

# 物理海洋学笔记

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<https://github.com/Cuiyingzhe/UUC-Physical-Oceanography-Notes>

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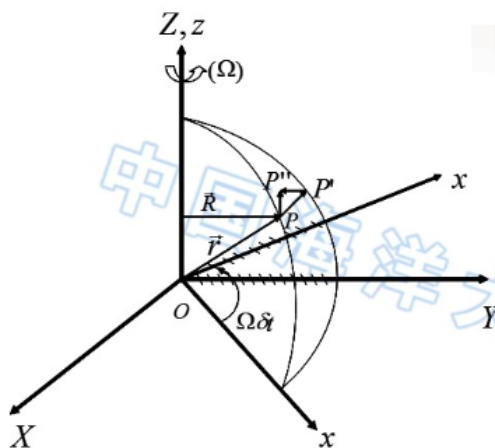
# 1 基本方程

## 1.1 旋转坐标系的速度和加速度

惯性坐标系: 静止的或是匀速直线运动的坐标系, 固定在恒星上的坐标系可以被看成惯性坐标系.

固定在地球上的坐标系: 地球对恒星的加速度主要是由地球自转引起的, 于是可以把地球当作一个对惯性坐标系作纯粹地转运动的物体.

### 1.1.1 旋转坐标系和惯性坐标系中的速度



惯性坐标系  $(XYZ)$  绝对位移:  $p\vec{p}'' = \vec{V}_a \delta t$ ,  $\vec{V}_a$  为绝对速度

旋转坐标系  $(xyz)$  相对位移:  $p'\vec{p}'' = \vec{V} \delta t$ ,  $\vec{V}$  为相对速度

$\therefore p\vec{p}'' = p'\vec{p}'' + p\vec{p}'$

$\therefore \vec{V}_a \delta t = \vec{V} \delta t + \vec{V}_e \delta t \Rightarrow \vec{V}_a = \vec{V} + \vec{V}_e$  (绝对速度等于相对速度与牵连速度的向量和)

其中,  $\vec{V}_e = \vec{\Omega} \times \vec{r} \Rightarrow \vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r}$

$$\Rightarrow \frac{d_a \vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

$$\frac{d_a \vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

### 1.1.2 旋转坐标系和惯性坐标系中的加速度

令  $\vec{A} = \vec{V}_a = \vec{V}_e + \vec{V} = \vec{V} + \vec{\Omega} \times \vec{r}$

$$\begin{aligned} \frac{d\vec{V}_a}{dt} &= \frac{d_a}{dt} (\vec{V} + \vec{V}_e) = \frac{d_a}{dt} (\vec{V} + \vec{\Omega} \times \vec{r}) \\ &= \frac{d}{dt} (\vec{V} + \vec{\Omega} \times \vec{r}) + \vec{\Omega} \times (\vec{V} + \vec{\Omega} \times \vec{r}) \\ &= \frac{d\vec{V}}{dt} + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\ &= \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R} \end{aligned}$$

## 1.2 作用在海水微团上的外力运动方程的向量形式

压强梯度力:  $\frac{1}{\rho} \nabla p$

分子粘性力 (摩擦力):

$$\begin{cases} F_x = \frac{1}{\rho} \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \frac{\mu}{\rho} \Delta u \\ F_y = \frac{1}{\rho} \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = \frac{\mu}{\rho} \Delta v \\ F_z = \frac{1}{\rho} \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \frac{\mu}{\rho} \Delta w \end{cases} \Rightarrow \vec{F} = \frac{\mu}{\rho} \Delta \vec{V} = \gamma \Delta \vec{v}$$

重力 (地球引力与地球自转产生的惯性离心力的合力):  $\vec{g} = -G \frac{M_g}{r^2} \cdot \left( \frac{\vec{r}}{r} \right)$

科氏力:  $-2\vec{\Omega} \times \vec{V}$

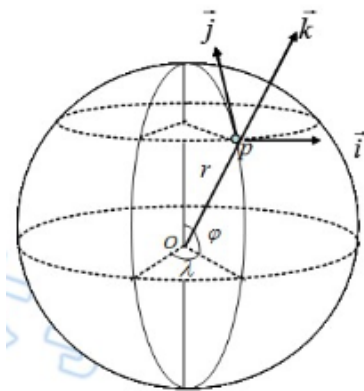
天体引潮力 (受其他天体万有引力与惯性力离心力的合力):  $\vec{F}_M = -G \frac{M_M}{L^2} + G \frac{M_M}{D^2} \cdot \left( \frac{\vec{D}}{D} \right)$

由牛顿第二定律和坐标系转换关系:

$$\begin{cases} \frac{d_a \vec{V}_a}{dt} = \sum_i \vec{F}_i \\ \frac{d_a \vec{A}_a}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R} \end{cases}$$

$$\Rightarrow \frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla P - 2\vec{\Omega} \times \vec{V} + \vec{g} + \nu \Delta \vec{V} + \vec{F}_T$$

### 1.3 运动方程在球坐标系的标量形式



速度:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\Rightarrow \begin{cases} u = r \cos \varphi \frac{d\lambda}{dt} \\ v = r \frac{d\varphi}{dt} \\ w = \frac{dr}{dt} \end{cases}$$

加速度:

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{\frac{\partial \vec{A}}{\partial t} dt + \frac{\partial \vec{A}}{\partial \lambda} d\lambda + \frac{\partial \vec{A}}{\partial \varphi} d\varphi + \frac{\partial \vec{A}}{\partial r} dr}{dt} \\ &= \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{A}}{\partial \lambda} \frac{d\lambda}{dt} + \frac{\partial \vec{A}}{\partial \varphi} \frac{d\varphi}{dt} + \frac{\partial \vec{A}}{\partial r} \frac{dr}{dt} \\ &= \frac{\partial \vec{A}}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial \vec{A}}{\partial \lambda} + \frac{v}{r} \frac{\partial \vec{A}}{\partial \varphi} + w \frac{\partial \vec{A}}{\partial r} \\ \Rightarrow \frac{d}{dt} &= \frac{\partial}{\partial t} + u \frac{\partial}{r \cos \varphi \partial \lambda} + v \frac{\partial}{r \partial \varphi} + w \frac{\partial}{\partial r} \\ \Rightarrow \frac{d}{dt} &= \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla) \\ \Rightarrow \nabla &= \frac{\partial}{r \cos \varphi \partial \lambda} \vec{i} + \frac{\partial}{r \partial \varphi} \vec{j} + \frac{\partial}{\partial r} \vec{k} \\ \Rightarrow \begin{cases} \frac{d\vec{i}}{dt} = \frac{\partial \vec{i}}{\partial t} + u \frac{\partial \vec{i}}{r \cos \varphi \partial \lambda} + v \frac{\partial \vec{i}}{r \partial \varphi} + w \frac{\partial \vec{i}}{\partial r} \\ \frac{d\vec{j}}{dt} = \frac{\partial \vec{j}}{\partial t} + u \frac{\partial \vec{j}}{r \cos \varphi \partial \lambda} + v \frac{\partial \vec{j}}{r \partial \varphi} + w \frac{\partial \vec{j}}{\partial r} \\ \frac{d\vec{k}}{dt} = \frac{\partial \vec{k}}{\partial t} + u \frac{\partial \vec{k}}{r \cos \varphi \partial \lambda} + v \frac{\partial \vec{k}}{r \partial \varphi} + w \frac{\partial \vec{k}}{\partial r} \end{cases} \end{aligned}$$

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\vec{i} + \frac{dv}{dt}\vec{j} + \frac{dw}{dt}\vec{k} + u\frac{d\vec{i}}{dt} + v\frac{d\vec{j}}{dt} + w\frac{d\vec{k}}{dt}$$

$$\Rightarrow \frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv \tan \varphi}{r} + \frac{uw}{r}\right)\vec{i} + \left(\frac{dv}{dt} + \frac{u^2 \tan \varphi}{r} + \frac{vw}{r}\right)\vec{j} + \left(\frac{dw}{dt} - \frac{u^2 + v^2}{r}\right)\vec{k}$$

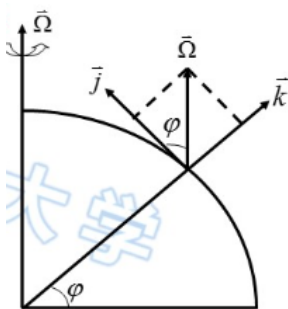
压强梯度力:

$$\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \left( \frac{1}{r \cos \varphi} \frac{\partial p}{\partial \lambda} \vec{i} + \frac{1}{r} \frac{\partial p}{\partial \varphi} \vec{j} + \frac{\partial p}{\partial r} \vec{k} \right)$$

重力:

$$\vec{g} = -g\vec{k}$$

科氏力:



$$\vec{\Omega} = \Omega \sin \varphi \vec{k} + \Omega \cos \varphi \vec{j}$$

$$-2\vec{\Omega} \times \vec{V} = -2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \Omega \cos \varphi & \Omega \sin \varphi \\ u & v & w \end{vmatrix}$$

$$= -2[(w\Omega \cos \varphi - v\Omega \sin \varphi)\vec{i} + (u\Omega \sin \varphi)\vec{j} + (-u\Omega \cos \varphi)\vec{k}]$$

$$\Rightarrow -2\vec{\Omega} \times \vec{V} = (fv - \tilde{f}w)\vec{i} - (fu)\vec{j} + (\tilde{f}u)\vec{k}$$

其中,  $\begin{cases} f = 2\Omega \sin \varphi \\ \tilde{f} = 2\Omega \cos \varphi \end{cases}$

$$\Rightarrow \begin{cases} \frac{du}{dt} = -\frac{1}{\rho r \cos \varphi} \frac{\partial p}{\partial \lambda} + fv - \tilde{f}w + \frac{uv \tan \varphi}{r} - \frac{uw}{r} + \gamma(\Delta \vec{v})_\lambda - \frac{1}{r \cos \varphi} \frac{\partial \phi_T}{\partial \lambda} \\ \frac{dv}{dt} = -\frac{1}{\rho r} \frac{\partial p}{\partial \varphi} - fu - \frac{u^2 \tan \varphi}{r} - \frac{vw}{r} + \gamma(\Delta \vec{v})_\varphi - \frac{1}{r} \frac{\partial \phi_T}{\partial \varphi} \\ \frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \tilde{f}u = g + \frac{u^2 + v^2}{r} + \gamma(\Delta \vec{v})_r - \frac{\partial \phi_T}{\partial r} \end{cases}$$

