物理海洋学笔记

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https://github.com/Cuiyingzhe/OUC-Physical-Oceanography-Notes

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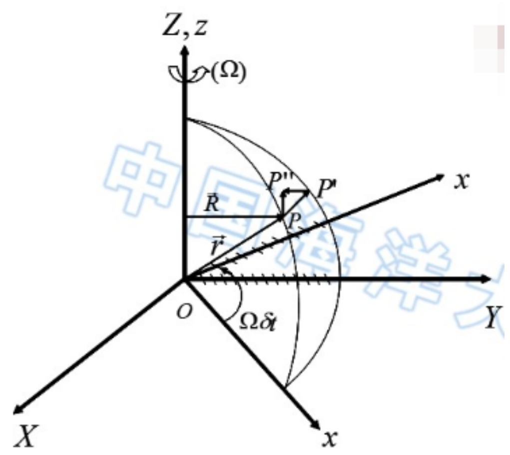
1 基本方程

1.1 旋转坐标系的速度和加速度

惯性坐标系: 静止的或是匀速直线运动的坐标系，固定在恒星上的坐标系可以被看成惯性坐标系.
固定在地球上的坐标系: 地球对恒星的加速度主要是由地球自转引起的，于是可以把地球当作一个对惯性坐标系作纯粹地转运动的物体.

1.1.1 旋转坐标系和惯性坐标系中的速度



惯性坐标系 (XYZ) 绝对位移：$\vec{pp''} = \vec{V}_a\delta t, \vec{V}_a$ 为绝对速度
旋转坐标系 (xyz) 相对位移：$\vec{p'p''} = \vec{V}\delta t, \vec{V}$ 为相对速度
$\because \vec{pp''} = \vec{p'p''} + \vec{pp'}$
$\therefore \vec{V}_a\delta t = \vec{V}\delta t + \vec{V}_e\delta t \Rightarrow \vec{V}_a = \vec{V} + \vec{V}_e$(绝对速度等于相对速度与牵连速度的向量和)
其中，$\vec{V}_e = \vec{\Omega}\times\vec{r} \Rightarrow \vec{V}_a = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\Rightarrow \frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}$$

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

1.1.2 旋转坐标系和惯性坐标系中的加速度

令 $\vec{A} = \vec{V}_a = \vec{V}_e + \vec{V} = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\begin{aligned}
\frac{d\bar{V}_a}{dt} &= \frac{d_a}{dt}\left(\vec{V} + \vec{V}_e\right) = \frac{d_a}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) + \vec{\Omega}\times(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R}
\end{aligned}$$

1.2 作用在海水微团上的外力运动方程的向量形式

压强梯度力：$\frac{1}{\rho}\nabla p$
分子粘性力 (摩擦力)：

$$\begin{cases}
F_x = \frac{1}{\rho}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta u \\
F_y = \frac{1}{\rho}\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta v \\
F_2 = \frac{1}{\rho}\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta w
\end{cases} \quad \Rightarrow \vec{F} = \frac{\mu}{\rho}\Delta\vec{V} = \gamma\Delta\vec{v}$$

重力 (地球引力与地球自转产生的惯性离心力的合力)：$\vec{g} = -G\frac{M_g}{r^2}\cdot\left(\frac{\vec{r}}{r}\right)$
科氏力：$-2\vec{\Omega}\times\vec{V}$

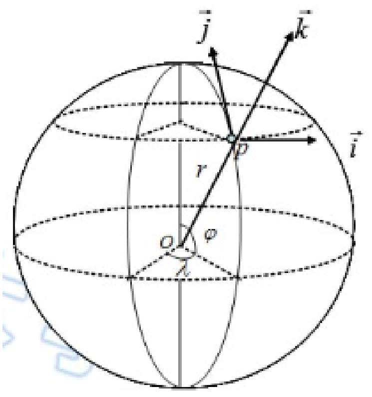
天体引潮力 (受其他天体万有引力与惯性力离心力的合力)：$\vec{F_M} = -G\frac{M_M}{L^2} + G\frac{M_M}{D^2}\cdot\left(\frac{\vec{D}}{D}\right)$

由牛顿第二定律和坐标系转换关系：

$$\begin{cases} \dfrac{d_a\vec{V}_a}{dt} = \sum_i \vec{F}_t \\ \dfrac{d_a\vec{A}_a}{dt} = \dfrac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} \end{cases}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla P - 2\vec{\Omega}\times\vec{V} + \vec{g} + \nu\Delta\vec{V} + \vec{F_T}}$$

1.3 运动方程在球坐标系的标量形式



速度：

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\Rightarrow \begin{cases} u = r\cos\varphi\frac{d\lambda}{dt} \\ v = r\frac{d\varphi}{dt} \\ w = \frac{dr}{dt} \end{cases}$$

加速度：

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{\frac{\partial\vec{A}}{\partial t}dt + \frac{\partial\vec{A}}{\partial\lambda}d\lambda + \frac{\partial\vec{A}}{\partial\varphi}d\varphi + \frac{\partial\vec{A}}{\partial r}dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial\lambda}\frac{d\lambda}{dt} + \frac{\partial\vec{A}}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial\vec{A}}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\vec{A}}{\partial\lambda} + \frac{v}{r}\frac{\partial\vec{A}}{\partial\varphi} + w\frac{\partial\vec{A}}{\partial r} \end{aligned}$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{r\cos\varphi\partial\lambda} + v\frac{\partial}{r\partial\varphi} + w\frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)}}$$

$$\Rightarrow \boxed{\boldsymbol{\nabla = \frac{\partial}{r\cos\varphi\partial\lambda}\vec{i} + \frac{\partial}{r\partial\varphi}\vec{j} + \frac{\partial}{\partial r}\vec{k}}}$$

$$\Rightarrow \begin{cases} \dfrac{d\vec{i}}{dt} = \dfrac{\partial\vec{i}}{\partial t} + u\dfrac{\partial\vec{i}}{r\cos\varphi\partial\lambda} + v\dfrac{\partial\vec{i}}{r\partial\varphi} + w\dfrac{\partial\vec{i}}{\partial r} \\ \dfrac{\partial\vec{j}}{dt} = \dfrac{\partial\vec{j}}{\partial t} + u\dfrac{\partial\vec{j}}{r\cos\varphi\partial\lambda} + v\dfrac{\partial\vec{j}}{r\partial\varphi} + w\dfrac{\partial\vec{j}}{\partial r} \\ \dfrac{d\vec{k}}{dt} = \dfrac{\partial\vec{k}}{\partial t} + u\dfrac{\partial\vec{k}}{r\cos\varphi\partial\lambda} + v\dfrac{\partial\vec{k}}{r\partial\varphi} + w\dfrac{\partial\vec{k}}{\partial r} \end{cases}$$

$$\frac{d\vec{V}}{dt}=\frac{du}{dt}\vec{i}+\frac{dv}{dt}\vec{j}+\frac{dv}{dt}\vec{k}+u\frac{d\vec{i}}{dt}+v\frac{d\vec{j}}{dt}+w\frac{d\vec{k}}{dt}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt}=\left(\frac{du}{dt}-\frac{uvtg\varphi}{r}+\frac{uw}{r}\right)\vec{i}+\left(\frac{dv}{dt}+\frac{u^2\operatorname{tg}\varphi}{r}+\frac{vw}{r}\right)\vec{j}+\left(\frac{dw}{dt}-\frac{u^2+v^2}{r}\right)}$$

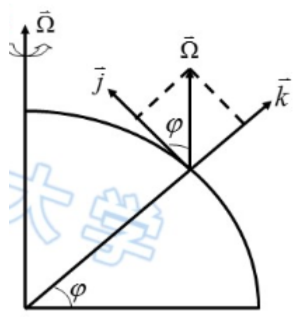
压强梯度力：

$$\frac{1}{\rho}\nabla p=-\frac{1}{\rho}\left(\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}\vec{i}+\frac{1}{r}\frac{\partial p}{\partial\varphi}\vec{j}+\frac{\partial p}{\partial r}\vec{k}\right)$$

重力：

$$\vec{g}=-g\vec{k}$$

科氏力：



$$\vec{\Omega}=\Omega\sin\varphi\vec{k}+\Omega\cos\varphi\vec{j}$$

$$\begin{aligned}
-2\vec{\Omega}\times\vec{V}&=-2\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & \Omega\cos\varphi & \Omega\sin\varphi\\ u & v & w\end{vmatrix}\\
&=-2[(w\Omega\cos\varphi-v\Omega\sin\varphi)\vec{i}+(u\Omega\sin\varphi)\vec{j}+(-u\Omega\cos\varphi)\vec{k}]
\end{aligned}$$

$$\Rightarrow -2\vec{\Omega}\times\vec{V}=(fv-\tilde{f}w)\vec{i}-(fu)\vec{j}+(\tilde{f}u)\vec{k}$$

其中，$\begin{cases} f=2\Omega\sin\varphi \\ \tilde{f}=2\Omega\cos\varphi \end{cases}$

$$\Rightarrow \boxed{\begin{cases}
\dfrac{du}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{r\cos\varphi\partial\lambda}+fv-\tilde{f}w+\dfrac{uv\tan\varphi}{r}-\dfrac{uw}{r}+\gamma(\Delta\vec{v})_\lambda-\dfrac{1}{r\cos\varphi}\dfrac{\partial\phi_T}{\partial\lambda}\\
\dfrac{dy}{dt}=7\dfrac{1}{\rho}\dfrac{\partial p}{r\partial\varphi}-fu-\dfrac{u^2\tan\varphi}{r}-\dfrac{vw}{r}+\gamma(\Delta\bar{v})_\varphi-\dfrac{1}{r}\dfrac{\partial\phi_T}{\partial\varphi}\\
\dfrac{dw}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}+\tilde{f}u=g+\dfrac{u^2+v^2}{r}+\gamma(\Delta\vec{v})_r-\dfrac{\partial\phi_T}{\partial r}
\end{cases}}$$

1.4 直角坐标系的运动方程

略去地球曲率的影响

$$\Rightarrow \boxed{\begin{cases}
\dfrac{du}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+fv+\tilde{f}w+F_{N\lambda}+F_{T\lambda}\\
\dfrac{dv}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}-fu+F_{Ny}+F_{Ty}\\
\dfrac{dw}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\tilde{f}u-g+F_{Nz}+F_{Tz}
\end{cases}}$$

1.5 海水层流运动的基本方程组

1.5.1 连续方程

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0\Leftrightarrow\frac{d\rho}{dt}+\rho\nabla\cdot\vec{V}=0$$

特别地，对于不可压缩流体：

$$\nabla\cdot\vec{V}=0$$

1.5.2 盐量扩散方程

$$
\begin{aligned}
&\overset{\text{盐量增加量}}{\frac{\partial}{\partial t}\iiint\limits_\tau \rho s d\tau} = \overset{\text{平流作用}}{-\oiint\limits_\sigma \rho s V_n d\sigma} + \overset{\text{分子扩散作用}}{-\oiint\limits_\sigma S_n d\sigma} \\
&\iiint\limits_\tau \frac{\partial(\rho s)}{\partial t} d\tau = \iiint\limits_\tau \nabla\cdot(\rho s\vec{V})d\tau - \iiint\limits_\tau \nabla\cdot\vec{S}d\tau \\
\Rightarrow &\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\vec{V}) + \nabla\cdot\vec{S} = 0 \\
\Rightarrow &\rho\frac{\partial s}{\partial t} + s\frac{\partial \rho}{\partial t} + s\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla s + \nabla\cdot\vec{S} = 0 \\
\Rightarrow &\left(\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s\right) + \frac{s}{\rho}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] = -\frac{1}{\rho}\nabla\cdot\vec{S} \\
\Rightarrow &\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = \frac{k}{\rho}\Delta s = k_D\Delta s
\end{aligned}
$$

