物理海洋学笔记

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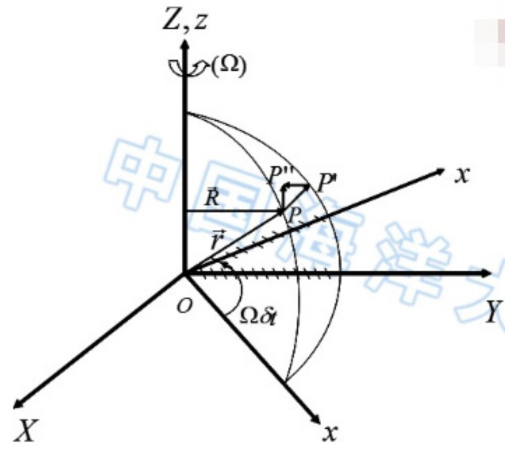
1 基本方程

1.1 旋转坐标系的速度和加速度

惯性坐标系: 静止的或是匀速直线运动的坐标系，固定在恒星上的坐标系可以被看成惯性坐标系.
固定在地球上的坐标系: 地球对恒星的加速度主要是由地球自转引起的，于是可以把地球当作一个对惯性坐标系作纯粹地转运动的物体.

1.1.1 旋转坐标系和惯性坐标系中的速度



惯性坐标系(XYZ)绝对位移：$\vec{pp''} = \vec{V}_a \delta t, \vec{V}_a$为绝对速度
旋转坐标系(xyz)相对位移：$\vec{p'p''} = \vec{V}\delta t, \vec{V}$为相对速度
$\because \vec{pp''} = \vec{p'p''} + \vec{pp'}$
$\therefore \vec{V}_a \delta t = \vec{V}\delta t + \vec{V}_e \delta t \Rightarrow \vec{V}_a = \vec{V} + \vec{V}_e$(绝对速度等于相对速度与牵连速度的向量和)
其中，$\vec{V}_e = \vec{\Omega} \times \vec{r} \Rightarrow \vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r}$

$$\Rightarrow \frac{d_a \vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

$$\frac{d_a \vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

1.1.2 旋转坐标系和惯性坐标系中的加速度

令$\vec{A} = \vec{V}_a = \vec{V}_e + \vec{V} = \vec{V} + \vec{\Omega} \times \vec{r}$

$$\begin{aligned}
\frac{d\bar{V}_a}{dt} &= \frac{d_a}{dt}\left(\vec{V} + \vec{V}_e\right) = \frac{d_a}{dt}(\vec{V} + \vec{\Omega} \times \vec{r}) \\
&= \frac{d}{dt}(\vec{V} + \vec{\Omega} \times \vec{r}) + \vec{\Omega} \times (\vec{V} + \vec{\Omega} \times \vec{r}) \\
&= \frac{d\vec{V}}{dt} + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\
&= \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R}
\end{aligned}$$

1.2 作用在海水微团上的外力运动方程的向量形式

压强梯度力：$\frac{1}{\rho}\nabla p$
分子粘性力(摩擦力)：

$$\begin{cases}
F_x = \frac{1}{\rho}\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta u \\
F_y = \frac{1}{\rho}\mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta v \quad \Rightarrow \vec{F} = \frac{\mu}{\rho}\Delta\vec{V} = \gamma\Delta\vec{v} \\
F_2 = \frac{1}{\rho}\mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta w
\end{cases}$$

重力(地球引力与地球自转产生的惯性离心力的合力)：$\vec{g} = -G\frac{M_g}{r^2} \cdot \left(\frac{\vec{r}}{r}\right)$
科氏力：$-2\vec{\Omega} \times \vec{V}$

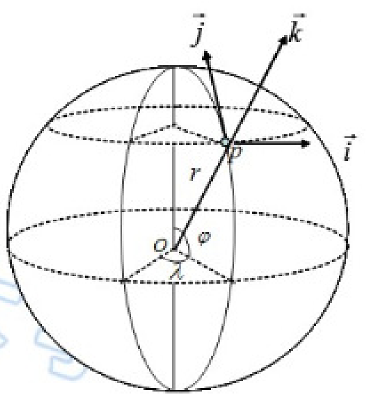
天体引潮力(受其他天体万有引力与惯性力离心力的合力)：$\vec{F_M} = -G\frac{M_M}{L^2} + G\frac{M_M}{D^2}\cdot\left(\frac{\vec{D}}{D}\right)$

