Tutorial Sheet 3

1 (a)

$$\text{pascal}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} =: A \text{ (let)}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\Rightarrow \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 3x_3 + 4x_4 \\ x_1 + 3x_2 + 6x_3 + 10x_4 \\ x_1 + 4x_2 + 10x_3 + 20x_4 \end{bmatrix}$$

$$\begin{aligned}
= \; & x_1^2 + \underline{x_1x_2} + \underline{x_1x_3 + x_1x_4} \\
& + \underline{x_1x_2} + 2x_2^2 + 3x_2x_3 + 4x_2x_4 \\
& + \underline{x_1x_3} + 3x_2x_3 + 6x_3^2 + 10x_3x_4 \\
& + \underline{x_1x_4} + 4x_2x_4 + 10x_3x_4 + 20x_4^2
\end{aligned}$$

$$\begin{aligned}
&= x_1^2 + 2x_1(x_2 + x_3 + x_4) + 2x_2^2 + 6x_3^2 + 20x_4^2 + 6x_2x_3 + 8x_2x_4 + 20x_3x_4 \\
&= [x_1 + (x_2 + x_3 + x_4)]^2 - (x_2 + x_3 + x_4)^2 + 2x_2^2 + 6x_3^2 + 20x_4^2 + 6x_2x_3 + 8x_2x_4 + 20x_3x_4 \\
&= (x_1 + x_2 + x_3 + x_4)^2 + \underline{x_2^2} + 5x_3^2 + 19x_4^2 + \underline{4x_2x_3 + 6x_2x_4} + 18x_3x_4 \\
&= (x_1 + x_2 + x_3 + x_4)^2 + \underline{x_2^2 + 2x_2(2x_3 + 3x_4)} + 5x_3^2 + 19x_4^2 + 18x_3x_4 \\
&= (x_1 + x_2 + x_3 + x_4)^2 + [x_2 + (2x_3 + 3x_4)]^2 - (2x_3 + 3x_4)^2 + 5x_3^2 + 19x_4^2 + 18x_3x_4 \\
&= (x_1 + x_2 + x_3 + x_4)^2 + (x_2 + 2x_3 + 3x_4)^2 + \underline{x_3^2} + 10x_4^2 + \underline{6x_3x_4} \\
&= (x_1 + x_2 + x_3 + x_4)^2 + (x_2 + 2x_3 + 3x_4)^2 + (x_3 + 3x_4)^2 + x_4^2 \geq 0
\end{aligned}$$

and $\vec{x}^T A \vec{x} = 0$ only when $\vec{x} = \vec{0}$.

Therefore the matrix pascal(4) is positive definite.

(b) Cholesky decomposition

Let
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix}$$
$$A \qquad\qquad L \qquad\qquad L^T$$

Multiplying L and L^T and comparing the ~~ele~~ elements on both sides.

$l_{11}^2 = 1 \Rightarrow \boxed{l_{11} = 1}$ $\quad l_{11} l_{21} = 1 \Rightarrow \boxed{l_{21} = 1}$ $\quad l_{11} l_{31} = 1 \Rightarrow \boxed{l_{31} = 1}$ $\quad l_{11} l_{41} = 1 \Rightarrow \boxed{l_{41} = 1}$

$l_{21}^2 + l_{22}^2 = 2 \Rightarrow \boxed{l_{22} = 1}$ $\quad l_{21} l_{31} + l_{22} l_{32} = 3 \Rightarrow \boxed{l_{32} = 2}$ $\quad l_{21} l_{41} + l_{22} l_{42} = 4 \quad \boxed{l_{42} = 3}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 6 \Rightarrow \boxed{l_{33} = 1}$ $\quad l_{31} l_{41} + l_{32} l_{42} + l_{33} l_{43} = 10 \quad \boxed{l_{43} = 3}$

$l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 = 20 \Rightarrow \boxed{l_{44} = 1}$

$$\Rightarrow \text{pascal}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

~~which~~ This factorization is exactly the same as that obtained in problem 5 of tutorial sheet 2.

/

(2) Let the rotation matrix be

$$P = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$PA = \begin{bmatrix} 3\cos\theta + \sin\theta & \cos\theta + 3\sin\theta & \sin\theta \\ -3\sin\theta + \cos\theta & -\sin\theta + 3\cos\theta & \cos\theta \\ 0 & 1 & 3 \end{bmatrix}$$

The entry in the second row and first column of PA should be zero

$$\therefore \quad -3\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = \frac{1}{3}.$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{10}}, \quad \cos\theta = \frac{3}{\sqrt{10}}$$

Hence $$P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} \sqrt{10} & \frac{6}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ 0 & 8/\sqrt{10} & 3/\sqrt{10} \\ 0 & 1 & 3 \end{bmatrix}$$

