Tutorial Sheet 2.

1. $$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 \rightarrow R_2 - \frac{3}{2} R_1$

$R_3 \rightarrow R_3 - \frac{1}{2} R_1$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & \frac{7}{2} & \frac{17}{2} & 11 \end{array}\right]$$

$R_3 \rightarrow R_3 - 7R_2$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$R_1 \rightarrow \frac{1}{2} R_1$

$R_2 \rightarrow 2 R_2$

$R_3 \rightarrow -\frac{1}{2} R_3$

$$\left[\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_3$

$$\left[\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{1}{2}R_2 - \frac{1}{2}R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

By Back substitution

$x_1 = 7$
$x_2 = -9$
$x_3 = 5.$

2. $\text{Pascal}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$

$R_2 \to R_2 - R_1$ | $m_{21} = 1$

$R_3 \to R_3 - R_1$ | $m_{31} = 1$

| $m_{41} = 1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ | $m_{32} = 2$

$R_4 \to R_4 - 3R_2$ | $m_{42} = 3.$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$ | $m_{43} = 3.$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

$\Rightarrow$ Pascal (4) $= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$

3. (a) Use Gauss Elimination and note down the multiplier

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$ $\qquad m_{21} = 0$, $m_{31} = 3$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$ which is in upper triangular form

$$\therefore E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

It is to be noted that $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

(b) $[E : I]$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 3R_1$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\therefore E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = L$$

(c) Verification is a straightforward only

ie $$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}}_{U}$$

4. Use Gauss Elimination

$$\begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 5-3c & 1 \end{bmatrix}$$

$$R_3 \to R_3 - \left[\frac{5-3c}{4-2c}\right] R_2$$

$$\begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 0 & 1 - \left[\frac{5-3c}{4-2c}\right] \end{bmatrix}$$

Second pivot will be zero if $4-2c=0$ or $c=2$.

For $c=2$, we have

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

We cannot proceed furthur without row exchange.

However, if we exchange rows 2 & 3 using the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ we have}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ | $m_{21} = 3$

$R_3 \to R_3 - 2R_1$ | $m_{31} = 2$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \; ; \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow PA = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

(b) From part (a),

the third pivot will be zero when

$$1 - \frac{5-3c}{4-2c} = 0 \Rightarrow 4-2c = 5-3c$$

$$\Rightarrow c = 1$$

For $c=1$, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The elimination has lead to a zero at pivot position, which is not possible, hence elimination fails.

5. $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}}_{U}$

Multiplying LU and comparing the components, we get.

$\boxed{u_{11}=1}$ $\boxed{u_{12}=1}$ $\boxed{u_{13}=0}$ $\boxed{u_{14}=3}$

$l_{21}u_{11}=2 \Rightarrow \boxed{l_{21}=2}$

$l_{21}u_{12}+u_{22}=1 \Rightarrow \boxed{u_{22}=-1}$

$l_{21}u_{13}+u_{23}=-1 \Rightarrow \boxed{u_{23}=-1}$

$l_{21}u_{14}+u_{24}=1 \Rightarrow \boxed{u_{24}=-5}$

$l_{31}u_{11}=3 \Rightarrow \boxed{l_{31}=3}$

$l_{31}u_{12}+l_{32}u_{22}=-1 \Rightarrow \boxed{l_{32}=4}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=-1 \Rightarrow \boxed{u_{33}=3}$

$l_{31}u_{14}+l_{32}u_{24}+u_{34}=2$

$9-20+u_{34}=2$

$\Rightarrow \boxed{u_{34}=13}$

$l_{41}u_{11}=-1 \Rightarrow \boxed{l_{41}=-1}$

$l_{41}u_{12}+l_{42}u_{22}=2 \Rightarrow \boxed{l_{42}=-3}$

$l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = 3$

$0 + 3 + 3l_{43} = 3$

$\Rightarrow \boxed{l_{43} = 0}$

$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} = -1$

$\Rightarrow -3 + 15 + 0 + u_{44} = -1$

$\Rightarrow \boxed{u_{44} = -13}$

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}} = LU$$

To solve

$$A\vec{x} = LU\bar{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix} = \vec{b}$$

Let $U\bar{x} = \vec{y}$. First we solve $L\vec{y} = \vec{b}$ ie

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}$$

By Forward - Substitution process:

$$\boxed{y_1 = 8}$$

$$2y_1 + y_2 = 7 \Rightarrow \boxed{y_2 = -9}$$

$$y_3 = 14 - 3y_1 - 4y_2 \Rightarrow \boxed{y_3 = 26}$$

$$y_4 = -7 + y_1 + 3y_2 \Rightarrow \boxed{y_4 = -26}$$

$$\therefore \vec{y} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}$$

Now solve $U\vec{x} = \vec{y}$ by back-substitution process:

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}$$

$$\boxed{x_4 = 2}$$

$$3x_3 + 13x_4 = 26 \Rightarrow \boxed{x_3 = 0}$$

$$-x_2 - x_3 - 5x_4 = -9 \Rightarrow x_2 = -1$$

$$x_1 + x_2 + 3x_4 = 8 \Rightarrow \boxed{x_1 = 3}$$

The solution of the given system is $\bar{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