由牛顿第二定律和坐标系转换关系：

$$\begin{cases} \frac{d_a\vec{V}_a}{dt} = \sum_i \vec{F}_t \\ \frac{d_a\vec{A}_a}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} \end{cases}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla P - 2\vec{\Omega}\times\vec{V} + \vec{g} + \nu\Delta\vec{V} + \vec{F_T}}$$

1.3 运动方程在球坐标系的标量形式



速度：

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\Rightarrow \begin{cases} u = r\cos\varphi\frac{d\lambda}{dt} \\ v = r\frac{d\varphi}{dt} \\ w = \frac{dr}{dt} \end{cases}$$

加速度：

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{\frac{\partial\vec{A}}{\partial t}dt + \frac{\partial\vec{A}}{\partial\lambda}d\lambda + \frac{\partial\vec{A}}{\partial\varphi}d\varphi + \frac{\partial\vec{A}}{\partial r}dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial\lambda}\frac{d\lambda}{dt} + \frac{\partial\vec{A}}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial\vec{A}}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\vec{A}}{\partial\lambda} + \frac{v}{r}\frac{\partial\vec{A}}{\partial\varphi} + w\frac{\partial\vec{A}}{\partial r} \end{aligned}$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{r\cos\varphi\partial\lambda} + v\frac{\partial}{r\partial\varphi} + w\frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)}}$$

$$\Rightarrow \boxed{\boldsymbol{\nabla = \frac{\partial}{r\cos\varphi\partial\lambda}\vec{i} + \frac{\partial}{r\partial\varphi}\vec{j} + \frac{\partial}{\partial r}\vec{k}}}$$

$$\Rightarrow \begin{cases} \frac{d\vec{i}}{dt} = \frac{\partial\vec{i}}{\partial t} + u\frac{\partial\vec{i}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{i}}{r\partial\varphi} + w\frac{\partial\vec{i}}{\partial r} \\ \frac{\partial\vec{j}}{dt} = \frac{\partial\vec{j}}{\partial t} + u\frac{\partial\vec{j}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{j}}{r\partial\varphi} + w\frac{\partial\vec{j}}{\partial r} \\ \frac{d\vec{k}}{dt} = \frac{\partial\vec{k}}{\partial t} + u\frac{\partial\vec{k}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{k}}{r\partial\varphi} + w\frac{\partial\vec{k}}{\partial r} \end{cases}$$

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\vec{i} + \frac{dv}{dt}\vec{j} + \frac{dv}{dt}\vec{k} + u\frac{d\vec{i}}{dt} + v\frac{d\vec{j}}{dt} + w\frac{d\vec{k}}{dt}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uvtg\varphi}{r} + \frac{uw}{r}\right)\vec{i} + \left(\frac{dv}{dt} + \frac{u^2\,\mathrm{tg}\,\varphi}{r} + \frac{vw}{r}\right)\vec{j} + \left(\frac{dw}{dt} - \frac{u^2+v^2}{r}\right)}$$

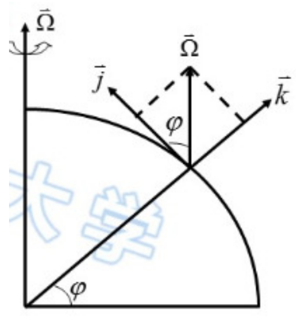
压强梯度力：

$$\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\left(\frac{1}{r\cos\varphi}\frac{\partial p}{\partial \lambda}\vec{i} + \frac{1}{r}\frac{\partial p}{\partial \varphi}\vec{j} + \frac{\partial p}{\partial r}\vec{k}\right)$$

重力：

$$\vec{g} = -g\vec{k}$$

科氏力：



$$\vec{\Omega} = \Omega\sin\varphi\vec{k} + \Omega\cos\varphi\vec{j}$$

$$-2\vec{\Omega}\times\vec{V} = -2\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ u & v & w \end{vmatrix}$$

$$= -2[(w\Omega\cos\varphi - v\Omega\sin\varphi)\vec{i} + (u\Omega\sin\varphi)\vec{j} + (-u\Omega\cos\varphi)\vec{k}]$$

$$\Rightarrow -2\vec{\Omega}\times\vec{V} = (fv - \tilde{f}w)\vec{i} - (fu)\vec{j} + (\tilde{f}u)\vec{k}$$

