物理海洋学笔记

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https://github.com/Cuiyingzhe/OUC-Physical-Oceanography-Notes

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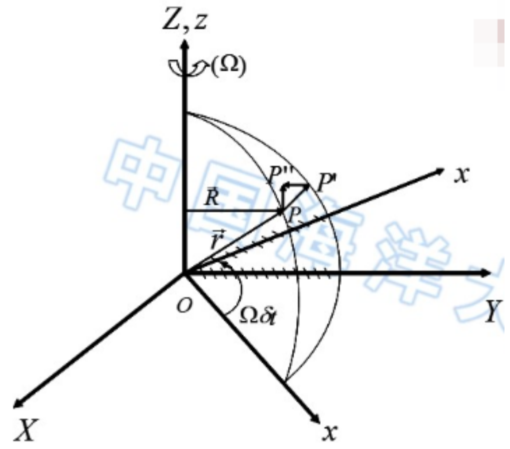
1 基本方程

1.1 旋转坐标系的速度和加速度

惯性坐标系: 静止的或是匀速直线运动的坐标系，固定在恒星上的坐标系可以被看成惯性坐标系.
固定在地球上的坐标系: 地球对恒星的加速度主要是由地球自转引起的，于是可以把地球当作一个对惯性坐标系作纯粹地转运动的物体.

1.1.1 旋转坐标系和惯性坐标系中的速度



惯性坐标系 (XYZ) 绝对位移：$\vec{pp''} = \vec{V}_a\delta t, \vec{V}_a$ 为绝对速度
旋转坐标系 (xyz) 相对位移：$\vec{p'p''} = \vec{V}\delta t, \vec{V}$ 为相对速度
$\because \vec{pp''} = \vec{p'p''} + \vec{pp'}$
$\therefore \vec{V}_a\delta t = \vec{V}\delta t + \vec{V}_e\delta t \Rightarrow \vec{V}_a = \vec{V} + \vec{V}_e$(绝对速度等于相对速度与牵连速度的向量和)
其中，$\vec{V}_e = \vec{\Omega}\times\vec{r} \Rightarrow \vec{V}_a = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\Rightarrow \frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}$$

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

1.1.2 旋转坐标系和惯性坐标系中的加速度

令 $\vec{A} = \vec{V}_a = \vec{V}_e + \vec{V} = \vec{V} + \vec{\Omega}\times\vec{r}$

$$\begin{aligned}
\frac{d\bar{V}_a}{dt} &= \frac{d_a}{dt}\left(\vec{V} + \vec{V}_e\right) = \frac{d_a}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) + \vec{\Omega}\times(\vec{V} + \vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \\
&= \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R}
\end{aligned}$$

1.2 作用在海水微团上的外力运动方程的向量形式

压强梯度力：$\frac{1}{\rho}\nabla p$
分子粘性力 (摩擦力)：

$$\begin{cases}
F_x = \frac{1}{\rho}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta u \\
F_y = \frac{1}{\rho}\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta v \\
F_2 = \frac{1}{\rho}\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\mu}{\rho}\Delta w
\end{cases} \Rightarrow \vec{F} = \frac{\mu}{\rho}\Delta\vec{V} = \gamma\Delta\vec{v}$$

重力 (地球引力与地球自转产生的惯性离心力的合力)：$\vec{g} = -G\frac{M_g}{r^2}\cdot\left(\frac{\vec{r}}{r}\right)$
科氏力：$-2\vec{\Omega}\times\vec{V}$

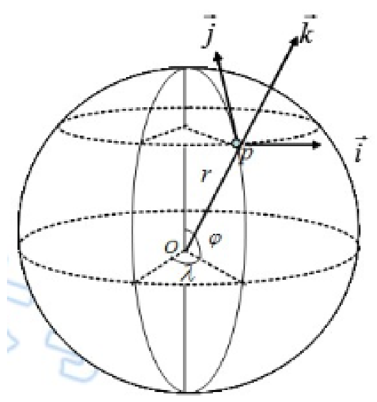
天体引潮力 (受其他天体万有引力与惯性力离心力的合力)：$\vec{F_M} = -G\frac{M_M}{L^2} + G\frac{M_M}{D^2} \cdot \left(\frac{\vec{D}}{D}\right)$

由牛顿第二定律和坐标系转换关系：

$$\begin{cases} \frac{d_a\vec{V_a}}{dt} = \sum_i \vec{F_t} \\ \frac{d_a\vec{A_a}}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} \end{cases}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla P - 2\vec{\Omega}\times\vec{V} + \vec{g} + \nu\Delta\vec{V} + \vec{F_T}}$$

1.3 运动方程在球坐标系的标量形式



速度：

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\Rightarrow \begin{cases} u = r\cos\varphi\frac{d\lambda}{dt} \\ v = r\frac{d\varphi}{dt} \\ w = \frac{dr}{dt} \end{cases}$$

加速度：

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{\frac{\partial\vec{A}}{\partial t}dt + \frac{\partial\vec{A}}{\partial\lambda}d\lambda + \frac{\partial\vec{A}}{\partial\varphi}d\varphi + \frac{\partial\vec{A}}{\partial r}dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial\lambda}\frac{d\lambda}{dt} + \frac{\partial\vec{A}}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial\vec{A}}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial\vec{A}}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\vec{A}}{\partial\lambda} + \frac{v}{r}\frac{\partial\vec{A}}{\partial\varphi} + w\frac{\partial\vec{A}}{\partial r} \end{aligned}$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{r\cos\varphi\partial\lambda} + v\frac{\partial}{r\partial\varphi} + w\frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)}}$$

$$\Rightarrow \boxed{\boldsymbol{\nabla = \frac{\partial}{r\cos\varphi\partial\lambda}\vec{i} + \frac{\partial}{r\partial\varphi}\vec{j} + \frac{\partial}{\partial r}\vec{k}}}$$

$$\Rightarrow \begin{cases} \frac{d\vec{i}}{dt} = \frac{\partial\vec{i}}{\partial t} + u\frac{\partial\vec{i}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{i}}{r\partial\varphi} + w\frac{\partial\vec{i}}{\partial r} \\ \frac{\partial\vec{j}}{dt} = \frac{\partial\vec{j}}{\partial t} + u\frac{\partial\vec{j}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{j}}{r\partial\varphi} + w\frac{\partial\vec{j}}{\partial r} \\ \frac{d\vec{k}}{dt} = \frac{\partial\vec{k}}{\partial t} + u\frac{\partial\vec{k}}{r\cos\varphi\partial\lambda} + v\frac{\partial\vec{k}}{r\partial\varphi} + w\frac{\partial\vec{k}}{\partial r} \end{cases}$$

$$\frac{d\vec{V}}{dt}=\frac{du}{dt}\vec{i}+\frac{dv}{dt}\vec{j}+\frac{dv}{dt}\vec{k}+u\frac{d\vec{i}}{dt}+v\frac{d\vec{j}}{dt}+w\frac{d\vec{k}}{dt}$$

$$\Rightarrow \boxed{\frac{d\vec{V}}{dt}=\left(\frac{du}{dt}-\frac{uvtg\varphi}{r}+\frac{uw}{r}\right)\vec{i}+\left(\frac{dv}{dt}+\frac{u^2\operatorname{tg}\varphi}{r}+\frac{vw}{r}\right)\vec{j}+\left(\frac{dw}{dt}-\frac{u^2+v^2}{r}\right)}$$

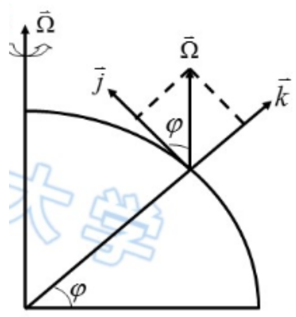
压强梯度力：

$$\frac{1}{\rho}\nabla p=-\frac{1}{\rho}\left(\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}\vec{i}+\frac{1}{r}\frac{\partial p}{\partial\varphi}\vec{j}+\frac{\partial p}{\partial r}\vec{k}\right)$$

重力：

$$\vec{g}=-g\vec{k}$$

科氏力：



$$\vec{\Omega}=\Omega\sin\varphi\vec{k}+\Omega\cos\varphi\vec{j}$$

$$\begin{aligned}-2\vec{\Omega}\times\vec{V}&=-2\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\0&\Omega\cos\varphi&\Omega\sin\varphi\\u&v&w\end{vmatrix}\\&=-2[(w\Omega\cos\varphi-v\Omega\sin\varphi)\vec{i}+(u\Omega\sin\varphi)\vec{j}+(-u\Omega\cos\varphi)\vec{k}]\\&\Rightarrow-2\vec{\Omega}\times\vec{V}=(fv-\tilde{f}w)\vec{i}-(fu)\vec{j}+(\tilde{f}u)\vec{k}\end{aligned}$$

其中，$\begin{cases}f=2\Omega\sin\varphi\\\tilde{f}=2\Omega\cos\varphi\end{cases}$

$$\Rightarrow\boxed{\begin{cases}\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{r\cos\varphi\partial\lambda}+fv-\tilde{f}w+\frac{uv\tan\varphi}{r}-\frac{uw}{r}+\gamma(\Delta\vec{v})_\lambda-\frac{1}{r\cos\varphi}\frac{\partial\phi_T}{\partial\lambda}\\\frac{dy}{dt}=7\frac{1}{\rho}\frac{\partial p}{r\partial\varphi}-fu-\frac{u^2\tan\varphi}{r}-\frac{vw}{r}+\gamma(\Delta\vec{v})_\varphi-\frac{1}{r}\frac{\partial\phi_T}{\partial\varphi}\\\frac{dw}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\tilde{f}u=g+\frac{u^2+v^2}{r}+\gamma(\Delta\vec{v})_r-\frac{\partial\phi_T}{\partial r}\end{cases}}$$

1.4 直角坐标系的运动方程

略去地球曲率的影响

$$\Rightarrow\boxed{\begin{cases}\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+\tilde{f}w+F_{N\lambda}+F_{T\lambda}\\\frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-fu+F_{Ny}+F_{Ty}\\\frac{dw}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\tilde{f}u-g+F_{Nz}+F_{Tz}\end{cases}}$$

1.5 海水层流运动的基本方程组

1.5.1 连续方程

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0\Leftrightarrow\frac{d\rho}{dt}+\rho\nabla\cdot\vec{V}=0$$

特别地，对于不可压缩流体：

$$\nabla\cdot\vec{V}=0$$

1.5.2 盐量扩散方程

$$
\begin{aligned}
&\overset{\text{盐量增加量}}{\frac{\partial}{\partial t}\iiint\limits_{\tau}\rho s d\tau} = \overset{\text{平流作用}}{-\oiint\limits_{\sigma}\rho s V_n d\sigma} + \overset{\text{分子扩散作用}}{-\oiint\limits_{\sigma} S_n d\sigma} \\
&\iiint\limits_{\tau}\frac{\partial(\rho s)}{\partial t}d\tau = \iiint\limits_{\tau}\nabla\cdot(\rho s\vec{V})d\tau - \iiint\limits_{\tau}\nabla\cdot\vec{S}d\tau \\
\Rightarrow&\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\vec{V}) + \nabla\cdot\vec{S} = 0 \\
\Rightarrow&\rho\frac{\partial s}{\partial t} + s\frac{\partial\rho}{\partial t} + s\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla s + \nabla\cdot\vec{S} = 0 \\
\Rightarrow&\left(\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s\right) + \frac{s}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] = -\frac{1}{\rho}\nabla\cdot\vec{S} \\
\Rightarrow&\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = \frac{k}{\rho}\Delta s = k_D\Delta s
\end{aligned}
$$

其中，$k_D = \frac{k}{\rho} \sim 1.1\times10^{-9}\left(\mathrm{m}^2/\mathrm{s}\right)$

1.5.3 热传导方程

与上面类似：

$$
\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \frac{\kappa}{\rho c_p}\Delta\theta = k_\theta\Delta\theta
$$

其中，$k_\theta = \frac{\kappa}{\rho c_p} \sim 1.4\times10^{-7}\left(\mathrm{m}^2/\mathrm{s}\right)$

1.5.4 热膨胀方程-状态方程

热膨胀方程：

$$
\rho = \overset{0^\circ\text{C 时的海水密度}}{\rho_0}(1 - \overset{\text{海水的热膨胀系数}}{k}\theta)
$$

EOS80 国际海水状态方程：

$$
\rho(s,t,p) = \rho(s,t,0)\left[1 - \frac{np}{k(s,t,p)}\right]^{-1}
$$

1.6 基本方程的矢量形式和标量形式

矢量形式：

$$
\boxed{\begin{cases}
\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\Omega\times\vec{V} + \vec{g} + \gamma\Delta\vec{V} - \nabla\phi_T \\
\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{V} = 0 \\
\frac{\partial s}{\partial t} + \vec{V}\cdot\nabla s = k_D\Delta s \\
\frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = k_\theta\Delta\theta \\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

标量形式 (直角坐标系)：

$$
\boxed{\begin{cases}
\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \gamma\Delta v - \frac{\partial\phi_T}{\partial y} \\
\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \tilde{f}u - g + \gamma\Delta w - \frac{\partial\phi_T}{\partial z} \\
\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \\
\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} = k_D\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}\right) \\
\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = k_\theta\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right) \\
\rho = \rho(\theta,s,p)
\end{cases}}
$$

1.7 边界条件

无质量交换的运动学边界条件：

$$\frac{\partial F}{\partial t}+\vec{c}\cdot\nabla F=0$$

例：

(1) 海面 $(z=\zeta(x,y,t))$：$\dfrac{\partial\zeta}{\partial t}+\vec{V}_H\cdot\nabla_H\zeta-w=0$

