Gauss-Jordan Elimination

Example: Solve

$$8x_2 + 2x_3 = -7$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$6x_1 + 2x_2 + 8x_3 = 26$$

$$\left[\begin{array}{ccc|c} \underline{0} & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$R_1 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

$|4| < |8|$, $R_2 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{2} R_2$.

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \end{array}\right. \longrightarrow$$

$$
\left[\begin{array}{ccc|c}
0 & 8 & 2 & -7 \\
0 & 0 & -3 & -3/2
\end{array}\right] \rightarrow
$$

Gauss - Jordan Stn.

$$R_1 \to \frac{1}{6} R_1 , \quad R_2 \to \frac{1}{8} R_2 , \quad R_3 \to \cancel{R_2} -\frac{1}{3} R_3$$

$$
\left[\begin{array}{ccc|c}
1 & 1/3 & 4/3 & 13/3 \\
0 & 1 & 1/4 & -7/8 \\
0 & 0 & 1 & 1/2
\end{array}\right]
$$

$$R_1 \to R_1 - \frac{4}{3} R_3 , \quad R_2 \to R_2 - \frac{1}{4} R_3$$

$$
\left[\begin{array}{ccc|c}
1 & 1/3 & 0 & 11/3 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1/2
\end{array}\right]
$$

$$R_1 \to R_1 - \frac{1}{3} R_2 .$$

$O(n^2)$

$$
\begin{array}{c}
\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1/2
\end{array}\right]
\end{array}
\Rightarrow \quad
\begin{aligned}
x_1 &= 4 \\
x_2 &= -1 \\
x_3 &= 1/2
\end{aligned}
$$

not recommended to solve linear system of equations, since it involves more arithmetic operations than Gauss elimination with back substitution.

LU decomposition $A = LU \rightarrow$ upper triangular matrix.

$\uparrow$ lower triangular

Lower triangular matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a non-singular square matrix.

Then, A can be factorized into the form $\underbrace{A = LU}$,

if $a_{11} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \vdots \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \cdots$, $\det(A) \neq 0$

where

$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots \\ l_{n1} & l_{n2} & \cdots & & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots \\ 0 & 0 & & u_{nn} \end{bmatrix}$

Also,

$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots \\ l_{n1} & l_{n2} & \cdots & & l_{nn} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

not unique $L = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots \\ l_{n1} & l_{n2} & \cdots & & l_{nn} \end{pmatrix}$ $U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$

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(1) Gaussian Elimination Method.

4x4

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \longrightarrow U = \Big[$

$$\begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad L$$

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} \end{bmatrix} \qquad m_{21} = \frac{a_{21}}{a_{11}}$$

$$L_3 L_2 L_1 A = U$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_{21} & 1 & 0 & 0 \\ -m_{31} & 0 & 1 & 0 \\ -m_{41} & 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 \\ 0 & -m_{42} & 0 & 1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_{43} & 1 \end{bmatrix}$$

$$L_3 L_2 L_1 A = U \quad \text{--- (1)}$$

$$A = (L_3 L_2 L_1)^{-1} U$$

$$= \underbrace{(L_1^{-1} L_2^{-1} L_3^{-1})}_{L} U$$

$L \rightarrow$ Lower triangular matrix.

$$L = L_1^{-1} L_2^{-1} L_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{pmatrix}$$

$$A = LU$$

Ex $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$

Gauss Elimination

$$A \xrightarrow[\substack{m_{21}=1 \\ m_{31}=-2}]{\substack{R_2 \to R_2 - m_{21}R_1 \\ R_3 \to R_3 - m_{31}R_1}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow[m_{32}=3]{R_3 \to R_3 - m_{32}R_2} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \qquad \boxed{A = L.U}$$

LU: $A\underline{x} = \underline{b}$ — $O\left(\frac{n^3}{3}\right)$

$LU\underline{x} = \underline{b}$ — $O(n^2)$

let $\underline{y} = U\underline{x}$

$\boxed{L\underline{y} = \underline{b}}$ — $O(n^2)$

$O(2n^2)$

(2) Doolittle method

10 systems of LE with

$O\left(\frac{n^3}{3}\right)$ $\quad A\underline{x} = [\underline{b_1}\ \underline{b_2} \cdots \underline{b_{10}}]$

$LU\underline{x}$

(2) Doolittle's Method.

$$A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = L.U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \begin{bmatrix} & & \\ & & \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$L.U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

all 9 unknowns $\rightarrow L\ U$

(3) Crout's Method. $A = LU$.

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$LU = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{pmatrix}$$

$A =$

Doolittle's method.

Ex $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

$u_{11} = 1, \quad u_{12} = 1, \quad u_{13} = -1$

$l_{21}u_{11} = 1 \Rightarrow l_{21} = 1$

$l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = 2 - 1 = 1$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+ \end{pmatrix}$$

$l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = 2 - 1 = 1$ $\quad \left| \; l_{31}u_{11} \quad l_{31}u_{12} + l_{32}u_{22} \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} \right)$

$\underbrace{l_{21}u_{13}}_{-1} + u_{23} = -2 \Rightarrow u_{23} = -2 + 1 = -1$

$l_{31}u_{11} = -2 \Rightarrow l_{31} = -2.$

$$l_{32} = 3, \quad u_{33} = 2.$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}.$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}.$$

Crout's method

Solve the system of LE by LU decomposition.

__Step-1__: First let $\underline{y} = U\underline{x}$ and solve the lower triangular system $L\underline{y} = \underline{b}$ for $\underline{y}$.

(we only required $O(n^2)$ operations.)

__Step-2__: Once y is known, the upper triangular

System $U\underline{x} = \underline{y}$ requires only an additional $O(n^2)$ operations to determine the sol^n $\underline{x}$.

Ex

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}.$$

Doolittle method, we obtain the LU decomposition as

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{pmatrix}$$

Let $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. $\quad L\underline{y} = \underline{b}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$$

$$y_1 = 9$$
$$y_2 = 1.5$$
$$y_3 = 5$$

we solve, $U\underline{x} = \underline{y}$

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1.5 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & \underline{18} \end{array}\right) \begin{pmatrix} \underline{z} \end{pmatrix} \begin{pmatrix} \underline{\underline{5}} \end{pmatrix}$$

$$z = 5/18$$

$$\begin{aligned} 2x + 3y + z &= 9 \\ 0.5y + 2.5z &= 1.5 \\ 18z &= 5 \end{aligned}$$

Gauss.

$$\boxed{x = \frac{35}{18}, \quad y = \frac{29}{18}, \quad z = \frac{5}{18}}$$

LU

n	$n^3/3$	$2n^2$	% Reduction
10	$3.\bar{3} \times 10^2$	2×10^2	40
100	$3.\bar{3} \times 10^5$	2×10^4	94
1000	$3.\bar{3} \times 10^8$	2×10^6	99.4

3.33