

## Eigenvalues and Eigenvectors:

If  $A$  is a square matrix, the characteristic polynomial of  $A$  is defined by

$$p(\lambda) = \det(A - \lambda I).$$

It is easy to see that  $p(\lambda)$  is an  $n^{\text{th}}$  degree polynomial and, consequently, has at most  $n$  distinct zeros, some of which might be complex. The zeros of  $p(\lambda)$  are eigenvalues, or characteristic values of the matrix  $A$ . If  $\lambda$  is an eigenvalue of  $A$  and  $\underline{x} \neq \underline{0}$  satisfies  $(A - \lambda I)\underline{x} = \underline{0}$ , then  $\underline{x}$  is an eigenvector, or characteristic vector, of  $A$  corresponding to the eigenvalue  $\lambda$ .

There is a close connection between these eigenvalues and the likelihood that an iterative method will converge. We will discuss it in the later sections.

## Spectral Radius:

The spectral radius  $P(A)$  of a matrix  $A$  is defined by  $P(A) = \max |\lambda|$ , where  $\lambda$  is an eigenvalue of  $A$ .

For complex  $\lambda = \alpha + i\beta$ , we define  $|\lambda| = \sqrt{\alpha^2 + \beta^2}$ .

Note that if  $A$  is  $n \times n$  matrix, then  $P(A) \leq \|A\|$ , for any natural  $\|\cdot\|$  norm.

## Convergent Matrices:

An  $n \times n$  matrix  $A$  is convergent if

$$\lim_{K \rightarrow \infty} (A^K)_{ij} = 0, \text{ for each } i=1, 2, \dots, n \\ j=1, 2, \dots, n.$$

Example: Consider

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Computing powers of  $A$ , we obtain

$$A^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, A^3 = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{pmatrix}, A^4 = \begin{pmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{pmatrix},$$

and, in general

$$A^K = \begin{bmatrix} \left(\frac{1}{2}\right)^K & 0 \\ \frac{K}{2^{K+1}} & \left(\frac{1}{2}\right)^K \end{bmatrix}$$

So  $A$  is a convergent matrix because

$$\lim_{K \rightarrow \infty} \left(\frac{1}{2}\right)^K = 0, \text{ and } \lim_{K \rightarrow \infty} \frac{K}{2^{K+1}} = 0.$$

Notice that the convergent matrix  $A$  (given above) has  $P(A) = \frac{1}{2}$ , because  $\frac{1}{2}$  is the only eigenvalue of  $A$ .

This illustrates an important connection that exists between the spectral radius of a matrix and the convergence of the matrix, as ~~described~~ given in the following result.

" $A$  is convergent iff  $P(A) < 1$ ."

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## Iterative Techniques

An iterative technique to solve the  $n \times n$  linear system  $A\bar{x} = \bar{b}$  starts with an initial approximation  $\bar{x}^{(0)}$  to the solution  $\bar{x}$  and generates a sequence of vectors  $\{\bar{x}^{(k)}\}_{k=0}^{\infty}$  that converges to  $\bar{x}$ .

We will describe the Jacobi and the Gauss-Seidel iterative methods. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type ~~are~~ arise frequently in circuit analysis and in the numerical solutions of boundary-value problems and partial-differential equations.

### Jacobi Method:

Example: Solve the following equations by ~~the~~ Jacobi method.

$$9x_1 + x_2 + x_3 = 10$$

$$2x_1 + 10x_2 + 3x_3 = 19$$

$$3x_1 + 4x_2 + 11x_3 = 0$$

Note that the exact solution of this system is

$$x_1 = 1, x_2 = 2, x_3 = -1$$

Sol<sup>y</sup> Re-write the equations as

$$x_1 = \frac{1}{9} [10 - x_2 - x_3]$$

$$x_2 = \frac{1}{10} [19 - 2x_1 - 3x_3]$$

$$x_3 = \frac{1}{11} [-3x_1 - 4x_2]$$

Let  $\underline{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$  be the initial guess.

Define

$$x_1^{(k+1)} = \frac{1}{9} [10 - x_2^{(k)} - x_3^{(k)}],$$

$$x_2^{(k+1)} = \frac{1}{10} [19 - 2x_1^{(k)} - 3x_3^{(k)}],$$

$$x_3^{(k+1)} = \frac{1}{11} [-3x_1^{(k)} - 4x_2^{(k)}],$$

where  $k=0, 1, 2, \dots$

Let us start with the initial guess  $\underline{x}^{(0)} = (0, 0, 0)^T$ .

