2^{12} \arrow[drr] & E_2^{22} & E_2^{32} \\
E_2^{01} \arrow[drr] & E_2^{11} \arrow[drr] & E_2^{21} & E_2^{31} \\
E_2^{00} & E_2^{10} & E_2^{20} & E_2^{30}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_3^{03} \arrow[ddrrr] & E_3^{13} & E_3^{23} & E_3^{33} \\
E_3^{02} \arrow[ddrrr] & E_3^{12} & E_3^{22} & E_3^{32} \\
E_3^{01} & E_3^{11} & E_3^{21} & E_3^{31} \\
E_3^{00} & E_3^{10} & E_3^{20} & E_3^{30}
\end{tikzcd}$$

1.3. For a spectral sequence (E_r), we say E_r has an **algebraic limit**, if for $r \gg 0$, the differential of E_r is zero. In this case, we write its limit by E_∞.

For a spectral sequence (E_r), and an object H, if there is a bounded filtration over H such that the associated graded object $\operatorname{gr} H = E_\infty$. Then we say E_r **converges** to H. It is usually denoted by $E_r \Rightarrow H$.

Usually the filtration is decreasing if our convension is cohomological.

The $H = (H^n)$ is filtered by $F^p H^n$ which is smaller when p is bigger. The $\operatorname{gr}^p H^n = F^p H^n / F^{p+1} H^n$. If $E_r \Rightarrow H$, it means $E_\infty^{pq} = \operatorname{gr}^p H^{p+q}$. As a result, the left most is a quotient, the right most is a subobject. We write

$$E_r^{pq} \Longrightarrow H^{p+q}.$$

1.4. In practice, there would be more different sense of limit and convergence, see 1.20.

Filtered Complex

1.5. Modular property For three sub-objects A, B, C of some big object in abelian category. If $B \subseteq A$, then $B + (C \cap A) = (B + C) \cap A$. Thus it makes sense to write $B + C \cap A$.

1.6. For two bounded filtrations F_1 and F_2 on C, we can refine F_1 by F_2 by adding $\{F_1^{p+1} + F_2^\bullet \cap F_1^p\}$ between $F_1^{p+1} \subseteq F_1^p$. We can also refine F_2 by F_1 by adding $\{F_2^{q+1} + F_1^\bullet \cap F_2^q\}$ between $F_2^{q+1} \subseteq F_2^q$. They have the isomorphic associated graded objects by the following Zassenhaus' lemma.

1.7. Zassenhaus' lemma If $B \subseteq A$ and $D \subseteq C$ of some big object in abelian category, then the following maps are all isomorphisms,

$$\begin{tikzcd}
 & \dfrac{A \cap C}{(A \cap D) + (B \cap C)} \arrow[dl] \arrow[dr] & \\
\dfrac{B + C \cap A}{B + D \cap A} \arrow[dr] & & \dfrac{D + A \cap C}{D + B \cap C} \arrow[dl] \\
 & \dfrac{(A + D) \cap (B + C)}{B + D} &
\end{tikzcd}$$

1.8. Spectral Sequences for Filtered Complexes Assume we have a bounded filtration F of complex on a complex (C, d), namely each $F^* = F^* C$ is a complex. Then there is a spectral sequence E with $E_0 = \operatorname{gr} C$ with induced differential,

$$E_1 = H(\operatorname{gr} C) \Longrightarrow H(C),$$

with the filtration on $H(C)$ given by the image of $\{\operatorname{im} d + F^* \cap \ker d\}$ in $H(C)$.

If $C = (C^i, d)$, with filtration $F^p C^i$. Then $E_1^{pq} = H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet)$.

Proof The techenique is to refine F with the filtration with $\{d(F^*)\}$ between $0 \subseteq \operatorname{im} d$, and $\{d^{-1}(F^*)\}$ between $\ker d \subseteq C$. Define

$$\begin{cases} Z^p_{r-1} = F^{p+1} + d^{-1}(F^{p+r}) \cap F^p & \supseteq F^{p+1} + \ker d \cap F^p \\ B^p_{r-1} = F^{p+1} + d(F^{p+1-r}) \cap F^p & \subseteq F^{p+1} + \operatorname{im} d \cap F^p \end{cases}$$

Then we have

$$\begin{aligned} \frac{Z^p_{r-1}}{Z^p_r} &= \frac{F^{p+1} + d^{-1}(F^{p+r}) \cap F^p}{F^{p+1} + d^{-1}(F^{p+r+1}) \cap F^p} \\ &\overset{(*)}{=} \frac{d(F^{p+1}) + F^{p+r} \cap d(F^p)}{d(F^{p+1}) + F^{p+r+1} \cap d(F^p)} = \frac{F^{p+r} + d(F^p) \cap F^{p+r+1}}{F^{p+r} + d(F^{p+1}) \cap F^{p+r+1}} = \frac{B^{p+r}_r}{B^{p+r}_{r-1}} \end{aligned}$$

The $\overset{(*)}{=}$ follows from the lemma 1.9 below. Define

$$E^p_r = \frac{Z^p_{r-1}}{B^p_{r-1}} = \frac{F^{p+1} + d^{-1}(F^{p+r}) \cap F^p}{F^{p+1} + d(F^{p+1-r}) \cap F^p}$$

with

$$d = \left[E^p_r = \frac{Z^p_{r-1}}{B^p_{r-1}} \twoheadrightarrow \frac{Z^p_{r-1}}{Z^p_r} \cong \frac{B^{p+r}_r}{B^{p+r}_{r-1}} \hookrightarrow \frac{Z^{p+r}_{r-1}}{B^{p+r}_{r-1}} = E^{p+r}_r \right].$$

Then it is clear that $\ker \left[E^p_r \xrightarrow{d} \cdots \right] = \frac{Z^p_r}{B^p_{r-1}}$, and $\operatorname{im} \left[\cdots \xrightarrow{d} E^p_r \right] = \frac{B^p_r}{B^p_{r-1}}$. So the cohomology is $\frac{Z^p_r}{B^p_r} = E^p_{r+1}$. Thus above construction gives a spectral sequence. Now

$$\begin{aligned} E^p_\infty &= \frac{\bigcap Z^p_r}{\bigcup B^p_r} = \frac{\bigcap F^{p+1} + d^{-1}(F^{p+r}) \cap F^p}{\bigcup F^{p+1} + d(F^{p-r-1}) \cap F^p} \\ &\overset{(*)}{=} \frac{F^{p+1} + \bigcap d^{-1}(F^{p+r}) \cap F^p}{F^{p+1} + \bigcup d(F^{p-r-1}) \cap F^p} = \frac{F^{p+1} + \ker d \cap F^p}{F^{p+1} + \operatorname{im} d \cap F^p} \\ &= \frac{\operatorname{im} d + F^p \cap \ker d}{\operatorname{im} d + F^{p+1} \cap \ker d} \end{aligned}$$

The equality $\overset{(*)}{=}$ uses the assumption of being a bounded filtration. To complete the proof, we need to compute the case $r = 1$,

$$E^p_1 = \frac{Z^p_0}{B^p_0} = \frac{F^{p+1} + d^{-1}(F^{p+1}) \cap F^p}{F^{p+1} + d^{-1}(F^p) \cap F^p} = H(F^p / F^{p+1}).$$

This what asserted in the theorem. Q.E.D.

1.9. Lemma Let $B \subseteq A \subseteq X$, and $D \subseteq C \subseteq Y$ with a morphism $X \to Y$, then the natural map

$$\frac{B + f^{-1}(C) \cap A}{B + f^{-1}(D) \cap A} \longrightarrow \frac{f(B) + C \cap f(A)}{f(B) + D \cap f(A)}$$

is an isomorphism. See also 1.16.

1.10. Actually, the differential on E_1 is given by

$$H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet) \xrightarrow{\delta} H^{p+q+1}(F^{p+1} C^\bullet) \longrightarrow H^{p+1+q}(F^{p+1} C^\bullet / F^p C^\bullet).$$

This follows from the diagram chasing — the map is induced by d.

Examples

1.11. Long exact sequence Consider a short exact sequence of complex

$$0 \longrightarrow D \longrightarrow C \longrightarrow Q \longrightarrow 0.$$

Consider the filtration on C by $C \supseteq D \supseteq 0$. Then the spectral sequence looks like

$$\begin{tikzcd} Q^{i+1} & D^{i+2} \\ Q^i \arrow[u] & D^{i+1} \arrow[u] \\ Q^{i-1} \arrow[u] & D^i \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} H^{i+1}(Q) \arrow[r] & H^{i+2}(D) \\ H^i(Q) \arrow[r] & H^{i+1}(D) \\ H^{i-1}(Q) \arrow[r] & H^i(D) \end{tikzcd} \qquad \begin{tikzcd} \ker^{i+1} & \operatorname{cok}^{i+2} \\ \ker^i & \operatorname{cok}^{i+1} \\ \ker^{i-1} & \operatorname{cok}^i \end{tikzcd}$$

Thus $E_2 = E_\infty$. We have exact sequences

$$0 \longrightarrow \ker^i \longrightarrow H^i(Q) \longrightarrow H^{i+1}(D) \longrightarrow \operatorname{cok}^{i+1} \longrightarrow 0$$

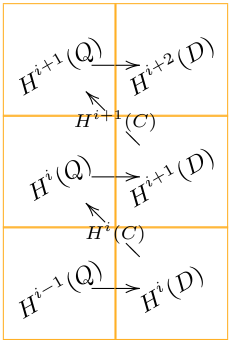
The convergence gives short exact sequences

$$0 \longrightarrow \operatorname{cok}^i \longrightarrow H^i(C) \longrightarrow \ker^i \longrightarrow 0.$$

Thus we can connect them to get the classic long exact sequence

$$\cdots \longrightarrow H^i(D) \longrightarrow H^i(C) \longrightarrow H^i(Q) \longrightarrow H^{i+1}(D) \longrightarrow \cdots .$$

Actually, the differential of E_1 coincides with the connecting morphism δ in the long exact sequence. This can be remembered by the following diagram



1.12. Simplicial Cohomology Denote $\mathsf{Sing}^\bullet(-)$ the complex computing singular cohomology. We have a surjective map $\mathsf{Sing}(X) \to \mathsf{Sing}(U)$ for any $U \subseteq X$ by restriction. Denote the kernel to be $\mathsf{Sing}(X, U)$. Note that it computes relative cohomology $H(X, U)$.

Let X be a CW complex. We firstly assume X is of finite dimensional. Denote X_k the union of cells of dimension $\leq k$ and $X_{-1} = \varnothing$. Then $\mathsf{Sing}(X, X_*)$ forms a filtration on $\mathsf{Sing}(X)$, with the associated graded complex to be $\mathsf{Sing}(X_*, X_{*+1})$. It is known that

$$H^{p+q}(X_p, X_{p-1}) = \begin{cases} \mathbb{Z}^{f_p}, & q = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad f_p = \#\{p\text{-cells}\}.$$

So the spectral sequences looks like

$$\boxed{\mathbb{Z}^{f_0}} \to \boxed{\mathbb{Z}^{f_1}} \to \boxed{\mathbb{Z}^{f_2}} \to \boxed{\mathbb{Z}^{f_3}}$$

It turns out, it coincides with the simplicial complex. This inspires the Leray–Serre spectral sequence 4.6.

1.13. Remark Remind the following homological algebra,

For any complex morphism $B \to C$, we can factor through C', with $C' \to C$ a homotopy-equivalence, and $B \to C'$ is injective.

$$\begin{tikzcd} & C' \arrow[d, "\wr"] \\ B \arrow[ur, hook] \arrow[r, "f"'] & C \end{tikzcd}$$

Actually, C' is the mapping cylinder, and the resulting quptient is the mapping cone, see 1.21.

For any complex morphism $B \xrightarrow{f} C$, we can find a complex D, with morphism $C \xrightarrow{g} D$ and $D \xrightarrow{h} B[1]$, such that the induced map in cohomology is a long exact sequence

$$\begin{tikzcd} H(B) \arrow[rr, "f"] & & H(C) \arrow[dl, "g"] \\ & H(D) \arrow[ul, "h"] & \end{tikzcd}$$

Actually, D is exactly the mapping cone, see 1.22. Actually, such $H(D)$ is uniquely determined by 5-lemma.

From this point of view, the assumption to be filtered complex is techenique, we can even deal with a series of morphisms of complexes $\cdots \to F^2 \to F^1 \to C$. Or, a series of triangles (triple of morphisms inducing long exact sequence as above). This can be generalized to exact couple 3.5.

Exercises

1.14. Prove 1.5, 1.7, 1.9.

1.15. Generalized Modular Property Show the projective formula for abelian sub-objects

$$f(f^{-1}(A') \cap C) = A' \cap f(C) \qquad f^{-1}(B' + f(D)) = f^{-1}(B') + D.$$

1.16. Functorial Zassenhaus' Lemma Assume we have two sets (A, B, C, D) and (A', B', C', D') in Zassenhaus' Lemma 1.7. If there is a morphism f between the big objects with $f(A) \subseteq A'$, etc. Then we have

$$\frac{B + C \cap A}{B + C \cap A} \longrightarrow \frac{B' + C' \cap A'}{B' + D' \cap A'}$$

Show that when $f(A \cap C) = A' \cap C'$, then this map is surjective; when $B + D = f^{-1}(B' + D')$, this map is injective.

1.17. Boundedness We say a filtraion F of a module C is **exhaustive** if $\bigcup F = C$; is **bounded below** if $C^i = 0$ for some i. Note that this does **NOT** make sense in arbitrary abstract abelian category.

1.18. Exchange of Limit For two submodules $B \subseteq A$ of some big object, and a filtraion $C^\bullet$, prove that

$$\bigcup (B + C^\bullet \cap A) = B + \bigcup C^\bullet \cap A.$$

If $C^\bullet$ is bounded below, show that

$$\bigcap (B + C^\bullet \cap A) = B + \bigcap C^\bullet \cap A.$$

1.19. Classic Limit For a spectral sequence $(E_r)_{r=r_0}^\infty$ of modules, E_r is a subquotient of E_{r_0}, writing $E_r = Z_{r-1}/B_{r-1}$, we define the **classic limit** $E_\infty = \bigcap Z_r / \bigcup B_r$.

For a spectral sequence (E_r), and an object H. If there is an exhaustive and bounded below filtration over H such that the associated graded object $\operatorname{gr} H = E_\infty$. Then we say E_r **converges** to H (in the classic sense).

Note that these do **NOT** make sense in arbitrary abstract abelian category.

1.20. Classic Convergence Check that 1.8 still holds in the classic sense for modules for exhaustive and bounded below filtered complex.

1.21. Mapping Cylinder Let $f : B \to C$ be a morphism of complexes. Define the **mapping cylinder**

$$\mathrm{cyl}(f) : \quad \begin{tikzcd}
& i-1 & i & i+1 & \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d"] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r] & \cdots \\
& \oplus & \oplus & \oplus & \\
\cdots \arrow[r] \arrow[uur] \arrow[ddr] & B^i \arrow[r, "-d"] \arrow[uur, "\mathrm{id}"] \arrow[ddr, "-f"'] & B^{i+1} \arrow[r, "-d"] \arrow[uur, "\mathrm{id}"] \arrow[ddr, "-f"'] & B^{i+2} \arrow[r] \arrow[uur] \arrow[ddr] & \cdots \\
& \oplus & \oplus & \oplus & \\
\cdots \arrow[r] & C^{i-1} \arrow[r, "d"'] & C^i \arrow[r, "d"'] & C^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Show that the map

$$\begin{tikzcd} C \arrow[r, shift left, "x \mapsto (0,0,x)"] & \mathrm{cyl}(f) \arrow[l, shift left, "(b,*,x) \mapsto f(b)+x"] \end{tikzcd}$$

gives the homotopoy equivalence. Actually, $(b, b', x) \mapsto (0, b, x)$ gives the homotopy for $\mathrm{cyl}(f) \to \mathrm{cyl}(f)$.

Now $B \to C$ factors through $\mathrm{cyl}(f)$ by $b \mapsto (b, 0, 0)$, and it is obviously an injective.

1.22. Mapping Cone The resulting quotient is the **mapping cone**

$$\mathrm{cone}(f): \quad \begin{tikzcd} & i-1 & i & i+1 & \\ \cdots \arrow[r] \arrow[dr] & B^i \arrow[r, "-d"] \arrow[dr, "-f"] & B^{i+1} \arrow[r, "-d"] \arrow[dr, "-f"] & B^{i+2} \arrow[r] \arrow[dr] & \cdots \\ & \oplus & \oplus & \oplus & \\ \cdots \arrow[r] & C^{i-1} \arrow[r, "d"'] & C^i \arrow[r, "d"'] & C^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

Then, we have

$$B \xrightarrow{f} C \xrightarrow{x \mapsto (0,x)} \mathrm{cone}(f) \xrightarrow{(b,x) \mapsto -b} B[1].$$

Show that it induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H(B) \arrow[r, "f"] \arrow[d, "\wr"] & H(\mathrm{cyl}\, f) \arrow[r] \arrow[d, "\wr"] & H(\mathrm{cone}\, f) \arrow[r, "\delta"] \arrow[d, "\wr"] & H(B) \arrow[r] \arrow[d, "\wr"] & \cdots \\ \cdots \arrow[r] & H(B) \arrow[r, "f"'] & H(C) \arrow[r] & H(\mathrm{cone}\, f) \arrow[r] & H(B) \arrow[r] & \cdots \end{tikzcd}$$

To do this, we need to remind how δ is given in long exact sequence.

2 Construction (II)

Double Complexes

2.1. Consider a double complex $C = (C^{pq})_{pq}$. We define its **total** (under the Koszul convention, see 2.12)

$$\mathrm{Tot}\, C : \quad \begin{tikzcd} p+q-1 & p+q & p+q+1 \\ C^{p-1,q} \arrow[dr, "d"] & \oplus & C^{p,q+1} \\ \oplus & C^{pq} \arrow[ur, "(-1)^p d"] \arrow[dr, "d"] & \oplus \\ C^{p,q-1} \arrow[ur, "(-1)^p d"] \arrow[dr, "d"] & \oplus & C^{p+1,q} \\ \oplus & C^{p+1,q-1} \arrow[ur, "(-1)^{p+1} d"] \arrow[dr, "d"] & \oplus \\ C^{p+1,q-2} \arrow[ur, "(-1)^{p+1} d"] & \oplus & C^{p+2,q-1} \end{tikzcd}$$

Formally, the differential restricted on the C^{pq} summand is

$$d = d_{(1,0)} + (-1)^p d_{(0,1)}.$$

The purpose of next theorem is to analyse the cohomology of $\mathrm{Tot}\, C$ using spectral sequences.

2.2. Spectral Sequences for Double Complexes For a double complex C, if $C^{pq} = 0$ for $|p| \gg 0$, then there is a spectral sequence E with $E_0 = (C, d_{(0,1)})$, and the differential of E_1 induced by $\pm d_{(1,0)}$ (under the Koszul convention),

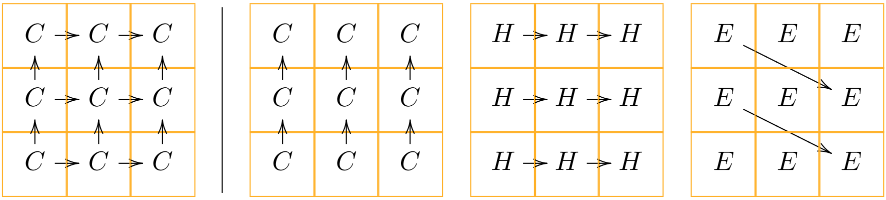
$$E_2 = H(H(C, d_{(0,1)}), d_{(1,0)}) \Longrightarrow H(\mathrm{Tot}\, C).$$

Note that we do not assert any information about the differential of E_2.

Under the cohomological convention, $E_2^{pq} = H^p(H^q(C, d_{(0,1)}), d_{(1,0)}) \Rightarrow H^{p+q}(\mathrm{Tot}\, C)$.

Proof We have the "column filtration" $\mathrm{Tot}(C^{pq})_{p \geq *}$ for $\mathrm{Tot}\, C$. The associated graded complex (for each $*$) is exactly $(C^{pq}, d_{(0,1)})_{p=*}$. This is E_0. By the proof of 1.8, a little diagram chasing, the map of E_1 is given by $\pm d_{(1,0)}$. We see that any sign exchanging is harmless. This is the proof. Q.E.D.

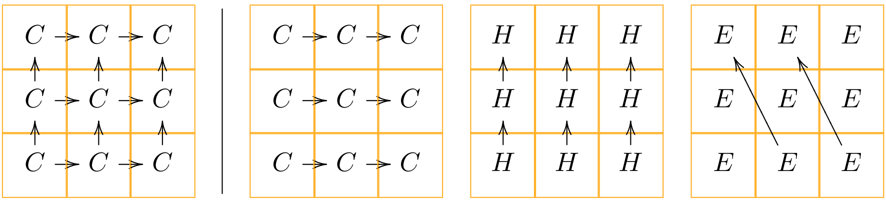
2.3. For a double complex, E_2 is obtained by computing the cohomology of $d_{(0,1)}$, and then computing the cohomology of $d_{(1,0)}$



2.4. For a double complex C, if $C^{pq} = 0$ for $|q| \gg 0$, then there is another spectral sequence E' with $E_0 = (C, d_{(1,0)})$, and the differential of E_1 induced by $d_{(0,1)}$,

$$E_2' = H(H(C, d_{(1,0)}), d_{(0,1)}) \Longrightarrow H(\operatorname{Tot} C).$$

But in this case, the convention will be modified



To avoid this, formally we take the transposition firstly.

> Under the cohomological convention, $E_2^{qp} = H^q(H^p(C, d_{(1,0)}), d_{(0,1)}) \Rightarrow H^{p+q}(C)$. Note that q is the first entry.

But in practice, we will not restrict ourselves in this convention.

Examples

2.5. Snake Lemma Assume we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "s"] & B \arrow[r, "g"] \arrow[d, "t"] & C \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & X \arrow[r, "k"] & Y \arrow[r, "h"] & Z &
\end{tikzcd}$$

with each row exact. We can think it as a double complex. Then the spectral sequence of it gives

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \ker f \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 & 0 & \operatorname{cok} h \end{tikzcd}$$

from which we know the cohomlogy of total is

$$\ker f, \qquad 0, \qquad 0, \qquad \operatorname{cok} h.$$

On the other hand, the spectral sequence for the other direction gives

$$\begin{tikzcd} A \arrow[d] & B \arrow[d] & C \arrow[d] \\ X & Y & Z \end{tikzcd} \qquad \begin{tikzcd} \ker s \arrow[r] & \ker t \arrow[r] & \ker r \\ \operatorname{cok} s \arrow[r] & \operatorname{cok} t \arrow[r] & \operatorname{cok} r \end{tikzcd} \qquad \begin{tikzcd} K_1 & M & C_1 \\ K_2 \arrow[urr] & N & C_2 \end{tikzcd}$$

Since we have computed the total, we know $K_1 = \ker f$, $C_2 = \operatorname{cok} h$, $M = 0$, $N = 0$, and $K_2 \to C_1$ is an isomorphism. This gives the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] \arrow[d, equal] & \ker s \arrow[r] \arrow[d] & \ker t \arrow[r] \arrow[d] & \ker r \arrow[d] \arrow[dddll, rounded corners] & & \\ 0 \arrow[r] & \ker f \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\ & 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \operatorname{cok} h \arrow[r] \arrow[d, equal] & 0 \\ & & \operatorname{cok} s \arrow[r] & \operatorname{cok} t \arrow[r] & \operatorname{cok} r \arrow[r] & \operatorname{cok} h \arrow[r] & 0 \end{tikzcd}$$

2.6. We can view a commutative square as a double complex

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

We denote the cohomology of its total by H^{-1}, H^0 and H^1

$$\begin{array}{ccccccccc} & & -1 & & 0 & & 1 & & \\ 0 & \longrightarrow & A & \xrightarrow{f+g} & B\oplus C & \xrightarrow{h-k} & D & \longrightarrow & 0. \end{array}$$

The spectral sequence looks like

$$\begin{tikzcd} A \arrow[d] & B \arrow[d] \\ C & D \end{tikzcd} \qquad \begin{tikzcd} \ker g \arrow[r] & \ker h \\ \operatorname{cok} g \arrow[r] & \operatorname{cok} h \end{tikzcd} \qquad \begin{tikzcd} K & M_1 \\ M_2 & C \end{tikzcd}$$

So we obtain an exact sequence

$$0\to H^{-1}\to \ker g\to \ker h\to H^0\to \operatorname{cok} g\to \operatorname{cok} h\to H^1\to 0.$$

Actually each braid of the following diagram is exact

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr] & & \ker f \arrow[rr, bend left] \arrow[dr] & & \ker k \arrow[rr, "B"] \arrow[dr] & & \operatorname{cok} g \arrow[rr, bend left] \arrow[dr] & & \operatorname{cok} h \arrow[rr, bend left] \arrow[dr] & & 0 \\
& H^{-1} \arrow[ur] \arrow[dr] & & \ker d \arrow[ur] \arrow[dr] & & H^0 \arrow[ur] \arrow[dr] & & \operatorname{cok} d \arrow[ur] \arrow[dr] & & H^1 \arrow[ur] \arrow[dr] & \\
0 \arrow[rr, bend right] \arrow[ur] & & \ker g \arrow[rr, bend right] \arrow[ur] & & \ker h \arrow[rr, "C"'] \arrow[ur] & & \operatorname{cok} f \arrow[rr, bend right] \arrow[ur] & & \operatorname{cok} k \arrow[rr, bend right] \arrow[ur] & & 0
\end{tikzcd}$$

See 2.13 for the exactness of the rest two braids.

