→ System of linear equation

Guass Elimination
Gauss Seidel method
Partial pivoting
row echelon form
LU factorization

Cholesky's method.
ill-conditioning System
Matrix norms.
Eigen-value problems
Power method.
Q.R. method
Gershgorian's theorem.

Book: (1) Numerical Analysis by Richard L. Burden and J. D. Faires.

(2) Introductory Methods for Numerical Analysis by S.S Sastry.

Solution of System of linear equations:

A linear system of n equations in n unknowns $x_1, x_2, \ldots x_n$ is a set of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

a_{ij} and b_i's are given coefficients and $x_1, x_2, \ldots x_n$ are unknown.

$$\underset{n\times n}{A}\ \underset{n\times 1}{\underline{x}} = \underset{n\times 1}{\underline{b}} \quad \text{——— (1)}$$

Where
$$A = \underset{n\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}, \quad \underline{x} = \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \quad \underline{b} = \underset{n\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}$$

This system is called homogeneous system if $\underline{b} = \underline{0}$, i.e. $b_j = 0 \ \forall j = 1, 2, \ldots n$; otherwise it is a non-homogeneous system.

A solution of system (1) is a set of numbers $x_1, x_2, \ldots x_n$ that satisfy all n equations.

Let $\underline{x} = (x_1, x_2, \ldots x_n)$ s.t $\underset{n\times n}{A}\ \underset{n\times 1}{\underline{x}} = \underline{b}$

then $\underline{x} \rightarrow$ solution vector.

Theorem: If A is real matrix of order $n\times n$, then the following statements are equivalent.

(i) $A\underline{x} = 0$ has only trivial solution.

(ii) For each $\underline{b}$, $A\underline{x} = \underline{b}$ has a solution.

(iii) A is invertible ✓

(iv) $\det(A) \neq 0$ ✓

If any of the four above conditions are satisfied, then one can find the solution of the system $A\underline{x} = \underline{b}$.

$$A^{-1} \underline{A} \underline{x} = A^{-1} \underline{b} \qquad A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

$$\Rightarrow \boxed{\underline{x} = A^{-1} \underline{b}}$$

Practical method: Gauss elimination. $\underline{A} \underline{x} = b$

$$A \longrightarrow T \rightarrow \text{triangular form.}$$

Recall that a square matrix is said to be triangular if the elements above (or below) of the main diagonal are zero. For example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3} \rightarrow \text{upper triangular matrix.}$$

$a_{ij} = 0$ for $i > j$

zero

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3} \rightarrow \text{Lower triangular matrix}$$

$b_{ij} = 0$ for $i < j$

$\Rightarrow$ A triangular matrix is non-singular only when all its diagonal elements are non-zero.

$\Rightarrow$ If A_1 and A_2 are two upper triangular matrices of same order, then $A_1 + A_2$ and $A_1 A_2$ are also upper triangular. Similar result is true for lower triangular.

→ The inverse of a non-singular upper triangular matrix is also an upper triangular.

Gauss Elimination:

Step-1: Reduces the System to 'triangular form'

Step-2: Solve by back substitution method.

Ex Suppose we want to solve

Pivot equation

$$x_1 + 2x_2 + x_3 = 0 \quad \text{---} \ (1)$$
$$2x_1 + 2x_2 + 3x_3 = 3. \quad \text{---} \ (2)$$
$$-x_1 - 3x_2 = 2 \quad \text{---} \ (3).$$

Eliminate x_1 from eq^n (2) & (3),

$$x_1 + 2x_2 + x_3 = 0 \quad \text{---} \ (4)$$
$$-2x_2 + x_3 = 3 \quad \text{---} \ (5)$$
$$-x_2 + x_3 = 2. \quad \text{---} \ (6)$$

$R_2 \to R_2$

Eliminate x_2 from eq^n (3)

$$x_1 + 2x_2 + x_3 = 0 \quad \text{---} \ (7)$$
$$-2x_2 + x_3 = 3 \quad \text{---} \ (8)$$
$$\frac{1}{2} x_3 = \frac{1}{2} \quad \text{---} \ (9)$$

Elimination steps.

→ upper triangular form.

Back substitution.

$$\frac{1}{2} x_3 = \frac{1}{2} \Rightarrow x_3 = 1 \checkmark$$

$$-2x_2 + x_3 = 3 \Rightarrow x_2 = -1 \checkmark$$

$$x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = 1 \checkmark$$

$$A\underline{x} = \underline{b}$$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$$[A|b] = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right] \rightarrow \text{Augmented matrix.}$$

Elementary Row operations:

① Interchange ~~of~~ of two rows.

② Addition of a constant multiple of one row to another row.

③ Multiplication of a row by a non-zero constant.

$$R_2 \to \underline{R_2 - 2R_1}, \quad R_3 \to R_3 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$R_3 \to R_3 - \frac{1}{2}R_2.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & \underline{-2} & \underline{1} & 3 \\ 0 & 0 & \underline{1/2} & \underline{1/2} \end{array}\right] \rightarrow \text{upper triangular form.}$$

Back substitution: $\frac{1}{2}x_3 = \frac{1}{2} \Rightarrow x_3 = 1$

$$-2x_2 + x_3 = 3 \Rightarrow x_2 = -1$$

$$x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = 1$$

Formal Structure of Gauss Elimination

$$A\underline{x} = \underline{b}$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \rightarrow \text{Augmented matrix.}$$

Goal: To reduce A to upper triangular form.

Step-1: To make entries a_{21} and a_{31} zeros.

Define $m_{21} = \dfrac{a_{21}}{a_{11}}$, $m_{31} = \dfrac{a_{31}}{a_{11}}$

$\rightarrow$ $R_2 \rightarrow R_2 - m_{21}R_1$, $R_3 \rightarrow R_3 - m_{31}R_1$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

where $a_{ij}^{(2)} = a_{ij} - m_{i1}a_{ij}$, $i = 2, 3$

$$b_i^{(2)} = b_i - m_{i1}b_1, \quad i = 2, 3.$$

Step-2: To make entry $a_{32}^{(2)}$ zero.

Define $m_{32} = \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}}$

$R_3 \to R_3 - m_{32} R_2$.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & \underline{a_{33}^{(3)}} & b_3^{(3)} \end{array}\right] \to \text{triangular form}$$

where $a_{33}^{(3)} = a_{33}^{(2)} - m_{32}\, a_{23}^{(2)}$.

$$b_3^{(3)} = b_3^{(2)} - m_{32}\, b_2^{(2)}.$$

Step-3: Back substitution.

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}} \checkmark$$

$$x_2 = \frac{b_2^{(2)} - a_{23}^{(2)} x_3}{a_{22}^{(2)}}$$

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

Generalization to a general non-singular system of n linear equations.

$\underline{\underline{3\times 3}}$

$$\left[\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & & & & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right]$$