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\ x^{(k+1)} - x^{(k)}\ _\infty$
0.	0	0	0	—
1.	$\frac{10}{9} = 1.1111$	$\frac{19}{10} = 1.9000$	0	1.9000
2.	$\frac{8.1}{9} = 0.9000$	$\frac{16.7778}{10} = 1.6778$	-0.9939	0.9939
3.	1.0351	2.0182	-0.8556	0.3404
4.	0.9819	1.9496	-1.0162	0.1606
5.	1.0074	2.0085	-0.9768	0.0589 (55)

$K$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	error = $\ x^{(k+1)} - x^{(k)}\ _\infty$
6.	0.9965	1.9915	-1.0051	0.0283
7.	1.0015	2.0022	-0.9960	0.0107
8.	0.9993	1.9985	-1.0012	0.0052
9.	1.0003	2.0005	-0.9993	0.0020
10.	0.9999	1.9997	-1.0003	0.0010
11.	1.0001	2.0001	-0.9999	0.0004

We stopped after eleven iterations because

$$\|x^{(11)} - x^{(10)}\|_\infty = 0.0004 < 10^{-3}. \text{ (we fixed } \epsilon = 10^{-3})$$

$$\text{In fact, } \|x^{(11)} - x\|_\infty = \|(1.0001, 2.0001, -0.9999)$$

$$- (1, 2, -1)\|_\infty = 0.0001 = 10^{-4}.$$

Therefore, the approximate solution of the given system of equations by Jacobi's technique is

$$\tilde{x}_1 = 1.0001 \approx 1$$

$$\tilde{x}_2 = 2.0001 \approx 2$$

$$\tilde{x}_3 = -0.9999 \approx -1$$

## Numerical Algorithm of Jacobi Method

Input:  $A = [a_{ij}]$ ,  $b$ ,  $\underline{x}^0 = (x_1^0, x_2^0, \dots, x_n^0)$ , tolerance =  $\epsilon$ ,  
maximum no. of iterations =  $N$ .

Step 1: Set  $K=1$  //

Step 2: While ( $K \leq N$ ) do Steps 3-6 //

Step 3: for  $i=1, 2, \dots, n$

$$\underline{x}_i = \frac{1}{a_{ii}} \left[ \sum_{\substack{j=1 \\ j \neq i}}^n (-a_{ij} \underline{x}_j^0) + b_i^0 \right],$$

Step 4: If  $\|\underline{x} - \underline{x}^0\| < \epsilon$ , then //

OUTPUT  $\underline{x} = (x_1, x_2, \dots, x_n)$ ; STOP.

Step 5: Set  $K = K+1$  //

Step 6: for  $i=1, 2, \dots, n$

Set  $\underline{x}^0 = \underline{x}$ .

Step 7: OUTPUT  $\underline{x} = (x_1, x_2, \dots, x_n)$ ; STOP.

\* Another stopping criterion in Step 4 is

$$\frac{\|\underline{x}^{(K+1)} - \underline{x}^{(K)}\|}{\|\underline{x}^{(K+1)}\|} < \epsilon.$$

# Gauss-Seidel Method:

Example:

$$\begin{array}{l}
 9x_1 + x_2 + x_3 = 10 \\
 2x_1 + 10x_2 + 3x_3 = 19 \\
 3x_1 + 4x_2 + 11x_3 = 0
 \end{array} \quad \left| \begin{array}{l} \text{exact solution} \\ \underline{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \end{array} \right.$$

$\downarrow$       Gauss-Seidel

$$\begin{aligned}
 \underline{x}_1^{(k+1)} &= \frac{1}{9} [10 - \underline{x}_2^{(k)} - \underline{x}_3^{(k)}], \\
 \underline{x}_2^{(k+1)} &= \frac{1}{10} [19 - 2\underline{x}_1^{(k+1)} - 3\underline{x}_3^{(k)}], \\
 \underline{x}_3^{(k+1)} &= \frac{1}{11} [-3\underline{x}_1^{(k+1)} - 4\underline{x}_2^{(k+1)}].
 \end{aligned}$$

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\text{Error} = \ \underline{x}^{(k+1)} - \underline{x}^{(k)}\ _\infty$
0	0	0	0	-
1	1.1111	1.6778	-0.9131	1.6778
2	1.0262	1.9687	-0.9958	0.0849
3	1.0030	1.9981	-1.0001	0.0294
4	1.0002	2.0000	-1.0000	0.0028
5	1.0000	2.0000	-1.0000	0.0002

We stopped after five iterations because  $\|\underline{x}^{(5)} - \underline{x}^{(4)}\|_\infty = 0.0002 < 10^{-3}$ . Moreover, the approximate solution of the given ~~statis~~ system of equations is the same as the exact solution..

