物理海洋学笔记

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https://github.com/Cuiyingzhe/Physical-Oceanography-Notes/

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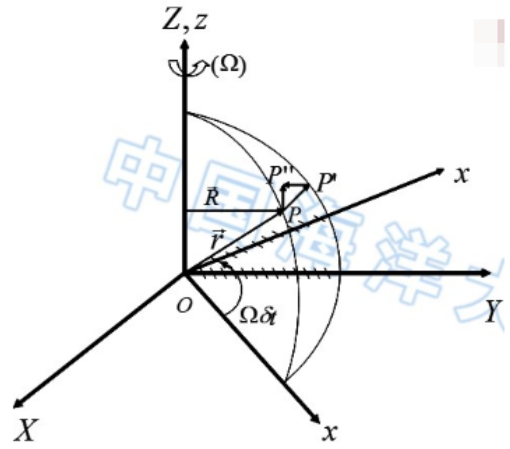
1 基本方程

1.1 旋转坐标系的速度和加速度

惯性坐标系: 静止的或是匀速直线运动的坐标系，固定在恒星上的坐标系可以被看成惯性坐标系.
固定在地球上的坐标系: 地球对恒星的加速度主要是由地球自转引起的，于是可以把地球当作一个对惯性坐标系作纯粹地转运动的物体.

1.1.1 旋转坐标系和惯性坐标系中的速度



惯性坐标系 (XYZ) 绝对位移：$\vec{pp''} = \vec{V}_a\delta t, \vec{V}_a$ 为绝对速度
旋转坐标系 (xyz) 相对位移：$\vec{p'p''} = \vec{V}\delta t, \vec{V}$ 为相对速度
$\because \vec{pp''} = \vec{p'p''} + \vec{pp'}$
$\therefore \vec{V}_a\delta t = \vec{V}\delta t + \vec{V}_e\delta t \Rightarrow \vec{V}_a = \vec{V} + \vec{V}_e$(绝对速度等于相对速度与牵连速度的向量和)
其中，$\vec{V}_e = \vec{\Omega}\times\vec{r} \Rightarrow \vec{V}_a = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\Rightarrow \frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}$$

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

1.1.2 旋转坐标系和惯性坐标系中的加速度

令 $\vec{A} = \vec{V}_a = \vec{V}_e + \vec{V} = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\begin{aligned}
\frac{d\bar{V}_a}{dt} &= \frac{d_a}{dt}\left(\vec{V} + \vec{V}_e\right) = \frac{d_a}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) + \vec{\Omega}\times(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R}
\end{aligned}$$

1.2 作用在海水微团上的外力运动方程的向量形式

压强梯度力：$\frac{1}{\rho}\nabla p$
分子粘性力 (摩擦力)：

$$\begin{cases}
F_x = \frac{1}{\rho}\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta u \\
F_y = \frac{1}{\rho}\mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta v \\
F_2 = \frac{1}{\rho}\mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] = \frac{\mu}{\rho}\Delta w
\end{cases} \Rightarrow \vec{F} = \frac{\mu}{\rho}\Delta\vec{V} = \gamma\Delta\vec{v}$$

重力 (地球引力与地球自转产生的惯性离心力的合力)：$\vec{g} = -G\frac{M_g}{r^2}\cdot\left(\frac{\vec{r}}{r}\right)$
科氏力：$-2\vec{\Omega}\times\vec{V}$

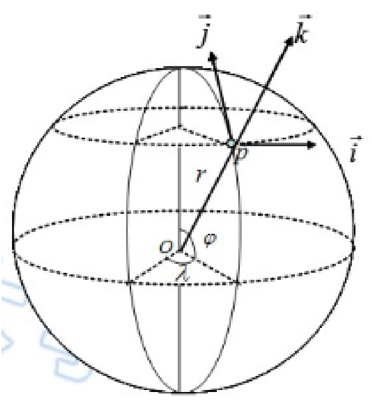
天体引潮力 (受其他天体万有引力与惯性力离心力的合力)：$\vec{F_M} = -G\frac{M_M}{L^2} + G\frac{M_M}{D^2} \cdot \left(\frac{\vec{D}}{D}\right)$

由牛顿第二定律和坐标系转换关系：

$$\begin{cases} \frac{d_a\vec{V_a}}{dt} = \sum_i \vec{F_t} \\ \frac{d_a\vec{A_a}}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2\vec{R} \end{cases}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla P - 2\vec{\Omega} \times \vec{V} + \vec{g} + \nu\Delta\vec{V} + \vec{F_T}}$$

1.3 运动方程在球坐标系的标量形式



速度：

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\Rightarrow \begin{cases} u = r\cos\varphi\frac{d\lambda}{dt} \\ v = r\frac{d\varphi}{dt} \\ w = \frac{dr}{dt} \end{cases}$$