(2) 海底 $(z=-h(x,y))$: $\vec{V}_H\cdot\nabla_H h+w=0$

动力学边界条件：

由牛顿第三定律，在界面法线方向有：

$$(\vec{p}_n)_1=(\vec{p}_n)_2$$

1.8 ※ 时间平均的基本方程和边界条件 (直角坐标系)

连续方程:

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0$$

运动方程：

$$\begin{cases}\dfrac{\partial\bar{u}}{\partial t}+\bar{u}\dfrac{\partial\bar{u}}{\partial x}+\bar{v}\dfrac{\partial\bar{u}}{\partial y}+\bar{w}\dfrac{\partial\bar{u}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial x}+f\bar{v}-\tilde{f}w+\gamma\Delta\bar{u}-\dfrac{\partial\bar{\phi}_T}{\partial x}+\dfrac{\partial}{\partial x}\left(A_{x\alpha}\dfrac{\partial\bar{u}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial\bar{u}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial\bar{u}}{\partial z}\right)\\ \dfrac{\partial\bar{v}}{\partial t}+\vec{u}\dfrac{\partial\bar{v}}{\partial x}+\vec{v}\dfrac{\partial\bar{v}}{\partial y}+\vec{w}\dfrac{\partial\bar{v}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial y}-f\bar{u}+\gamma\Delta\bar{v}-\dfrac{\partial\bar{\phi}_T}{\partial y}+\dfrac{\partial}{\partial x}\left(A_{yx}\dfrac{\partial\bar{v}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{yy}\dfrac{\partial\bar{v}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{yz}\dfrac{\partial\bar{v}}{\partial z}\right)\\ \dfrac{\partial\bar{w}}{\partial t}+\bar{u}\dfrac{\partial\bar{w}}{\partial x}+\bar{v}\dfrac{\partial\bar{w}}{\partial y}+\bar{w}\dfrac{\partial\bar{w}}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial\bar{p}}{\partial z}+\tilde{f}\bar{u}-g+\gamma\Delta\bar{w}-\dfrac{\partial\bar{\phi}_T}{\partial z}+\dfrac{\partial}{\partial x}\left(A_{2x}\dfrac{\partial\bar{w}}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(A_{zy}\dfrac{\partial\bar{w}}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(A_{zz}\dfrac{\partial\bar{w}}{\partial z}\right)\end{cases}$$

盐量扩散方程：

$$\frac{\partial\bar{s}}{\partial t}+\bar{u}\frac{\partial\bar{s}}{\partial x}+\bar{v}\frac{\partial\bar{s}}{\partial y}+\bar{w}\frac{\partial\bar{s}}{\partial z}$$

热传导方程：

$$\frac{\partial\bar{\theta}}{\partial t}+\vec{u}\frac{\partial\bar{\theta}}{\partial x}+\vec{v}\frac{\partial\bar{\theta}}{\partial y}+\vec{w}\frac{\partial\bar{\theta}}{\partial z}=k_\theta\Delta\bar{\theta}+\frac{\partial}{\partial x}\left(K_{\theta_x}\frac{\partial\bar{\theta}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{\theta y}\frac{\partial\bar{\theta}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{\theta z}\frac{\partial\bar{\theta}}{\partial z}\right)$$

状态方程：

$$\bar{\rho}=\bar{\rho}(\bar{s},\bar{\theta},\bar{p})$$

1.9 铅直向平均的基本方程

$$\frac{\partial}{\partial x}[(h+\zeta)\langle u\rangle]+\frac{\partial}{\partial y}[(h+\zeta)\langle v\rangle]-\left[u|_\zeta\frac{\partial\zeta}{\partial x}+v|_\zeta\frac{\partial\zeta}{\partial y}-w|_\zeta\right]-\left[u|_{-h}\frac{\partial h}{\partial x}+v|_{-h}\frac{\partial h}{\partial y}+w|_{-h}\right]=\mathbf{0}$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x}+\frac{\partial[(h+\zeta)\langle v\rangle]}{\partial y}=0$$

1.10 尺度分析

Rossby 数 Ro=$\dfrac{U}{FL}\begin{cases}\gg 1:\text{平流非线性项比 Coriolis 力重要, 大尺度运动}\\ =1:\text{平流非线性项与 Coriolis 力同等重要}\\ \ll 1:\text{平流非线性项可以忽略, 小尺度运动}\end{cases}$

水平 Ekman 数 $\mathrm{E_l}=\dfrac{A_l}{FL^2}$ 水平湍流摩擦项与 Coriolis 力比值

垂直 Ekman 数 $\mathrm{E_z}=\dfrac{A_z}{FD^2}$ 垂直湍流摩擦项与 Coriolis 力比值

准静力近似 f 平面近似 β 平面近似 Boussinesq 近似

2 海流

2.1 地转流

地转流：不考虑海面风的作用，远离沿岸的大洋中部大尺度、准水平、定常的海水流动.
产生原因：海水受热力和动力因素导致压力 (和密度) 在水平方向分布不均匀.

$$p = p_a + \rho g h \quad \rho \begin{cases} \neq \rho_0 \Rightarrow \text{梯度流} \\ = \rho_0 \Rightarrow \text{倾斜流} \end{cases}$$

2.1.1 梯度流

假定和方程

(1) 在相当长一段时间里海面温度变化和降水蒸发变化都不大，于是可以认为已形成的海水密度场、温度场和盐度场近似于定常，从而相应的海水运动也近似于定常：$\dfrac{\partial}{\partial t} = 0$.
(2) 海洋深而宽广，在远离海岸及海底的大洋中部海区，大尺度运动：$\mathrm{Ro} \ll 1$.
(3) 不考虑海底摩擦及边界摩擦的影响，且海面无风力作用，则流动属一种无摩擦流动：$\mathrm{E_l}, \mathrm{E_z} \ll 1$.
(4) β 平面近似准静力近似
x 方向基本方程：

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{\partial}{\partial x}\left(A_{xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_{xy}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_{xz}\frac{\partial u}{\partial z}\right)$$

假定 (1)$\Rightarrow \dfrac{\partial u}{\partial t} = 0$

假定 (2)$\Rightarrow u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = 0$

假定 (3)$\Rightarrow \dfrac{\partial}{\partial x}\left(A_{xx}\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(A_{xy}\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(A_{xz}\dfrac{\partial u}{\partial z}\right) = 0$

可得梯度流的控制方程：

$$\begin{cases} 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + fv \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} + w\dfrac{\partial \theta}{\partial z} = 0 \\ u\dfrac{\partial s}{\partial x} + v\dfrac{\partial s}{\partial y} + w\dfrac{\partial s}{\partial z} = 0 \\ \rho = \rho(s, \theta) \end{cases}$$

特征

水平速度：

$$\begin{cases} 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + fv & (1) \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} - fu & (2) \end{cases}$$

$$\Longrightarrow \begin{cases} v = \dfrac{1}{\rho f}\dfrac{\partial p}{\partial x} \\ u = -\dfrac{1}{\rho f}\dfrac{\partial p}{\partial y} \end{cases}$$

(1) 水平速度和压强梯度成正比；
(2) 与密度和科氏参数成反比；
(3) 地转关系在赤道不成立 ($f=0$).
垂向速度：

$$
\begin{aligned}
\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x}\Leftrightarrow\frac{\partial(\rho f v)}{\partial v}+\frac{\partial(\rho f u)}{\partial x}=0 \\
&\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+f\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial\rho}{\partial z}+\beta\rho v=0 \\
&\Leftrightarrow f\rho\overset{=0}{\boxed{\nabla\vec{V}}}+f\vec{V}\cdot\nabla\rho-f\rho\frac{\partial w}{\partial z}-fw\frac{\partial\rho}{\partial z}+\beta\rho v=0 \qquad (3)
\end{aligned}
$$

$$
\vec{V}\cdot\nabla\rho=\vec{V}\cdot\nabla\rho(s,\theta)=\vec{V}\cdot\left(\nabla s\frac{\partial\rho}{\partial s}+\nabla\theta\frac{\partial\rho}{\partial\theta}\right)=0
$$

$$
(3)\Leftrightarrow f\left(\rho\frac{\partial w}{\partial z}+w\frac{\partial\rho}{\partial z}\right)=\beta\rho v\overset{\text{尺度分析}}{\Rightarrow}f\frac{\partial w}{\partial z}=\beta v\overset{\text{尺度分析}}{\Rightarrow}W=\frac{\beta D}{F}U\sim 2\times10^{-4}U
$$

垂向流速比水平流速小得多，地转流为准水平运动.

运动特性

$$
(1)\times u+(2)\times v\Leftrightarrow u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0\Leftrightarrow\vec{V}_H\cdot\nabla_H p=0
$$

(1) 梯度流平行于等压线；
(2) 北半球，流向右侧为高压，南半球相反；

密度特性

$$
\frac{\partial(1)\times\rho}{\partial y}-\frac{\partial(2)\times\rho}{\partial x}\Leftrightarrow f\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}\right)+\rho u\frac{\partial f}{\partial x}+\rho v\frac{\partial f}{\partial y}=0\Leftrightarrow\vec{V}_H\cdot\nabla_H\rho=0
$$

(1) 梯度流近似平行于等密线；
(2) 在北半球，流向右侧密度小；
(3) 等压面倾斜与等密面倾斜方向相反.

温盐特性
忽略垂向运动：

$$
\begin{aligned}
\vec{V}_H\cdot\nabla_H\theta=0 \\
\vec{V}_H\cdot\nabla_H s=0
\end{aligned}
$$

(1) 梯度流平行于等温线和等盐线；
(2) 在北半球，流向右侧温度高，盐度低.

2.1.2 倾斜流

假定和方程 (1) 海水密度为常数；
(2) 水平方向的压强梯度是由海面倾斜引起的.

$$
\Rightarrow p=p_a+\int_z^{\zeta}\rho g dz=p_a+\rho g(\zeta-z)
$$

倾斜流的控制方程：

$$
\begin{cases}
fv=g\dfrac{\partial\zeta}{\partial x} & (4)\\
fu=-g\dfrac{\partial\zeta}{\partial y} & (5)
\end{cases}
$$

性质：

$$(4)\times u-(5)\times v \Leftrightarrow u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}=0 \Leftrightarrow \vec{V}_H\cdot\nabla\zeta=0$$

(1) 倾斜流平行于等水位线；
(2) 在北半球，流向右侧水位高；
(3) 倾斜流从表至底流速流向相同，压强梯度相同.

2.2 Ekman 漂流

由恒速定常的风长时间驱动大尺度、均匀密度的海洋，所产生的处于稳定状态的海流.

2.2.1 无限深海漂流

物理背景

Ekman 的老师 Nansen 在海洋调查时发现，冰山不是顺风漂移，而是沿着风向右方 $20^\circ\sim 40^\circ$ 的方向移动.Ekman 在 1905 年研究了这种现象并提出风海流理论 [Ekman, 1905].

假定

无限深海 Ekman 漂流中用到了以下假定：
1) 海洋无限广阔，海洋无限深.
即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋无限深，$z\to\infty, \vec{V}=0$
2) 定常均匀风场长时间作用.
即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.
3) 密度分布均匀，ρ 为常数，不考虑热盐性质.
4) 采用 f 平面近似.

方程推导

控制方程和边界条件

首先给出一般的控制方程：

$$\begin{cases}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+fv+A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+A_z\dfrac{\partial^2 u}{\partial z^2}\\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}-fu+A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+A_z\dfrac{\partial^2 v}{\partial z^2}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}\tag{6}$$

由假定 1)，$A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=0,\ A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)=0$；
由假定 2)，$\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=0,\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=0$；
由假定 3)，$-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=0,-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$

则 (6) 化为：

$$\begin{cases}0=fv+A_z\dfrac{\partial^2 u}{\partial z^2} & (7a)\\ 0=-fu+A_z\dfrac{\partial^2 v}{\partial z^2} & (7b)\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 & (7c)\end{cases}$$

不失一般性地，假定风力仅沿 y 方向作用，即 $\tau_x = 0, \tau_y = const.$ 再结合假定 1)，控制方程的边界条件为：

$$\begin{cases} z = 0, \rho A_z \dfrac{\partial u}{\partial z} = 0 & (8a) \\ z = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y & (8b) \\ z = \infty, u = v = 0 & (8c) \end{cases}$$

方程求解

$$(7a) + (7b) \times i \Leftrightarrow A_z \frac{\partial^2 (u + iv)}{\partial z^2} = if(u + iv)$$

令 $W = u + iv$，得：

$$A_z \frac{\partial^2 W}{\partial z^2} = ifW \Rightarrow \frac{\partial^2 W}{\partial z^2} = \frac{(1+i)^2 \Omega \sin\varphi}{A_z} W$$

令 $a = \sqrt{\Omega \sin\varphi / A_z}$，$j^2 = (1+i)^2 a^2$，得：

$$\frac{d^2 W}{dz^2} - j^2 W = 0 \tag{9}$$

(9) 式通解为：$W = Ae^{jz} + Be^{-jz}$

结合边界条件：$(8a) + (8b) \times i \Rightarrow z = 0, \rho A_z \dfrac{\partial W}{\partial z} = -\tau_y, z \to \infty, W = 0$

$$z \to \infty \Rightarrow A = 0, W = Be^{-jz}; z = 0, \rho A_z \left.\frac{\partial W}{\partial z}\right|_{z=0} = \rho A_z \left.\frac{\partial \left(Be^{-jz}\right)}{\partial z}\right|_{z=0} = -\tau_y, \Rightarrow B = \tau_y / \left(j\rho A_z\right)$$

因此，方程的解为：

$$W = \frac{\tau_y}{j\rho A_z} e^{-jz} = \frac{i\tau_y}{(1+i)a\rho A_z} e^{-(1+i)az} = \frac{e^{i\frac{\pi}{2}} \tau_y}{\sqrt{2} e^{i\frac{\pi}{4}} a\rho A_z} e^{-(1+i)az}$$

令 $D_0 = \pi / a$, 得到最终解的形式为：

$$W = \frac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z + i\left(\frac{\pi}{4} - \frac{\pi}{D_0} z\right)} \tag{10}$$

物理性质

运动速度

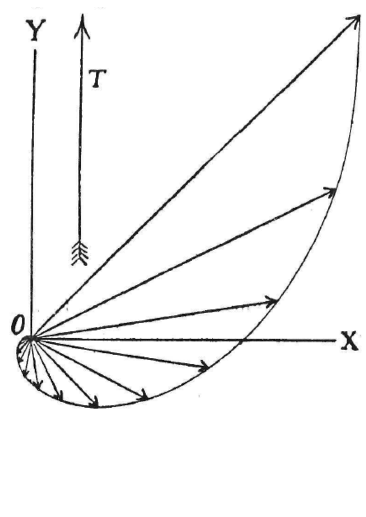
在海面 $(z = 0)$ 处，$W_0 = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{i\frac{\pi}{4}}$. 大小为 $|W_0| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z}$，方向与 x 轴成 45°，即与风向向右偏 45°.