其中，$k_D = \frac{k}{\rho} \sim 1.1\times 10^{-9}\,(\mathrm{m^2/s})$

1.5.3 热传导方程

与上面类似：

$$\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \frac{\kappa}{\rho c_p}\Delta\theta = k_\theta\Delta\theta$$

其中，$k_\theta = \frac{\kappa}{\rho c_p} \sim 1.4\times 10^{-7}\,(\mathrm{m^2/s})$

1.5.4 热膨胀方程-状态方程

热膨胀方程：

$$\rho = \overset{\text{0°C 时的海水密度}}{\rho_0}(1 - \overset{\text{海水的热膨胀系数}}{k}\theta)$$

EOS80 国际海水状态方程：

$$\rho(s,t,p) = \rho(s,t,0)\left[1 - \frac{np}{k(s,t,p)}\right]^{-1}$$

1.6 基本方程的矢量形式和标量形式

矢量形式：

$$
\boxed{
\begin{cases}
\dfrac{d\vec{V}}{dt} = -\dfrac{1}{\rho}\nabla p - 2\Omega\times\vec{V} + \vec{g} + \gamma\Delta\vec{V} - \nabla\phi_T \\
\dfrac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{V} = 0 \\
\dfrac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = k_D\Delta s \\
\dfrac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = k_\theta\Delta\theta \\
\rho = \rho(\theta, s, p)
\end{cases}
}
$$

标量形式 (直角坐标系)：

$$
\boxed{
\begin{cases}
\dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu + \gamma\Delta v - \dfrac{\partial\phi_T}{\partial y} \\
\dfrac{dw}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \tilde{f}u - g + \gamma\Delta w - \dfrac{\partial\phi_T}{\partial z} \\
\dfrac{d\rho}{dt} + \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0 \\
\dfrac{\partial s}{\partial t} + u\dfrac{\partial s}{\partial x} + v\dfrac{\partial s}{\partial y} + w\dfrac{\partial s}{\partial z} = k_D\left(\dfrac{\partial^2 s}{\partial x^2} + \dfrac{\partial^2 s}{\partial y^2} + \dfrac{\partial^2 s}{\partial z^2}\right) \\
\dfrac{\partial\theta}{\partial t} + u\dfrac{\partial\theta}{\partial x} + v\dfrac{\partial\theta}{\partial y} + w\dfrac{\partial\theta}{\partial z} = k_\theta\left(\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} + \dfrac{\partial^2\theta}{\partial z^2}\right) \\
\rho = \rho(\theta, s, p)
\end{cases}
}
$$

1.7 边界条件

无质量交换的运动学边界条件：

$$\frac{\partial F}{\partial t}+\vec{c}\cdot\nabla F=0$$

例：

(1) 海面 ($z=\zeta(x,y,t)$)：$\dfrac{\partial\zeta}{\partial t}+\vec{V}_H\cdot\nabla_H\zeta-w=0$

(2) 海底 ($z=-h(x,y)$): $\vec{V}_H\cdot\nabla_H h+w=0$

动力学边界条件：

由牛顿第三定律，在界面法线方向有：

$$(\vec{p}_n)_1=(\vec{p}_n)_2$$

1.8 ※ 时间平均的基本方程和边界条件 (直角坐标系)

连续方程:

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0$$

运动方程：

$$\begin{cases}\dfrac{\partial\bar{u}}{\partial t}+\bar{u}\dfrac{\partial\bar{u}}{\partial x}+\bar{v}\dfrac{\partial\bar{u}}{\partial y}+\bar{w}\dfrac{\partial\bar{u}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial x}+f\bar{v}-\tilde{f}w+\gamma\Delta\bar{u}-\dfrac{\partial\bar{\phi}_T}{\partial x}+\dfrac{\partial}{\partial x}\left(A_{x\alpha}\dfrac{\partial\bar{u}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial\bar{u}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial\bar{u}}{\partial z}\right)\\ \dfrac{\partial\bar{v}}{\partial t}+\vec{u}\dfrac{\partial\bar{v}}{\partial x}+\vec{v}\dfrac{\partial\bar{v}}{\partial y}+\vec{w}\dfrac{\partial\bar{v}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial y}-f\bar{u}+\gamma\Delta\bar{v}-\dfrac{\partial\bar{\phi}_T}{\partial y}+\dfrac{\partial}{\partial x}\left(A_{yx}\dfrac{\partial\bar{v}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{yy}\dfrac{\partial\bar{v}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{yz}\dfrac{\partial\bar{v}}{\partial z}\right)\\ \dfrac{\partial\bar{w}}{\partial t}+\bar{u}\dfrac{\partial\bar{w}}{\partial x}+\bar{v}\dfrac{\partial\bar{w}}{\partial y}+\bar{w}\dfrac{\partial\bar{w}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial z}+\tilde{f}\bar{u}-g+\gamma\Delta\bar{w}-\dfrac{\partial\bar{\phi}_T}{\partial z}+\dfrac{\partial}{\partial x}\left(A_{2x}\dfrac{\partial\bar{w}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{zy}\dfrac{\partial\bar{w}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{zz}\dfrac{\partial\bar{w}}{\partial z}\right)\end{cases}$$

盐量扩散方程：

$$\frac{\partial\bar{s}}{\partial t}+\bar{u}\frac{\partial\bar{s}}{\partial x}+\bar{v}\frac{\partial\bar{s}}{\partial y}+\bar{w}\frac{\partial\bar{s}}{\partial z}$$

热传导方程：

$$\frac{\partial\bar{\theta}}{\partial t}+\vec{u}\frac{\partial\bar{\theta}}{\partial x}+\vec{v}\frac{\partial\bar{\theta}}{\partial y}+\vec{w}\frac{\partial\bar{\theta}}{\partial z}=k_\theta\Delta\bar{\theta}+\frac{\partial}{\partial x}\left(K_{\theta_x}\frac{\partial\bar{\theta}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{\theta y}\frac{\partial\bar{\theta}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{\theta z}\frac{\partial\bar{\theta}}{\partial z}\right)$$

状态方程：

$$\bar{\rho}=\bar{\rho}(\bar{s},\bar{\theta},\bar{p})$$

1.9 铅直向平均的基本方程

$$\frac{\partial}{\partial x}[(h+\zeta)\langle u\rangle]+\frac{\partial}{\partial y}[(h+\zeta)\langle v\rangle]-\left[u|_\zeta\frac{\partial\zeta}{\partial x}+v|_\zeta\frac{\partial\zeta}{\partial y}-w|_\zeta\right]-\left[u|_{-h}\frac{\partial h}{\partial x}+v|_{-h}\frac{\partial h}{\partial y}+w|_{-h}\right]=\mathbf{0}$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x}+\frac{\partial[(h+\zeta)\langle v\rangle]}{\partial y}=0$$

1.10 尺度分析

Rossby 数 Ro=$\dfrac{U}{FL}\begin{cases}\gg 1:\text{平流非线性项比 Coriolis 力重要, 大尺度运动}\\ =1:\text{平流非线性项与 Coriolis 力同等重要}\\ \ll 1:\text{平流非线性项可以忽略, 小尺度运动}\end{cases}$

水平 Ekman 数 $\mathrm{E_l}=\dfrac{A_l}{FL^2}$ 水平湍流摩擦项与 Coriolis 力比值

垂直 Ekman 数 $\mathrm{E_z}=\dfrac{A_z}{FD^2}$ 垂直湍流摩擦项与 Coriolis 力比值

准静力近似 f 平面近似 β 平面近似 Boussinesq 近似

2 海流

2.1 地转流

地转流：不考虑海面风的作用，远离沿岸的大洋中部大尺度、准水平、定常的海水流动.
产生原因：海水受热力和动力因素导致压力 (和密度) 在水平方向分布不均匀.

$$p = p_a + \rho g h \quad \rho \begin{cases} \neq \rho_0 \Rightarrow \text{梯度流} \\ = \rho_0 \Rightarrow \text{倾斜流} \end{cases}$$

2.1.1 梯度流

假定和方程

(1) 在相当长一段时间里海面温度变化和降水蒸发变化都不大，于是可以认为已形成的海水密度场、温度场和盐度场近似于定常，从而相应的海水运动也近似于定常：$\dfrac{\partial}{\partial t} = 0$.
(2) 海洋深而宽广，在远离海岸及海底的大洋中部海区，大尺度运动：$\mathrm{Ro} \ll 1$.
(3) 不考虑海底摩擦及边界摩擦的影响，且海面无风力作用，则流动属一种无摩擦流动：$\mathrm{E_l}, \mathrm{E_z} \ll 1$.
(4) β 平面近似准静力近似
x 方向基本方程：

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{\partial}{\partial x}\left(A_{xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{xy}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_{xz}\frac{\partial u}{\partial z}\right)$$

假定 (1)$\Rightarrow \dfrac{\partial u}{\partial t} = 0$

假定 (2)$\Rightarrow u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = 0$

假定 (3)$\Rightarrow \dfrac{\partial}{\partial x}\left(A_{xx}\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial u}{\partial z}\right) = 0$

可得梯度流的控制方程：

$$\begin{cases} 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + fv \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} + w\dfrac{\partial \theta}{\partial z} = 0 \\ u\dfrac{\partial s}{\partial x} + v\dfrac{\partial s}{\partial y} + w\dfrac{\partial s}{\partial z} = 0 \\ \rho = \rho(s, \theta) \end{cases}$$

特征

水平速度：

$$\begin{cases} 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + fv & (1) \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu & (2) \end{cases}$$

$$\Longrightarrow \begin{cases} v = \dfrac{1}{\rho f}\dfrac{\partial p}{\partial x} \\ u = -\dfrac{1}{\rho f}\dfrac{\partial p}{\partial y} \end{cases}$$

(1) 水平速度和压强梯度成正比；
(2) 与密度和科氏参数成反比；
(3) 地转关系在赤道不成立 ($f=0$).
垂向速度：

$$
\begin{aligned}
\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x} &\Leftrightarrow \frac{\partial(\rho f v)}{\partial v}+\frac{\partial(\rho f u)}{\partial x}=0 \\
&\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+f\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial \rho}{\partial z}+\beta\rho v=0 \\
&\Leftrightarrow f\rho\overset{=0}{\boxed{\nabla\vec{V}}}+f\vec{V}\cdot\nabla\rho-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial \rho}{\partial z}+\beta\rho v=0
\end{aligned}
\tag{3}
$$

$$
\vec{V}\cdot\nabla\rho=\vec{V}\cdot\nabla\rho(s,\theta)=\vec{V}\cdot\left(\nabla s\frac{\partial\rho}{\partial s}+\nabla\theta\frac{\partial\rho}{\partial\theta}\right)=0
$$

$$
(3)\Leftrightarrow f\left(\rho\frac{\partial w}{\partial z}+w\frac{\partial\rho}{\partial z}\right)=\beta\rho v\overset{\text{尺度分析}}{\Rightarrow}f\frac{\partial w}{\partial z}=\beta v\overset{\text{尺度分析}}{\Rightarrow}W=\frac{\beta D}{F}U\sim 2\times10^{-4}U
$$

垂向流速比水平流速小得多，地转流为准水平运动.