2.7. Transgression Let $f: C\to D$ be a morphism of two exact complexes. Then $\ker f$ and $\operatorname{cok} f$ are both complexes. There is a natural isomorphism

$$H^{i-1}(\operatorname{cok} f)\longrightarrow H^{i+1}(\ker f).$$

On one hand,

$$\begin{tikzcd} \cdots \arrow[r] & C \arrow[r] & C \arrow[r] & C \arrow[r] & \cdots \\ \cdots \arrow[r] & D \arrow[r] & D \arrow[r] & D \arrow[r] & \cdots \end{tikzcd} \qquad \begin{tikzcd} \cdots & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{tikzcd}$$

So the total is zero. On the other hand,

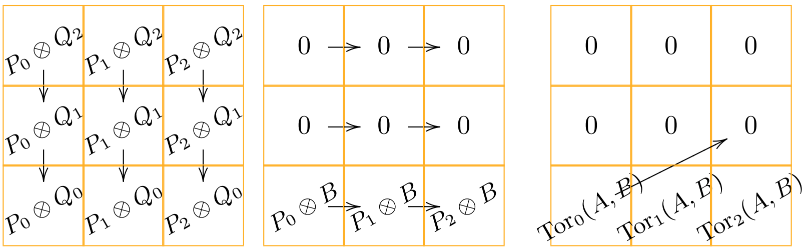
$$\begin{tikzcd} \cdots & C \arrow[d] & C \arrow[d] & C \arrow[d] & \cdots \\ \cdots & D & D & D & \cdots \end{tikzcd} \qquad \begin{tikzcd} \cdots \arrow[r] & \ker f \arrow[r] & \ker f \arrow[r] & \ker f \arrow[r] & \cdots \\ \cdots \arrow[r] & \operatorname{cok} f \arrow[r] & \operatorname{cok} f \arrow[r] & \operatorname{cok} f \arrow[r] & \cdots \end{tikzcd} \qquad \begin{tikzcd} \cdots & H(\ker f) & H(\ker f) & H(\ker f) & \cdots \\ \cdots & H(\operatorname{cok} f) \arrow[ur] & H(\operatorname{cok} f) \arrow[ur] & H(\operatorname{cok} f) \arrow[ur] & \cdots \end{tikzcd}$$

By our computation of total, all maps $H(\operatorname{cok} f)\to H(\ker f)$ is an isomorphism. This recovers above two examples.

2.8. Balanced Tor and Ext Let A, B be two (right and left) modules (for simplicity). We can pick projective resolutions $P \to A$ and $Q \to B$. Then $P \otimes Q$ is a double complex. Then

$$H_n(\mathrm{Tot}(P \otimes Q)) = \mathrm{Tor}_n(A, B).$$

Actually,



Similarly, we have the similar result for Ext. Pick an injective resolution $B \to I$. Then

$$H^n(\mathrm{Tot}(\mathrm{Hom}(P, I))) = \mathrm{Ext}^n(A, B).$$

2.9. Derived Functor, Acyclic Object For a left exact functor F. Recall the definition of the **derived functor** $R^i F$.

> For any object A, picking an injective resolution, $A \to I$, we define the **derived functor** $R^i F = H^i(F(I))$. We say an object A is **F-acyclic** if $R^i F(A) = 0$ for $i \geq 1$.

It is standard homological algebra to check it does not depend on the choice of resolutions up to isomorphisms, and literally gives a functor.

Up to taking the opposite category, Ext, Tor are examples of it. Note that for an injective object I, it is always F-acyclic by definition. In the case of Tor, flag object is Tor-acyclic (in the opposite category).

2.10. Acyclicity Enough If $A \to I$ is a resolution with each I^i being F-acyclic, then $R^i F(A) = H^i(F(I))$. That is, to compute $R^i F$, it suffices to use F-acyclic resolution.

Actually, we can resolve each I^i by an injective resolution $I^i \to J^i$. Then J forms a bicomplex. Note that $\mathrm{Tot}\, J$ is an injective resolution for A by the following spectral sequence argument

$$\begin{tikzcd} J^{02} & J^{12} & J^{22} \\ J^{01} \arrow[u] & J^{11} \arrow[u] & J^{21} \arrow[u] \\ J^{00} \arrow[u] & J^{10} \arrow[u] & J^{20} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & 0 \\ I^0 \arrow[r] & I^1 \arrow[r] & I^2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A \arrow[urr] & 0 & 0 \end{tikzcd}$$

Then $R^iF(A) = H^i(\operatorname{Tot} F(J))$ by definition. Now apply F, and use the spectral sequence argument,

$$\begin{tikzcd} F(J^{02}) & F(J^{12}) & F(J^{22}) \\ F(J^{01}) \arrow[u] & F(J^{11}) \arrow[u] & F(J^{21}) \arrow[u] \\ F(J^{00}) \arrow[u] & F(J^{10}) \arrow[u] & F(J^{20}) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} R^2F(I^0) \arrow[r] & R^2F(I^1) \arrow[r] & R^2F(I^2) \\ R^1F(I^0) \arrow[r] & R^1F(I^1) \arrow[r] & R^1F(I^2) \\ F(I^0) \arrow[r] & F(I^1) \arrow[r] & F(I^2) \end{tikzcd} = \begin{tikzcd} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F(I^0) \arrow[r] & F(I^1) \arrow[r] & F(I^2) \end{tikzcd}$$

This inspires the Grothendieck spectral sequence.

2.11. Čech Cohomology Let X be a topological space with a finite open covering $\mathcal{U}$. We use $\mathsf{Sing}^\bullet(-)$ to stand the complex computing singular cohomology. Denote for $p \geq 0$

$$U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}, \qquad U^p = \coprod_{i_0 < \ldots < i_p} U_{i_0,\ldots,i_p}.$$

Here the disjoint union is formal. We denote the restriction

$$\mathrm{res}_{i_\ell} : \quad \mathsf{Sing}(U_{i_0,\ldots,\hat{i_\ell},\ldots,i_p}) \longrightarrow \mathsf{Sing}(U_{i_0,\ldots,i_p}).$$

Consider the double complex $\check{C}$ with

$$\check{C}^{pq} = \mathsf{Sing}^q(U^p) = \prod_{i_0 < \ldots < i_p} \mathsf{Sing}^q(U_{i_0,\ldots,i_p}), \qquad d_{(1,0)} = \prod_{i_0 < \ldots < i_p} \sum_{\ell=0}^{p} (-1)^\ell \, \mathrm{res}_{i_\ell}.$$

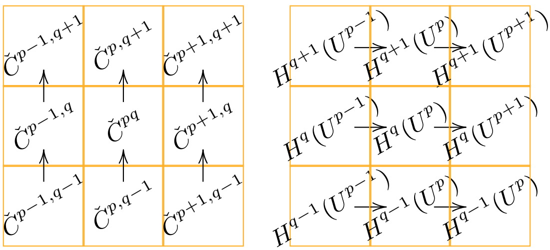
For example, for three open sets A, B, C

$$\begin{tikzcd}
 & 0 & 1 & 2 \\
 & \mathsf{Sing}(A) \arrow[r, dotted] \arrow[dr, dotted] & \mathsf{Sing}(A\cap B) \arrow[dr] & \\
\mathsf{Sing}(X) \arrow[ur] \arrow[r] \arrow[dr] & \mathsf{Sing}(B) \arrow[ur] \arrow[dr, dotted] & \mathsf{Sing}(A\cap C) \arrow[r, dotted] & \mathsf{Sing}(A\cap B\cap C) \\
 & \mathsf{Sing}(C) \arrow[ur] \arrow[r] & \mathsf{Sing}(B\cap C) \arrow[ur] &
\end{tikzcd}$$

where $\longrightarrow$ is positive, and $\dashrightarrow$ is negative.

On one hand, from algebraic topology, the p-th cohomology of $(\check{C}, d_{(1,0)})$ is homotopy (as complex induced by $d_{(0,1)}$) to zero for $p > 0$, and to $\mathsf{Sing}^\bullet$ for $p = 0$. Thus the total complex computes singular cohomology $H^\bullet(X)$.

On the other hand,



We get a spectral sequence

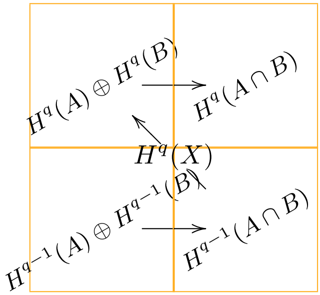
$$E_1^{pq} = H^q(U^p) \Longrightarrow H^{p+q}(X).$$

For the case all U^p are acyclic, E_1 only rests $H^0(U_p) = \mathbb{Z}^{f_p}$ where f_p is the number of connected components of U_p. This is known as **Čech cohomology**. Actually, the sheaf version is more common to see. After more efforts, the simplicial cohomology 1.12 can be included by this.

If there are only two open subsets A and B. Then

$$H^q(U^0) = H^q(A) \oplus H^q(B), \qquad H^q(U^1) = H^q(A \cap B).$$

This recovers the Mayer–Vietoris sequence.



So this spectral sequence is also known as **Mayer–Vietoris spectral sequence**.

Exercises

2.12. The Koszul Convention Consider the transposition C^{t} of a double complex C. Then there is an isomorphism of complex

$$\operatorname{Tot} C \longrightarrow \operatorname{Tot} C^{\mathsf{t}}$$

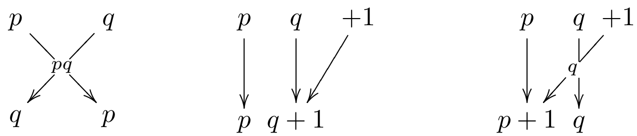
given by $(-1)^{pq}\,\mathrm{id}$ over C^{pq}.

$$\begin{tikzcd}
C^{pq} \arrow[r, shift left, "(-1)^p"] \arrow[r, shift right] \arrow[d, "(-1)^{pq}"'] & C^{p,q+1} \oplus C^{p+1,q} \arrow[d, "{(-1)^{p(q+1)} \oplus (-1)^{(p+1)q}}"'] \\
C^{pq} \arrow[r, shift left] \arrow[r, shift right, "(-1)^q"'] & C^{p,q+1} \oplus C^{p+1,q}
\end{tikzcd}$$

The degree shifting is also important, we see

$$\operatorname{Tot}(C[0,1]) \xrightarrow{\mathrm{id}} (\operatorname{Tot} C)[1] \qquad \operatorname{Tot}(C[1,0]) \xrightarrow{(-1)^q\,\mathrm{id}} (\operatorname{Tot} C)[1]$$

We can remember this convention by the following diagrams



No matter how our convension is taken, we always have the following diagram

$$
\begin{tikzcd}
\mathrm{Tot}(C[1,1]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\substack{\text{anti}\\ \text{commutative}}"] & \mathrm{Tot}(C[1,0])[1] \arrow[d] \\
\mathrm{Tot}(C[0,1])[1] \arrow[r] & \mathrm{Tot}(C[0,0])[1]
\end{tikzcd}
$$

2.13. For a morphism $A \xrightarrow{f} B \xrightarrow{g} C$, prove that there is a long exact sequence

$$0 \to \ker f \to \ker gf \to \ker g \overset{(*)}{\to} \operatorname{cok} f \to \operatorname{cok} gf \to \operatorname{cok} g \to 0.$$

The $\overset{(*)}{\to}$ is the map through B.

$$
\begin{tikzcd}
 & & \ker g \arrow[rr, bend left] \arrow[dr] & & \operatorname{cok} f \arrow[ddrr] & & \\
 & & & B \arrow[ur] \arrow[dr] & & & \\
\ker gf \arrow[uurr] \arrow[rr] & & A \arrow[ur] \arrow[rr] & & C \arrow[rr] \arrow[dr] & & \operatorname{cok} gf \arrow[dl, bend left] \\
 & \ker f \arrow[ul, bend left] \arrow[ur] & & 0 \arrow[ll] & & \operatorname{cok} g \arrow[ll] &
\end{tikzcd}
$$

2.14. 5-Lemma Prove 5-lemma using spectral sequence. Assume we have the following commutative diagram with each row exact

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & C \arrow[r] \arrow[d, "k"'] & C \arrow[d, "\ell"'] \\
D \arrow[r] & D \arrow[r] & D \arrow[r] & D \arrow[r] & D
\end{tikzcd}
$$

Then when ℓ is mono, and g, k are epi, then h is epi; when f is epi, and g, k are mono, then h is mono.

2.15. 4-lemma If we have the following diagram with rows exact

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & 0
\end{tikzcd}
$$

Prove that

$$\beta \text{ is injective} \Rightarrow \alpha \text{ is injective}$$

$$\left.\begin{array}{r} \beta \text{ is surjective} \\ \gamma \text{ is injective} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha \text{ is surjective} \\ \delta \text{ is injective} \end{array}\right.$$

$$\gamma \text{ is surjective} \Rightarrow \delta \text{ is surjective}$$

3 Construction (III)

Exact Couples

3.1. Exact Couple An **exact couple** (D, E, i, j, k) is a long exact sequence

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

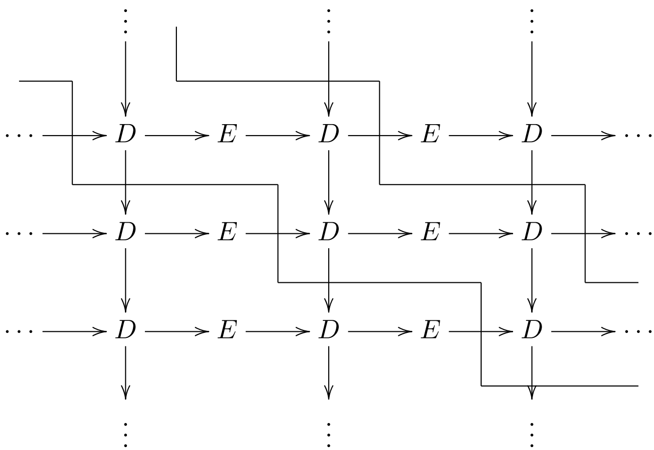
We say an exact couple is **nilpotent** if $i^r(D) = 0$ for $r \gg 0$.

3.2. Derived Couple We define its **derived couple** to be

$D' = \operatorname{im} i$, $E' = H(E, j \circ k)$, and i' induced by i, j' induced by $j \circ i^{-1}$, and k' induced by k.

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

One can check that (D', E', i', j', k') is still an exact couple.



3.3. We denote $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$ the n-th iterated derived couple. One can check that $D^{(n)} = \operatorname{im} i^n$, and $i^{(n)}$ induced by i, $j^{(n)}$ induced by ji^{-n}, and $k^{(n)}$ induced by k.

Usually, the degree convention is $\deg i = (-1, 1)$, $\deg j = (n, -n)$, $\deg k = (1, 0)$ for n-th iterated derived couple.

3.4. Cohomology Define the **cohomology** of an exact couple to be

$$H = \varinjlim \left[\cdots \to D \xrightarrow{i} D \to \cdots \right].$$

There is a filtration over H is given by the $\operatorname{im}\left[D \xrightarrow{i^r} H\right]$.

3.5. Spectral Sequences for Exact Couples For a nilpotent exact couple (D, E), there is a spectral sequence $E_r = E^{(r-1)}$ with differential $j^{(r-1)} \circ k^{(r-1)}$, converges to H in the classic sense 1.20.

Remark In the proof, we use the explicit description of the limit, this does not hold in general abstract abelian category.

3.6. Filtered Complex Let C be a filtered complex of modules. Consider the short exact sequence

$$0 \longrightarrow F^{p+1}C \longrightarrow F^pC \longrightarrow F^pC/F^{p+1}C \longrightarrow 0$$

which gives rise to

$$\begin{tikzcd}
H^{p+q}(F^{p+1}C) \arrow[rr] & & H^{p+q}(F^pC) \arrow[dddddl] \\
& \cdots & \\
H^{p+q+1}(F^{p+1}C) \arrow[rr] & & H^{p+q+1}(F^pC) \arrow[dddddl] \\
& H^{p+q-1}(F^pC/F^{p+1}C) \arrow[uuul] & \\
H^{p+q+2}(F^{p+1}C) \arrow[rr] & & H^{p+q+2}(F^pC) \\
& H^{p+q}(F^pC/F^{p+1}C) \arrow[uuul] & \\
\cdots & & \cdots \\
& H^{p+q+1}(F^pC/F^{p+1}C) \arrow[uuul] & \\
\cdots & & \cdots
\end{tikzcd}$$

It forms an exact couple. Actually, the spectral sequence coincides with what we get in 1.8 for $r \geq 1$ by tough diagram chasing. From the remarks below, we see it also recovers the classic convergence 1.20.

3.7. In the case of modules, we may use the fact that

$$\varinjlim : \mathcal{C}^{\{0\to 1\to 2\to \cdots\}} \longrightarrow \mathcal{C} \qquad \big(M_0 \xrightarrow{\rho_0} M_1 \xrightarrow{\rho_1} \cdots\big) \longmapsto \varinjlim_i M_i$$

is exact. But in the case of modules, its dual

$$\varprojlim : \mathcal{C}^{\{\cdots\to 2\to 1\to 0\}} \longrightarrow \mathcal{C} \qquad \big(\cdots \xrightarrow{\rho_2} M_1 \xrightarrow{\rho_1} M_0\big) \longmapsto \varprojlim_i M_i$$

is not exact (but left exact).

As a result, $\varinjlim$ commutes with homology groups of complex. In particular, if C is some filtered complex,

$$\varinjlim_p H^n(F^pC) = H^n\big(\bigcup F^pC\big).$$

3.8. Simplicial Cohomology Let X be a CW complex. Denote X_k the union of cells of dimension $\leq k$, and $X_{-1} = \varnothing$. We have an long exact sequence

$$\cdots \longrightarrow H^n(X, X_p) \longrightarrow H^n(X, X_{p-1}) \longrightarrow H^n(X_p, X_{p-1}) \longrightarrow \cdots$$

Then

$$\begin{cases} D = \bigoplus D_{pq}, & D_{pq} = H^{p+q}(X, X_p), \\ E = \bigoplus E_{pq}, & E_{pq} = H^{p+q}(X_p, X_{p-1}). \end{cases}$$

forms an exact couple. The cohomology

$$H^n = \varinjlim_p H^n(X, X_p) = H^n(X).$$

This exact couple is nilpotent since for $p \geq n$, $H^n(X, X_p) = 0$. This recovers 1.12.

3.9. K-theory Analogy The exact couple helps to understand "cohomology theory" not computed by a complex, for example K-theory. The topological K-theory has the same exact sequence for CW complex as above example,

$$\cdots \longrightarrow K^n(X, X_p) \longrightarrow K^n(X, X_{p-1}) \longrightarrow K^n(X_p, X_{p-1}) \longrightarrow \cdots$$

When X is finite (i.e. built by finite cells),

$$E_1^{pq} = K^{p+q}(X_p, X_{p-1}) \Longrightarrow K^{p+q}(X).$$

We can prove

$$K^{p+q}(X_p, X_{p-1}) = K^q(\mathrm{pt})^{\oplus f_p} \qquad f_p = \#\{k\text{-cells}\}$$

As a result, $K^{p+\bullet}(X_p, X_{p-1}) = H^{p+\bullet}(X_p, X_{p-1}) \otimes K^\bullet(\mathrm{pt})$. By direct computation, it coincides with the simplicial cohomology, so

$$E_2^{pq} = H^p(X, K^q(\mathrm{pt})) \Longrightarrow K^{p+q}(X).$$

This is a special case of Atiyah–Hirzebruch spectral sequences 5.8.

The proof

Proof of 3.5 We can compute (under the notation in 1.19)

$$\begin{aligned}
B_r^{pq} &= \operatorname{im} d^{(r-1)} = j^{(r-1)}(\operatorname{im} k^{(r-1)})) \\
&= j(i^{-(r-1)}(\ker i^{(r-1)})) = j(i^{-r}(0)), \\
Z_r^{pq} &= \ker d^{(r-1)} = (k^{(r-1)})^{-1}(\ker j^{(r-1)}) \\
&= (k^{(r-1)})^{-1}(\operatorname{im} i^{(r-1)}) = k^{-1}(i^r(D)) \cap Z_{r-1}^{pq} \\
&= k^{-1}(i^r(D)) \cap k^{-1}(i^{r-1}(D)) \cap Z_{r-2}^{pq} = k^{-1}(i^r(D)) \cap Z_{r-2}^{pq} \\
&= \cdots = k^{-1}(i^r(D)).
\end{aligned}$$

Thus

$$\begin{aligned}
B_\infty^{pq} &= \textstyle\bigcup_r B_r^{pq} = \bigcup_r j(i^{-r}(0)) = j\big(\bigcup_r i^{-r}(0)\big), \\
Z_\infty^{pq} &= \textstyle\bigcap_r Z_r^{pq} = \bigcap_r k^{-1}(i^r(D)) = k^{-1}\big(\bigcap_r i^r(D)\big).
\end{aligned}$$

Denote the image of D^{pq} in the H^{p+q} by $\tilde{D}^{pq}$. Now we assume the exact couple is nilpotent, then

$$\bigcap_r i^r(D) = 0, \qquad \bigcup_r i^{-r}(0) = \ker[D \to \tilde{D}].$$

Thus $Z_\infty^{pq} = k^{-1}(0) = \ker k = \operatorname{im} j$. Now, consider the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & \ker \pi_{p-1,q+1} \arrow[r] \arrow[d] & D^{p-1,q+1} \arrow[r, "\pi"] \arrow[d] & \tilde{D}^{p-1,q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \pi_{pq} \arrow[r] \arrow[d, "j"] & D^{pq} \arrow[r, "\pi"] \arrow[d, "j"] & \tilde{D}^{pq} \arrow[r] \arrow[d] & 0 \\
 & B_\infty^{pq} \arrow[r] & Z_\infty^{pq} \arrow[r] & \text{factors} \arrow[r] & 0
\end{tikzcd}$$

The exactness of leftmost row. If $x \in \ker \pi_{pq}$, with $j(x) = 0$, then $x = i(y)$ for some $y \in D^{p-1,q+1}$. But $x \in \ker \pi_{pq} = \bigcup_r i^{-r}(0)$, so $x \in \bigcup_r i^{-r}(0) = \ker \pi_{p-1,q+1}$. The rest exactness is clear. Q.E.D.