在任意深度处，$|W_z| = \dfrac{\tau_y}{\sqrt{2} a\rho A_z} e^{-\frac{\pi}{D_0} z}$，方向为 $\frac{\pi}{4} - \frac{\pi}{D_0} z$，即流速随深度增加呈指数形式减小，流向随深度的增加而逐渐向右偏.

在摩擦深度 $z = D_0$ 处，$|W_{D_0}| = \dfrac{\tau_y e^{-\pi}}{\sqrt{2} a\rho A_z} = e^{-\pi} |W_0| = 0.043 |W_0|$，方向 $-\frac{3}{4}\pi$，即与 x 轴成 $-135°$，与表面流向正好相反.

Ekman 螺旋和 Ekman 螺线

根据速度的垂向分布，表层流速最大，流向偏向风向的右方 45°；随深度增加，流速逐渐减小，流向逐渐右偏；到摩擦深度，流速是表面流速的 4.3%，流向与表面流向相反，运动可以忽略. 连连接各层流速的矢量端点，构成 Ekman 螺旋；Ekman 螺旋在平面上的投影，称为 Ekman 螺线 [Ekman, 1905].



水平体积输运

体积输运：

$$
\begin{aligned}
S &= \int_0^\infty W dz \\
&= \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{i\frac{\pi}{4}} \int_0^\infty e^{-\frac{\pi}{D_0}(1+i)z} dz \\
&= \frac{\tau_y}{\sqrt{2}a\rho A_z} e^{i\frac{\pi}{4}} \left[-\frac{D_0}{\pi}\frac{1}{(1+i)}\right] e^{-\frac{\pi}{D_0}(1+i)z}\Big|_0^\infty \\
&= \frac{\tau_y}{2\Omega\sin\varphi\rho} = \frac{\tau_y}{f\rho}
\end{aligned}
$$

可以发现，得到的输运结果只有实部，没有虚部，说明体积输运方向为 x 轴正向，即在北半球水体向风向右侧 $90°$ 输运.

2.2.2 有限深海漂流

假定

有限深海 Ekman 漂流中用到了以下假定：

1) 海区无限广阔、有限深，远离海岸.

即无侧边界效应，仅有垂直湍流所生水平湍流摩擦力，并假定垂直湍流粘滞系数 A_z 为常量. 由于海洋有限深，$z \to h, \vec{V} = 0$

2) 定常均匀风场长时间作用.

即运动的基本参量与时间和水平坐标无关且海面无升降、无水平压强梯度.

3) 密度分布均匀，ρ 为常数，不考虑热盐性质.

4) 采用 f 平面近似.

控制方程和边界条件：

$$
\begin{cases}
0 = fv + A_z\dfrac{\partial^2 u}{\partial z^2} \\
0 = -fu + A_z\dfrac{\partial^2 v}{\partial z^2} \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\
z = 0 : \rho A_z\dfrac{\partial u}{\partial z} = 0, \rho A_z\dfrac{\partial v}{\partial z} = -\tau_y \\
z \to h : u = v = 0
\end{cases}
$$

方程求解　令 $\xi = h - z$，定解问题化为：

$$
\begin{cases}
-fv = A_z\dfrac{\partial^2 u}{\partial \xi^2} & (11) \\
fu = A_z\dfrac{\partial^2 v}{\partial \xi^2} & (12) \\
\xi = h : \rho A_z\dfrac{\partial u}{\partial \xi} = 0, \rho A_z\dfrac{\partial v}{\partial \xi} = \tau_y \\
\xi \to 0 : u = v = 0
\end{cases}
$$

令 $W = u + iv, \tau = \tau_x + i\tau_y$, 控制方程：

$$
(11) + (12) \times i \Leftrightarrow \frac{d^2 W}{d\xi^2} - j^2 W = 0
$$

边界条件：

$$
\xi = h : \rho A_z\frac{\partial W}{\partial \xi} = \tau
$$
$$
\xi = 0 : W = 0
$$

与无限深海漂流解法类似，解得：

$$
W = \frac{(1+i)\tau_y}{2a\rho A_z}\frac{sh(1+i)a\xi}{ch(1+i)ah}
$$

物理性质

与水深的关系

(1) $h\ 2D_0$ 时，有限深海漂流流速流向与无限深海相同；(2) 水深越浅，流向随深度增加右偏 (北半球) 越缓慢；(3) 从上层到下层的流速矢量越是趋近风矢量的方向.

体积输运

(1) 在 x, y 方向 (平行和垂直风向) 都有输送；

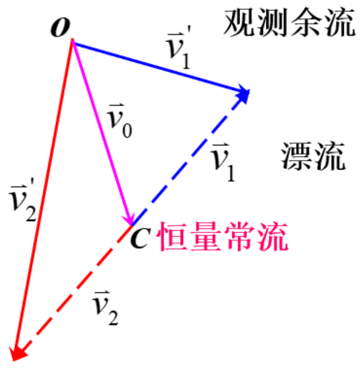
(2) 运输方向为风向右端，$\pm 90°$ 之间：

$$S_x > 0; 0 < h < D_0, ah < \pi, S_y > 0; D_0 < h < 2D_0, \pi < ah < 2\pi, S_y < 0; h > 2D_0, S_y = 0$$

2.2.3 漂流分离

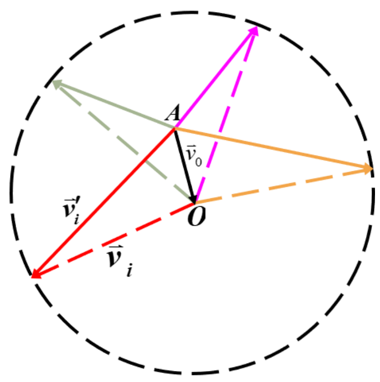
利用风速大小相等、方向相反的两次观测余流分离漂流

⁎ 余流＝漂流＋恒量常流



利用一组风速大小相等、方向不同的实测余流分离漂流

⁎ 漂流速度矢量端点落在同一圆周上

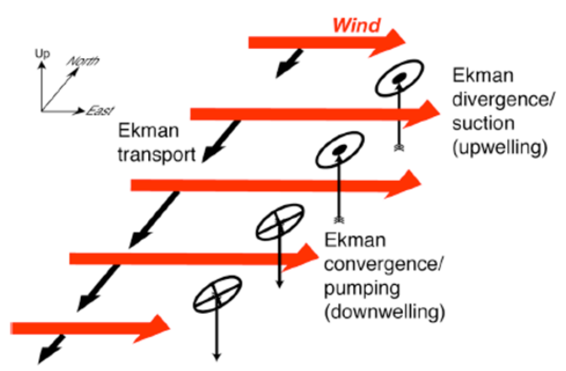


2.2.4 升降流

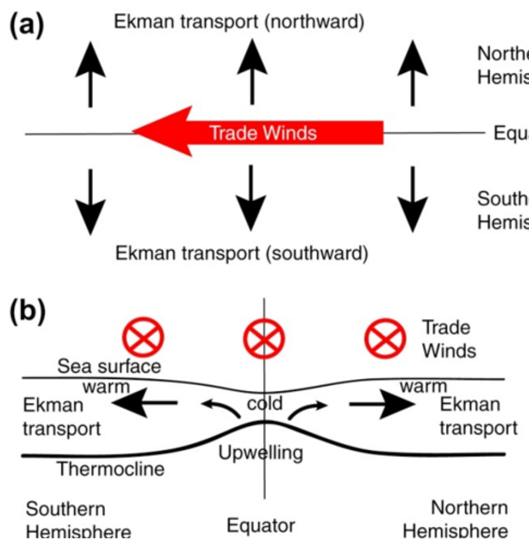
由不均匀风场或风场和地形配合产生的“较强烈”的铅直向流动。

物理背景

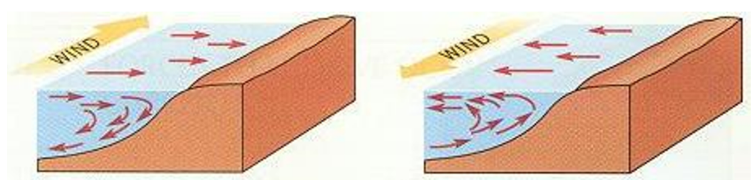
$\boxed{\text{非均匀风场}} \Rightarrow \boxed{\text{非均匀 Ekman 漂流}} \Rightarrow \boxed{\text{非均匀体积输运}} \Rightarrow \boxed{\text{辐聚辐散}} \Rightarrow \boxed{\text{升降流}}$



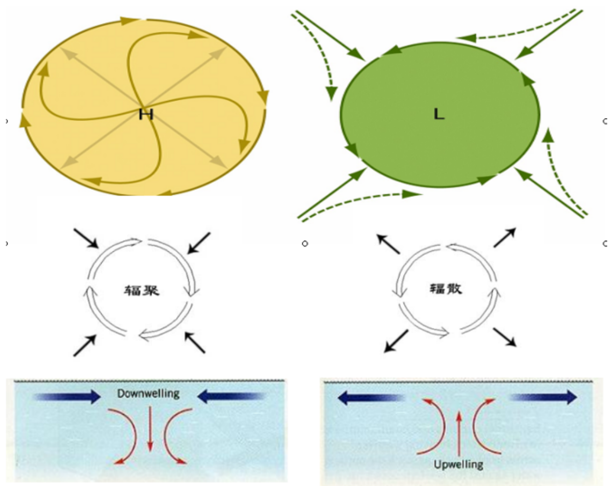
赤道附近的升降流



顺 (沿) 岸风产生的升降流



气旋和反气旋产生的海洋升降流



假定

(1) ρ 为常数；

(2) 直线风系，风仅沿 x 方向有变化；风区内为恒定的均匀风场；风区外无风；$\dfrac{\partial}{\partial y}=0$

(3) 定常风场；$\dfrac{\partial}{\partial t}=0$

(4) 大尺度；$Ro \ll 1$

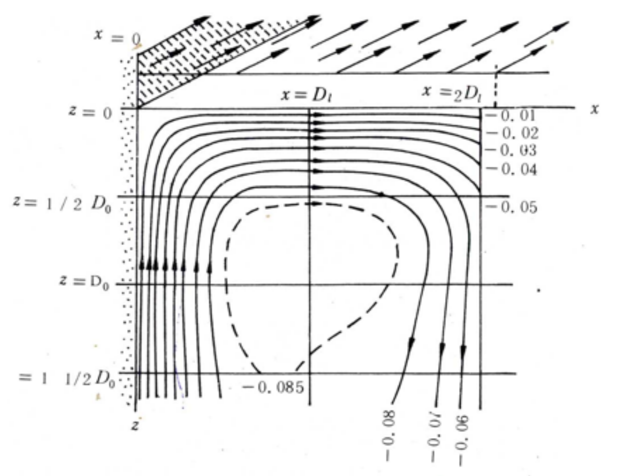
(5) 有限深度.$h \geq 2D_0$

控制方程及边界条件

$$\begin{cases} A_l\dfrac{\partial^2 u}{\partial x^2}+A_z\dfrac{\partial^2 u}{\partial z^2}+fv+g\dfrac{\partial \zeta}{\partial x}=0 \\ A_l\dfrac{\partial^2 v}{\partial x^2}+A_z\dfrac{\partial^2 v}{\partial z^2}-fu=0 \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}=0 \end{cases}$$

$$\begin{cases} z=\zeta:\rho A_z\dfrac{\partial u}{\partial z}=0,\quad \rho A_z\dfrac{\partial v}{\partial z}=-\tau_y\quad (0\le x\le L) \\ z=h:u=v=0 \\ x=0:u=v=0 \\ x\to\infty:u=v=0,\dfrac{\partial \zeta}{\partial x}=0 \end{cases}$$

结果讨论



(1) 近岸产生上升流 $x\le 0.5D_l$；

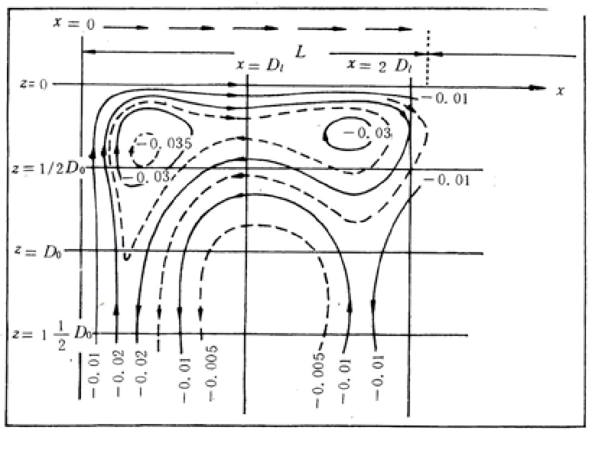
(2) 风区外延附近下降流 $x=2D_l$；

(3) 上升流来自 $z=1.5D_0$ 或更深；

(4) 最大 w 出现在 $z=D_0$；

(5) 上层离岸流，下层向岸流，构成一个循环.