运动特性

$$
(1)\times u+(2)\times v\Leftrightarrow u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0\Leftrightarrow \vec{V}_H\cdot\nabla_H p=0
$$

(1) 梯度流平行于等压线；
(2) 北半球，流向右侧为高压，南半球相反；

密度特性

$$
\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x}\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}\right)+\rho u\frac{\partial f}{\partial x}+\rho v\frac{\partial f}{\partial y}=0\Leftrightarrow\vec{V}_H\cdot\nabla_H\rho=0
$$

(1) 梯度流近似平行于等密线；
(2) 在北半球，流向右侧密度小；
(3) 等压面倾斜与等密面倾斜方向相反.

温盐特性
忽略垂向运动：

$$
\begin{aligned}
\vec{V}_H\cdot\nabla_H\theta=0 \\
\vec{V}_H\cdot\nabla_H s=0
\end{aligned}
$$

(1) 梯度流平行于等温线和等盐线；
(2) 在北半球，流向右侧温度高，盐度低.

2.1.2 倾斜流

假定和方程 (1) 海水密度为常数；
(2) 水平方向的压强梯度是由海面倾斜引起的.

$$
\Rightarrow p=p_a+\int_z^\zeta\rho g dz=p_a+\rho g(\zeta-z)
$$

倾斜流的控制方程：

$$
\begin{cases}
fv=g\dfrac{\partial\zeta}{\partial x} & (4)\\
fu=-g\dfrac{\partial\zeta}{\partial y} & (5)
\end{cases}
$$

性质：

$$(4)\times u-(5)\times v \Leftrightarrow u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}=0 \Leftrightarrow \vec{V}_H\cdot\nabla\zeta=0$$

(1) 倾斜流平行于等水位线；
(2) 在北半球，流向右侧水位高；
(3) 倾斜流从表至底流速流向相同，压强梯度相同.

2.2 Ekman 漂流

由恒速定常的风长时间驱动大尺度、均匀密度的海洋，所产生的处于稳定状态的海流.

2.2.1 无限深海漂流

物理背景

Ekman 的老师 Nansen 在海洋调查时发现，冰山不是顺风漂移，而是沿着风向右方 $20^\circ\sim 40^\circ$ 的方向移动.Ekman 在 1905 年研究了这种现象并提出风海流理论 [?].

假定

无限深海 Ekman 漂流中用到了以下假定：
1) 海洋无限广阔，海洋无限深.
即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋无限深，$z\to\infty,\vec{V}=0$
2) 定常均匀风场长时间作用.
即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.
3) 密度分布均匀，ρ 为常数，不考虑热盐性质.
4) 采用 f 平面近似.

方程推导

控制方程和边界条件

首先给出一般的控制方程：

$$\begin{cases}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+fv+A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+A_z\dfrac{\partial^2 u}{\partial z^2}\\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}-fu+A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+A_z\dfrac{\partial^2 v}{\partial z^2}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}\tag{6}$$

由假定 1)，$A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=0,\ A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)=0$；
由假定 2)，$\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=0,\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=0$；
由假定 3)，$-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=0,-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$

则 (6) 化为：

$$\begin{cases}0=fv+A_z\dfrac{\partial^2 u}{\partial z^2} & (7a)\\ 0=-fu+A_z\dfrac{\partial^2 v}{\partial z^2} & (7b)\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 & (7c)\end{cases}$$

不失一般性地，假定风力仅沿 y 方向作用，即 $\tau_x = 0, \tau_y = const.$ 再结合假定 1)，控制方程的边界条件为：

$$\begin{cases} z = 0, \rho A_z \dfrac{\partial u}{\partial z} = 0 & (8a) \\ z = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y & (8b) \\ z = \infty, u = v = 0 & (8c) \end{cases}$$

方程求解

$$(7a) + (7b) \times i \Leftrightarrow A_z \frac{\partial^2 (u + iv)}{\partial z^2} = if(u + iv)$$

令 $W = u + iv$，得：

$$A_z \frac{\partial^2 W}{\partial z^2} = ifW \Rightarrow \frac{\partial^2 W}{\partial z^2} = \frac{(1+i)^2 \Omega \sin\varphi}{A_z} W$$

令 $a = \sqrt{\Omega \sin\varphi / A_z}$，$j^2 = (1+i)^2 a^2$，得：

$$\frac{d^2 W}{dz^2} - j^2 W = 0 \tag{9}$$

(9) 式通解为：$W = Ae^{jz} + Be^{-jz}$

结合边界条件：$(8a) + (8b) \times i \Rightarrow z = 0, \rho A_z \dfrac{\partial W}{\partial z} = -\tau_y, z \to \infty, W = 0$

$$z \to \infty \Rightarrow A = 0, W = Be^{-jz}; z = 0, \rho A_z \left.\frac{\partial W}{\partial z}\right|_{z=0} = \rho A_z \left.\frac{\partial \left(Be^{-jz}\right)}{\partial z}\right|_{z=0} = -\tau_y, \Rightarrow B = \tau_y / \left(j\rho A_z\right)$$

因此，方程的解为：

$$W = \frac{\tau_y}{j\rho A_z} e^{-jz} = \frac{i\tau_y}{(1+i)a\rho A_z} e^{-(1+i)az} = \frac{e^{i\frac{\pi}{2}} \tau_y}{\sqrt{2} e^{i\frac{\pi}{4}} a\rho A_z} e^{-(1+i)az}$$

令 $D_0 = \pi/a$, 得到最终解的形式为：

$$W = \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z + i\left(\frac{\pi}{4} - \frac{\pi}{D_0} z\right)} \tag{10}$$

物理性质

运动速度

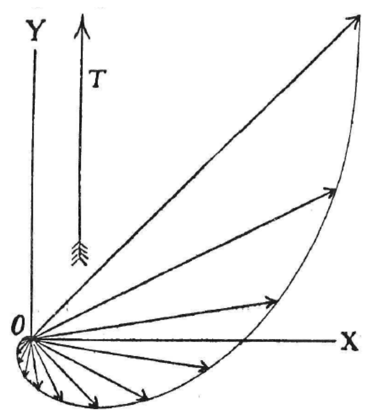
在海面 $(z = 0)$ 处，$W_0 = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}}$. 大小为 $|W_0| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z}$，方向与 x 轴成 45°，即与风向向右偏 45°.

在任意深度处，$|W_z| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z}$，方向为 $\frac{\pi}{4} - \frac{\pi}{D_0} z$，即流速随深度增加呈指数形式减小，流向随深度的增加而逐渐向右偏.

在摩擦深度 $z = D_0$ 处，$|W_{D_0}| = \dfrac{\tau_y e^{-\pi}}{\sqrt{2} a\rho A_z} = e^{-\pi} |W_0| = 0.043 |W_0|$，方向 $-\frac{3}{4}\pi$，即与 x 轴成 $-135°$，与表面流向正好相反.

Ekman 螺旋和 Ekman 螺线

根据速度的垂向分布，表层流速最大，流向偏向风向的右方 45°；随深度增加，流速逐渐减小，流向逐渐右偏；到摩擦深度，流速是表面流速的 4.3%，流向与表面流向相反，运动可以忽略. 连连接各层流速的矢量端点，构成 Ekman 螺旋；Ekman 螺旋在平面上的投影，称为 Ekman 螺线 [**?**].



水平体积输运

体积输运：

$$
\begin{aligned}
S &= \int_0^\infty W dz \\
&= \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}} \int_0^\infty e^{-\frac{\pi}{D_0}(1+i)z} dz \\
&= \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}} \left[-\frac{D_0}{\pi} \frac{1}{(1+i)} \right] e^{-\frac{\pi}{D_0}(1+i)z} \Big|_0^\infty \\
&= \frac{\tau_y}{2\Omega \sin\varphi\rho} = \frac{\tau_y}{f\rho}
\end{aligned}
$$

可以发现，得到的输运结果只有实部，没有虚部，说明体积输运方向为 x 轴正向，即在北半球水体向风向右侧 90° 输运.

2.2.2 有限深海漂流

假定

有限深海 Ekman 漂流中用到了以下假定：

1) 海区无限广阔、有限深，远离海岸.

即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋有限深，$z \to h, \vec{V} = 0$

2) 定常均匀风场长时间作用.

即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.

3) 密度分布均匀，ρ 为常数，不考虑热盐性质.

4) 采用 f 平面近似.

控制方程和边界条件：

$$
\begin{cases}
0 = fv + A_z \dfrac{\partial^2 u}{\partial z^2} \\
0 = -fu + A_z \dfrac{\partial^2 v}{\partial z^2} \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\
z = 0 : \rho A_z \dfrac{\partial u}{\partial z} = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y \\
z \to h : u = v = 0
\end{cases}
$$

方程求解　令 $\xi = h - z$，定解问题化为：

$$
\begin{cases}
-fv = A_z \dfrac{\partial^2 u}{\partial \xi^2} & (11) \\
fu = A_z \dfrac{\partial^2 v}{\partial \xi^2} & (12) \\
\xi = h : \rho A_z \dfrac{\partial u}{\partial \xi} = 0, \rho A_z \dfrac{\partial v}{\partial \xi} = \tau_y \\
\xi \to 0 : u = v = 0
\end{cases}
$$

令 $W = u + iv, \tau = \tau_x + i\tau_y$, 控制方程：

$$
(11) + (12) \times i \Leftrightarrow \frac{d^2 W}{d\xi^2} - j^2 W = 0
$$

边界条件：

$$
\xi = h : \rho A_z \frac{\partial W}{\partial \xi} = \tau
$$

$$
\xi = 0 : W = 0
$$

与无限深海漂流解法类似，解得：

$$
W = \frac{(1+i)\tau_y}{2a\rho A_z} \frac{sh(1+i)a\xi}{ch(1+i)ah}
$$

物理性质

与水深的关系

(1) $h\ 2D_0$ 时，有限深海漂流流速流向与无限深海相同；(2) 水深越浅，流向随深度增加右偏 (北半球) 越缓慢；(3) 从上层到下层的流速矢量越是趋近风矢量的方向.