Proof of the claim in 3.6 Consider

$$E_{r+1}^{pq} = \frac{Z_r^{pq}}{B_r^{pq}} = \frac{k^{-1}(i^r(D))}{j(i^{-r}(0))}$$

Pick $x \bmod (\cdots) \in E_1^{pq} = H^p(F^pC/F^{p+1}C)$, where $x \in F^pC$ with $dx \in F^{p+1}C$,

$$\begin{aligned} x \bmod (\cdots) \in k^{-1}(i^r(D)) &\iff dx \in \operatorname{im} i^r + \operatorname{im}[F^{p+1}C \xrightarrow{d} F^{p+1}C] \qquad (*) \\ &\iff dx \in F^{p+r+1}C + \operatorname{im}[F^{p+1}C \xrightarrow{d} F^{p+1}C] \\ &\iff x \in d^{-1}(F^{p+r+1}C) + F^{p+1}C. \end{aligned}$$

where the i in $(*)$ is by the definition of connected morphism that $k(x \bmod (\cdots)) = dx \in H(F^{p+1}C)$.

$$\begin{aligned} x \bmod (\cdots) \in j(i^{-r}(0)) &\iff \exists y \begin{cases} y \in \ker[F^pC \xrightarrow{d} F^pC], \\ i^r(y) \in \operatorname{im}[F^{p-r}C \xrightarrow{d} F^{p-r}C], \\ x \equiv y \bmod F^{p+1}C. \end{cases} \\ &\iff \exists y \begin{cases} i^r(y) \in \operatorname{im}[F^{p-r}C \xrightarrow{d} F^{p-r}C], \\ x \equiv y \bmod F^{p+1}C. \end{cases} \\ &\iff \exists y \begin{cases} y \in d(F^{p-r}C), \\ x \equiv y \bmod F^{p+1}C. \end{cases} \\ &\iff x \in d(F^{p-r}C) + F^{p+1}C. \end{aligned}$$

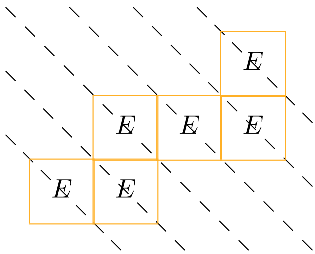
So

$$\begin{cases} Z_r^{pq} = \dfrac{F^{p+1}C + d^{-1}(F^{p+r+1}C) \cap F^pC}{d^{-1}(F^{p+1}C)} \\ B_r^{pq} = \dfrac{F^{p+1}C + d(F^{p-r}C) \cap F^pC}{d^{-1}(F^{p+1}C)} \end{cases}$$

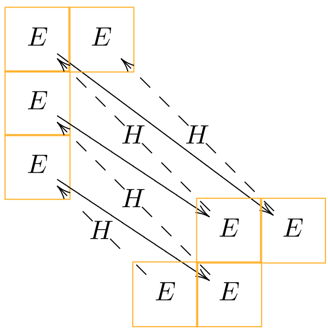
coincides what we defined for filtered complex (where it starts from 0-th page). The differntial is induced by d, thus the same.

Computations

3.10. If for each n, there is only one $E_r^{p+q} \neq 0$ with $p+q=n$, and the differential is zero, then the nonzero term is H^n.

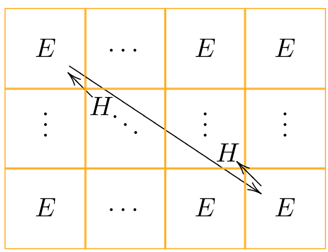


3.11. If for each n, there is only two $E_r^{p+q} \neq 0$ with $p+q=n$, then usually we can get a long exact sequence involing H^n. The direction goes as follows (under cohomological convention)

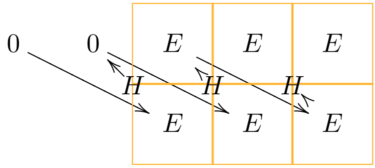


To be exact, it happens when there is no differentials between upper nonzero terms.

3.12. In general, we only know a four term exact sequence before achieving the limit

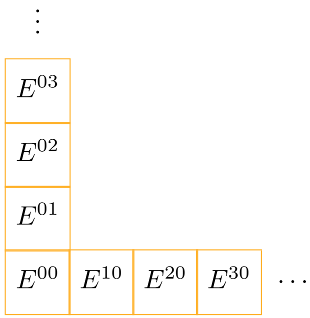


3.13. First Five Terms If the spectral sequence lies in a corner, then we can have **first five term exact sequence**



Exercises

3.14. Write the long exact sequence for the spectral sequence whose E_2 page looks like



Answer $0 \to E^{10} \to H^1 \to H^{01} \to E^{20} \to H^2 \to E^{02} \to \cdots.$

3.15. Prove 3.2.

3.16. Rees system Historically, the following commutative diagram

$$\begin{tikzcd}
& D \arrow[dl, "\beta"'] \arrow[rr, "i"] & & D \arrow[dl, "j"'] \arrow[dr, "\beta"] & \\
F \arrow[dr, "\gamma"'] \arrow[r, phantom, "\blacktriangleleft"] & & E \arrow[ul, "k"'] \arrow[dl, "\bar{k}"] \arrow[u, phantom, "\blacktriangledown"] \arrow[d, phantom, "\blacktriangle"] & & F \arrow[dl, "\gamma"] \arrow[l, phantom, "\blacktriangleright"] \arrow[llll, bend left=60, "="] \\
& \bar{D} \arrow[uu, "\alpha"] \arrow[rr, "\bar{\imath}"'] & & \bar{D} \arrow[uu, "\alpha"'] \arrow[ul, "\bar{\jmath}"'] &
\end{tikzcd}$$

where ▲'s all long exact sequences, is called a **Rees system**. It is easy to see $j \circ k = \bar{\jmath} \circ k$. Show that the following diagram is also a Rees system

$$\begin{tikzcd}
 & D' \arrow[rr, "i'"] \arrow[dl, "\beta'"'] & & D' \arrow[dl, "j'"] \arrow[dr, "\beta'"] \arrow[dd, phantom, ""] & \\
F \arrow[dr, "\gamma'"'] & & E' \arrow[ul, "k'"'] \arrow[dl, "\overline{k}'"] \arrow[u, phantom, "\blacktriangledown"] \arrow[d, phantom, "\blacktriangle"] & & F \arrow[dl, "\gamma'"] \arrow[llll, bend left, "="] \\
 & \overline{D}' \arrow[uu, "\alpha'"] \arrow[rr, "\overline{i}'"'] & & \overline{D}' \arrow[uu, "\alpha'"'] \arrow[ul, "\overline{\jmath}'"'] &
\end{tikzcd}$$

with the two ▲'s the derived couples, and α' induced by α, β' by βi^{-1}, γ' by γ.

3.17. Given the commutative diagram with each row exact, prove the exactness of the sequence.

$$\begin{tikzcd}
 & D \arrow[d, "g"] \arrow[dr, "h"] & \\
A \arrow[ur] \arrow[r, "f"'] & B \arrow[r, "\dagger"'] & C
\end{tikzcd} \qquad 0 \to \operatorname{im} f \to \operatorname{im} g \xrightarrow{\dagger} \operatorname{im} h \to 0.$$

$$\begin{tikzcd}
 & D \arrow[dr, "\dagger"] & \\
A \arrow[ur, "f"] \arrow[r] & B \arrow[u, "g"] \arrow[r, "h"'] & C
\end{tikzcd} \qquad 0 \to \operatorname{im} f \to \operatorname{im} g \xrightarrow{\dagger} \operatorname{im} h \to 0.$$

(Frankly, the author do not know how to prove this by spectral sequences)

4 Topology (I)

4.1. In this section, the coefficient of cohomology groups can be any commutative ring, even we sometimes write $\mathbb{Z}$. In this section, we assume all the spaces appearing are paracompact (every open cover has a locally finite open refinement) which admits partitions of unity. For example, manifolds, CW complexes, algebraic varieties under complex topology.

Fibre Bundles

4.2. Fibre Bundle Let X and F be two topological spaces, consider $E = X \times F$. Denote the projection $\pi = \left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$. Note that at each point $x \in X$, the fibre $\pi^{-1}(x)$ is a copy of F. We say $\left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$ is a **trivial fibre bundle** with fibre F. In general, a map $\xi = \left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$ is said to be a **fibre bundle** with fibre F if

For each point $x \in X$, there exists an open neighborhood U, such that the restriction $\xi|_{\xi^{-1}(U)} = \left[\begin{smallmatrix} \xi^{-1}(U) \\ \downarrow \\ U \end{smallmatrix}\right]$ is a trivial fibre bundle with fibre F.

$$\begin{tikzcd}
U \times F \arrow[r, "\sim"] \arrow[d, "\mathrm{proj}"] & \xi^{-1}(U) \arrow[r, "\subseteq"] \arrow[d, "\xi|_{\xi^{-1}(U)}"] & E \arrow[d, "\xi"] \\
U \arrow[r, equal] & U \arrow[r, "\subseteq"'] & X
\end{tikzcd}$$

We will say X is the **base space**, E is the **total space**, F the **fibre**, and denote $E_x = \xi^{-1}(x)$ for $x \in X$ the **fibre at x**. The isomorphism $U \times F \to \xi^{-1}(U)$ is called a **local trialization**.

The computation of cohomology of fibre bundles is very important. Leray–Serre spectral sequence gives a tool to analyse it. Before the statement, we firstly see two theorems on cohomology of fibre bundles.

4.3. Künneth theorem For trivial bundle $E = X \times F$, two natural projections π_1, π_2 induce a ring homomorphism

$$H^*(X) \otimes H^*(F) \longrightarrow H^*(E), \qquad \alpha \otimes \beta \longmapsto \pi_1^* \alpha \smile \pi_2^* \beta.$$

When $H^*(F)$ is a free module (with respect to the coefficient), then this map is an isomorphism.

4.4. Leray–Hirsch theorem For a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ with fibre F with $H^*(F)$ a free module (with respect to the coefficient), assume that

there is a linear lifting $H^*(F) \to H^*(E)$, such that for any $x \in X$, the restriction of it gives an isomorphism to $H^*(F)$.

$$\begin{tikzcd} & H^*(F) \arrow[dl, "\sim"'] \arrow[d] \\ H^*(E_x) & H^*(E) \arrow[l] \end{tikzcd}$$

Denote $\tilde{\beta} \in H^*(E)$ the lifting of $\beta \in H^*(F)$. Then the $H^\bullet(X)$-module homomorphism

$$H^\bullet(X) \otimes H^\bullet(F) \longrightarrow H^\bullet(E), \qquad \alpha \otimes \beta \longmapsto \xi^*\alpha \smile \tilde{\beta}$$

is an isomorphism.

Actually, it suffices to check the asserted property for an x from each path-connected component of X.

4.5. Hopf Fibration Recall that the complex projective line $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C} \cong \mathbb{R}^2$, thus $\mathbb{C}P^1 = S^2$ **the Riemann sphere**. On the other hand, the natural map

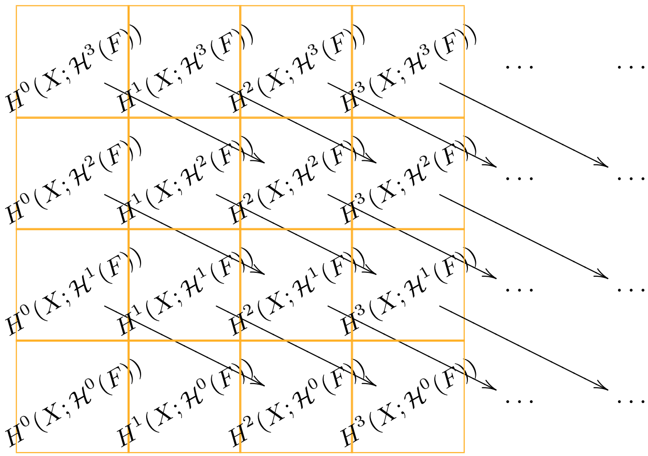
$$S^3 \subseteq \mathbb{R}^4 \setminus 0 \cong \mathbb{C}^2 \setminus 0 \longrightarrow \mathbb{C}P^1 \cong S^2$$

gives a fibre bundle with fibre S^1. This fibre bundle is called the **Hopf fibration**. This is an example Leray–Hirsch theorem cannot analyse.

4.6. Leray–Serre Spectral Sequences Assume we have a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ with fibre F. Then there exists a spectral sequence

$$E_2^{pq} = H^p(X; \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(E)$$

where $\mathcal{H}^q(F)$ is the local system of cohomology of fibres (4.7). In particular, when X is simply-connected, $\mathcal{H}^q(F) = H^q(F)$ the constant coefficient.



Proof We can assume X is a CW complex by approximation. We can also assume over each cell, the fibre bundle is trivial. Denote X_p the union of all cells of dimension $\leq p$. Denote E_p the preimage of X_p. Now, we have a filtraion on $\mathsf{Sing}^\bullet(E)$ by $\mathsf{Sing}^\bullet(E, E_p)$. We can compute the associative graded complex

$$\operatorname{gr}\mathsf{Sing}^\bullet(E) = \mathsf{Sing}^\bullet(E_p, E_{p-1}).$$

As a result, the spectral sequence

$$E_1^{pq} = H^{p+q}\Big(\mathsf{Sing}^\bullet(E_p, E_{p-1})\Big) = H^p(X_p, X_{p-1}) \otimes \mathcal{H}^q(F),$$

by relative Künneth theorem. We can check that the following diagram commutes

$$\begin{tikzcd}
H^{p+q}(E_p, E_{p-1}) \arrow[r] \arrow[d] & H^{p+q}(E_p) \arrow[r, "\delta"] & H^{p+q+1}(E_{p+1}, E_p) \arrow[d] \\
H^p(X_p, X_{p-1}) \otimes \mathcal{H}^q(F) \arrow[rr] & & H^{p+1}(X_{p+1}, X_p) \otimes \mathcal{H}^q(F)
\end{tikzcd}$$

The up map is the differential for E_1 (since it is induced by d), and the below map is the differential of the complex computing the simplicial cohomology of local coefficient, see 4.7). Thus,

$$E_2^{pq} = H^p(X; \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(X).$$

The proof is complete. Q.E.D.

4.7. Local system Denote $\mathsf{Map}(\Delta^p, X)$ the set of all continuous maps from the p-simplex Δ^p to X. Recall

$$\mathsf{Sing}^p(X) = \mathbb{Z} \cdot \mathsf{Map}(\Delta^p, X)$$

the space of formal combinations of nf with $n \in \mathbb{Z}$ and $f \in \mathsf{Map}(\Delta^p, X)$.

A **local system** $\mathcal{L}$ is a functor $\Pi_1(X) \longrightarrow \mathsf{Ab}$ from fundamental groupoid to the category of abelian groups. We can twist $\mathsf{Sing}^p(X)$ by $\mathcal{L}$

$$\mathsf{Sing}^p(X; \mathcal{L}) = \mathcal{L} \cdot \mathsf{Map}(\Delta^p, X) := \mathcal{L} \otimes \mathbb{Z} \cdot \mathsf{Map}(\Delta^p, X)$$

the space of formal combinations of nf with $n \in \mathcal{L}_x$ and $f \in \mathsf{Map}(\Delta^p, X)$ where $x \in X$ corresponds to the centre of Δ^p. We can define differential, by the straight line joint the centre of each face to the centre of Δ^p. This defines the **cohomology group of local coefficient** $H^p(X; \mathcal{L})$.

We can also define **simplicial cohomology group of local coefficient**. By the spectral sequence argument as in 1.12, it coincides with the singular (above) one.

4.8. Let us analyse the Hopf fibration where

$$\begin{aligned} E_2^{pq} &= H^p(S^2) \otimes H^q(S^1) \\ &= \begin{cases} \mathbb{Z} & (p,q) \in \{0,2\} \times \{0,1\} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[drr] & & \mathbb{Z} \\ \mathbb{Z} & & \mathbb{Z} \end{tikzcd}$$

Since $E = S^3$, we know the map is an isomorphism. Actually, the map

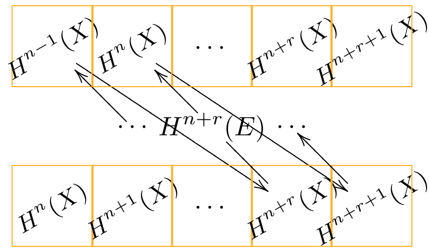
$$E_2^{01} = H^1(F) \longrightarrow H^2(X) = E_2^{20}$$

can be described, see 4.17.

4.9. Acyclic Cases We call a space X is **acyclic** if $H^i(X) = 0$ for $i \geq 1$. For example, a contractible space is acyclic. When the fibre F is acyclic, then the spectral sequence has only one row. Thus $H^n(E) = H^n(X)$. When the base space X is acyclic, then the spectral sequence has only one column. Thus $H^n(E) = H^n(F)$.

4.10. Gysin Sequences A fibre bundle with fibre $F = S^r$ is called a **sphere bundle**. For a sphere bundle $\begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$, we have **Gysin sequence**

$$\cdots \longrightarrow H^{n-1}(X) \longrightarrow H^{n+r}(X) \longrightarrow H^{n+r}(E) \longrightarrow H^n(X) \longrightarrow \cdots$$

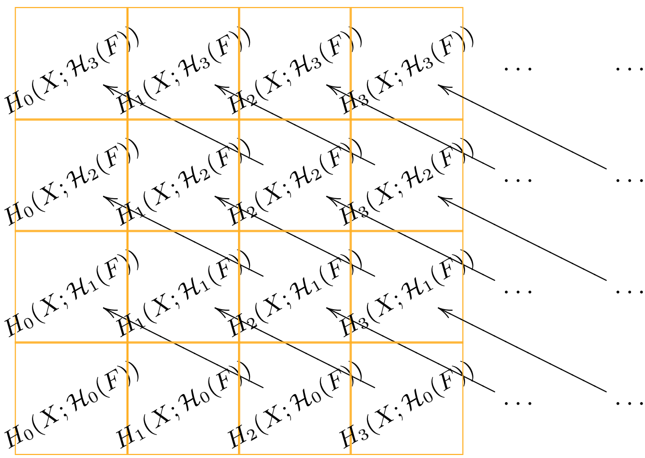


More on Leray–Serre Spectral Sequences

4.11. Homological Version We also have homological version.

$$E^2_{pq} = H_p(X; \mathcal{H}_q(F)) \Longrightarrow H_{p+q}(E)$$

where $\mathcal{H}_q(F)$ is the local system of homology of fibres.



4.12. Functoriality We define the morphism between spectral sequences in the obvious way, i.e. a morphism from (E_r) to (E'_r) is a complex morphism $E_r \to E'_r$ for each r, such that $E_{r+1} \to E_{r+1}$ is induced by this morphism.

Then for a morphism of fibre bundles $\begin{bmatrix} E' \\ \downarrow \\ X' \end{bmatrix} \to \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$, namely commutative diagram, it induces a morphism of corresponding spectral sequences. This follows from the approximation by cellular maps. On E_2, it is given by the obvious one. Similar property holds for homology.

4.13. Multiplicative Structure A multiplicative structure over a spectral sequence (E_r) is a complex morphism $\mathrm{Tot}(E_r \otimes E_r) \to E_r$ for each r, such that $\mathrm{Tor}(E_{r+1} \otimes E_{r+1}) \to E_{r+1}$ is induced by this morphism. By the functoriality, Serre–Leray spectral sequencee has a multiplicative structure (induced by the diagonal map). On E_2, it is given by the obvious map

$$H^p(X; \mathcal{H}^q(F)) \otimes H^{p'}(X; \mathcal{H}^{q'}(F)) \longrightarrow H^{p+p'}(X; \mathcal{H}^{q+q'}(F)).$$

by $(\alpha \otimes \phi) \otimes (\beta \otimes \psi) \mapsto (-1)^{p'q}(\alpha \smile \beta) \otimes (\phi \smile \psi)$ under Koszul convention 2.12.

Similar property holds for homology, and cap product (after defining monoidal structure for spectral sequences).

4.14. Gysin Sequence revised Recall the Gysin sequence 4.10.

- The map $H^{n-1}(X) \to H^{n+r}(X)$ is given by cup product by an element in $H^{r+1}(X)$, called the **Euler class**. This follows from the existence of multiplicative structure.
- The map $H^{n+r}(X) \to H^{n+r}(X)$ is given by the natural pull back.
- When X is compact and smooth, then $H^{n+r}(E) \to H^n(X)$ is given by

$$\begin{tikzcd} H^{n+r}(E) \arrow[r] \arrow[d, "\text{Poincaré duality}"'] & H^n(X) \arrow[d, "\text{Poincaré duality}"] \\ H_{\dim E-n-r}(E) \arrow[r, "\text{push forward}"'] & H_{\dim X-n}(X) \end{tikzcd}$$

  Since the Poincaré duality is given by a cap product.

4.15. Relative Version We also have relative version with respect to base space

$$E_2^{pq} = H^p(X, X_0; \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(E; E_0)$$

with E_0 the preimage of X_0; as well as to fibre

$$E_2^{pq} = H^p(X; \mathcal{H}^q(F; F_0)) \Longrightarrow H^{p+q}(E; E_0)$$

if $E_0 \subseteq E$ cuts each fibre by $F_0 \subseteq F$.

4.16. Thom Isomorphisms Consider the case $\left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$ with fibre $F \cong \mathbb{R}^r$. After one-point compactification at each fibre, we get an sphere bundle $\left[\begin{smallmatrix} \hat{E} \\ \downarrow \\ X \end{smallmatrix}\right]$. Denote ∞ the union of infinity points at each fibre. Since

$$H^n(\mathbb{R}^r \cup \{\infty\}, \infty) = \begin{cases} \mathbb{Z}, & n = r, \\ 0, & \text{otherwise.} \end{cases}$$

Then $H^n(X) = H^{n+r}(\hat{E}, \infty)$ where $\infty = \{\infty\} \times X$ is the infinity section, the union of all infinity points.

4.17. Transgression We can describe $E_r^{0,r-1} \to E_r^{r0}$, the so-called **transgression**. We have the following commutative diagram with three long rows exact

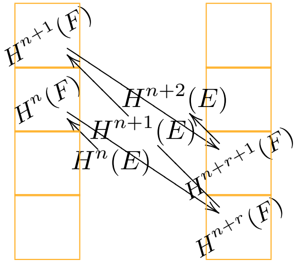
$$\begin{tikzcd}
 & & H^{r-1}(*) \arrow[r] & H^r(X,*) \arrow[dr] \arrow[dd] & & \\
\cdots \arrow[r] & H^{r-1}(X) \arrow[ur] \arrow[dd] & & & H^r(X) \arrow[r] \arrow[dl, two heads] \arrow[dd] & \cdots \\
 & & E_r^{0,r-1} \arrow[r, "\text{transgression}"] \arrow[dd, hook] & E_r^{r0} \arrow[dr] \arrow[dd, hook] & & \\
\cdots \arrow[r] & H^{r-1}(E) \arrow[ur] \arrow[dr] & & & H^r(E) \arrow[r] & \cdots \\
 & & H^{r-1}(F) \arrow[r, "\delta"'] & H^r(E,F) \arrow[ur] & &
\end{tikzcd}$$

The middle square commutes by our construction. The rest part commutes by the functoriality of $\left[\begin{smallmatrix} F \\ \downarrow \\ * \end{smallmatrix}\right] \to \left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} X \\ \downarrow \\ X \end{smallmatrix}\right] \to \left[\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right]$. The first row and the last row is the long exact sequence for the couple (E, F) and $(B, *)$. Actually, the map $E_r^{0,r-1} \to E_r^{r0}$ is also induced by the transgression between exact complexes (the first and the last) introduced in 2.7 (since it is induced by d).