其中，$\begin{cases} f = 2\Omega sin\varphi \\ \tilde{f} = 2\Omega cos\varphi \end{cases}$

$$\Rightarrow \boxed{\begin{cases} \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{r\cos\varphi\partial\lambda} + fv - \tilde{f}w + \frac{uv\tan\varphi}{r} - \frac{uw}{r} + \gamma(\Delta\vec{v})_\lambda - \frac{1}{r\cos\varphi}\frac{\partial\phi_T}{\partial\lambda} \\ \frac{dy}{dt} = 7\frac{1}{\rho}\frac{\partial p}{r\partial\varphi} - fu - \frac{u^2\tan\varphi}{r} - \frac{vw}{r} + \gamma(\Delta\bar{v})_\varphi - \frac{1}{r}\frac{\partial\phi_T}{\partial\varphi} \\ \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \tilde{f}u = g + \frac{u^2+v^2}{r} + \gamma(\Delta\vec{v})_r - \frac{\partial\phi_T}{\partial r} \end{cases}}$$

1.4 直角坐标系的运动方程

略去地球曲率的影响

$$\Rightarrow \boxed{\begin{cases} \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \tilde{f}w + F_{N\lambda} + F_{T\lambda} \\ \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - fu + F_{Ny} + F_{Ty} \\ \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \tilde{f}u - g + F_{Nz} + F_{Tz} \end{cases}}$$

1.5 海水层流运动的基本方程组

1.5.1 连续方程

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0 \Leftrightarrow \frac{d\rho}{dt} + \rho\nabla\cdot\vec{V} = 0$$

特别地，对于不可压缩流体：

$$\nabla\cdot\vec{V} = 0$$

1.5.2 盐量扩散方程

$$
\begin{aligned}
&\overset{\text{盐量增加量}}{\frac{\partial}{\partial t}\iiint\limits_{\tau}\rho s d\tau} = \overset{\text{平流作用}}{-\oiint\limits_{\sigma}\rho s V_n d\sigma} + \overset{\text{分子扩散作用}}{-\oiint\limits_{\sigma}S_n d\sigma}\\
&\iiint\limits_{\tau}\frac{\partial(\rho s)}{\partial t}d\tau = \iiint\limits_{\tau}\nabla\cdot(\rho s\vec{V})d\tau - \iiint\limits_{\tau}\nabla\cdot\vec{S}d\tau\\
\Rightarrow&\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\vec{V}) + \nabla\cdot\vec{S} = 0\\
\Rightarrow&\rho\frac{\partial s}{\partial t} + s\frac{\partial\rho}{\partial t} + s\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla s + \nabla\cdot\vec{S} = 0\\
\Rightarrow&\left(\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s\right) + \frac{s}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] = -\frac{1}{\rho}\nabla\cdot\vec{S}\\
\Rightarrow&\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = \frac{k}{\rho}\Delta s = k_D\Delta s
\end{aligned}
$$

其中，$k_D = \frac{k}{\rho} \sim 1.1\times 10^{-9}\,(\mathrm{m^2/s})$

1.5.3 热传导方程

与上面类似：

$$\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \frac{\kappa}{\rho c_p}\Delta\theta = k_\theta\Delta\theta$$

其中，$k_\theta = \frac{\kappa}{\rho c_p} \sim 1.4\times 10^{-7}\,(\mathrm{m^2/s})$

1.5.4 热膨胀方程-状态方程

热膨胀方程：

$$\rho = \overset{\text{0℃时的海水密度}}{\rho_0}(1 - \overset{\text{海水的热膨胀系数}}{k}\theta)$$

EOS80国际海水状态方程：

$$\rho(s,t,p) = \rho(s,t,0)\left[1 - \frac{np}{k(s,t,p)}\right]^{-1}$$

1.6 基本方程的矢量形式和标量形式

矢量形式：

$$
\boxed{\begin{cases}
\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\Omega\times\vec{V} + \vec{g} + \gamma\Delta\vec{V} - \nabla\phi_T\\
\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{V} = 0\\
\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = k_D\Delta s\\
\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = k_\theta\Delta\theta\\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

标量形式(直角坐标系)：

$$
\boxed{\begin{cases}
\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \gamma\Delta v - \frac{\partial\phi_T}{\partial y}\\
\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \tilde{f}u - g + \gamma\Delta w - \frac{\partial\phi_T}{\partial z}\\
\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\\
\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} = k_D\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}\right)\\
\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = k_\theta\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right)\\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

1.7 边界条件

无质量交换的运动学边界条件：

$$\frac{\partial F}{\partial t} + \vec{c} \cdot \nabla F = 0$$

例：

(1) 海面($z = \zeta(x, y, t)$)：$\dfrac{\partial \zeta}{\partial t} + \vec{V}_H \cdot \nabla_H \zeta - w = 0$

