Eigenvalue Problem

When multiplied with $A\underline{x}$, almost all vectors change directions. Certain exceptional vectors $\underline{x}$ are in the same direction as $A\underline{x}$

eigen vectors.

$$A\underline{x} = \lambda \underline{x}$$

$\lambda \to$ eigenvalue.

λ may be zero. If $\lambda = 0$, $A\underline{x} = 0.\underline{x} = \underline{0}$, then eigen vec $\underline{x}$ lies in the null space of A. ~~A is~~ In this ~~case~~ case, A is singular. The eigen vector $\underline{x}$ never be zero.

$$\det(A - \lambda I) = 0$$

Caley Hamilton Theorem: Every square matrix satisfies its characteristic polynomial.

Let $\underline{x}$ be an eigenvector corresponding to eigen value λ. Then

$$A\underline{x} = \lambda \underline{x}$$

$$\Rightarrow A^2\underline{x} = A.A\underline{x} = A.\lambda\underline{x} = \lambda A\underline{x} = \lambda^2\underline{x}$$

$\Rightarrow \lambda^2$ is an eigen value of A^2 with same eigen vector $\underline{x}$

Similarly,

If $\underline{x}$ is ~~even~~ eigen vector, then $\alpha\underline{x}$ is also eigen vector.

~~Prote~~ Properties:

① Elimination (row exchange, row ~~do~~ addition) does not preserve eigenvalues (Bad news)

Ex $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 1 = 0$

$\Leftrightarrow \lambda^2 - 2\lambda + 1 - 1 = 0$

$\Leftrightarrow \lambda^2 - 2\lambda = 0$

$\lambda : 0, 2$

$R_2 \to R_2 - R_1$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow |B-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\lambda = 0$

$\Rightarrow \lambda = 0, 1$

② The product of n eigenvalues equals to the determinant of matrix

$$\det(A) = \prod_{i=1}^{n} \lambda_i .$$

③ The sum of n eigenvalues equals to the sum of n diagonal elements of A.

$$\mathrm{trac}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i .$$

④ A matrix may have repeated eigenvalues but it may or may not have the full set of linearly independent eigenvectors.

$$A_{n\times n} \longrightarrow \underbrace{\lambda_1 \ \lambda_2}\ \cdots\ \lambda_n$$

$$\lambda_1 = \lambda_2$$

$$\underline{x}_1, \underline{x}_2 \text{———} \underline{x}_n$$

<u>Ex</u>

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A-\lambda I)$$

$$= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$= (-1-\lambda)\left[(1-\lambda)^2 - 4\right]$$

$$= -(1+\lambda)(1+\lambda^2-2\lambda-4)$$

$$= -(1+\lambda)(\lambda^2-2\lambda-3)$$

$$(\lambda^2-3\lambda+\lambda-3)$$

$$\lambda(\lambda-3)+1(\lambda-3)$$

$$= -(1+\lambda)(\lambda+1)(\lambda-3) = 0.$$

$$\Rightarrow \boxed{\lambda = -1, -1, 3.}$$

For $\lambda = -1$,

$$(A-\lambda I)\underline{x} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{r} 2x_1 + 2x_2 = 0. \\ 0x_3 = 0 \end{array}\right\} \quad \underline{x} = \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{LI}$$

$\lambda = 3$

$$(A-\lambda I)\underline{x} = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{r} -2x_1 + 2x_2 = 0 \\ -4x_3 = 0 \end{array}\right\} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \checkmark$$

Diagonalization: Suppose that $n \times n$ matrix A has n linearly independent eigen vectors $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$. Put

$$S = (\underbrace{\overset{n\times 1}{\underline{x}_1}, \overset{n\times 1}{\underline{x}_2}, \ldots \overset{n\times 1}{\underline{x}_n}}_{\text{Column vector}})_{n \times n}$$

Then, $S^{-1}AS$ is the eigen matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & - & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & & - & \lambda_n \end{pmatrix}$$

Proof

$$A\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$AS = A\left[\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\right]$$

$$= \left[\lambda_1 \underline{x}_1, \lambda_2 \underline{x}_2, \ldots, \lambda_n \underline{x}_n\right]$$

$$= \left[\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n\right] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= S\Lambda$$

$$\Rightarrow \quad AS = S\Lambda$$

$$\Rightarrow \quad \boxed{S^{-1}AS = \Lambda} \quad \text{or} \quad \underline{A = S\Lambda S^{-1}}$$

$$A^2 = A.A = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$$

$$\Lambda^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \ \lambda_n^2 \end{pmatrix}$$

$$\vdots$$

$$A^K = S\Lambda^K S^{-1} = S.\begin{pmatrix} \lambda_1^K & & 0 \\ & \lambda_2^K & \\ 0 & & \ddots \ \lambda_n^K \end{pmatrix} S^{-1}$$

Ex $\quad A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad$ eigen values of A

$\lambda = -1, 0$

$\lambda = -1,$

~~$(A+I) = 0 \Rightarrow \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \Rightarrow$~~

$(A+I)\underline{x} = \underline{0} \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0$

