Solution of a System of linear equations

A linear System of n equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \quad (1)$$

Where the Coefficients a_{ij}'s and the b_i's are given numbers. The system is called homogeneous if all the b_i's are zero, otherwise it is non-homogeneous. This System Can also be written in matrix form as

$$A\underline{x} = \underline{b} \qquad \text{———} \qquad (2)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A solution of the System (1) is a set of numbers $x_1, x_2, \ldots, x_n$ that Satisfy all n equations, and a Solution vector is a vector $\underline{x}$ whose elements/components Constitute a Solution of the System (1).

For the Solution of the System of linear equations (2), we recall an important theorem ~~of~~ from Linear algebra.

Theorem (1): If A is real matrix of order $n \times n$, then the following statements are equivalent.

(a) $A\underline{x} = 0$ has only trivial Solution.

(b) For each $\underline{b}$, $A\underline{x} = \underline{b}$ has a Solution.

(c) A is invertible.

(d) $\det(A) \neq 0$.

Remark (1): If any of the four equivalent conditions are satisfied in the above theorem, then one can find the solution of the System of linear equations by multiplying the inverse of the matrix A on left of both side of the equations $A\underline{x} = \underline{b}$ to get

$$\underline{x} = A^{-1}\underline{b}.$$

To find A^{-1}, we know ~~that~~ the standard method of finding the adjoint of the matrix A and $A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$.

The method of Solving Such a System by determinants is not practical, even with efficient methods for evaluating the determinant.

Practical Method: Gauss-elimination

The Gauss elimination method reduces the System to triangular form. Recall that a Square matrix is Said to be triangular if the elements above (or below) of the main diagonal are zero. For example, the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

are triangular matrices, where A is called an upper triangular matrix and B is a lower triangular matrix.

- It is clear that in A (upper triangular) $a_{ij} = 0$ for $i > j$, and $b_{ij} = 0$ for $j > i$ in B (lower triangular).
- A triangular matrix is non-Singular only when all its diagonal elements are non-zero.
- If A_1 and A_2 are two upper triangular matrices of the Same order, then $A_1 + A_2$ and $A_1 A_2$ are also upper triangular matrices of the Same order. Similar results hold good for lower triangular matrices also.
- The inverse of a non-singular lower tringular matrix is also a lower triangular. Similar result holds good for an upper triangular matrix also.

Gauss Elimination:

Step-1: Reduces the System to "triangular form"

Step-2: Solve by back Substitution.

Ex: Suppose we want to Solve

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (1)}$$
$$2x_1 + 2x_2 + 3x_3 = 3 \quad \text{--- (2)}$$
$$-x_1 - 3x_2 = 2 \quad \text{--- (3)}$$

Eliminate x_1 from equations (2) and (3),

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (4)}$$
$$-2x_2 + x_3 = 3 \quad \text{--- (5)}$$
$$-x_2 + x_3 = 2 \quad \text{--- (6)}$$

Eliminate x_2 from equation (6)

$$x_1 + 2x_2 + x_3 = 0 \quad \text{--- (7)}$$
$$-2x_2 + x_3 = 3 \quad \text{--- (8)}$$
$$\frac{1}{2}x_3 = \frac{1}{2} \quad \text{--- (9)}$$

upper triangular form

Elimination Steps

Back Substitution:

$$x_3 = 1$$
$$-2x_2 + x_3 = 3 \Rightarrow x_2 = -1$$
$$x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = 1$$

The elimination steps are more conveniently carried out using the matrix notation

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right] \quad \text{Augmented Matrix}$$

Elementary Row operations:

(1) Interchange of two rows.

(2) Addition of a constant multiple of one row to another row.

(3) Multiplication of a row by a non-zero constant.

We turn 'A' into upper triangular form by elementary row operations:

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - (-1)R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$R_3 \to R_3 - \tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1/2 & 1/2 \end{array}\right]$$

Now apply backward substitution. We get the solution of the given system of linear equations.

Formal Structure of Gauss Elimination:

3×3 Case

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix Form

$$[A \mid \underline{b}] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Goal: To reduce A to upper triangular form.

