Norms of Vectors and Matrices:

Let $\mathbb{R}^n$ denote the set of all n-dimensional column vectors with real-number components. To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

Definition: A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$ from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\underline{x}\| \geq 0$ for all $\underline{x} \in \mathbb{R}^n$,

(ii) $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$,

(iii) $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$ for all $\alpha \in \mathbb{R}$, and $\underline{x} \in \mathbb{R}^n$,

(iv) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$.

Specific Norms: The l_p and l_∞ norms for the vector $\underline{x} = (x_1, x_2, \cdots, x_n)^T$ are defined by

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i^p| \right)^{1/p}$$

and

$$\|\underline{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| .$$

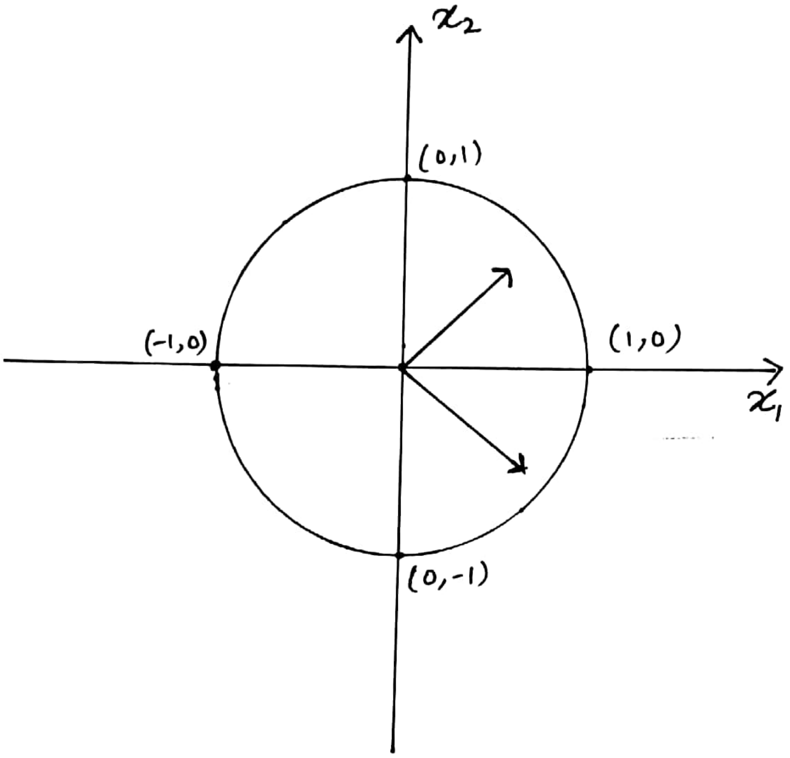
For $p = 1$, we have $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$.

For $p = 2$, we have $\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

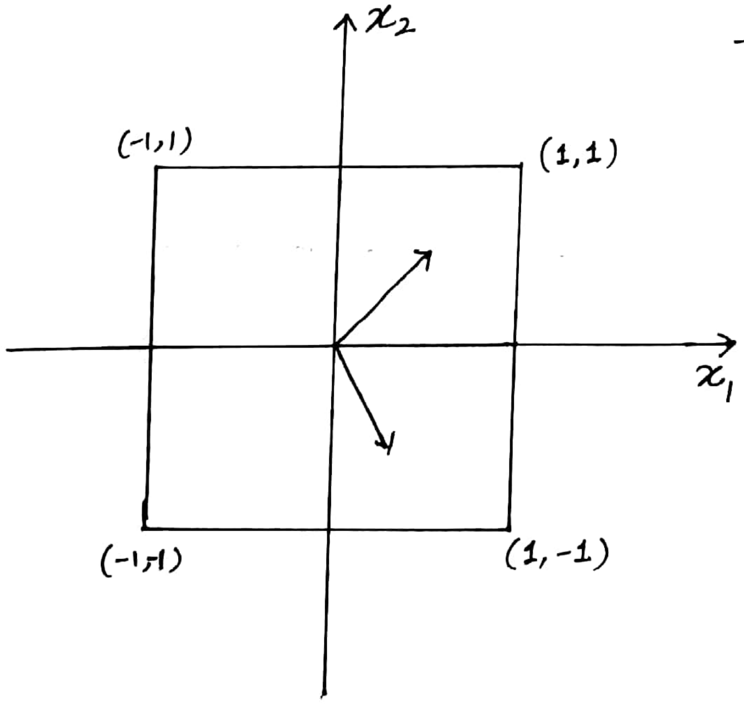
Note that each of these norms reduces to the absolute value in the case $n = 1$.