$\Rightarrow \underline{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

$\lambda = 0,$

$(A + 0I)\underline{x} = 0 \Rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = 0.$

$\underline{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

$S = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$

$S^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

$\Lambda^{100} = \begin{pmatrix} (-1)^{100} & 0 \\ 0 & (0)^{100} \end{pmatrix}$

$A^{100} = S\Lambda^{100}S^{-1}$

$= \begin{pmatrix} -1 & 0 \\ \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \end{pmatrix}\begin{pmatrix} -1 & 0 \\ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Results A $n \times n$ matrix B is called similar to $n \times n$ matrix A if there is a non-singular matrix T such that $\boxed{B = T^{-1} A T.}$

→ Similar matrices have the same:

(i) Characteristic equation and eigen values

(ii) determinant and invertibility

(iii) trace.

(iv) rank and nullity.

Ex Two matrices may have equal eigen values but not similar.

Ex $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P_A(x) = P_B(x) = (x-1)^2.$ ⟹ A & B both have eigen values $1, 1$.

Conversely, if we assume that A and B are similar, i.e, $\exists$ a non-singular matrix S such that $A = S^{-1} B S$. Then

$$A = S^{-1} I S = S^{-1} S = I$$

But $A \neq I$

This contradict our assumption.

Dominant Eigenvalue & eigenvector

Let $\lambda_1, \lambda_2, \cdots \lambda_n$ be the eigen values of $A_{n\times n}$.

λ_1 is called the dominant eigen value of A

if $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$.

The eigen vector ~~con~~ corresponding to the dominant eigen value λ_1 is called the dominant eigen vector.

* Not every matrix has a dominant eigen value.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \lambda_1 = 1, \quad \lambda_2 = -1$$

has no dominant eigenvalue.

Power Method.

Assumptions:

1. A has a dominant eigen value ~~an~~ with a corresponding eigen vector.
2. A has n linearly independent eigen vectors.

Note that if A does not have n linearly independent eigenvectors, the power method may still be successful but it is not guaranteed.

<u>Procedure:</u> Choose an initial approximation $\underline{x}_0 \neq \underline{0}$ for a dominant eigen vector.

$$A \underline{x}_0 = C_1 \underline{x}_1 \quad (\text{say})$$

$$A \underline{x}_1 = C_2 \underline{x}_2$$

$$\vdots$$

$$A \underline{x}_K = C_{K+1} \underline{x}_{K+1}$$

$C_K \to \lambda$ dominant eⁿ

For large K, and by <u>properly scaling this sequence</u>, we see ~~that~~ a good approximation of the <u>dominant eigen vector</u> of A

<u>Ex</u>

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$\underline{x} \to \frac{\underline{x}}{\|\underline{x}\|_\infty}$$

we begin with $\underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A \underline{x}_0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underset{\underline{x}}{\begin{pmatrix} -10 \\ -4 \end{pmatrix}} = \underset{C_1}{-10} \underset{\underline{x}_1}{\begin{pmatrix} 1 \\ 0.4 \end{pmatrix}}$$

$$A \underline{x}_1 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -2.8 \\ -1 \end{pmatrix} = \underset{C_2}{-2.8} \underset{\underline{x}_2}{\begin{pmatrix} 1 \\ 0.357 \end{pmatrix}}$$

$$A \underline{x}_2 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 0.357 \end{pmatrix} = \begin{pmatrix} -2.28 \\ [illegible] \end{pmatrix} = -2.28 \begin{pmatrix} 1 \\ [illegible] \end{pmatrix}$$

$$A\underline{x}_2 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.357 \end{pmatrix} = \begin{pmatrix} -2.28 \\ -0.785 \end{pmatrix} = -2.28 \underbrace{\begin{pmatrix} 1 \\ 0.344 \end{pmatrix}}_{x_3}$$

$$A\underline{x}_3 = \underbrace{-2.13}_{c_4}\underbrace{\begin{pmatrix} 1 \\ 0.338 \end{pmatrix}}_{x_4}$$

$$A\underline{x}_4 = -2.06 \underbrace{\begin{pmatrix} 1 \\ 0.335 \end{pmatrix}}_{x_5} \checkmark$$

$$\rightarrow \begin{pmatrix} 1 \\ 0.335 \end{pmatrix} \text{ ~~appro x~~}$$

$\lambda = -2$ exact dominant eigen value

$\begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$ ✓

Theorem: If $\underline{x}$ is an eigenvector of A, then its corresponding eigenvalue is given by

$$\lambda = \frac{\underline{x}^T A \underline{x}}{\underline{x}^T \underline{x}}, \quad \text{(Rayleigh Quotient)}$$

$$\underline{x}_5 = \begin{pmatrix} 1 \\ 0.335 \end{pmatrix}$$

$$\tilde{\lambda} = \frac{\underline{x}_5^T A \underline{x}_5}{\underline{x}_5^T \underline{x}_5} = \frac{-2.25}{1.11} \equiv -2.03 \checkmark$$

$$\lambda \underline{x} \Rightarrow a\lambda \quad \frac{x}{a}$$