3 (a) Jacobi Method:

$$x_1^{(k+1)} = \frac{1}{10}\left(6 + x_2^{(k)} - 2x_3^{(k)}\right),$$

$$x_2^{(k+1)} = \frac{1}{11}\left(25 + x_1^{(k)} + x_3^{(k)} - 3x_4^{(k)}\right),$$

$$x_3^{(k+1)} = \frac{1}{10}\left(-11 - 2x_1^{(k)} + x_2^{(k)} + x_4^{(k)}\right),$$

$$x_4^{(k+1)} = \frac{1}{8}\left(15 - 3x_2^{(k)} + x_3^{(k)}\right).$$

Start with $\vec{x}^{(0)} = (0,0,0,0)^T$.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	Relative error $= \frac{\lVert \vec{x}^{(k)} - \vec{x}^{(k-1)} \rVert_\infty}{\lVert \vec{x} \rVert_\infty}$
0	0	0	0	0	
1	0.6000	2.2727	-1.1000	1.8750	1
2	1.0473	1.7159	-0.8052	0.8852	0.4375
3	0.9326	2.0533	-1.0493	1.1309	0.1420
4	1.0152	1.9537	-0.9681	0.9738	0.0654
5	0.9890	2.0114	-1.0103	1.0214	0.0232
6	1.0032	1.9922	-0.9945	0.9944	0.0107
7	0.9981	2.0023	-1.0020	1.0036	0.0039
8	1.0006	1.9987	-0.9990	0.9989	0.0018
9	0.9997	2.0005	-1.0004	1.0006	0.6649×10^{-3}
10	1.0001	1.9998	-0.9998	0.9998	0.3096×10^{-3}
11	0.9999	2.0001	-1.0001	1.0001	0.1160×10^{-3}
12	1.0000	2.0000	-1.0000	1.0000	0.5425×10^{-4}
13	0.9999	2.0000	-1.0000	1.0000	0.2052×10^{-4}
14	1.0000	2.0000	-1.0000	1.0000	9.6222×10^{-6}
15	1.0000	2.0000	-1.0000	1.0000	3.7509×10^{-6}
16	1.0000	2.0000	-1.0000	1.0000	1.7209×10^{-6}
					6.9204×10^{-7} STOP

b) Gauss – Seidel Method:

$$x_1^{(k+1)} = \frac{1}{10}\left(6 + x_2^{(k)} - 2x_3^{(k)}\right),$$

$$x_2^{(k+1)} = \frac{1}{11}\left(25 + x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)}\right),$$

$$x_3^{(k+1)} = \frac{1}{10}\left(-11 - 2x_1^{(k+1)} + x_2^{(k+1)} + x_4^{(k)}\right),$$

$$x_4^{(k+1)} = \frac{1}{8}\left(15 - 3x_2^{(k+1)} + x_3^{(k+1)}\right).$$

Start with $\vec{x}^{(0)} = (0, 0, 0, 0)^T$.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	relative error $= \frac{\lVert \vec{x} - \vec{x}^{(k)} \rVert_\infty}{\lVert \vec{x} \rVert_\infty}$
0	0	0	0	0	1
1	0.6000	2.3273	−0.9873	0.8789	0.2
2	1.0302	2.0369	−1.0145	0.9843	0.018469 $= 1.8469 \times 10^{-2}$
3	1.0066	2.0036	−1.0025	0.9984	3.2925×10^{-3}
4	1.0009	2.0003	−1.0003	0.9999	4.3049×10^{-4}
5	1.0001	2.0000	−1.0000	1.0000	4.5640×10^{-5}
6	1.0000	2.0000	−1.0000	1.0000	4.1818×10^{-6}
7	1.0000	2.0000	−1.0000	1.0000	3.3317×10^{-7} STOP

4) We can rewrite the scheme as

$$x_i^{(new)} = x_i^{(old)} + \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(new)} - \sum_{j=i}^{n} a_{ij}\, x_j^{(old)}\right]$$

if $x_i^{(new)} = x_i^{(old)}$, the quantity in the square bracket vanishes.

$$\Rightarrow b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(old)} - \sum_{j=i}^{n} a_{ij}\, x_j^{(old)} = 0 \Rightarrow \sum_{j=1}^{n} a_{ij}\, x_j^{(old)} = b_i \quad \forall\ i=1,2,\dots,n$$

which is exactly $A\vec{x} = \vec{b}$. This means that this $[x_i^{(old)}]$ or $[x_i^{(new)}]$ satisfies the original system and hence ~~it is a~~ this is the correct solution of $A\vec{x} = \vec{b}$.

5 (a) $A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$

$\|A\|_\infty = \max\{|1| + |2|, |1.00001| + |2|\}$

$= 3.00001$

$[A : I] = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 1.00001 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 1.00001 R_1} \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -0.00002 & -1.00001 & 1 \end{array}\right]$

$R_1 \longrightarrow R_1 + 10^5 R_2$

$\left[\begin{array}{cc:cc} 1 & 0 & -100000 & 100000 \\ 0 & -0.00002 & -1.00001 & 1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -100000 & 100000 \\ 50000.5 & -50000 \end{bmatrix}$ $\|A^{-1}\|_\infty = 200000$

$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = \cancel{6000002}$ $\underline{\underline{600002}}$

(b) $B = \begin{bmatrix} 58.9 & 0.03 \\ -6.10 & 5.31 \end{bmatrix}$ $\|B\|_\infty = 58.93$