## 1.4 直角坐标系的运动方程

略去地球曲率的影响

$$\Rightarrow \begin{cases} \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv + \tilde{f}w + F_{N\lambda} + F_{T\lambda} \\ \frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu + F_{Ny} + F_{Ty} \\ \frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \tilde{f}u - g + F_{Nz} + F_{Tz} \end{cases}$$

## 1.5 海水层流运动的基本方程组

### 1.5.1 连续方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \Leftrightarrow \frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0$$

特别地, 对于不可压缩流体:

$$\nabla \cdot \vec{V} = 0$$

### 1.5.2 盐量扩散方程

$$\begin{aligned}
& \begin{array}{ccc} \text{盐量增加量} & \text{平流作用} & \text{分子扩散作用} \\ \frac{\partial}{\partial t} \iiint_{\tau} \rho s d\tau & = - \oiint_{\sigma} \rho s V_n d\sigma + & - \oiint_{\sigma} S_n d\sigma \end{array} \\
& \iiint_{\tau} \frac{\partial(\rho s)}{\partial t} d\tau = \iiint_{\tau} \nabla \cdot (\rho s \vec{V}) d\tau - \iiint_{\tau} \nabla \cdot \vec{S} d\tau \\
& \Rightarrow \frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s \vec{V}) + \nabla \cdot \vec{S} = 0 \\
& \Rightarrow \rho \frac{\partial s}{\partial t} + s \frac{\partial \rho}{\partial t} + s \nabla \cdot (\rho \vec{V}) + \rho \vec{V} \cdot \nabla s + \nabla \cdot \vec{S} = 0 \\
& \Rightarrow \left( \frac{\partial s}{\partial t} + \vec{V} \cdot \nabla s \right) + \frac{s}{\rho} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] = -\frac{1}{\rho} \nabla \cdot \vec{S} \\
& \Rightarrow \frac{\partial s}{\partial t} + \vec{V} \cdot \nabla s = \frac{k}{\rho} \Delta s = k_D \Delta s
\end{aligned}$$

其中,  $k_D = \frac{k}{\rho} \sim 1.1 \times 10^{-9} \text{ (m}^2/\text{s)}$

### 1.5.3 热传导方程

与上面类似:

$$\frac{\partial \theta}{\partial t} + \vec{V} \cdot \nabla \theta = \frac{\kappa}{\rho c_p} \Delta \theta = k_{\theta} \Delta \theta$$

其中,  $k_{\theta} = \frac{\kappa}{\rho c_p} \sim 1.4 \times 10^{-7} \text{ (m}^2/\text{s)}$

### 1.5.4 热膨胀方程-状态方程

热膨胀方程:

$$\rho = \rho_0 \left( 1 - \frac{\theta}{k} \right)$$

0°C 时的海水密度      海水的膨胀系数

EOS80 国际海水状态方程:

$$\rho(s, t, p) = \rho(s, t, 0) \left[ 1 - \frac{np}{k(s, t, p)} \right]^{-1}$$

## 1.6 基本方程的矢量形式和标量形式

矢量形式:

$$\begin{cases} \frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla p - 2\Omega \times \vec{V} + \vec{g} + \gamma \Delta \vec{V} - \nabla \phi_T \\ \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{V} = 0 \\ \frac{\partial s}{\partial t} + \vec{V} \cdot \nabla s = k_D \Delta s \\ \frac{\partial \theta}{\partial t} + \vec{V} \cdot \nabla \theta = k_{\theta} \Delta \theta \\ \rho = \rho(\theta, s, p) \end{cases}$$

标量形式 (直角坐标系):

$$\begin{cases} \frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu + \gamma \Delta v - \frac{\partial \phi_T}{\partial y} \\ \frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \tilde{f}u - g + \gamma \Delta w - \frac{\partial \phi_T}{\partial z} \\ \frac{d\rho}{dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \\ \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} = k_D \left( \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \right) \\ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = k_{\theta} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \\ \rho = \rho(\theta, s, p) \end{cases}$$

## 1.7 边界条件

无质量交换的运动学边界条件：

$$\frac{\partial F}{\partial t} + \vec{c} \cdot \nabla F = 0$$

例：

$$(1) \text{ 海面 } (z = \zeta(x, y, t)): \frac{\partial \zeta}{\partial t} + \vec{V}_H \cdot \nabla_H \zeta - w = 0$$

$$(2) \text{ 海底 } (z = -h(x, y)): \vec{V}_H \cdot \nabla_H h + w = 0$$

动力学边界条件：

由牛顿第三定律，在界面法线方向有：

$$(\vec{p}_n)_1 = (\vec{p}_n)_2$$

## 1.8 \* 时间平均的基本方程和边界条件 (直角坐标系)

连续方程：

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

运动方程：

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + f\bar{v} - \bar{f}\bar{w} + \gamma \Delta \bar{u} - \frac{\partial \bar{\phi}_T}{\partial x} + \frac{\partial}{\partial x} \left( A_{x\alpha} \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left( A_{xy} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial z} \left( A_{xz} \frac{\partial \bar{u}}{\partial z} \right) \\ \frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} - f\bar{u} + \gamma \Delta \bar{v} - \frac{\partial \bar{\phi}_T}{\partial y} + \frac{\partial}{\partial x} \left( A_{yx} \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left( A_{yy} \frac{\partial \bar{v}}{\partial y} \right) + \frac{\partial}{\partial z} \left( A_{yz} \frac{\partial \bar{v}}{\partial z} \right) \\ \frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} + \bar{f}\bar{u} - g + \gamma \Delta \bar{w} - \frac{\partial \bar{\phi}_T}{\partial z} + \frac{\partial}{\partial x} \left( A_{2x} \frac{\partial \bar{w}}{\partial x} \right) + \frac{\partial}{\partial y} \left( A_{zy} \frac{\partial \bar{w}}{\partial y} \right) + \frac{\partial}{\partial z} \left( A_{zz} \frac{\partial \bar{w}}{\partial z} \right) \end{cases}$$

盐量扩散方程：

$$\frac{\partial \bar{s}}{\partial t} + \bar{u} \frac{\partial \bar{s}}{\partial x} + \bar{v} \frac{\partial \bar{s}}{\partial y} + \bar{w} \frac{\partial \bar{s}}{\partial z}$$

热传导方程：

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{u} \frac{\partial \bar{\theta}}{\partial x} + \bar{v} \frac{\partial \bar{\theta}}{\partial y} + \bar{w} \frac{\partial \bar{\theta}}{\partial z} = k_\theta \Delta \bar{\theta} + \frac{\partial}{\partial x} \left( K_{\theta x} \frac{\partial \bar{\theta}}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_{\theta y} \frac{\partial \bar{\theta}}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_{\theta z} \frac{\partial \bar{\theta}}{\partial z} \right)$$

状态方程：

$$\bar{\rho} = \bar{\rho}(\bar{s}, \bar{\theta}, \bar{p})$$

## 1.9 铅直向平均的基本方程

$$\begin{aligned} \frac{\partial}{\partial x} [(h + \zeta) \langle u \rangle] + \frac{\partial}{\partial y} [(h + \zeta) \langle v \rangle] - \left[ u|_\zeta \frac{\partial \zeta}{\partial x} + v|_\zeta \frac{\partial \zeta}{\partial y} - w|_\zeta \right] - \left[ u|_{-h} \frac{\partial h}{\partial x} + v|_{-h} \frac{\partial h}{\partial y} + w|_{-h} \right] = 0 \\ \frac{\partial \zeta}{\partial t} + \frac{\partial [(h + \zeta) \langle u \rangle]}{\partial x} + \frac{\partial [(h + \zeta) \langle v \rangle]}{\partial y} = 0 \end{aligned}$$

## 1.10 尺度分析

$$\text{Rossby 数 } \text{Ro} = \frac{U}{FL} \begin{cases} \gg 1: \text{平流非线性项比 Coriolis 力重要, 大尺度运动} \\ = 1: \text{平流非线性项与 Coriolis 力同等重要} \\ \ll 1: \text{平流非线性项可以忽略, 小尺度运动} \end{cases}$$

$$\text{水平 Ekman 数 } E_l = \frac{A_l}{FL^2} \text{ 水平湍流摩擦项与 Coriolis 力比值}$$

$$\text{垂直 Ekman 数 } E_z = \frac{A_z}{FD^2} \text{ 垂直湍流摩擦项与 Coriolis 力比值}$$

准静力近似  $f$  平面近似  $\beta$  平面近似 Boussinesq 近似

## 2 海流

### 2.1 地转流

地转流：不考虑海面风的作用，远离沿岸的大洋中部大尺度、准水平、定常的海水流动。

产生原因：海水受热力和动力因素导致压力 (和密度) 在水平方向分布不均匀。

$$p = p_a + \rho gh \quad \rho \begin{cases} \neq \rho_0 \Rightarrow \text{梯度流} \\ = \rho_0 \Rightarrow \text{倾斜流} \end{cases}$$

#### 2.1.1 梯度流

假定和方程

(1) 在相当长一段时间里海面温度变化和降水蒸发变化都不大，于是可以认为已形成的海水密度场、温度场和盐度场近似于定常，从而相应的海水运动也近似于定常： $\frac{\partial}{\partial t} = 0$ 。

(2) 海洋深而宽广，在远离海岸及海底的大洋中部海区，大尺度运动： $Ro \ll 1$ 。

(3) 不考虑海底摩擦及边界摩擦的影响，且海面无风力作用，则流动属一种无摩擦流动： $E_l, E_z \ll 1$ 。

(4)  $\beta$  平面近似准静力近似

$x$  方向基本方程：

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv + \frac{\partial}{\partial x} \left( A_{xx} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( A_{xy} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( A_{xz} \frac{\partial u}{\partial z} \right)$$

$$\text{假定 (1)} \Rightarrow \frac{\partial u}{\partial t} = 0$$

$$\text{假定 (2)} \Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 0$$

$$\text{假定 (3)} \Rightarrow \frac{\partial}{\partial x} \left( A_{xx} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( A_{xy} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( A_{xz} \frac{\partial u}{\partial z} \right) = 0$$

可得梯度流的控制方程：

$$\begin{cases} 0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = 0 \\ u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} = 0 \\ \rho = \rho(s, \theta) \end{cases}$$

特征

水平速度：

$$\begin{cases} 0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \end{cases} \quad (1)$$

$$\Rightarrow \begin{cases} v = \frac{1}{\rho f} \frac{\partial p}{\partial x} \\ u = -\frac{1}{\rho f} \frac{\partial p}{\partial y} \end{cases} \quad (2)$$

- (1) 水平速度和压强梯度成正比；
- (2) 与密度和科氏参数成反比；
- (3) 地转关系在赤道不成立 ( $f = 0$ ).