若风向与海岸成 θ 角：



(1) 三个升降流系统：两个顺时针，一个逆时针；

(2) 大顺时针循环；

(3) $\theta=21.5^\circ$ 时，升降流达最大强度；

(4) 纬度越低，升降流越强.

2.3 非定常运动

2.3.1 漂流的发展

假定

(1) 远离海岸和海底的开阔大洋；

(2) 风场均匀恒定；

(3) ρ 为常数；

(4) 海面无倾斜；

(5) 运动非定常.

控制方程

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv = A_z \dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial v}{\partial t} + fu = A_z \dfrac{\partial^2 v}{\partial z^2} \end{cases}$$

初边值条件

$$\begin{cases} z = 0 : A_z \dfrac{\partial u}{\partial z} = 0, \rho A_z \dfrac{\partial v}{\partial z} = -\tau_y (t > 0) \\ z \to \infty : u = v = 0 \\ t = 0 : u = v = 0 / u = C_1, v = C_2 \end{cases}$$

解的讨论

$$\begin{cases} u = \dfrac{2\pi\tau_y}{\rho f D_0} \displaystyle\int_0^{t'} \dfrac{\sin(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \\ v = \dfrac{2\pi\tau_y}{\rho f D_0} \displaystyle\int_0^{t'} \dfrac{\cos(2\pi\eta)}{\sqrt{\eta}} e^{\frac{\pi z^2}{4D_0^2}} d\eta \end{cases}$$

根据下图 [Ekman, 1905]：

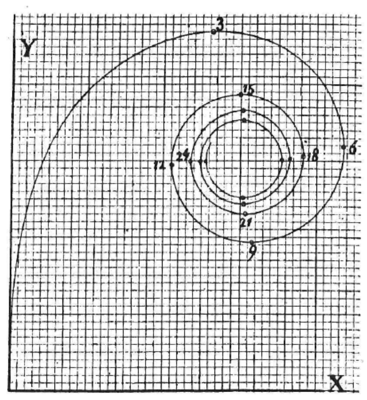


Fig. 3. z=0.

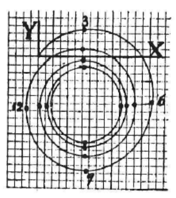


Fig. 4. z=0,5 D.

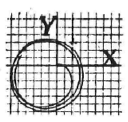


Fig. 5. z=D.

Fig. 6. z=2D.

随时间增加，空间某点流苏端点顺时针旋转 (北半球)，逐渐趋向一个极限值 (即漂流).

2.3.2 惯性流

假定和方程

(1) 风场消失或者流动离开风区；

(2) 外部驱动小时，湍摩擦失去作用；

(3) 流动转为由惯性项维持平衡；

(4) 强制流动转变为自由流动.

$$\begin{cases} \dfrac{du}{dt} - fv = 0 & (13) \\ \dfrac{dv}{dt} + fu = 0 & (14) \end{cases}$$

求解

$$\begin{aligned} (13) \times u + (14 \times v) &\Leftrightarrow u\frac{du}{dt} + v\frac{dv}{dt} = \frac{1}{2}\frac{du^2}{dt} + \frac{1}{2}\frac{dv^2}{dt} = 0 \\ &\Leftrightarrow \frac{d}{dt}\left(u^2 + v^2\right) = 0 \\ &\Leftrightarrow u^2 + v^2 = c = V_0^2 \end{aligned}$$

$$\begin{aligned} (13), (14) &\Rightarrow \begin{cases} v = \dfrac{dy}{dt} = \dfrac{1}{f}\dfrac{du}{dt} \\ u = \dfrac{dx}{dt} = \dfrac{1}{f}\dfrac{dv}{dt} \end{cases} \\ &\Rightarrow \begin{cases} y - y' = \dfrac{1}{f}\left(u - u'\right) \\ x - x' = -\dfrac{1}{f}\left(v - v'\right) \end{cases} \\ &\Rightarrow \begin{cases} y - \left(y' - \dfrac{u'}{f}\right) = \dfrac{u}{f} \\ x - \left(x' + \dfrac{v'}{f}\right) = -\dfrac{v}{f} \end{cases} \\ &\Rightarrow \boxed{\left(y - y_0\right)^2 + \left(x - x_0\right)^2 = \frac{1}{f^2}\left(u^2 + v^2\right) = \frac{V^2}{f^2} = r^2} \end{aligned}$$

流体质点沿半径为 r 的圆周作匀速运动，这个圆称之为惯性圆，对应的流动为惯性流.

惯性圆半径

科氏力充当向心力：$\dfrac{V_0^2}{r} = fV_0 \Rightarrow V_0 = fr \Rightarrow r = \dfrac{V_0}{f} = \dfrac{V_0}{2\omega\sin\varphi}$

随纬度增加而减小；赤道 $r \to \infty$，水质点作直线运动.

周期

$$T_i = \frac{2\pi r}{V_0} = \frac{2\pi r}{fr} = \frac{2\pi}{f} = \frac{\pi}{\omega\sin\varphi}$$

运动方向

北半球，顺时针；南半球，逆时针.

背景流

(1) 当无其他外加流动时，所有惯性圆的圆心位于同一条铅直线上，因而海水就像以角速度 $2\omega\sin\varphi$ 旋转的刚体一样；

(2) 当有其他外加流动时，除了在同一水平面上所有海水质点皆在同一时刻由同一流速速率外，还依外加流动速度方向移动.

2.4 风生大洋环流

2.4.1 Sverdrup 理论

假定

(1) 远离海岸的大洋中部海区，Ro≪ 1 大尺度、等深大洋 h 为常数；
(2) 远离边界、无侧边界影响，无水平湍摩擦应力 $\mathrm{E}_l \ll 1$；
(3) 定常风定常流动；
(4) ρ 为常数；
(5) β 平面近似.

控制方程

$$\begin{cases} -fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + A_z\dfrac{\partial^2 u}{\partial^2 z} \\ fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + A_z\dfrac{\partial^2 v}{\partial^2 z} \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

边界条件

$$\begin{cases} z = \zeta(\text{海面}) : \rho A_z\dfrac{\partial u}{\partial z} = \tau_{x\zeta}, \rho A_z\dfrac{\partial v}{\partial z} = \tau_{y\zeta} \\ z = -h(\text{足够深}) : u = v = 0, \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0, \dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial y} = 0 \end{cases}$$

求解

对上式进行垂直积分：

$$\begin{cases} -fM_y = -\dfrac{\partial P}{\partial x} + \tau_{x\zeta} & (15) \\ fM_x = -\dfrac{\partial P}{\partial y} + \tau_{y\zeta} & (16) \\ \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_y}{\partial y} = 0 \end{cases}$$

其中，$M_x = \int_{-h}^{0} \rho u dz, M_y = \int_{-h}^{0} \rho v dz, P = \int_{-h}^{0} p dz, \tau_{x\zeta} = \int_{-h}^{0} \rho A_z\dfrac{\partial^2 u}{\partial z^2} = \rho A_z\left(\dfrac{\partial u}{\partial z}\Big|_{z=0} - \dfrac{\partial u}{\partial z}\Big|_{z=-h}\right) \quad (\zeta \ll h)$

$$\begin{aligned} \frac{\partial(1)}{\partial y} - \frac{\partial(2)}{\partial x} &\Leftrightarrow -M_y\frac{\partial f}{\partial y} \overset{0}{\boxed{-f\frac{\partial M_y}{\partial y} - f\frac{\partial M_x}{\partial x}}} = \frac{\partial \tau_{x\zeta}}{\partial y} - \frac{\partial \tau_{y\zeta}}{\partial x} \\ &\Leftrightarrow M_y\frac{\partial f}{\partial y} = \frac{\partial \tau_{y\zeta}}{\partial x} - \frac{\partial \tau_{x\zeta}}{\partial y} \\ &\Leftrightarrow \underbrace{\boxed{\beta M_y = \mathrm{rot}_z\, \tau_\zeta}}_{\text{Sverdrup 方程}} \end{aligned}$$

Sverdrup 方程的物理意义 1：海水南北向的输运由风应力旋度所驱动.

讨论

将质量输运分为 Ekman 漂流输运与地转流输运两部分：

$$M_x = M_{xE}(\text{Ekman 漂流}) + M_{xg}(\text{地转流})$$
$$M_y = M_{yE}(\text{Ekman 漂流}) + M_{yg}(\text{地转流})$$

$$
\begin{cases}
-fM_{yE} = \tau_{x\zeta} & (17)\\
fM_{xE} = \tau_{y\zeta} & (18)\\
-fM_{yg} = -\dfrac{\partial P}{\partial x} & (19)\\
fM_{yg} = -\dfrac{\partial P}{\partial y} & (20)
\end{cases}
$$

$$
\frac{\partial(17)}{\partial y} - \frac{\partial(18)}{\partial x} \Leftrightarrow \frac{\partial M_{xE}}{\partial x} + \frac{\partial M_{yE}}{\partial y} = \boxed{\nabla \cdot \vec{M}_E = \left(\mathrm{rot}_z\, \vec{\tau}_\zeta - \beta M_{yE}\right)/f} \tag{21}
$$

$$
\frac{\partial(19)}{\partial y} - \frac{\partial(20)}{\partial x} \Leftrightarrow \frac{\partial M_{xg}}{\partial x} + \frac{\partial M_{yg}}{\partial y} = \boxed{\nabla \cdot \vec{M}_g = -\beta M_{yg}/f} \tag{22}
$$

(1) Ekman 漂流质量输运的水平散度与 ① 风应力旋度 ② f ③ β 有关.
(2) 地转流质量输运的水平辐散引起南北向的地转运动.

$$
(21) + (22) \Leftrightarrow \mathrm{rot}_z\, t\vec{au}_\zeta - \beta M_y = 0 \Leftrightarrow \boxed{\beta M_y = \mathrm{rot}_z\, \vec{\tau}_\zeta}
$$

Sverdrup 方程的物理意义 2: 地转流流量的散度和 Ekman 漂流流量的散度相平衡，所以 Sverdrup 方程又称 Sverdrup 平衡.
(1) 所有南北向的地转运动，必须显示水平散度；
(2) 虽然 Ekman 漂流流量的散度与地转流流量的散度本身不为 0，但是它们的和，即总流量的水平散度必须为 0，说明 Ekman 漂流流量的散度和地转流流量的散度刚好取得平衡；
(3) $\mathrm{rot}_z\, \vec{\tau}_\zeta = 0$：只存在东西方向的输运，$\mathrm{rot}_z\, \vec{\tau}_\zeta > 0$：质量输运向北 (北半球)，$\mathrm{rot}_z\, \vec{\tau}_\zeta < 0$：质量输运向南 (南半球)；
(4) 地转流引起的南北质量运输量比 Ekman 漂流引起的大.

缺陷

设仅有纬向风，且 $\tau_{x\zeta}$ 仅为 y 的函数：

$$
M_y = \frac{1}{\beta}\,\mathrm{rot}_z\, \vec{\tau}_\zeta = \frac{1}{\beta}\left(\cancel{\frac{\partial \tau_{y\zeta}}{\partial x}} - \frac{\partial \tau_{x\zeta}}{\partial y}\right) = -\frac{a}{2\omega\cos\varphi}\frac{\partial \tau_{x\zeta}}{\partial y}
$$

$$
\frac{\partial M_x}{\partial x} = -\frac{\partial M_y}{\partial y} \Rightarrow M_x = \frac{x}{2\omega\cos\varphi}\left(a\frac{\partial^2 \tau_{xf}}{\partial y^2} + \frac{\partial \tau_{xf}}{\partial y} tg\varphi\right) + c(y)
$$

在东、西边界，$M_x = 0$ 而 Sverdrup 理论不能同时满足.

