Formal Structure of Gauss Elimination:

A general non-singular system of n linear equations.

$$\left[\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right]$$

For $K = 1, 2, 3, \ldots (n-1)$, carry out the following elimination steps:

Step-K: ✓ To eleminate coefficients of x_K from row $(K+1)$ through n. The results of preceeding steps $1, 2, \ldots (K-1)$ will have yielded.

$$\begin{array}{c} \\ \\ K^{th} \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1K}^{(1)} & \cdots\ a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2K}^{(2)} & \cdots\ a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & a_{KK}^{(K)} & \cdots\ a_{Kn}^{(K)} & b_K^{(K)} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & & a_{nK}^{(K)} & \cdots\ a_{nn}^{(K)} & b_n^{(K)} \end{array}\right]$$

Assume $a_{KK}^{(K)} \neq 0$ and define multiplier

$$m_{iK} = \frac{a_{iK}^{(K)}}{a_{KK}^{(K)}} \checkmark \quad \text{for } i = \underbrace{(K+1), (K+2), \ldots, n}_{(n-K)}$$

$$R_i \to R_i - m_{iK} R_K \quad \text{for } i = (K+1), (K+2), \ldots, n$$

Now, the new coefficients and right hand side are.

$$a_{ij}^{(K+1)} = a_{ij}^{(K)} - \underline{m_{iK}} \cdot a_{Kj}^{(K)} \quad , \quad i, j = (K+1), (K+2) \cdots n.$$

and

$$b_i^{(K+1)} = b_i^{(K)} - m_{iK} \cdot b_i^{(K)} , \quad i = (K+1), (K+2) \cdots , n.$$

When Step-$(n-1)$ is completed, the linear system will be in upper triangular form.

$$u_{n-1,n-1} x_{n-1} + u_{n-1,n} x_n = g_{n-1}$$

$$x_{n-1} = \frac{g_{n-1} - u_{n-1,n} x_n}{u_{n-1,n-1}}$$

$$\left[\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1n} & g_1 \\ 0 & u_{22} & \cdots & u_{2n} & g_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} & g_n \end{array}\right]$$

Back Substitution:

$$\boxed{\begin{aligned} x_n &= \frac{g_n}{u_{nn}} \\ x_i &= \frac{g_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}} , \quad i = (n-1), (n-2) \cdots, 2, 1 \end{aligned}}$$

Operations Count:

For Gauss elimination, the operations count for a full matrix (a matrix with relatively many non-zero entries) is as follows:

In Step-K, we eliminate x_K from $(n-K)$ equations. This needs $(n-K)$ divisions in computing m_{jK}, and $(n-K)\cdot(n-K+1)$ multiplications and $(n-K)(n-K+1)$

for b_j's

substractions.

Since we do $(n-1)$ steps, k goes from 1 to $(n-1)$. Hence, total number of operations in this forward elimination is

$$f(n) = \sum_{K=1}^{n-1} (n-K) + 2 \sum_{K=1}^{n-1} (n-K)(n-K+1)$$

$$= n(n-1) - \frac{n(n-1)}{2} + 2 \sum_{K=1}^{n-1} \left[n(n+1) - (2n+1)K + K^2 \right]$$

$$= n(n-1) - \frac{n(n-1)}{2} + 2 \cdot \left[n(n+1)(n-1) - (2n+1)\frac{n(n-1)}{2} + \frac{n(n-1)(2n-1)}{6} \right]$$

$$= \frac{n(n-1)}{2} + 2n(n-1)\left[(n+1) - \frac{1}{2}(2n+1) + \frac{2n-1}{6} \right]$$

$$= \frac{n(n-1)}{2} + 2n(n-1)\left(\frac{n+1}{3}\right)$$

$$= \frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6}$$

10^4 ✓
10^8 ✓
10^{12}

$$\simeq \frac{2n^3}{3} \quad \text{(dropping lower powers of } n\text{)}$$

We say that $f(n) = O(n^3)$, is order of n^3.

In the back substitution of x_i, we make $(n-i)$ multiplications and $(n-i)$ substractions and 1 division

Hence,

$$b(n) = 2\sum_{i=1}^{n-1}(n-i) + \sum_{i=1}^{n} 1.$$

$$= 2\left[n(n-1) - \frac{n(n-1)}{2}\right] + n.$$

$$= n(n-1) + n$$

$$= n^2.$$

For example, If an operation takes 10^{-9} sec, then the time needed are

Algorithm	$n = 1000$	$n = 10000$ ✓
Elimination (forward)	0.7 sec	11 min ✓
Back Substitution	0.001 sec.	0.1 sec ✓

We see that no. of operations in the back Substitution goes slower than that in the forward elimination of Gauss elimination, so that it is neglible for large systems because it is smaller by a factor n.

$a_{KK}^{(K)} = 0$??

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}} \checkmark$$

Partial Pivoting in Gauss Elimination

we assume that a_{KK} (in step-K) are different from zero. what if we obtain $a_{KK} = 0$ at some step? or $|a_{KK}|$ is very small.

→ At a given step, one equation remains unaltered. we refer to this equation as the 'pivot equation'. ✓

→ A 'pivot' in the corresponding row is the element, which is used to make all the elements below it zero.

→ Ex

$$0x_1 + 8x_2 + 2x_3 = -7 \quad \text{—} (1)$$
$$3x_1 + 5x_2 + 2x_3 = 8 \quad \text{—} (2)$$
$$6x_1 + 2x_2 + 8x_3 = 26 \quad \text{—} (3)$$

$$m_{jK} = \frac{\quad}{a_{KK}}$$

$|6| > |3|$

⇒

we choose as our pivot equation one that has the absolutely largest a_{jK} in column K on or below the main diagonal

Here, we exchange (1) ⟷ (3).

$$\left.\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_2 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7 \end{aligned}\right] \to$$ Pivot equation

$$\left[\begin{array}{ccc|c} \boxed{6} & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Step-1: Eliminate the cofficients of x_1.

$$R_2 \to R_2 - \frac{3}{6} \cdot R_1$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & \boxed{4} & -2 & -5 \\ 0 & \boxed{8} & 2 & -7 \end{array}\right]$$

Step-2: Elimination of x_2.

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & \boxed{8} & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right]$$ $\to$ ~~Poivo~~ Pivot equation

$$R_3 \to R_3 - \frac{4}{8} R_2.$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right]$$

Back Substitution:

$$-3x_3 = -3/2 \Rightarrow x_3 = \frac{1}{2}$$

$$8x_2 + 2x_3 = -7$$

$$x_2 = \frac{-7 - 2 \times \frac{1}{2}}{8} = -1 \checkmark$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

$$\boxed{x_1 = 4}$$

Total ✓ ~~poi~~ pivoting ✗ ×

Things to Remember:

(1) If $a_{KK} = 0$ in step-K, we must pivot

(2) If $|a_{KK}|$ is small, we should pivot to avoid magnification of round-off errors that may seriously ~~af~~ affect accuracy or even produce non-sensical results.