垂向速度：

$$\begin{aligned}
 \frac{\partial(1) \times \rho}{\partial y} - \frac{\partial(2) \times \rho}{\partial x} &\Leftrightarrow \frac{\partial(\rho f v)}{\partial v} + \frac{\partial(\rho f u)}{\partial x} = 0 \\
 &\Leftrightarrow f \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + f \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) - f \rho \frac{\partial w}{\partial z} - f w \frac{\partial \rho}{\partial z} + \beta \rho v = 0 \\
 &\Leftrightarrow f \rho \stackrel{=0}{\boxed{\nabla \vec{V}}} + f \vec{V} \cdot \nabla \rho - f \rho \frac{\partial w}{\partial z} - f w \frac{\partial \rho}{\partial z} + \beta \rho v = 0
 \end{aligned} \tag{3}$$

$$\vec{V} \cdot \nabla \rho = \vec{V} \cdot \nabla \rho(s, \theta) = \vec{V} \cdot \left( \nabla_s \frac{\partial \rho}{\partial s} + \nabla_\theta \frac{\partial \rho}{\partial \theta} \right) = 0$$

$$(3) \Leftrightarrow f \left( \rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z} \right) = \beta \rho v \stackrel{\text{尺度分析}}{\Rightarrow} f \frac{\partial w}{\partial z} = \beta v \stackrel{\text{尺度分析}}{\Rightarrow} W = \frac{\beta D}{F} U \sim 2 \times 10^{-4} U$$

垂向流速比水平流速小得多，地转流为准水平运动.

### 运动特性

$$(1) \times u + (2) \times v \Leftrightarrow u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0 \Leftrightarrow \vec{V}_H \cdot \nabla_H p = 0$$

- (1) 梯度流平行于等压线；
- (2) 北半球，流向右侧为高压，南半球相反；

### 密度特性

$$\frac{\partial(1) \times \rho}{\partial y} - \frac{\partial(2) \times \rho}{\partial x} \Leftrightarrow f \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + f \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) + \rho u \frac{\partial f}{\partial x} + \rho v \frac{\partial f}{\partial y} = 0 \Leftrightarrow \vec{V}_H \cdot \nabla_H \rho = 0$$

- (1) 梯度流近似平行于等密线；
- (2) 在北半球，流向右侧密度小；
- (3) 等压面倾斜与等密面倾斜方向相反.

### 温盐特性

忽略垂向运动：

$$\vec{V}_H \cdot \nabla_H \theta = 0$$

$$\vec{V}_H \cdot \nabla_H s = 0$$

- (1) 梯度流平行于等温线和等盐线；
- (2) 在北半球，流向右侧温度高，盐度低.

### 2.1.2 倾斜流

假定和方程 (1) 海水密度为常数；

(2) 水平方向的压强梯度是由海面倾斜引起的.

$$\Rightarrow p = p_a + \int_z^\zeta \rho g dz = p_a + \rho g(\zeta - z)$$

倾斜流的控制方程：

$$\begin{cases} f v = g \frac{\partial \zeta}{\partial x} \\ f u = -g \frac{\partial \zeta}{\partial y} \end{cases} \tag{4}$$

$$\tag{5}$$



性质：

$$(4) \times u - (5) \times v \Leftrightarrow u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = 0 \Leftrightarrow \vec{V}_H \cdot \nabla \zeta = 0$$

- (1) 倾斜流平行于等水位线；
- (2) 在北半球，流向右侧水位高；
- (3) 倾斜流从表至底流速流向相同，压强梯度相同。

## 2.2 Ekman 漂流

由恒速定常的风长时间驱动大尺度、均匀密度的海洋，所产生的处于稳定状态的海流。

### 2.2.1 无限深海漂流

#### 物理背景

Ekman 的老师 Nansen 在海洋调查时发现，冰山不是顺风漂移，而是沿着风向右方  $20^\circ \sim 40^\circ$  的方向移动。Ekman 在 1905 年研究了这种现象并提出风海流理论 [Ekman, 1905]。

#### 假定

无限深海 Ekman 漂流中用到了以下假定：

- 1) 海洋无限广阔，海洋无限深。

即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数  $A_z$  为常量。由于海洋无限深， $z \rightarrow \infty, \vec{V} = 0$

- 2) 定常均匀风场长时间作用。

即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度。

- 3) 密度分布均匀， $\rho$  为常数，不考虑热盐性质。

- 4) 采用  $f$  平面近似。

#### 方程推导

#### 控制方程和边界条件

首先给出一般的控制方程：

$$\begin{cases} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv + A_l \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + A_z \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu + A_l \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + A_z \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases} \quad (6)$$

由假定 1)， $A_l \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$ ， $A_l \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$ ；

由假定 2)， $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 0$ ， $\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = 0$ ；

由假定 3)， $-\frac{1}{\rho} \frac{\partial p}{\partial x} = 0$ ， $-\frac{1}{\rho} \frac{\partial p}{\partial y} = 0$

则 (6) 化为：

$$\begin{cases} 0 = fv + A_z \frac{\partial^2 u}{\partial z^2} \end{cases} \quad (7a)$$

$$\begin{cases} 0 = -fu + A_z \frac{\partial^2 v}{\partial z^2} \end{cases} \quad (7b)$$

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases} \quad (7c)$$

不失一般性地，假定风力仅沿  $y$  方向作用，即  $\tau_x = 0, \tau_y = \text{const.}$  再结合假定 1)，控制方程的边界条件为：

$$\begin{cases} z = 0, \rho A_z \frac{\partial u}{\partial z} = 0 & (8a) \\ z = 0, \rho A_z \frac{\partial v}{\partial z} = -\tau_y & (8b) \\ z = \infty, u = v = 0 & (8c) \end{cases}$$

方程求解

$$(7a) + (7b) \times i \Leftrightarrow A_z \frac{\partial^2(u + iv)}{\partial z^2} = if(u + iv)$$

令  $W = u + iv$ ，得：

$$A_z \frac{\partial^2 W}{\partial z^2} = ifW \Rightarrow \frac{\partial^2 W}{\partial z^2} = \frac{(1+i)^2 \Omega \sin \varphi}{A_z} W$$

令  $a = \sqrt{\Omega \sin \varphi / A_z}$ ， $j^2 = (1+i)^2 a^2$ ，得：

$$\frac{d^2 W}{dz^2} - j^2 W = 0 \quad (9)$$

(9) 式通解为： $W = Ae^{jz} + Be^{-jz}$

结合边界条件： $(8a) + (8b) \times i \Rightarrow z = 0, \rho A_z \frac{\partial W}{\partial z} = -\tau_y, z \rightarrow \infty, W = 0$

$$z \rightarrow \infty \Rightarrow A = 0, W = Be^{-jz}; z = 0, \rho A_z \frac{\partial W}{\partial z} \Big|_{z=0} = \rho A_z \frac{\partial (Be^{-jz})}{\partial z} \Big|_{z=0} = -\tau_y, \Rightarrow B = \tau_y / (j\rho A_z)$$

因此，方程的解为：

$$W = \frac{\tau_y}{j\rho A_z} e^{-jz} = \frac{i\tau_y}{(1+i)a\rho A_z} e^{-(1+i)az} = \frac{e^{i\frac{\pi}{2}} \tau_y}{\sqrt{2} e^{i\frac{\pi}{4}} a\rho A_z} e^{-(1+i)az}$$

令  $D_0 = \pi/a$ ，得到最终解的形式为：

$$W = \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{-\frac{\pi}{D_0}z + i(\frac{\pi}{4} - \frac{\pi}{D_0}z)} \quad (10)$$

物理性质

运动速度

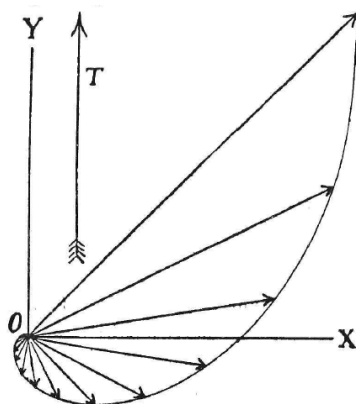
在海面 ( $z = 0$ ) 处， $W_0 = \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{i\frac{\pi}{4}}$ 。大小为  $|W_0| = \frac{\tau_y}{\sqrt{2}a\rho A_z}$ ，方向与  $x$  轴成  $45^\circ$ ，即与风向向右偏  $45^\circ$ 。

在任意深度处， $|W_z| = \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{-\frac{\pi}{D_0}z}$ ，方向为  $\frac{\pi}{4} - \frac{\pi}{D_0}z$ ，即流速随深度增加呈指数形式减小，流向随深度的增加而逐渐向右偏。

在摩擦深度  $z = D_0$  处， $|W_{D_0}| = \frac{\tau_y e^{-\pi}}{\sqrt{2}a\rho A_z} = e^{-\pi}|W_0| = 0.043|W_0|$ ，方向  $-\frac{3}{4}\pi$ ，即与  $x$  轴成  $-135^\circ$ ，与表面流向正好相反。

**Ekman 螺旋和 Ekman 螺线**

根据速度的垂向分布，表层流速最大，流向偏向风向的右方  $45^\circ$ ；随深度增加，流速逐渐减小，流向逐渐右偏；到摩擦深度，流速是表面流速的 4.3%，流向与表面流向相反，运动可以忽略。连接各层流速的矢量端点，构成 Ekman 螺旋；Ekman 螺旋在平面上的投影，称为 Ekman 螺线 [Ekman, 1905]。



水平体积输运

体积输运：

$$\begin{aligned}
 S &= \int_0^\infty W dz \\
 &= \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{i\frac{\pi}{4}} \int_0^\infty e^{-\frac{\pi}{D_0}(1+i)z} dz \\
 &= \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{i\frac{\pi}{4}} \left[ -\frac{D_0}{\pi} \frac{1}{(1+i)} \right] e^{-\frac{\pi}{D_0}(1+i)z} \Big|_0^\infty \\
 &= \frac{\tau_y}{2\Omega \sin \varphi \rho} = \frac{\tau_y}{f\rho}
 \end{aligned}$$