2.4.2 Stommel 理论

假定

(1) 远离海岸的大洋中部海区，Ro$\ll 1$ 大尺度、等深大洋 h 为常数；
(2) 远离边界、无侧边界影响，无水平湍摩擦应力 $\mathrm{E}_l \ll 1$；
(3) 定常风定常流动；
(4) ρ 为常数；
(5) β 平面近似；
(6) 考虑底摩擦

控制方程

$$
\begin{cases}
-fv = -g\dfrac{\partial \zeta}{\partial x} + A_z\dfrac{\partial^2 u}{\partial^2 z}\\
fu = -g\dfrac{\partial \zeta}{\partial y} + A_z\dfrac{\partial^2 v}{\partial^2 z}\\
0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g\\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0
\end{cases}
$$

边界条件

$$
\begin{cases}
z=\zeta(\text{海面}):\tau_{x,\zeta}=\rho A_z\dfrac{\partial u}{\partial z}\Big|_{t=\zeta}=-F\cos(\pi y/b),\tau_{x,\zeta}=0\\
z=-h(\text{海底}):\tau_{x,-h}=\rho+\dfrac{\partial u}{\partial z}\Big|_{t=-h}=\rho ku,\tau_{y,-h}=\rho+\dfrac{\partial v}{\partial z}\Big|_{t=-h}=\rho kv
\end{cases}
$$

求解

对上式进行垂直平均：

$$
\begin{cases}
-f\langle v\rangle=-g\dfrac{\partial\zeta}{\partial x}+\dfrac{A_z}{h+\zeta}\dfrac{\partial u}{\partial z}\Big|_{\zeta}-\dfrac{A_z}{h+\zeta}\dfrac{\partial u}{\partial z}\Big|_{-h}\\
f\langle u\rangle=-g\dfrac{\partial\zeta}{\partial y}+\dfrac{A_z}{h+\zeta}\dfrac{\partial v}{\partial z}\Big|_{\zeta}-\dfrac{A_z}{h+\zeta}\dfrac{\partial v}{\partial z}\Big|_{-h}\\
\dfrac{\partial}{\partial x}[(n+\zeta)\langle u\rangle]+\dfrac{\partial}{\partial y}[(n+\zeta)\langle v\rangle]=0
\end{cases}
$$

将边界条件代入方程：

$$
\begin{cases}
0=f\rho hv-F\cos(\pi y/b)-\rho ku-\rho gh\dfrac{\partial\zeta}{\partial x}\quad(\zeta\ll k) & (23)\\
0=-f\rho hu-\rho kv-\rho gh\dfrac{\partial\zeta}{\partial y} & (24)\\
\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0
\end{cases}
$$

$$
\frac{\partial(23)}{\partial y}-\frac{\partial(24)}{\partial x}\Rightarrow\frac{h}{k}[\beta v+r\sin(\pi y/b)]+\overbrace{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}^{\text{来源于底摩擦}}=0
$$

引入流函数 $\psi:u=\dfrac{\partial\psi}{\partial y},v=-\dfrac{\partial\psi}{\partial x}$：

$$
\overbrace{\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}}^{\text{来源于底摩擦}}+\frac{h}{k}\beta\frac{\partial\psi}{\partial x}=\frac{h}{k}r\sin\frac{\pi y}{b}
$$

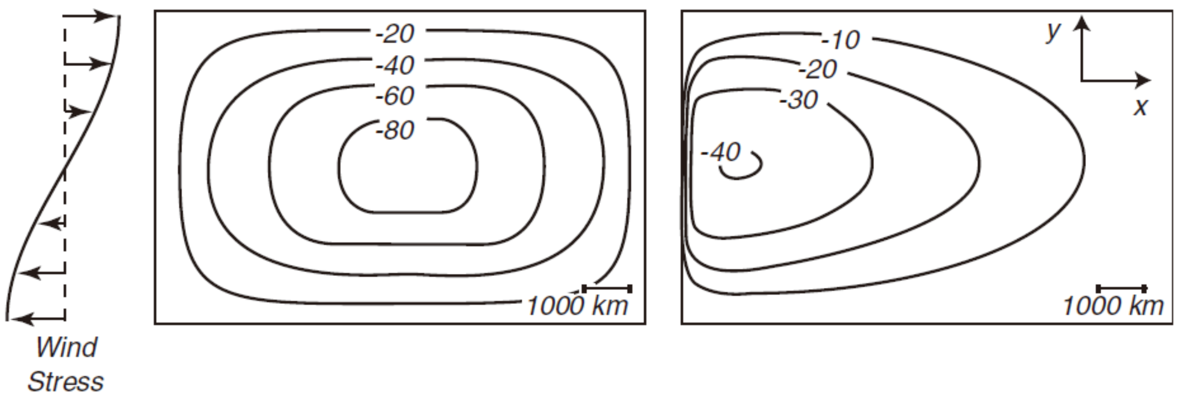
边界条件：

$$
\psi(0,y)=\psi(a,y)=\psi(x,0)=\psi(x,b)=0
$$

$$
\Rightarrow\psi(x,y)=\frac{Fb}{k\pi}\sin\frac{\pi y}{b}\left[\frac{e^{\frac{h\beta}{2k}(a-x)}\operatorname{sh}\alpha x+e^{\frac{h\beta}{2k}x}\operatorname{sh}\alpha(a-x)}{\operatorname{sh}\alpha a}-1\right]
$$

特别地，若 $\beta=0$：$\psi(x,y)=\dfrac{Fb}{k\pi}\sin\dfrac{\pi y}{b}\left[\dfrac{\operatorname{sh}\frac{\pi}{b}x+\operatorname{sh}\frac{\pi}{b}(a-x)}{\operatorname{sh}\frac{\pi}{b}a}-1\right]$

讨论　图片来自 *Introduction to Physical Oceanography*(Robert H. Stewart,2008 pp190)



※β效应导致了西向强化现象.

$\beta=0$(非旋转坐标系或 f 平面近似):	$\beta\neq0$(β 平面近似):
(1) 流线南北对称；	(1) 流线南北对称；
(2) 流线东西对称.	(2) 流线东西不对称，西部密集，东部稀疏.

2.4.3 惯性理论

方程

在 Sverdrup 理论的控制方程中引入惯性项：

$$\begin{cases}\dfrac{du}{dt}-fv=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+A_z\dfrac{\partial^2 u}{\partial^2 z}\\ \dfrac{dv}{dt}+fu=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+A_z\dfrac{\partial^2 v}{\partial^2 z}\\ 0=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}-g\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}$$

求解

内区 (中部区)

满足 Sverdrup 理论：$\beta M_y=\mathrm{rot}_z\,\tau_\zeta, M_y=\dfrac{\partial\varphi}{\partial x}=\dfrac{1}{\beta}\mathrm{rot}_z\,\tau_\zeta=\dfrac{1}{\beta}\left(\dfrac{\partial\tau_{y\zeta}}{\partial x}-\dfrac{\partial\tau_{x\zeta}}{\partial y}\right)$

又设风应力：$\tau_{x\zeta}=-W\left(1-\dfrac{y^2}{s^2}\right), \tau_{y\zeta}=0\quad(0\le y\le s)$

$$\begin{aligned}&\Rightarrow\frac{\partial\varphi}{\partial x}=-\frac{2W}{\beta s^2}y\\ &\Rightarrow\varphi=-\frac{2W}{\beta s^2}yx+C(y)\\ (x=r:\varphi=0)\quad &0=-\frac{2W}{\beta s^2}yr+C(y)\\ &\Rightarrow\varphi=\frac{2W}{\beta s^2}y(r-x)\end{aligned}$$

大洋西部海域

不考虑湍摩擦效应：

$$\begin{cases}\dfrac{du}{dt}-fv=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & (25)\\ \dfrac{dv}{dt}+fu=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & (26)\\ 0=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}-g\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0\end{cases}$$

$$\begin{aligned}\frac{\partial(26)}{\partial x}-\frac{\partial(25)}{\partial y}&\Leftrightarrow\frac{d}{dt}(\xi_r+f)+(\xi_r+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\\ &\Leftrightarrow\frac{d}{dt}(\xi_r+f)=-(\xi_r+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\end{aligned}\tag{27}$$

$\xi_r=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ 相对涡度，f 行星涡度，$\xi_a=\xi_r+f$ 绝对涡度.

对连续方程进行垂向平均：

$$\begin{aligned}&\frac{\partial\zeta}{\partial t}+\frac{\partial[(h+\zeta)\langle u\rangle]}{\partial x}+\frac{\partial[(h+\zeta)(v)]}{\partial y}=0\\ &\Leftrightarrow\frac{d(\zeta+h)}{dt}+(\zeta+h)\left(\frac{\partial(u)}{\partial x}+\frac{\partial(v)}{\partial y}\right)=0\\ (\text{令}H=h+\zeta)\ &\Leftrightarrow\frac{dH}{dt}+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\end{aligned}\tag{28}$$

将 (28) 代入 (27) 中：

$$\begin{aligned}
&\Rightarrow \frac{d}{dt}(\xi_r+f)=\frac{1}{H}(\xi_r+f)\frac{dH}{dt}\\
&\Rightarrow \frac{1}{H}\frac{d}{dt}(\xi_r+f)=\frac{1}{H^2}(\xi_r+f)\frac{dH}{dt}\\
&\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\xi_r+f}{H}\right)=0}
\end{aligned} \tag{29}$$

(29) 为 位势涡度守恒方程.

假设在西边界区 $\frac{\partial u}{\partial y}=0$，则:

$$\xi_r=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial \varphi}{\partial x}\right)$$

代入 (29)：

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\frac{\partial}{\partial x}\left(\frac{1}{M}\frac{\partial \varphi}{\partial x}\right)+f}{H}\right]=0\\
\Rightarrow&\frac{\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial \varphi}{\partial x}\right)+f}{H}=F(\varphi)\\
\Rightarrow&\frac{1}{H}\frac{\partial^2\varphi}{\partial x^2}+f_0+\beta y=HF(\varphi)=G(\varphi)
\end{aligned}$$

在西边界层的边缘

内部的解即为西边界的解，惯性项可以忽略：

$$x=L: f_0+\beta y=G(\phi_i)$$

在内区的边缘 ($L\ll r$)

$$\varphi_i=\frac{2W}{\beta s^2}y(r-x)=\frac{2W}{\beta s^2}y(r-L)=\frac{2W}{\beta s^2}yr=u^*y\quad\left(u^*=\frac{2W}{\beta s^2}r\right)$$

上面两解应该等价，因此：

$$\begin{aligned}
&f_0+\beta y=G(u^*y)\\
\Rightarrow&G(\varphi)=f_0+\beta\frac{\varphi}{u^*}\\
\Rightarrow&\frac{\partial^2\varphi}{\partial x^2}+H(f_0+\beta y)=H\left(f_0+\beta\frac{\varphi}{u^*}\right)\\
\Rightarrow&\frac{\partial^2\varphi}{\partial x^2}-\frac{H\beta}{u^*}\varphi=-H\beta y
\end{aligned}$$

再结合两个边界条件：$x=0:\begin{cases}\varphi(0,y)=0\\ \frac{\partial\varphi}{\partial x}=0\end{cases}$，上面的二阶常系数线性微分方程的解为：

$$\varphi=u^*y\left[1-e^{-(H\beta/u^*)^{1/2}x}\right]$$

海水南北输运：

$$\boxed{M_y=\frac{\partial\varphi}{\partial x}=(u^*H\beta)^{1/2}ye^{-(H\beta u^*)^{1/2}x}}$$

在西边界区域有一强烈的北向流动，近岸处质量运输很大，随着离岸距离的增加，质量输运迅速减少.

2.4.4 Munk 理论

假定

(1) 矩形大洋 $r\times 2s$；

(2) 远离海岸的等深封闭矩形大洋，静止时水深为常量 h；

(3) 增加了侧向摩擦.

控制方程

$$
\begin{cases}
-fv=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+A_l\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+A_z\dfrac{\partial^2 u}{\partial z^2}\\
fu=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+A_l\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+A_z\dfrac{\partial^2 v}{\partial z^2}\\
0=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}-g\\
\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0
\end{cases}
$$

边界条件

$$
\begin{cases}
z=\zeta,\quad \rho A_z\dfrac{\partial u}{\partial z}=\tau_{x\zeta},\rho A_z\dfrac{\partial v}{\partial z}=\tau_{y\zeta}\\
z=-h,\quad u=v=0,\dfrac{\partial u}{\partial z}=\dfrac{\partial v}{\partial z}=0
\end{cases}
$$

求解

全流方程 (垂直积分)：

$$
\begin{cases}
-fM_y=-\dfrac{\partial P}{\partial x}+A_l\left(\dfrac{\partial^2 M_x}{\partial x^2}+\dfrac{\partial^2 M_x}{\partial y^2}\right)+\tau_{x\zeta} & (30)\\
fM_x=-\dfrac{\partial P}{\partial y}+A_l\left(\dfrac{\partial^2 M_y}{\partial x^2}+\dfrac{\partial^2 M_y}{\partial y^2}\right)+\tau_{y\zeta} & (31)\\
\dfrac{\partial M_x}{\partial x}+\dfrac{\partial M_y}{\partial y}=0
\end{cases}
$$

其中，$M_x=\int_{-h}^{0}\rho u dz, M_y=\int_{-h}^{0}\rho v dz, P=\int_{-h}^{0}p dz, \tau_{x\zeta}=\int_{-h}^{0}\rho A_z\frac{\partial^2 u}{\partial z^2}=\rho A_z\left(\frac{\partial u}{\partial z}\Big|_{z=0}-\frac{\partial u}{\partial z}\Big|_{z=-h}\right)$ $(\zeta\ll h)$ 引入流函数：$M_x=-\frac{\partial \psi}{\partial y}, M_y=\frac{\partial \psi}{\partial x}$

$$
\begin{aligned}
\frac{\partial(30)}{\partial y}-\frac{\partial(31)}{\partial x} &\Leftrightarrow A_l\left(\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}\right)-\beta\frac{\partial\psi}{\partial x}=-\left(\frac{\partial\tau_{y\zeta}}{\partial x}-\frac{\partial\tau_{x\zeta}}{\partial y}\right)\\
&\Leftrightarrow A_l\nabla^4\psi-\beta\frac{\partial\psi}{\partial x}=-\mathrm{rot}_z\,\vec{\tau}_\zeta
\end{aligned}
$$

边界条件：

$$
\begin{cases}
x=0,r:\psi=0,\dfrac{\partial\psi}{\partial x}=0\\
y=-s,s:\psi=0,\dfrac{\partial\psi}{\partial y}=0
\end{cases}
$$

内区 (中部区)　满足 Sverdrup 理论：$\beta\frac{\partial\psi}{\partial x}=\mathrm{rot}_z\,\tau_\zeta=-\frac{\partial\tau_{x\zeta}}{\partial y}$

假定纬向风系：$\begin{cases}\tau_{y\zeta}=0\quad(-s<y<s)\\ \tau_{x\zeta}=a\cos ny+b\sin ny+c\\ n=\dfrac{j\pi}{s},j=1,2,3\cdots\end{cases}$

对 x 积分：$\psi=-\frac{1}{\beta}\frac{\partial\tau_{x\zeta}}{\partial y}x+f_1(y)$

结合 $\psi|_r=0\Rightarrow\psi=-\frac{1}{\beta}\frac{\partial\tau_{x\zeta}}{\partial y}(x-r)$ 因此，$x=0$ 处：

$$
\begin{cases}
\psi(0,y)=\dfrac{r}{\beta}\dfrac{\partial\tau_{x\zeta}}{\partial y}\\
\dfrac{\partial\psi}{\partial x}\Big|_{x=0}=M_y(0,y)=-\dfrac{1}{\beta}\dfrac{\partial\tau_{x\zeta}}{\partial y}
\end{cases}
$$