The l_2 norm is called the Euclidean norm of the vector $\underline{x}$ because it represents the usual notion of distance from the origin in case $\underline{x}$ is in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$.

The most used ~~use~~ norms are the l_2 and l_∞.



The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside the unit circle centred at origin. (See the figure)



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside a unit square as shown in this figure.

Example: Determine the l_1, l_2 and l_∞ norms of the vector $\underline{x} = (-1, 1, -2)^T$.

Sol^n:

$$\|\underline{x}\|_1 = |-1| + |1| + |-2| = 4$$

$$\|\underline{x}\|_2 = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$\|\underline{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = \max\{1, 1, 2\} = 2.$$

Cauchy-Bunyakovsky-Schwarz Inequality for sums:

For each $\underline{x} = (x_1, x_2, \ldots x_n)^T$ and $\underline{y} = (y_1, y_2, \ldots y_n)^T$ in $\mathbb{R}^n$,

$$\underline{x}^T \underline{y} = \sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2} = \|\underline{x}\|_2 . \|\underline{y}\|_2 .$$

Distance between vectors in $\mathbb{R}^n$:

If $\underline{x} = (x_1, x_2, \ldots, x_n)^T$ and $\underline{y} = (y_1, y_2, \ldots, y_n)^T$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\underline{x}$ and $\underline{y}$ are defined by

$$\|\underline{x} - \underline{y}\|_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2},$$

and $$\|\underline{x} - \underline{y}\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i| .$$

Example: Suppose a linear system $A\underline{x} = \underline{b}$ has the exact solution $\underline{x} = (x_1, x_2, x_3)^T = (1, 1, 1)^T$, and Gaussian elimination performed using five-digit rounding arithmetic and partial pivoting, produces the approximate solution $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = (1.2001, 0.99991, 0.92538)^T$.

Determine the l_2 and l_∞ distances between the exact and approximate solutions.

Sol^n: Measurements of $\underline{x}-\tilde{x}$ are given by

$$\|\underline{x}-\tilde{x}\|_2 = \left[(1-1.2001)^2+(1-0.99991)^2+(1-0.92538)^2\right]^{1/2}$$

$$= \left[(0.2001)^2+(0.00009)^2+(0.07462)^2\right]^{1/2}$$

$$= 0.21356$$

and

$$\|\underline{x}-\tilde{x}\|_\infty = \max\{|1-1.2001|, |1-0.99991|, |1-0.92538|\}$$

$$= \max\{0.2001, 0.00009, 0.07462\}$$

$$= 0.2001.$$

Although the components $\tilde{x}_2$ and $\tilde{x}_3$ are good approximations to x_2 and x_3, the component $\tilde{x}_1$ is a poor approximation to x_1, and $|x_1-\tilde{x}_1|$ dominates both norms.

Convergence of Sequence of vectors in $\mathbb{R}^n$:

A sequence $\{\underline{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to $\underline{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon>0$, there exists an integer $N(\varepsilon)$ such that

$$\|\underline{x}^{(k)}-\underline{x}\| < \varepsilon \quad \text{for all } k \geq N(\varepsilon).$$

Theorem: The sequance of vectors $\{\underline{x}^{(K)}\}$ converges to $\underline{x}$ in $\mathbb{R}^n$ with respect to l_∞ norm iff $\lim_{K\to\infty} x_i^{(K)} = x_i$ for each $i = 1, 2, \ldots, n$.

Example: Show that

$$\underline{x}^{(K)} = (x_1^{(K)}, x_2^{(K)}, x_3^{(K)}, x_4^{(K)})^T = \left(1,\ 2+\frac{1}{K},\ \frac{3}{K^2},\ e^{-K}\cdot\sin K\right)^T$$

converges to $\underline{x} = (1, 2, 0, 0)^T$ with respect to l_∞ norm.

Sol^n:

$$\lim_{K\to\infty} x_1^{(K)} = \lim_{K\to\infty} 1 = 1$$

$$\lim_{K\to\infty} x_2^{(K)} = \lim_{K\to\infty} \left(2+\frac{1}{K}\right) = 2$$

$$\lim_{K\to\infty} x_3^{(K)} = \lim_{K\to\infty} \frac{3}{K^2} = 0$$

$$\lim_{K\to\infty} x_4^{(K)} = \lim_{K\to\infty} e^{-K} \sin K = 0.$$

This implies that the given sequence $\{\underline{x}^{(K)}\}$ converges to $(1, 2, 0, 0)^T$ w.r.t. the l_∞ norm.

→ To show directly that the sequence given in the above example converges to $(1, 2, 0, 0)^T$ with respect to the l_2-norm is quite complicated. It is better to use ~~prove~~ the next result to prove the same.