加速度：

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{\frac{\partial\vec{A}}{\partial t}dt + \frac{\partial\vec{A}}{\partial\lambda}d\lambda + \frac{\partial\vec{A}}{\partial\varphi}d\varphi + \frac{\partial\vec{A}}{\partial r}dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial\lambda}\frac{d\lambda}{dt} + \frac{\partial\vec{A}}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial\vec{A}}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\vec{A}}{\partial\lambda} + \frac{v}{r}\frac{\partial\vec{A}}{\partial\varphi} + w\frac{\partial\vec{A}}{\partial r} \end{aligned}$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{r\cos\varphi\partial\lambda} + v\frac{\partial}{r\partial\varphi} + w\frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)}}$$

$$\Rightarrow \boxed{\boldsymbol{\nabla = \frac{\partial}{r\cos\varphi\partial\lambda}\vec{i} + \frac{\partial}{r\partial\varphi}\vec{j} + \frac{\partial}{\partial r}\vec{k}}}$$

$$\Rightarrow \begin{cases} \frac{d\vec{i}}{dt} = \frac{\partial\vec{i}}{\partial t} + u\frac{\partial\vec{i}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{i}}{r\partial\varphi} + w\frac{\partial\vec{i}}{\partial r} \\ \frac{\partial\vec{j}}{dt} = \frac{\partial\vec{j}}{\partial t} + u\frac{\partial\vec{j}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{j}}{r\partial\varphi} + w\frac{\partial\vec{j}}{\partial r} \\ \frac{d\vec{k}}{dt} = \frac{\partial\vec{k}}{\partial t} + u\frac{\partial\vec{k}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{k}}{r\partial\varphi} + w\frac{\partial\vec{k}}{\partial r} \end{cases}$$

$$\frac{d\vec{V}}{dt}=\frac{du}{dt}\vec{i}+\frac{dv}{dt}\vec{j}+\frac{dv}{dt}\vec{k}+u\frac{d\vec{i}}{dt}+v\frac{d\vec{j}}{dt}+w\frac{d\vec{k}}{dt}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt}=\left(\frac{du}{dt}-\frac{uvtg\varphi}{r}+\frac{uw}{r}\right)\vec{i}+\left(\frac{dv}{dt}+\frac{u^2\operatorname{tg}\varphi}{r}+\frac{vw}{r}\right)\vec{j}+\left(\frac{dw}{dt}-\frac{u^2+v^2}{r}\right)}$$

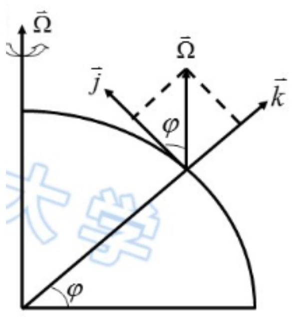
压强梯度力：

$$\frac{1}{\rho}\nabla p=-\frac{1}{\rho}\left(\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}\vec{i}+\frac{1}{r}\frac{\partial p}{\partial\varphi}\vec{j}+\frac{\partial p}{\partial r}\vec{k}\right)$$

重力：

$$\vec{g}=-g\vec{k}$$

科氏力：



$$\vec{\Omega}=\Omega\sin\varphi\vec{k}+\Omega\cos\varphi\vec{j}$$

$$\begin{aligned}-2\vec{\Omega}\times\vec{V}&=-2\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\0&\Omega\cos\varphi&\Omega\sin\varphi\\u&v&w\end{vmatrix}\\&=-2[(w\Omega\cos\varphi-v\Omega\sin\varphi)\vec{i}+(u\Omega\sin\varphi)\vec{j}+(-u\Omega\cos\varphi)\vec{k}]\\&\Rightarrow -2\vec{\Omega}\times\vec{V}=(fv-\tilde{f}w)\vec{i}-(fu)\vec{j}+(\tilde{f}u)\vec{k}\end{aligned}$$

其中，$\begin{cases}f=2\Omega\sin\varphi\\\tilde{f}=2\Omega\cos\varphi\end{cases}$

$$\Rightarrow\boxed{\begin{cases}\dfrac{du}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{r\cos\varphi\partial\lambda}+fv-\tilde{f}w+\dfrac{uv\tan\varphi}{r}-\dfrac{uw}{r}+\gamma(\Delta\vec{v})_\lambda-\dfrac{1}{r\cos\varphi}\dfrac{\partial\phi_T}{\partial\lambda}\\\dfrac{dy}{dt}=7\dfrac{1}{\rho}\dfrac{\partial p}{r\partial\varphi}-fu-\dfrac{u^2\tan\varphi}{r}-\dfrac{vw}{r}+\gamma(\Delta\vec{v})_\varphi-\dfrac{1}{r}\dfrac{\partial\phi_T}{\partial\varphi}\\\dfrac{dw}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}+\tilde{f}u=g+\dfrac{u^2+v^2}{r}+\gamma(\Delta\vec{v})_r-\dfrac{\partial\phi_T}{\partial r}\end{cases}}$$

1.4 直角坐标系的运动方程

略去地球曲率的影响

$$\Rightarrow\boxed{\begin{cases}\dfrac{du}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+fv+\tilde{f}w+F_{N\lambda}+F_{T\lambda}\\\dfrac{dv}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}-fu+F_{Ny}+F_{Ty}\\\dfrac{dw}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\tilde{f}u-g+F_{Nz}+F_{Tz}\end{cases}}$$

1.5 海水层流运动的基本方程组

1.5.1 连续方程

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0\Leftrightarrow\frac{d\rho}{dt}+\rho\nabla\cdot\vec{V}=0$$