### Numerical Algorithm of Gauss-Seidel Method :

Input :  $A = [a_{ij}]$ ,  $b$ ,  $\underline{x}^0 = (x_1^0, x_2^0, \dots, x_n^0)$ , tolerance =  $\epsilon$ , max. no. of iterations =  $N$ .

Step 1: Set  $K=1$

Step 2: While ( $K \leq N$ ) do Steps 3-6

Step 3: for  $i=1, 2, \dots, n$

$$x_i^0 = \frac{1}{a_{ii}} \left[ - \sum_{j=1}^{i-1} (a_{ij} x_j) - \sum_{j=i+1}^n (a_{ij} x_j^0) + b_i \right],$$

Step 4: If  $\|\underline{x} - \underline{x}^0\| < \epsilon$ , then OUTPUT  $\underline{x}$ , STOP.

Step 5: Set  $K = K+1$ .

Step 6: for  $i=1, 2, \dots, n$

Set  $\underline{x}^0 = \underline{x}$ .

Step 7: OUTPUT  $\underline{x} = (x_1, x_2, \dots, x_n)$ ; STOP.

\* Another stopping criterion is

$$\frac{\|\underline{x}^{(K+1)} - \underline{x}^{(K)}\|}{\|\underline{x}^{(K+1)}\|} < \epsilon.$$

# Convergence of Iterative Methods

In general, iterative techniques for solving linear systems involving a process that converts the system  $A \underline{x} = b$  into an equivalent system of the form  $\underline{x} = T \underline{x} + c$  for some fixed matrix  $T$  and vector  $c$ . After the initial vector  $\underline{x}^{(0)}$  is selected, the sequence of approximate solution vectors is generated by computing

$$\underline{x}^{(k+1)} = T \underline{x}^{(k)} + c,$$

for each  $k = 1, 2, 3, \dots$

Notation: The matrix  $A$  can be splitted into three matrices as

$$A = L + D + U$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix} + \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$L$  is a lower triangular matrix with zeros above the main diagonal.

$D$  is a diagonal matrix.

$U$  is an upper triangular matrix with zeros below the main diagonal.

## Jacobi Method:

$$\begin{aligned} & \underline{A = D + L + U} \\ \Rightarrow & \underline{(D + L + U) \underline{x} = \underline{b}} \\ \Rightarrow & \underline{D \underline{x} = - (L + U) \underline{x} + \underline{b}} \\ \Rightarrow & \underline{\underline{x} = - D^{-1} (L + U) \underline{x} + D^{-1} \underline{b}} \end{aligned}$$

Jacobi :  $\underline{x}^{(k+1)} = - D^{-1} (L + U) \underline{x}^{(k)} + D^{-1} \underline{b}$

$$\underline{T_{Jacobi}} = - D^{-1} (L + U), \quad \underline{C_{Jacobi}} = D^{-1} \underline{b}$$

## Gauss-Seidel Method:

$$\begin{aligned} & (D + L) \underline{\underline{x}}^{(k+1)} = - U \underline{\underline{x}}^{(k)} + \underline{\underline{b}} \\ \Rightarrow & \underline{\underline{x}}^{(k+1)} = - (D + L)^{-1} U \underline{\underline{x}}^{(k)} + (D + L)^{-1} \underline{\underline{b}} \\ \underline{T_{G-S}} & = - (D + L)^{-1} U, \quad \underline{C_{G-S}} = (D + L)^{-1} \underline{\underline{b}}. \end{aligned}$$

Theorem :- For any  $\underline{x}^{(0)} \in \mathbb{R}^n$ , the sequence  $\{\underline{x}^{(k)}\}_{k=0}^{\infty}$

defined by  $\underline{\underline{x}}^{(k+1)} = \underline{T} \underline{\underline{x}}^{(k)} + \underline{C}$ , for each  $k \geq 0$ ,

Converges to the unique solution of  $\underline{x} = T \underline{x} + C$   
iff the spectral radius of  $T$  is less than unity,  
i.e.,  $R(T) < 1$ .