Exercises

4.18. Wang Sequences Let $\begin{bmatrix} E \\ \downarrow \\ S^n \end{bmatrix}$ be bundle over sphere with $n \neq 0, 1$. Show that there is the following **Wang sequence**

$$\cdots \to H^n(E) \to H^n(F) \to H^{n+r}(F) \to H^{n+1}(E) \to \cdots$$



4.19. Recall the construction of Hopf fibration 4.5 If we exchange $\mathbb{C}$ by quaternion $\mathbb{H}$, we will get

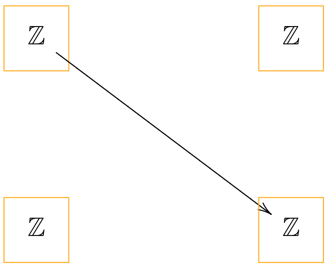
$$S^7 \subseteq \mathbb{R}^8 \setminus 0 \cong \mathbb{H}^2 \setminus 0 \longrightarrow \mathbb{H}P^1 \cong S^4$$

whose fibre is S^3. If we exchange by octonion $\mathbb{O}$, we will get

$$S^{15} \subseteq \mathbb{R}^{16} \setminus 0 \cong \mathbb{O}^2 \setminus 0 \longrightarrow \mathbb{O}P^1 \cong S^8$$

whose fibre is S^7. Draw the spectral sequence for them.

Answer:



4.20. For a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ whose fibre F and bases space X both have finite betti numbers, show that $\chi(E) = \chi(X)\chi(F)$ where $\chi(-)$ is the Euler characteristic.

5 Topology (II)

5.1. In this section, we take the same convention as last section.

Eilenberg–Moore Spectral Sequences

5.2. Pull Back For a continuous map $f : X \to Y$ and a fibre bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ Y \end{bmatrix}$, the pull back $f^*\xi = \begin{bmatrix} E_f \\ \downarrow \\ X \end{bmatrix}$ with $E_f = \{(x, v) \in X \times E : f(x) = \xi(v)\}$ forms a fibre bundle over X. Intuitively, the fibre of $f^*\xi$ at x is a copy of the fibre of ξ at $f(x)$.

We hope to say something on cohomology of E_f. For example, when ξ satisfies the condition of Leray–Hirsch 4.4, then so is $f^*\xi$.

5.3. Eilenberg–Moore Spectral Sequences Consider the pull back of fibre bundle

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "f^*\xi"'] & E \arrow[d, "\xi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

when X and Y are both simply connected, then there is a spectral sequnece

$$E_2^{pq} = \text{ deg } q \text{ part of } \mathrm{Tor}_{-p}^{H^\bullet(Y)}(H^\bullet(E), H^\bullet(X)) \Longrightarrow H^{p+q}(E_f).$$

Proof Assume $X \to Y$ is cellular map between CW complexes. Let Y_k (resp. X_k) be the union of cells of Y (resp. X) of dimension $\le k$. We define E_k ($(E_f)_k$) the preimage of Y_k (resp. X_k).

Take a $\mathsf{Sing}^\bullet(Y)$-projective resolution $\wp_\bullet(X) \to \mathsf{Sing}(X)$. We denote $P^\bullet$ by its total, in particular $H(P) = H(\mathsf{Sing}(X)) = H(X)$. There is a natural map

$$\mathsf{Sing}^\bullet(X) \underset{\mathsf{Sing}^\bullet(Y)}{\otimes} \mathsf{Sing}^\bullet(E) \xrightarrow{\times} \mathsf{Sing}^\bullet(E_f)$$

by taking product of pull backs. Denote

$$T^\bullet = \mathrm{Tot}(P \underset{\mathsf{Sing}(Y)}{\otimes} \mathsf{Sing}(E)) \xrightarrow{*} \mathsf{Sing}^\bullet(E_f)$$

induced by $\times$.

Consider the filtrations

$$\begin{array}{ll} F^k T^\bullet & = \bigoplus_{u+v=\bullet} P^u(X) \underset{\mathsf{Sing}^\bullet(Y)}{\otimes} \mathsf{Sing}^v(E, E_k) \\ \big\downarrow & \\ F^k\,\mathsf{Sing}^\bullet(X) & = \mathsf{Sing}^\bullet(E_f, (E_f)_k) \end{array}$$

which is compactible with $*$. So there is a morphism of spectral seqeunces. Note that

$$\begin{array}{ll} \boxed{E_0^{pq}} & = \bigoplus_{u+v=p+q} P^u(X) \underset{\mathsf{Sing}^\bullet(Y)}{\otimes} \mathsf{Sing}^v(E_p, E_{p-1}) \\ \boxed{E_0^{pq}} & = \mathsf{Sing}^{p+q}((E_f)_p, (E_f)_{p-1}) \end{array}$$

Thus,

$$\begin{array}{ll} \boxed{E_1^{pq}} & = \bigoplus_{u+v=p+q} H^u(X) \underset{H^\bullet(Y)}{\otimes} H^c(E_p, E_{p-1}) \\ \boxed{E_1^{pq}} & = H^{p+q}((E_f)_p, (E_f)_{p-1}) = H^p(X_p, X_{p-1}) \otimes H^q(F) \end{array}$$

Here we use the projectivity of $P_\bullet$, see 5.5 below. Then

$$\begin{array}{ll} \boxed{E_2^{pq}} & = \bigoplus_{u+k=p} H^u(X) \underset{H^\bullet(Y)}{\otimes} H^k(Y; H^q(F)) = H^p(X; H^q(F)) \\ \boxed{E_2^{pq}} & = H^p(X; H^q(F)) \end{array}$$

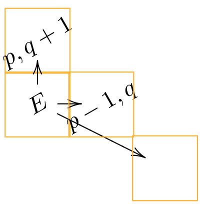
which is identity. Then, by 5-lemma, (i.e. comparison lemma 5.4)

$$H^\bullet(T^\bullet) = H^\bullet(\mathsf{Sing}^\bullet(E_f)) = H^\bullet(E_f).$$

Now, rewrite

$$T = \mathrm{Tot}(C), \qquad C^{p\bullet} = \wp_p^\bullet(X) \underset{\mathsf{Sing}^\bullet(Y)}{\otimes} \mathsf{Sing}^\bullet(X)$$

Then let us compute $T^\bullet$ by spectral sequence of a double complex



$$\begin{aligned} E_0 &= \text{degree } q \text{ part of } \wp_p^\bullet(X) \underset{\mathsf{Sing}^\bullet(Y)}{\otimes} \mathsf{Sing}^\bullet(E) \\ E_1 &= \text{degree } q \text{ part of } H^\bullet(\wp_p(X)) \underset{H^\bullet(Y)}{\otimes} H^\bullet(E) \\ E_2 &= \text{degree } q \text{ part of } \mathrm{Tor}^p_{H^\bullet(Y)}(H^\bullet(X), H^\bullet(E)) \end{aligned}$$

This proves the theorem. Q.E.D.

5.4. Comparison Lemma For a morphism of complex between two bounded filtered complexes $C \to D$ compatible with filtration, if the induced map $E_r \to E_r$ is isomorphic for $r \gg 0$, then C are quasi-isomorphic to D, (i.e. the induced map $H(C) \to H(D)$ is an isomorphism).

5.5. For an differential algebra (C, d), and two differential C-algebras D, E, if D is projective over C, then

$$H(D \otimes_C E) = H(D) \otimes_{H(C)} H(E).$$

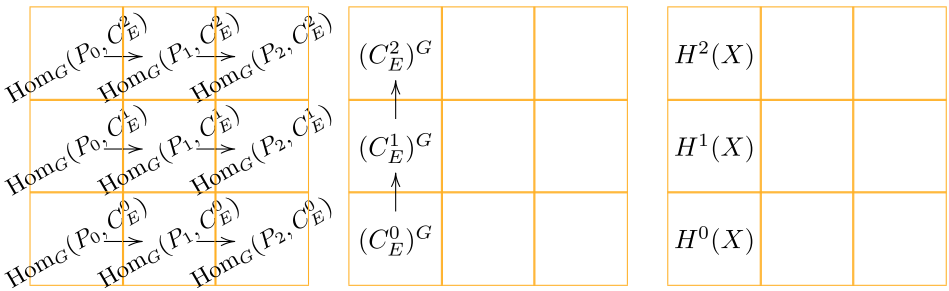
Cartan–Leray Spectral Sequences

5.6. Cartan–Leray Spectral Sequences Let $\pi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ be a normal (Galois) covering, that is, the discrete group $G = \mathrm{Aut}_X(\pi)$ acts freely on E, and $X = E/G$. There is a spectral sequence

$$E_2^{pq} = H^p(G; H^q(E)) \Longrightarrow H^{p+q}(X).$$

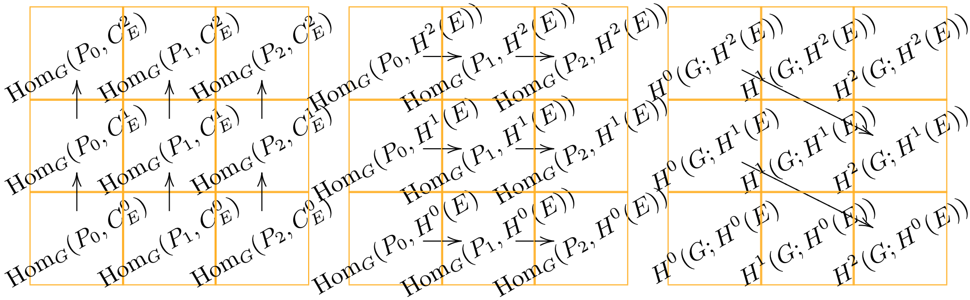
Here $H^p(G; -)$ is the group cohomology.

Proof Let us assume X is a CW complex with each cells locally trivial. This equips E a CW complex structure. If we denote $C^\bullet_-$ the complex computing simplicial cohomology, we have $\left(C^\bullet_E\right)^G = C^\bullet_X$. Since the action is free, so each $C^\bullet_E$ is co-induced $\mathbb{Z}[G]$-module (7.10). Pick a $\mathbb{Z}[G]$-resolution of $P \to \mathbb{Z}$. Then use the double complexes $\mathrm{Hom}_G(P_\bullet, C^\bullet_E)$. On one hand



Note that $H^0(G; X) = X^G$, and when X is a free $\mathbb{Z}[G]$-module, $H^i(G; K) = 0$

for $i \geq 0$. On the other hand,



the spectral sequence asserted. Q.E.D.

Atiyah–Hirzebruch Spectral Sequences

5.7. We denote the algebraic K-theory by $K^q(-)$. Note that q can be negative.

5.8. **Atiyah–Hirzebruch Spectral Sequence** Assume we have a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ with fibre F. When X is finite (i.e. built by finite cells), there exists a spectral sequence

$$E_2^{pq} = H^p(X; \mathcal{K}^q(F)) \Longrightarrow K^{p+q}(E)$$

where $\mathcal{K}^q(F)$ is the local system of K-theory of fibres (4.7).

Proof We have

$$\cdots \longrightarrow K^n(E, E_p) \longrightarrow K^n(E, E_{p-1}) \longrightarrow K^n(E_p, E_{p-1}) \longrightarrow \cdots$$

Then

$$E_1^{pq} = K^{p+q}(E_p, E_{p-1}) \Longrightarrow K^{p+q}(X).$$

We can prove

$$K^{p+q}(E_p, E_{p-1}) = H^p(X_p, X_{p-1}) \otimes K^q(F).$$

Actually, this follows from the fact that (X_p, X_{p-1}) is the suspension of discrete points. By direct computation as in the proof 4.6, it coincides with the simplicial cohomology, so

$$E_2^{pq} = H^p(X, K^q(F)) \Longrightarrow K^{p+q}(X).$$

Finally, we need X to be finite to ensure the convergence. Q.E.D.

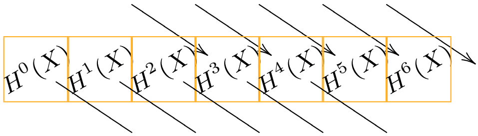
5.9. In particular, as we see before,

$$H^p(X; \mathcal{K}^q(\mathrm{pt})) \Longrightarrow K^{p+q}(X).$$

Taking complex K-theory, the K-theory is periodic $K^q = K^{q+2}$ by Bott periodicity theorem, and

$$K^{\mathrm{even}}(\mathrm{pt}) = \mathbb{Z}, \qquad K^{\mathrm{odd}}(\mathrm{pt}) = 0$$

So the nontrivial spectral sequence starts from E_3, and looks like



Postnikov Tower

5.10. Long Exact Sequence for Homotopy Groups For a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ with fibre F, we have a long exact sequence for homotopy group

$$\cdots \longrightarrow \pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \cdots.$$

5.11. Eilenberg–MacLane Spaces Recall the Eilenberg–MacLane space $K(G,n)$ for abelian group G and $n \geq 0$ is defined to be the only space with the property

$$\pi_p(K(G,n)) = \begin{cases} G & n = p \\ 0 & \text{otherwise} \end{cases}$$

- Firstly

  $$\Omega K(G,n) = \begin{cases} K(G,n-1) & n > 0 \\ \mathrm{pt} & n = 0 \end{cases},$$

  where Ω is the pointed loop space.

- Thus we can consider the fibre bundle $\begin{bmatrix} EK(G,n+1) \\ \downarrow \\ K(G,n+1) \end{bmatrix}$ whose fibre is $\Omega K(G,n+1) = K(G,n)$, where E the space of pointed path space which is always contractible.

- Secondly, by Hurewicz theorem,

i	0	1	$\cdots$	$n-1$	n	$n+1$	$n+2$
$H_i(K(G,n))$	$\mathbb{Z}$	0		0	G	?	$\cdots$

  By the universal coefficient theorem 6.11, $H^n(K(G,n);G) = G \cdot \mathrm{id}_G$.

- Actually, $K(G,n)$ presents the functor $H^n(-,G)$, i.e. we have a bijection

$$\pi(X, K(G,n)) = H^n(X;G),$$

  natural in X, where $\pi(-,-) = \mathrm{Map}(-,-)/\text{Homotopy}$ is the homotopy classes of maps. To be precise,

  > For any $\alpha \in H^n(X;G)$, it is pull back of id_G by some map $X \to K(G,n)$.

5.12. Postnikov Approximation There is a standard trick of "dévissage". We can construct the **Postnikov approximation**

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & X_0 \\ & & & X \arrow[ull] \arrow[ul] \arrow[u] \end{tikzcd}$$

such that

For any k, $\pi_i(X_k) = 0$ for $i > k$ and $\pi_i(X) \to \pi_i(X_k)$ is an isomorphism for $i \le k$; each $X_k \to X_{k-1}$ is a fibration with fibre $K(\pi_k(X), k)$.

	π_0	π_1	π_2	$\cdots$
X_0	$\pi_0(X)$	0	0	$\cdots$
X_1	$\pi_0(X)$	$\pi_1(X)$	0	$\cdots$
X_2	$\pi_0(X)$	$\pi_1(X)$	$\pi_2(X)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fibration (map satisfying homotopy lifting property) is the topological generalization of fibre bundle where we also have Leray–Serre spectral sequence and long exact sequence of homotopy groups.

5.13. Here lists some examples of Eilenberg–MacLane Spaces

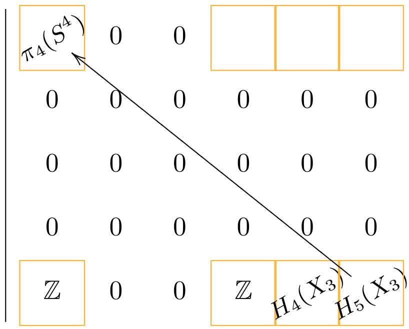
- For any group, $K(G,0) = G$ in discrete topology.
- By direct computation, $S^1 = K(\mathbb{Z},1)$.
- Note that $(\mathbb{C}^\infty)\setminus 0 \simeq S^\infty$ is a contractible CW complex, then using the long exact sequence of fibre $\begin{bmatrix} S^\infty \\ \downarrow \\ \mathbb{C}P^\infty \end{bmatrix}$, we see $\mathbb{C}P^\infty = K(\mathbb{Z},2)$.
- Similarly, the infinite lens space S^∞/C_m is $K(\mathbb{Z}/m,1)$, where $C_m = \{z \in \mathbb{C} : z^m = 1\} \subseteq S^1 \subseteq \mathbb{C}^\times$.

5.14. Consider the case $X = S^3$, then $X_0 = X_1 = X_2$ is just a point, Consider the fibration $\begin{bmatrix} X_4 \\ \downarrow \\ X_3 \end{bmatrix}$ with fibre $K(\pi_4(S^3),4)$.

Note that $X_3 = K(\mathbb{Z},3)$.

By relative Hurewicz theorem for $X \to X_4$, we have $H^4(X_4) = H^5(X_4) = 0$. Note that $X_3 = K(\mathbb{Z},3)$, thus

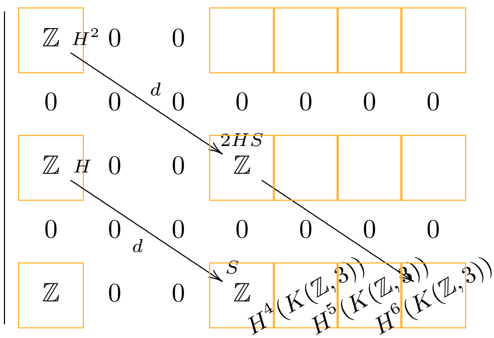
$$\pi_4(S^3) = H_5(X_3) = H_5(K(\mathbb{Z},3)).$$



Consider the fibre bundle $\begin{bmatrix} EK(\mathbb{Z},3) \\ \downarrow \\ K(\mathbb{Z},3) \end{bmatrix}$ whose fibre is $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ (the computation of its cohomology, see 8.6). Note that $E_\infty = 0$ except $E_\infty^{00} = \mathbb{Z}$ since $EK(\mathbb{Z},3)$ is contractible.

By contractiblity of $EK(\mathbb{Z},3)$, we can conclude the exactness of most positions marked in the diagram.

Assume the image of H is S under the below d. Then by the multiplicative structure, the up d is given by $H^2 \mapsto d(H^2) = 2HS$. Thus the up d is injective.



Furthermore,

$$H^4(K(\mathbb{Z},3)) = H^5(K(\mathbb{Z},3)) = 0 \qquad H^6(K(\mathbb{Z},3)) = \mathbb{Z}/2.$$

Finally, by universal coefficient theorem 6.11, $H_5(K(\mathbb{Z}),3) = \mathbb{Z}/2$. Thus we can conclude that $\pi_4(S^3) = \mathbb{Z}/2$.

Adams Spectral Sequences

5.15. Steenrod Algebra The **Steenrod Algebra**

$$\mathbb{A}_G^\bullet = \varprojlim_n \left[\cdots \xleftarrow{\Omega} \pi(K(G,n), K(G,n+\bullet)) \xleftarrow{\Omega} \cdots \right] = \varprojlim_n H^{n+\bullet}(K(G,n);G).$$

It acts on $H^*(-,G) = \pi(-,K(G,*))$, and commutes with the connection homomorphism δ in any long exact sequence. We denote $\mathbb{A}_p$ the Steenrod Algebra for $\mathbb{Z}/p$. Actually, it is generated by known cohomology operations, say **Steenrod squares** and **Bockstein homomorphism** (not necessary if $p=2$).

5.16. Stable Homotopy Group For a space X, define the **stable homotopy group**

$$\pi_p^s(X) = \varinjlim_n \left[\cdots \xrightarrow{\Sigma} \pi_{p+n}(\Sigma^n X) \xrightarrow{\Sigma} \cdots \right],$$

where Σ is the pointed suspension.

5.17. Adams Spectral Sequences For space X of finite type (i.e. each homotopy group is finite generated), there is a spectral sequence

$$E_2^{\rho q} = \text{degree } q \text{ part of } \operatorname{Ext}^\rho_{\mathbb{A}_p^\bullet}(\tilde{H}^\bullet(X), \mathbb{Z}/p) \Longrightarrow \pi^s_{\rho+q}(X) \otimes \mathbb{Z}_{(p)},$$

where $\mathbb{Z}_{(p)}$ is the ring of p-adic integers, and $\tilde{H}^\bullet(X) = H^\bullet(X,\mathrm{pt})$ the reduced cohomology group.

Proof Let us pick a free resolution of $\mathbb{A}_p^\bullet$

$$\cdots \to P_2^\bullet(X) \to P_1^\bullet(X) \to P_0^\bullet(X) \to \tilde{H}^\bullet(X;\mathbb{Z}/p) \to 0.$$

Pick an $N \gg 0$, note that $\tilde{H}^{\bullet+N}(\Sigma^N X) = \tilde{H}^{\bullet}(X)$. The choice of $P_0 \to \tilde{H}$ defines $\Sigma^N X \longrightarrow \prod_i K(\mathbb{Z}/p, N+d_i)$. Define $X(1)$ by pull back

$$\begin{tikzcd} X(1) \arrow[r] \arrow[d] & \prod_i EK(\mathbb{Z}/p, N+d_i) \arrow[d] \\ \Sigma^N X \arrow[r] & \prod_i K(\mathbb{Z}/p, N+d_i) \end{tikzcd} \qquad \text{fibre } = \prod_i K(\mathbb{Z}/p, N+d_i-1).$$

Using the spectral seqeunce for $\begin{bmatrix} X(1) \\ \downarrow \\ \Sigma^N X \end{bmatrix}$,

$$\begin{array}{ccccccccc} \boxed{\cdots} & 0 & \cdots & 0 & 0 & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{\cdots} & 0 & \cdots & 0 & 0 & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{\cdots} & 0 & \cdots & 0 & 0 & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{\mathbb{Z}} & 0 & \cdots & 0 & 0 & \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{array}$$

when $\bullet < 2N-3$, we have

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}^{\bullet-1}(X(1)) \arrow[r] & H^{\bullet-1}(K(\mathbb{Z}/p, N+d_i-1);\mathbb{Z}/p) \arrow[r, "\text{transgression}"] & \tilde{H}^{\bullet}(\Sigma^N X;\mathbb{Z}/p) \\ 0 \arrow[r] & \ker^{\bullet-N} \arrow[r] \arrow[u, dashed] & P_0^{\bullet-N} \arrow[r] \arrow[u] & \tilde{H}^{\bullet}(\Sigma^N X;\mathbb{Z}/p) \arrow[u] \end{tikzcd}$$

this follows from the description of the transgression 4.17. Then we can continuous this process on $X(1)$ and the image of generators of ker, to get

$$\cdots \longrightarrow X(2) \xrightarrow{\prod K(\mathbb{Z}/p, N+d'-1)} X(1) \xrightarrow{\prod K(\mathbb{Z}/p, N+d-1)} X(0) = \Sigma^N X\ .$$

From the long exact sequence of homotopy group

$$\cdots \longrightarrow \pi_{n+N}(X(k)) \longrightarrow \pi_{n+N}(X(k-1)) \longrightarrow \prod \pi_{n+N-1}(K(\mathbb{Z}/p, d^{(k-1)})) \to \pi_{n-1}(X(k$$

we get an exact couple

$$E_1^{\rho q} = \prod \pi_{\rho+q}(K(\mathbb{Z}/p, d^{(\rho)})).$$

Take $N \to \infty$, the $K(\mathbb{Z}/p, d^{(k)})$ exhaust generators of all degrees, we get

$$E_1^{\rho q} = \text{degree } q \text{ part of } \operatorname{Hom}_{\mathbb{A}_p^\bullet}\left(P_\rho^\bullet, \mathbb{Z}/p\right).$$

From the description above, the differential is exactly induced by P_*. So

$$E_2^{\rho q} = \text{degree } q \text{ part of } \operatorname{Ext}^\rho_{\mathbb{A}_p^\bullet}(\tilde{H}^\bullet(X), \mathbb{Z}/p).$$

But this exact couple is not nilpotent. If we restrict ourselves over the p-adic integer, then for fixed N, $\pi_N(X(k)) \otimes \mathbb{Z}_{(p)} = 0$ for $k \gg 0$. This follows from induction using spectral sequence and assumption of finite type. Then we can conclude the theorem. Q.E.D.