西边界区

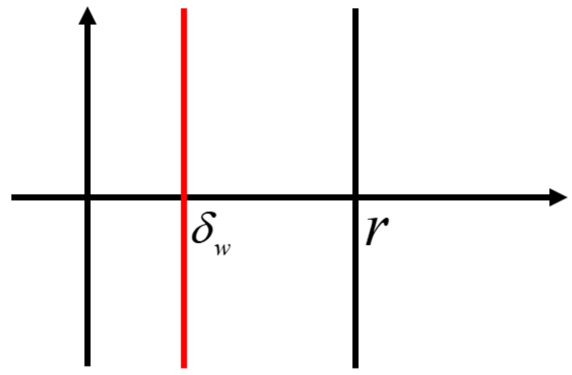
保留侧向湍流应力，结合 $L_x\ll L_y$：

$$
A_l\nabla^4\psi-\beta\frac{\partial\psi}{\partial x}=-\mathrm{rot}_z\,\vec{\tau}_\zeta\Rightarrow A_l\frac{\partial\psi}{\partial x^4}-\beta\frac{\partial\psi}{\partial x}=\frac{\partial\tau_{x\zeta}}{\partial y}
$$

设试解：$\psi = X(x)\dfrac{\partial \tau_{x\zeta}}{\partial y} \Rightarrow A_l X^{(4)} - \beta X' = 1 \Rightarrow X(x) = A + Be^{kx} + De^{-\frac{k}{2}x}\cos\left(\dfrac{\sqrt{3}}{2}kx + E\right) - \dfrac{x}{\beta}$ 自然边界条件：

$$x = 0: \begin{cases} \psi = 0 \\ \dfrac{\partial \psi}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} X(0) = 0 \\ X'(0) = 0 \end{cases}$$

衔接边界条件：$x = \delta_w: \begin{cases} \psi = \dfrac{r}{\beta}\dfrac{\partial \tau_{x\zeta}}{\partial y} \\ \dfrac{\partial \psi}{\partial x} = -\dfrac{1}{\beta}\dfrac{\partial \tau_{x\zeta}}{\partial y} \end{cases} \Rightarrow \begin{cases} X(0) = \dfrac{r}{\beta} \\ X'(0) = -\dfrac{1}{\beta} \end{cases}$



解得：$\psi(x,y) = \dfrac{r}{\beta}\left[1 - \dfrac{2}{\sqrt{3}}e^{-\frac{1}{2}kx}\cos\left(\dfrac{\sqrt{3}}{2}kx - \dfrac{\pi}{6}\right)\right]\dfrac{\partial \tau_{x\zeta}}{\partial y}$

东边界区

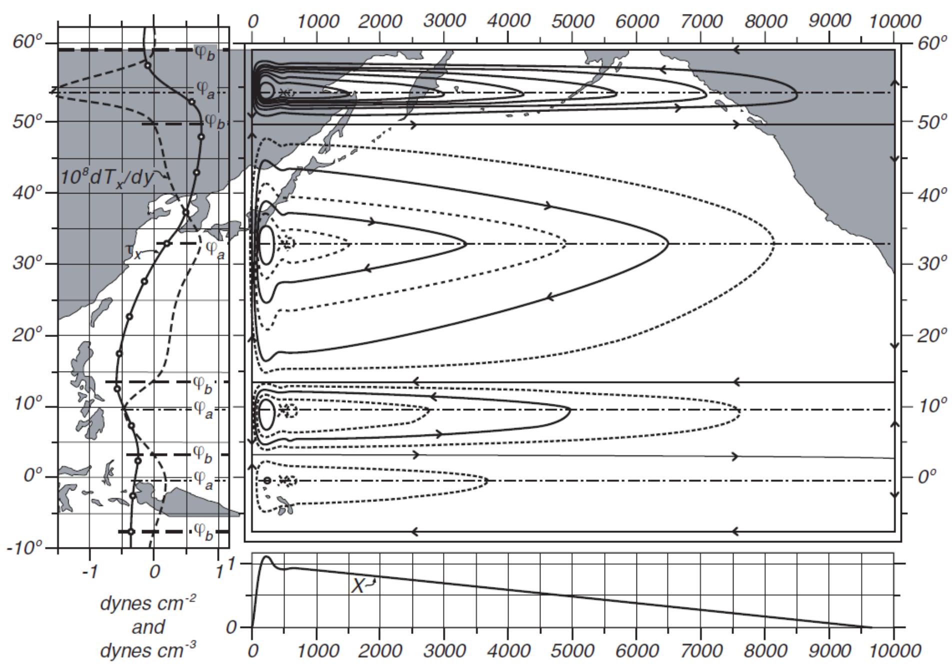
$$\psi(x,y) = \frac{r}{\beta}\left[1 - \frac{x}{r} + \frac{1}{kr}\left(e^{-k(r-x)} - 1\right)\right]\frac{\partial \tau_{x\zeta}}{\partial y}$$

综合内区、西边界区和东边界区的解可得统一的解的形式：

$$\boxed{\psi = \frac{r}{\beta} f(x) \frac{\partial x_{x\zeta}}{\partial y}}$$

讨论

环流空间分布特征　图片来自 *Introduction to Physical Oceanography*(Robert H. Stewart,2008 pp191)



(1) Gyres 之间的分界处位于：$M_y = \dfrac{\partial \psi}{\partial x} = 0$，只有东西向的流动：$f'(x) = 0/\dfrac{\partial \tau_{x\zeta}}{\partial y} = 0$；

(2) Gyres 主轴位于：$M_x = \dfrac{\partial \psi}{\partial x} = 0$ 只有南北向的流动：$f(x) = 0/\dfrac{\partial^2 \tau_{x\zeta}}{\partial y^2} = 0$；
(3) 流动西强东弱.

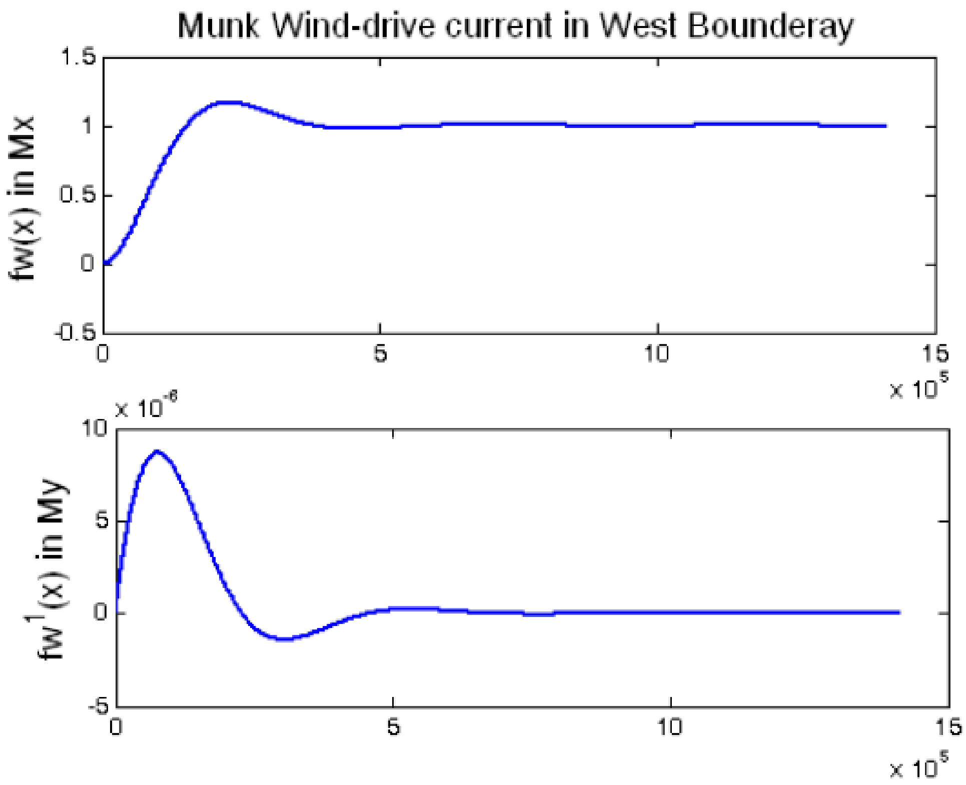
大洋西边界区的物质输运特点

$$\psi_W = \frac{r}{\beta} f_m(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

质量输运：

$$\begin{cases} M_{xW} = -\dfrac{\partial \psi}{\partial y} = -\dfrac{r}{\beta} f_W(x) \dfrac{\partial^2 \tau_{x\zeta}}{\partial y^2} \\ M_{yW} = \dfrac{\partial \psi}{\partial x} = \dfrac{r}{\beta} f'_W(x) \dfrac{\partial \tau_{x\zeta}}{\partial y} \end{cases}$$

其中，$\begin{cases} f_W(x) = 1 - \dfrac{2}{\sqrt{3}} e^{-\frac{1}{2}kx} \cos\left(\dfrac{\sqrt{3}}{2}kx - \dfrac{\pi}{6}\right) \\ f'_W(x) = \dfrac{2}{\sqrt{3}} k e^{-\frac{1}{2}kx} \sin \dfrac{\sqrt{3}}{2} kx \end{cases}$



随 x 增大而衰减的阻尼振动

对于任意纬度：

$$M_{yW} = \frac{\partial \psi}{\partial x} = \frac{r}{\beta} f'_w(x) \frac{\partial \tau_{x\zeta}}{\partial y}$$

因此，$f'_W(x)$ 的极值决定了 M_{yW} 的极值：

$$f''_W(x) = 0$$

$$(\text{令}\lambda_W = \frac{2\pi}{\text{波数}} = \frac{4\pi}{\sqrt{3}k}) \Rightarrow x_{a,b} = \frac{2\pi}{3\sqrt{3}k}\left(\frac{1}{6}\lambda_W\right), \quad \frac{8\pi}{3\sqrt{3}k}\left(\frac{4}{6}\lambda_W\right)$$

$$\Rightarrow \begin{cases} f'_W(x_a) = ke^{-\pi/3\sqrt{3}} \\ f'_W(x_b) = -ke^{-4\pi/3\sqrt{3}} \end{cases}$$

$$\frac{f'_W(x_b)}{f'_W(x_a)} = -e^{-\pi/3} = -0.17$$

在西边界内，以 $x = x_a(1/6)$ 波长为主轴处有一主流为北向的流动；以 $x = x_b(4/6)$ 波长为主轴处存在一逆流，逆流的量值为主流的 0.17 倍.
对于任意给定纬度，$f_W(x)$ 的极值决定 M_{xW} 的极值：

$$f'_W(x) = 0 \Rightarrow x_{1,2} = \frac{2\pi}{\sqrt{3}k}, \frac{4\pi}{\sqrt{3}k} \Rightarrow \begin{cases} f_W(x_1) = 1 - e^{-\pi/\sqrt{3}} \\ f_W(x_2) = 1 + e^{-2\pi/\sqrt{3}} \end{cases}$$

极值在 1 附近变动.

缺陷

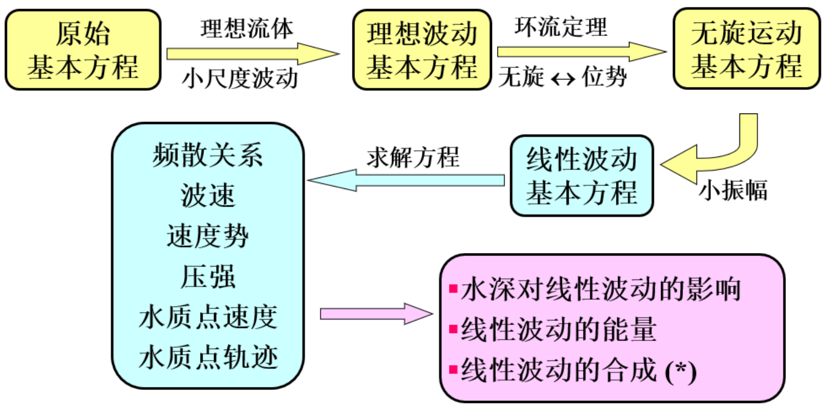
(1) 若取 $A_l = 5\times 10^3 m^2 s^{-1}$，西部阻尼振荡的波长比实际大约 3 倍：$\lambda = \dfrac{4\pi}{\sqrt{3}k} = \dfrac{4\pi}{\sqrt{3}}\dfrac{1}{\sqrt[3]{\beta/A_l}} \approx 200km$；符合实际主流和逆流宽度时，$A_l = 10^2 m^2 s^{-1}$

(2) 西部总流量比实测值小一半.

3 海浪

3.1 线性波动理论

理论框架：



3.1.1 无旋运动的基本方程

假定及方程

(1) 海水均匀不可压缩；

(2) 理想流体；

(3) 短周期小尺度波动；

(4) 重力为唯一的外力；

(5) 忽略分子粘性项、科氏力、引潮力和湍摩擦力.

