Special Types of Matrices

We now turn attention to two classes of matrices for which Gaussian elimination can be performed effectively without row interchanges.

(1) Positive Definite Matrices:

A real Square matrix A is said to be positive definite if $\det(A) > 0$ and all leading principal minors are positive.

For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, A is positive definite if

(i) $a_{11} > 0$, (ii) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, (iii) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$.

Example: (i) $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$ is positive definite because

$a_{11} = 1 > 0$, $\begin{vmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{vmatrix} = \frac{1}{12} > 0$, $\begin{vmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{vmatrix} = \frac{1}{2160} > 0$.

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ is not positive definite because

$$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2-6 = -4 < 0.$$

An equivalent definition of Positive Definite:

A matrix A is positive definite if $\underline{x}^T A \underline{x} > 0$ for all n-dimensional vectors $\underline{x} \neq 0$.

<u>Example:</u>

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

<u>Sol^n</u>: Suppose $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is any three-dimensional column vector. Then

$$\underline{x}^T A \underline{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\underline{x}^T A \underline{x} = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0 \ \forall\ \underline{x} \neq 0$$

which implies that

$$\underline{x}^T A \underline{x} > 0 \ \forall\ \underline{x} \neq 0.$$

Thus, A is positive definite matrix.

Some necessary conditions that can be used to eliminate certain matrices from consideration of positive definite matrices.

<u>Result:</u> If A is an $n \times n$ positive definite matrix, then

(i) A has an inverse;

(ii) $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$;

(iii) $\max\limits_{1 \le k, j \le n} |a_{kj}| \le \max\limits_{1 \le i \le n} |a_{ii}|$;

(iv) $a_{ij}^2 < a_{ii} a_{jj}$, for each $i \neq j$.

(2) Diagonally Dominant Matrices:

The $n \times n$ matrix A is said to be diagonally dominant when

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ holds for each } i = 1, 2, \ldots n. \quad \text{— (1)}$$

A diagonally dominant matrix is said to be strictly diagonally dominant when the inequality in (1) is strict for each i, that is, when

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ holds for each } i = 1, 2, \ldots n. \quad \text{— (2)}$$

Illustration: Consider the matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

The matrix A is strictly diagonally dominant because

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad |-5| > |0| + |5|.$$

The matrix B is not strictly diagonally dominant because, for example, in the first row

$$|6| < |4| + |-3| = 7.$$

It is interesting to note that $A^T = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}$ is not strictly diagonally dominant because the middle row of A^T is $[2, 5, 5]$, and $|5| < |2| + |5| = 7$.

Result: A strictly diagonally dominant matrix A is non-singular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\underline{x} = \underline{b}$ to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off-errors.

Cholesky's Decomposition Method

Cholesky's method is applicable for symmetric and positive definite matrix A. In this case, the decomposition of A is $A = LL^T$, where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad \text{so that}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LL^T = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Here we only need to solve six equations in six unknowns. To solve a linear system $A\underline{x} = \underline{b}$, we first solve $L\underline{y} = \underline{b}$ for $\underline{y}$, and then solve $L^T\underline{x} = \underline{y}$ for $\underline{x}$.

For 3x3 matrix, we have

$$l_{11}^2 = a_{11} \Rightarrow l_{11} = \sqrt{a_{11}}$$

$$l_{11}l_{21} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{l_{11}}$$

$$l_{11}l_{31} = a_{31} \Rightarrow l_{31} = \frac{a_{31}}{l_{11}}.$$

$$l_{21}^2 + l_{22}^2 = a_{22} \Rightarrow l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$$l_{21}l_{31} + l_{22}l_{32} = a_{32} \Rightarrow l_{32} = \frac{a_{32} - l_{21}l_{31}}{l_{22}}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33} \Rightarrow l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}.$$

For a general symmetric matrix

$$l_{11} = \sqrt{a_{11}}$$

$$l_{j1} = \frac{a_{j1}}{l_{11}}, \quad j = 2, 3, \ldots, n$$

$$l_{jj} = \sqrt{a_{jj} - \sum_{s=1}^{j-1} l_{js}^2}, \quad j = 2, 3, \ldots, n$$

$$l_{pj} = \frac{1}{l_{jj}} \left(a_{pj} - \sum_{s=1}^{j-1} l_{js} l_{ps} \right), \quad p = j+1, \ldots, n$$
$$j \geq 2.$$

Remark: If A is symmetric but not positive definite this method could still be applied, but then it leads to a complex matrix L, so that the method becomes impractical.

Example: Solve the linear equations

$$4x_1 + 2x_2 + 14x_3 = 14$$
$$2x_1 + 17x_2 - 5x_3 = -101$$
$$14x_1 - 5x_2 + 83x_3 = 155$$

Solⁿ. To solve $A\underline{x} = \underline{b}$, where

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \underline{b} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}.$$

We write $A = LL^T$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

This implies that

$l_{11}^2 = 4 \Rightarrow l_{11} = 2$ (we may choose −ve sign while taking square root but we stick to positive root for simplicity).

$l_{11} l_{21} = 2 \Rightarrow l_{21} = 1$

$l_{11} l_{31} = 14 \Rightarrow l_{31} = 7$

$l_{21}^2 + l_{22}^2 = 17 \Rightarrow l_{22} = 4$

$l_{21} l_{31} + l_{22} l_{32} = -5 \Rightarrow l_{32} = -3$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83 \Rightarrow l_{33} = 5$.

Thus, $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$.

Let $L^T \underline{x} = \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then $L\underline{y} = \underline{b}$.

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix} \Rightarrow y_1 = 7,\ y_2 = -27,\ y_3 = 5.$$

Now solve $L^T \underline{x} = \underline{y}$.

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix} \Rightarrow x_1 = 3,\ x_2 = -6,\ x_3 = 1.$$