Exercises

5.18. Prove the comparison lemma 5.4.

5.19. Prove that under the assumption of 5.3, show that

$$\operatorname{Tor}^{\mathsf{Sing}(X)}(\mathsf{Sing}(E), \mathsf{Sing}(Y)) = H(E_f).$$

5.20. Deduct from the Hopf fibration 4.5 that $\pi_k(S^2) = \pi_k(S^2)$ for $k \geq 3$.

	π_1	π_2	π_3	π_4	$\cdots$
S^1	$\mathbb{Z}$	0	0	0	$\cdots$
S^2		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\cdots$
S^3			$\mathbb{Z}$	$\mathbb{Z}/2$	$\cdots$
S^4				$\mathbb{Z}$	$\cdots$

Actually, by Freudenthal suspension theorem, $\pi_{n+1}(S^n) = \mathbb{Z}/2$ for $n \geq 3$.

6 Algebra (I)

Hypercohomology

6.1. A complex C is said to be **split** if it is isomorphic to direct sum of two kinds of complexes

$$\cdots \longrightarrow 0 \longrightarrow C \longrightarrow 0 \longrightarrow \cdots \qquad \cdots \longrightarrow 0 \longrightarrow C \xrightarrow{\mathrm{id}} C \longrightarrow 0 \longrightarrow \cdots$$

Note that in this case

$$C \cong \left[\begin{tikzcd}[row sep=tiny]
 & & & C/\ker d \arrow[r, "\sim"] & \cdots \\
 & & & \oplus & \\
 & & & H(C) & \\
 & & & \oplus & \\
 & & C/\ker d \arrow[r, "\sim"] & \operatorname{im} d & \\
 & & \oplus & & \\
 & & H(C) & & \\
 & & \oplus & & \\
 & C/\ker d \arrow[r, "\sim"] & \operatorname{im} d & & \\
 & \oplus & & & \\
 & H(C) & & & \\
 & \oplus & & & \\
\cdots \arrow[r, "\sim"'] & \operatorname{im} d & & &
\end{tikzcd}\right]$$

6.2. Hyper-resolution Let C be a complex, we can find a double complex I and a morphism $C \to I$ (C viewed as a double complex C supported in $(0, *)$) such that for any p, the complex $(I^{p\bullet}, d_{(0,1)})$ is split, and itself, as well as $H^*, \ker, \operatorname{cok}, \operatorname{im}$ form injective resolutions of the counterparts of C, say, for any q,

$$\begin{array}{rcll}
C^q & \longrightarrow & I^{\bullet q} & \\
H^q(C, d) & \longrightarrow & H^q(I, d_{(0,1)}) & \\
\ker[C^q \xrightarrow{d} C^{q+1}] & \longrightarrow & \ker[I^{\bullet q} \xrightarrow{d_{(0,1)}} I^{\bullet, q+1}] & \quad \text{are all injective resolutions} \\
\operatorname{cok}[C^q \xrightarrow{d} C^{q+1}] & \longrightarrow & \operatorname{cok}[I^{\bullet, q} \xrightarrow{d_{(0,1)}} I^{\bullet, q+1}] & \\
\operatorname{im}[C^q \xrightarrow{d} C^{q+1}] & \longrightarrow & \operatorname{im}[I^{\bullet, q} \xrightarrow{d_{(0,1)}} I^{\bullet, q+1}] &
\end{array}$$

We call such double complex a **hyper-resolution**.

Proof The existence more of less follows from definition of cohomology 1.1. Firstly, find an injective resolution for $H(C)$ and $\operatorname{im} d$; then we can get a resolution for $\ker d$ by horseshoe lemma on $0 \to \operatorname{im} d \to \ker d \to H(C) \to 0$;

then we can get a resolution for C by horseshoe lemma again on $0 \to \ker d \to C \to \operatorname{im} d \to 0$. Lastly, from $0 \to \operatorname{im} d \to C \to \operatorname{cok} d \to 0$, the condition on cok follows automatically. Q.E.D.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \\
C^{i+2} \arrow[u] \arrow[r] & I^{0,i+2} \arrow[u] \arrow[r] & I^{1,i+2} \arrow[u] \arrow[r] & I^{2,i+2} \arrow[u] \arrow[r] & \cdots \\
C^{i+1} \arrow[u] \arrow[r] & I^{0,i+1} \arrow[u] \arrow[r] & I^{1,i+1} \arrow[u] \arrow[r] & I^{2,i+1} \arrow[u] \arrow[r] & \cdots \\
C^{i} \arrow[u] \arrow[r] & I^{0i} \arrow[u] \arrow[r] & I^{1i} \arrow[u] \arrow[r] & I^{2i} \arrow[u] \arrow[r] & \cdots \\
C^{i-1} \arrow[u] \arrow[r] & I^{0,i-1} \arrow[u] \arrow[r] & I^{1,i-1} \arrow[u] \arrow[r] & I^{2,i-1} \arrow[u] \arrow[r] & \cdots \\
C^{i-2} \arrow[u] \arrow[r] & I^{0,i-2} \arrow[u] \arrow[r] & I^{1,i-1} \arrow[u] \arrow[r] & I^{2,i-2} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

Künneth Spectral Sequences

6.3. For a hyper-resolution $C \to I$, the induced map $C \to \operatorname{Tot} I$ is a quasi-isomorphism (i.e. inducing isomorphism on cohomology). This facts follows from a common usage of spectral sequence, which is left as an exercise.

6.4. Hyper-derived Functors For a left exact functor F. We define the **hyper-derived functor** $\mathbf{R}^i F$ on lower bounded complexe C (i.e. $C^p = 0$ for $p \ll 0$) as follows. We can find a quasi-isomorphism $C \to I$ with each I^i injective (the existence is established above). We define

$$\mathbf{R}^i F(C) = H^i(F(I)).$$

Note that different choice of I does not affect $\mathbf{R}^i F(C)$.

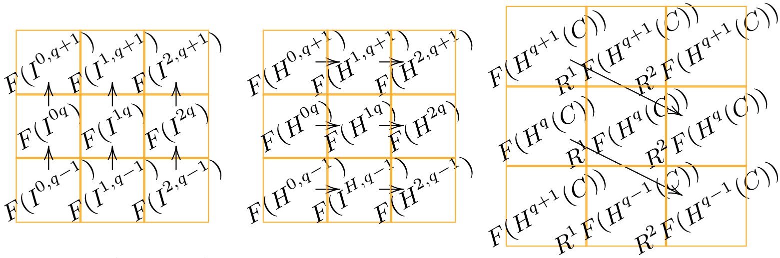
6.5. For the readers who are familiar with the derived category $\mathcal{D}(-)$, this diagram probably helps

$$\underbrace{\mathcal{A} \xrightarrow{\subseteq} \overbrace{\mathcal{D}(\mathcal{A}) \xrightarrow{\mathbf{R}F} \mathcal{D}(\mathcal{B}) \xrightarrow{H^i(-)} \mathcal{B}}^{\mathbf{R}^i F}}_{=R^i F}$$

6.6. Künneth Spectral Sequences For a left exact functor F, and C a lower bounded complex, there is a spectral sequence

$$E_2^{pq} = R^p F(H^q(C)) \Longrightarrow \mathbf{R}^{p+q} F(C).$$

Proof Let us pick a hyper-resolution $C \to I$. Then the cohomology $\operatorname{Tot} F(I)$ is by definition $\mathbf{R}^{p+q} F(C)$.



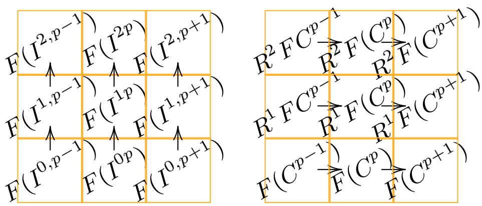
where $H = H(I, d_{(1,0)})$. The computation holds since the complex splits. Note that

$$R^q A = H^q(C) \longrightarrow H^q(I) = H^{\bullet q}$$

is assumed to be an injective resolution. Q.E.D.

6.7. Another direction in 6.6 Under the setting of 6.6, another direction gives spectral sequence

$$E_2^{pq} = H^p(R^q F C^\bullet, d) \Longrightarrow \mathbf{R}^{p+q} F(C).$$



In particular, when each C^i is F-acyclic, $H^i(F(C)) = \mathbf{R}^i F(C)$, a complex version of 2.10.

6.8. More general, there would be some functor sending complex to complex (for example, $\mathrm{Tot}(- \otimes C^\bullet)$), there is also a spectral sequence to control them under some condition, see for example 6.21.

6.9. Classic Künneth Spectral Sequences Let $C^\bullet$ a bounded complex with each C^i flat. There exists a spectral sequence

$$E_2^{pq} = \mathrm{Tor}_{-p}(H^q(C^\bullet), M) \Longrightarrow H^{p+q}(C^\bullet \otimes M).$$

Let $C^\bullet$ a bounded complex with each C^i projective. There exists a spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p(H^{-q}(C^\bullet), M) \Longrightarrow H_{-p-q}(\mathrm{Hom}(C^\bullet, M)).$$

6.10. Universal Coefficient Theorem Assume that in $C^\bullet$, each C and $\operatorname{im} d$ are both flat (respectively, projective). The short exact sequence

$$0 \longrightarrow \ker d \longrightarrow C \longrightarrow \operatorname{im} d \longrightarrow 0$$

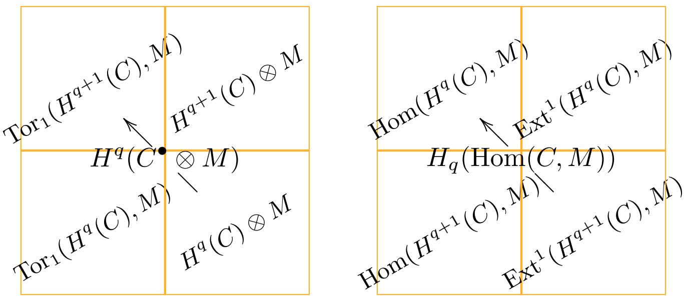
shows that $\ker d$ is also flat (respectively, projective). The short exact sequence

$$0 \longrightarrow \operatorname{im} d \longrightarrow \ker d \longrightarrow H(C) \longrightarrow 0$$

shows that $\mathrm{Tor}_i(H(C), -) = 0$ for $i \geq 2$. Thus we get a short exact sequence

$$0 \longrightarrow H^q(C) \otimes M \longrightarrow H^q(C \otimes M) \longrightarrow \mathrm{Tor}(H^{q+1}(C), M) \longrightarrow 0.$$

$$0 \longrightarrow \mathrm{Ext}(H^{q+1}(C), M) \otimes M \longrightarrow H^q(\mathrm{Hom}(C, M)) \longrightarrow \mathrm{Hom}(H^q(C), M) \longrightarrow 0.$$



Assume that in $C^\bullet$, each C and $\operatorname{im} d$ are both projective. Then $\ker d$ is a direct summand of C, thus $\ker d \otimes M$ is a direct summand of $C \otimes M$, from the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & \operatorname{im} d \otimes M \arrow[r] \arrow[d, equal] & \ker d \otimes M \arrow[r] \arrow[d] & H(C) \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}(d \otimes \mathrm{id}_M) \arrow[r] & \ker(d \otimes \mathrm{id}_M) \arrow[r] & H(C \otimes M) \arrow[r] & 0
\end{tikzcd}$$

the middle map splits, and thus the last map. Similarly, the sequence for Ext splits.

6.11. Universal Coefficient Theorem This gives the **universal coefficient theorem** in topology

$$0 \longrightarrow H^q(X;\mathbb{Z}) \otimes R \longrightarrow H^q(X;R) \longrightarrow \operatorname{Tor}(H^{q+1}(X;\mathbb{Z}), R) \longrightarrow 0.$$

$$0 \longrightarrow H_q(X;\mathbb{Z}) \otimes R \longrightarrow H_q(X;R) \longrightarrow \operatorname{Tor}(H_{q-1}(X;\mathbb{Z}), R) \longrightarrow 0.$$

$$0 \longrightarrow \operatorname{Ext}(H_{q-1}(X;\mathbb{Z}), M) \longrightarrow H^q(X;M) \longrightarrow \operatorname{Hom}(H_q(X;\mathbb{Z}), M) \longrightarrow 0.$$

$$0 \longrightarrow \operatorname{Ext}(H^{q+1}(X;\mathbb{Z}), M) \longrightarrow H_q(X;M) \longrightarrow \operatorname{Hom}(H^q(X;\mathbb{Z}), M) \longrightarrow 0.$$

Auslander–Reiten Transpose

6.12. Auslander–Reiten Transpose For a finitely generated left module M over some ring R. Define the right module by $M^* = \operatorname{Hom}_R(M, R)$. Pick a projective resolution

$$P_1 \to P_0 \to M \to 0$$

Define the **Auslander–Reiten transpose** $\operatorname{Tr} M$ by the right module satisfying the exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to \operatorname{Tr} M \to 0.$$

6.13. Theorem Let N be a right R-module. We have a four-term exact sequence

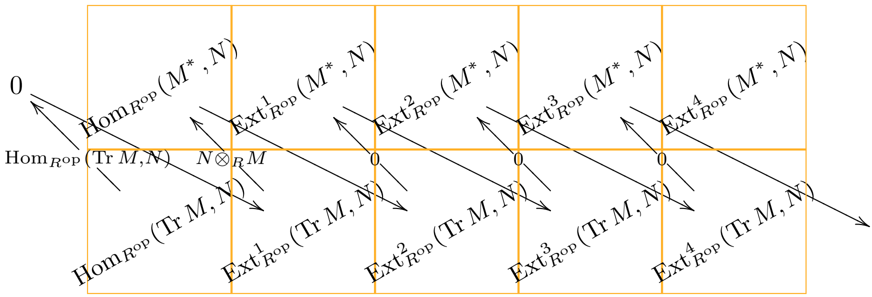
$$0 \to \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} M, N) \to N \otimes_R M \to \operatorname{Hom}_{R^{\mathrm{op}}}(M^*, N) \to \operatorname{Ext}^2_{R^{\mathrm{op}}}(\operatorname{Tr} M, N) \to 0$$

In particular, M is reflexible i.e. $M \cong (M^*)^*$ by the natural map if and only $\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Tr} M, R^{\mathrm{op}}) = 0$ for $i = 1, 2$.

Proof Apply $\mathrm{Hom}_{R^{\mathrm{op}}}(-,N)$ to the two-term exact sequence $P_0^* \to P_1^*$, then by 6.6, the following spectral sequence converges to the cohomology of

$$\underbrace{\mathrm{Hom}_{R^{\mathrm{op}}}(P_1^*, M)}_{\cong N\otimes_R P_1} \longrightarrow \underbrace{\mathrm{Hom}_{R^{\mathrm{op}}}(P_0^*, M)}_{\cong N\otimes_R P_0}$$

i.e. $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Tr}\, M, N)$ and $N \otimes_R M$.



This reads the computation in the theorem. Q.E.D.

6.14. Stable Hom For two left modules M, N of some ring R. We define the **stable Hom**

$$\underline{\mathrm{Hom}}_R(M,N) = \frac{\mathrm{Hom}_R(M,N)}{\{M \to P \to N : \text{for some projective } P\}}$$

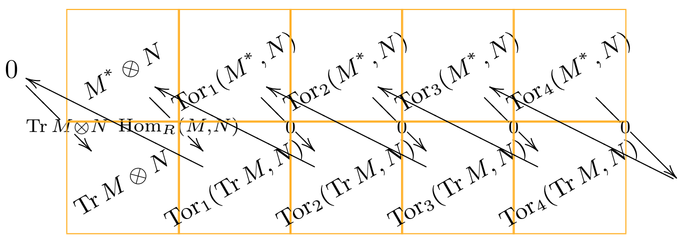
6.15. Theorem Let N be a right R-module. Then

$$\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Tor}_1^R(\mathrm{Tr}\, M, N)$$

Proof Apply $- \otimes N$. We see

$$\underbrace{P_0^* \otimes_R N}_{\cong \mathrm{Hom}_R(P_0,N)} \to \underbrace{P_1^* \otimes_R N}_{\cong \mathrm{Hom}_R(P_1,N)}$$

whose homology is $\mathrm{Ext}^1(M,N)$ and $\mathrm{Tr}\, M \otimes N$.



Note that for any surjective $P \twoheadrightarrow N$,

$$\underline{\mathrm{Hom}}_R(M,N) = \mathrm{cok}\left[\mathrm{Hom}_R(M,P) \to \mathrm{Hom}_R(M,N)\right].$$

This factors through the surjection $\mathrm{Hom}(M,P) = M^* \otimes P \to M^* \otimes N$. Q.E.D.

6.16. Actually, we get a four-term exact sequence

$$0 \to \mathrm{Tor}_2^R(\mathrm{Tr}\, M, N) \to M^* \otimes N \to \mathrm{Hom}_R(M,N) \to \mathrm{Tor}_1^R(\mathrm{Tr}\, M, N) \to 0.$$

6.17. Further more, if R and M, N are all finite-dimensional over a field $\Bbbk$, then we can define a functor

$$D = \mathrm{Hom}_{\Bbbk}(-,\Bbbk) : R\text{-}\mathsf{mod} \longrightarrow \mathsf{mod}\text{-}R$$

where mod stands the finite-dimensional module. Note that

$$D\,\mathrm{Hom}_R(M,N) = DN \otimes M, \qquad D(M^* \otimes N) = \mathrm{Hom}_{R^{\mathrm{op}}}(M^*, DN).$$

This sets a duality between two four-term exact sequences, thus we have

$$\underline{\mathrm{Hom}}_R(M,N) = D\,\mathrm{Ext}^1_{R^{\mathrm{op}}}(\mathrm{Tr}\, M, DN) = D\,\mathrm{Ext}^1_R(N, D\,\mathrm{Tr}\, M).$$

This is known as **Auslander–Reiten formula**. The functor $D\,\mathrm{Tr}$ is called **Auslander–Reiten translation**. It can be understood as a generalized Serre functor. We can similarly define $\overline{\mathrm{Hom}}$ by factoring all morphisms through injective modules. We have the dual version of Auslander–Reiten formula,

$$D\overline{\mathrm{Hom}}_R(M,N) = \mathrm{Ext}^1_R(\mathrm{Tr}\, DN, M).$$

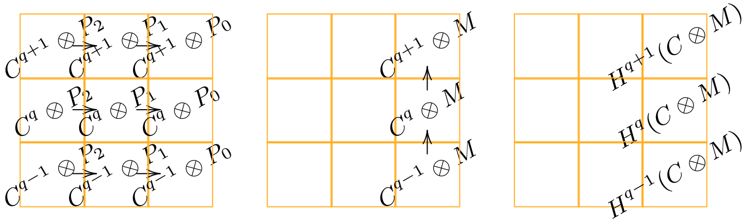
Exercises

6.18. Prove 6.3.

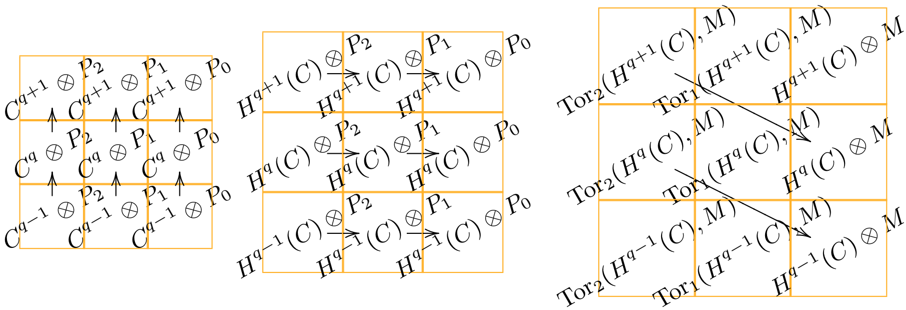
6.19. Prove that for hyper-resolution $C \to I$, the induced map $C \to \mathrm{Tot}\, I$ is a quasi-isomorphism as claimed in 6.3.

6.20. Prove the classic Künneth spectral sequences 6.9 by picking a resolution for M.

Hint Pick a resolution $P \to M$. Then $C \otimes P$ forms a double complex,



Here we use the fact that C^i are all flat. On the other hand,



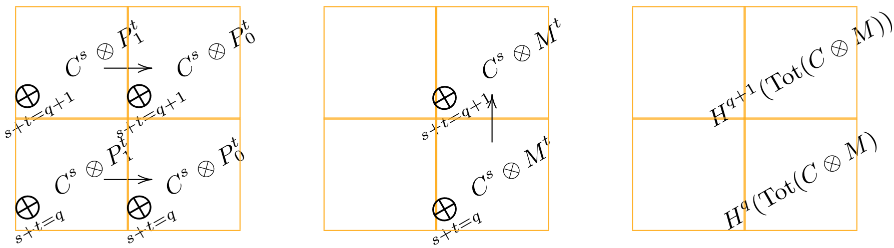
6.21. Künneth Spectral Sequences Let $C^\bullet$ a bounded complex with each C^i flat. Prove that there exists a spectral sequence

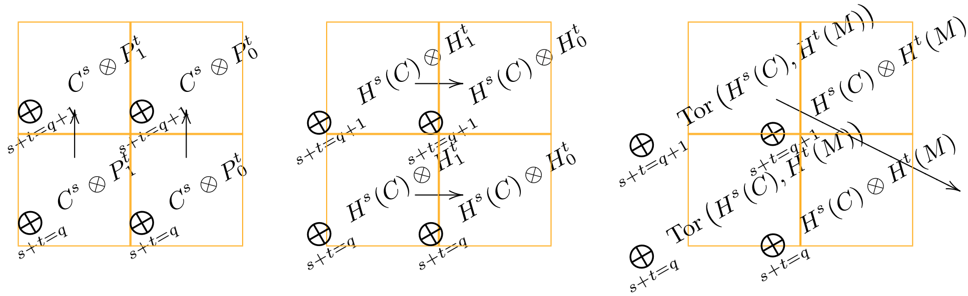
$$E_2^{pq} = \bigoplus_{s+t=q} \mathrm{Tor}_{-p}(H^s(C), H^t(M)) \Longrightarrow H^{p+q}(\mathrm{Tor}(C \otimes M)).$$

Hint We can find a (projective) hyper-resolution $P \to M$. Then let us compute

$$\mathrm{Tot}\, K, \qquad K_p^\bullet = \mathrm{Tot}\, C^\bullet \otimes P_p^\bullet$$

We can compute that




$C^s \otimes P_1^t$
$C^s \otimes P_0^t$
$C^s \otimes P_1^t$
$C^s \otimes P_0^t$
$\bigoplus_{s+t=q+1}$
$\bigoplus_{s+t=q}$
$H^s(C) \otimes H_1^t$
$H^s(C) \otimes H_0^t$
$\mathrm{Tor}\left(H^s(C), H^t(M)\right)$
$H^s(C) \otimes H^t(M)$

7 Algebra (II)

Grothendieck Spectral Sequences

7.1. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three categories with enough injectives. Consider two left exact functors F and G

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "G"'] & \end{tikzcd}$$

Assume for any injective object I in $\mathcal{A}$, $F(I)$ is G-acyclic, that is $R^iG(FI) = 0$ for $i \geq 1$. $(*)$

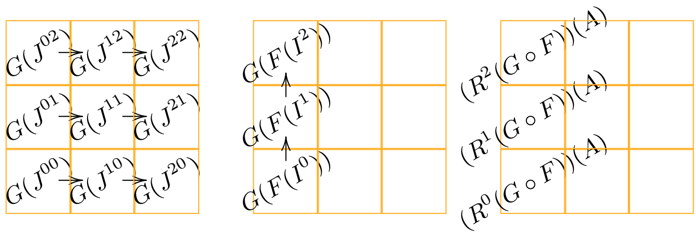
In terms of derived category, $\mathbf{R}G \circ \mathbf{R}F = \mathbf{R}(G \circ F)$.

