Example:

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$R_2 \to R_2 - 0.2R_1$, $R_3 \to R_3 - 0.4R_1$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad 3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$
$$0 = 0$$

$$\Rightarrow \quad x_2 = 1 - x_3 + 4x_4$$
$$x_1 = \frac{8 - 2(1 - x_3 + 4x_4) - 2x_3 + 5x_4}{3}$$
$$= 2 - x_4$$

Since x_3 and x_4 remain arbitrary, the System has infinitely many solutions.

Note that when $m \neq n$, the Gauss elimination does note yield the upper triangular matrix, rather it reduces the coefficient matrix as well as the augmented matrix into the so-called "Echelon Form".

Echelon Form:

Properties:

① All non-zero rows are above any rows of all zeros.

② Each leading entry (i.e., the left most non-zero entry) of a row is in a column to the right of the leading entry of the row above it.

③ All entries in a column below a leading entry are zero.

Examples: $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & -\frac{3}{2} & 1 \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ $\begin{bmatrix} 0 & 5 & -1 & 7 & 0 & 5 & 4 \\ 0 & 0 & 3 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Examples: Not in Echelon Form

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 3 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Property - 1   Property - 2   Property 2 & 3

→ Echelon form of a matrix is not unique.

→ At the end of Gauss elimination, the Coefficient matrix and the augmented matrix are always in Echelon form.

Reduced Row Echelon Form:

In addition to the above 3 properties, if matrix also satisfies the following two properties, it is said to be in reduced row echelon form.

1. The leading entry in each non-zero row is 1.
2. Each leading 1 is the only non-zero entry in its Column.

Example:

$$\begin{bmatrix} 0 & 1 & -1 & 0 & -7 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

Gauss-Jordan Elimination:-

Example: Solve

$$8x_2 + 2x_3 = -7$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$6x_1 + 2x_2 + 8x_3 = 26$$

$$\left[\begin{array}{ccc|c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

$|4| < |8|$, $\quad R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right]$$

$R_3 \rightarrow R_3 - \frac{1}{2} R_2$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right]$$

Gauss-Jordan

$R_1 \rightarrow \frac{1}{6} R_1$, $\quad R_2 \rightarrow \frac{1}{8} R_2$, $\quad R_3 \rightarrow -\frac{1}{3} R_3$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 4/3 & 13/3 \\ 0 & 1 & 1/4 & -7/8 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{4}{3} R_3$, $\quad R_2 \rightarrow R_2 - \frac{1}{4} R_3$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 0 & 11/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{1}{3} R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 4 \\ x_2 = -1 \\ x_3 = 1/2. \end{array}$

Gauss-Jordan is not recommended to solve linear system of equation, since it involves more arithmetic operations than those involved in back substitution.

LU Decomposition of Matrix:

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$
be a non-Singular Square matrix.

Then, A can be factorized into the form LU, where

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix},$$

if $a_{11} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, and so on.

It is a Standard result of linear algebra that such a factorization, when it exists, is unique.

Similarly, the factorization LU, where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}, \text{ and } U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

is also unique factorization.

→ Note that $A = LU$ factorization is not unique if the diagonal entries of L (or U) are not set to 1.

Now, we will discuss a few methods in this regard.

(1) Gaussian Elimination Method:

We assume that no interchange of rows has taken place at any stage. Recall that in Gaussian elimination method, matrix A is reduced to an upper triangular matrix U by a series of elementary row operations

For a 4×4 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

these operations may be expressed in following ways

$$L_3 L_2 L_1 A = U \quad \text{———} \quad (1)$$

where L_1, L_2, and L_3 are Lower triangular matrices as

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_{21} & 1 & 0 & 0 \\ -m_{31} & 0 & 1 & 0 \\ -m_{41} & 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 \\ 0 & -m_{42} & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_{43} & 1 \end{bmatrix}$$

and U is the final upper triangular matrix, i.e.,

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} \end{bmatrix}$$

From equation (1), we have

$$A = (L_3 L_2 L_1)^{-1} U = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{L \text{ (say)}} U \quad \text{—— (2)}$$

It is easy to see that the inverse of L_1, L_2 and L_3 can be obtained simply by changing the signs of multipliers. It can be shown by the logic of elementary transformations that

$$L = L_1^{-1} L_2^{-1} L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix}.$$

Thus, we have the desired decomposition $A = LU$.

Example:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

Gauss Elimination:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} \xrightarrow[m_{21}=1,\; m_{31}=-2]{R_2 \to R_2 - m_{21}R_1,\; R_3 \to R_3 - m_{31}R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow[m_{32}=3]{R_3 \to R_3 - m_{32}R_2}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = U \text{ (say)}$$

Here,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \text{ and } L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}.$$

Thus, we get $A = LU$.