Example: Consider the System

$$\left[ \begin{array}{ccc|c} 9 & 1 & 1 & 10 \\ 2 & 10 & 3 & 19 \\ 3 & 4 & 11 & 0 \end{array} \right].$$

Here,

$$A = \left[ \begin{array}{ccc} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{array} \right] = \left[ \begin{array}{ccc} 9 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 11 \end{array} \right] + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{array} \right] + \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

D                    L                    U

$$\underline{T_{\text{Jacobi}}} = -D^{-1}(L+U)$$

$$= - \left[ \begin{array}{ccc} 1/9 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/11 \end{array} \right] \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 4 & 0 \end{array} \right] = - \left[ \begin{array}{ccc} 0 & 1/9 & 1/9 \\ 2/10 & 0 & 3/10 \\ 3/11 & 4/11 & 0 \end{array} \right].$$

Eigenvalues of  $T_{\text{Jacobi}}$  : -0.4472, 0.3314, 0.1159

Spectral radius of  $T_{\text{Jacobi}} = \max |\lambda| = 0.4472 < 1$ .

Hence, the method will converge to the exact solution for any initial guess.

$$\underline{T_{G-S}} = -(D+L)^{-1}U = - \left[ \begin{array}{ccc} 9 & 0 & 0 \\ 2 & 10 & 0 \\ 3 & 4 & 11 \end{array} \right]^{-1} \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

$$T_{G-S} = - \begin{bmatrix} 1/9 & 0 & 0 \\ -1/45 & 1/10 & 0 \\ -1/45 & -2/55 & 1/11 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/9 & -1/9 \\ 0 & 1/45 & -5/18 \\ 0 & 1/45 & 13/99 \end{bmatrix}$$

$$\lambda_{G-S} = \left\{ 0, \frac{76 \pm i\sqrt{3134}}{990} \right\}$$

$$|\lambda_{G-S}| = \left\{ 0, \frac{1}{\sqrt{110}}, \frac{1}{\sqrt{110}} \right\}$$

$$P(T_{G-S}) = \frac{1}{\sqrt{110}} = 0.0953 < 1.$$

Hence, the method will converge to the exact solution for any initial guess.

Theorem: If A is strictly diagonally dominant, then for any choice of  $\underline{x}^{(0)}$ , both the Jacobi and Gauss-Seidel methods give sequences  $\{\underline{x}^{(k)}\}_{k=0}^{\infty}$  that converge to the unique solution of  $A\underline{x} = \underline{b}$ .

Remarks:

\* If one of the  $a_{ii}$  entries is zero and the system is non-singular, a reordering of equations can be performed so that no  $a_{ii} = 0$ .

\* To speed-up convergence, the equations should be arranged in such a way that  $a_{ii}$  is as large as possible.

## ILL-CONDITIONING:

A computational problem is called 'ill-conditioned' (or ill-posed) if 'small' changes in the data (input) cause 'large' ~~large~~ changes in the solution (output). The problem is called 'well-conditioned' (or well-posed) if 'small' changes in the data cause only 'small' changes in the solution.

Example: The system

$$0.9999 x_1 - 1.0001 x_2 = 1 \\ x_1 - x_2 = \textcircled{1}$$

has the exact solution  $(x_1, x_2)^T = (0.5, -0.5)$ . Let us make small change in the given system

$$0.9999 x_1 - 1.0001 x_2 = 1 \\ x_1 - x_2 = 1 + \underline{\epsilon}.$$

We see that the solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \underline{0.5 + 5000.5 \epsilon} \\ \underline{-0.5 + 4999.5 \epsilon} \end{bmatrix} \quad \text{Huge change.}$$

e.g.,  $\epsilon = 10^{-4}$  will change b only by a small number but

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.00005 \\ -0.00005 \end{bmatrix}. \quad \text{Hence, the system is ill-conditioned.}$$

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Let  $\underline{\tilde{x}}$  be an approximate solution of the system  $A \underline{x} = \underline{b}$ . We denote the error vector by  $\underline{e} = \underline{x} - \underline{\tilde{x}}$ , and call  $\underline{b} - A \underline{\tilde{x}} := \underline{r}$  as the residual.