Step-1: To make entries a_{21} and a_{31} zeros.

Define $m_{21} = \dfrac{a_{21}}{a_{11}}$, $m_{31} = \dfrac{a_{31}}{a_{11}}$

$$R_2 \to R_2 - m_{21}R_1 \,, \quad R_3 \to R_3 - m_{31}R_1$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

Where $a_{ij}^{(2)} = a_{ij} - m_{i1}a_{1j}$, $i, j = 2, 3$,

and $b_i^{(2)} = b_i - m_{i1}b_1$, $i = 2, 3$.

Step-2: To make entry $a_{32}^{(2)}$ zero.

Define $m_{32} = \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}}$

$$R_3 \to R_3 - m_{32} R_2$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{array}\right]$$

where $a_{33}^{(3)} = a_{33}^{(2)} - m_{32} a_{23}^{(2)}$,

and $b_3^{(3)} = b_3^{(2)} - m_{32} b_2^{(2)}$.

Step-3: Back Substitution

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}.$$

$$x_2 = \frac{b_2^{(2)} - a_{23}^{(2)} x_3}{a_{22}^{(2)}}.$$

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}.$$

Generalization to a general non-Singular System of n linear equations.

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & & & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right]$$

For $K = 1, 2, 3, \ldots, (n-1)$, carry out the following elimination steps.

Step-K: To eliminate coefficient of x_K from row $(K+1)$ through n. The results of preceding steps $1, 2, \ldots, (K-1)$ will have yielded.

$$\left[\begin{array}{cccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1K}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2K}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & \ddots & & & & \vdots \\ 0 & 0 & & a_{KK}^{(K)} & \cdots & a_{Kn}^{(K)} & b_K^{(K)} \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & & a_{nK}^{(K)} & \cdots & a_{nn}^{(K)} & b_n^{(K)} \end{array}\right]$$

Assume $a_{KK}^{(K)} \neq 0$, and define multiplier

$$m_{iK} = \frac{a_{iK}^{(K)}}{a_{KK}^{(K)}} \quad \text{for } i = (K+1), (K+2), \ldots, n$$

$$R_i \longrightarrow R_i - m_{iK} R_K \quad \text{for } i = (K+1), (K+2), \ldots, n$$

New coefficients and right-hand sides are

$$a_{ij}^{(K+1)} = a_{ij}^{(K)} - m_{iK}\, a_{Kj}^{(K)}, \quad i, j = (K+1), (K+2), \ldots, n$$

and

$$b_i^{(K+1)} = b_i^{(K)} - m_{iK}\, b_i^{(K)}, \quad i = (K+1), (K+2), \ldots, n.$$

When Step-$(n-1)$ is completed, the linear system will be in upper triangular form.

$$\left[\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1n} & g_1 \\ 0 & u_{22} & \cdots & u_{2n} & g_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & u_{nn} & g_n \end{array}\right]$$

Back substitution:-

$$x_n = \frac{g_n}{u_{nn}}$$

$$x_i = \frac{g_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}}, \quad i = (n-1), (n-2), \cdots, 1.$$

Operations Count:

Important factors to judge the quality of a numerical method are

(i) amount of Storage

(ii) amount of time ($\equiv$ no. of operations)

(iii) Effect of round-off error.

For Gauss elimination, the operations count for a full matrix (a matrix with relatively many non-zero entries) is as follows.

In step-K, we eliminate x_K from $(n-K)$ equations. This needs $(n-K)$ divisons in computing the m_{jK}, and $(n-K)(n-K+1)$ multiplications and $(n-K)(n-K+1)$ Substractions.

for b_j's

Since we do $(n-1)$ steps, K goes from 1 to $(n-1)$. Hence, total number of operations in this forward elimination is

$$f(n) = \sum_{K=1}^{n-1} (n-K) + 2\sum_{K=1}^{n-1} (n-K)(n-K+1)$$

$$= n(n-1) - \frac{n(n-1)}{2} + 2\sum_{K=1}^{n-1}\left[n(n+1) - (2n+1)K + K^2\right]$$

$$= n(n-1) - \frac{n(n-1)}{2} + 2\left[n(n+1)(n-1) - (2n+1).\frac{n(n-1)}{2} + \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \frac{n(n-1)}{2} + 2n(n-1)\left[(n+1) - \frac{1}{2}(2n+1) + \frac{2n-1}{6}\right]$$

$$= \frac{n(n-1)}{2} + 2n(n-1).\frac{(n+1)}{3}$$

$$= \frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6}$$

$$\simeq \frac{2n^3}{3} \text{ (dropping lower powers of } n)$$

We say that $f(n) = O(n^3)$, is order of n^3.