体积输运

(1) 在 x, y 方向 (平行和垂直风向) 都有输送；

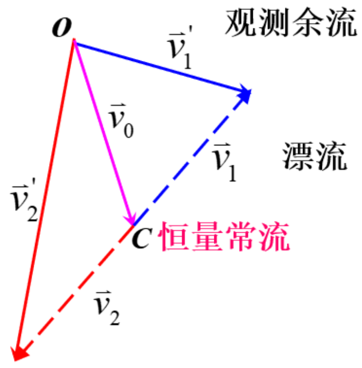
(2) 运输方向为风向右端，±90° 之间：

$$S_x > 0; 0 < h < D_0, ah < \pi, S_y > 0; D_0 < h < 2D_0, \pi < ah < 2\pi, S_y < 0; h > 2D_0, S_y = 0$$

2.2.3 漂流分离

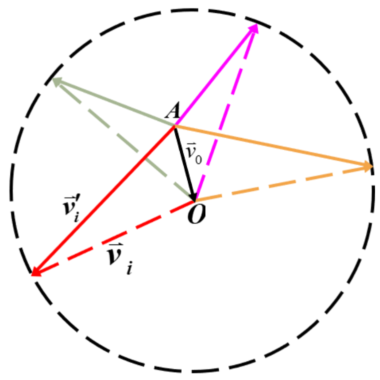
利用风速大小相等、方向相反的两次观测余流分离漂流

※ 余流＝漂流＋恒量常流



利用一组风速大小相等、方向不同的实测余流分离漂流

※ 漂流速度矢量端点落在同一圆周上

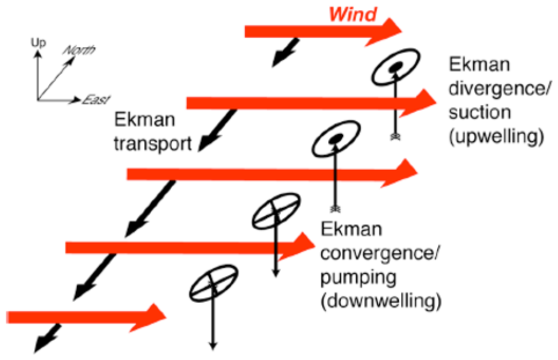


2.2.4 升降流

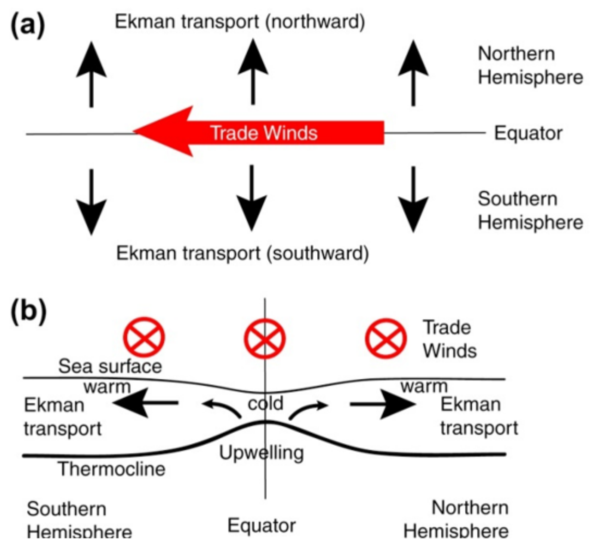
由不均匀风场或风场和地形配合产生的“较强烈”的铅直向流动。

物理背景

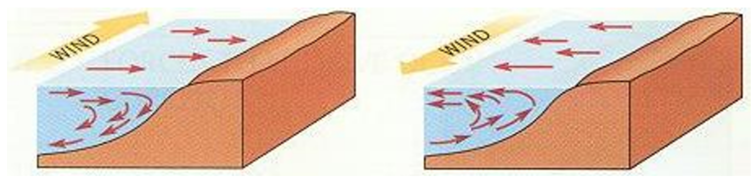
$\boxed{\text{非均匀风场}} \Rightarrow \boxed{\text{非均匀 Ekman 漂流}} \Rightarrow \boxed{\text{非均匀体积输运}} \Rightarrow \boxed{\text{辐聚辐散}} \Rightarrow \boxed{\text{升降流}}$



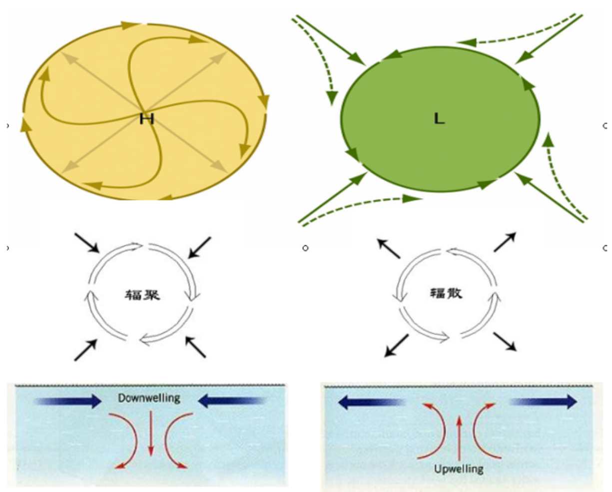
赤道附近的升降流



顺 (沿) 岸风产生的升降流



气旋和反气旋产生的海洋升降流



假定

(1) ρ 为常数；

(2) 直线风系，风仅沿 x 方向有变化；风区内为恒定的均匀风场；风区外无风；$\dfrac{\partial}{\partial y}=0$

(3) 定常风场；$\dfrac{\partial}{\partial t}=0$

(4) 大尺度；$Ro \ll 1$

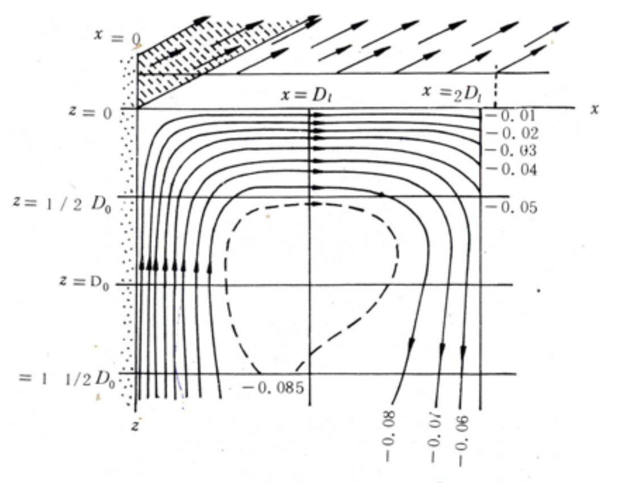
(5) 有限深度.$h \geq 2D_0$

控制方程及边界条件

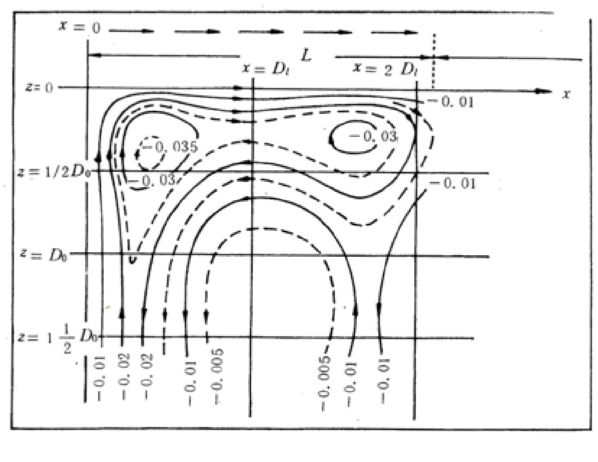
$$\begin{cases} A_l\dfrac{\partial^2 u}{\partial x^2}+A_z\dfrac{\partial^2 u}{\partial z^2}+fv+g\dfrac{\partial\zeta}{\partial x}=0\\ A_l\dfrac{\partial^2 v}{\partial x^2}+A_z\dfrac{\partial^2 v}{\partial z^2}-fu=0\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}=0 \end{cases}$$

$$\begin{cases} z=\zeta:\rho A_z\dfrac{\partial u}{\partial z}=0,\quad \rho A_z\dfrac{\partial v}{\partial z}=-\tau_y\quad (0\le x\le L)\\ z=h:u=v=0\\ x=0:u=v=0\\ x\to\infty:u=v=0,\dfrac{\partial\zeta}{\partial x}=0 \end{cases}$$

结果讨论



(1) 近岸产生上升流 $x\le 0.5D_l$；
(2) 风区外延附近下降流 $x=2D_l$；
(3) 上升流来自 $z=1.5D_0$ 或更深；
(4) 最大 w 出现在 $z=D_0$；
(5) 上层离岸流，下层向岸流，构成一个循环.
若风向与海岸成 θ 角：



(1) 三个升降流系统：两个顺时针，一个逆时针；
(2) 大顺时针循环；
(3) $\theta=21.5^\circ$ 时，升降流达最大强度；
(4) 纬度越低，升降流越强.

2.3 非定常运动

2.3.1 漂流的发展

假定

(1) 远离海岸和海底的开阔大洋；

(2) 风场均匀恒定；

(3) ρ 为常数；

(4) 海面无倾斜；

(5) 运动非定常.

控制方程

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv = A_z \dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial v}{\partial t} + fu = A_z \dfrac{\partial^2 v}{\partial z^2} \end{cases}$$

初边值条件

$$\begin{cases} z = 0 : A_z \dfrac{\partial u}{\partial z} = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y (t > 0) \\ z \to \infty : u = v = 0 \\ t = 0 : u = v = 0 / u = C_1, v = C_2 \end{cases}$$

解的讨论

$$\begin{cases} u = \dfrac{2\pi \tau_y}{\rho f D_0} \displaystyle\int_0^{t'} \dfrac{\sin(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \\ v = \dfrac{2\pi \tau_y}{\rho f D_0} \displaystyle\int_0^{t'} \dfrac{\cos(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \end{cases}$$

根据下图 [?]：

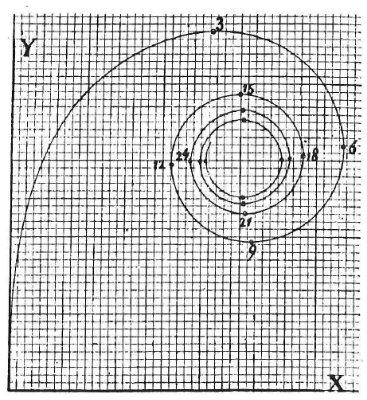


Fig. 3. z = 0.

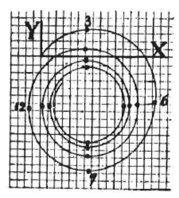


Fig. 4. z = 0,5 D.

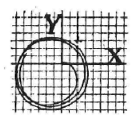


Fig. 5. z = D.

Fig. 6. z = 2D.

随时间增加，空间某点流苏端点顺时针旋转 (北半球)，逐渐趋向一个极限值 (即漂流).

2.3.2 惯性流

假定和方程

(1) 风场消失或者流动离开风区；

(2) 外部驱动小时，湍摩擦失去作用；

(3) 流动转为由惯性项维持平衡；

(4) 强制流动转变为自由流动.

$$\begin{cases} \dfrac{du}{dt} - fv = 0 & (13) \\ \dfrac{dv}{dt} + fu = 0 & (14) \end{cases}$$

求解

$$\begin{aligned}(13) \times u + (14 \times v) &\Leftrightarrow u\frac{du}{dt} + v\frac{dv}{dt} = \frac{1}{2}\frac{du^2}{dt} + \frac{1}{2}\frac{dv^2}{dt} = 0 \\ &\Leftrightarrow \frac{d}{dt}\left(u^2 + v^2\right) = 0 \\ &\Leftrightarrow u^2 + v^2 = c = V_0^2\end{aligned}$$

$$\begin{aligned}(13),(14) &\Rightarrow \begin{cases} v = \dfrac{dy}{dt} = \dfrac{1}{f}\dfrac{du}{dt} \\ u = \dfrac{dx}{dt} = \dfrac{1}{f}\dfrac{dv}{dt} \end{cases} \\ &\Rightarrow \begin{cases} y - y' = \dfrac{1}{f}\left(u - u'\right) \\ x - x' = -\dfrac{1}{f}\left(v - v'\right) \end{cases} \\ &\Rightarrow \begin{cases} y - \left(y' - \dfrac{u'}{f}\right) = \dfrac{u}{f} \\ x - \left(x' + \dfrac{v'}{f}\right) = -\dfrac{v}{f} \end{cases} \\ &\Rightarrow \boxed{\left(y - y_0\right)^2 + \left(x - x_0\right)^2 = \frac{1}{f^2}\left(u^2 + v^2\right) = \frac{V^2}{f^2} = r^2}\end{aligned}$$

流体质点沿半径为 r 的圆周作匀速运动，这个圆称之为惯性圆，对应的流动为惯性流.