Tip Assume a class of objects $\mathcal{J} \subseteq \mathcal{A}$ is both $G \circ F$-acyclic and F-acyclic, such that for any $A \in \mathcal{A}$ there is an injection $A \to J$ for $J \in \mathcal{J}$. Then any injective objective is a direct summand of J for some $J \in \mathcal{J}$. Thus it suffices to check the conditioin for I in $\mathcal{J}$.

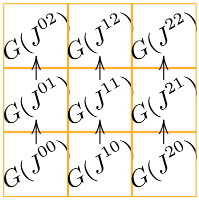
7.2. Grothendieck Spectral Sequences Under the assumption of $(*)$, for any $A \in \mathcal{A}$, we have a spectral sequence

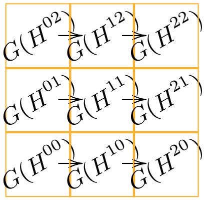
$$E_2^{pq} = R^pG(R^qF(A)) \Longrightarrow (R^{p+q}(G \circ F))(A).$$

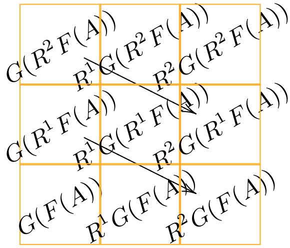
Proof Pick an injective resolution $A \to I$, and apply the Künneth spectral sequence 6.6 to $F(I)$. But to prove them directly is not hard. Find a hyper-resolution for $F(I) \to J$. Let us compute the cohomology of the double complex $G(J)$.



Here we use the assumption that $F(I^q)$ are all G-acyclic. One the other hand,





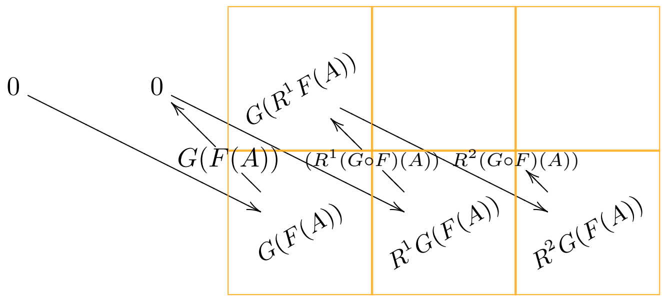


where $H = H(J, d_{(1,0)})$, this computation follows from the fact $J^{p\bullet}$ splits. Note that

$$R^q A = H^q(F(I)) \longrightarrow H^q(J) = H^{\bullet q}$$

is assumed to be an injective resolution. Q.E.D.

7.3. First Five Terms As suggested 3.13, we have **first five terms**



7.4. Change of Ring Let $A \xrightarrow{\varphi} B$ be a ring homomorphism. Let M be an A module, N be a B module (left or right indicated by notations).

$$\begin{tikzcd} A\text{-Mod} \arrow[rr, "-\otimes_A N"] \arrow[dr, "-\otimes_A B"'] & & \mathsf{Ab} \\ & B^{op}\text{-Mod} \arrow[ur, "-\otimes_B N"'] & \end{tikzcd}$$

$$E^2_{pq} = \mathrm{Tor}^B_p(\mathrm{Tor}^A_q(M, B), N) \Longrightarrow \mathrm{Tor}^A_{p+q}(M, N)$$

$$\begin{tikzcd} A\text{-Mod} \arrow[rr, "N\otimes_A -"] \arrow[dr, "B\otimes_A -"'] & & \mathsf{Ab} \\ & B\text{-Mod} \arrow[ur, "N\otimes_B -"'] & \end{tikzcd}$$

$$E^2_{pq} = \mathrm{Tor}^B_p(N, \mathrm{Tor}^A_q(B, M)) \Longrightarrow \mathrm{Tor}^A_{p+q}(N, M)$$

$$\begin{tikzcd} A\text{-Mod} \arrow[rr, "\mathrm{Hom}_A(-,N)"] \arrow[dr, "B\otimes_A -"'] & & \mathsf{Ab}^{\mathrm{op}} \\ & B\text{-Mod} \arrow[ur, "\mathrm{Hom}_B(-,N)"'] & \end{tikzcd} \qquad E_2^{pq} = \mathrm{Ext}^p_B(\mathrm{Tor}^A_{-q}(B,M),N) \Longrightarrow \mathrm{Tor}^A_{-p-q}(N,M)$$

$$\begin{tikzcd} A\text{-Mod} \arrow[rr, "\mathrm{Hom}_A(N,-)"] \arrow[dr, "\mathrm{Hom}_A(B,-)"'] & & \mathsf{Ab} \\ & B\text{-Mod} \arrow[ur, "\mathrm{Hom}_B(N,-)"'] & \end{tikzcd} \qquad E_2^{pq} = \mathrm{Ext}^p_B(N, \mathrm{Ext}^q_A(B,M)) \Longrightarrow \mathrm{Ext}^*_A(N,M)$$

7.5. Local Hom Recall the local Hom $\mathcal{H}om_X(\mathcal{F},\mathcal{G})$ for two sheaves $\mathcal{F}$, $\mathcal{G}$ over X. We can define $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ as derived functor of the second variable G. Then we have sepctral sequence

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q(\mathcal{F},\mathcal{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{F},\mathcal{G}).$$

A similar result holds for coherent sheaves over scheme X.

This comes from the Grothendieck spectral sequence on $\Gamma(X, \mathcal{H}om_X(\mathcal{F},-)) = \mathrm{Hom}_X(\mathcal{F},-)$. By definition, $\mathcal{H}om_X(\mathcal{F},\bullet)$ is flabby if $\bullet$ is.

Group Cohomology

7.6. Group cohomology Let G be a disrecte group, and M a $\mathbb{Z}[G]$-module, we define

$$\begin{aligned} M^G &= \mathrm{Hom}_G(\mathbb{Z}, M) = \{m \in G : {}^{\forall g \in G,}\, gm = m\}, \\ M_G &= \mathbb{Z} \otimes_G M = M/(gm - m : g \in G). \end{aligned}$$

where $\mathbb{Z} = \mathbb{Z}_{\mathrm{tri}}$ with trivial G-action. We call their derived functors by **group (co)homology**

$$H^n(G; M) = \mathrm{Ext}^n_G(\mathbb{Z}; M), \qquad H_n(G; M) = \mathrm{Tor}^G_n(\mathbb{Z}; M).$$

7.7. Group cohomology enough For two G modules M, N, $\mathrm{Hom}(M,N)$ is also a G-module by $(gf)(m) = gf(g^{-1}m)$; $M \otimes N$ is also a G-module by $g(m \otimes n) = gm \otimes gn$. So

$$\mathrm{Hom}_G(M,N) = \mathrm{Hom}(M,N)^G, \qquad M \otimes_G N = (M \otimes N)_G.$$

More general,

$$\mathrm{Ext}^n_G(M,N) = H^n(G; \mathrm{Hom}(M,N)), \qquad \mathrm{Tor}^G_n(M,N) = H_n(G; M \otimes N).$$

7.8. When M is free $\mathbb{Z}[G]$-module, then

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(M, N_{\mathrm{tri}}), \qquad M \otimes N \cong M \otimes N_{\mathrm{tri}}$$

where N_{tri} is the abelian group N but with trivial G-action. Note that this does not mean $H^n(G; N) = H^n(G; N_{\mathrm{tri}})$, since above isomorphism is not natural in M.

7.9. Shapiro Lemma Let H be a subgroup of a discrete group G and M a $\mathbb{Z}[H]$-module. Then

$$H^n(G; M{\Uparrow}_H^G) = H^n(H; M), \qquad H_n(G; M{\uparrow}_H^G) = H_n(H; M),$$

where

$$M{\Uparrow}_H^G := \mathrm{Hom}_H(\mathbb{Z}[G], M), \qquad M{\uparrow}_H^G := \mathbb{Z}[G] \otimes_H M.$$

Proof Let $P_\bullet \to \mathbb{Z}$ be a $\mathbb{Z}[G]$-projective resolution. Then it is also an $\mathbb{Z}[H]$-projective resolution. Then

$$\begin{aligned} H^n(G; M{\Uparrow}_H^G) &= H^n(\mathrm{Hom}_G(P_\bullet, \mathrm{Hom}_H(\mathbb{Z}[G], M))) \\ &= H^n(\mathrm{Hom}_H(P_\bullet, M)) = H^n(H; M). \\ H_n(G; M{\uparrow}_H^G) &= H_n(P_\bullet \otimes_G \mathbb{Z}[G] \otimes_H M)) \\ &= H_n(P_\bullet \otimes_H M) = H_n(H; M). \end{aligned}$$

Actually, this can also be seen from the fact both side are derived functor of

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{Z}, -{\Uparrow}_H^G) &= \mathrm{Hom}_G(\mathbb{Z}, \mathrm{Hom}_H(\mathbb{Z}[G], -)) = \mathrm{Hom}_H(\mathbb{Z}, -) \\ \mathbb{Z} \otimes_G -{\uparrow}_H^G &= \mathbb{Z} \otimes_G \mathbb{Z}[G] \otimes_H - = \mathbb{Z} \otimes_H - \end{aligned}$$

We need to use the fact ${\Uparrow}_H^G$ and ${\uparrow}_H^G$ are exact. Q.E.D.

7.10. (Co)induced G-module In particular, when H is the trivial subgroup. Then the **coinduced and induced G-modules**

$$M{\Uparrow}^G := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M), \qquad M{\uparrow}^G := \mathbb{Z}[G] \otimes_{\mathbb{Z}} M,$$

are G-(co)cyclic. That is, for $n \geq 1$

$$H^n(G; M{\Uparrow}^G) = 0, \qquad H_n(G; M{\uparrow}^G) = 0.$$

7.11. For a $\mathbb{Z}[G]$-projective resolution $P_\bullet \to \mathbb{Z}$.

- For a subgroup H, $P_\bullet \to \mathbb{Z}$ is also a $\mathbb{Z}[H]$-projective resolution.
- For a normal subgroup N, $(P_\bullet)_N \to \mathbb{Z}$ is a $\mathbb{Z}[G/N]$-projective resolution.

7.12. Restriction For a subgroup $H \subseteq G$, we can define **(co)restriction**

$$H^n(G;M) \xrightarrow{\text{res}} H^n(H;M), \qquad H_n(H;M) \xrightarrow{\text{cores}} H_n(G;M).$$

It is induced by

$$M^G \hookrightarrow M^H, \qquad M_H \twoheadrightarrow M_G.$$

7.13. Inflation For a normal subgroup $N \subseteq G$, we can define **(co)inflation**

$$H^n(G/N;M^N) \xrightarrow{\text{inf}} H^n(G;M), \qquad H_n(G;M) \xrightarrow{\text{coinf}} H_n(G/N;M_N).$$

It is induced by

$$(M^N)^{G/N} = M^G, \qquad M_G = (M_N)_{G/N}.$$

7.14. Hochschild Spectral Sequences Let G be a discrete group, and N be a normal subgroup. For any G-module M, there a spectral sequence

$$E_2^{pq} = H^p(G/N; H^q(N;M)) \Longrightarrow H^{p+q}(G;M).$$

$$E^2_{pq} = H_p(G/N; H_q(N;M)) \Longrightarrow H_{p+q}(G;M).$$

Proof Note that

$$\begin{tikzcd}
G\text{-Mod} \arrow[rr, "(-)^G"] \arrow[dr, "(-)^N"'] & & \mathsf{Ab} \\
& G/N\text{-Mod} \arrow[ur, "(-)^{G/N}"'] &
\end{tikzcd}$$

Firstly, the derived functor of $(-)^N$ coincides with $H(N,-)$ (this is not trivial), since a projective $\mathbb{Z}[G]$-module is also a projective $\mathbb{Z}[N]$-module, thus the computation of cohomology is the same.

Secondly, we see that any G-module M admits an injective G-map to coinduced module $M \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$, and coinduced G-modules are also coinduced N-modules. Thus coinduced module is enough to compute the $H^i(G;-)$ and $H^i(N;-)$. Now

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],M)^N &= \mathrm{Hom}_N\left(\mathbb{Z},\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],M)\right)\\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]\otimes_N\mathbb{Z},M)=\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/N],M)\end{aligned}$$

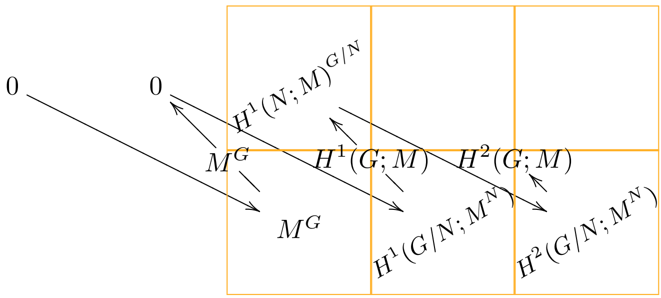
is also coinduced. Thus we can apply the Grothendieck spectral sequence 7.2.

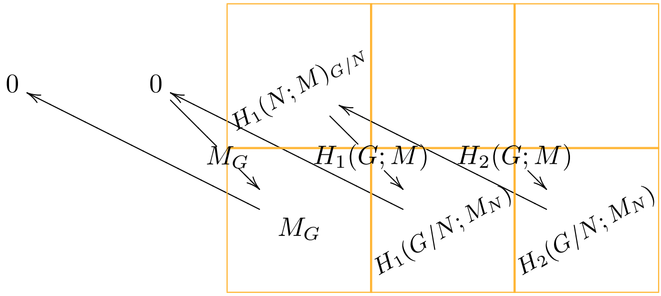
For homology, we can argue similarly, for any G-module M, we have a surjective G-map $\mathbb{Z}[G]\otimes_{\mathbb{Z}}M\to M$. The details are left to readers. Q.E.D.

7.15. First Five Terms In particular, we have first five term sequence 3.13

$$0\to H^1(G/N;M^N)\xrightarrow{\text{inf}}H^1(G;M)\xrightarrow{\text{res}}H^1(N;M)^{G/N}\xrightarrow{d}H^2(G/N;M^N)\xrightarrow{\text{inf}}H^2(G;M)\to ??$$

$$??\to H_2(G;M)\xrightarrow{\text{coinf}}H_2(G/N;M_N)\xrightarrow{d}H_1(N;M)_{G/N}\xrightarrow{\text{cores}}H_1(G;M)\xrightarrow{\text{coinf}}H_1(G/N;M_N)\to 0$$





7.16. Equivariant Cohomology Geometrically,

$$H^n(G;\mathbb{Z}) = H^n(K(G,1)), \qquad H_n(G;\mathbb{Z}) = H_n(K(G,1)).$$

Slightly generally, for a G-module M, it induced a local system $\mathcal{M}$ over $K(G,1)$, actually, $H^n(G;M) = H^n(K(G,1);\mathcal{M})$.

In general, for continuous group G, we should replace $K(G,1)$ by the classifying space $BG = EG/G$ constructed by Milnor. For any G-space X, and an equivariant sheaf $\mathcal{F}$ over X, it induces a sheaf $\mathcal{F}_G$ over the **Borel construction** $EG \times_G X$. We define **equivariant cohomology**

$$H^\bullet_G(X;\mathcal{F}) = H^\bullet(EG \times_G X;\mathcal{F}_G).$$

By 9.5 there is a spectral sequence

$$E_2^{pq} = H^p(BG;\mathcal{H}^q(X;\mathcal{F})) \Longrightarrow H^{p+q}_G(X;\mathcal{F}).$$

Note that the group cohoomology is the case when G is discrete and X is a point.

7.17. Bar resolution For any G-module M, there is a standard free resolution $P_\bullet \to M$ where $P_n = \big(\mathbb{Z}[G]^{\otimes n} \otimes M\big)\!\uparrow^G$ with differentials given by

$$P_n \longrightarrow P_{n-1} \qquad g_0(g_1|\cdots|g_n) \longmapsto \begin{array}{c} g_0g_1(g_2|\cdots|g_n|m) \\ +\sum_{i=i}^{n-1}(-1)^i(\cdots|g_ig_{i+1}|\cdots|m) \\ +(-1)^n(\cdots|g_nm) \end{array}$$

Here we use $|$ rather than $\otimes$ to save places, the reason it is called the **bar** resolution of M. It is exact since has a $\mathbb{Z}$-homotopy $g_0(g_1|\cdots|g_n|m) \mapsto (g_0|g_1|\cdots|g_n|m)$.

7.18. In terms of Cycles We will use the case of $M = \mathbb{Z}$. The first several terms are

$$\begin{array}{ccccccccccc}
\cdots & & 3 & & 2 & & 1 & & 0 & & \\
\cdots & \longrightarrow & \mathbb{Z}[G]\otimes\mathbb{Z}[G]^{\otimes 3} & \longrightarrow & \mathbb{Z}[G]\otimes\mathbb{Z}[G]^{\otimes 2} & \longrightarrow & \mathbb{Z}[G]\otimes\mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}
\end{array}$$

$$\begin{aligned}
g &\longmapsto 1 \\
g_0\otimes g_1 &\longrightarrow g_0g_1 - g_0 \\
g_0\otimes g_1\otimes g_2 &\longmapsto g_0g_1\otimes g_2 - g_0\otimes g_1g_2 + g_0\otimes g_1 \\
g_0\otimes g_1\otimes g_2\otimes g_3 &\longmapsto g_0g_1\otimes g_2\otimes g_3 - g_0\otimes g_1g_2\otimes g_3 + g_0\otimes g_1\otimes g_2g_3 - g_0\otimes g
\end{aligned}$$

We see that

$$H^1(G;M) = \frac{\{G\xrightarrow{f} M : f(g_1g_2) = g_1f(g_2) + f(g_1)\}}{\{G\xrightarrow{f} M : \exists x\in M, f(g_1) = g_1x - x\}} := \frac{\mathrm{Der}(G,M)}{\mathrm{Der}_{\mathrm{Inn}}(G,M)}.$$

$$H^2(G;M) = \frac{\{G\times G\xrightarrow{f} M : g_1f(g_2,g_3) + f(g_1,g_2g_3) = f(g_1g_2,g_3) + f(g_1,g_2)\}}{\{G\times G\xrightarrow{f} M : \exists G\xrightarrow{h} M, f(g_1,g_2) = g_1h(g_2) - h(g_1g_2) + h(g_1)\}}.$$

Actually, the five terms sequences 7.15 above can be proved directly by diagram chasing using above presentation.

7.19. In particular, under the condition of 7.14, when M is a G/N-module, the sequence can be modified to be

$$0\to \mathrm{Der}(G/N,M)\to\mathrm{Der}(G,M)\to\mathrm{Hom}_{G/N}(N_{\mathrm{ab}},M)\to H^2(G/N;M)\to H^2(G;M),$$

where the G/N-module action on the $N_{\mathrm{ab}} = N/[N,N]$ is induced from the conjugation action of G.

Hochschild Cohomology

7.20. Hochschild Cohomology Let $\Bbbk$ be a field for simplicity. Let R be an $\Bbbk$-algebra. Denote $A^e = R\otimes_{\Bbbk} R^{\mathrm{op}}$ the **enveloping algebra**. Let M be a A^e-module. We define

$$\begin{aligned}
M^R &= \mathrm{Hom}_{R^e}(R,M) = \{m\in M : {}^{\forall r\in R,}\, rm = mr\} \\
M_R &= R\otimes_{R^e} M = M/(rm - mr : r\in R)
\end{aligned}$$

We define the **Hochschild (co)homology** by its derived functor

$$\mathrm{HH}^n(R;M) = \mathrm{Ext}^n_{R^e}(R,M), \qquad \mathrm{HH}_n(R;M) = \mathrm{Tor}^{R^e}_n(R,M).$$

7.21. For two R-modules M, N, $\mathrm{Hom}(M, N)$ is an R^e-module by $(rfs)(m) = rf(sm)$. For right R-module M and left R-module N, $M \otimes_{\Bbbk} N$ is also a R^e-module by $r(m \otimes n)s = ms \otimes rg$. So

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}(M, N)^R, \qquad M \otimes_R N = (M \otimes N)_R.$$

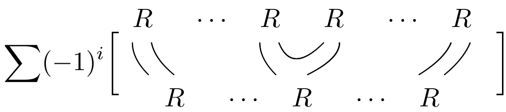
More general,

$$\mathrm{Ext}^n_R(M, N) = \mathrm{HH}^n(R; \mathrm{Hom}(M, N)), \qquad \mathrm{Tor}^R_n(M, N) = \mathrm{HH}_n(R; M \otimes N).$$

7.22. Bar Resolutions We have a bar resolution $B_\bullet \to R$ by $B_n = R \otimes_{\Bbbk} R^{\otimes n} \otimes_{\Bbbk} R$, with

$$d(x_0 \mid \cdots \mid x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \mid \cdots \mid x_i x_{i+1} \mid \cdots x_{n+1}.$$

Here $\mid$ the "bar" is a abbreviation of $\otimes$. Actually, $S(x_0 \mid \cdots \mid x_n) \longmapsto 1 \mid x_0 \mid \cdots \mid x_n$ provides a homotopy. Diagrammatically,



In particular,

$$\mathrm{HH}^1(R; M) = \frac{\left\{R \overset{\text{linear } f}{\longrightarrow} M : f(x_1 x_2) = x_1 f(x_2) + f(x_1) x_2\right\}}{\left\{R \overset{\text{linear } f}{\longrightarrow} M : \exists x \in M, f(x_1) = x_1 x - x x_1\right\}} := \frac{\mathrm{Der}(R, M)}{\mathrm{Der}_{\mathrm{Inn}}(R, M)}.$$

$$\mathrm{HH}^2(R; M) = \frac{\left\{R \otimes R \overset{\text{linear } f}{\longrightarrow} M : \begin{array}{c} x_1 f(x_2, x_3) + f(x_1, x_2 x_3) \\ = f(x_1 x_2, x_3) + f(x_1, x_2) x_3 \end{array}\right\}}{\left\{R \times R \overset{\text{linear } f}{\longrightarrow} M : \begin{array}{l} \exists R \overset{\text{linear } h}{\longrightarrow} M \quad f(x_1, x_2) \\ = x_1 h(x_2) - h(x_1 x_2) + h(x_1) x_2 \end{array}\right\}}.$$

7.23. Similar to group cohomology 7.19, we also have

$$0 \to \mathrm{Der}(R/I, M) \to \mathrm{Der}(R, M) \to \mathrm{Hom}_{R/I}(I_{\mathrm{ab}}, M) \to \mathrm{HH}^2(R/I; M) \to \mathrm{HH}^2(R; M),$$

where the R/I-module action on the $I_{\mathrm{ab}} = I/[I, I]$ is induced from the multiplication action of R. But the author does not know to prove it using spectral sequence argument.

Exercises

7.24. Grothendieck Spectral Sequences Prove that we have complex version of Grothendieck spectral sequence

$$E_2^{pq} = R^p G(\mathbf{R}^q F(C)) \Longrightarrow (\mathbf{R}^{p+q}(G \circ F))(C).$$

7.25. Prove 7.11.

7.26. Prove Cartan–Leray spectral sequence 5.6 by Künneth spectral sequence 6.6.

7.27. Algebra Structure There is a natural algebra structure over $H^\bullet(G;\mathbb{Z}) = \operatorname{Ext}^\bullet_G(\mathbb{Z},\mathbb{Z})$ or $\operatorname{HH}^\bullet(R;R) = \operatorname{Ext}^\bullet_{R^e}(R,R)$ by **Yoneda pairing**. Show that they are graded-commutative

$$xy = (-1)^{\deg x \deg y} yx$$

by **Eckmann–Hilton argument**.

