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Distance between vectors in $\mathbb{R}^n$.

Let $\underline{x} = (x_1, x_2, \ldots x_n)^T$ & $\underline{y} = (y_1, y_2, \ldots y_n)^T$

l_2 distance b/w $\underline{x}$ & $\underline{y}$

$$\|\underline{x} - \underline{y}\|_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

l_∞ distance b/w $\underline{x}$ & $\underline{y}$

$$\|\underline{x} - \underline{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

Ex $A\underline{x} = \underline{b}$ $\rightarrow$ exact $\underline{x} = (x_1, x_2, x_3)^T = (1, 1, 1)^T$.

a approximate solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T = (1.2001, 0.99991, 0.92538)^T$.

$$\|\underline{x} - \tilde{x}\|_2 = \left[(1-1.2001)^2 + (1-0.99991)^2 + (1-0.92538)^2 \right]^{1/2}$$

$$= 0.21356.$$

$$\|\underline{x} - \tilde{x}\|_\infty = \max \{ |1-1.2001|, |1-0.99991|, |1-0.92538| \}$$

$$= 0.2001$$

Convergence of Sequence of vectors in $\mathbb{R}^n$.

A sequence $\{\underline{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to be convergent (with limit $\underline{x}$) with resp to norm $\|\cdot\|$ if ~~$\forall \varepsilon > 0$~~ given $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$\|\underline{x}^{(k)} - \underline{x}\| < \varepsilon$ for all $k \geq N(\varepsilon)$.

Theorem: The sequence of vectors $\{\underline{x}^{(k)}\}$ converges to $\underline{x}$ in $\mathbb{R}^n$ with respect l_∞ norm iff

$$\lim_{k\to\infty} x_i^{(k)} = x_i \quad \forall\ i = 1, 2, \ldots n.$$

Ex

$$\underline{x}^{(k)} = \left(1,\ 2+\frac{1}{k},\ \frac{3}{k^2},\ e^{-k}\sin k\right)^T$$

$$\underline{x} = (1, 2, 0, 0)^T$$

Sol^n

$$\lim_{k\to\infty} x_1^{(k)} = \lim_{k\to\infty} 1 = 1 \checkmark$$

$$\lim_{k\to\infty} x_2^{(k)} = \lim_{k\to\infty} \left(2+\frac{1}{k}\right) = 2. \checkmark$$

$$\lim_{k\to\infty} x_3^{(k)} = \lim_{k\to\infty} \frac{3}{k^2} = 0. \checkmark$$

$$\lim_{k\to\infty} x_4^{(k)} = \lim_{k\to\infty} e^{-k}\sin k = 0. \checkmark$$

$\Rightarrow \quad \underline{x}^{(k)} \longrightarrow \underline{x}$ w.r.t l_∞ norm.

$l_\infty \rightarrow l_2$ ✓

Theorem: For each $x \in \mathbb{R}^n$

$$\|\underline{x}\|_\infty \leq \|\underline{x}\|_2 \leq \sqrt{n}\ \|x\|_\infty.$$

If $\|\cdot\|$, $\|\cdot\|'$ two norms in $\mathbb{R}^n$

If $\underline{x}^{(k)} \longrightarrow \underline{x}$ in with r.s.t to $\|\cdot\|$ then $\underline{x}^{(k)} \rightarrow x$

with r.s.t to $\|\cdot\|$ then $x^{(k)} \to x$ with r. to $\|\cdot\|'$.

For above example, we know that

$$\underline{x}^{(k)} \longrightarrow \underline{x} \quad \text{with resp. to } l_\infty\text{-norm.}$$

Given any $\varepsilon > 0$, $\exists$ an integer $N(\frac{\varepsilon}{2})$ such that

$$\|\underline{x}^{(k)} - \underline{x}\|_\infty < \frac{\varepsilon}{2} \quad \forall\ k \geq N(\tfrac{\varepsilon}{2}).$$

Using the above theorem, we get

$$\|\underline{x}^{(k)} - \underline{x}\|_2 \leq \sqrt{n}\ \|\underline{x}^{(k)} - \underline{x}\|_\infty$$

$$\leq \sqrt{4} \cdot \frac{\varepsilon}{2} \quad \forall\ k \geq N(\tfrac{\varepsilon}{2})$$

$$\Rightarrow \quad \|\underline{x}^{(k)} - \underline{x}\|_2 < \varepsilon \quad \forall\ k \geq N(\tfrac{\varepsilon}{2})$$

$$\underline{x}^{(k)} \longrightarrow \underline{x} \quad \text{with. to } l_2\text{-norm.}$$

Matrix Norms and Distances

Definition: A matrix norm on the set of all $n \times n$ matrices is a real-valued function $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α.

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$ iff A is the matrix with all zero entries.

(iii) $\|\alpha A\| = |\alpha| . \|A\|$,

(iv) $\|A + B\| \leq \|A\| + \|B\|$,

(iv) $\|A+B\| \le \|A\| + \|B\|$,

(v) $\|AB\| \le \|A\| . \|B\|$.

The distance between $n \times n$ matrices A & B with r. to $\|\cdot\|$ is $\underbrace{\|A-B\|}$.