可以发现，得到的输运结果只有实部，没有虚部，说明体积输运方向为  $x$  轴正向，即在北半球水体向风向右侧  $90^\circ$  输运。

## 2.2.2 有限深海漂流

假定

有限深海 Ekman 漂流中用到了以下假定：

1) 海区无限广阔、有限深，远离海岸。

即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数  $A_z$  为常量。由于海洋有限深， $z \rightarrow h, \vec{V} = 0$

2) 定常均匀风场长时间作用。

即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度。

3) 密度分布均匀， $\rho$  为常数，不考虑热盐性质。

4) 采用  $f$  平面近似。

控制方程和边界条件：

$$\begin{cases}
 0 = fv + A_z \frac{\partial^2 u}{\partial z^2} \\
 0 = -fu + A_z \frac{\partial^2 v}{\partial z^2} \\
 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\
 z = 0 : \rho A_z \frac{\partial u}{\partial z} = 0, \rho A_z \frac{\partial v}{\partial z} = -\tau_y \\
 z \rightarrow h : u = v = 0
 \end{cases}$$

方程求解 令  $\xi = h - z$ ，定解问题化为：

$$\begin{cases}
 -fv = A_z \frac{\partial^2 u}{\partial \xi^2} \\
 fu = A_z \frac{\partial^2 v}{\partial \xi^2}
 \end{cases} \quad (11)$$

$$\begin{cases}
 \xi = h : \rho A_z \frac{\partial u}{\partial \xi} = 0, \rho A_z \frac{\partial v}{\partial \xi} = \tau_y \\
 \xi \rightarrow 0 : u = v = 0
 \end{cases} \quad (12)$$

令  $W = u + iv, \tau = \tau_x + i\tau_y$ ，控制方程：

$$(11) + (12) \times i \Leftrightarrow \frac{d^2 W}{d\xi^2} - j^2 W = 0$$

边界条件：

$$\xi = h : \rho A_z \frac{\partial W}{\partial \xi} = \tau$$

$$\xi = 0 : W = 0$$

与无限深海漂流解法类似，解得：

$$W = \frac{(1+i)\tau_y}{2a\rho A_z} \frac{sh(1+i)a\xi}{ch(1+i)ah}$$

物理性质

与水深的关系

(1)  $h \geq 2D_0$  时，有限深海漂流流速流向与无限深海相同；(2) 水深越浅，流向随深度增加右偏（北半球）越缓慢；(3) 从上层到下层的流速矢量越是趋近风矢量的方向。

体积输运

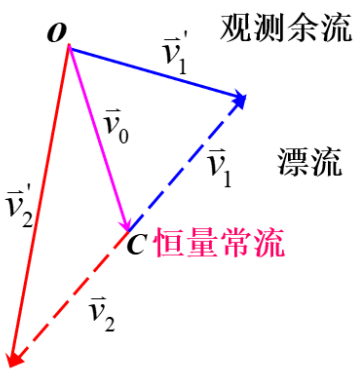
- (1) 在  $x, y$  方向（平行和垂直风向）都有输送；
- (2) 运输方向为风向右端， $\pm 90^\circ$  之间：

$$S_x > 0; 0 < h < D_0, ah < \pi, S_y > 0; D_0 < h < 2D_0, \pi < ah < 2\pi, S_y < 0; h > 2D_0, S_y = 0$$

2.2.3 漂流分离

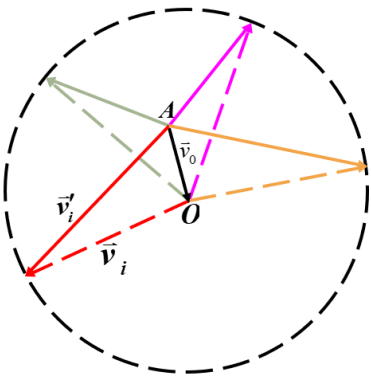
利用风速大小相等、方向相反的两组观测余流分离漂流

※ 余流 = 漂流 + 恒量常流



利用一组风速大小相等、方向不同的实测余流分离漂流

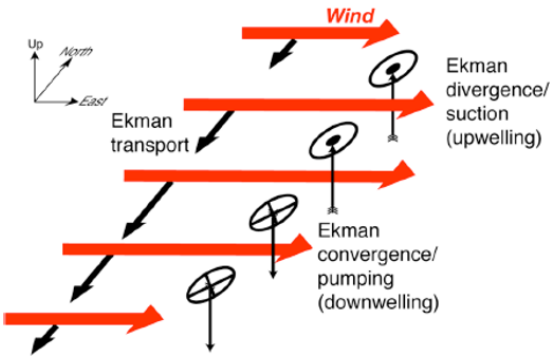
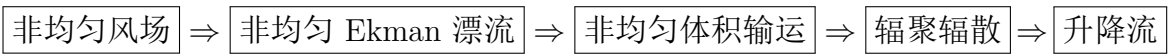
※ 漂流速度矢量端点落在同一圆周上



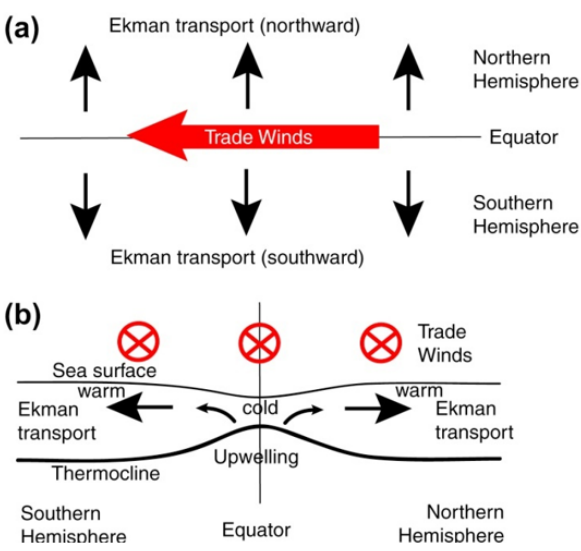
2.2.4 升降流

由不均匀风场或风场和地形配合产生的“较强烈”的铅直向流动。

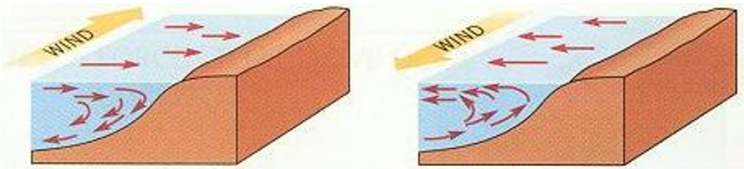
物理背景



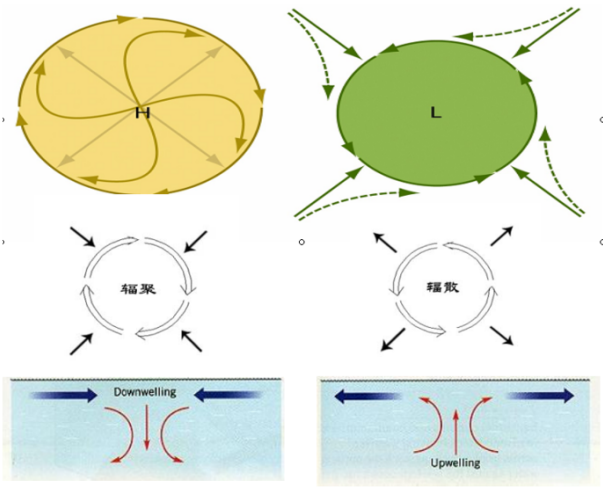
赤道附近的升降流



顺 (沿) 岸风产生的升降流



气旋和反气旋产生的海洋升降流



假定

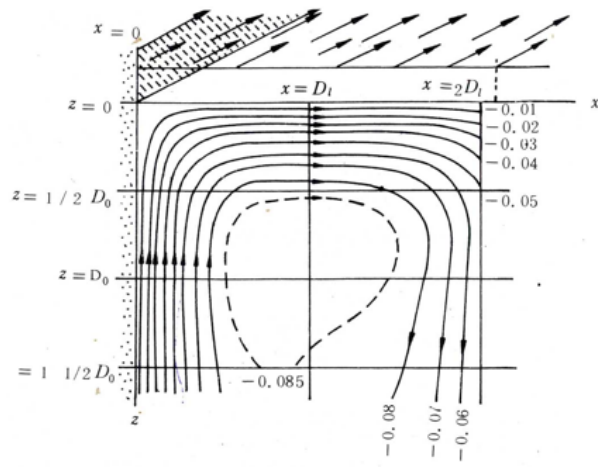
- (1)  $\rho$  为常数;
- (2) 直线风系, 风仅沿  $x$  方向有变化; 风区内为恒定的均匀风场; 风区外无风;  $\frac{\partial}{\partial y} = 0$
- (3) 定常风场;  $\frac{\partial}{\partial t} = 0$
- (4) 大尺度;  $Ro \ll 1$
- (5) 有限深度.  $h \geq 2D_0$

控制方程及边界条件

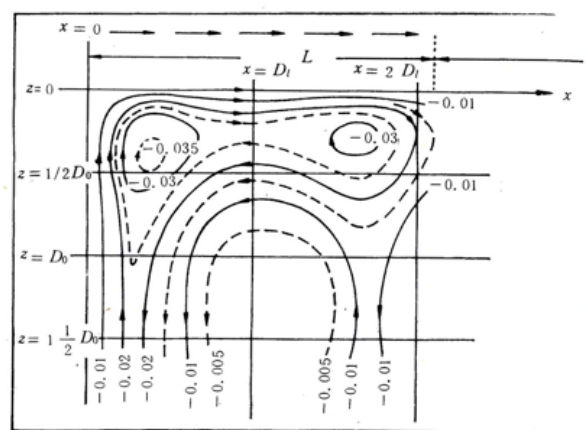
$$\begin{cases} A_l \frac{\partial^2 u}{\partial x^2} + A_z \frac{\partial^2 u}{\partial z^2} + f v + g \frac{\partial \zeta}{\partial x} = 0 \\ A_l \frac{\partial^2 v}{\partial x^2} + A_z \frac{\partial^2 v}{\partial z^2} - f u = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

$$\begin{cases} z = \zeta : \rho A_z \frac{\partial u}{\partial z} = 0, \quad \rho A_z \frac{\partial v}{\partial z} = -\tau_y \quad (0 \leq x \leq L) \\ z = h : u = v = 0 \\ x = 0 : u = v = 0 \\ x \rightarrow \infty : u = v = 0, \quad \frac{\partial \zeta}{\partial x} = 0 \end{cases}$$

结果讨论



- (1) 近岸产生上升流  $x \leq 0.5D_l$ ;
  - (2) 风区外延附近下降流  $x = 2D_l$ ;
  - (3) 上升流来自  $z = 1.5D_0$  或更深;
  - (4) 最大  $w$  出现在  $z = D_0$ ;
  - (5) 上层离岸流, 下层向岸流, 构成一个循环.
- 若风向与海岸成  $\theta$  角:



- (1) 三个升降流系统: 两个顺时针, 一个逆时针;
- (2) 大顺时针循环;
- (3)  $\theta = 21.5^\circ$  时, 升降流达最大强度;
- (4) 纬度越低, 升降流越强.