$[B : I] = \left[\begin{array}{cc|cc} 58.9 & 0.03 & 1 & 0 \\ -6.10 & 5.31 & 0 & 1 \end{array}\right]$

$$R_2 \to R_2 + \frac{6.10}{58.9} R_1$$

$$\left[\begin{array}{cc|cc} 58.9 & 0.03 & 1 & 0 \\ 0 & 5.3069 & 0.1036 & 1 \end{array}\right]$$

$$R_1 \to R_1 - \frac{0.03}{5.3069} R_2$$

$$\left[\begin{array}{cc|cc} 58.9 & 0 & 0.9994 & -0.0057 \\ 0 & 5.3069 & 0.1036 & 1 \end{array}\right]$$

$$R_1 \to R_1/58.9 \qquad R_2 \to R_2/5.3069$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0.0170 & -0.001 \\ 0 & 1 & 0.0195 & 0.1884 \end{array}\right]$$

$$B^{-1} \approx \begin{bmatrix} 0.0170 & -0.001 \\ 0.0195 & 0.1884 \end{bmatrix}$$

$$\Rightarrow \|B^{-1}\|_\infty = |0.0195| + |0.1884| = 0.2079$$

$$K(B) = \|B\|_\infty \|B^{-1}\|_\infty \approx \underline{\underline{12.2515}}$$

(c) $C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ $\|C\|_\infty = 3$

$$[C|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_1 \to R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \xleftarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ $C^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ $\Rightarrow$ $\|C^{-1}\|_\infty = 4$

$K(C) = \|C\|_\infty \|C^{-1}\|_\infty = 12$

(d) $D = \begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}$

$\|D\|_\infty = 7$

$$[D : I] = \left[\begin{array}{ccc|ccc} 0.04 & 0.01 & -0.01 & 1 & 0 & 0 \\ 0.2 & 0.5 & -0.2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 - 25R_1$

$$\left[\begin{array}{ccc|ccc} 0.04 & 0.01 & -0.01 & 1 & 0 & 0 \\ 0 & 0.45 & -0.15 & -5 & 1 & 0 \\ 0 & 1.75 & 4.25 & -25 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - \frac{175}{45} R_2$

$$\left[\begin{array}{ccc|ccc} 0.04 & 0.01 & -0.01 & 1 & 0 & 0 \\ 0 & 0.45 & -0.15 & -5 & 1 & 0 \\ 0 & 0 & 29/6 & \frac{-50}{9} & \frac{-35}{9} & 1 \end{array}\right]$$

$$R_1 \to R_1 + \frac{6}{2900} R_3$$

$$R_2 \to R_2 + \frac{6 \times 15}{2900} R_3 .$$

$$\left[\begin{array}{ccc|ccc} 0.04 & 0.01 & 0 & \frac{86}{87} & -\frac{7}{870} & \frac{3}{1450} \\ 0 & 0.45 & 0 & \frac{-150}{29} & \frac{51}{8} & \frac{9}{290} \\ 0 & 0 & \frac{29}{6} & \frac{-50}{9} & \frac{-35}{9} & 1 \end{array}\right]$$

$$R_1 \to R_1 - \frac{1}{45} R_2$$

$$\left[\begin{array}{ccc|ccc} 0.04 & 0 & 0 & \frac{32}{29} & \frac{-4}{145} & \frac{1}{725} \\ 0 & 0.45 & 0 & \frac{-150}{29} & \frac{51}{58} & \frac{9}{290} \\ 0 & 0 & \frac{29}{6} & \frac{-50}{9} & -\frac{35}{9} & 1 \end{array}\right]$$

$$R_1 \to 25 R_1$$

$$R_2 \to \frac{100}{45} R_2$$

$$R_3 \to \frac{6}{29} R_3 .$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{800}{29} & \frac{-20}{29} & \frac{1}{29} \\ 0 & 1 & 0 & \frac{-1000}{87} & \frac{170}{87} & \frac{2}{29} \\ 0 & 0 & 1 & \frac{-100}{87} & \frac{-70}{87} & \frac{6}{29} \end{array}\right]$$

$$\Rightarrow D^{-1} = \begin{bmatrix} \frac{800}{29} & -\frac{20}{29} & \frac{1}{29} \\ -\frac{1000}{87} & \frac{170}{87} & \frac{2}{29} \\ -\frac{100}{87} & -\frac{70}{87} & \frac{6}{29} \end{bmatrix} = \begin{bmatrix} 27.5862 & -0.6897 & 0.0345 \\ -11.4943 & 1.9540 & 0.0690 \\ -1.1494 & -0.8046 & 0.2069 \end{bmatrix}$$

$$\|D^{-1}\|_\infty = \frac{821}{29} \approx 28.3103$$

$$K(D) = \|D\|_\infty \, \|D^{-1}\|_\infty = 7 \times \frac{821}{29} \approx 198.172.$$

6 (a)

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 6 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Let $\vec{x_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{y_i} = A\vec{x_i}$

Scaling
$$\vec{x_{i+1}} = \frac{\vec{y_i}}{\|\vec{y_i}\|_\infty}$$

$\vec{y_0} = A\vec{x_0} = \begin{bmatrix} 9 \\ 11 \\ 5 \end{bmatrix}$

$\vec{x_1} = \dfrac{\vec{y_0}}{\|\vec{y_0}\|_\infty} = \begin{bmatrix} 0.818 \\ 1.000 \\ 0.455 \end{bmatrix}$