Theorem: For each $\underline{x} \in \mathbb{R}^n$,

$$\|\underline{x}\|_\infty \le \|\underline{x}\|_2 \le \sqrt{n}\, \|\underline{x}\|_\infty.$$

Proof. Let x_j be a coordinate of $\underline{x}$ such that

$$\|\underline{x}\|_\infty = \max_{1\le i\le n} |x_i| = |x_j|.$$

Then
$$\|\underline{x}\|_\infty^2 = |x_j|^2 = x_j^2 \le \sum_{i=1}^{n} x_i^2 = \|\underline{x}\|_2^2,$$

and
$$\|\underline{x}\|_\infty \le \|\underline{x}\|_2.$$

So,
$$\|\underline{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 \le \sum_{i=1}^{n} x_j^2 = n x_j^2 = n\|\underline{x}\|_\infty^2$$

$$\Rightarrow \quad \|\underline{x}\|_2 \le \sqrt{n}\, \|\underline{x}\|_\infty .$$

Example: Consider
$$\underline{x}^{(K)} = \left(1,\ 2+\frac{1}{K},\ \frac{3}{K^2},\ e^{-K}\sin K\right)^T,$$

and $\underline{x} = (1, 2, 0, 0)^T$.

We know that $\underline{x}^{(K)} \to \underline{x}$ with r. to l_∞ norm. (See the last examp.

That is, given any $\varepsilon > 0$, $\exists$ an integer $N(\frac{\varepsilon}{2})$ such that

$$\|\underline{x}^{(K)} - \underline{x}\|_\infty < \frac{\varepsilon}{2} \quad \forall\ K \ge N\left(\frac{\varepsilon}{2}\right).$$

Now, using the above Theorem, we get

$$\|\underline{x}^{(K)} - \underline{x}\|_2 \le \sqrt{4}\, \|\underline{x}^{(K)} - \underline{x}\|_\infty$$
$$\le 2 \cdot \frac{\varepsilon}{2} \quad \forall\ K \ge N\left(\frac{\varepsilon}{2}\right).$$
$$= \varepsilon.$$

$$\Rightarrow \|\underline{x}^{(K)} - \underline{x}\|_2 \le \varepsilon \quad \forall\ K \ge N\left(\frac{\varepsilon}{2}\right).$$

Thus, $\underline{x}^{(K)} \to \underline{x}$ w.r.t. l_2 norm..

Remark: It can be shown that all norms on $\mathbb{R}^n$ are equivalent with respect to convergence; i.e., if $\|\cdot\|$ and $\|\cdot\|'$ are any two norms on $\mathbb{R}^n$ and $\{\underline{x}^{(k)}\}$ has the limit $\underline{x}$ with respect to $\|\cdot\|$, then $\{\underline{x}^{(k)}\}$ also has the limit $\underline{x}$ with respect to $\|\cdot\|'$.

Matrix Norms and Distances:

We would need methods to determine the distances between $n\times n$ matrices.

Definition: A matrix norm on the set of all $n\times n$ matrices is a real-valued function $\|\cdot\|$, defined on this set, satisfying for all $n\times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$,

(ii) $\|A\| = 0$ iff A is the matrix with all zero entries,

(iii) $\|\alpha A\| = |\alpha|\,\|A\|$,

(iv) $\|A+B\| \leq \|A\| + \|B\|$,

(v) $\|AB\| \leq \|A\|.\|B\|$.

The distance between $n\times n$ matrices A and B with respect to this matrix norm is $\|A-B\|$.

⇒ Although matrix norm can be obtained in various ways, the norms considered most frequently are those that are natural consequences of the vectors norms l_2 and l_∞.

Theorem: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

(a) $\|A\|_2 = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$;

(b) $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$; (along the rows)

(c) $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$, (along the columns).

Example: Determine $\|A\|_2$, $\|A\|_\infty$ and $\|A\|_1$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}.$$

Sol^n. We have

$$\|A\|_2 = \left(1^2 + 2^2 + (-1)^2 + 0^2 + 3^2 + (-1)^2 + 5^2 + (-1)^2 + 1^2\right)^{\frac{1}{2}}$$

$$= \sqrt{43}.$$

Along the rows, we have

$$\sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4$$

$$\sum_{j=1}^{3} |a_{2j}| = |0| + |3| + |-1| = 4$$

$$\sum_{j=1}^{3} |a_{3j}| = |5| + |-1| + |1| = 7$$

$$\|A\|_\infty = \max\{4, 4, 7\} = 7.$$

Similarly, along the columns, we get

$$\|A\|_1 = \max\{6, 6, 3\} = 6.$$