惯性圆半径

科氏力充当向心力：$\dfrac{V_0^2}{r} = fV_0 \Rightarrow V_0 = fr \Rightarrow r = \dfrac{V_0}{f} = \dfrac{V_0}{2\omega\sin\varphi}$

随纬度增加而减小；赤道 $r \to \infty$，水质点作直线运动.

周期

$$T_i = \frac{2\pi r}{V_0} = \frac{2\pi r}{fr} = \frac{2\pi}{f} = \frac{\pi}{\omega\sin\varphi}$$

运动方向

北半球，顺时针；南半球，逆时针.

背景流

(1) 当无其他外加流动时，所有惯性圆的圆心位于同一条铅直线上，因而海水就像以角速度 $2\omega\sin\varphi$ 旋转的刚体一样；

(2) 当有其他外加流动时，除了在同一水平面上所有海水质点皆在同一时刻由同一流速速率外，还依外加流动速度方向移动.

2.4 风生大洋环流

2.4.1 Sverdrup 理论

假定

(1) 远离海岸的大洋中部海区，Ro$\ll 1$ 大尺度、等深大洋 h 为常数；
(2) 远离边界、无侧边界影响，无水平湍摩擦应力 $\mathrm{E}_l \ll 1$；
(3) 定常风定常流动；
(4) ρ 为常数；
(5) β 平面近似.

控制方程

$$\begin{cases} -fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + A_z\dfrac{\partial^2 u}{\partial^2 z} \\ fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + A_z\dfrac{\partial^2 v}{\partial^2 z} \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta(\text{海面}) : \rho A_z\dfrac{\partial u}{\partial z} = \tau_{x\zeta}, \rho A_z\dfrac{\partial v}{\partial z} = \tau_{y\zeta} \\ z = -h(\text{足够深}) : u = v = 0, \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0, \dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial y} = 0 \end{cases}$$

求解

对上式进行垂直积分：

$$\begin{cases} -fM_y = -\dfrac{\partial P}{\partial x} + \tau_{x\zeta} & (15) \\ fM_x = -\dfrac{\partial P}{\partial y} + \tau_{y\zeta} & (16) \\ \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_y}{\partial y} = 0 \end{cases}$$

其中，$M_x = \displaystyle\int_{-h}^{0} \rho u dz, M_y = \int_{-h}^{0} \rho v dz, P = \int_{-h}^{0} p dz, \tau_{x\zeta} = \int_{-h}^{0} \rho A_z\frac{\partial^2 u}{\partial z^2} = \rho A_z\left(\frac{\partial u}{\partial z}\bigg|_{z=0} - \frac{\partial u}{\partial z}\bigg|_{z=-h}\right) \quad (\zeta \ll h)$

$$\begin{aligned} \frac{\partial(1)}{\partial y} - \frac{\partial(2)}{\partial x} &\Leftrightarrow -M_y\frac{\partial f}{\partial y}\overset{0}{\boxed{-f\frac{\partial M_y}{\partial y} - f\frac{\partial M_x}{\partial x}}} = \frac{\partial \tau_{x\zeta}}{\partial y} - \frac{\partial \tau_{y\zeta}}{\partial x} \\ &\Leftrightarrow M_y\frac{\partial f}{\partial y} = \frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y} \\ &\Leftrightarrow \underbrace{\boxed{\beta M_y = \mathrm{rot}_z\, \tau_\zeta}}_{\text{Sverdrup 方程}} \end{aligned}$$

Sverdrup 方程的物理意义 1：海水南北向的输运由风应力旋度所驱动.

讨论

将质量输运分为为 Ekman 漂流输运与地转流输运两部分：

$$M_x = M_{xE}(\text{Ekman 漂流}) + M_{xg}(\text{地转流})$$
$$M_y = M_{yE}(\text{Ekman 漂流}) + M_{yg}(\text{地转流})$$

$$
\begin{cases}
-fM_{yE} = \tau_{x\zeta} & (17) \\
fM_{xE} = \tau_{y\zeta} & (18) \\
-fM_{yg} = -\dfrac{\partial P}{\partial x} & (19) \\
fM_{yg} = -\dfrac{\partial P}{\partial y} & (20)
\end{cases}
$$

$$
\frac{\partial(17)}{\partial y} - \frac{\partial(18)}{\partial x} \Leftrightarrow \frac{\partial M_{xE}}{\partial x} + \frac{\partial M_{yE}}{\partial y} = \boxed{\nabla\cdot\vec{M}_E = (\mathrm{rot}_z\,\vec{\tau}_\zeta - \beta M_{yE})/f} \tag{21}
$$

$$
\frac{\partial(19)}{\partial y} - \frac{\partial(20)}{\partial x} \Leftrightarrow \frac{\partial M_{xg}}{\partial x} + \frac{\partial M_{yg}}{\partial y} = \boxed{\nabla\cdot\vec{M}_g = -\beta M_{yg}/f} \tag{22}
$$

(1) Ekman 漂流质量输运的水平散度与 ① 风应力旋度 ② f ③ β 有关.
(2) 地转流质量输运的水平辐散引起南北向的地转运动.

$$
(21)+(22) \Leftrightarrow \mathrm{rot}_z\,t\vec{au}_\zeta - \beta M_y = 0 \Leftrightarrow \boxed{\beta M_y = \mathrm{rot}_z\,\vec{\tau}_\zeta}
$$

Sverdrup 方程的物理意义 2: 地转流流量的散度和 Ekman 漂流流量的散度相平衡，所以 Sverdrup 方程又称 Sverdrup 平衡.
(1) 所有南北向的地转运动，必须显示水平散度；
(2) 虽然 Ekman 漂流流量的散度与地转流流量的散度本身不为 0，但是它们的和，即总流量的水平散度必须为 0，说明 Ekman 漂流流量的散度和地转流流量的散度刚好取得平衡；
(3) $\mathrm{rot}_z\,\vec{\tau}_\zeta = 0$：只存在东西方向的输运，$\mathrm{rot}_z\,\vec{\tau}_\zeta > 0$：质量输运向北 (北半球)，$\mathrm{rot}_z\,\vec{\tau}_\zeta < 0$：质量输运向南 (南半球)；
(4) 地转流引起的南北质量运输量比 Ekman 漂流引起的大.

缺陷
设仅有纬向风，且 $\tau_{x\zeta}$ 仅为 y 的函数：

$$
M_y = \frac{1}{\beta}\mathrm{rot}_z\,\vec{\tau}_\zeta = \frac{1}{\beta}\left(\cancel{\frac{\partial \tau_{y\zeta}}{\partial x}} - \frac{\partial \tau_{x\zeta}}{\partial y}\right) = -\frac{a}{2\omega\cos\varphi}\frac{\partial \tau_{x\zeta}}{\partial y}
$$

$$
\frac{\partial M_x}{\partial x} = -\frac{\partial M_y}{\partial y} \Rightarrow M_x = \frac{x}{2\omega\cos\varphi}\left(a\frac{\partial^2 \tau_{xf}}{\partial y^2} + \frac{\partial \tau_{xf}}{\partial y}tg\varphi\right) + c(y)
$$

在东、西边界，$M_x = 0$ 而 Sverdrup 理论不能同时满足.

2.4.2 Stommel 理论

假定
(1) 远离海岸的大洋中部海区，Ro$\ll 1$ 大尺度、等深大洋 h 为常数；
(2) 远离边界、无侧边界影响，无水平湍摩擦应力 $\mathrm{E}_l \ll 1$；
(3) 定常风定常流动；
(4) ρ 为常数；
(5) β 平面近似；
(6) 考虑底摩擦

控制方程

$$
\begin{cases}
-fv = -g\dfrac{\partial \zeta}{\partial x} + A_z\dfrac{\partial^2 u}{\partial^2 z} \\
fu = -g\dfrac{\partial \zeta}{\partial y} + A_z\dfrac{\partial^2 v}{\partial^2 z} \\
0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0
\end{cases}
$$

边界条件

$$\begin{cases} z=\zeta(\text{海面}): \tau_{x,\zeta}=\rho A_z \dfrac{\partial u}{\partial z}\Big|_{t=\zeta} = -F\cos(\pi y/b), \tau_{x,\zeta}=0 \\ z=-h(\text{海底}): \tau_{x,-h}=\rho+\dfrac{\partial u}{\partial z}\Big|_{t=-h} = \rho k u, \tau_{y,-h}=\rho+\dfrac{\partial v}{\partial z}\Big|_{t=-h} = \rho k v \end{cases}$$

求解

对上式进行垂直平均：

$$\begin{cases} -f\langle v\rangle = -g\dfrac{\partial \zeta}{\partial x} + \dfrac{A_z}{h+\zeta}\dfrac{\partial u}{\partial z}\Big|_{\zeta} - \dfrac{A_z}{h+\zeta}\dfrac{\partial u}{\partial z}\Big|_{-h} \\ f\langle u\rangle = -g\dfrac{\partial \zeta}{\partial y} + \dfrac{A_z}{h+\zeta}\dfrac{\partial v}{\partial z}\Big|_{\zeta} - \dfrac{A_z}{h+\zeta}\dfrac{\partial v}{\partial z}\Big|_{-h} \\ \dfrac{\partial}{\partial x}[(n+\zeta)\langle u\rangle] + \dfrac{\partial}{\partial y}[(n+\zeta)\langle v\rangle] = 0 \end{cases}$$

将边界条件代入方程：

$$\begin{cases} 0 = f\rho h v - F\cos(\pi y/b) - \rho k u - \rho g h\dfrac{\partial \zeta}{\partial x} \quad (\zeta \ll k) & (23) \\ 0 = -f\rho h u - \rho k v - \rho g h \dfrac{\partial \zeta}{\partial y} & (24) \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

$$\frac{\partial(23)}{\partial y} - \frac{\partial(24)}{\partial x} \Rightarrow \frac{h}{k}[\beta v + r\sin(\pi y/b)] + \overbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}^{\text{来源于底摩擦}} = 0$$

引入流函数 $\psi: u = \dfrac{\partial \psi}{\partial y}, v = -\dfrac{\partial \psi}{\partial x}$：

$$\overbrace{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}}^{\text{来源于底摩擦}} + \frac{h}{k}\beta\frac{\partial \psi}{\partial x} = \frac{h}{k} r \sin\frac{\pi y}{b}$$

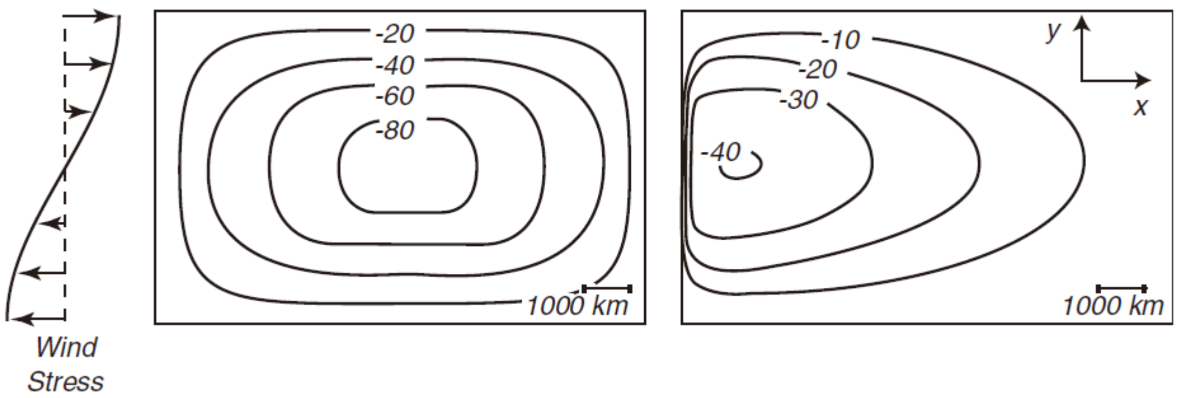
边界条件：

$$\psi(0,y) = \psi(a,y) = \psi(x,0) = \psi(x,b) = 0$$

$$\Rightarrow \psi(x,y) = \frac{Fb}{k\pi}\sin\frac{\pi y}{b}\left[\frac{e^{\frac{h\beta}{2k}(a-x)}\operatorname{sh}\alpha x + e^{\frac{h\beta}{2k}x}\operatorname{sh}\alpha(a-x)}{\operatorname{sh}\alpha a} - 1\right]$$

特别地，若 $\beta=0$：$\psi(x,y) = \dfrac{Fb}{k\pi}\sin\dfrac{\pi y}{b}\left[\dfrac{\operatorname{sh}\frac{\pi}{b}x + \operatorname{sh}\frac{\pi}{b}(a-x)}{\operatorname{sh}\frac{\pi}{b}a} - 1\right]$

讨论 图片来自 *Introduction to Physical Oceanography*(Robert H. Stewart,2008 pp190)



※β效应导致了西向强化现象.