(2) Doolittle's Method

In this method, for 3×3 matrix, A is decomposed as $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

So that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Now, one has to solve nine equations to find the all total nine unknown coefficients of lower triangular matrix L and upper triangular matrix U. But these are easy to solve more or less only substitutions are needed.

(3) Crout's Method:

Here one decomposes $A = LU$, where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Here, again need to solve nine equations to determine all the nine unknown coefficients.

Example: $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$, Obtain $A = LU$ by Doolittle's method.

$$A = LU$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_{11} = 1, & u_{12} = 1, & u_{13} = -1 \\ l_{21}u_{11} = 1, & l_{21}u_{12} + u_{22} = 2, & l_{21}u_{13} + u_{23} = -2 \\ l_{31}u_{11} = -2, & l_{31}u_{12} + l_{32}u_{22} = 1, & l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} u_{11} = 1, & u_{12} = 1, & u_{13} = -1 \\ l_{21} = 1, & u_{22} = 1, & u_{23} = -1 \\ l_{31} = -2, & l_{32} = 3, & u_{33} = 2 \end{cases}$$

we get

$$\underset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}.$$

Example: $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$, obtain $A = LU$ by Crout's method.

$A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{cases} l_{11} = 1, \; l_{21} = 1, \; l_{31} = -2 \\ l_{11}u_{12} = 1, \; l_{21}u_{12} + l_{22} = 2, \; l_{31}u_{12} + l_{32} = 1 \\ l_{11}u_{13} = -1, \; l_{21}u_{13} + l_{22}u_{23} = -2, \; l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} l_{11} = 1, \; l_{21} = 1, \; l_{31} = -2 \\ u_{12} = 1, \; l_{22} = 1, \; l_{32} = 3 \\ u_{13} = -1, \; u_{23} = -1, \; l_{33} = 2 \end{cases}$$

we get

$$\underset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Solution of System of Linear Equations using LU factorization:

To solve a system of the form $A\underline{x} = \underline{b}$ can be used to factor a matrix. The factorization is particularly useful when it has the form $A = LU$, where L is lower triangular and U is upper triangular.

Although not all matrices have this type of representation, many do that occurs frequently in the application of numerical techniques.

We know that Gaussian elimination applied to an arbitrary linear system $A\underline{x} = \underline{b}$ requires $O(\frac{n^3}{3})$ arithmetic operations to determine $\underline{x}$. However, to solve a linear system that involves an upper-triangular system requires only backward substitution, which takes $O(n^2)$ operations. The number of operations required to solve a lower-triangular system is similar.

→ Suppose that A has been factorized into the triangular form $A = LU$, where L - lower tringular, U - uppertriangular.

Then, we can solve for $\underline{x}$ more easily by using a two-step process:

Step-1: First we let $\underline{y} = U\underline{x}$ and solve the lower triangular system $L\underline{y} = \underline{b}$ for $\underline{y}$. Since L is triangular, determining $\underline{y}$ from this equation requires only $O(n^2)$ operations.

Step-2: Once $\underline{y}$ is known, the upper triangular system $U\underline{x} = \underline{y}$ requires only an additional $O(n^2)$ operations to determine the solution $\underline{x}$.

→ Solving a linear system $A\underline{x} = \underline{b}$ in factored form means that the no. of operations needed to solve the system $A\underline{x} = \underline{b}$ is ~~required~~ reduced from $O(n^3/3)$ to $O(2n^2)$.

Example:
→ Approximate no. of operations required to determine the solutions to a linear system using a technique requiring $O(n^3/3)$ operations and one requiring $O(n^2)$.

n	$n^3/3$	$2n^2$	% Reduction
10	$3.\bar{3} \times 10^2$	2×10^2	40
100	$3.\bar{3} \times 10^5$	2×10^4	94
1000	$3.\bar{3} \times 10^8$	2×10^6	99.4

As the example illustrates, the reduction factor increases dramatically with the size of the matrix. Not surprisingly, the reductions from the factorization come at a cost; determining the specific matrices L and U requires $O(n^3/3)$ operations. But once the factorization is determined, systems involving the matrix A can be solved in this simplified manner for any number of vectors $\underline{b}$.

Example: Solve the equations

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by LU decomposition method.

Sol^n: We have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Now, using Doolittle's method, we obtained the LU decomposition of A, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{bmatrix}.$$

Let $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then the equation $L\underline{y} = \underline{b}$ can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\Rightarrow \quad y_1 = 9, \; y_2 = 1.5, \; y_3 = 5.$

Now, we solve $U\underline{x} = \underline{y}$, i.e.,

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1.5 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x + 3y + z = 9 \\ 0.5y + 2.5z = 1.5 \\ 18z = 5 \end{cases}$$

$\Rightarrow \quad x = \frac{35}{18}, \; y = \frac{29}{18}, \; z = \frac{5}{18}.$