6 (a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(i) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - 1\right] + (-1)(2-\lambda) = 0$

$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - 2\right] = 0$

$\Rightarrow \lambda = 2$ or $(2-\lambda)^2 = 2$

$\Rightarrow 2 - \lambda = \pm 2 \Rightarrow \lambda = 2 \pm \sqrt{2}$

The eigen values of A are $2-\sqrt{2}, 2, 2+\sqrt{2}$, and all of them are positive

(ii) $|2| > 0$, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(4-1) + -2 = 4 > 0$

(iii) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$R_2 \to R_2 + \frac{1}{2} R_1$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 + \frac{2}{3} R_2$

$$\begin{bmatrix} \boxed{2} & -1 & 0 \\ 0 & \boxed{3/2} & -1 \\ 0 & 0 & \boxed{4/3} \end{bmatrix}$$

The pivots are $2, \frac{3}{2}$ and $\frac{4}{3}$, all are positive.

(iv) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\vec{x}^T A \bar{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - x_1x_2 + -x_1x_2 + 2x_2^2 - x_2x_3 + -x_2x_3 + 2x_3^2$$

$$= 2\left[x_1^2 - x_1x_2 + x_2^2 + x_3^2 - x_2x_3\right]$$

$$= 2\left[\left(x_1 - \frac{x_2}{2}\right)^2 - \frac{x_2^2}{4} + x_2^2 - x_2x_3 + x_3^2\right]$$

$$= 2\left[\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{3}{4}\left[x_2^2 - \frac{4}{3}x_2x_3\right] + x_3^2\right]$$

$$= 2\left[\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{3}{4}\left(x_2 - \frac{2}{3}x_3\right)^2 - \frac{1}{3}x_3^2 + x_3^2\right]$$

$$= 2\left[\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{3}{4}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2\right] > 0$$

$$\forall \vec{x} \neq \vec{0}$$

(b) We use the determinant test

(i) $|2| > 0$, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

For positive definiteness

$$\begin{vmatrix} 2 & -1 & a \\ -1 & 2 & -1 \\ a & -1 & 2 \end{vmatrix} > 0 \Rightarrow 2(4-1) + (-2+a) + a(1-2a) > 0$$

$$\Rightarrow 4 + 2a - 2a^2 > 0$$

$\Rightarrow a^2 - a - 2 < 0$

$\Rightarrow (a+1)(a-2) < 0$

$\Rightarrow \boxed{-1 < a < 2}$

(ii) $|1| > 0$ $\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix} \Rightarrow b-4 > 0 \Rightarrow b > 4$

$\begin{vmatrix} 1 & 2 & 3 \\ 2 & b & 4 \\ 3 & 4 & 5 \end{vmatrix} > 0 \Rightarrow (5b-16) - 2(10-12) + 3(8-3b) > 0$

$\Rightarrow -4b + 12 > 0 \Rightarrow b < 3.$

One determinant is +ve for $b > 4$ and the other for $b < 3$. $\therefore$ Both determinants can't be positive simultaneously for any value of b.

$\therefore$ The matrix B is NOT positive definite for any value of b.

(iii) $|c| > 0 \Rightarrow c > 0$

$\begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} > 0 \Rightarrow c^2 - 1 > 0 \Rightarrow (c-1)(c+1) > 0$

$\Rightarrow c < -1$ or $c > 1$

$$\begin{vmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{vmatrix} = c(c^2-1)-(c-1)+(1-c)>0$$

$$\Rightarrow (c-1)\left[c^2+c-1-1\right]>0$$

$$\Rightarrow (c-1)\ (c^2+c-2)>0$$

$$\Rightarrow \ (c-1)(c+2)(c-1)>0$$

$$\Rightarrow \ c<-2 \quad \text{or } c>1$$

From all the conditions above, all the determinants will be positive if $c>1$. Hence the matrix C is +ve definite for $c>1$.

7 (a) Let x_j be a coordinate of $\bar{x}$ such that

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| = |x_j|.$$

Then $\|\vec{x}\|_\infty^2 = |x_j|^2 = x_j^2 \le \sum_{i=1}^{n} x_i^2 = \|\vec{x}\|_2^2$

$$\Rightarrow \|\vec{x}\|_\infty \le \|\vec{x}\|_2 \quad -(1)$$

Now $\|\vec{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 \le \sum_{i=1}^{n} x_j^2 = n x_j^2$
$= n\|x\|_\infty^2$

$$\Rightarrow \|\vec{x}\|_2 \le \sqrt{n}\ \|\vec{x}\|_\infty \quad -(2)$$

Combining ① and ②, we get

$$\|\vec{x}\|_\infty \le \|\vec{x}\|_2 \le \sqrt{n}\,\|\vec{x}\|_\infty$$

b) $\|\vec{x}+\vec{y}\|_\infty = \max\limits_{1\le i\le n} |x_i + y_i| \le \max\limits_{1\le i\le n} (|x_i| + |y_i|)$

$$\le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i|$$

$$= \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$$