$\beta=0$(非旋转坐标系或 f 平面近似): (1) 流线南北对称; (2) 流线东西对称.	$\beta\neq 0$(β 平面近似): (1) 流线南北对称; (2) 流线东西不对称，西部密集，东部稀疏.

2.4.3 惯性理论

方程

在 Sverdrup 理论的控制方程中引入惯性项：

$$\begin{cases}\dfrac{du}{dt}-fv=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+A_z\dfrac{\partial^2 u}{\partial^2 z}\\ \dfrac{dv}{dt}+fu=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+A_z\dfrac{\partial^2 v}{\partial^2 z}\\ 0=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}-g\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}$$

求解

内区 (中部区)

满足 Sverdrup 理论：$\beta M_y=\mathrm{rot}_z\,\tau_\zeta, M_y=\dfrac{\partial\varphi}{\partial x}=\dfrac{1}{\beta}\mathrm{rot}_z\,\tau_\zeta=\dfrac{1}{\beta}\left(\dfrac{\partial\tau_{y\zeta}}{\partial x}-\dfrac{\partial\tau_{x\zeta}}{\partial y}\right)$

又设风应力：$\tau_{x\zeta}=-W\left(1-\dfrac{y^2}{s^2}\right), \tau_{y\zeta}=0\quad(0\le y\le s)$

$$\begin{aligned}&\Rightarrow\frac{\partial\varphi}{\partial x}=-\frac{2W}{\beta s^2}y\\ &\Rightarrow\varphi=-\frac{2W}{\beta s^2}yx+C(y)\\ (x=r:\varphi=0)\quad&0=-\frac{2W}{\beta s^2}yr+C(y)\\ &\Rightarrow\varphi=\frac{2W}{\beta s^2}y(r-x)\end{aligned}$$

大洋西部海域

不考虑湍摩擦效应：

$$\begin{cases}\dfrac{du}{dt}-fv=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & (25)\\ \dfrac{dv}{dt}+fu=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & (26)\\ 0=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}-g\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}$$

$$\begin{aligned}\frac{\partial(26)}{\partial x}-\frac{\partial(25)}{\partial y}&\Leftrightarrow\frac{d}{dt}(\xi_r+f)+(\xi_r+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\\ &\Leftrightarrow\frac{d}{dt}\left(\xi_r+f\right)=-\left(\xi_r+f\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\end{aligned}\tag{27}$$

$\xi_r=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ 相对涡度，f 行星涡度，$\xi_a=\xi_r+f$ 绝对涡度.

对连续方程进行垂向平均：

$$\begin{aligned}&\frac{\partial\zeta}{\partial t}+\frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x}+\frac{\partial[(h+\zeta)(v)]}{\partial y}=0\\ &\Leftrightarrow\frac{d(\zeta+h)}{dt}+(\zeta+h)\left(\frac{\partial(u)}{\partial x}+\frac{\partial(v)}{\partial y}\right)=0\\ (\text{令}H=h+\zeta)\ &\Leftrightarrow\frac{dH}{dt}+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\end{aligned}\tag{28}$$

将 (28) 代入 (27) 中：

$$
\begin{aligned}
&\Rightarrow \frac{d}{dt}(\xi_r+f)=\frac{1}{H}(\xi_r+f)\frac{dH}{dt}\\
&\Rightarrow \frac{1}{H}\frac{d}{dt}(\xi_r+f)=\frac{1}{H^2}(\xi_r+f)\frac{dH}{dt}\\
&\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\xi_r+f}{H}\right)=0}
\end{aligned}
\tag{29}
$$

(29) 为 位势涡度守恒方程.

假设在西边界区 $\frac{\partial u}{\partial y}=0$，则:

$$
\xi_r=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial \varphi}{\partial x}\right)
$$

代入 (29)：

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\frac{\partial}{\partial x}\left(\frac{1}{M}\frac{\partial\varphi}{\partial x}\right)+f}{H}\right]=0\\
\Rightarrow&\frac{\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial\varphi}{\partial x}\right)+f}{H}=F(\varphi)\\
\Rightarrow&\frac{1}{H}\frac{\partial^2\varphi}{\partial x^2}+f_0+\beta y=HF(\varphi)=G(\varphi)
\end{aligned}
$$

在西边界层的边缘

内部的解即为西边界的解，惯性项可以忽略：

$$
x=L: f_0+\beta y=G(\phi_i)
$$

在内区的边缘 ($L\ll r$)

$$
\varphi_i=\frac{2W}{\beta s^2}y(r-x)=\frac{2W}{\beta s^2}y(r-L)=\frac{2W}{\beta s^2}yr=u^*y\quad\left(u^*=\frac{2W}{\beta s^2}r\right)
$$

上面两解应该等价，因此：

$$
\begin{aligned}
&f_0+\beta y=G(u^*y)\\
\Rightarrow&G(\varphi)=f_0+\beta\frac{\varphi}{u^*}\\
\Rightarrow&\frac{\partial^2\varphi}{\partial x^2}+H(f_0+\beta y)=H\left(f_0+\beta\frac{\varphi}{u^*}\right)\\
\Rightarrow&\frac{\partial^2\varphi}{\partial x^2}-\frac{H\beta}{u^*}\varphi=-H\beta y
\end{aligned}
$$

再结合两个边界条件：$x=0:\begin{cases}\varphi(0,y)=0\\ \dfrac{\partial\varphi}{\partial x}=0\end{cases}$，上面的二阶常系数线性微分方程的解为：

$$
\varphi=u^*y\left[1-e^{-(H\beta/u^*)^{1/2}x}\right]
$$

海水南北输运：

$$
\boxed{M_y=\frac{\partial\varphi}{\partial x}=(u^*H\beta)^{1/2}ye^{-(H\beta u^*)^{1/2}x}}
$$

在西边界区域有一强烈的北向流动，近岸处质量运输很大，随着离岸距离的增加，质量输运迅速减少.

2.4.4 Munk 理论

假定

(1) 矩形大洋 $r\times 2s$；

(2) 远离海岸的等深封闭矩形大洋，静止时水深为常量 h；

(3) 增加了侧向摩擦.

控制方程

$$\begin{cases} -fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + A_l\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + A_z\dfrac{\partial^2 u}{\partial z^2} \\ fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + A_l\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + A_z\dfrac{\partial^2 v}{\partial z^2} \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta, \quad \rho A_z\dfrac{\partial u}{\partial z} = \tau_{x\zeta}, \rho A_z\dfrac{\partial v}{\partial z} = \tau_{y\zeta} \\ z = -h, \quad u = v = 0, \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 \end{cases}$$

求解

全流方程 (垂直积分)：

$$\begin{cases} -fM_y = -\dfrac{\partial P}{\partial x} + A_l\left(\dfrac{\partial^2 M_x}{\partial x^2} + \dfrac{\partial^2 M_x}{\partial y^2}\right) + \tau_{x\zeta} & (30) \\ fM_x = -\dfrac{\partial P}{\partial y} + A_l\left(\dfrac{\partial^2 M_y}{\partial x^2} + \dfrac{\partial^2 M_y}{\partial y^2}\right) + \tau_{y\zeta} & (31) \\ \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_y}{\partial y} = 0 \end{cases}$$

其中，$M_x = \int_{-h}^{0} \rho u dz, M_y = \int_{-h}^{0} \rho v dz, P = \int_{-h}^{0} p dz, \tau_{x\zeta} = \int_{-h}^{0} \rho A_z \frac{\partial^2 u}{\partial z^2} = \rho A_z\left(\left.\frac{\partial u}{\partial z}\right|_{z=0} - \left.\frac{\partial u}{\partial z}\right|_{z=-h}\right)$ $(\zeta \ll h)$ 引入流函数：$M_x = -\frac{\partial \psi}{\partial y}, M_y = \frac{\partial \psi}{\partial x}$

$$\begin{aligned} \frac{\partial(30)}{\partial y} - \frac{\partial(31)}{\partial x} &\Leftrightarrow A_l\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) - \beta\frac{\partial \psi}{\partial x} = -\left(\frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y}\right) \\ &\Leftrightarrow A_l\nabla^4\psi - \beta\frac{\partial \psi}{\partial x} = -\operatorname{rot}_z \vec{\tau}_\zeta \end{aligned}$$

边界条件：

$$\begin{cases} x = 0, r : \psi = 0, \dfrac{\partial \psi}{\partial x} = 0 \\ y = -s, s : \psi = 0, \dfrac{\partial \psi}{\partial y} = 0 \end{cases}$$

内区 (中部区)　满足 Sverdrup 理论：$\beta\frac{\partial \psi}{\partial x} = \operatorname{rot}_z \tau_\zeta = -\frac{\partial \tau_{x\zeta}}{\partial y}$

假定纬向风系：$\begin{cases} \tau_{y\zeta} = 0 \quad (-s < y < s) \\ \tau_{x\zeta} = a\cos ny + b\sin ny + c \\ n = \dfrac{j\pi}{s}, j = 1, 2, 3\cdots \end{cases}$

对 x 积分：$\psi = -\frac{1}{\beta}\frac{\partial \tau_{x\zeta}}{\partial y}x + f_1(y)$

结合 $\psi|_r = 0 \Rightarrow \psi = -\frac{1}{\beta}\frac{\partial \tau_{x\zeta}}{\partial y}(x - r)$ 因此，$x = 0$ 处：

$$\begin{cases} \psi(0, y) = \dfrac{r}{\beta}\dfrac{\partial \tau_{x\zeta}}{\partial y} \\ \left.\dfrac{\partial \psi}{\partial x}\right|_{x=0} = M_y(0, y) = -\dfrac{1}{\beta}\dfrac{\partial \tau_{x\zeta}}{\partial y} \end{cases}$$

