Fixed point System → IIII.FFFF (decimal number)

20.3401, 351.3241

9999.9999 → Largest number

0000.0001 → Smallest number.

Floating point System

↓

$x = \pm m . 10^{e}$ → exponent

↑ mantisa

$0.1 \leq |m| < 1$, and $e \in I$

$0.6247 . 10^{3}$, $0.1731 . 10^{-2}$

123456.00732 →

0.1234×10^{6}

Significant Digits (Rules)

① Non-zero digits are always Significant.

② Any zero between two Significant digits are Significant

③ A final zero or trailing Zeros in the decimal portion ONLY are Significant.

408 → 3

0.00500 → 3

0.030400 → 5

136000 → 6

0.0013400 → 5

0.136000×10^{6}

$$x = 0.d_1 d_2 d_3 \ldots d_k d_{k+1} d_{k+2} \ldots \times 10^e$$

Chopping ~~&~~ rounding.

An error caused by chopping or rounding is called rounding error or round-off error.

Chopping Procedure (K-digits decimal)

$$fl(x) = 0.d_1 d_2 \ldots d_k \cdot 10^e$$

$$0.54321720059 \times 10^4$$

5-decimal digits chopping

$$fl(x) = 0.54321 \cdot 10^4$$

Rounding 5-digits

$$fl(x) = 0.54322 \cdot 10^4$$

In rounding adds $5 \times 10^{e-(k+1)}$ to x and then chops the result

$$fl(x) = 0.\delta_1 \delta_2 \ldots \delta_k \times 10^e.$$

If $d_{k+1} \geq 5$, we add 1 to d_k, and discards the digits after k^{th} place. (Round up)

If $d_{k+1} < 5$, we simply chop off all digits after d_k (round down).

$$\pi = 3.141592 65 \ldots$$

$$= 0.314159265 \ldots \times 10^1$$

5-digit chopping $fl(x) = 0.31415 \times 10^1$

5-digit chopping $\quad fl(x) = 0.31415 \times 10^1$

$= 3.1415$

5-digit rounding $\quad fl(x) = 0.31416 \times 10^1$

$= 3.1416.$

$P \quad P^*$

Ex

Absolute error $= |P - P^*|$

relative error $= \dfrac{|P-P^*|}{|P|}$

Ex ~~0.0004 x_1~~

$$0.0004\, x_1 + 1.402\, x_2 = 1.406 \quad \text{---} (1)$$

$$0.4003\, x_1 - 1.502\, x_2 = 2.501 \quad \text{---} (2)$$

Exact solution is $x_1 = 10, \quad x_2 = 1$

Gauss elimination method using four-digit floating point arithmatic.

$$m_{21} = \frac{0.4003}{0.0004} = 1001$$

$$(-1.502 - 1.402 \times 1001)\, x_2 = 2.501 - 1.406 \times 1001$$

$$\Rightarrow \quad -1405\, x_2 = -1404$$

$$x_2 = \frac{-1404}{-1405} = 0.9993$$

From the first equation,

$$x_1 = \frac{1.406 - 1.402 \times 0.9993}{0.0004}$$

$$= 1.406 - 1.401$$

$$= \frac{1.406 - 1.401}{0.0004}$$

$$= \frac{0.005}{0.0004} = \underline{12.5} \quad \text{large error.}$$

Using Pivoting: $|0.4003| > |0.004|$

Exchange equations 1 and 2

$$0.4003x_1 - 1.502x_2 = 2.501$$
$$0.0004\,x_1 + 1.402\,x_2 = 1.406$$

$$m_{21} = \frac{0.004}{0.4003} = 0.000\underbrace{9993}_{\text{4- significant digits}}$$

$R_2 \to R_2 - m_{21} R_1$

$$(1.402 + 0.0009993 \times 1.502)\, x_2 = 1.406 - 0.0009993 \times 2.501$$

$$\boxed{x_2 = 1}$$

$$0.4003\, x_1 = 2.501 + 1.502$$

$$\boxed{x_1 = 10}$$

For example, if we had $x_2 = 0.9993$

$$x_1 = \frac{2.501 + 1.502 \times 0.9993}{0.4003}$$

$$= \underline{9.998}$$

Application of Gauss Elimination (To find inverse of A).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

of A).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}.$$

Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be the inverse of A.

$$AX = I \quad \text{(by definition)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left[\begin{array}{c|cc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{array}\right] = \begin{bmatrix} \boxed{1} & 0 & 0 \\ \boxed{0} & 1 & 0 \\ \boxed{0} & 0 & 1 \end{bmatrix}$$

$$A\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Ex $\quad A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$

Augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - (-2)R_1$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - (-2)R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

$$\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{matrix}$$

Continue to apply elementary row operations untill A in augmented matrix becomes identity matrix.

$R_3 \to \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7/2 & -3/2 & 1/2 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{\text{identity}}\middle|\underbrace{\begin{array}{ccc} 2 & -1 & 0 \\ 3/2 & -1/2 & 1/2 \\ 5/2 & -3/2 & 1/2 \end{array}}_{\text{Inverse of } A}\right]$$

identity Inverse of A

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 3/2 & -1/2 & 1/2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$A_{m\times n}$ $(m \neq n)$ $\rightarrow$ Gauss elimination does not yield an upper triangular matrix, rather it reduces the coefficient matrix/argument matrix into the so-called "Echelon Form"

Properties of Echelon Form

① All non-zero rows are above any rows of all zeros.

② Each leading entry (i.e., the left most non-zero entry) of a row is in a column to the right of the leading entry of the row above it.

③ All entries in a column below a leading entry are zero.

Ex
$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 3/2 & 1 \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Not in ~~Ech~~ Echelon Form

$$\begin{bmatrix} 4 & 0 & 5 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Property – 1   Property – 2.

$$\begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Reduced Row Echelon Form

In addition to the above 3 properties, if matrix also satisfies the following two properties, it is said to be in reduced row echelon form.

① The leading entry in each non-zero row is 1 ✓

② Each leading 1 is the only non-zero entry in its column.

Ex

$$\begin{bmatrix} 0 & 1 & -1 & 0 & -7 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 \end{bmatrix} \checkmark$$

Gauss-Jorden Elimination

$\underline{1.00} \rightarrow 3$

$0.\underbrace{100} \times 10^1$

$57.200 = 0.\underbrace{57200} \times 10^2$

$\Rightarrow 0.\underline{00}\underline{57200} \times 10^4$