Tutorial Sheet 1

1.
$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 3 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & -1 \\ 1 & 2 & 3 & 4 & -1 \end{array}\right]$$

$R_2 \to R_2 - \frac{3}{4} R_1$

$R_3 \to R_3 - \frac{1}{2} R_1$

$R_4 \to R_4 - \frac{1}{4} R_1$

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & \frac{7}{4} & \frac{3}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & \frac{3}{2} & 3 & \frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{5}{4} & \frac{5}{2} & \frac{15}{4} & -\frac{5}{4} \end{array}\right]$$

$R_3 \to R_3 - \frac{6}{7} R_2$; $R_4 \to R_4 - \frac{5}{7} R_2$

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & \frac{7}{4} & \frac{3}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & 0 & \frac{12}{7} & \frac{10}{7} & -\frac{12}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} & -\frac{10}{7} \end{array}\right]$$

$R_4 \to R_4 - \frac{5}{6} R_3$

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & \frac{7}{4} & \frac{3}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & 0 & \frac{12}{7} & \frac{10}{7} & \frac{-12}{7} \\ 0 & 0 & 0 & \frac{5}{3} & 0 \end{array}\right]$$

Using Back substitution

$\frac{5}{3} x_4 = 0 \Rightarrow x_4 = 0$

$\frac{12}{7} x_3 + \frac{10}{7} \times 0 = \frac{-12}{7} \Rightarrow x_3 = -1$

$\frac{7}{4} x_2 - \frac{3}{2} = \frac{1}{4} \Rightarrow x_2 = 1$

$4 x_1 + 3 - 2 = 1 \Rightarrow x_1 = 0$

2.

ⓐ Without Pivoting

$$\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

$$m_{21} = \frac{5.291}{0.003000} \approx 1764$$

$R_2 \to R_2 - 1764\ R_1$

$$\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 0 & -104300 & -104400 \end{array}\right]$$

$$\Rightarrow x_2 = \frac{-104400}{-104300} = 1.001$$

Error in x_2 is 0.001

$$x_1 = \frac{1}{0.003000}\left[59.17 - 59.14 \times 1.001\right]$$

$$= \frac{1}{0.003000}\left[59.17 - 59.20\right]$$

$$= \frac{1}{0.003000}\left[-0.03\right]$$

$$= -10$$

Error in $x_1 = |-10-10| = 20$

Huge error.

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With pivoting

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0.0003000 & 59.14 & 59.17 \end{array}\right]$$

$$m_{21} = \frac{0.0003000}{5.291} = 0.00005670$$

$R_2 \rightarrow R_2 - m_{21} R_1$

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0 & 59.14 & 59.14 \end{array}\right]$$

$x_2 = 1$

$$x_1 = \frac{1}{5.291}\left[46.78 + 6.130\right]$$

$$= \frac{1}{5.291}\left[52.91\right] = 10$$

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3. Verification is straight forward only

Without pivoting

$$\left[\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 2 & \frac{2}{3} & \frac{1}{3} & 1 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 2.000 & 0.6667 & 0.3333 & 1.000 \\ 1.000 & 2.000 & -1.000 & 0.000 \end{array}\right]$$

$m_{21} = \frac{2}{6} = 0.3333$ $\quad m_{31} = \frac{1}{6} = 0.1667$

$R_2 \rightarrow R_2 - m_{21} R_1$

$R_3 \rightarrow R_3 - m_{31} R_1$

$$\left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.000 & 0.0001000 & -0.3333 & 1.667 \\ 0.000 & 1.667 & -1.333 & 0.3334 \end{array}\right]$$

$m_{32} = 16670$ $\quad R_3 \rightarrow R_3 - m_{32} R_2$

$$\left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.000 & 0.0001000 & -0.3333 & 1.667 \\ 0.000 & 0.000 & 5555 & -27790 \end{array}\right]$$

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Back substitution

$x_1 = 1.335 \qquad x_2 = 0.0000 \qquad x_3 = -5.003$

Huge errors in x_1 and x_2

Pivoting

$$\left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 2.000 & 0.6667 & 0.3333 & 1.000 \\ 1.000 & 2.000 & -1.000 & 0.000 \end{array}\right]$$

$m_{21} = \frac{2}{6} = 0.3333 \qquad m_{31} = \frac{1}{6} = 0.1667$

$R_2 \Rightarrow R_2 - m_{21} R_1$

$R_3 \Rightarrow R_3 - m_{31} R_1$

$$\left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.000 & 0.0001000 & -0.3333 & 1.667 \\ 0.000 & 1.667 & -1.333 & 0.3334 \end{array}\right]$$

$R_3 \Leftrightarrow R_2$

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.000 & 1.667 & -1.333 & 0.3334 \\ 0.000 & 0.0001000 & -0.3333 & 1.667 \end{bmatrix}$$

$$m_{32} = \frac{0.0001000}{1.667}$$

$$= 0.00005999$$

$R_3 \to R_3 - m_{32} R_2$

$$\left[\begin{array}{ccc|c} 6.000 & 2.000 & 2.000 & -2.000 \\ 0.000 & 1.667 & -1.333 & 0.3334 \\ 0.000 & 0.000 & -0.3332 & 1.667 \end{array}\right]$$

Back substitution

$x_1 = 2.602$

$x_2 = -3.801$

$x_3 = -5.003$.