$\vec{y_1} = A\vec{x_1} = \begin{bmatrix} 8.273 \\ 9.909 \\ 2.273 \end{bmatrix}$

$\vec{x_2} = \begin{bmatrix} 0.835 \\ 1.000 \\ 0.115 \end{bmatrix}$

$\vec{y_2} = A\vec{x_2} = \begin{bmatrix} 8.339 \\ 10.009 \\ 1.147 \end{bmatrix}$

$\vec{x_3} = \begin{bmatrix} 0.8333 \\ 1.000 \\ 0.115 \end{bmatrix}$

$\vec{y_3} = A\vec{x_3} = \begin{bmatrix} 8.333 \\ 9.999 \\ 0.573 \end{bmatrix}$

$\vec{x_4} = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.057 \end{bmatrix}$

$$\vec{y}_4 = A\vec{x}_4 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.286 \end{bmatrix} \qquad \vec{x}_5 = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.029 \end{bmatrix}$$

$$\vec{y}_5 = A\vec{x}_5 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.143 \end{bmatrix} \qquad \vec{x}_6 = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.014 \end{bmatrix}$$

$$\vec{y}_6 = A\vec{x}_6 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.072 \end{bmatrix} \qquad \vec{x}_7 = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.007 \end{bmatrix}$$

$$\vec{y}_7 = A\vec{x}_7 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.036 \end{bmatrix} \qquad \vec{x}_8 = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.004 \end{bmatrix}$$

$$\vec{y}_8 = A\vec{x}_8 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.018 \end{bmatrix} \qquad \vec{x}_9 = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.002 \end{bmatrix}$$

$$\vec{y}_9 = A\vec{x}_9 = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.009 \end{bmatrix} \qquad \vec{x}_{10} = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.001 \end{bmatrix}$$

$$\vec{y}_{10} = A\vec{x}_{10} = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.004 \end{bmatrix} \qquad \vec{x}_{11} = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.000 \end{bmatrix}$$

$$\vec{y}_{11} = A\vec{x}_{11} = \begin{bmatrix} 8.333 \\ 10.000 \\ 0.000 \end{bmatrix} \qquad \vec{x}_{12} = \begin{bmatrix} 0.833 \\ 1.000 \\ 0.000 \end{bmatrix}$$

Eigen value:

$$\lambda = \frac{A\vec{x}_{12} \cdot \vec{X}_{12}}{\vec{X}_{12} \cdot \vec{X}_{12}} = \frac{\vec{Y}_{12} \, \vec{X}_{12}}{\vec{X}_{12} \cdot \vec{X}_{12}}$$

$$\approx \frac{16.944}{1.694}$$

$$= 10.000$$

Hence the largest eigen value of A in magnitude (dominant eigen value) is $\underline{\underline{10}}$

(b) $$[A:I] = \left[\begin{array}{ccc|ccc} 4 & 5 & 0 & 1 & 0 & 0 \\ 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{3}{2}R_1} \left[\begin{array}{ccc|ccc} 4 & 5 & 0 & 1 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -2 & 2 & 0 \\ 0 & -\frac{5}{2} & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to -\frac{2}{5}R_2,\ R_3 \to \frac{R_3}{5}]{R_1 \to \frac{R_1}{4}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 3/5 & -2/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 3/5 & -2/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

The smallest Eigen value of A (in absolute value) will lead to the dominant ~~largest~~ eigen value of A^{-1}

ie If A has $|\lambda_1| > |\lambda_2| \cdots |\lambda_n| \Rightarrow A^{-1}$ has $\frac{1}{|\lambda_n|} > \frac{1}{|\lambda_{n-1}|} \cdots > \frac{1}{|\lambda_1|}$

Thus if λ_n is the smallest eigen value in magnitude $\Rightarrow \frac{1}{\lambda_n}$ is the largest eigen value in magnitude.

Let $\vec{x_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B

Scaling $\vec{x}_{i+1} = \dfrac{\vec{y}_i}{\|y_i\|_\infty}$

$\vec{y_i} = A^{-1} \vec{x_0}$

$\vec{y_0} = A^{-1} \vec{x_0} = \begin{bmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$

$\vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\vec{y_1} = \begin{bmatrix} \frac{1}{2} \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$