Hint For a resolution $P_\bullet \to \mathbb{Z}$, the product $P_\bullet \otimes P_\bullet$ is also a projective resolution of $\mathbb{Z}$. Assume $x = [f]$ and $y = [g]$ with $\deg x = m$ and $\deg y = n$. Then

$$\begin{tikzcd}
P_\bullet \arrow[d, "f"] & P_\bullet \otimes P_\bullet \arrow[d, "\mathrm{id}\otimes f"] & P_\bullet \otimes P_\bullet \arrow[d, "\mathrm{id}\otimes f"] & P_\bullet \otimes P_\bullet \arrow[d, "g\otimes \mathrm{id}"] & P_\bullet \arrow[d, "g"] \\
P_\bullet[m] \arrow[d, "g"] \arrow[r] & P_\bullet \otimes P_\bullet[m] \arrow[d, "\mathrm{id}\otimes g"] \arrow[r] & P_\bullet \otimes P_\bullet[m] \arrow[d, "g\otimes \mathrm{id}"] \arrow[r] & P_\bullet[n] \otimes P_\bullet \arrow[d, "\mathrm{id}\otimes f"] \arrow[r] & P_\bullet[m] \arrow[d, "f"] \\
P_\bullet[m+n] & P_\bullet \otimes P_\bullet[m+n] & P_\bullet[n] \otimes P_\bullet[m] & P_\bullet[n] \otimes P_\bullet[m] & P_\bullet[m+n]
\end{tikzcd}$$

The above $\to$ are all homotopy up to sign. Then by Kozsul convention 2.12, the sign is just reflected by the graded commutativity.

8 Geometry (I)

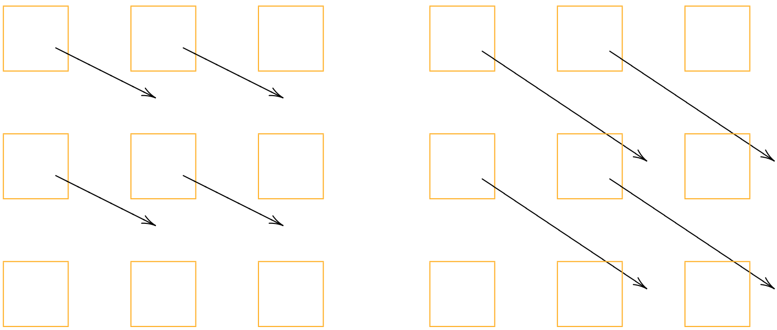
Degeneration

8.1. Degeneration We say a spectral sequence **degenerates** at r-th stage if $E_r = E_\infty$, i.e. there is no nonzero differential $\geq r$.

8.2. Relation to Leray–Hirsch theorem Assume the fibre bundle satisfies the condition of Leray–Hirsch theorem 4.4. Then the spectral sequence degenerates at E_2, i.e. $E_2 = E_\infty$. Since the cohomology cannot be "less than" E_2.

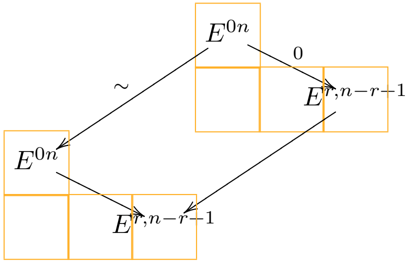
Conversely, for path-connected B, if the spectral sequence degenerates at E_2, and $H^\bullet(F)$ is a free module, then we can lift a set of generators to $H^\bullet(X)$ (since $E_\infty^{0r} = E_2^{0r} = H^r(F)$ is a quotient of $H^r(X)$), then it satisfies the condition of Leray–Hirsch theorem.

8.3. For a fibre bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ with fibre F, if $H^{\mathrm{odd}}(F) = H^{\mathrm{odd}}(X) = 0$, then the spectral sequence for ξ degenerates.



8.4. Degeneration Theorem If the Leray–Serre spectral sequence for ξ degenerates at E_2, then so is its pull back.

Proof Firstly, the differential of E_2 for $f^*\xi$ are all zero. Due to the multiplicative structure, it suffices to show the differential from E_2^{0n}. By the functoriality, it is zero



Then the differential of E_3 for $f^*\xi$ are all zero. Due to the multiplicative structure (the same structure as E_2 since the differentials of E_2 are zero), it still suffices to show the differential from E_3^{0n}. So the general case has no difference and can proved by induction. Q.E.D.

Flags, Grassmannians, etc.

8.5. Denote $\mathbb{C}^\infty = \bigoplus_{i=1}^\infty \mathbb{C}e_i$, under the inductive topology (topology for inductive limit).

8.6. Projective Spaces For any complex vector space V (not necessarily finite dimensional), we define the projective space

$$\mathbb{P}V = \{\text{linear subspace } \ell \subseteq V : \dim \ell = 1\}.$$

For $V = \mathbb{C}^{N+1}$, it is usually denoted by $\mathbb{C}P^N$. Then

$$H^\bullet(\mathbb{P}(V)) = \begin{cases} \mathbb{Z}[H]/(H^{\dim V+1}) & \dim V < \infty \\ \mathbb{Z}[H] & \dim V = \infty \end{cases} \qquad \deg H = 2.$$

The generator $H = H_V \in H^2(\mathbb{P}(V))$ is universal in the following sense, for any linear subspace $W \subseteq V$, the natural map $H^2(\mathbb{P}(V)) \to H^2(\mathbb{P}(W))$ induced by $\mathbb{P}W \subseteq \mathbb{P}V$ sending H_V to H_W.

Proof For any choice of $V \cong \mathbb{C}^{N+1}$, it defines a stratification (cellular) structure

$$\mathbb{P}V = \mathrm{pt} \sqcup \mathbb{C} \sqcup \mathbb{C}^2 \sqcup \cdots \sqcup \mathbb{C}^N$$

Thus,

i	0	1	2	3	4	$\cdots$	$2N$
$H^i(\mathbb{P}V)$	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	$\cdots$	$\mathbb{Z}$
$H_i(\mathbb{P}V)$	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	$\cdots$	$\mathbb{Z}$

In homology,

$$H_{2i}(\mathbb{P}V) = \mathbb{Z} \cdot [L^i],$$

where L^i is the closure of $\mathbb{C}^i$ in the stratification, an i-plane. Actually, this can be any i-plane since $\mathrm{GL}_{N+1}(\mathbb{C})$ is connected and acts on i-planes transitively. Then by Poincaré duality, when $\dim V < \infty$,

$$H^{2i}(\mathbb{P}V) = \mathbb{Z} \cdot [H^i],$$

where H^i is any $(n-i)$-plane. In particular $[H] \in H^2(\mathbb{P}(V))$ is the hyperplane, and $[H^i] = [H]^i$ by linear algebra. Thus

$$H^\bullet(\mathbb{P}(V)) = \mathbb{Z}[H]/(H^{\dim V+1}).$$

For the infinite case, note that $\mathbb{P}V \subseteq \mathbb{P}\mathbb{C}^\infty$ induces algebra homormohphism $H^\bullet(\mathbb{P}\mathbb{C}^\infty) \to H^\bullet(\mathbb{P}V)$ which is isomorphic for $\bullet < 2\dim V$. Thus

$$H^\bullet(\mathbb{P}(V)) = \mathbb{Z}[H].$$

From the construction, we see the choice of H is universal. Q.E.D.

8.7. Partial Flag Varieties Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}_0^n$ with $|\mathbf{d}| := d_1 + \cdots + d_n = d$. Denote

$$\mathcal{F}\ell(\mathbf{d}, V) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V : \dim V_*/V_{*-1} = d_* \right\}.$$

It suffices to consider two cases, $\dim V = \infty$, and $\dim V = d$. Then

$$H^\bullet(\mathcal{F}\ell(\mathbf{d}, V)) = \begin{cases} \dfrac{\mathbb{Z}[x_1, \ldots, x_d]^{\mathfrak{S}_{\mathbf{d}}}}{\left\langle \mathbb{Z}[x_1, \ldots, x_d]^{\mathfrak{S}_d}_{\deg \geq 1} \right\rangle} & \dim V = d \\ \mathbb{Z}[x_1, \ldots, x_d]^{\mathfrak{S}_{\mathbf{d}}} & \dim V = \infty \end{cases}$$

where $\deg x_i = 2$, and $\mathfrak{S}_{\mathbf{d}} = \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n} \subseteq \mathfrak{S}_d$.

8.8. Here is some other notations for $\mathcal{F}\ell(\mathbf{d}, V)$

- When $V = \mathbb{C}^\infty$, we will just write $\mathcal{F}\ell(\mathbf{d}, \infty)$.
- When $V = \mathbb{C}^d$, we will just write $\mathcal{F}\ell(\mathbf{d})$.
- For $\mathbf{d} = (1, \dots, 1)$ we will simply denote $\mathcal{F}\ell(\mathbf{d}, \infty)$ by $\mathcal{F}\ell(d, \infty)$, the infinite **flag variety**. By above theorem, its cohomology is $\mathbb{Z}[x_1, \dots, x_d]$ the algebra of polynomials of d variables.
- We also denote when $\dim V = d$, $\mathcal{F}\ell(\mathbf{d}, V) = \mathcal{F}\ell(V)$ and $\mathcal{F}\ell(\mathbf{d})$ by $\mathcal{F}\ell(d)$, called the **flag varieties**.
- For $\mathbf{d} = (d)$, we will denote $\mathcal{F}\ell(\mathbf{d}, \infty)$ by $\mathcal{G}r(d, \infty)$, the infinite **Grassmannian**. By above theorem, its cohomology is $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_d}$ the algebra of symmetric polynomialn of d variables.
- For $\mathbf{d} = (k, d-k)$, we will denote $\mathcal{F}\ell(\mathbf{d})$ by $\mathcal{G}r(k, d)$, the **Grassmannian**.

Sketch of the Proof of 8.7 We pick a unitary product on V. Then by picking unitary basis, we see

$$\mathcal{F}\ell(\mathbf{d}, V) = \left\{ (\ell_1, \dots, \ell_n) : \begin{array}{l} {}^{\forall i}, \dim \ell_i = d_i, \\ {}^{\forall i \neq j}, \ \ell_i \perp \ell_j \end{array} \right\}.$$

Thus we have

$$\begin{tikzcd}
\boxed{\begin{matrix}\mathcal{F}\ell(d_1)\\ \times \\ \vdots \\ \times \\ \mathcal{F}\ell(d_n)\end{matrix}} \arrow[r] \arrow[dr] & \mathcal{F}\ell(d) \arrow[r] \arrow[d] & \mathcal{F}\ell(\mathbf{d}) \arrow[d] \\
 & \mathcal{F}\ell(d, \infty) \arrow[r] \arrow[d] \arrow[dl] \arrow[ddl] \arrow[dd, bend left] & \mathcal{F}\ell(\mathbf{d}, \infty) \arrow[dl] \\
\mathbb{C}P^\infty \arrow[d, dashed, leftrightarrow] & \boxed{\mathcal{G}r(d, \infty)} & \\
\mathbb{C}P^\infty \arrow[r, dashed, leftrightarrow, "\mathfrak{S}_n \text{ action}"'] & \mathbb{C}P^\infty \arrow[ul, dashed, leftrightarrow] &
\end{tikzcd}$$

We can define $x_i \in H^\bullet(\mathcal{F}\ell(d, \infty))$ the pull back of hyperplane section through the i-th projection $\mathcal{F}\ell(d, \infty) \to \mathbb{C}P^\infty$.

- By inductively using Leray–Hirsch theorem 4.4, we can prove that $\mathcal{F}\ell(d,\infty)$ satisfies the theorem. Note that the fibre of each projection is $\mathcal{F}\ell(d-1,\ell_i^\perp)\cong\mathcal{F}\ell(d-1,\infty)$, and one should use the universality of H in 8.6.
- By induction on d, We can prove the vanishinig of odd cohomology and the Poincaré polynomials for $\mathcal{F}\ell(d,\infty)$, $\mathcal{F}\ell(d)$. Thus we can do so for $\mathcal{G}r(d,\infty)$, $\mathcal{F}\ell(\mathbf{d},\infty)$ and $\mathcal{F}\ell(\mathbf{d})$ since the degeneration of spectral sequence can be implied by Leray–Hirsch theorem.
- Note that $\mathfrak{S}_d$ acts on n-projections, and the cohomology of $\mathcal{F}\ell(\mathbf{d},\infty)$ is exactly the $\mathfrak{S}_{\mathbf{d}}$-invariant part. The inclusion follows directly, and the equality follows from the computation of Poincaré polynomials.
- Finally, by spectral sequence
$$H^\bullet(\mathcal{F}\ell(\mathbf{d}))=\mathbb{Z}\underset{H^\bullet(\mathcal{G}r(n,\infty))}{\otimes}H^\bullet(\mathcal{F}\ell(\mathbf{d},\infty)).$$
This gives the description in the theorem. Q.E.D.

8.9. Remarks We know $BGL_r=\mathcal{G}r(r,\infty)$. Acually, the computation in the proof can be generalized to the computation of $H^\bullet(BG;\mathbb{Q})$ for a lie group G. But a nice description for coefficient $\mathbb{Z}$ cannot be generalized. On the other hand, $\mathcal{F}\ell(\mathbf{d},V)$ has a cellular structure of only even cells, thus no odd cohomology. It is the topic of classic Schubert calculus.

Vector Bundles

8.10. Vector Bundles We can rewrite the definition of fibre bundle in terms of coordinate.

A map $\xi=\begin{bmatrix}E\\\downarrow\\X\end{bmatrix}$ is said to be a **fibre bundle** if there exists an open covering $\mathcal{U}$, and coordinates $\left\{(U,\varphi_U): \begin{array}{c}U\in\mathcal{U},\\ \xi^{-1}(U)\xrightarrow[\sim]{\varphi_U}U\times F\end{array}\right\}$ with
$$\varphi_V\circ\varphi_U^{-1}:(U\cap V)\times F\to(U\cap V)\times F$$
induced by a continuous map $(U\cap V)\to\mathrm{Aut}(F)=\{\text{self-homoemorphism of }F\}$.

A **vector bundle** is the case F is a r-dimensional vector space, and $\mathrm{Aut}(F)$ replaced by GL_r. We call r the **rank** of the vector bundle. When the rank is 1, it is usually referred as a **line bundle**.

We will mainly focus on $\mathbb{C}$-vector spaces. A morphism between vector bundles is locally linear, i.e. induced by $U\cap V\to\mathrm{Hom}(F_1,F_2)$.

8.11. Tangent Bundles For a manifold M, denote the tangent bundle $TM = \bigcup_{x\in M} T_x M$. By the theory of manifold, it is a manifold of dimension $2\dim M$. The projection $\begin{bmatrix} TM \\ \downarrow \\ M \end{bmatrix}$ is a vector bundle, called the **tangent bundle**.

8.12. Tautological Bundle Recall that the the projective space $\mathbb{C}P^N$ is the space of all lines in $\mathbb{C}^{N+1}$. We define $P = \{(\ell, x) \in \mathbb{C}P^N \times \mathbb{C}^{N+1} : x \in \ell\}$. Then $\begin{bmatrix} P \\ \downarrow \\ \mathbb{C}P^N \end{bmatrix}$ is a rank 1 vector bundle. This is known as tautological bundle, since the fibre at ℓ is ℓ itself. The same construction can be done for $\mathcal{F}\ell(\mathbf{d}, V)$ (but with n many).

8.13. Classifying Theorem We have a bijection

$$\mathrm{Vec}^r_{\mathbb{C}}(X) \longrightarrow \pi(X, \mathcal{G}r(r,\infty))$$

where $\pi(-,-) = \mathrm{Map}(-,-)/\mathrm{Homotopy}$ is the homotopy classes of maps. Moreover, this bijection is natural in X. To be exact,

> any rank r vector bundle over X is isomorphic to the pull back of the tautological bundle over $\mathcal{G}r(r,\infty)$ for some map $X \to \mathcal{G}r(r,\infty)$.

We say ξ is **classified by this map**.

Sketch of the Proof By an argument of partition of unity, we can embed any vector bundle ξ into the trivial vector bundle of infinite rank $\begin{bmatrix} X\times\mathbb{C}^\infty \\ \downarrow \\ X \end{bmatrix}$. Then we define the classifying map $X \to \mathcal{G}r(r,\infty)$ by sending $x \in X$ to its fibre in $\mathbb{C}^\infty$. It is clear that the pull back of tautological bundle of $\mathcal{G}r(r,\infty)$ gives back the vector bundle. Lastly, using the vector bundle over $X \times I$, we prove the bijection. Q.E.D.

8.14. Chern Classes Recall that

$$H^\bullet(\mathcal{G}r(r,\infty)) = \mathbb{Z}[x_1,\ldots,x_n]^{\mathfrak{S}_n} = \mathbb{Z}[e_1,\ldots,e_n]$$

with $e_i \in H^\bullet(\mathcal{G}r(r,\infty))$ is the i-th elementary polynomial in $x_1,\ldots,x_r$. For a vector bundle ξ over X, assume it is classified by $f : X \to \mathcal{G}r(r,\infty)$, we define the **Chern classes** $c_i(\xi) \in H^{2i}(X)$ to be the pull back of $(-1)^i e_i \in H^{2i}(\mathcal{G}r(r,\infty))$. We define the total Chern class

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_r(\xi).$$

Then by definition, Chern Class commutes with pull back, that is, $c(f^*\xi) = f^*c(\xi)$ for vector bundle ξ and continuous map f.

8.15. Consider the tautological bundle V_i of $\mathcal{F}\ell(\mathbf{d}, V)$, i.e. at each $(0 = V_0 \subseteq V_1 \subseteq \cdots V_d \subseteq V)$, the fibre is V_i. For the case $\mathcal{F}\ell(d, \infty)$, $c(V_i/V_{i-1}) = 1 - x_i$. Actually, $V_i = \ell_i \oplus \cdots \oplus \ell_1$, where ℓ_i is the pull back of tautological bundle over $\mathbb{C}P^\infty$ through the i-th projection in the proof of 8.7.

8.16. For a vector bundle ξ and a sub-vector bundle η, we have $c(\xi) = c(\xi/\eta)c(\eta)$.

Proof Assume $\operatorname{rank} \xi = r$, and $\operatorname{rank} \eta = s \leq r$. Actually the pair $(\eta \subseteq \xi)$ is classified by two-setp Grassmannian (just as the proof of 8.13)

$$X \longrightarrow \mathcal{G}r(s, r-s, \infty) = \mathcal{F}\ell(\mathbf{d}, \infty), \qquad \mathbf{d} = (s, r-s).$$

So it suffices to deal with the universal case — two tautological bundles $\eta \subseteq \xi$ over $\mathcal{G}r(s, r-s, \infty)$. Now the following maps

$$\begin{tikzcd}
 & \mathcal{G}r(s, r-s, \infty) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{G}r(s, \infty) & \mathcal{G}r(r, \infty) & \mathcal{G}r(r-s, \infty)
\end{tikzcd}$$

classify η, ξ and ξ/η respectively. The cohomology maps are all injective, and by the proof of 8.7,

$$\begin{aligned}
c(\eta) &= 1 - e_1(x_1, \ldots, x_s) + \cdots = (1 - x_1) \cdots (1 - x_s). \\
c(\xi/\eta) &= 1 - e_1(x_{s+1}, \ldots, x_r) + \cdots = (1 - x_{s+1}) \cdots (1 - x_r). \\
c(\xi) &= 1 - e_1(x_1, \ldots, x_r) = (1 - x_1) \cdots (1 - x_r)
\end{aligned}$$

This proves the theorem. Q.E.D.

8.17. Associated Flag Bundle For a vector bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$, the associative projective bundle $\mathcal{F}\ell(\xi) = \begin{bmatrix} \mathcal{F}\ell(E) \\ \downarrow \\ X \end{bmatrix}$ is obtained by exchanging each fibre E_x by the corresponding flag variety $\mathcal{F}\ell(E_x)$ of it.

8.18. Splitting Principle The spectral sequence for the associated projective bundle $\begin{bmatrix} \mathcal{F}\ell(E) \\ \downarrow \\ X \end{bmatrix}$ of a vector bundle degenerates, i.e. $E_2 = E_\infty$. In particular, $H^\bullet(X) \to H^\bullet(\mathcal{F}\ell(X))$ is injective.

Note that, the pull back of ξ on $\mathcal{F}\ell(\xi)$ has a filtration of line bundles (rank 1 vector bundles). This is known as **splitting principle**.

Proof It suffices to deal with the universal case that the vector bundle is tautological bundle $\mathcal{T}$ over $\mathcal{G}r(r, \infty)$. By definition,

$$\mathcal{F}\ell(\mathcal{T}) = \mathcal{F}\ell(r, \infty).$$

Then $\begin{bmatrix} \mathcal{F}\ell(\mathcal{T}) \\ \downarrow \\ \mathcal{G}r(r,\infty) \end{bmatrix}$ degenerates by our computation. Q.E.D.

8.19. The theory of Chern classes can be reformulated in differential geometry and algebraic geometry where spaces of infinite dimensional such as $\mathcal{G}r(r, \infty)$ no longer exist. But we can try to prove the properties asserted as above.

Exercises

8.20. Definition (Associated Projective Bundle) For a vector bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$, the associative projective bundle $\mathbb{P}(\xi) = \begin{bmatrix} \mathbb{P}(E) \\ \downarrow \\ X \end{bmatrix}$ is obtained by exchanging each fibre E_x by the corresponding projective space $\mathbb{P}E_x$ of it. We can define the tautological bundle over $\mathbb{P}(E)$ whose fibre at $\ell \subseteq E_x$ is ℓ itself.

8.21. Degeneration Theorem The spectral sequence for the associated projective bundle $\begin{bmatrix} \mathbb{P}(E) \\ \downarrow \\ X \end{bmatrix}$ of a vector bundle degenerates, i.e. $E_2 = E_\infty$.

Furthermore, as algebra,

$$H^\bullet(\mathbb{P}(E)) = H^\bullet(X)\,[H] \Big/ \big(H^r + c_1(\xi)H^{r-1} + \cdots + c_r(\xi)\big),$$

where $H = c_1(\tau) \in H^2(\mathbb{P}(E))$ with τ the tautological bundle of $\mathbb{P}(E)$.

Actually, this is Grothendieck's way to define Chern classes in algebraic geometry.

8.22. Express the Chern class of $\mathsf{S}^r\xi$ and $\Lambda^r\xi$ in terms of the Chern classes of ξ for a vector bundle ξ.

9 Geometry (II)

Sheaf-theoretic Leray Spectral Sequences

9.1. Push Forward Let $X \to Y$ be a continuous map. If we have a sheaf $\mathcal{F}$ over X, then we can define the **push forward**

$$f_*\mathcal{F} = \Big[U \longmapsto \mathcal{F}(f^{-1}(U))\Big] \quad \text{a sheaf over } Y.$$

It turns out that f_* is left exact, we define $R^i f_*$ by its derived functor, the **higher push forward**.

9.2. For example, when $Y = \mathrm{pt}$, $f_* = \Gamma(X, -)$ is the same as the functor of taking global sections. Thus $R^i f_* = H^i(X; -)$ the functor taking i-th cohomology.

9.3. Higher Direct Image The higher push forward admits a explicit description

$$R^i f_*\mathcal{F} = \text{ associated sheaf of } \Big[U \longmapsto H^i\big(f^{-1}(U); \mathcal{F}|_{f^{-1}(U)}\big)\Big].$$

This techenique is known as **higher direct image**.

9.4. Leray Spectral Sequences For continous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a spectral sequence

$$E_2^{pq} = R^p g_*(R^q f_*(\mathcal{F})) \Longrightarrow R^{p+q}(g \circ f)\mathcal{F},$$

natural in $\mathcal{F} \in \mathsf{Coh}\, X$.

Proof Note that flabby (flasque, en français) sheaves are preserved by f_*, thus it satisfies the condition of Grothendieck spectral sequences 7.2. Q.E.D.

9.5. Sheaf-theoretic Leray Spectral Sequences For any continous map $X \xrightarrow{f} Y$, there is a spectral sequence

$$E_2^{pq} = H^p(Y; R^q f_*\mathcal{F}) = H^{p+q}(X, \mathcal{F}),$$

natural in $\mathcal{F} \in \mathsf{Coh}\, X$.