特别地，对于不可压缩流体：

$$\nabla\cdot\vec{V}=0$$

1.5.2 盐量扩散方程

$$
\begin{aligned}
&\overset{\text{盐量增加量}}{\frac{\partial}{\partial t}\iiint_\tau \rho s d\tau} = \overset{\text{平流作用}}{-\oint\!\!\!\oint_\sigma \rho s V_n d\sigma} + \overset{\text{分子扩散作用}}{-\oint\!\!\!\oint_\sigma S_n d\sigma} \\
&\iiint_\tau \frac{\partial(\rho s)}{\partial t} d\tau = \iiint_\tau \nabla\cdot(\rho s\vec{V})d\tau - \iiint_\tau \nabla\cdot\vec{S}d\tau \\
&\Rightarrow \frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\vec{V}) + \nabla\cdot\vec{S} = 0 \\
&\Rightarrow \rho\frac{\partial s}{\partial t} + s\frac{\partial\rho}{\partial t} + s\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla s + \nabla\cdot\vec{S} = 0 \\
&\Rightarrow \left(\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s\right) + \frac{s}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] = -\frac{1}{\rho}\nabla\cdot\vec{S} \\
&\Rightarrow \frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = \frac{k}{\rho}\Delta s = k_D\Delta s
\end{aligned}
$$

其中，$k_D = \dfrac{k}{\rho} \sim 1.1\times 10^{-9}\ (\mathrm{m^2/s})$

1.5.3 热传导方程

与上面类似：

$$\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \frac{\kappa}{\rho c_p}\Delta\theta = k_\theta\Delta\theta$$

其中，$k_\theta = \dfrac{\kappa}{\rho c_p} \sim 1.4\times 10^{-7}\ (\mathrm{m^2/s})$

1.5.4 热膨胀方程-状态方程

热膨胀方程：

$$\rho = \underset{\rho_0}{0^\circ\text{C 时的海水密度}}(1 - \underset{k}{\text{海水的热膨胀系数}}\theta)$$

EOS80 国际海水状态方程：

$$\rho(s,t,p) = \rho(s,t,0)\left[1 - \frac{np}{k(s,t,p)}\right]^{-1}$$

1.6 基本方程的矢量形式和标量形式

矢量形式：

$$
\boxed{\begin{cases}
\dfrac{d\vec{V}}{dt} = -\dfrac{1}{\rho}\nabla p - 2\Omega\times\vec{V} + \vec{g} + \gamma\Delta\vec{V} - \nabla\phi_T \\
\dfrac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{V} = 0 \\
\dfrac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = k_D\Delta s \\
\dfrac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = k_\theta\Delta\theta \\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

标量形式 (直角坐标系)：

$$
\boxed{\begin{cases}
\dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu + \gamma\Delta v - \dfrac{\partial\phi_T}{\partial y} \\
\dfrac{dw}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \tilde{f}u - g + \gamma\Delta w - \dfrac{\partial\phi_T}{\partial z} \\
\dfrac{d\rho}{dt} + \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0 \\
\dfrac{\partial s}{\partial t} + u\dfrac{\partial s}{\partial x} + v\dfrac{\partial s}{\partial y} + w\dfrac{\partial s}{\partial z} = k_D\left(\dfrac{\partial^2 s}{\partial x^2} + \dfrac{\partial^2 s}{\partial y^2} + \dfrac{\partial^2 s}{\partial z^2}\right) \\
\dfrac{\partial\theta}{\partial t} + u\dfrac{\partial\theta}{\partial x} + v\dfrac{\partial\theta}{\partial y} + w\dfrac{\partial\theta}{\partial z} = k_\theta\left(\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} + \dfrac{\partial^2\theta}{\partial z^2}\right) \\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

1.7 边界条件

无质量交换的运动学边界条件：

$$\frac{\partial F}{\partial t}+\vec{c}\cdot\nabla F=0$$

例：

(1) 海面 $(z=\zeta(x,y,t))$：$\dfrac{\partial\zeta}{\partial t}+\vec{V}_H\cdot\nabla_H\zeta-w=0$

(2) 海底 $(z=-h(x,y))$: $\vec{V}_H\cdot\nabla_H h+w=0$

动力学边界条件：

由牛顿第三定律，在界面法线方向有：

$$(\vec{p}_n)_1=(\vec{p}_n)_2$$

1.8 ※ 间平均的基本方程和边界条件 (直角坐标系)

连续方程:

