① Positive Definite Matrices:

A real square matrix A is said to be positive definite if $\det(A) > 0$ and all leading principal minors are positive.

For example, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is positive definite

if (i) $a_{11} > 0$, (ii) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ (iii) $\det(A) > 0$.

Ex

(i) $A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$ $\qquad a_{11} = 1 > 0$, $\begin{vmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{vmatrix} = \frac{1}{3} - \frac{1}{4} > 0$.

$\det(A) = \frac{1}{2160} > 0$.

$\Rightarrow$ A is positive definite

(ii) $B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 4 & 5 & 6 \end{pmatrix}$ $\qquad b_{11} = 1 > 0$

$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 < 0$.

$\Rightarrow$ B is not positive definite.

An equivalent definition.

A matrix A is positive definite if $\underline{x}^T A \underline{x} > 0$ for all n-dimensional vectors $\underline{x} \neq 0$.

Ex

$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}_{3\times3}$.

Sol^n Suppose $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then

Sol: Suppose $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then

$$\underline{x}^T A \underline{x} = (x_1, x_2, x_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (x_1, x_2, x_3) \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix}$$

$$= \underbrace{2x_1^2 - 2x_1x_2 + 2x_2^2} - 2x_2x_3 + 2x_3^2$$

$$= x_1^2 + \underbrace{x_1^2 - 2x_1x_2 + x_2^2} + \underbrace{x_2^2 - 2x_2x_3 + x_3^2} + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2.$$

$$> 0 \quad \forall \ \underline{x} = (x_1, x_2, x_3) \neq (0,0,0).$$

$\Rightarrow$ matrix A is positive definite.

Result: If A is an $n \times n$ symmetric and positive definite matrix, then

(i) A has an inverse ✓

(ii) $a_{ii} > 0$ for each $i = 1, 2, \dots n$.

(iii) $\max\limits_{1 \le k, j \le n} |a_{kj}| \le \max\limits_{1 \le i \le n} |a_{ii}|$

(iv) $a_{ij}^2 < a_{ii} \cdot a_{jj}$ for each $i \neq j$.

(2) Diagonally Dominant Matrices.

A is said to be diagonally dominant if

n

A is said to be diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ holds for each } i = 1, 2 \ldots n.$$

example,

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$$

$|7| > |2| + |0|$ ✓

$|5| > |3| + |-1|$ ✓

$|-6| > |0| + |5|$ ✓

$\Rightarrow$ A is strictly diagonally dominant.

A is <u>strictly diagonally dominant</u> if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for each } i = 1, 2 \ldots n.$$

$$A^T = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}$$

$\rightarrow |7| > |3| + |0|$

$|5| < |2| + |5|$ ✗

A^T is not diagonally dominant.

A strictly diagonally dominant matrix A is non-singular.

Cholesky's Decomposition Method.

Cholesky's decomposition is applicable for symmetric and positive definite matrix A. In this case, the decomposition of A is $A = LL^T$,

where

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

$$A = LL^T = \begin{pmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ \end{pmatrix}$$

$$LL^T = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$$

we only need to solve six equations in six unknown.

$A\underline{x} = \underline{b}$

First we solve $L\underline{y} = \underline{b}$ for $\underline{y}$ $\longrightarrow$

Second we solve $L^T\underline{x} = \underline{y}$ for $\underline{x}$. $\longrightarrow$

For 3×3 matrix,

$$l_{11}^2 = a_{11} \Rightarrow l_{11} = \sqrt{a_{11}} \checkmark$$

$$l_{11}l_{21} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{l_{11}}$$

$$l_{11}l_{31} = a_{31} \Rightarrow l_{31} = \frac{a_{31}}{l_{11}}$$

$$l_{21}^2 + l_{22}^2 = a_{22} \Rightarrow l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$$l_{21}l_{31} + l_{22}l_{32} = a_{32} \Rightarrow l_{32} = \frac{a_{32} - l_{21}l_{31}}{l_{22}}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33} \Rightarrow l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}. \checkmark$$

Ex Solve the linear equations

$$4x_1 + 2x_2 + 14x_3 = 14$$
$$2x_1 + 17x_2 - 5x_3 = -101$$
$$14x_1 - 5x_2 + 83x_3 = 155$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{bmatrix} \cdot \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ -101 \\ 155 \end{bmatrix}$$

$$\quad A \qquad\qquad \underline{x} \qquad\qquad \underline{b}$$

$$A = LL^T = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} & & \\ 0 & & \\ 0 & - & - \end{bmatrix}^T$$

$$l_{11}^2 = 4 \Rightarrow l_{11} = 2$$

$$l_{11} l_{21} = 2 \Rightarrow l_{21} = 1$$

$$l_{11} l_{31} = 14 \Rightarrow l_{31} = 7$$

$$l_{21}^2 + l_{22}^2 = 17 \Rightarrow l_{22} = 4$$

$$l_{21} l_{31} + l_{22} l_{32} = -5 \Rightarrow l_{32} = -3$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83 \Rightarrow l_{33} = 5$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}.$$