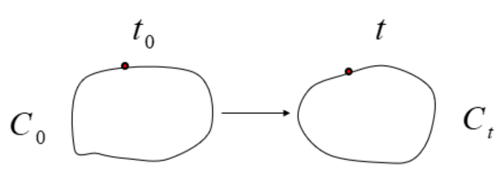
控制方程：

$$
\begin{cases}
\dfrac{d\vec{V}}{dt} = \dfrac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = -\dfrac{1}{\rho}\nabla p - \vec{g} \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}
\end{cases}
$$

边界条件：

$$
\begin{cases}
z = \zeta(\text{海面}) : \dfrac{\partial \zeta}{\partial t} + u\dfrac{\partial \zeta}{\partial x} + v\dfrac{\partial \zeta}{\partial y} = w, p_I = p_a(x,y,t) \\
\text{固体边界处} : V_n = 0
\end{cases}
$$

环流定理



C_t 上的环流：$\Gamma(t)=\oint\limits_{ct} udx+vdy+wdz$

$$\Rightarrow \frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\left(\frac{du}{dt}dx+\frac{dv}{dt}dy+\frac{dw}{dt}dz\right)+\underbrace{\oint\limits_{ct}(udu+vdv+wdw)}_{=\frac{1}{2}\oint\limits_{ct} du^2+dv^2+dw^2=0}$$

$$\Rightarrow \frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\frac{d\vec{V}}{dt}\cdot d\vec{l}$$

环流的实质微商等于加速度的环流.

$$\frac{d\Gamma(t)}{dt}=\oint\limits_{ct}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)dx+\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right)dy+\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\right)dz=\oint\limits_{ct} d\underbrace{\left(-\frac{p}{\rho}-gz\right)}_{\text{单值函数}}=0$$

环流定理：对不可压缩的理想流体，由相同质点构成的封闭曲线上的环流不随时间而变。

无旋运动的基本方程和边界条件

若在重力场中，理想流体于起始时刻为静止或匀速运动，则任何时刻，对任何封闭曲线有：$\Gamma(t)=0$，根据 Stokes 定理：

$$\Gamma(t)=\oint\limits_{ct}\vec{V}\cdot d\vec{l}=\iint\limits_{s}\nabla\times\vec{V}d\sigma=0\Rightarrow\nabla\times\vec{V}\Longrightarrow\vec{V}=\nabla\varphi$$

运动方程：

$$\frac{d\vec{V}}{dt}=\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-\vec{g}$$

$$\Rightarrow\frac{d\vec{V}}{dt}=\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})=-\nabla\frac{p-p_0}{\rho}-\nabla(gz)$$

$$\Rightarrow\nabla\frac{\partial\varphi}{\partial t}+\frac{1}{2}\nabla(\nabla\varphi\cdot\nabla\varphi)=-\nabla\frac{p-p_0}{\rho}-\nabla(gz)$$

$$\Rightarrow\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)(\nabla\varphi)+\frac{p-p_0}{\rho}+gz=0$$

连续方程：

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

运动学边界条件：

$$\text{海面}:\left(\frac{\partial\zeta}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\zeta}{\partial y}\right)\bigg|_{z=\zeta}=\frac{\partial\varphi}{\partial z}\bigg|_{z=\zeta}$$

$$\text{固定边界}:\frac{\partial\varphi}{\partial n}=0$$

动力学边界条件：

$$p_I=p_a(x,y,t)$$

$$\left[\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)(\nabla\varphi)\right]\bigg|_{z=\zeta}+g\zeta=0$$

3.1.2 线性波动

假定

(1) 均质不可压理想流体；

(2) 波动的振幅相对波长很小；

(3) 设水域广阔等深；

(4) 波动只沿 x 方向传播.

$$\text{小振幅假定}:\Rightarrow\begin{cases}\varphi\text{的微商乘积项可忽略}\\ \dfrac{\partial\varphi}{\partial z}\bigg|_{z=\zeta}\approx\dfrac{\partial\varphi}{\partial z}\bigg|_{z=0},\ \dfrac{\partial\varphi}{\partial t}\bigg|_{z=\zeta}\approx\dfrac{\partial\varphi}{\partial t}\bigg|_{z=0}\end{cases}$$

方程简化

运动方程：$\dfrac{\partial\varphi}{\partial t}+\cancel{\dfrac{1}{2}(\nabla\varphi)(\nabla\varphi)}+\dfrac{p-p_0}{\rho}+gz=0$

连续方程：$\Delta\varphi=\dfrac{\partial^2\varphi}{\partial x^2}+\cancel{\dfrac{\partial^2\varphi}{\partial y^2}}+\dfrac{\partial^2\varphi}{\partial z^2}=0$

边界条件：$\begin{cases}\left(\dfrac{\partial\zeta}{\partial t}+\cancel{\dfrac{\partial\varphi}{\partial x}\dfrac{\partial\zeta}{\partial x}}+\cancel{\dfrac{\partial\varphi}{\partial y}\dfrac{\partial\zeta}{\partial y}}\right)\Big|_{z=\zeta}=\dfrac{\partial\varphi}{\partial z}\Big|_{z=\zeta}\\ \dfrac{\partial\varphi}{\partial n}=0\Rightarrow\dfrac{\partial\varphi}{\partial z}\Big|_{z=-d}=0\\ \left[\dfrac{\partial\varphi}{\partial t}+\cancel{\dfrac{1}{2}(\nabla\varphi)(\nabla\varphi)}\right]\Big|_{z=\zeta}+g\zeta=0\end{cases}$

因此，线性波动的基本方程：

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}+\dfrac{p-p_0}{\rho}+gz=0 & (32)\\ \dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\partial^2\varphi}{\partial z^2}=0 & (33)\\ \dfrac{\partial\varphi}{\partial z}\Big|_{==0}=\dfrac{\partial\zeta}{\partial t} & (34)\\ \dfrac{\partial\varphi}{\partial z}\Big|_{z=-d}=0 & (35)\\ \dfrac{\partial\varphi}{\partial t}\Big|_{z=0}+g\zeta=0 & (36)\end{cases}$$

(36) 代入 (??) 中：

$$\left(\frac{\partial\varphi}{\partial z}+\frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\right)\Big|_{z=0}=0$$

求解

对前进波：

$$\varphi=\varphi_0(z)\cos(kx-\omega t) \tag{37}$$

(37) 代入 (33) 中：

$$\varphi_0(z)=Ae^{kz}+Be^{-kz} \tag{38}$$

(37) 代入 (3.1.2) 中 (海面)：

$$\left[\frac{\partial\varphi_0(z)}{\partial z}-\frac{\omega^2}{g}\varphi_0(z)\right]\Big|_{z=0}=0 \tag{39}$$

(37) 代入 (35) 中 (海底)：

$$\frac{d\varphi_0(z)}{dz}\Big|_{z=-d}=0 \tag{40}$$

(38) 代入 (39) 中：

$$\left(\omega^2-gk\right)A+\left(\omega^2+gk\right)B=0 \tag{41}$$

(38) 代入 (40) 中：

$$e^{-kd}A-e^{kd}B=0 \tag{42}$$

联立 (41) (42)，有非零解的条件为：

$$\begin{vmatrix}e^{-kd} & -e^{kd}\\ (\omega^2-gk) & (\omega^2+gk)\end{vmatrix}=0\Rightarrow\underset{\text{频散关系}}{\omega^2=gk\,\mathrm{th}\,kd}$$

频散关系表示波动频率和波数之间的关系，代表某种波动的性质.

波速：$c=\dfrac{\lambda}{T}=\dfrac{2\pi/k}{2\pi/\omega}=\dfrac{\omega}{k}\Rightarrow c^2=\dfrac{\omega^2}{k^2}=\dfrac{g}{k}\ \mathrm{th}\ kd$

波速与水深和波动性质有关系.

由 (42) : $Ae^{-kd}=Be^{kd}=\frac{1}{2}D\Rightarrow A=\frac{D}{2}/e^{-kd};\quad B=\frac{D}{2}/e^{kd}$

代入 (37):

$$\varphi = D\,\mathrm{ch}[k(z+d)]\cos(kx-\omega t) \tag{43}$$

(43) 代入 (36) 中：

$$\zeta = -\frac{\omega}{g} D\,\mathrm{ch}\,kd \sin(kx-\omega t) = a\sin(kx-\omega t)$$

速度势的解：

$$\varphi = -\frac{ag}{\omega}\frac{\mathrm{ch}[k(z+d)]}{\mathrm{ch}\,kd}\cos(kx-\omega t) \tag{44}$$

(44) 代入 (32) 中，可得压强分布：

$$p = p_0 + \rho g a \frac{\mathrm{ch}[k(z+d)]}{\mathrm{ch}\,kd}\sin(kx-\omega t) - \rho g z$$

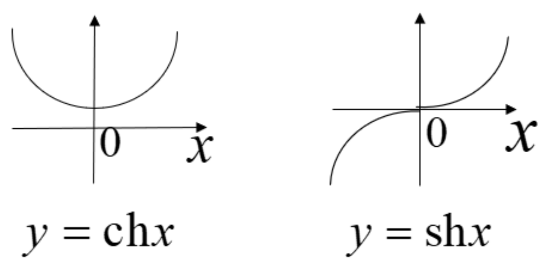
水质点速度：

$$u = \frac{\partial\varphi}{\partial x} = \frac{agk}{\omega}\frac{\mathrm{ch}[k(z+d)]}{\mathrm{ch}\,kd}\sin(kx-\omega t)$$
$$w = \frac{\partial\varphi}{\partial z} = -\frac{agk}{\omega}\frac{\mathrm{sh}[k(z+d)]}{\mathrm{ch}\,kd}\cos(kx-\omega t)$$

水质点运动轨迹 (x_0, z_0) 为平衡位置:

$$\frac{(x-x_0)^2}{\left[a\frac{\mathrm{ch}[k(d+z_0)]}{\mathrm{sh}\,kd}\right]^2} + \frac{(z-z_0)^2}{\left[a\frac{\mathrm{sh}[k(d+z_0)]}{\mathrm{sh}\,kd}\right]^2} = 1$$

解的讨论



(1) 线性波动水质点轨迹为椭圆，长轴为 x 方向；
(2) 椭圆长轴和短轴均与 z_0 有关，与 x_0 无关；
(3) 长轴和短轴均随平衡位置的加深而减小；
(4) 在海底水质点做水平运动.

3.1.3 水深对线性波动的影响

深水波 ($d > \frac{1}{2}\lambda$)

$kd = \dfrac{2\pi}{\lambda}d \to \infty$ $\mathrm{sh}\,kd = \mathrm{ch} \approx \dfrac{1}{2}e^{kd}, \mathrm{th}\,kd \approx 1$

频散关系：$\omega^2 = gk$ th $kd \Rightarrow \omega^2 = gk$

波速：$c^2 = \dfrac{g}{k}\,\mathrm{th}\,kd \Rightarrow c^2 = \dfrac{g}{k}$

深水波的角频率和波速与水深无关，只与波动性质 (波数或波长) 有关.

速度势：

$$\begin{aligned} \varphi &= -\frac{ag}{\omega}\frac{\mathrm{ch}[k(z+d)]}{\mathrm{ch}\,kd}\cos(kx-\omega t) \\ \left(\mathrm{ch}[k(z+d)] \approx \frac{1}{2}e^{kd}(\mathrm{ch}\,kz + \mathrm{sh}\,kz)\right) \quad &= -\frac{ag}{\omega}e^{kz}\cos(kx-\omega t) \end{aligned}$$

压强分布：$p = p_0 + \rho g a \dfrac{\mathrm{ch}[k(z+d)]}{\mathrm{ch}\,kd}\sin(kx-\omega t) - \rho g z$

水质点速度：$u = a\omega e^{kz}\sin(kx-\omega t), w = -a\omega e^{kz}\cos(kx-\omega t)$

水质点轨迹：$(x-x_0)^2+(z-z_0)^2=\left(ae^{k_0}\right)^2$
深水波水质点运动轨迹为圆，半径随水质点平衡位置的深度增加而减小，当达到很大深度时，半径无限小，运动消失——表面波性质.

浅水波（$d<\frac{1}{20}\lambda$）　$kd=\dfrac{2\pi}{\lambda}d\to 0$

幂级数展开：
$$\operatorname{sh}x=x+\frac{x^3}{3!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots(-\infty<x<\infty)$$
$$\operatorname{ch}x=1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots$$

$\operatorname{sh}kd\approx kd,\operatorname{ch}kd\approx 1+\dfrac{k^2d^2}{2}\approx 1,\operatorname{th}\approx kd$

频散关系：$\omega^2=gk\ \operatorname{th}\ kd\Rightarrow\omega^2=gdk^2$

波速：$c^2=\dfrac{g}{k}\operatorname{th}kd\Rightarrow c^2=gd$

波速只与水深有关，而与波动性质无关.

速度势：$\varphi=-\dfrac{ag}{\omega}\left[1+\dfrac{k^2(d+z)^2}{2}\right]\cos(kx-\omega t)$

压强分布：$p=p_0+\rho g(\zeta-z)$

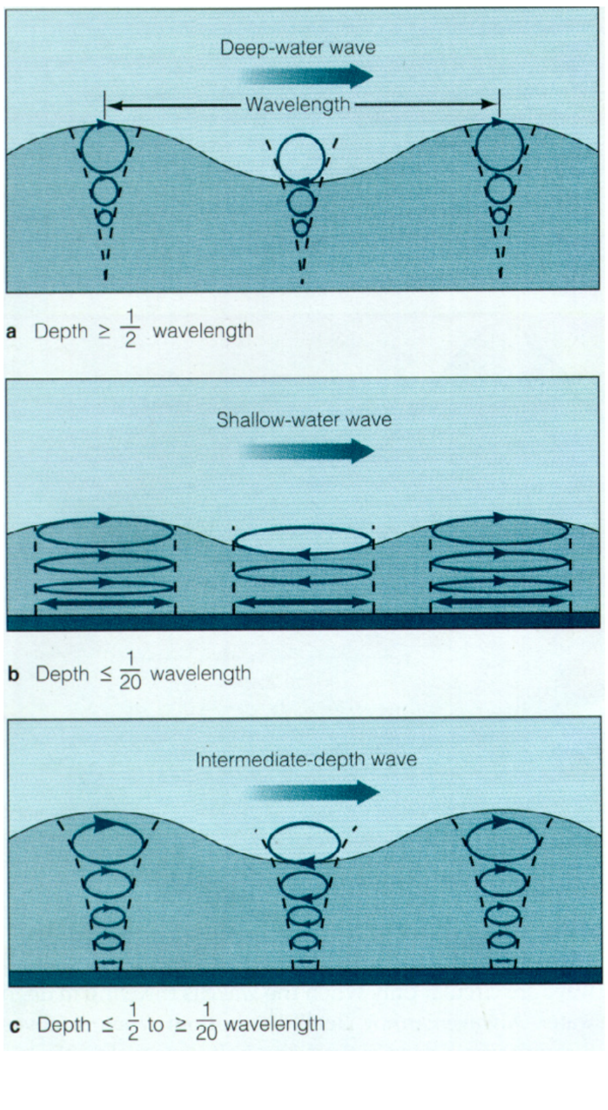
压强分布近似为静压分布.