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0$$

运动方程：

$$\begin{cases}\dfrac{\partial\bar{u}}{\partial t}+\bar{u}\dfrac{\partial\bar{u}}{\partial x}+\bar{v}\dfrac{\partial\bar{u}}{\partial y}+\bar{w}\dfrac{\partial\bar{u}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial x}+f\bar{v}-\tilde{f}w+\gamma\Delta\bar{u}-\dfrac{\partial\bar{\phi}_T}{\partial x}+\dfrac{\partial}{\partial x}\left(A_{x\alpha}\dfrac{\partial\bar{u}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial\bar{u}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial\bar{u}}{\partial z}\right)\\ \dfrac{\partial\bar{v}}{\partial t}+\vec{u}\dfrac{\partial\bar{v}}{\partial x}+\vec{v}\dfrac{\partial\bar{v}}{\partial y}+\vec{w}\dfrac{\partial\bar{v}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial y}-f\bar{u}+\gamma\Delta\bar{v}-\dfrac{\partial\bar{\phi}_T}{\partial y}+\dfrac{\partial}{\partial x}\left(A_{yx}\dfrac{\partial\bar{v}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{yy}\dfrac{\partial\bar{v}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{yz}\dfrac{\partial\bar{v}}{\partial z}\right)\\ \dfrac{\partial\bar{w}}{\partial t}+\bar{u}\dfrac{\partial\bar{w}}{\partial x}+\bar{v}\dfrac{\partial\bar{w}}{\partial y}+\bar{w}\dfrac{\partial\bar{w}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial z}+\tilde{f}\bar{u}-g+\gamma\Delta\bar{w}-\dfrac{\partial\bar{\phi}_T}{\partial z}+\dfrac{\partial}{\partial x}\left(A_{2x}\dfrac{\partial\bar{w}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{zy}\dfrac{\partial\bar{w}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{zz}\dfrac{\partial\bar{w}}{\partial z}\right)\end{cases}$$

盐量扩散方程：

$$\frac{\partial\bar{s}}{\partial t}+\bar{u}\frac{\partial\bar{s}}{\partial x}+\bar{v}\frac{\partial\bar{s}}{\partial y}+\bar{w}\frac{\partial\bar{s}}{\partial z}$$

热传导方程：

$$\frac{\partial\bar{\theta}}{\partial t}+\vec{u}\frac{\partial\bar{\theta}}{\partial x}+\vec{v}\frac{\partial\bar{\theta}}{\partial y}+\vec{w}\frac{\partial\bar{\theta}}{\partial z}=k_\theta\Delta\bar{\theta}+\frac{\partial}{\partial x}\left(K_{\theta_x}\frac{\partial\bar{\theta}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{\theta y}\frac{\partial\bar{\theta}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{\theta z}\frac{\partial\bar{\theta}}{\partial z}\right)$$

状态方程：

$$\bar{\rho}=\bar{\rho}(\bar{s},\bar{\theta},\bar{p})$$

1.9 铅直向平均的基本方程

$$\frac{\partial}{\partial x}[(h+\zeta)\langle u\rangle]+\frac{\partial}{\partial y}[(h+\zeta)\langle v\rangle]-\left[u|_\zeta\frac{\partial\zeta}{\partial x}+v|_\zeta\frac{\partial\zeta}{\partial y}-w|_\zeta\right]-\left[u|_{-h}\frac{\partial h}{\partial x}+v|_{-h}\frac{\partial h}{\partial y}+w|_{-h}\right]=\mathbf{0}$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x}+\frac{\partial[(h+\zeta)\langle v\rangle]}{\partial y}=0$$

1.10 尺度分析

Rossby 数 $\mathrm{Ro}=\dfrac{U}{FL}\begin{cases}\gg 1: \text{平流非线性项比 Coriolis 力重要, 大尺度运动}\\ =1: \text{平流非线性项与 Coriolis 力同等重要}\\ \ll 1: \text{平流非线性项可以忽略, 小尺度运动}\end{cases}$

水平 Ekman 数 $\mathrm{E_l}=\dfrac{A_l}{FL^2}$ 水平湍流摩擦项与 Coriolis 力比值

垂直 Ekman 数 $\mathrm{E_z}=\dfrac{A_z}{FD^2}$ 垂直湍流摩擦项与 Coriolis 力比值

准静力近似 f 平面近似 β 平面近似 Boussinesq 近似

2 海流

2.1 地转流

地转流：不考虑海面风的作用，远离沿岸的大洋中部大尺度、准水平、定常的海水流动.
产生原因：海水受热力和动力因素导致压力 (和密度) 在水平方向分布不均匀.

$$p = p_a + \rho g h \quad \rho \begin{cases} \neq \rho_0 \Rightarrow \text{梯度流} \\ = \rho_0 \Rightarrow \text{倾斜流} \end{cases}$$

2.1.1 梯度流

假定和方程

(1) 在相当长一段时间里海面温度变化和降水蒸发变化都不大，于是可以认为已形成的海水密度场、温度场和盐度场近似于定常，从而相应的海水运动也近似于定常：$\frac{\partial}{\partial t} = 0$.
(2) 海洋深而宽广，在远离海岸及海底的大洋中部海区，大尺度运动：$\mathrm{Ro} \ll 1$.
(3) 不考虑海底摩擦及边界摩擦的影响，且海面无风力作用，则流动属一种无摩擦流动：$\mathrm{E_l}, \mathrm{E_z} \ll 1$.
(4) β 平面近似准静力近似
x 方向基本方程：

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{\partial}{\partial x}\left(A_{xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{xy}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_{xz}\frac{\partial u}{\partial z}\right)$$

假定 (1)$\Rightarrow \frac{\partial u}{\partial t} = 0$

假定 (2)$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0$

假定 (3)$\Rightarrow \frac{\partial}{\partial x}\left(A_{xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{xy}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_{xz}\frac{\partial u}{\partial z}\right) = 0$

可得梯度流的控制方程：

$$\begin{cases} 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0 \\ u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} = 0 \\ \rho = \rho(s, \theta) \end{cases}$$