西边界区

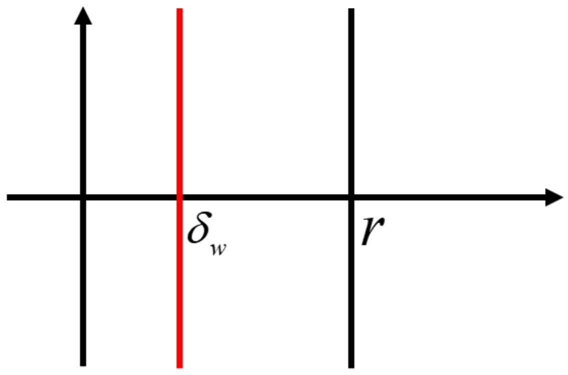
保留侧向湍流应力，结合 $L_x \ll L_y$：

$$A_l\nabla^4\psi - \beta\frac{\partial \psi}{\partial x} = -\operatorname{rot}_z \vec{\tau}_\zeta \Rightarrow A_l\frac{\partial \psi}{\partial x^4} - \beta\frac{\partial \psi}{\partial x} = \frac{\partial \tau_{x\zeta}}{\partial y}$$

设试解：$\psi = X(x)\dfrac{\partial \tau_{x\zeta}}{\partial y} \Rightarrow A_l X^{(4)} - \beta X' = 1 \Rightarrow X(x) = A + Be^{kx} + De^{-\frac{k}{2}x}\cos\left(\dfrac{\sqrt{3}}{2}kx + E\right) - \dfrac{x}{\beta}$ 自然边界条件：

$$x=0:\begin{cases}\psi = 0\\ \dfrac{\partial\psi}{\partial x} = 0\end{cases} \Rightarrow \begin{cases}X(0)=0\\ X'(0)=0\end{cases}$$

衔接边界条件：$x=\delta_w:\begin{cases}\psi = \dfrac{r}{\beta}\dfrac{\partial\tau_{x\zeta}}{\partial y}\\ \dfrac{\partial\psi}{\partial x} = -\dfrac{1}{\beta}\dfrac{\partial\tau_{x\zeta}}{\partial y}\end{cases} \Rightarrow \begin{cases}X(0)=\dfrac{r}{\beta}\\ X'(0)=-\dfrac{1}{\beta}\end{cases}$



解得：$\psi(x,y) = \dfrac{r}{\beta}\left[1 - \dfrac{2}{\sqrt{3}}e^{-\frac{1}{2}kx}\cos\left(\dfrac{\sqrt{3}}{2}kx - \dfrac{\pi}{6}\right)\right]\dfrac{\partial\tau_{x\zeta}}{\partial y}$

东边界区

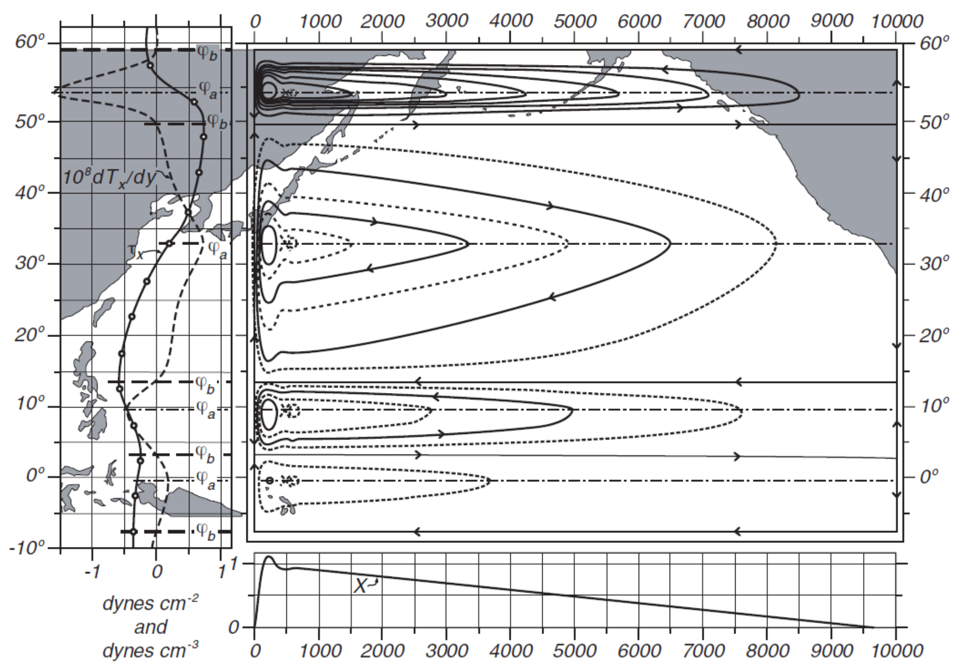
$$\psi(x,y) = \frac{r}{\beta}\left[1 - \frac{x}{r} + \frac{1}{kr}\left(e^{-k(r-x)} - 1\right)\right]\frac{\partial\tau_{x\zeta}}{\partial y}$$

综合内区、西边界区和东边界区的解可得统一的解的形式：

$$\boxed{\psi = \frac{r}{\beta}f(x)\frac{\partial x_{x\zeta}}{\partial y}}$$

讨论

环流空间分布特征　图片来自 *Introduction to Physical Oceanography*(Robert H. Stewart,2008 pp191)



(1) Gyres 之间的分界处位于：$M_y = \dfrac{\partial\psi}{\partial x} = 0$，只有东西向的流动：$f'(x) = 0/\dfrac{\partial\tau_{x\zeta}}{\partial y} = 0$；

(2) Gyres 主轴位于：$M_x = \dfrac{\partial \psi}{\partial x} = 0$ 只有南北向的流动：$f(x) = 0/\dfrac{\partial^2 \tau_{x\zeta}}{\partial y^2} = 0$；
(3) 流动西强东弱.

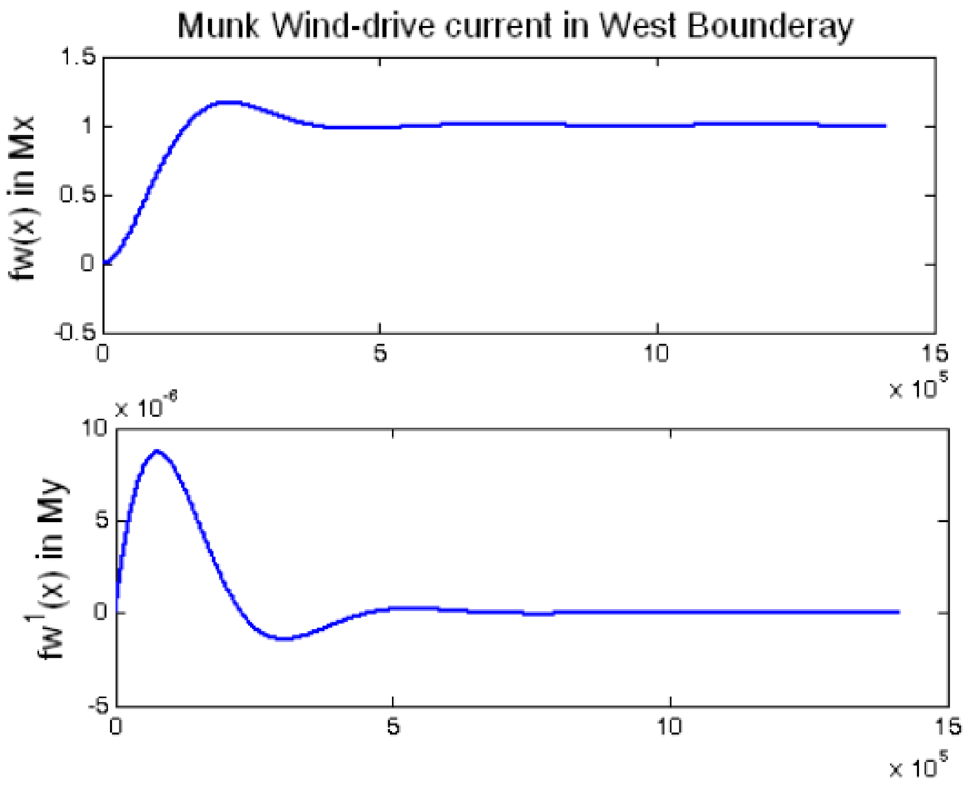
大洋西边界区的物质输运特点

$$\psi_W = \frac{r}{\beta} f_m(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

质量输运：

$$\begin{cases} M_{xW} = -\dfrac{\partial \psi}{\partial y} = -\dfrac{r}{\beta} f_W(x) \dfrac{\partial^2 \tau_{x\zeta}}{\partial y^2} \\ M_{yW} = \dfrac{\partial \psi}{\partial x} = \dfrac{r}{\beta} f'_W(x) \dfrac{\partial \tau_{x\zeta}}{\partial y} \end{cases}$$

其中，$\begin{cases} f_W(x) = 1 - \dfrac{2}{\sqrt{3}} e^{-\frac{1}{2}kx} \cos\left(\dfrac{\sqrt{3}}{2}kx - \dfrac{\pi}{6}\right) \\ f'_W(x) = \dfrac{2}{\sqrt{3}} k e^{-\frac{1}{2}kx} \sin \dfrac{\sqrt{3}}{2} kx \end{cases}$



随 x 增大而衰减的阻尼振动

对于任意纬度：

$$M_{yW} = \frac{\partial \psi}{\partial x} = \frac{r}{\beta} f'_w(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

因此，$f'_W(x)$ 的极值决定了 M_{yW} 的极值：

$$f''_W(x) = 0$$

$$(\text{令}\lambda_W = \frac{2\pi}{\text{波数}} = \frac{4\pi}{\sqrt{3}k}) \Rightarrow x_{a,b} = \frac{2\pi}{3\sqrt{3}k}\left(\frac{1}{6}\lambda_W\right), \quad \frac{8\pi}{3\sqrt{3}k}\left(\frac{4}{6}\lambda_W\right)$$

$$\Rightarrow \begin{cases} f'_W(x_a) = ke^{-\pi/3\sqrt{3}} \\ f'_W(x_b) = -ke^{-4\pi/3\sqrt{3}} \end{cases}$$

$$\frac{f'_W(x_b)}{f'_W(x_a)} = -e^{-\pi/3} = -0.17$$

在西边界内，以 $x = x_a(1/6)$ 波长为主轴处有一主流为北向的流动；以 $x = x_b(4/6)$ 波长为主轴处存在一逆流，逆流的量值为主流的 0.17 倍.
对于任意给定纬度，$f_W(x)$ 的极值决定 M_{xW} 的极值：

$$f'_W(x) = 0 \Rightarrow x_{1,2} = \frac{2\pi}{\sqrt{3}k}, \frac{4\pi}{\sqrt{3}k} \Rightarrow \begin{cases} f_W(x_1) = 1 - e^{-\pi/\sqrt{3}} \\ f_W(x_2) = 1 + e^{-2\pi/\sqrt{3}} \end{cases}$$

极值在 1 附近变动.

缺陷

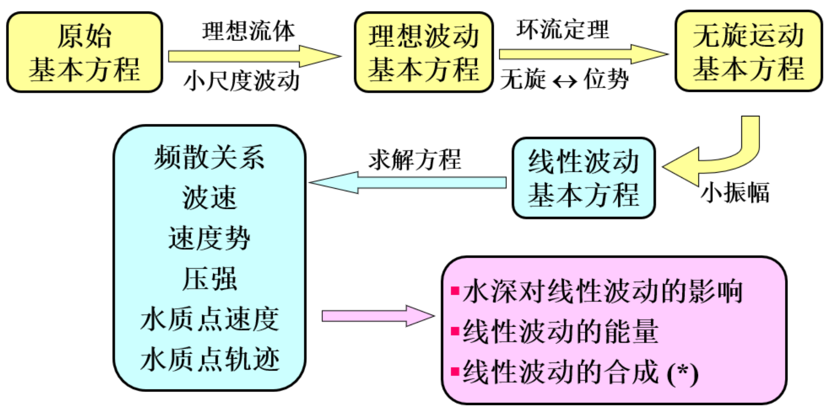
(1) 若取 $A_l = 5\times 10^3 m^2 s^{-1}$，西部阻尼振荡的波长比实际大约 3 倍：$\lambda = \dfrac{4\pi}{\sqrt{3}k} = \dfrac{4\pi}{\sqrt{3}}\dfrac{1}{\sqrt[3]{\beta/A_l}} \approx 200km$；符合实际主流和逆流宽度时，$A_l = 10^2 m^2 s^{-1}$

(2) 西部总流量比实测值小一半.