## 2.3 非定常运动

### 2.3.1 漂流的发展

假定

- (1) 远离海岸和海底的开阔大洋；
- (2) 风场均匀恒定；
- (3)  $\rho$  为常数；
- (4) 海面无倾斜；
- (5) 运动非定常。

控制方程

$$\begin{cases} \frac{\partial u}{\partial t} - fv = A_z \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial v}{\partial t} + fu = A_z \frac{\partial^2 v}{\partial z^2} \end{cases}$$

初边值条件

$$\begin{cases} z = 0 : A_z \frac{\partial u}{\partial z} = 0, \rho A_z \frac{\partial v}{\partial z} = -\tau_y (t > 0) \\ z \rightarrow \infty : u = v = 0 \\ t = 0 : u = v = 0 / u = C_1, v = C_2 \end{cases}$$

解的讨论

$$\begin{cases} u = \frac{2\pi\tau_y}{\rho f D_0} \int_0^{t'} \frac{\sin(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \\ v = \frac{2\pi\tau_y}{\rho f D_0} \int_0^{t'} \frac{\cos(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \end{cases}$$

根据下图 [Ekman, 1905]:

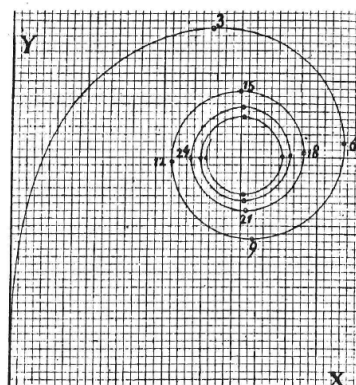


Fig. 3.  $z=0$ .

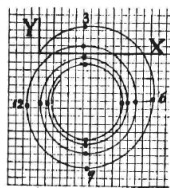


Fig. 4.  $z=0,5 D$ .

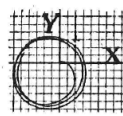


Fig. 5.  $z=D$ .



Fig. 6.  $z=2D$ .

随时间增加，空间某点流苏端点顺时针旋转（北半球），逐渐趋向一个极限值（即漂流）。

### 2.3.2 惯性流

假定和方程

- (1) 风场消失或者流动离开风区；
- (2) 外部驱动小时，湍摩擦失去作用；
- (3) 流动转为由惯性项维持平衡；

(4) 强制流动转变为自由流动.

$$\begin{cases} \frac{du}{dt} - fv = 0 \\ \frac{dv}{dt} + fu = 0 \end{cases} \quad (13)$$

$$\quad (14)$$

求解

$$\begin{aligned} (13) \times u + (14) \times v &\Leftrightarrow u \frac{du}{dt} + v \frac{dv}{dt} = \frac{1}{2} \frac{du^2}{dt} + \frac{1}{2} \frac{dv^2}{dt} = 0 \\ &\Leftrightarrow \frac{d}{dt} (u^2 + v^2) = 0 \\ &\Leftrightarrow u^2 + v^2 = c = V_0^2 \\ (13), (14) &\Rightarrow \begin{cases} v = \frac{dy}{dt} = \frac{1}{f} \frac{du}{dt} \\ u = \frac{dx}{dt} = \frac{1}{f} \frac{dv}{dt} \end{cases} \\ &\Rightarrow \begin{cases} y - y' = \frac{1}{f} (u - u') \\ x - x' = -\frac{1}{f} (v - v') \end{cases} \\ &\Rightarrow \begin{cases} y - \left( y' - \frac{u'}{f} \right) = \frac{u}{f} \\ x - \left( x' + \frac{v'}{f} \right) = -\frac{v}{f} \end{cases} \\ &\Rightarrow \boxed{(y - y_0)^2 + (x - x_0)^2 = \frac{1}{f^2} (u^2 + v^2) = \frac{V^2}{f^2} = r^2} \end{aligned}$$

流体质点沿半径为  $r$  的圆周作匀速运动，这个圆称之为惯性圆，对应的流动为惯性流.

惯性圆半径

$$\text{科氏力充当向心力: } \frac{V_0^2}{r} = fV_0 \Rightarrow V_0 = fr \Rightarrow r = \frac{V_0}{f} = \frac{V_0}{2\omega \sin \varphi}$$

随纬度增加而减小；赤道  $r \rightarrow \infty$ ，水质点作直线运动.

周期

$$T_i = \frac{2\pi r}{V_0} = \frac{2\pi r}{fr} = \frac{2\pi}{f} = \frac{\pi}{\omega \sin \varphi}$$

运动方向

北半球，顺时针；南半球，逆时针.

背景流

- (1) 当无其他外加流动时，所有惯性圆的圆心位于同一条铅直线上，因而海水就像以角速度  $2\omega \sin \varphi$  旋转的刚体一样；
- (2) 当有其他外加流动时，除了在同一水平面上所有海水质点皆在同一时刻由同一流速速率外，还依外加流动速度方向移动.



## 2.4 风生大洋环流

### 2.4.1 Sverdrup 理论

假定

- (1) 远离海岸的大洋中部海区， $Ro \ll 1$  大尺度、等深大洋  $h$  为常数；
- (2) 远离边界、无侧边界影响，无水平湍摩擦应力  $E_l \ll 1$ ；
- (3) 定常风定常流动；
- (4)  $\rho$  为常数；
- (5)  $\beta$  平面近似。

控制方程

$$\begin{cases} -fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_z \frac{\partial^2 u}{\partial z^2} \\ fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_z \frac{\partial^2 v}{\partial z^2} \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta(\text{海面}) : \rho A_z \frac{\partial u}{\partial z} = \tau_{x\zeta}, \rho A_z \frac{\partial v}{\partial z} = \tau_{y\zeta} \\ z = -h(\text{足够深}) : u = v = 0, \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \end{cases}$$

求解

对上式进行垂直积分：

$$\begin{cases} -fM_y = -\frac{\partial P}{\partial x} + \tau_{x\zeta} & (15) \\ fM_x = -\frac{\partial P}{\partial y} + \tau_{y\zeta} & (16) \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \end{cases}$$

$$\text{其中, } M_x = \int_{-h}^0 \rho u dz, M_y = \int_{-h}^0 \rho v dz, P = \int_{-h}^0 p dz, \tau_{x\zeta} = \int_{-h}^0 \rho A_z \frac{\partial^2 u}{\partial z^2} = \rho A_z \left( \frac{\partial u}{\partial z} \Big|_{z=0} - \frac{\partial u}{\partial z} \Big|_{z=-h} \right) \quad (\zeta \ll h)$$

$$\begin{aligned} \frac{\partial(1)}{\partial y} - \frac{\partial(2)}{\partial x} &\Leftrightarrow -M_y \frac{\partial f}{\partial y} \overset{0}{\boxed{-f \frac{\partial M_y}{\partial y} - f \frac{\partial M_x}{\partial x}}} = \frac{\partial \tau_{x\zeta}}{\partial y} - \frac{\partial \tau_{y\zeta}}{\partial x} \\ &\Leftrightarrow M_y \frac{\partial f}{\partial y} = \frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y} \\ &\Leftrightarrow \boxed{\beta M_y = \text{rot}_z \tau_\zeta} \\ &\text{Sverdrup 方程} \end{aligned}$$

**Sverdrup 方程的物理意义 1：** 海水南北向的输运由风应力旋度所驱动。

讨论

将质量输运分为 Ekman 漂流输运与地转流输运两部分：

$$\begin{aligned} M_x &= M_{xE}(\text{Ekman 漂流}) + M_{xg}(\text{地转流}) \\ M_y &= M_{yE}(\text{Ekman 漂流}) + M_{yg}(\text{地转流}) \end{aligned}$$

$$\begin{cases} -fM_{yE} = \tau_{x\zeta} & (17) \\ fM_{xE} = \tau_{y\zeta} & (18) \\ -fM_{yg} = -\frac{\partial P}{\partial x} & (19) \\ fM_{yg} = -\frac{\partial P}{\partial y} & (20) \end{cases}$$

$$\frac{\partial(17)}{\partial y} - \frac{\partial(18)}{\partial x} \Leftrightarrow \frac{\partial M_{xE}}{\partial x} + \frac{\partial M_{yE}}{\partial y} = \boxed{\nabla \cdot \vec{M}_E = (\text{rot}_z \vec{\tau}_\zeta - \beta M_{yE}) / f} \quad (21)$$

$$\frac{\partial(19)}{\partial y} - \frac{\partial(20)}{\partial x} \Leftrightarrow \frac{\partial M_{xg}}{\partial x} + \frac{\partial M_{yg}}{\partial y} = \boxed{\nabla \cdot \vec{M}_g = -\beta M_{yg} / f} \quad (22)$$

(1) Ekman 漂流质量输运的水平散度与 ① 风应力旋度 ②  $f$  ③  $\beta$  有关.