特征

水平速度：

$$\begin{cases} 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv & (1) \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu & (2) \end{cases}$$

$$\Longrightarrow \begin{cases} v = \frac{1}{\rho f}\frac{\partial p}{\partial x} \\ u = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \end{cases}$$

(1) 水平速度和压强梯度成正比；
(2) 与密度和科氏参数成反比；
(3) 地转关系在赤道不成立 ($f=0$).
垂向速度：

$$\begin{aligned}\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x}\Leftrightarrow\frac{\partial(\rho f v)}{\partial v}+\frac{\partial(\rho f u)}{\partial x}=0\\ &\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+f\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial\rho}{\partial z}+\beta\rho v=0\\ &\Leftrightarrow f\rho\overset{=0}{\boxed{\nabla\vec{V}}}+f\vec{V}\cdot\nabla\rho-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial\rho}{\partial z}+\beta\rho v=0\end{aligned}\tag{3}$$

$$\vec{V}\cdot\nabla\rho=\vec{V}\cdot\nabla\rho(s,\theta)=\vec{V}\cdot\left(\nabla s\frac{\partial\rho}{\partial s}+\nabla\theta\frac{\partial\rho}{\partial\theta}\right)=0$$

$$(3)\Leftrightarrow f\left(\rho\frac{\partial w}{\partial z}+w\frac{\partial\rho}{\partial z}\right)=\beta\rho v\overset{\text{尺度分析}}{\Rightarrow}f\frac{\partial w}{\partial z}=\beta v\overset{\text{尺度分析}}{\Rightarrow}W=\frac{\beta D}{F}U\sim 2\times10^{-4}U$$

垂向流速比水平流速小得多，地转流为准水平运动.

运动特性

$$(1)\times u+(2)\times v\Leftrightarrow u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0\Leftrightarrow\vec{V_H}\cdot\nabla_H p=0$$

(1) 梯度流平行于等压线；
(2) 北半球，流向右侧为高压，南半球相反；

密度特性

$$\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x}\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}\right)+\rho u\frac{\partial f}{\partial x}+\rho v\frac{\partial f}{\partial y}=0\Leftrightarrow\vec{V_H}\cdot\nabla_H\rho=0$$

(1) 梯度流近似平行于等密线；
(2) 在北半球，流向右侧密度小；
(3) 等压面倾斜与等密面倾斜方向相反.

温盐特性
忽略垂向运动：

$$\begin{aligned}\vec{V_H}\cdot\nabla_H\theta=0\\ \vec{V_H}\cdot\nabla_H s=0\end{aligned}$$

(1) 梯度流平行于等温线和等盐线；
(2) 在北半球，流向右侧温度高，盐度低.

2.1.2 倾斜流

假定和方程 (1) 海水密度为常数；
(2) 水平方向的压强梯度是由海面倾斜引起的.

$$\Rightarrow p=p_a+\int_z^{\zeta}\rho g dz=p_a+\rho g(\zeta-z)$$

倾斜流的控制方程：

$$\begin{cases}fv=g\dfrac{\partial\zeta}{\partial x} & (4)\\ fu=-g\dfrac{\partial\zeta}{\partial y} & (5)\end{cases}$$

性质：

$$(4)\times u-(5)\times v \Leftrightarrow u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=0 \Leftrightarrow \vec{V}_H\cdot\nabla\zeta=0$$

(1) 倾斜流平行于等水位线；
(2) 在北半球，流向右侧水位高；
(3) 倾斜流从表至底流速流向相同，压强梯度相同.

2.2 Ekman 漂流

由恒速定常的风长时间驱动大尺度、均匀密度的海洋，所产生的处于稳定状态的海流.

2.2.1 无限深海漂流

物理背景

Ekman 的老师 Nansen 在海洋调查时发现，冰山不是顺风漂移，而是沿着风向右方 $20^\circ\sim 40^\circ$ 的方向移动.Ekman 在 1905 年研究了这种现象并提出风海流理论 [1].

假定

无限深海 Ekman 漂流中用到了以下假定：
1) 海洋无限广阔，海洋无限深.
即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋无限深，$z\to\infty, \vec{V}=0$
2) 定常均匀风场长时间作用.
即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.
3) 密度分布均匀，ρ 为常数，不考虑热盐性质.
4) 采用 f 平面近似.

方程推导

控制方程和边界条件

首先给出一般的控制方程：

$$\begin{cases}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+fv+A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+A_z\dfrac{\partial^2 u}{\partial z^2}\\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}-fu+A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+A_z\dfrac{\partial^2 v}{\partial z^2}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases} \tag{6}$$

由假定 1)，$A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=0,\ A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)=0$；
由假定 2)，$\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=0,\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=0$；
由假定 3)，$-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=0,-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$

则 (6) 化为：

$$\begin{cases}0=fv+A_z\dfrac{\partial^2 u}{\partial z^2} & \text{(7a)}\\ 0=-fu+A_z\dfrac{\partial^2 v}{\partial z^2} & \text{(7b)}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 & \text{(7c)}\end{cases}$$

不失一般性地，假定风力仅沿 y 方向作用，即 $\tau_x = 0, \tau_y = const.$ 再结合假定 1)，控制方程的边界条件为：

$$\begin{cases} z=0, \rho A_z \dfrac{\partial u}{\partial z} = 0 & (8a) \\ z=0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y & (8b) \\ z=\infty, u=v=0 & (8c) \end{cases}$$