In the back substitution of x_i, we make $(n-i)$ multiplications and $(n-i)$ substractions and 1 division. Hence, the number of operations in the back substitution is

$$b(n) = 2\sum_{i=1}^{n-1}(n-i) + n \leftarrow \text{divisons}$$

$$= 2\left[n(n-1) - \frac{n(n-1)}{2}\right] + n$$

$$= n^2.$$

We see that no. of operations in the back substitution goes slower than that in the forward elimination of Gauss algorithm, so that it is neglible for large systems because it is smaller by a factor n, approximately, e.g, if an operation takes 10^{-9} sec, then the times needed are

Algorithm	$n=1000$	$n=10000$
Elimination	0.7 sec	11 min
Back Substitution	0.001 sec	0.1 sec

Gauss Elimination: Partial Pivoting

Recall

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

For Gauss elimination, we assumed that a_{KK} (in step-K) are different from zero. What if we obtain $a_{KK}=0$ at some step?

→ At a given step, one equation remains unaltered. We refer to this equation as the ~~pi~~ pivot equation.

→ A pivot in the corresponding row of the matrix is the element, which is used to make all the elements below it zero.

→ A pivot must be different from zero.

(obvious: because we need to divide by that in the elimination step).

→ It should be large in absolute value to avoid magnification of round-off error.

For this, ~~as~~ we choose as our pivot equation one that has the absolutely largest a_{jK} in column K on or below the main diagonal. (actually the uppermost if there are several such equations). This is called <u>Partial Pivoting</u> (achived by exchange of rows).

→ There is also <u>total pivoting</u> which involves both row and column exchanges but is hardly used in Practice.

<u>Example:</u>

$$8x_2 + 2x_3 = -7 \quad \text{— (1)}$$
$$3x_1 + 5x_2 + 2x_3 = 8 \quad \text{— (2)}$$
$$6x_1 + 2x_2 + 8x_3 = 26 \quad \text{— (3)}$$

Here $|6| > |3|$, $(1) \longleftrightarrow (3)$

$$6x_1 + 2x_2 + 8x_3 = 26 \rightarrow \text{Pivot equation}$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$8x_2 + 2x_3 = -7$$

Convert into matrix form

$$\left[\begin{array}{ccc|c} \boxed{6} & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Step-1: Elimination of x_1

$$R_2 \rightarrow R_2 - \frac{3}{6} R_1$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Step-2: Elimination of x_2

The largest element in Column (2) is 8. Therefore,

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & \boxed{8} & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right] \longrightarrow \text{Pivot row}$$

$R_3 \rightarrow R_3 - \frac{4}{8} R_2$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right]$$

Back Substitution:

$$-3x_3 = -\frac{3}{2} \Rightarrow x_3 = \frac{1}{2}.$$

$$8x_2 + 2x_3 = -7 \Rightarrow x_2 = -1$$

$$6x_1 + 2x_2 + 8x_3 = 26 \Rightarrow x_1 = 4.$$

Things to Remember:-

→ If $a_{KK} = 0$ in step-K, we must pivot.

→ If $|a_{KK}|$ is small, we should pivot to avoid magnification of round-off errors that may seriously affect accuracy or even produce non-sensical results.

Difficulty with Small pivots :-

Example: $0.0004\, x_1 + 1.402\, x_2 = 1.406$ — (1)

$0.4003\, x_1 - 1.502\, x_2 = 2.501$ — (2)

The exact solution of this system of equations is

$$x_1 = 10, \quad x_2 = 1.$$

Solve by Gauss elimination method using four-digit floating-point arithmatic.