(2) 地转流质量输运的水平辐散引起南北向的地转运动.

$$(21) + (22) \Leftrightarrow \text{rot}_z \vec{\tau}_\zeta - \beta M_y = 0 \Leftrightarrow \boxed{\beta M_y = \text{rot}_z \vec{\tau}_\zeta}$$

**Sverdrup 方程的物理意义 2:** 地转流流量的散度和 Ekman 漂流流量的散度相平衡, 所以 Sverdrup 方程又称 Sverdrup 平衡.

(1) 所有南北向的地转运动, 必须显示水平散度;

(2) 虽然 Ekman 漂流流量的散度与地转流流量的散度本身不为 0, 但是它们的和, 即总流量的水平散度必须为 0, 说明 Ekman 漂流流量的散度和地转流流量的散度刚好取得平衡;

(3)  $\text{rot}_z \vec{\tau}_\zeta = 0$ : 只存在东西方向的输运,  $\text{rot}_z \vec{\tau}_\zeta > 0$ : 质量输运向北 (北半球),  $\text{rot}_z \vec{\tau}_\zeta < 0$ : 质量输运向南 (南半球);

(4) 地转流引起的南北质量运输量比 Ekman 漂流引起的大.

### 缺陷

设仅有纬向风, 且  $\tau_{x\zeta}$  仅为  $y$  的函数:

$$M_y = \frac{1}{\beta} \text{rot}_z \vec{\tau}_\zeta = \frac{1}{\beta} \left( \cancel{\frac{\partial \tau_{x\zeta}}{\partial x}} - \frac{\partial \tau_{x\zeta}}{\partial y} \right) = -\frac{a}{2\omega \cos \varphi} \frac{\partial \tau_{x\zeta}}{\partial y}$$

$$\frac{\partial M_x}{\partial x} = -\frac{\partial M_y}{\partial y} \Rightarrow M_x = \frac{x}{2\omega \cos \varphi} \left( a \frac{\partial^2 \tau_{xf}}{\partial y^2} + \frac{\partial \tau_{xf}}{\partial y} \tan \varphi \right) + c(y)$$

在东、西边界,  $M_x = 0$  而 Sverdrup 理论不能同时满足.

### 2.4.2 Stommel 理论

#### 假定

(1) 远离海岸的大洋中部海区,  $\text{Ro} \ll 1$  大尺度、等深大洋  $h$  为常数;

(2) 远离边界、无侧边界影响, 无水平湍摩擦应力  $E_l \ll 1$ ;

(3) 定常风定常流动;

(4)  $\rho$  为常数;

(5)  $\beta$  平面近似;

(6) 考虑底摩擦

#### 控制方程

$$\begin{cases} -fv = -g \frac{\partial \zeta}{\partial x} + A_z \frac{\partial^2 u}{\partial z^2} \\ fu = -g \frac{\partial \zeta}{\partial y} + A_z \frac{\partial^2 v}{\partial z^2} \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta(\text{海面}) : \tau_{x,\zeta} = \rho A_z \frac{\partial u}{\partial z} \Big|_{t=\zeta} = -F \cos(\pi y/b), \tau_{x,\zeta} = 0 \\ z = -h(\text{海底}) : \tau_{x,-h} = \rho + \frac{\partial u}{\partial z} \Big|_{t=-h} = \rho k u, \tau_{y,-h} = \rho + \frac{\partial v}{\partial z} \Big|_{t=-h} = \rho k v \end{cases}$$

求解

对上式进行垂直平均：

$$\begin{cases} -f \langle v \rangle = -g \frac{\partial \zeta}{\partial x} + \frac{A_z}{h+\zeta} \frac{\partial u}{\partial z} \Big|_{\zeta} - \frac{A_z}{h+\zeta} \frac{\partial u}{\partial z} \Big|_{-h} \\ f \langle u \rangle = -g \frac{\partial \zeta}{\partial y} + \frac{A_z}{h+\zeta} \frac{\partial v}{\partial z} \Big|_{\zeta} - \frac{A_z}{h+\zeta} \frac{\partial v}{\partial z} \Big|_{-h} \\ \frac{\partial}{\partial x} [(n+\zeta) \langle u \rangle] + \frac{\partial}{\partial y} [(n+\zeta) \langle v \rangle] = 0 \end{cases}$$

将边界条件代入方程：

$$\begin{cases} 0 = f \rho h v - F \cos(\pi y/b) - \rho k u - \rho g h \frac{\partial \zeta}{\partial x} \quad (\zeta \ll k) \end{cases} \quad (23)$$

$$\begin{cases} 0 = -f \rho h u - \rho k v - \rho g h \frac{\partial \zeta}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases} \quad (24)$$

$$\frac{\partial(23)}{\partial y} - \frac{\partial(24)}{\partial x} \Rightarrow \frac{h}{k} [\beta v + r \sin(\pi y/b)] + \overbrace{\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}^{\text{来源于底摩擦}} = 0$$

引入流函数  $\psi : u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$ ：

$$\overbrace{\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)}^{\text{来源于底摩擦}} + \frac{h}{k} \beta \frac{\partial \psi}{\partial x} = \frac{h}{k} r \sin \frac{\pi y}{b}$$

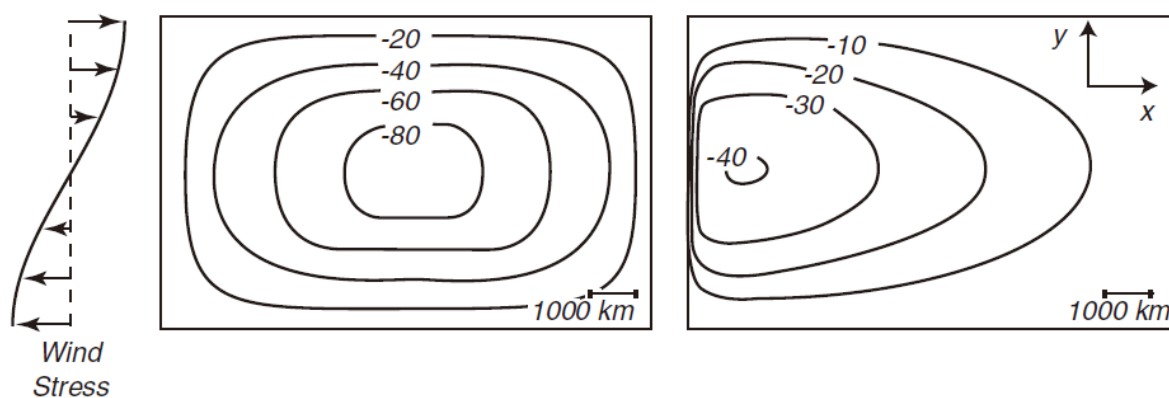
边界条件：

$$\psi(0, y) = \psi(a, y) = \psi(x, 0) = \psi(x, b) = 0$$

$$\Rightarrow \psi(x, y) = \frac{Fb}{k\pi} \sin \frac{\pi y}{b} \left[ \frac{e^{\frac{h\beta}{2k}(a-x)} \text{sh } \alpha x + e^{\frac{h\beta}{2k}x} \text{sh } \alpha(a-x)}{\text{sh } \alpha a} - 1 \right]$$

$$\text{特别地，若 } \beta = 0: \psi(x, y) = \frac{Fb}{k\pi} \sin \frac{\pi y}{b} \left[ \frac{\text{sh } \frac{\pi}{b} x + \text{sh } \frac{\pi}{b} (a-x)}{\text{sh } \frac{\pi}{b} a} - 1 \right]$$

讨论 图片来自 *Introduction to Physical Oceanography* (Robert H. Stewart, 2008 pp190)



※ $\beta$ 效应导致了西向强化现象.

$\beta = 0$  (非旋转坐标系或 f 平面近似):

- (1) 流线南北对称;
- (2) 流线东西对称.

$\beta \neq 0$  ( $\beta$  平面近似):

- (1) 流线南北对称;
- (2) 流线东西不对称, 西部密集, 东部稀疏.

### 2.4.3 惯性理论

#### 方程

在 Sverdrup 理论的控制方程中引入惯性项：

$$\begin{cases} \frac{du}{dt} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_z \frac{\partial^2 u}{\partial^2 z} \\ \frac{dv}{dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_z \frac{\partial^2 v}{\partial^2 z} \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

求解

内区 (中部区)

满足 Sverdrup 理论：  $\beta M_y = \text{rot}_z \tau_\zeta$ ,  $M_y = \frac{\partial \varphi}{\partial x} = \frac{1}{\beta} \text{rot}_z \tau_\zeta = \frac{1}{\beta} \left( \frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y} \right)$

又设风应力：  $\tau_{x\zeta} = -W \left( 1 - \frac{y^2}{s^2} \right)$ ,  $\tau_{y\zeta} = 0$  ( $0 \leq y \leq s$ )

$$\begin{aligned} \Rightarrow \frac{\partial \varphi}{\partial x} &= -\frac{2W}{\beta s^2} y \\ \Rightarrow \varphi &= -\frac{2W}{\beta s^2} yx + C(y) \\ (x = r : \varphi = 0) \quad 0 &= -\frac{2W}{\beta s^2} yr + C(y) \\ \Rightarrow \varphi &= \frac{2W}{\beta s^2} y(r - x) \end{aligned}$$

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不考虑湍摩擦效应：

$$\begin{cases} \frac{du}{dt} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} & (25) \\ \frac{dv}{dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} & (26) \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

$$\begin{aligned} \frac{\partial(26)}{\partial x} - \frac{\partial(25)}{\partial y} &\Leftrightarrow \frac{d}{dt}(\xi_r + f) + (\xi_r + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \\ &\Leftrightarrow \frac{d}{dt}(\xi_r + f) = -(\xi_r + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{aligned} \quad (27)$$

$\xi_r = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$  相对涡度,  $f$  行星涡度,  $\xi_a = \xi_r + f$  绝对涡度.

对连续方程进行垂向平均：

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{\partial[(h + \zeta)\langle u \rangle]}{\partial x} + \frac{\partial[(h + \zeta)\langle v \rangle]}{\partial y} &= 0 \\ \Leftrightarrow \frac{d(\zeta + h)}{dt} + (\zeta + h) \left( \frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y} \right) &= 0 \\ (\text{令 } H = h + \zeta) \Leftrightarrow \frac{dH}{dt} + H \left( \frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y} \right) &= 0 \end{aligned} \quad (28)$$

将 (28) 代入 (27) 中：

$$\begin{aligned}
 &\Rightarrow \frac{d}{dt}(\xi_r + f) = \frac{1}{H}(\xi_r + f) \frac{dH}{dt} \\
 &\Rightarrow \frac{1}{H} \frac{d}{dt}(\xi_r + f) = \frac{1}{H^2}(\xi_r + f) \frac{dH}{dt} \\
 &\Rightarrow \boxed{\frac{d}{dt} \left( \frac{\xi_r + f}{H} \right) = 0}
 \end{aligned} \tag{29}$$

(29) 为 **位势涡度守恒方程**.