方程求解

$$(7a) + (7b) \times i \Leftrightarrow A_z \frac{\partial^2 (u+iv)}{\partial z^2} = if(u+iv)$$

令 $W = u + iv$，得：

$$A_z \frac{\partial^2 W}{\partial z^2} = ifW \Rightarrow \frac{\partial^2 W}{\partial z^2} = \frac{(1+i)^2 \Omega \sin\varphi}{A_z} W$$

令 $a = \sqrt{\Omega \sin\varphi / A_z}$，$j^2 = (1+i)^2 a^2$，得：

$$\frac{d^2 W}{dz^2} - j^2 W = 0 \tag{9}$$

(9) 式通解为：$W = Ae^{jz} + Be^{-jz}$

结合边界条件：$(7a) + (7b) \times i \Rightarrow z = 0, \rho A_z \dfrac{\partial W}{\partial z} = -\tau_y, z \to \infty, W = 0$

$$z \to \infty \Rightarrow A = 0, W = Be^{-jz}; z = 0, \rho A_z \frac{\partial W}{\partial z}\bigg|_{z=0} = \rho A_z \frac{\partial \left(Be^{-jz}\right)}{\partial z}\bigg|_{z=0} = -\tau_y, \Rightarrow B = \tau_y / \left(j\rho A_z\right)$$

因此，方程的解为：

$$W = \frac{\tau_y}{j\rho A_z} e^{-jz} = \frac{i\tau_y}{(1+i)a\rho A_z} e^{-(1+i)az} = \frac{e^{i\frac{\pi}{2}} \tau_y}{\sqrt{2} e^{i\frac{\pi}{4}} a\rho A_z} e^{-(1+i)az}$$

令 $D_0 = \pi / a$, 得到最终解的形式为：

$$W = \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z + i\left(\frac{\pi}{4} - \frac{\pi}{D_0} z\right)} \tag{10}$$

物理性质

运动速度

在海面 $(z = 0)$ 处，$W_0 = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}}$. 大小为 $|W_0| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z}$，方向与 x 轴成 45°，即与风向向右偏 45°.

在任意深度处，$|W_z| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z}$，方向为 $\frac{\pi}{4} - \frac{\pi}{D_0} z$，即流速随深度增加呈指数形式减小，流向随深度的增加而逐渐向右偏.

在摩擦深度 $z = D_0$ 处，$|W_{D_0}| = \dfrac{\tau_y e^{-\pi}}{\sqrt{2} a\rho A_z} = e^{-\pi} |W_0| = 0.043 |W_0|$，方向 $-\frac{3}{4}\pi$，即与 x 轴成 $-135°$，与表面流向正好相反.

Ekman 螺旋和 Ekman 螺线

根据速度的垂向分布，表层流速最大，流向偏向风向的右方 45°；随深度增加，流速逐渐减小，流向逐渐右偏；到摩擦深度，流速是表面流速的 4.3%，流向与表面流向相反，运动可以忽略. 连连接各层流速的矢量端点，构成艾克曼螺旋；艾艾克曼螺旋在平面上的投影，称为艾克曼螺线.

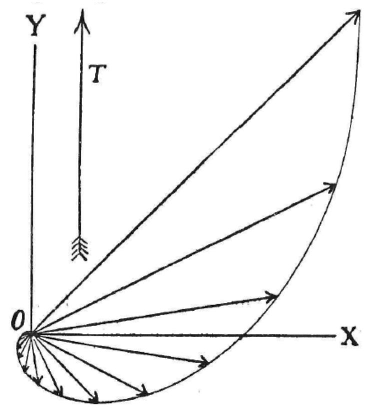


图 1 Ekman 螺线 [1]

水平体积输运

体积输运：

$$
\begin{aligned}
S &= \int_0^\infty W dz \\
&= \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}} \int_0^\infty e^{-\frac{\pi}{D_0}(1+i)z} dz \\
&= \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}} \left[-\frac{D_0}{\pi} \frac{1}{(1+i)} \right] e^{-\frac{\pi}{D_0}(1+i)z} \Big|_0^\infty \\
&= \frac{\tau_y}{2\Omega \sin\varphi\rho} = \frac{\tau_y}{f\rho}
\end{aligned}
$$

可以发现，得到的输运结果只有实部，没有虚部，说明体积输运方向为 x 轴正向，即在北半球水体向风向右侧 $90°$ 输运.

2.2.2 有限深海漂流

假定

有限深海 Ekman 漂流中用到了以下假定：

1) 海区无限广阔、有限深，远离海岸.

即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋有限深，$z \to h, \vec{V} = 0$

2) 定常均匀风场长时间作用.

即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.

3) 密度分布均匀，ρ 为常数，不考虑热盐性质.

4) 采用 f 平面近似.