Let first equation be the pivot equation. we need to multiply the second equation with

$$m_{21} = \frac{0.4003}{0.0004} = 1001$$

and substract the result from the second equation

$$(-1.502 - 1.402 \times 1001)\, x_2 = 2.501 - 1.406 \times 1001$$

$$\Rightarrow (-1.502 - 1403.\cancel{4})\, x_2 = 2.501 - 1407.\cancel{4}\cancel{1} \quad (\text{four digit})$$

$$\Rightarrow -1405\, x_2 = -1404$$

$$\Rightarrow x_2 = \frac{-1404}{-1405} = 0.9993$$

From the first equation

$$x_1 = \frac{1.406 - 1.402 \times 0.9993}{0.0004}$$

$$\Rightarrow x_1 = \frac{1.406 - 1.401}{0.0004}$$

$$\Rightarrow x_1 = \frac{0.005}{0.0004} = 12.5$$

This failure occured because $|a_{11}|$ is very small as compared to $|a_{12}|$, so that a small error in x_2 led to a large error in x_1.

Same Example with pivoting:

Exchange equations (1) and (2).

$$0.4003\, x_1 - 1.502\, x_2 = 2.501$$
$$0.0004\, x_1 + 1.402\, x_2 = 1.406$$

$$m_{21} = \frac{0.0004}{0.4003} = 0.000\underbrace{9993}_{\text{4 significant digits}}$$

$R_2 \to R_2 - m_{21} R_1$

$$(1.402 + 0.0009993 \times 1.502)\, x_2 = 1.406 - 0.0009993 \times 2.501$$

$$\Rightarrow \quad (1.402 + 0.001501)\, x_2 = 1.406 - 0.002499$$

$$\Rightarrow \quad 1.404\, x_2 = 1.404$$

$$\Rightarrow \quad x_2 = 1.$$

Now, from pivot equation, $0.4003\, x_1 = 2.501 + 1.502$

$$\Rightarrow \quad x_1 = \frac{4.003}{0.4003} = 10$$

Here, $|a_{11}|$ is not very small in comparison with $|a_{12}|$, so that a small round-off error in x_2 would not led to a large error in x_1.

For instance, if we had $x_2 = 1.002$, we would still have from the pivot equation the good value of x_1.

$$x_1 = \frac{2.501 + 1.502 * 1.002}{0.4003}$$

$$= \frac{2.501 + 1.505}{0.4003}$$

$$= \frac{4.006}{0.4003}$$

$$= 10.0075.$$

Even, if we had older value for $x_2 = 0.9993$, the value of x_1 is

$$x_1 = \frac{2.501 + 1.502 \times 0.9993}{0.4003}$$

$$= \frac{2.501 + 1.501}{0.4003}$$

$$= \frac{4.002}{0.4003}$$

$$= 9.998.$$

Application of Gauss Elimination (To find inverse of A)

Let A be a 3x3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be the inverse of A.

By definition, $AX = I$

This implies that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can find the values of x_{ij} $(i=1,2,3,\ j=1,2,3)$ by solving these three systems of equations via Gauss elimination method.

Example: Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

Sol^n: Augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - (-2)R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \quad \text{——— (1)}$$

Continue to apply elementary row operations until A in augmented matrix ~~because~~ becomes identity matrix.

$R_3 \to \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7/2 & -3/2 & 1/2 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 3/2 & -1/2 & 1/2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Alternatively, we can use the back substitution method from System (1).

$$x_{11} + x_{21} - x_{31} = 1$$

$$x_{21} - x_{31} = -1$$

$$2x_{31} = 5$$

This implies that $x_{11} = 2$, $x_{21} = \frac{3}{2}$, $x_{31} = \frac{5}{2}$.

Similarly, we can obtain

$$x_{12} = -1, \quad x_{22} = -\frac{1}{2}, \quad x_{32} = -\frac{3}{2}$$

$$x_{13} = 0, \quad x_{23} = \frac{1}{2}, \quad x_{33} = \frac{1}{2}.$$

Note that Gauss elimination can take care of all three possible cases that a system has infinitely many solutions, a unique solution or no solutions. (Examples with unique solution, we have already seen).