$\vec{x_2} = \begin{bmatrix} 1 \\ -4/5 \\ 2/5 \end{bmatrix}$

$\vec{y_2} = \begin{bmatrix} -0.900 \\ 0.920 \\ 0.080 \end{bmatrix}$

$\vec{x_3} = \begin{bmatrix} -0.978 \\ 1.000 \\ 0.087 \end{bmatrix}$

$\vec{y_3} = \begin{bmatrix} 0.989 \\ -0.987 \\ 0.017 \end{bmatrix}$

$\vec{x_4} = \begin{bmatrix} 1.000 \\ -0.998 \\ 0.018 \end{bmatrix}$

$\vec{y_4} = \begin{bmatrix} -0.999 \\ 0.999 \\ 0.004 \end{bmatrix}$

$\vec{x_5} = \begin{bmatrix} -1.000 \\ 1.000 \\ 0.004 \end{bmatrix}$

$\vec{y_5} = \begin{bmatrix} 1.000 \\ -1.000 \\ 0.001 \end{bmatrix}$

$\vec{x_6} = \begin{bmatrix} 1.000 \\ -1.000 \\ 0.001 \end{bmatrix}$

$$\vec{Y_6} = \begin{bmatrix} -1.000 \\ 1.000 \\ 0.000 \end{bmatrix} \qquad \vec{X_7} = \begin{bmatrix} -1.000 \\ 1.000 \\ 0.000 \end{bmatrix}$$

$$\vec{Y_7} = \begin{bmatrix} 1.000 \\ -1.000 \\ 0.000 \end{bmatrix} \qquad \vec{X_8} = \begin{bmatrix} 1.000 \\ -1.000 \\ 0.000 \end{bmatrix}$$

$$\vec{Y_8} = \begin{bmatrix} -1.000 \\ 1.000 \\ 0.000 \end{bmatrix}$$

$\therefore$ The largest eigen value for A^{-1}

$$= \frac{(A^{-1}\vec{X_8})\,\vec{X_8}}{\vec{X_8}\cdot\vec{X_8}} = \frac{\vec{Y_8}\,\vec{X_8}}{\vec{X_8}\cdot\vec{X_8}}$$

$$= \frac{-2.000}{2.000}$$

$$= -1.000$$

$\therefore$ Smallest eigen value of A is

$$\frac{1}{-1} = -1.$$

(c) Trace of A = sums of Eigen value

$\Rightarrow 4+5+5 = 10+\lambda+-1$

$\Rightarrow \lambda = 5$

$\Rightarrow$ Third Eigen value is 5

$\therefore$ Eigen values of A are 10, 5, −1.

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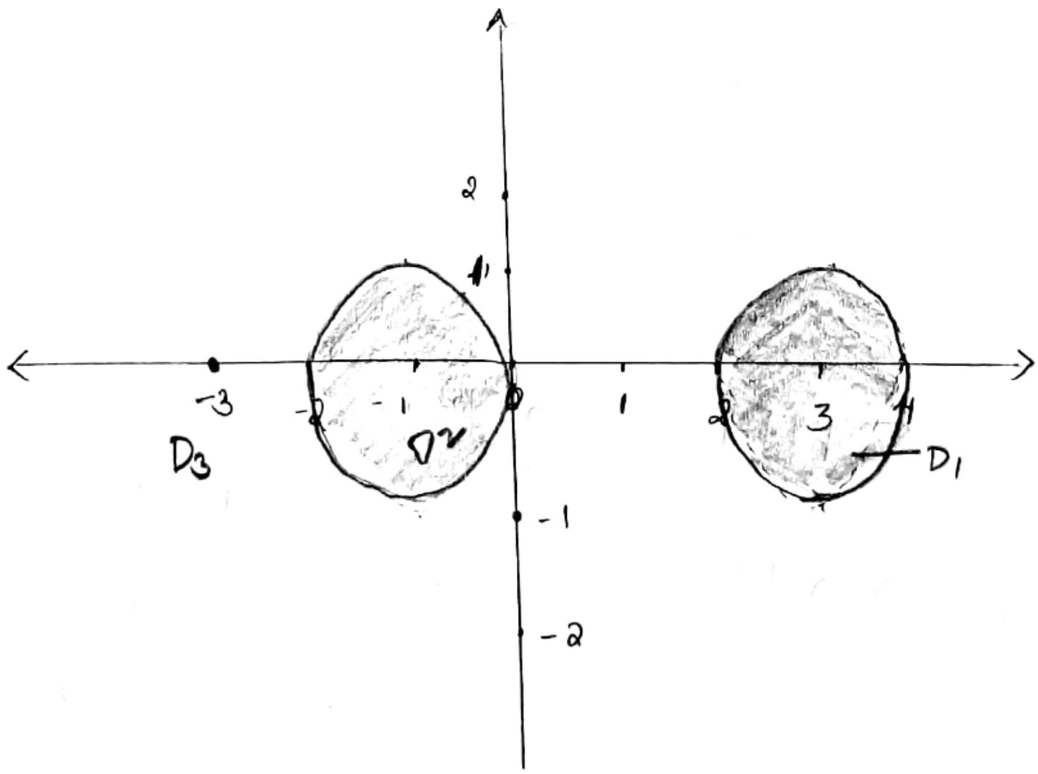
7. a

$$D_1 = \{z \in C : |z - a_{11}| \leq \sum_{\substack{j=1 \\ j \neq 1}}^{n} |a_{1j}|\}$$

$$= \{z \in C : |z - 3| \leq 1\}$$

$$D_2 = \{z \in C : |z - -1| \leq 1\}$$

$$D_3 = \{z \in C : |z - -3| \leq 0\}$$



(b) The Gersgorin discs for A are disjoint, therefore it is diagonalizable.