Theorem: If $A = [a_{ij}]_{n\times n}$, then

(a) $\|A\|_2 = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$; ✓

(b) $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$, (along the rows)

(c) $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$, (along the columns)

Ex

$$A = \begin{array}{c c} \begin{array}{ccc} 6 & 6 & 3 \end{array} & \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix} & \begin{array}{l} \to 4 \\ \to 4 \\ \to \cancel{6}\,7 \end{array} \end{array}$$

$$\|A\|_2 = \left(1^2 + 2^2 + (-1)^2 + 0^2 + 3^2 + (-1)^2 + 5^2 + (-1)^2 + 1^2\right)^{1/2}.$$

$$= \sqrt{43}.$$

$$\|A\|_\infty = \max_{1 \le i \le 3} \sum_{j=1}^{3} |a_{ij}|$$ ✓

$$= \max\{4,4,7\} = 7.$$

$$\|A\|_1 = \max_{1\le j\le 3} \sum_{i=1}^{3} |a_{ij}|$$

$$= \max\{6,6,3\}$$

$$= 6.$$

Spectral ~~Rada~~ Radius:

If A is a square matrix., the Characteristi polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I) \longrightarrow n^{th} \text{ degree polynomial.}$$

at most n distinct zeros,

The zeros of $p(\lambda)$ are called eigenvalues.

If λ is an eigen value of A and $\underline{x} \neq 0$ satisfying $(A - \lambda I)\underline{x} = 0$, then $\underline{x}$ is called the eigen vector ~~of~~ corresponding to λ..

Definition:

The spectral radius $\rho(A)$ of a matrix A is defined as

$$\rho(A) = \max |\lambda|,$$

where λ is an eigen value of A.

For complex $\lambda = \alpha + i\beta$, we define $|\lambda| = \sqrt{\alpha^2+\beta^2}$.

If $A_{n\times n}$ matrix, then

$$\rho(A) \le \|A\| \quad \text{for any norm } \|\cdot\|.$$

Convergent Matrices:

An matrix $A_{n\times n}$ is convergent if

$$\lim_{k\to\infty} (A^k)_{ij} = 0 \quad \text{for each } i = 1, 2, \dots n, \; j = 1, 2, \dots n.$$

Ex

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$A^2 = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{pmatrix}, \quad A^3 = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{pmatrix}, \quad A^4 = \begin{pmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$

and, in general,

$$A^k = \begin{pmatrix} \left(\frac{1}{2}\right)^k & 0 \\ \frac{k}{2^{k+1}} & \left(\frac{1}{2}\right)^k \end{pmatrix} \xrightarrow[k\to\infty]{} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

$\Rightarrow$ A is convergent matrix.

Note,

$$|A - \lambda I| = \begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{1}{2}-\lambda\right)^2 = 0$$

$$\Rightarrow \lambda = \frac{1}{2}, \frac{1}{2}.$$

$$\rho(A) = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1 \checkmark$$

Result:

"Matrix A is convergent iff $\rho(A) < 1$."

Iterative Techniques

$$A\underline{x} = \underline{b} \longrightarrow$$

$A_{n\times n}$ $\quad \underset{n\times 1}{\underline{x}}$ $\quad \underset{n\times 1}{\underline{b}}$

initial approximation $\underline{x}^{(0)}$ $\longrightarrow$ generate a sequence of vectors $\{\underline{x}^{(K)}\}$ such that

$$\underline{x}^{(K)} \longrightarrow \underline{x} \quad \text{(Solution of System)}.$$

Iterative method.

① Jacobi Method ② Gauss-Seidel method.

Jacobi Method

Ex Solve the system of equations.

$$9x_1 + x_2 + x_3 = 10 \quad \text{—①}$$
$$2x_1 + 10x_2 + 3x_3 = 19 \quad \text{—②}$$
$$3x_1 + 4x_2 + 11x_3 = 0 \quad \text{—③}$$

exact solⁿ. $x_1 = 1,\ x_2 = 2$
$x_3 = -1$

Solⁿ

$$x_1 = \frac{1}{9}\left[10 - x_2 - x_3\right]$$

$$x_2 = \frac{1}{10}\left[19 - 2x_1 - 3x_3\right]$$

$$x_3 = \frac{1}{11}\left[-3x_1 - 4x_2\right]$$

Let $\underline{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ be the initial guess.

Define,

$$x_1^{(k+1)} = \frac{1}{9}\left[10 - x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{10}\left[19 - 2x_1^{(k)} - 3x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{11}\left[-3x_1^{(k)} - 4x_2^{(k)}\right]$$

where $k = 0, 1, 2, 3, \ldots$

Let us start with the initial guess $\underline{x}^{(0)} = (0,0,0)^T$.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	error $= \lVert \underline{x}^{(k+1)} - \underline{x}^{(k)} \rVert_\infty$
0	0	0	0	—
1	$\frac{10}{9} = 1.1111$	$\frac{19}{10} = 1.9000$	0	1.9000
2	0.9000	1.6778	−0.9939	0.9939
3	1.0351	2.0182	−0.8556	0.3404
4	0.9819	1.9496	−1.0162	0.1606
⋮				
10.	0.9991	1.9997	−1.0003	0.0010
11.	1.0001	2.0001	−0.9999	0.0004.

10^{-2}

10^{-2}

$$\| \underline{x}^{(11)} - \underline{x}^{(10)} \|_\infty \neq 0.0004 < 10^{-3}$$

ε.

$$\frac{\| x^{(k+1)} - x^{(k)} \|}{\| x^{(k+1)} \|} < \varepsilon.$$