We expect that if  $\|\underline{r}\|$  is small, then  $\|\underline{e}\|$  should be small. However, this is not the case for ill-conditioned systems.

Theorem: Suppose that  $\underline{\tilde{x}}$  is an approximate solution of  $A \underline{x} = \underline{b}$ ,  $A$  is a non-singular matrix and  $\underline{r}$  is the residual vector for  $\underline{\tilde{x}}$ . Then, for any natural norm, absolute error

$$\|\underline{e}\| = \|\underline{x} - \underline{\tilde{x}}\| \leq \|A^{-1}\| \|\underline{r}\| \quad (1)$$

and if  $\underline{x} \neq 0$  and  $\underline{b} \neq 0$ ,

relative error: 
$$\frac{\|\underline{x} - \underline{\tilde{x}}\|}{\|\underline{x}\|} \leq \|A\| \cdot \|A^{-1}\| \frac{\|\underline{r}\|}{\|\underline{b}\|} \quad (2)$$

Condition number

Remark: Note that for  $\underline{x} \neq 0$ , and  $\underline{b} \neq 0$ ,

$$\|A \underline{x}\| \leq \|A\| \cdot \|\underline{x}\| \Rightarrow \frac{1}{\|A \underline{x}\|} \geq \frac{1}{\|A\| \|\underline{x}\|}. \quad (a)$$

$$\frac{\|\underline{x} - \underline{\tilde{x}}\|}{\|\underline{x}\|} \leq \frac{\|A^{-1}\| \|\underline{r}\|}{\|\underline{x}\|} = \frac{\|A\| \cdot \|A^{-1}\| \|\underline{r}\|}{\|A\| \cdot \|\underline{x}\|} \leq \frac{\|A\| \cdot \|A^{-1}\| \cdot \|\underline{x}\|}{\|A \underline{x}\|} = \frac{\|A\| \cdot \|A^{-1}\| \|\underline{r}\|}{\|\underline{b}\|} \quad (\text{using } (a))$$

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Definition: The condition number  $K(A)$  of a non-singular matrix  $A$  relative to a norm  $\|\cdot\|$  is

$$K(A) = \|A\| \|A^{-1}\|.$$

In this notation, inequalities ① & ② can be written as

$$\|\underline{x} - \tilde{x}\| \leq K(A) \frac{\|\underline{x}\|}{\|A\|},$$

and

$$\frac{\|\underline{x} - \tilde{x}\|}{\|\underline{x}\|} \leq K(A) \frac{\|\underline{x}\|}{\|b\|}.$$

For any non-singular matrix  $A$ , and natural norm  $\|\cdot\|$ , we have

$$\|AA^{-1}\| \leq \|A\| \cdot \|A^{-1}\| = K(A)$$

$$\Rightarrow K(A) \geq \|AA^{-1}\| = \|I_n\| = 1$$

$$\Rightarrow K(A) \geq 1$$

A matrix  $A$  is well-conditioned if  $K(A)$  is close to 1, and is ill-conditioned when  $K(A)$  is significantly greater than 1.

Example: From our previous example,

$$A = \begin{bmatrix} 0.9999 & -1.0001 \\ 1 & -1 \end{bmatrix}$$

$$\|A\|_{\infty} = \max \{|0.9999| + |-1.0001|, |1| + |-1|\} = 2.$$

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$$A^{-1} = \begin{bmatrix} -5000.0 & 5000.5 \\ -5000.0 & 4999.5 \end{bmatrix}$$

$$\underline{\|A^{-1}\|_{\infty}} = \max \{ |-5000.0| + |5000.5|, |-5000| + |4999.5| \} \\ = 10000.5$$

$$\text{Condition number } K(A) = \underline{\|A\|_{\infty} \cdot \|A^{-1}\|_{\infty}}$$

$$= 2 \times 10000.5$$

$$= \underline{20001} \text{ very large.}$$

Remark:

- (1) If  $K(A)$  is large in one norm, it will be large in any other norm.
- (2) When  $K(A)$  is very large, the solution of  $Ax = b$  will be very sensitive to relatively small changes in  $b$  or in  $A$ .
- (3) In this case, we will use Residual correction method, and Iterative refinement methods (not in Syllabus).