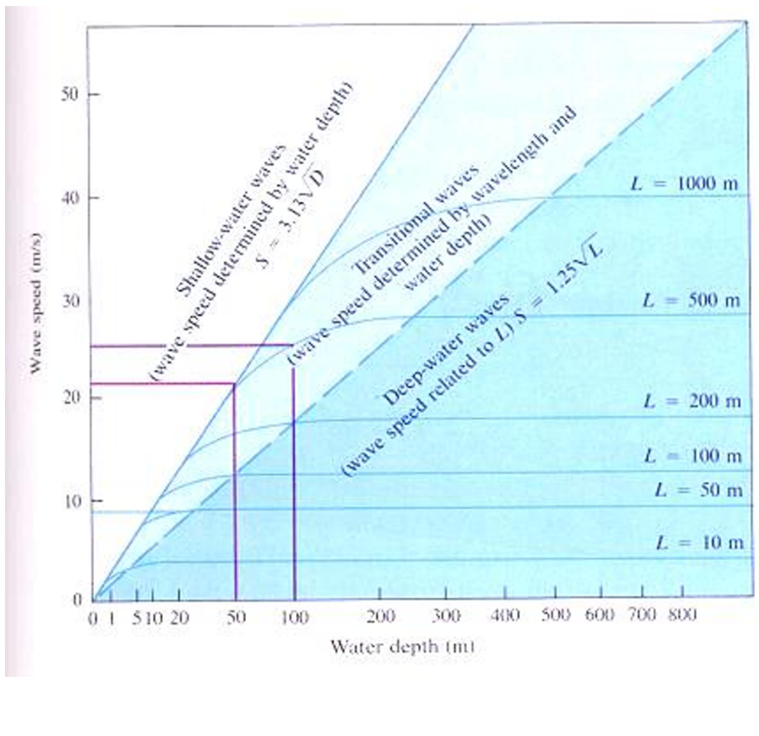
水质点速度：$u=\dfrac{a\omega}{kd}\sin(kx-\omega t),w=-a\omega\left(1+\dfrac{z}{d}\right)\cos(kx-\omega t)$

质点水平速度显著大于铅直速度.

水质点轨迹：$\dfrac{(x-x_0)^2}{\left(\frac{a}{kd}\right)^2}+\dfrac{(z-z_0)^2}{\left[a\left(1+\frac{z_0}{d}\right)\right]^2}=1$

浅水波水质点运动轨迹为椭圆，水平轴（长轴）不随深度改变，短轴随深度增加逐渐减小.





3.1.4 线性波动的能量

以下都是讨论一个波长范围内的线性波动能量，示意图如下：

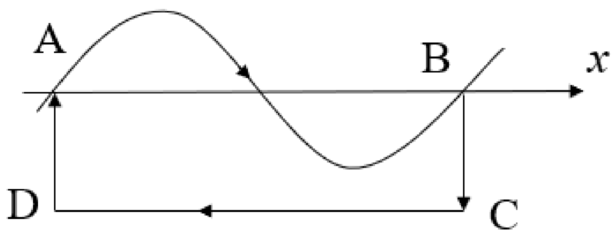


图 1 波动示意图

动能

根据动能定义，

$$\begin{aligned}E_k &= \frac{1}{2}mV^2 \\ &= \int_0^{\lambda}\left[\int_{-d}^{\zeta}\frac{1}{2}\rho\left(u^2+w^2\right)dz\right]dx \\ &= \frac{\rho}{2}\int_0^{\lambda}\int_{-d}^{\zeta}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]dzdx\end{aligned}$$

根据格林第一定理：

$$\iiint_{\tau}\varphi\Delta\varphi d\tau+\iiint_{\tau}\nabla\varphi\cdot\nabla\varphi d\tau=\oiint_{s}\varphi\frac{\partial\varphi}{\partial n}ds$$

又因为对于不可压流体 $\Delta\varphi=0$，因此：

$$\begin{aligned}E_k &= \frac{1}{2}mV^2 \\ &= \frac{\rho}{2}\iiint_{\tau}\nabla\varphi\cdot\nabla\varphi d\tau \\ &= \frac{\rho}{2}\oiint_{s}\varphi\frac{\partial\varphi}{\partial n}ds \\ &= \frac{\rho}{2}\oint_{c}\varphi\frac{\partial\varphi}{\partial n}dl\end{aligned}$$

其中，c 为图 2 中的闭合有向曲线。
根据对称性，$\overrightarrow{BC}$ 与 $\overrightarrow{DA}$ 上的积分值等大反向，根据边界条件，在海底 $\frac{\partial\phi}{\partial n}=\frac{\partial\phi}{\partial z}|_{z=-d}=0$。因此：

$$E_k=\frac{\rho}{2}\int_0^{\lambda}\left.(\varphi\frac{\partial\varphi}{\partial z})\right|_{z=\zeta}dx$$

再根据小振幅假定，振幅相对波长很小，$\zeta\approx 0$，因此：

$$E_k=\frac{\rho}{2}\int_0^{\lambda}\left.(\varphi\frac{\partial\varphi}{\partial z})\right|_{z=0}dx$$

带入线性波动的解：

$$\begin{aligned}\varphi &= -\frac{ag}{\omega}\frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd}\cos(kx-\omega t) \\ \Rightarrow\frac{\partial\varphi}{\partial z} &= -\frac{agk\mathrm{sh}\left[k\left(d+z\right)\right]}{\omega\mathrm{ch}\left(kd\right)}\cos\left(kx-\omega t\right)\end{aligned}$$

$$\begin{aligned}E_k &= \frac{\rho}{2}\int_0^{\lambda}\left.\frac{ag}{\omega}\frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd}\cos(kx-\omega t)\frac{agk\mathrm{sh}\left[k\left(d+z\right)\right]}{\omega\mathrm{ch}\left(kd\right)}\cos\left(kx-\omega t\right)\right|_{z=0}dx \\ &= \frac{\rho}{2\omega^2}a^2g^2k\mathrm{th}kd\int_0^{\lambda}\cos^2(kx-\omega t)dx \\ &= \frac{\rho}{2\omega^2}a^2g^2\mathrm{th}kd\int_0^{2\pi}\cos^2(\theta-\omega t)d\theta \\ &= \frac{\rho}{2\omega^2}a^2g^2\pi\mathrm{th}kd\end{aligned}$$

再带入频散关系：$\omega^2=gk\mathrm{th}kd$ 以及 $k=\frac{2\pi}{\lambda}$：

$$E_k=\frac{1}{4}\rho ga^2\lambda$$

势能

根据势能的定义：

$$
\begin{aligned}
E_p &= mgh \\
&= \int_0^\lambda \int_0^\zeta \rho g z dz dx \\
&= \rho g \int_0^\lambda \int_0^\zeta \frac{1}{2} dz^2 dx \\
&= \frac{1}{2}\rho g \int_0^\lambda \zeta^2 dx \\
&= \frac{1}{2}\rho g \int_0^\lambda a^2 \sin^2(kx-\omega t) dx \\
&= \frac{1}{4}\rho g a^2 \lambda
\end{aligned}
$$

总能量

一个波长内的总能量为动能和势能之和：

$$
E = E_K + E_P = \frac{1}{2}\rho g a^2 \lambda
$$

进而可以计算单位表面面积水柱的平均总能量：

$$
\bar{E} = \frac{E}{\lambda} = \frac{1}{2}\rho g a^2
$$

平均能量通量及水深影响　能量通量是单位时间内通过垂直于波动传播方向断面的能量，等于单位时间内通过某断面的波动能量等于该断面外侧有效压力所做的功。根据线性波动的势函数方程：

$$
\frac{\partial \varphi}{\partial t} + \frac{p - p_0}{\rho} + gz = 0
$$

则有效压强为：

$$
p_e = -\rho \frac{\partial \varphi}{\partial t}
$$

根据能量通量定义：

$$
P = \int_{-d}^{\zeta} p_e u dz = \int_{-d}^{\zeta} -\rho \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} dz
$$

带入线性波动的解：

$$
\begin{aligned}
\varphi &= -\frac{ag}{\omega} \frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd}\ cos(kx-\omega t) \\
\Rightarrow \frac{\partial \varphi}{\partial t} &= -ag \frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd} \sin(kx-\omega t), \frac{\partial \varphi}{\partial x} = \frac{agk}{\omega} \frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd} \sin(kx-\omega t)
\end{aligned}
$$

$$
\begin{aligned}
P &= \int_{-d}^{\zeta} \rho a g \frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd} \sin(kx-\omega t) \frac{agk}{\omega} \frac{\mathrm{ch}[k(d+z)]}{\mathrm{ch}kd} \sin(kx-\omega t) dz \\
&= \frac{\rho a^2 g^2}{\omega \mathrm{ch}^2 kd} \sin^2(kx-\omega t) \int_{-d}^{\zeta} \mathrm{ch}^2[k(d+\zeta)] dz \\
&= \frac{\rho a^2 g^2}{\omega \mathrm{ch}^2 kd} \sin^2(kx-\omega t) \left(\frac{1}{2} k(d+\zeta) + \frac{1}{4} \mathrm{sh}[2k(d+\zeta)] \right) \\
&= \frac{\rho a^2 g^2}{\sqrt{gk \mathrm{th} kd} \mathrm{ch}^2 kd} \sin^2(kx-\omega t) \left(\frac{1}{2} k(d+\zeta) + \frac{1}{4} \mathrm{sh}[2k(d+\zeta)] \right) \\
&= \sqrt{\frac{g}{k} \mathrm{th} kd} \frac{\rho a^2 g}{\mathrm{th} kd \mathrm{ch}^2 kd} \sin^2(kx-\omega t) \left(\frac{1}{2} k(d+\zeta) + \frac{1}{4} \mathrm{sh}[2k(d+\zeta)] \right) \\
&= \frac{\rho g a^2 c}{\mathrm{sh} kd \mathrm{ch} kd} \sin^2(kx-\omega t) \left(\frac{1}{2} k(d+\zeta) + \frac{1}{4} \mathrm{sh}[2k(d+\zeta)] \right) \\
&= \frac{2\rho g a^2 c}{\mathrm{sh} 2kd} \sin^2(kx-\omega t) \left(\frac{1}{2} k(d+\zeta) + \frac{1}{4} \mathrm{sh}[2k(d+\zeta)] \right) \\
&= \rho g a^2 c \left(\frac{1}{\mathrm{sh} 2kd} k(d+\zeta) + \frac{1}{2 \mathrm{sh} 2kd} \mathrm{sh}[2k(d+\zeta)] \right) \sin^2(kx-\omega t)
\end{aligned}
$$

令 $\zeta = 0$：

$$\boxed{P = \frac{\rho g a^2 c}{2}\left(1+\frac{2kd}{\mathrm{sh}2kd}\right)\sin^2(kx-\omega t)}$$

进而可以计算一个周期内的平均能量通量：

$$\begin{aligned}\bar{P} &= \frac{1}{T}\int_0^T P dt \\ &= \frac{1}{T}\int_0^T \frac{\rho g a^2 c}{2}\left(1+\frac{2kd}{\mathrm{sh}2kd}\right)\sin^2(kx-\omega t)dt \\ &= \frac{1}{2}\rho g a^2 \cdot \frac{c}{2}(1+\frac{2kd}{\mathrm{sh}2kd})\end{aligned}$$

令 $\bar{E} = \frac{1}{2}\rho g a^2$，则：

$$\bar{P} = \bar{E}\frac{c}{2}(1+\frac{2kd}{\mathrm{sh}2kd})$$

对于深水波：

$$kd \to \infty, \frac{2kd}{\mathrm{sh}2kd} \to 0, \bar{P} = \bar{E}\frac{c}{2}$$

对于浅水波：

$$kd \to 0, \frac{2kd}{\mathrm{sh}2kd} \to 1, \bar{P} = \bar{E}c$$

综上，深水波的平均能量通量等于单位表面积水柱内的总能量乘上波速的一半；浅水波的平均能量通量等于单位表面积水柱内的总能量乘上波速。

3.2 线性波动的合成

驻波

两列振幅、周期、波长相等，传播方向相反的前进波叠加形成的波动.

波面

$$\zeta = \overbrace{\frac{1}{2}a\sin(kx-\omega t)}^{+x} + \overbrace{\frac{1}{2}a\sin(kx+\omega t)}^{-x} = a\sin kx\cos\omega t$$

节点：$\sin kx = 0, kx = n\pi(n = 0, \pm 1, \pm 2, \cdots)$

腹点：$\sin kx = \pm 1, kx = (n + \frac{1}{2}\pi)$

速度势：$\varphi = -\frac{ag}{\omega}\frac{ch[k(d+z)]}{chkd}\sin kx\sin\omega t$

水质点轨迹：$\frac{z-z_0}{x-x_0} = \tan kx_0 th\left[k\left(d+z_0\right)\right]$

(1) 驻波中流体质点的轨迹为直线；

(2) 具有不同平衡位置的质点，振动方向不同：在节点，质点水平振动 (w=0)；在腹点，质点铅直振动 ($u = 0$)；kx_0 在一、三 (二、四) 象限时，斜率为正 (负)；(3) 振动幅度 (斜率) 随平衡位置的深度迅速减小.

波群

沿同一方向成群向外传播的波列所产生的波动叠加后，在某一固定点观测点，波动振幅由小到大，又由大到小的现象.

参考文献

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