Let $y = \begin{pmatrix} \\ \\ \end{pmatrix}$. Then $L\underline{y} = \underline{b}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$y_1 = 7, \quad y_2 = -27, \quad y_3 = 5.$$

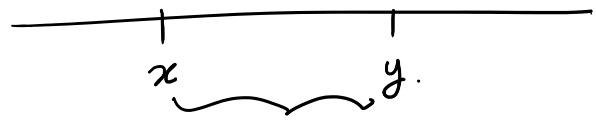
Now, we solve $L^T \underline{x} = \underline{y}$

$$\Rightarrow \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -27 \\ 5 \end{pmatrix}$$

$$x_3 = 1$$
$$x_2 = -6$$
$$x_1 = 3$$

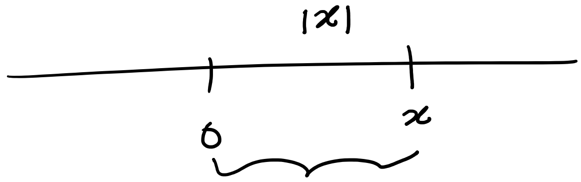
Norms of vectors and Matrices:

$\mathbb{R}$

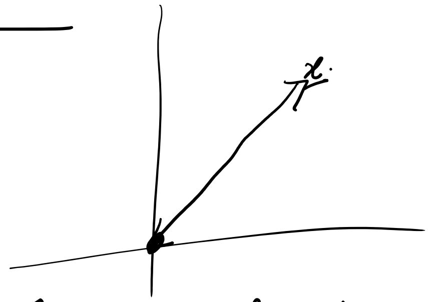


$|y - x| \rightarrow$

$|x| \rightarrow$



$\mathbb{R}^n$ $\quad \underline{x} = (x_1, x_2, \ldots x_n)$



Definition A vector norm on $\mathbb{R}^n$ is a function $\|\cdot\|$ from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties

(i) $\|\underline{x}\| \geq 0$ for all $\underline{x} \in \mathbb{R}^n$

(ii) $\|\underline{x}\| = 0$ if and only if $\underline{x} = (0, 0, \ldots, 0)$

(iii) $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$ for all $\alpha \in \mathbb{R}$ and $\underline{x} \in \mathbb{R}^n$

(iv) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$

Specific Norms: The l_p and l_∞ norms for the vector $\underline{x} = (x_1, x_2, \ldots x_n)^T$ are defined by

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

and

$$\|\underline{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| . \checkmark$$

$1 \le i \le n$

$p=1, \quad \|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$

$p=2, \quad \|\underline{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ → Euclidean norm

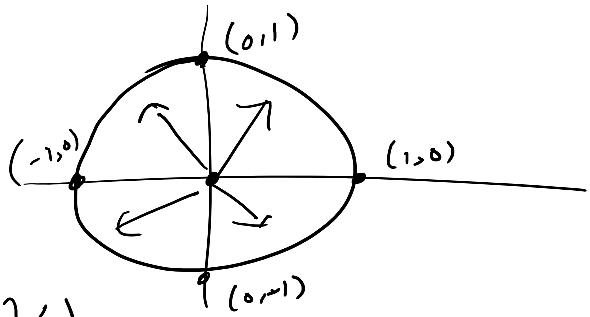
$\underline{\underline{\mathbb{R}^2}}$ The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are given by a unit circle centred at origin

l_2 -

$\underline{x} = (x_1, x_2)$.

$\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2} < 1$

$\iff x_1^2 + x_2^2 < 1$

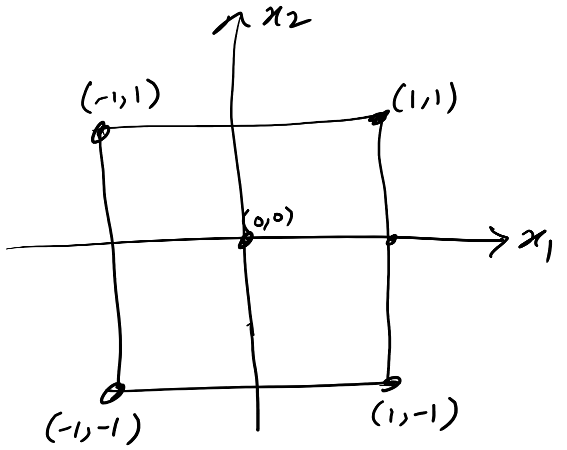


l_∞ - norm

$\|\underline{x}\|_\infty = \max\{|x_1|, |x_2|\} < 1$

$\iff |x_1| < 1, \; \& \; |x_2| < 1$

$-1 \le x_1 < 1, \; \& \; -1 < x_2 < 1$



$\underline{x} = (-1, 1, -2)^T$

$\|\underline{x}\|_2 = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$

$\|\underline{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2.$