控制方程和边界条件：

$$
\begin{cases}
0 = fv + A_z \dfrac{\partial^2 u}{\partial z^2} \\
0 = -fu + A_z \dfrac{\partial^2 v}{\partial z^2} \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\
z = 0 : \rho A_z \dfrac{\partial u}{\partial z} = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y \\
z \to h : u = v = 0
\end{cases}
$$

方程求解　令 $\zeta = h - z$，定解问题化为：

$$
\begin{cases}
-fv = A_z \dfrac{\partial^2 u}{\partial \zeta^2} & (11) \\
fu = A_z \dfrac{\partial^2 v}{\partial \zeta^2} & (12) \\
\zeta = h : \rho A_z \dfrac{\partial u}{\partial \zeta} = 0, \rho A_z \dfrac{\partial v}{\partial \zeta} = \tau_y \\
\zeta \to 0 : u = v = 0
\end{cases}
$$

令 $W = u + iv, \tau = \tau_x + i\tau_y$, 控制方程：

$$
(11) + (12) \times i \Leftrightarrow \frac{d^2 W}{d\xi^2} - j^2 W = 0
$$

边界条件：

$$
\zeta = h : \rho A_z \frac{\partial W}{\partial \zeta} = \tau
$$

$$
\zeta = 0 : W = 0
$$

与无限深海漂流解法类似，解得：

$$
W = \frac{(1+i)\tau_y}{2a\rho A_z} \frac{sh(1+i)a\xi}{ch(1+i)ah}
$$

物理性质

与水深的关系

(1) $h\ 2D_0$ 时，有限深海漂流流速流向与无限深海相同；(2) 水深越浅，流向随深度增加右偏 (北半球) 越缓慢；(3) 从上层到下层的流速矢量越是趋近风矢量的方向.

体积输运

(1) 在 x, y 方向 (平行和垂直风向) 都有输送；

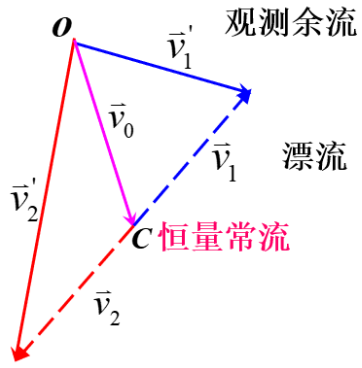
(2) 运输方向为风向右端，$\pm 90°$ 之间：

$$S_x > 0; 0 < h < D_0, ah < \pi, S_y > 0; D_0 < h < 2D_0, \pi < ah < 2\pi, S_y < 0; h > 2D_0, S_y = 0$$

2.2.3 漂流分离

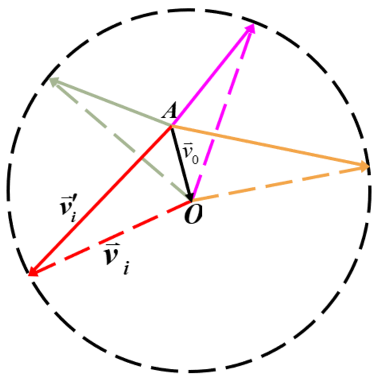
利用风速大小相等、方向相反的两次观测余流分离漂流

※ 余流＝漂流＋恒量常流



利用一组风速大小相等、方向不同的实测余流分离漂流

※ 漂流速度矢量端点落在同一圆周上

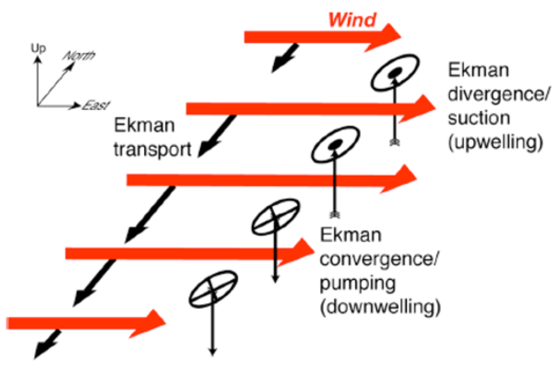


2.2.4 升降流

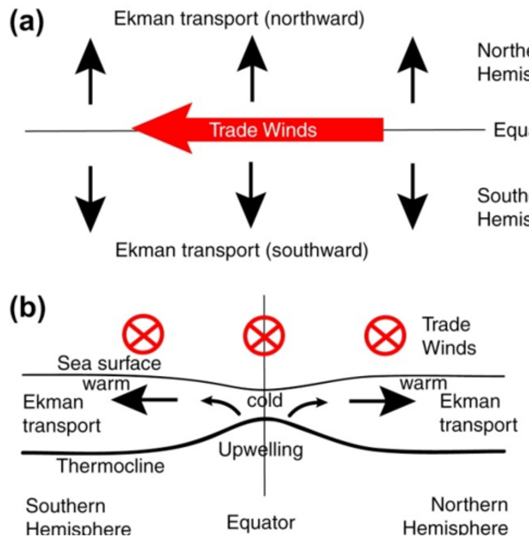
由不均匀风场或风场和地形配合产生的“较强烈”的铅直向流动。

物理背景

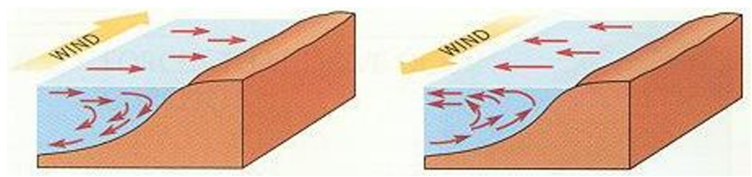
$\boxed{\text{非均匀风场}} \Rightarrow \boxed{\text{非均匀 Ekman 漂流}} \Rightarrow \boxed{\text{非均匀体积输运}} \Rightarrow \boxed{\text{辐聚辐散}} \Rightarrow \boxed{\text{升降流}}$