3 海浪

3.1 线性波动理论

理论框架：



3.1.1 无旋运动的基本方程

假定及方程

(1) 海水均匀不可压缩；

(2) 理想流体；

(3) 短周期小尺度波动；

(4) 重力为唯一的外力；

(5) 忽略分子粘性项、科氏力、引潮力和湍摩擦力.

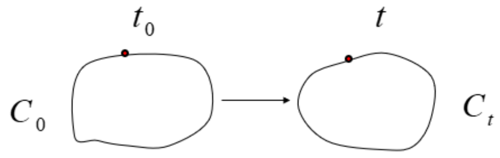
控制方程：

$$\begin{cases} \dfrac{d\vec{V}}{dt} = \dfrac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = -\dfrac{1}{\rho}\nabla p - \vec{g} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \end{cases}$$

边界条件：

$$\begin{cases} z = \zeta(\text{海面}) : \dfrac{\partial \zeta}{\partial t} + u\dfrac{\partial \zeta}{\partial x} + v\dfrac{\partial \zeta}{\partial y} = w, p_I = p_a(x,y,t) \\ \text{固体边界处} : V_n = 0 \end{cases}$$

环流定理



C_t 上的环流：$\Gamma(t)=\oint\limits_{ct} udx+vdy+wdz$

$$\Rightarrow \frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\left(\frac{du}{dt}dx+\frac{dv}{dt}dy+\frac{dw}{dt}dz\right)+\underbrace{\oint\limits_{ct}(udu+vdv+wdw)}_{=\frac{1}{2}\oint\limits_{ct} du^2+dv^2+dw^2=0}$$

$$\Rightarrow \frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\frac{d\vec{V}}{dt}\cdot d\vec{l}$$

环流的实质微商等于加速度的环流.

$$\frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)dx+\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right)dy+\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\right)dz=\oint\limits_{ct} d\underbrace{\left(-\frac{p}{\rho}-gz\right)}_{\text{单值函数}}=0$$

环流定理：对不可压缩的理想流体，由相同质点构成的封闭曲线上的环流不随时间而变。

无旋运动的基本方程和边界条件

若在重力场中，理想流体于起始时刻为静止或匀速运动，则任何时刻，对任何封闭曲线有：$\Gamma(t)=0$，根据 Stokes 定理：

$$\Gamma(t)=\oint\limits_{ct}\vec{V}\cdot d\vec{l}=\iint\limits_{s}\nabla\times\vec{V}d\sigma=0\Rightarrow\nabla\times\vec{V}\Longrightarrow\vec{V}=\nabla\varphi$$

运动方程：

$$\begin{aligned}\frac{d\vec{V}}{dt}&=\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-\vec{g}\\ \Rightarrow\frac{d\vec{V}}{dt}&=\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})=-\nabla\frac{p-p_0}{\rho}-\nabla(gz)\\ \Rightarrow\nabla\frac{\partial\varphi}{\partial t}&+\frac{1}{2}\nabla(\nabla\varphi\cdot\nabla\varphi)=-\nabla\frac{p-p_0}{\rho}-\nabla(gz)\\ \Rightarrow\frac{\partial\varphi}{\partial t}&+\frac{1}{2}(\nabla\varphi)(\nabla\varphi)+\frac{p-p_0}{\rho}+gz=0\end{aligned}$$

连续方程：

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

运动学边界条件：

$$\begin{aligned}\text{海面}:&\left.\left(\frac{\partial\zeta}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\zeta}{\partial y}\right)\right|_{z=\zeta}=\left.\frac{\partial\varphi}{\partial z}\right|_{z=\zeta}\\ \text{固定边界}:&\frac{\partial\varphi}{\partial n}=0\end{aligned}$$

动力学边界条件：

$$\begin{aligned}&p_I=p_a(x,y,t)\\ &\left.\left[\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)(\nabla\varphi)\right]\right|_{z=\zeta}+g\zeta=0\end{aligned}$$

3.1.2 线性波动

假定

(1) 均质不可压理想流体；

(2) 波动的振幅相对波长很小；

(3) 设水域广阔等深；

(4) 波动只沿 x 方向传播.

$$\text{小振幅假定}:\Rightarrow\begin{cases}\varphi\text{的微商乘积项可忽略}\\ \left.\frac{\partial\varphi}{\partial z}\right|_{z=\zeta}\approx\left.\frac{\partial\varphi}{\partial z}\right|_{z=0},\ \left.\frac{\partial\varphi}{\partial t}\right|_{z=\zeta}\approx\left.\frac{\partial\varphi}{\partial t}\right|_{z=0}\end{cases}$$

方程简化

运动方程：$\frac{\partial \varphi}{\partial t}+\cancel{\frac{1}{2}(\nabla\varphi)(\nabla\varphi)}+\frac{p-p_0}{\rho}+gz=0$

连续方程：$\Delta\varphi=\frac{\partial^2\varphi}{\partial x^2}+\cancel{\frac{\partial^2\varphi}{\partial y^2}}+\frac{\partial^2\varphi}{\partial z^2}=0$

边界条件：$\begin{cases}\left(\frac{\partial\zeta}{\partial t}+\cancel{\frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x}}+\cancel{\frac{\partial\varphi}{\partial y}\frac{\partial\zeta}{\partial y}}\right)\Big|_{z=\zeta}=\frac{\partial\varphi}{\partial z}\Big|_{z=\zeta}\\ \frac{\partial\varphi}{\partial n}=0\Rightarrow\frac{\partial\varphi}{\partial z}\Big|_{z=-d}=0\\ \left[\frac{\partial\varphi}{\partial t}+\cancel{\frac{1}{2}(\nabla\varphi)(\nabla\varphi)}\right]\Big|_{z=\zeta}+g\zeta=0\end{cases}$

因此，线性波动的基本方程：

$$\begin{cases}\frac{\partial\varphi}{\partial t}+\frac{p-p_0}{\rho}+gz=0 & (32)\\ \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial z^2}=0 & (33)\\ \frac{\partial\varphi}{\partial z}\Big|_{==0}=\frac{\partial\zeta}{\partial t} & (34)\\ \frac{\partial\varphi}{\partial z}\Big|_{z=-d}=0 & (35)\\ \frac{\partial\varphi}{\partial t}\Big|_{z=0}+g\zeta=0 & (36)\end{cases}$$

(36) 代入 (??) 中：

$$\left(\frac{\partial\varphi}{\partial z}+\frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\right)\Big|_{z=0}=0$$

求解

对前进波：

$$\varphi=\varphi_0(z)\cos(kx-\omega t) \tag{37}$$

(37) 代入 (33) 中：

$$\varphi_0(z)=Ae^{kz}+Be^{-kz} \tag{38}$$

(37) 代入 (3.1.2) 中 (海面)：

$$\left[\frac{\partial\varphi_0(z)}{\partial z}-\frac{\omega^2}{g}\varphi_0(z)\right]\Big|_{z=0}=0 \tag{39}$$

(37) 代入 (35) 中 (海底)：

$$\frac{d\varphi_0(z)}{dz}\Big|_{z=-d}=0 \tag{40}$$

(38) 代入 (39) 中：

$$\left(\omega^2-gk\right)A+\left(\omega^2+gk\right)B=0 \tag{41}$$

(38) 代入 (40) 中：

$$e^{-kd}A-e^{kd}B=0 \tag{42}$$

联立 (41) (42)，有非零解的条件为：

$$\begin{vmatrix}e^{-kd} & -e^{kd}\\ (\omega^2-gk) & (\omega^2+gk)\end{vmatrix}=0\Rightarrow\underbrace{\omega^2=gk\,\mathrm{th}\,kd}_{\text{频散关系}}$$

频散关系表示波动频率和波数之间的关系，代表某种波动的性质.

波速：$c=\frac{\lambda}{T}=\frac{2\pi/k}{2\pi/\omega}=\frac{\omega}{k}\Rightarrow c^2=\frac{\omega^2}{k^2}=\frac{g}{k}\ \mathrm{th}\ kd$

波速与水深和波动性质有关系.

由 (42) : $Ae^{-kd}=Be^{kd}=\frac{1}{2}D\Rightarrow A=\frac{D}{2}/e^{-kd};\quad B=\frac{D}{2}/e^{kd}$

代入 (37):

$$\varphi = D\operatorname{ch}[k(z+d)]\cos(kx-\omega t) \tag{43}$$

(43) 代入 (36) 中：

$$\zeta = -\frac{\omega}{g}D\operatorname{ch}kd\sin(kx-\omega t) = a\sin(kx-\omega t)$$

速度势的解：

$$\varphi = -\frac{ag}{\omega}\frac{\operatorname{ch}[k(z+d)]}{\operatorname{ch}kd}\cos(kx-\omega t) \tag{44}$$

(44) 代入 (32) 中，可得压强分布：

$$p = p_0 + \rho ga\frac{\operatorname{ch}[k(z+d)]}{\operatorname{ch}kd}\sin(kx-\omega t) - \rho gz$$

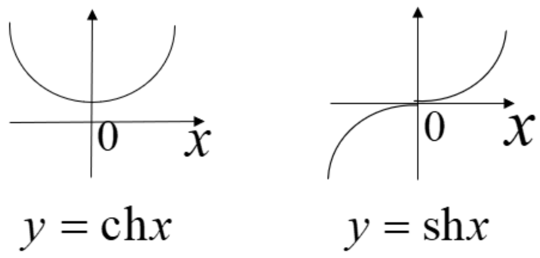
水质点速度：

$$u = \frac{\partial\varphi}{\partial x} = \frac{agk}{\omega}\frac{\operatorname{ch}[k(z+d)]}{\operatorname{ch}kd}\sin(kx-\omega t)$$
$$w = \frac{\partial\varphi}{\partial z} = -\frac{agk}{\omega}\frac{\operatorname{sh}[k(z+d)]}{\operatorname{ch}kd}\cos(kx-\omega t)$$

水质点运动轨迹 (x_0, z_0) 为平衡位置:

$$\frac{(x-x_0)^2}{\left[a\frac{\operatorname{ch}[k(d+z_0)]}{\operatorname{sh}kd}\right]^2} + \frac{(z-z_0)^2}{\left[a\frac{\operatorname{sh}[k(d+z_0)]}{\operatorname{sh}kd}\right]^2} = 1$$

解的讨论



(1) 线性波动水质点轨迹为椭圆，长轴为 x 方向；

(2) 椭圆长轴和短轴均与 z0 有关，与 x0 无关；

(3) 长轴和短轴均随平衡位置的加深而减小；

(4) 在海底水质点做水平运动.

参考文献

[Ekman, 1905] Ekman, V. W. (1905). *On the Influence of the Earth's Rotation on Ocean-Currents*, volume 2. University Microfilms INC.