Here errors are low

4 (1) $K = 0$; the matrix A has a zero row and hence is singular, ie, non invertible

(2) $K = 2$; the matrix A has two rows as the same, and the Gauss elimination will lead to a zero row. Hence for $k=2$, matrix A is non-invertible.

3) $k=7$, Matrix A has two columns as the same, and the Gauss Elimination will again lead to a zero row. Hence for $k=7$, the matrix A is non invertible.

For other values of k, let us do the Gauss Elimination on A

$$\begin{bmatrix} 2 & k & k \\ k & k & k \\ 8 & 7 & k \end{bmatrix}$$

$R_2 \rightarrow R_2 - \frac{k}{2} R_1$

$R_3 \rightarrow R_3 - 4 R_1$

$$\begin{bmatrix} 2 & k & k \\ 0 & k-\frac{k^2}{2} & k-\frac{k^2}{2} \\ 0 & 7-4k & -3k \end{bmatrix}$$

$R_3 \rightarrow \cancel{R_3} - \dfrac{7-4k}{k-\frac{k^2}{2}} R_2$

$$\begin{bmatrix} 2 & k & k \\ 0 & k-\frac{k^2}{2} & k-\frac{k^2}{2} \\ 0 & 0 & k-7 \end{bmatrix}$$

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Pivots are zero only when $k-\frac{k^2}{2}=0$ or $k-7=0$

ie only when $k=0, 2, 7$

For all other values of k, the pivots are non-zero and hence the matrix A is invertible for all other values of k.

5. $$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

Pivots can only be zero if $a=0$ or $a-b=0$ or $a=b$

Therefore the given matrix is invertible if $a\neq 0$ and $a\neq b$.

Finding Inverse.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{a}$; $R_2 \to \frac{1}{a-b} R_2$ $\quad R_3 \to \frac{1}{a-b} R_3$.

$$\left[\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{b}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$R_1 \to R_1 - \frac{b}{a} R_2 - \frac{b}{a} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & -\frac{b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$$\therefore A^{-1} = \frac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$$

6. $$[A \mid I] = \left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + \frac{1}{2} R_1$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + \frac{2}{3} R_2$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$R_2 \to R_2 + \frac{3}{4} R_3$.

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 + \frac{2}{3} R_2$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$R_1 \rightarrow \frac{1}{2} R_1$

$R_2 \rightarrow \frac{2}{3} R_2$

$R_3 \rightarrow \frac{3}{4} R_3.$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

7. A permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column and 0 otherwise.

There are $n!$ permutation matrices of size $n \times n$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

8 (a) In step-k we eliminate x_k from $(n-k)$ equations. This needs $(n-k)$ divisions in computing the m_{jk}, and $(n-k)(n-k+1)$ multiplications and $(n-k)(n-k+1)$ substractions

Since we do $(n-1)$ steps

$\therefore$ Total no. of multiplications / divisions operations in forward steps is

$$= \sum_{k=1}^{n-1} (n-k)(n-k+1) + (n-k)$$

$$= \sum_{k=1}^{n-1} (n-k)[n-k+2]$$

$$= \sum_{k=1}^{n-1} (n-k)^2 + 2\sum_{k=1}^{n-1} (n-k)$$

$$= \sum_{k=1}^{n-1} k^2 + 2\sum_{k=1}^{n-1} k$$

$$= \frac{(n-1)(2n-1)n}{6} + \frac{2(n-1)(n)}{2}$$

$$= \frac{2n^3+3n^2-5n}{6}$$

(b) Since $(n-k)(n-k+1)$ substractions required for eliminating x_k from $(n-k)$ equations

$\therefore$ Total number of additions / substractions

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Operations required for forward step is

$$\sum_{k=1}^{n-1} (n-k)(n-k+1) = \sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)$$

$$= \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2}$$

$$= \frac{n^3 - n}{3}$$

(c) Back substitution

To find x_n, one division is required and for each x_k; $k \neq n$ $(n-k)$ multiplications and for each summation term $(n-k-1)$ addition for each summation term and then one subtraction and one division.

The total number of multiplications/division operations for backward substitution is

$$1 + \sum_{k=1}^{n-1} [(n-k)+1] = 1 + \sum_{k=1}^{n-1} (n-k) + n-1$$

$$= n + \frac{n(n-1)}{2}$$

$$= \frac{n^2+n}{2}.$$

Total number of additions / substractions required for backward substitution is given by

$$\sum_{k=1}^{n-1}\left[(n-k-1)+1\right]$$

$$=\sum_{k=1}^{n-1}(n-k) = \frac{n^2-n}{2}.$$