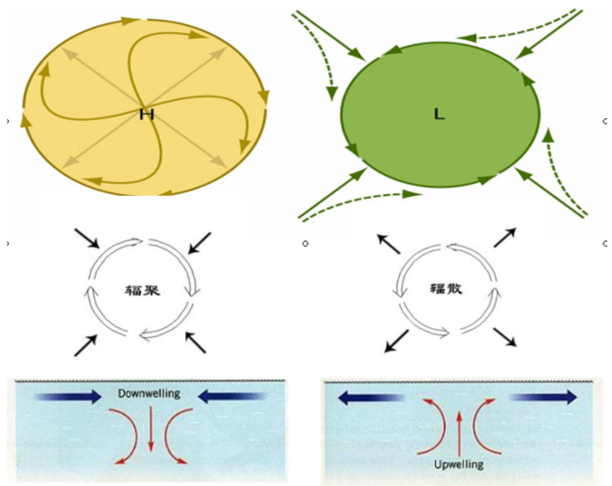
赤道附近的升降流



顺 (沿) 岸风产生的升降流



气旋和反气旋产生的海洋升降流



假定

(1) ρ 为常数；

(2) 直线风系，风仅沿 x 方向有变化；风区内为恒定的均匀风场；风区外无风；$\dfrac{\partial}{\partial y}=0$

(3) 定常风场；$\dfrac{\partial}{\partial t}=0$

(4) 大尺度；$Ro \ll 1$

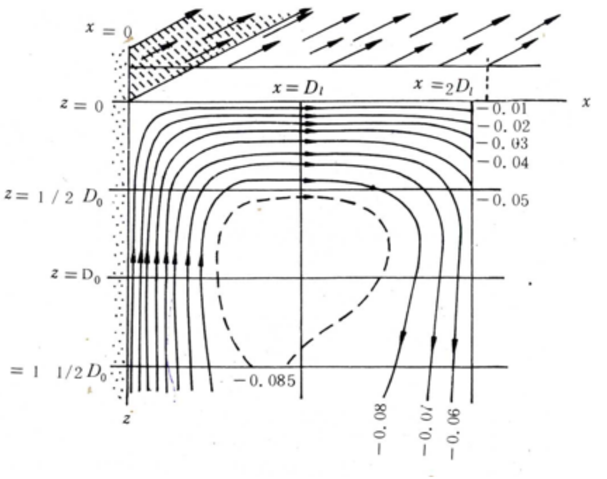
(5) 有限深度.$h \geq 2D_0$

控制方程及边界条件

$$\begin{cases} A_l\dfrac{\partial^2 u}{\partial x^2}+A_z\dfrac{\partial^2 u}{\partial z^2}+fv+g\dfrac{\partial \zeta}{\partial x}=0 \\ A_l\dfrac{\partial^2 v}{\partial x^2}+A_z\dfrac{\partial^2 v}{\partial z^2}-fu=0 \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}=0 \end{cases}$$

$$\begin{cases} z=\zeta: \rho A_z\dfrac{\partial u}{\partial z}=0, \quad \rho A_z\dfrac{\partial v}{\partial z}=-\tau_y \quad (0\le x\le L) \\ z=h: u=v=0 \\ x=0: u=v=0 \\ x\to\infty: u=v=0, \dfrac{\partial \zeta}{\partial x}=0 \end{cases}$$

结果讨论



(1) 近岸产生上升流 $x\le 0.5D_l$；

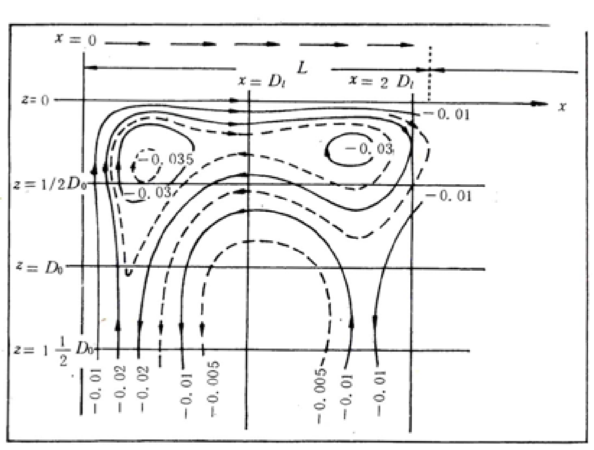
(2) 风区外延附近下降流 $x=2D_l$；

(3) 上升流来自 $z=1.5D_0$ 或更深；

(4) 最大 w 出现在 $z=D_0$；

(5) 上层离岸流，下层向岸流，构成一个循环.

若风向与海岸成 θ 角：



(1) 三个升降流系统：两个顺时针，一个逆时针；

(2) 大顺时针循环；

(3) $\theta=21.5^\circ$ 时，升降流达最大强度；

(4) 纬度越低，升降流越强.

参考文献

[1] V. W. Ekman, *On the Influence of the Earth's Rotation on Ocean-Currents*, vol. 2. University Microfilms INC, 01 1905.