Statement: If all the Gersgorin discs of a matrix are disjoint, then the matrix is diagonalizable

(c): $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 0 & 1 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)\left((3-\lambda)(-1-\lambda)\right] = 0 \Rightarrow \lambda = -3, -1, 3.$$

Eigen vector for $\lambda = -3$

$$\begin{bmatrix} 6 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 6x_1 + x_3 = 0 \\ -x_1 + 2x_2 = 0 \end{array}$$

$$\Rightarrow x_2 = \frac{x_1}{2}$$

$$x_3 = -6x$$

$$\Rightarrow \begin{bmatrix} 2 \\ 1 \\ -12 \end{bmatrix}$$

Eigen vector for $\lambda = -1$

$$\begin{bmatrix} 4 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 + x_3 = 0$$
$$-x_1 = 0$$
$$-2x_3 = 0$$
$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector $\lambda = 3$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= 0 \\ -x_1 - 4x_2 &= 0 \\ -6x_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 0 & -4 \\ 1 & 1 & 1 \\ -12 & 0 & 0 \end{bmatrix}$$

$$[S \vdots I] = \left[\begin{array}{ccc|ccc} 2 & 0 & -4 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -12 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \longrightarrow R_2 - \frac{1}{2} R_1$

$R_3 \longrightarrow R_3 + 6 R_1$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3 & \frac{-1}{2} & 1 & 0 \\ 0 & 0 & -24 & 6 & 0 & 1 \end{array}\right]$$

$R_1 \Rightarrow R_1 - \frac{1}{6} R_3$

$R_2 \Rightarrow R_2 + \frac{1}{8} R_3$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 0 & \frac{-1}{6} \\ 0 & 1 & 0 & \frac{1}{4} & 1 & \frac{1}{8} \\ 0 & 0 & -24 & 6 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \Rightarrow -\frac{1}{24} R_3]{R_1 \Rightarrow \frac{1}{2} R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{-1}{12} \\ 0 & 1 & 0 & \frac{1}{4} & 1 & \frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{-1}{24} \end{array}\right]$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 0 & 0 & \frac{-1}{12} \\ \frac{1}{4} & 1 & 1/8 \\ -\frac{1}{4} & 0 & \frac{-1}{24} \end{bmatrix}$$

$$\Rightarrow A = S \Lambda S^{-1}$$

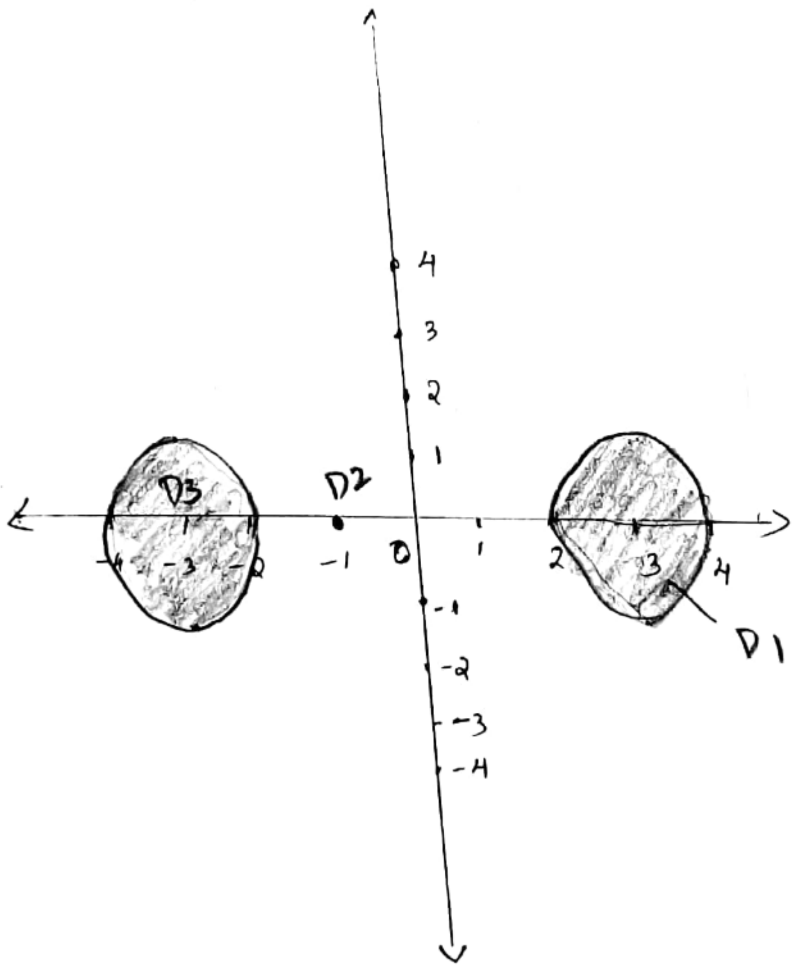
$$\begin{bmatrix} 3 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ 1 & 1 & 1 \\ -12 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{-1}{12} \\ \frac{1}{4} & 1 & \frac{1}{8} \\ -\frac{1}{4} & 0 & \frac{-1}{24} \end{bmatrix}$$

(d)

$D_1 = \{ z \in \mathbb{C} : |z-3| \leq 1 \}$

$D_2 = \{ z \in \mathbb{C} : |z - -1| \leq 0 \}$

$D_3 = \{ z \in \mathbb{C} : |z - -3| \leq 1 \}$



(8) The matrix A is not in symmetric tridiagonal form. So we shall have to apply Householder transformation to first get a symmetric tridiagonal matrix that is similar to A.