(2) 海底($z = -h(x, y)$)：$\vec{V}_H \cdot \nabla_H h + w = 0$

动力学边界条件：

由牛顿第三定律，在界面法线方向有：

$$(\vec{p}_n)_1 = (\vec{p}_n)_2$$

1.8 ※间平均的基本方程和边界条件(直角坐标系)

连续方程:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

运动方程：

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \bar{u}\dfrac{\partial \bar{u}}{\partial x} + \bar{v}\dfrac{\partial \bar{u}}{\partial y} + \bar{w}\dfrac{\partial \bar{u}}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x} + f\bar{v} - \tilde{f}w + \gamma\Delta\bar{u} - \dfrac{\partial \bar{\phi}_T}{\partial x} + \dfrac{\partial}{\partial x}\left(A_{x\alpha}\dfrac{\partial \bar{u}}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial \bar{u}}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial \bar{u}}{\partial z}\right) \\ \dfrac{\partial \bar{v}}{\partial t} + \vec{u}\dfrac{\partial \bar{v}}{\partial x} + \vec{v}\dfrac{\partial \bar{v}}{\partial y} + \vec{w}\dfrac{\partial \bar{v}}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial y} - f\bar{u} + \gamma\Delta\bar{v} - \dfrac{\partial \bar{\phi}_T}{\partial y} + \dfrac{\partial}{\partial x}\left(A_{yx}\dfrac{\partial \bar{v}}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(A_{yy}\dfrac{\partial \bar{v}}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(A_{yz}\dfrac{\partial \bar{v}}{\partial z}\right) \\ \dfrac{\partial \bar{w}}{\partial t} + \bar{u}\dfrac{\partial \bar{w}}{\partial x} + \bar{v}\dfrac{\partial \bar{w}}{\partial y} + \bar{w}\dfrac{\partial \bar{w}}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial z} + \tilde{f}\bar{u} - g + \gamma\Delta\bar{w} - \dfrac{\partial \bar{\phi}_T}{\partial z} + \dfrac{\partial}{\partial x}\left(A_{2x}\dfrac{\partial \bar{w}}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(A_{zy}\dfrac{\partial \bar{w}}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(A_{zz}\dfrac{\partial \bar{w}}{\partial z}\right) \end{cases}$$

盐量扩散方程：

$$\frac{\partial \bar{s}}{\partial t} + \bar{u}\frac{\partial \bar{s}}{\partial x} + \bar{v}\frac{\partial \bar{s}}{\partial y} + \bar{w}\frac{\partial \bar{s}}{\partial z}$$

热传导方程：

$$\frac{\partial \bar{\theta}}{\partial t} + \vec{u}\frac{\partial \bar{\theta}}{\partial x} + \vec{v}\frac{\partial \bar{\theta}}{\partial y} + \vec{w}\frac{\partial \bar{\theta}}{\partial z} = k_\theta\Delta\bar{\theta} + \frac{\partial}{\partial x}\left(K_{\theta_x}\frac{\partial \bar{\theta}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{\theta y}\frac{\partial \bar{\theta}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{\theta z}\frac{\partial \bar{\theta}}{\partial z}\right)$$

状态方程：

$$\bar{\rho} = \bar{\rho}(\bar{s}, \bar{\theta}, \bar{p})$$

1.9 铅直向平均的基本方程

$$\frac{\partial}{\partial x}[(h+\zeta)\langle u\rangle] + \frac{\partial}{\partial y}[(h+\zeta)\langle v\rangle] - \left[u|_\zeta\frac{\partial \zeta}{\partial x} + v|_\zeta\frac{\partial \zeta}{\partial y} - w|_\zeta\right] - \left[u|_{-h}\frac{\partial h}{\partial x} + v|_{-h}\frac{\partial h}{\partial y} + w|_{-h}\right] = \mathbf{0}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x} + \frac{\partial[(h+\zeta)\langle v\rangle]}{\partial y} = 0$$

1.10 尺度分析

Rossby数 Ro=$\dfrac{U}{FL}\begin{cases} \gg 1: \text{平流非线性项比Coriolis力重要,大尺度运动} \\ = 1: \text{平流非线性项与Coriolis力同等重要} \\ \ll 1: \text{平流非线性项可以忽略,小尺度运动} \end{cases}$

水平Ekman数 $\mathrm{E}_l = \dfrac{A_l}{FL^2}$ 水平湍流摩擦项与Coriolis力比值

垂直Ekman数 $\mathrm{E}_z = \dfrac{A_z}{FD^2}$ 垂直湍流摩擦项与Coriolis力比值

准静力近似 f平面近似 β平面近似 Boussinesq近似

2 地转流