9.6. Assume a space X is locally contractible.

It is known that $H^n(X;\mathbb{Z}_X)$ is the n-th singular cohomology of X, where $\mathbb{Z}_X$ is the constant sheaf. In general, $H^n(X;\mathcal{L}_X)$ is the n-th singular cohomology of X with coefficient in $\mathcal{L}$, see the remark 9.7 below.

Now consider a fibre bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$ with fibre F. Then using the higher direct image 9.3, we see that $R^q\xi_*\mathbb{Z}_E$ is the local system $\mathcal{H}^q(F)$ over X. Thus above spectral sequence recover Leray–Serre spectral seqeuences 4.6.

9.7. Locally Constant Sheaves Recall a local system 4.7 is a functor from the fundamental groupoid $\Pi(X)$ to the category of abelian group Ab. We can define a sheaf from a local system $\mathcal{L}$, by

$$\mathcal{L}_X(U) = \mathrm{Nat}_{\Pi(U)\to\mathsf{Ab}}(\mathbb{Z}, \mathcal{L}|_{\pi(U)}),$$

where $\mathbb{Z}$ is the constant functor to $\mathbb{Z} \in \mathsf{Ab}$. That is, assign each $x \in U$ an element $s_x \in \mathcal{L}_x$, such that for any path $x \to y$, the inducing map $\mathcal{L}_x \to \mathcal{L}_y$ sending s_x to s_y.

Assume X to be locally simply-connected, This construction defines a locally constant sheaf. Conversely, we can recover the local system by taking stalks. Actually, local system is the same thing as locally constant sheaf, and we will not differ them in notation.

Čech Cohomology

9.8. Čech Spectral Sequences For a sheaf $\mathcal{F}$ and an open covering $\mathcal{U}$, there is a spectral sequence

$$E_1^{pq} = H^q(U^p; \mathcal{F}|_{U^p}) \Longrightarrow H^{p+q}(X),$$

where U^p is the formal disjoint union of all intersections of $(p+1)$ different members of $\mathcal{U}$, and $\mathcal{F}|_{U^p}$ the pull back from X to U^p.

9.9. Before the proof, let us introduce a symbol convention. Pick a set of symbol $\{\mathbf{e}_i : i \in I\}$. We define the wedge product $\wedge$ which is associative with the properties

$$\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i, \qquad \mathbf{e}_i \wedge \mathbf{e}_i = 0.$$

We define the interior product $i_{\mathbf{e}_i}$ by

$$i_{\mathbf{e}_i}(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n}) = \sum (-1)^\ell \langle \mathbf{e}_i, \mathbf{e}_{i_\ell} \rangle \cdot \mathbf{e}_{i_1} \wedge \cdots \widehat{\mathbf{e}_{i_\ell}} \cdots \wedge \mathbf{e}_{i_n}$$

where we assume $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. Note that for all $i, j \in I$

$$i_{\mathbf{e}_i}(\mathbf{e}_j \wedge -) + \mathbf{e}_j \wedge (i_{\mathbf{e}_i} -) = \mathrm{id}\,.$$

9.10. Proof of 9.8 Assume $\mathcal{U}$ is totally ordered by $\{U_i : i \in I\}$. Denote for $p \geq 0$, $U_{i_0,\dots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, then $U^p = \bigsqcup_{i_0 < \dots < i_p} U_{i_0,\dots,i_p}$. Denote the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ by

$$\check{C}^p(\mathcal{U}; \mathcal{F}) = \mathcal{F}(U^p) = \prod_{i_0 < \dots < i_p} \mathbf{e}_{i_0} \wedge \cdots \wedge \mathbf{e}_{i_p} \cdot \mathcal{F}(U_{i_0,\dots,i_p}),$$

with differential $d\alpha = \prod_{i \in I} \mathbf{e}_i \wedge \alpha|_i$.

- Firstly, by definition of a sheaf,

$$H^0(\check{C}(\mathcal{U}; \mathcal{F})) = \ker \left[\prod_{U \in \mathcal{U}} \mathcal{F}(U) \longrightarrow \prod_{U,V \in \mathcal{U}} \mathcal{F}(U \cap V) \right] = \mathcal{F}(U).$$

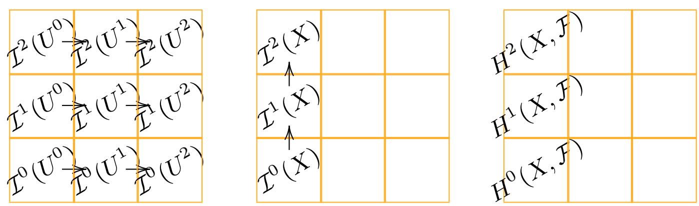
- Secondly, assume $\mathcal{F}$ is supported on one point x, then $\check{C}(\mathcal{U}; \mathcal{F})$ is acyclic. Actually, now

$$\left[\overset{-1}{\mathcal{F}} \longrightarrow \overset{\geq 0}{\check{C}(\mathcal{U}; \mathcal{F})} \right] = \bigotimes_{U_i \ni x} \left[\overset{0}{\mathbb{Z}} \longrightarrow \overset{1}{\mathbf{e}_i \mathbb{Z}} \right] \otimes \mathcal{F}[1].$$

  This is a general fact of Kozsul complex, any exactness of tensor factor kill the cohomology. Explicitly, we can pick one $U_k \ni x$, and define the homotopy by $\alpha \mapsto i_{\mathbf{e}_k} \alpha$ the interior product.

- Thirdly, assume $\mathcal{F} = \prod_x (i_x)_* F_x$ for some abelian group F_x at each point $x \xrightarrow{i_x} X$, then $\check{C}(\mathcal{U}; \mathcal{F})$ is acyclic.

Recall the construction of Godement resolution, we can pick a resolution $\mathcal{F} \to \mathcal{I}$ with $\check{C}(\mathcal{U}; \mathcal{I}^q)$ acyclic. Then



shows that $\operatorname{Tot}\check{C}(\mathcal{U},\mathcal{I})$ computes $H^n(X,\mathcal{F})$. On the other hand,

$$\begin{tikzcd}
\mathcal{I}^2(U^0) & \mathcal{I}^2(U^1) & \mathcal{I}^2(U^2) \\
\mathcal{I}^1(U^0) \arrow[u] & \mathcal{I}^1(U^1) \arrow[u] & \mathcal{I}^1(U^2) \arrow[u] \\
\mathcal{I}^0(U^0) \arrow[u] & \mathcal{I}^0(U^1) \arrow[u] & \mathcal{I}^0(U^2) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
H^2(U^0,\mathcal{F}\backslash_{U^0}) \arrow[r] & H^2(U^1,\mathcal{F}\backslash_{U^1}) \arrow[r] & H^2(U^2,\mathcal{F}\backslash_{U^0}) \\
H^1(U^0,\mathcal{F}\backslash_{U^0}) \arrow[r] & H^1(U^1,\mathcal{F}\backslash_{U^1}) \arrow[r] & H^1(U^2,\mathcal{F}\backslash_{U^0}) \\
H^0(U^0,\mathcal{F}\backslash_{U^0}) \arrow[r] & H^0(U^1,\mathcal{F}\backslash_{U^1}) \arrow[r] & H^0(U^2,\mathcal{F}\backslash_{U^0})
\end{tikzcd}$$

This is the spectral sequence claimed in the theorem. Q.E.D.

9.11. Čech Cohomology In particular, when $\mathcal{F}$ has no higher cohomology over U^p, the **Čech cohomology**

$$\check{H}^n(\mathcal{U},\mathcal{F}) = H^n\left(\check{C}^\bullet(\mathcal{U},\mathcal{F})\right)$$

computes the cohomology $H^n(X;\mathcal{F})$. In particular, when $\mathcal{F}$ is flabby, the Čech complex is acyclic. For example, when $\mathcal{F}=\mathbb{Z}_X$, and X is locally contractible topological space, $\varinjlim_{\mathcal{U}\text{ finer}} \check{H}^n(\mathcal{U};\mathcal{F}) = H^n(X;\mathbb{Z})$. For neotherian seperated scheme X, any open affine cover $\mathcal{U}$, we have $\check{H}^n(\mathcal{U};\mathcal{F}) = H^n(X;\mathcal{F})$.

Spectral Sequences for Stratifications

9.12. For a sheaf $\mathcal{F}$ over X, and $K\subseteq X$, the notation restriction $\mathcal{F}|_K$ stands the pull back of $\mathcal{F}$ to K. Note that for example, when K is a point, this notation stands the stalk at this point. Historically, this notation has different meanings.

9.13. Shriek Push Forward Let $f: X\to Y$ be continous. Let $\mathcal{F}$ be a sheaf, denote the **shriek push forward** $f_!\mathcal{F}$ to be the subsheaf of $f_*\mathcal{F}$ with section of proper support, that is

$$f_!\mathcal{F}(U) = \{s\in\mathcal{F}(f^{-1}(U)) : f|_{\operatorname{supp}s}\text{ is proper}\}.$$

It is known that $f_!$ is left exact, thus we can define its right derived functor $R^if_!$. It is known that the shriek push forward maps injective sheaves to c-soft sheaves, thus satisfies the condition of Grothendieck spectral sequences.

9.14. For example, when $Y = \mathrm{pt}$, $f_! = \Gamma_c(X, -)$ is the same as the functor of taking global sections of compact support. Thus $R^i f_! = H^i_c(X; -)$ the functor taking i-th cohomology of compact support.

9.15. Higher Direct Image of proper support The higher shriek push forward admits a explicit description on stalk

$$(R^i f_! \mathcal{F})_y = H^i_c(f^{-1}(y); \mathcal{F}|_{f^{-1}(y)}).$$

9.16. Excision Triangle For any open subset $U \subseteq X$, denote its complement $F := X \setminus U$, and two inclusions $j : U \to X$ and $i : F \to X$. For any sheaf $\mathcal{F}$, we have a long exact sequence called **excision long exact sequence**

$$\cdots \longrightarrow H^i_c(U; \mathcal{F}|_U) \longrightarrow H^i_c(X; \mathcal{F}) \longrightarrow H^i_c(F; \mathcal{F}|_F) \longrightarrow H^{i+1}_c(U; \mathcal{F}|_U) \longrightarrow \cdots$$

9.17. For example, for $\mathcal{F} = \mathbb{Z}_X$, this gives the long exact sequence of cohomology of compact support

$$\cdots \longrightarrow H^i_c(U) \longrightarrow H^i_c(X) \longrightarrow H^i_c(F) \longrightarrow H^{i+1}_c(U) \longrightarrow \cdots .$$

9.18. Stratification Let X be a topological space. A **stratification** of X is a finite set of subspaces $\mathcal{S}$ (**strata**) such that

$$X = \bigcup_{S \in \mathcal{S}} S \quad \text{(disjoint)} \qquad \overline{S_1} \cap S_2 = S_2 \text{ or } \varnothing \quad \text{for } S_1, S_2 \in \mathcal{S}.$$

We set

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X \qquad X_k = \bigcup_{\dim S \le k} S$$

with all X_k closed.

9.19. Spectral Sequences for Stratifications Assume $\mathcal{F}$ is sheaf over X with a stratification $\mathcal{S}$. Then there exists a spectral sequence

$$E_1^{pq} = H^{p+q}_c(S_p; \mathcal{F}|_{S_p}) \Longrightarrow H^{p+q}_c(X; \mathcal{F}),$$

where S_p is the disjoint union of all $\dim p$ strata.

Proof Set $X^k = X \setminus X_{k-1} = \bigcup_{\dim S \geq k} S$. Note that the excision long exact gives

$$\cdots \longrightarrow H_c^i(X^{k+1}; \mathcal{F}|_{X^{k-1}}) \longrightarrow H_c^i(X^k; \mathcal{F}|_{X^k}) \longrightarrow H_c^i(S_k; \mathcal{F}|_{S_k}) \longrightarrow \cdots$$

Thus we have an exact couple.

$$E_1^{pq} = H_c^{p+q}(S_p; \mathcal{F}|_F) \Longrightarrow H_c^{p+q}(X; \mathcal{F}).$$

This proves the theorem. Q.E.D.

9.20. Simplicial Cohomology For example, we apply this theorem on constant sheaf $\mathbb{Z}_X$. It tells

$$E_1^{pq} = H_c^{p+q}(S_p) \Longrightarrow H_c^{p+q}(X).$$

We know that for an open disc D^p of dimension p,

$$H_c^{p+q}(D^p; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus when X has an affine stratification, i.e. each stratum is homoemorphic to $\mathbb{R}^p$ for some p, then the cohomology of compact support can be computed as simplicial cohomology 1.12. Note that to be a CW complex, we also need to assume further that the boundary of each stratum is attached to lower dimensional strata.

9.21. Complex Version of 9.19 The same excision long exact sequence holds for hypercohomology $\mathbb{H}_c^i = \mathbf{R}^i\Gamma_c$ of compact support. Actually, in derived category, we have a triangle (under the notation of 9.19)

$$\mathbf{R}j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathbf{R}i_* i^* \mathcal{F} \xrightarrow{+1} .$$

Thus there is a spectral sequence

$$E_1^{pq} = \mathbb{H}_c^{p+q}(S_p; \mathcal{F}|_{S_p}) \Longrightarrow \mathbb{H}_c^{p+q}(X; \mathcal{F}).$$

9.22. Dual Version We have another excision triangle in the derived category (still under the notation of 9.19)

$$\mathbf{R}i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathbf{R}j_* j^* \mathcal{F} \xrightarrow{+1}$$

where $f^!$ is the shriek pull back defined by Verdier. For example, this gives the long exact sequence of **Borel–Moore homology**

$$\cdots \longrightarrow H_i^{\mathrm{BM}}(F) \longrightarrow H_i^{\mathrm{BM}}(X) \longrightarrow H_i^{\mathrm{BM}}(U) \longrightarrow H_{i-1}^{\mathrm{BM}}(F) \longrightarrow \cdots$$

Note that $H_i^{\mathrm{BM}}(X) = \mathbb{H}^{-i}(\omega_X)$ with $\omega_X = a_X^!\mathbb{Q}$ for the unique map $a_X : X \to \mathrm{pt}$. Hence, apply to

$$\cdots \longrightarrow \mathbb{H}_c^i(X_{k-1}; \mathcal{F}|_{X_{k-1}}) \longrightarrow \mathbb{H}_c^i(X_k; \mathcal{F}|_{X_k}) \longrightarrow \mathbb{H}_c^i(S_k; \mathcal{F}|_{S_k}) \longrightarrow \cdots$$

there is a spectral sequence

$$E_1^{-p,-q} = \mathbb{H}^{p+q}(S_p; i_p^!\mathcal{F}) \Longrightarrow \mathbb{H}^{p+q}(X; \mathcal{F}),$$

where $i_p : S_p \to X$ the inclusion. For example, there is a Borel–Moore homology version

$$E_{pq}^1 = H_{p+q}^{\mathrm{BM}}(S_p) \Longrightarrow H_{p+q}^{\mathrm{BM}}(X).$$

Hodge Theory

9.23. Dolbeault cohomology For a smooth algebraic variety X of dimension n, we have the **holomorphic de Rham complex**

$$\begin{array}{cccccccc} & 0 & & 1 & & & & n \\ \Omega_X^\bullet : & \mathscr{O}_X(X) & \xrightarrow{\partial} & \Omega_X(X) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \omega_X(X) \end{array}$$

where Ω_X the Kähler differential, and ω_X the canonical bundle. Note that the morphism in the complex is only sheaf morphisms rather than coherent.

Define the **Dolbeault cohomology**

$$H^{pq}(X) = H^q(X; \Omega_X^p).$$

9.24. Frölicher Spectral Sequences We have a spectral sequence

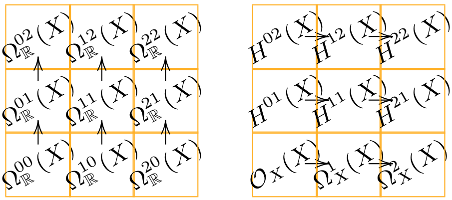
$$E_1^{pq} = H^q(X; \Omega_X^p) = H^{pq}(X) \Longrightarrow H^{p+q}(X; \mathbb{C}).$$

Proof For the differentiable de Rham complex $\Omega_{\mathbb{R}}^\bullet$, we have a decomposition $\mathbb{C} \otimes \Omega_{\mathbb{R}}^\bullet = \mathrm{Tot}\, \Omega_{\mathbb{R}}^{pq}$ where $\Omega_{\mathbb{R}}^{pq}$ is the direct summand of $\mathbb{C} \otimes \mathscr{C}^\infty$-sheaf $\mathbb{C} \otimes \Omega_{\mathbb{R}}^{p+q}$ locally spanned by

$$f(z) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}, \qquad f \in \mathbb{C} \otimes \mathscr{C}^\infty,$$

under a local coordinate $(z_1, \dots, z_n)$. The two direction differential $\bar{\partial}$ and ∂ is given by $\alpha \mapsto \sum_i d\bar{z}_i \wedge \frac{\partial}{\partial \bar{z}_i}\alpha$ and $\alpha \mapsto \sum_i dz_i \wedge \frac{\partial}{\partial z_i}\alpha$. By Dolbeault theorem, Dolbeault cohomology is also the cohomology of

$$\underset{\Omega^{p0}_{\mathbb{R}}(X)}{\overset{0}{}} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \underset{\Omega^{p,n-p}_{\mathbb{R}}(X)}{\overset{n-p}{}} .$$



It converges to the de Rham cohomolog with coefficient in $\mathbb{C}$ i.e. $H^{p+q}(X;\mathbb{C})$. Q.E.D.

9.25. Degeneration When X is projective, or in general is a compact Kähler manifold, the Frölicher spectral sequences degenerates at E_1, i.e. $E_1 = E_\infty$. Actually, this is equivalent to say that we have the following **Hodge decomposition**

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{pq}(X).$$

Theoretically, if we denote $F^p\Omega^\bullet = \Omega^{\geq p}$, then equivalently, the spectral sequence for this filtration degenerates at E_1.

9.26. Deligne Degeneration Let $\begin{bmatrix} X \\ \downarrow \\ Y \end{bmatrix}$ be a projective morphism of varieties which is a topological fibre bundle with smooth fibre F. Then the Leray–Serre spectral sequence 4.6 with coefficient $\mathbb{Q}$

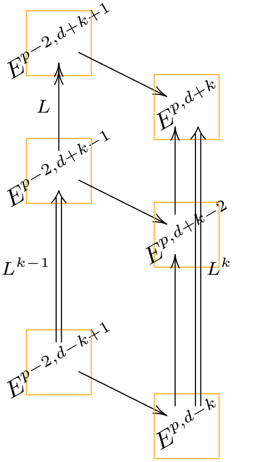
$$E_2^{pq} = H^p(X;\mathcal{H}^q(F;\mathbb{Q})) \Longrightarrow H^{p+q}(E;\mathbb{Q})$$

degenerates at E_2, that is, $E_2 = E_\infty$.

Proof By definition, it factors through $X \hookrightarrow \mathbb{P}^n \times Y \twoheadrightarrow Y$ with the first map a closed embedding and last map the natural projection. Denote $H \in H^2(E;\mathbb{Q})$ the restriction of the class of hyperplane section from $H^2(\mathbb{P}^2 \times Y)$. The restriction of H to E, each fibre F holds the **hard Lefschetz theorem**

$$L^k : H^{d-k}(F;\mathbb{Q}) \xrightarrow{\sim} H^{d+k}(F;\mathbb{Q}), \qquad \alpha \longmapsto \alpha \smile H^k,$$

where $d = \dim_{\mathbb{C}} F = \frac{1}{2}\dim_{\mathbb{R}} F$. At E_2, it looks like



$$
\begin{gathered}
[\cdots \xrightarrow{d} E^{p,d+k}] = 0 \\
\Downarrow \\
[\cdots \xrightarrow{d} E^{p,d-k}] = 0 \\
\Downarrow \\
[\cdots \xrightarrow{d} E^{p,d+k-2}] = 0
\end{gathered}
$$

Thus by induction, the differentials of E_2 all vanish. The same reason for E_3, etc. Q.E.D.

9.27. Actually, for smooth projective maps $X \to Y$, the sheaf-theoretic Leray spectral sequence 9.5 with coefficient $\mathbb{Q}_X$

$$E_2^{pq} = H^p(Y; R^q f_* \mathbb{Q}_X) \Longrightarrow H^{p+q}(X; \mathbb{Q})$$

degenerates at E_2, that is, $E_2 = E_\infty$.

Exercise

9.28. Complex Version of 9.8 The theorem 9.8 is also true when $\mathcal{F}$ is a complex of sheaves with respect to the hyper-cohomology $\mathbb{H}^i = \mathbf{R}^i\Gamma$

$$E_1^{pq} = \mathbb{H}^q(U^p; \mathcal{F}|_{U^p}) \Longrightarrow \mathbb{H}^{p+q}(X).$$

One may use the fact that $\check{C}(\mathcal{U}, \mathcal{I})$ is acyclic for injective $\mathcal{I}$.

9.29. Sheaf-theoretic Čech cohomology Under the notation of 9.8

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \Big[V \longmapsto \check{C}^p(\mathcal{U}|_V, \mathcal{F}|_V) \Big], \qquad \mathcal{U}|_V = \{U \cap V\}_{U \in \mathcal{U}}$$

forms a sheaf. Show that it is exact. Actually, we was equivalently doing this in the proof of 9.8.

9.30. The First Čech Cohomology Show that

$$\varinjlim_{\mathcal{U}\text{ finer}} \check{H}^1(\mathcal{U};\mathcal{F}) = H^1(X;\mathcal{F}).$$

Actually, $\check{C}(\mathcal{U},-)$ does not preserves exactness. But at least, the $\varinjlim H^0$ of a quotient of a flabby sheaf is the same by direct computation.

9.31. We can define $H^1(X;\mathcal{G})$ as above for any sheaf of group $\mathcal{G}$. Say,

$$H^1(X;\mathcal{G}) = \varinjlim_{\mathcal{U}\text{ finer}} \frac{\left\{ f_{ij} \in \prod_{i<j} \mathcal{F}(U_i \cap U_j) : \begin{array}{c} f_{ij}f_{jk} = f_{ik} \\ \text{over each } U_i \cap U_j \cap U_k \end{array} \right\}}{f_{ij} = f'_{ij} \iff \exists (\varphi_i)_i \in \prod_i \mathcal{F}(U_i) : \begin{array}{c} f_{ij}\varphi_j = \varphi_i f'_{ij} \\ \text{over each } U_i \cap U_j \end{array}}$$

It is the set of the equivalence classes of $\mathcal{G}$-principle bundle ($\mathcal{G}$-torsor), that is, the sheaf of right $\mathcal{G}$-set locally isomorphic to $\mathcal{G}_{\text{right}}$. A typical example is $H^1(X;\mathcal{O}_X^*) = \text{Pic}(X) = \{\text{equivalence classes of lines bundles}\}$.

9.32. Prove that for any subset $i : F \subseteq X$, the shriek push forward $i_!$ is an exact functor (extending by zero) with i^* a one-direction inverse.

9.33. For a bounded complex of sheaf $\mathcal{F}$ over $X = \mathbb{R}^n$. Assume the i-th cohomology of $\mathcal{F}$ is a constant sheaf, say H^i. Show that $\mathbb{H}^i(\mathcal{F}) = H^i$ and $\mathbb{H}^i_c(\mathcal{F}) = H^{i-n}$.

9.34. Assume X has an affine stratification $\{S \cong \mathbb{R}^{(\cdots)}\}$. Let $\mathcal{F}$ be a bounded complex over X such that the cohomology of $\mathcal{F}$ is a constant sheaf. Show that there is a spectral sequence

$$E_1 = \bigoplus_{\dim S = p} H_c^{p+q}(\mathcal{F}|_S) = \bigoplus_{\dim S = p} H^q(\mathcal{F}|_S) \Longrightarrow \mathbb{H}_c^{p+q}(X;\mathcal{F}).$$

Thank You For Your Reading