假设在西边界区  $\frac{\partial u}{\partial y} = 0$ ，则：

$$\xi_r = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{H} \frac{\partial \varphi}{\partial x} \right)$$

代入 (29)：

$$\begin{aligned}
 &\frac{d}{dt} \left[ \frac{\frac{\partial}{\partial x} \left( \frac{1}{H} \frac{\partial \varphi}{\partial x} \right) + f}{H} \right] = 0 \\
 &\Rightarrow \frac{\frac{\partial}{\partial x} \left( \frac{1}{H} \frac{\partial \varphi}{\partial x} \right) + f}{H} = F(\varphi) \\
 &\Rightarrow \frac{1}{H} \frac{\partial^2 \varphi}{\partial x^2} + f_0 + \beta y = HF(\varphi) = G(\varphi)
 \end{aligned}$$

在西边界层的边缘

内部的解即为西边界的解，惯性项可以忽略：

$$x = L : f_0 + \beta y = G(\phi_i)$$

在内区的边缘 ( $L \ll r$ )

$$\varphi_i = \frac{2W}{\beta s^2} y(r - x) = \frac{2W}{\beta s^2} y(r - L) = \frac{2W}{\beta s^2} yr = u^* y \quad \left( u^* = \frac{2W}{\beta s^2} r \right)$$

上面两解应该等价，因此：

$$\begin{aligned}
 &f_0 + \beta y = G(u^* y) \\
 &\Rightarrow G(\varphi) = f_0 + \beta \frac{\varphi}{u^*} \\
 &\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} + H(f_0 + \beta y) = H \left( f_0 + \beta \frac{\varphi}{u^*} \right) \\
 &\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} - \frac{H\beta}{u^*} \varphi = -H\beta y
 \end{aligned}$$

再结合两个边界条件： $x = 0 : \begin{cases} \varphi(0, y) = 0 \\ \frac{\partial \varphi}{\partial x} = 0 \end{cases}$ ，上面的二阶常系数线性微分方程的解为：

$$\varphi = u^* y \left[ 1 - e^{-(H\beta/u^*)^{1/2} x} \right]$$

海水南北输运：

$$\boxed{M_y = \frac{\partial \varphi}{\partial x} = (u^* H \beta)^{1/2} y e^{-(H\beta/u^*)^{1/2} x}}$$

在西边界区域有一强烈的北向流动，近岸处质量运输很大，随着离岸距离的增加，质量输运迅速减少。

#### 2.4.4 Munk 理论

假定

- (1) 矩形大洋  $r \times 2s$ ；
- (2) 远离海岸的等深封闭矩形大洋，静止时水深为常量  $h$ ；
- (3) **增加了侧向摩擦**。

控制方程

$$\begin{cases} -fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_l \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + A_z \frac{\partial^2 u}{\partial z^2} \\ fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_l \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + A_z \frac{\partial^2 v}{\partial z^2} \\ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta, & \rho A_z \frac{\partial u}{\partial z} = \tau_{x\zeta}, \rho A_z \frac{\partial v}{\partial z} = \tau_{y\zeta} \\ z = -h, & u = v = 0, \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \end{cases}$$

求解

全流方程 (垂直积分):

$$\begin{cases} -fM_y = -\frac{\partial P}{\partial x} + A_l \left( \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_x}{\partial y^2} \right) + \tau_{x\zeta} \end{cases} \quad (30)$$

$$\begin{cases} fM_x = -\frac{\partial P}{\partial y} + A_l \left( \frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} \right) + \tau_{y\zeta} \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \end{cases} \quad (31)$$

其中,  $M_x = \int_{-h}^0 \rho u dz, M_y = \int_{-h}^0 \rho v dz, P = \int_{-h}^0 p dz, \tau_{x\zeta} = \int_{-h}^0 \rho A_z \frac{\partial^2 u}{\partial z^2} = \rho A_z \left( \frac{\partial u}{\partial z} \Big|_{z=0} - \frac{\partial u}{\partial z} \Big|_{z=-h} \right)$  ( $\zeta \ll h$ ) 引入流函数:  $M_x = -\frac{\partial \psi}{\partial y}, M_y = \frac{\partial \psi}{\partial x}$

$$\begin{aligned} \frac{\partial(30)}{\partial y} - \frac{\partial(31)}{\partial x} &\Leftrightarrow A_l \left( \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \right) - \beta \frac{\partial \psi}{\partial x} = - \left( \frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y} \right) \\ &\Leftrightarrow A_l \nabla^4 \psi - \beta \frac{\partial \psi}{\partial x} = - \text{rot}_z \vec{\tau}_\zeta \end{aligned}$$

边界条件:

$$\begin{cases} x = 0, r : \psi = 0, \frac{\partial \psi}{\partial x} = 0 \\ y = -s, s : \psi = 0, \frac{\partial \psi}{\partial y} = 0 \end{cases}$$

内区 (中部区) 满足 Sverdrup 理论:  $\beta \frac{\partial \psi}{\partial x} = \text{rot}_z \tau_\zeta = -\frac{\partial \tau_{x\zeta}}{\partial y}$

假定纬向风系:  $\begin{cases} \tau_{y\zeta} = 0 & (-s < y < s) \\ \tau_{x\zeta} = a \cos ny + b \sin ny + c \\ n = \frac{j\pi}{s}, j = 1, 2, 3 \dots \end{cases}$

对  $x$  积分:  $\psi = -\frac{1}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} x + f_1(y)$

结合  $\psi|_r = 0 \Rightarrow \psi = -\frac{1}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} (x - r)$  因此,  $x = 0$  处:

$$\begin{cases} \psi(0, y) = \frac{r}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} \\ \frac{\partial \psi}{\partial x} \Big|_{x=0} = M_y(0, y) = -\frac{1}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} \end{cases}$$

西边界区

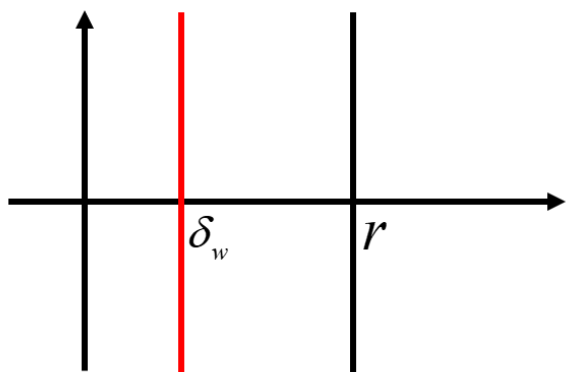
保留侧向湍流应力, 结合  $L_x \ll L_y$ :

$$A_l \nabla^4 \psi - \beta \frac{\partial \psi}{\partial x} = - \text{rot}_z \vec{\tau}_\zeta \Rightarrow A_l \frac{\partial^4 \psi}{\partial x^4} - \beta \frac{\partial \psi}{\partial x} = \frac{\partial \tau_{x\zeta}}{\partial y}$$

设试解：  $\psi = X(x) \frac{\partial \tau_{x\zeta}}{\partial y} \Rightarrow A_l X^{(4)} - \beta X' = 1 \Rightarrow X(x) = A + B e^{kx} + D e^{-\frac{k}{2}x} \cos\left(\frac{\sqrt{3}}{2} kx + E\right) - \frac{x}{\beta}$  自然边界条件：

$$x = 0 : \begin{cases} \psi = 0 \\ \frac{\partial \psi}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} X(0) = 0 \\ X'(0) = 0 \end{cases}$$

$$\text{衔接边界条件： } x = \delta_w : \begin{cases} \psi = \frac{r}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} \\ \frac{\partial \psi}{\partial x} = -\frac{1}{\beta} \frac{\partial \tau_{x\zeta}}{\partial y} \end{cases} \Rightarrow \begin{cases} X(0) = \frac{r}{\beta} \\ X'(0) = -\frac{1}{\beta} \end{cases}$$



$$\text{解得： } \psi(x, y) = \frac{r}{\beta} \left[ 1 - \frac{2}{\sqrt{3}} e^{-\frac{1}{2} kx} \cos\left(\frac{\sqrt{3}}{2} kx - \frac{\pi}{6}\right) \right] \frac{\partial \tau_{x\zeta}}{\partial y}$$

东边界区

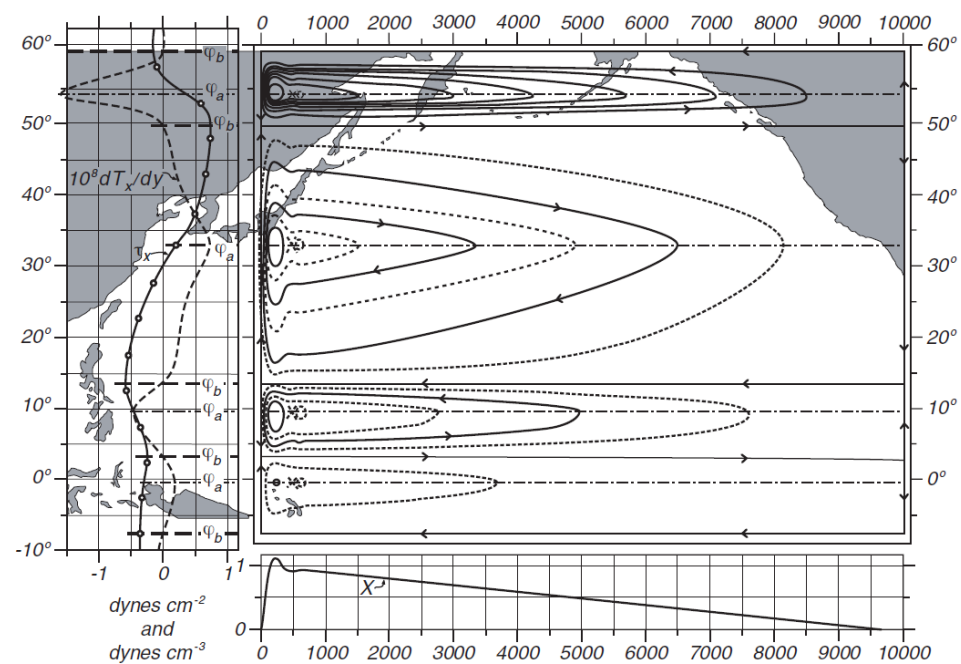
$$\psi(x, y) = \frac{r}{\beta} \left[ 1 - \frac{x}{r} + \frac{1}{kr} (e^{-k(r-x)} - 1) \right] \frac{\partial \tau_{x\zeta}}{\partial y}$$

综合内区、西边界区和东边界区的解可得统一的解的形式：

$$\psi = \frac{r}{\beta} f(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

讨论

环流空间分布特征 图片来自 *Introduction to Physical Oceanography* (Robert H. Stewart, 2008 pp191)



(1) Gyres 之间的分界处位于：  $M_y = \frac{\partial \psi}{\partial x} = 0$ ，只有东西向的流动：  $f'(x) = 0 / \frac{\partial \tau_{x\zeta}}{\partial y} = 0$ ；

- (2) Gyres 主轴位于:  $M_x = \frac{\partial \psi}{\partial x} = 0$  只有南北向的流动:  $f(x) = 0 / \frac{\partial^2 \tau_{x\zeta}}{\partial y^2} = 0$ ;  
 (3) 流动西强东弱.