Let $A = A_0$.

Step 1 $\quad S_1 = \sqrt{(4)^2 + (1)^2 + (1)^2} = \sqrt{18}$

$$v_{11} = 0$$

$$v_{21} = \sqrt{\frac{1}{2}\left(1 + \frac{|a_{21}|}{S_1}\right)} = \sqrt{\frac{1}{2}\left(1 + \frac{4}{\sqrt{18}}\right)} = \sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}$$

$$v_{31} = \frac{a_{31}\, sgn(a_{21})}{2\, v_{21}\, S_1} = \frac{1 \times 1}{2\sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}\sqrt{18}} = \frac{1}{2\sqrt{18}\sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}}$$

$$v_{41} = \frac{a_{41}\, sgn(a_{21})}{2\, v_{21}\, S_1} = \frac{1}{2\sqrt{18}\sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}}$$

$$\vec{v}_1 = \left[0,\ \sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}},\ \frac{1}{2\sqrt{18}\sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}},\ \frac{1}{2\sqrt{18}\sqrt{\frac{1}{2} + \frac{2}{\sqrt{18}}}}\right]^T$$

$$P_1 = I - 2\vec{v}_1\vec{v}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ 0 & -\frac{1}{3\sqrt{2}} & \frac{1}{6}(3+2\sqrt{2}) & \frac{1}{6}(2\sqrt{2}-3) \\ 0 & -\frac{1}{3\sqrt{2}} & \frac{1}{6}(2\sqrt{2}-3) & \frac{1}{6}(2\sqrt{2}+3) \end{bmatrix}$$

$$A_1 = P_1 A_0 P_1 = \begin{bmatrix} 6 & -3\sqrt{2} & 0 & 0 \\ -3\sqrt{2} & 7 & -1 & -1 \\ 0 & -1 & 9/2 & 3/2 \\ 0 & -1 & 3/2 & 9/2 \end{bmatrix}$$

Step 2 $S_2 = \sqrt{(a_{32}^{(1)})^2 + (a_{42}^{(1)})^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$v_{12} = v_{22} = 0$

$v_{32} = \sqrt{\frac{1}{2}\left(1 + \frac{|a_{32}^{(1)}|}{S_2}\right)} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

$v_{42} = \frac{a_{42}^{(1)} \operatorname{sgn}(a_{32}^{(1)})}{2 v_{32} S_2} = \frac{-1 \times (-1)}{2\sqrt{\frac{\sqrt{2}+1}{\cancel{2}\sqrt{2}}} \times \cancel{\sqrt{2}}} = \frac{1}{2}\sqrt{\frac{\sqrt{2}}{\sqrt{2}+1}}$

$$\vec{V}_2 = \left[0,\ 0,\ \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}},\ \frac{1}{2}\sqrt{\frac{\sqrt{2}}{\sqrt{2}+1}}\right]^T$$

$$P_2 = I - 2\vec{V}_2\vec{V}_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A_2 = P_2 A_1 P_2 = \begin{bmatrix} 6 & -3\sqrt{2} & 0 & 0 \\ -3\sqrt{2} & 7 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

which is the required tridiagonal matrix that is similar to the given A.

Clearly, A_2 has one eigenvalue as 3. Therefore, for simplicity, we can apply the QR-factorization to a smaller matrix;

$$B = \begin{bmatrix} 6 & -3\sqrt{2} & 0 \\ -3\sqrt{2} & 7 & \sqrt{2} \\ 0 & \sqrt{2} & 6 \end{bmatrix}.$$

To apply the QR-factorization to the matrix

$$B_0 = \begin{bmatrix} 6 & -3\sqrt{2} & 0 \\ -3\sqrt{2} & 7 & \sqrt{2} \\ 0 & \sqrt{2} & 6 \end{bmatrix}$$

Step 1

$$C_2 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0.816497 & -0.57735 & 0 \\ 0.57735 & 0.816497 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$b_{21}^{(1)}$ entry of $C_2 B$ should be zero. This gives

$$-6\sin\theta_2 - 3\sqrt{2}\cos\theta_2 = 0 \Rightarrow \tan\theta_2 = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\theta_2 = -\frac{1}{\sqrt{3}}, \quad \cos\theta_2 = \frac{\sqrt{2}}{\sqrt{3}}$$

$$C_2 B_0 = \begin{bmatrix} 7.34847 & -7.50555 & -0.816497 \\ 0 & 3.26599 & 1.1547 \\ 0 & 1.41421 & 6 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix}$$

$b_{32}^{(2)}$ entry of $C_3(C_2 B)$ should be zero. This gives

$$(-\sin\theta_3)(3.26599) + (\cos\theta_3)(1.41421) = 0 \Rightarrow \tan\theta_3 = 0.433013$$

$$\Rightarrow \sin\theta_3 = 0.39736, \quad \cos\theta_3 = 0.917663$$

$$\Rightarrow C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.917663 & 0.39736 \\ 0 & -0.39736 & 0.917663 \end{bmatrix}$$

$$R_0 = C_3 C_2 B_0 = \begin{bmatrix} 7.34847 & -7.50555 & -0.816497 \\ 0 & 3.55903 & 3.44378 \\ 0 & 0 & 5.04715 \end{bmatrix}$$

$$B_0 = \underbrace{C_2^T C_3^T}_{Q_0} R_0$$

$$B_1 = R_0 C_2^T C_3^T = \begin{bmatrix} 10.3333 & -2.0548 & 0 \\ -2.0548 & 4.03509 & 2.00553 \\ 0 & 2.00553 & 4.63158 \end{bmatrix}$$

Step 2

$\cos\theta_2 = \dfrac{1}{\sqrt{1+(b_{21}/b_{11})^2}}$, $\sin\theta_2 = \dfrac{b_{21}/b_{11}}{\sqrt{1+(b_{21}/b_{11})^2}}$. (Can be done similarly to step 1)

$$C_2 = \begin{bmatrix} 0.980797 & -0.195033 & 0 \\ 0.195033 & 0.980797 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 B_1 = \begin{bmatrix} 10.5357 & -2.80232 & -0.391146 \\ 0 & 3.55684 & 1.96702 \\ 0 & 2.00553 & 4.63158 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.871072 & 0.491155 \\ 0 & -0.491155 & 0.871072 \end{bmatrix}$$

$$R_1 = C_3 C_2 B_1 = \begin{bmatrix} 10.5357 & -2.80232 & -0.391146 \\ 0 & 4.0833 & 3.98324 \\ 0 & 0 & 3.06833 \end{bmatrix}$$

$$B_2 = R_1 C_2^T C_3^T = \begin{bmatrix} 10.8799 & -0.796379 & 0 \\ -0.796379 & 5.44739 & 1.50703 \\ 0 & 1.50703 & 2.67273 \end{bmatrix}$$

Step-3

$$C_2 = \begin{bmatrix} 0.997332 & -0.0730021 & 0 \\ 0.0730021 & 0.997332 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 B_2 = \begin{bmatrix} 10.999 & -1.19193 & -0.110016 \\ 0 & 5.37471 & 1.503 \\ 0 & 1.50703 & 2.67273 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.962866 & 0.26998 \\ 0 & -0.26998 & 0.962866 \end{bmatrix}$$

$$R_2 = C_3 C_2 B_2 = \begin{bmatrix} 10.909 & -1.19193 & -0.110016 \\ 0 & 5.582 & 2.16877 \\ 0 & 0 & 2.1677 \end{bmatrix}$$

$$B_3 = R_2 C_2^T C_3^T = \begin{bmatrix} 10.9669 & -0.407498 & 0 \\ -0.407498 & 5.9459 & 0.585236 \\ 0 & 0.585236 & 2.08721 \end{bmatrix}$$

Step-4

$$C_2 = \begin{bmatrix} 0.99931 & -0.0371314 & 0 \\ 0.0371314 & 0.99931 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 B_3 = \begin{bmatrix} 10.9745 & -0.627996 & -0.0217306 \\ 0 & 5.92667 & 0.584832 \\ 0 & 0.585236 & 2.08721 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.99516 & 0.0982683 \\ 0 & -0.0982683 & 0.99516 \end{bmatrix}$$

$$R_3 = C_3 C_2 B_3 = \begin{bmatrix} 10.9745 & -0.627996 & -0.0217306 \\ 0 & 5.95549 & 0.787108 \\ 0 & 0 & 2.01964 \end{bmatrix}$$

$$B_4 = R_3 C_2^T C_3^T = \begin{bmatrix} 10.9902 & -0.221136 & 0 \\ -0.221136 & 5.99993 & 0.198466 \\ 0 & 0.198466 & 2.00986 \end{bmatrix}$$

Step-5

$$C_2 = \begin{bmatrix} 0.999798 & -0.0201171 & 0 \\ 0.0201171 & 0.999798 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 B_4 = \begin{bmatrix} 10.9924 & -0.341792 & -0.00399256 \\ 0 & 5.99427 & 0.198426 \\ 0 & 0.198466 & 2.00986 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.999452 & 0.0330912 \\ 0 & -0.0330912 & 0.999452 \end{bmatrix}$$

$$\cancel{R_4 = \begin{bmatrix} \boxed{10.9971} & -0.120653 & 0 \\ -0.120653 & 6.00182 & 0.066255 \\ 0 & 0.066255 & 2.0011 \end{bmatrix}}$$

$$R_4 = \begin{bmatrix} \boxed{10.9924} & -0.341792 & -0.00399256 \\ 0 & \boxed{5.99755} & 0.264826 \\ 0 & 0 & \boxed{2.00219} \end{bmatrix}$$

After 5 steps of the QR-method, the eigenvalues of B turn out to be 10.9924, 5.99755, 2.00219. Hence the approximate eigenvalues of A are 10.9924, 5.99755, 2.00219 and 3. Note that the actual eigenvalues of A are 11, 6, 3, 2.