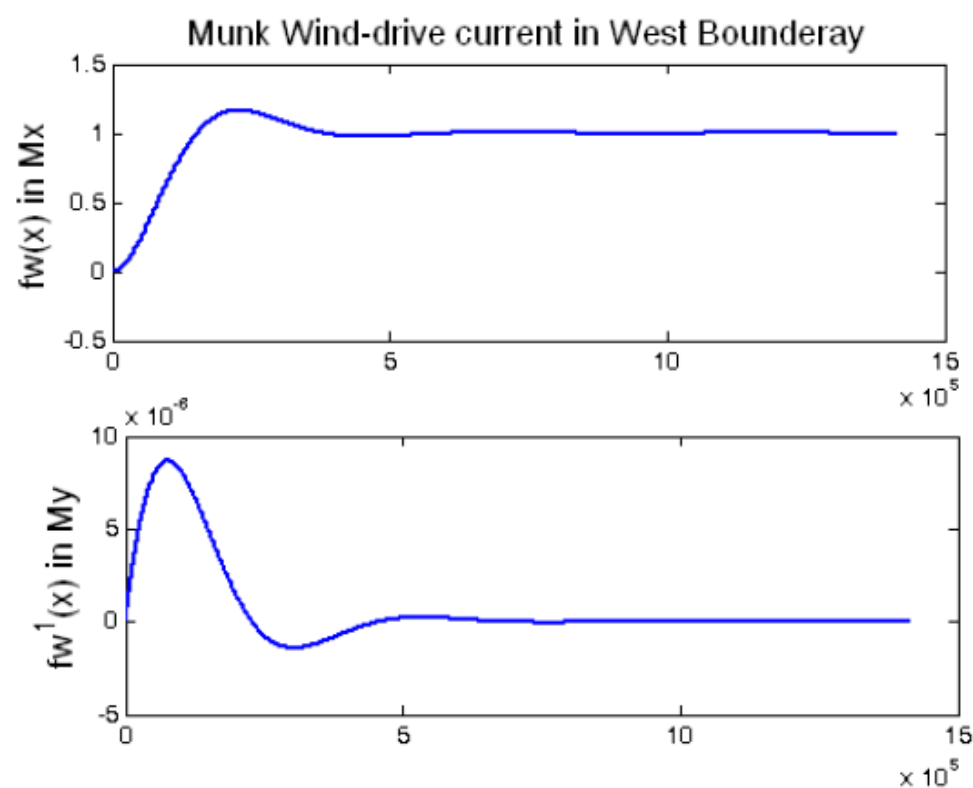
大洋西边界区的物质输运特点

$$\psi_W = \frac{r}{\beta} f_m(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

质量输运:

$$\begin{cases} M_{xW} = -\frac{\partial \psi}{\partial y} = -\frac{r}{\beta} f_W(x) \frac{\partial^2 \tau_{x\zeta}}{\partial y^2} \\ M_{yW} = \frac{\partial \psi}{\partial x} = \frac{r}{\beta} f'_W(x) \frac{\partial \tau_{x\zeta}}{\partial y} \end{cases}$$

其中, 
$$\begin{cases} f_W(x) = 1 - \frac{2}{\sqrt{3}} e^{-\frac{1}{2}kx} \cos\left(\frac{\sqrt{3}}{2}kx - \frac{\pi}{6}\right) \\ f'_W(x) = \frac{2}{\sqrt{3}} k e^{-\frac{1}{2}kx} \sin\frac{\sqrt{3}}{2}kx \end{cases}$$



随  $x$  增大而衰减的阻尼振动

对于任意纬度:

$$M_{yW} = \frac{\partial \psi}{\partial x} = \frac{r}{\beta} f'_W(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

因此,  $f'_W(x)$  的极值决定了  $M_{yW}$  的极值:

$$\begin{aligned} f''_W(x) &= 0 \\ (\text{令 } \lambda_W = \frac{2\pi}{\text{波数}} = \frac{4\pi}{\sqrt{3}k}) &\Rightarrow x_{a,b} = \frac{2\pi}{3\sqrt{3}k} \left( \frac{1}{6}\lambda_W \right), \quad \frac{8\pi}{3\sqrt{3}k} \left( \frac{4}{6}\lambda_W \right) \\ &\Rightarrow \begin{cases} f'_W(x_a) = k e^{-\pi/3\sqrt{3}} \\ f'_W(x_b) = -k e^{-4\pi/3\sqrt{3}} \end{cases} \end{aligned}$$

$$\frac{f'_W(x_b)}{f'_W(x_a)} = -e^{-\pi/3} = -0.17$$

在西边界内, 以  $x = x_a(1/6)$  波长为主轴处有一主流为北向的流动; 以  $x = x_b(4/6)$  波长为主轴处存在一逆流, 逆流的量值为主流的 0.17 倍.

对于任意给定纬度,  $f_W(x)$  的极值决定  $M_{xW}$  的极值:

$$f'_W(x) = 0 \Rightarrow x_{1,2} = \frac{2\pi}{\sqrt{3}k}, \frac{4\pi}{\sqrt{3}k} \Rightarrow \begin{cases} f_W(x_1) = 1 - e^{-\pi/\sqrt{3}} \\ f_W(x_2) = 1 + e^{-2\pi/\sqrt{3}} \end{cases}$$



极值在 1 附近变动.

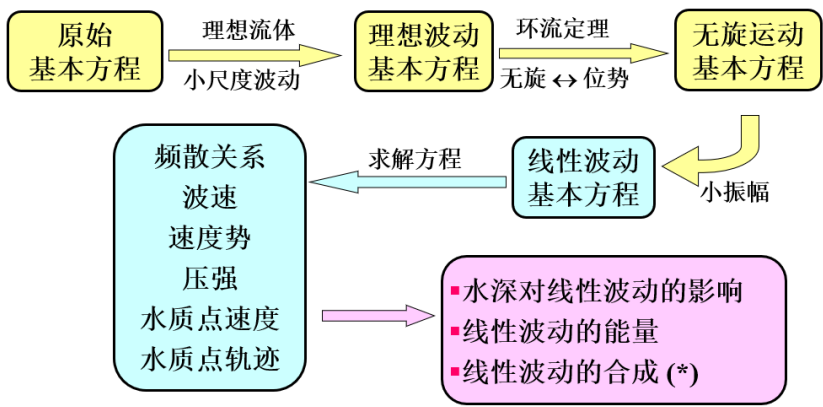
缺陷

- (1) 若取  $A_l = 5 \times 10^3 m^2 s^{-1}$ , 西部阻尼振荡的波长比实际大约 3 倍:  $\lambda = \frac{4\pi}{\sqrt{3}k} = \frac{4\pi}{\sqrt{3}} \frac{1}{\sqrt[3]{\beta/A_l}} \approx 200km$ ; 符合实际主流和逆流宽度时,  $A_l = 10^2 m^2 s^{-1}$
- (2) 西部总流量比实测值小一半.

### 3 海浪

#### 3.1 线性波动理论

理论框架:



##### 3.1.1 无旋运动的基本方程

假定及方程

- (1) 海水均匀不可压缩;
- (2) 理想流体;
- (3) 短周期小尺度波动;
- (4) 重力为唯一的外力;
- (5) 忽略分子粘性项、科氏力、引潮力和湍摩擦力.

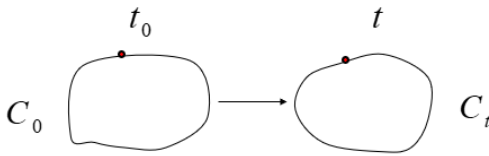
控制方程:

$$\begin{cases} \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\frac{1}{\rho} \nabla p - \vec{g} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件:

$$\begin{cases} z = \zeta(\text{海面}): \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = w, p_I = p_a(x, y, t) \\ \text{固体边界处}: V_n = 0 \end{cases}$$

环流定理



$C_t$  上的环流：  $\Gamma(t) = \oint_{ct} u dx + v dy + w dz$

$$\begin{aligned} \Rightarrow \frac{d\Gamma(t)}{dt} &= \oint_{ct} \left( \frac{du}{dt} dx + \frac{dv}{dt} dy + \frac{dw}{dt} dz \right) + \underbrace{\oint_{ct} (u du + v dv + w dw)}_{=\frac{1}{2} \oint_{ct} du^2 + dv^2 + dw^2 = 0} \\ \Rightarrow \frac{d\Gamma(t)}{dt} &= \oint_{ct} \frac{d\vec{V}}{dt} \cdot d\vec{l} \end{aligned}$$

环流的实质微商等于加速度的环流.

$$\frac{d\Gamma(t)}{dt} = \oint_{ct} \left( -\frac{1}{\rho} \frac{\partial p}{\partial x} \right) dx + \left( -\frac{1}{\rho} \frac{\partial p}{\partial y} \right) dy + \left( -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \right) dz = \oint_{ct} d \underbrace{\left( -\frac{p}{\rho} - gz \right)}_{\text{单值函数}} = 0$$

**环流定理：**对不可压缩的理想流体，由相同质点构成的封闭曲线上的环流不随时间而变。

## 参考文献

[Ekman, 1905] Ekman, V. W. (1905). *On the Influence of the Earth's Rotation on Ocean-Currents*, volume 2. University Microfilms INC.