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# TECHREP - Locking-Detection in Optics

**Shadi Masoumi** (shadi.masoumi@cervo.ulaval.ca)

This document is a summary of different references which I read about Locking-detection concept, its techniques and the application in optics. In the following, I will also go through different cases in terms of different forms of reference and signal, and try to make them easy to understand by driving the mathematics formulas.

*Anyone with this link can edit the project : https://www.overleaf.com/3155824314cphbtdphyxdd*

# Contents

# 1 Introduction

Lock-in amplifiers were invented in the 1930's [1] and commercialized [2] in the mid 20th century as electrical instruments. They are electronic devices that are often applied in the optical science such as low level light measurements. The purpose of a lock-in amplifier is to recover small or weak signals that would otherwise be lost in noise. This device serves to detect the amplitude of a AC signal $S$ that is superimposed by noise, fig 1. The instrumentation consists of a bandpass filter, the central frequency of which is determined by the reference signal. The signal and the reference should be modulated periodically in this device. Lock-in detection can be applied to the measurement of nonperiodic quantities by introducing an additional device into the measurement setup: CHOPPER. And The chopper wheel interrupts the light path periodically.[3]

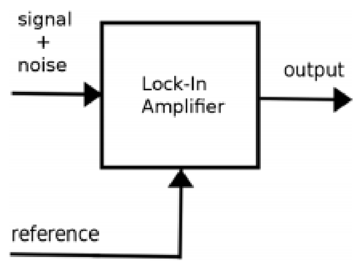


Figure 1: A basic diagram of any Lock-in Amplifier. To be able to discriminate the signal from noise, it is essential to provide a reference signal to the lock-in amplifier that carries information on the signal $S$ [3].

# 2 What does a Lock-In Amplifier Measure?

Using Fourier theorem, any input signal, including the noise accompanying it, can be represented as the sum of many sinewaves of different amplitudes, phases and frequencies. The Phase-Sensitive Detector (PSD) (main component of every Lock-in Amplifier which is a special rectifier), see section 3.3, in the lock-in amplifier multiplies all these components by a signal at the reference frequency [4]. see section 4.

A lock-in amplifier, in common with most AC indicating instruments, provides a DC output proportional to the AC signal under investigation [4]. (The rectifier rectifies just signal of interest and suppresses effect of noise and other interfering component). The noise at the input of lock in amplifier is not rectified but appears as the AC fluctuation which can be separated from the DC part using a low pass filter. The detector must be able to recognize the signal of interest through supplying it with reference voltage of the same frequency and a fixed phase relationship to that of the signal [4].

# 3 Basic Components and Definitions

The block diagram of a typical lock-in amplifier is shown in fig 4. It consists of different component which I will explain in the following.

## 3.1 Signal Channel

AS it is shown in Figure 4, input signal, including noise, is amplified by an adjustable-gain (AC-coupled amplifier) in order to match it more closely to the optimum input signal range of the PSD. The performance of the PSD is usually improved if the bandwidth of the noise voltages reaching it is reduced from that of the full frequency range of the instrument. To achieve this, the signal is passed through some form of filter, which may be simply a bandpass filter centered at the reference frequency [4].

## 3.2 Reference Channel

The proper operation of the PSD requires the generation of a precision reference signal within the instrument. In the reference channel, the internally generated reference is passed through a phase- shifter, which is used to compensate for phase differences that may have been introduced between the signal and reference inputs by the experiment, before being applied to the PSD, see fig 4 [4].

## 3.3 Phase-Sensitive Detection

### 3.3.1 General Concept

The phase-sensitive detector works by multiplying the the reference and signal and the entire process of mixing and low pass filtering is called phase sensitive detection or demodulation.

Figure 2a shows a situation in which the amplifier detects a noise free sinusoid, "Signal In" in the diagram. The instrument is also fed with a reference sinousoidal reference. The demodulated Output is yield from these two reference and signal. Since there is no relateive phase shift between the signal and reference phase, the demodulator output takes the form of a sinusoid at twice the reference frequency, but with a mean, or average, level which is positive. Figure 2b shows the same situation, except that the signal phase is now delayed by 90Âř with respect to the reference. It can been seen that although the output still contains a signal at twice the reference frequency, the mean level is now zero[4].

From this it can be seen that the mean level is:

- proportional to the product of the signal and reference frequency amplitudes
- related to the phase angle between the signal and reference.

It will be appreciated that if the reference signal amplitude is maintained at a fixed value, and the reference phase is adjusted to ensure a relative phase-shift of zero degrees, then by measuring the mean level, the input signal amplitude can be determined. The mean level is the DC component of the demodulator output, so it can be isolated using a low-pass filter. The filtered output is then measured using conventional DC voltmeter techniques.

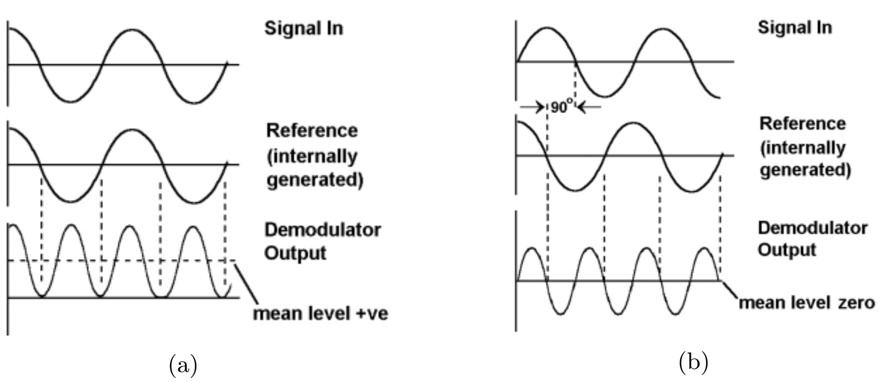


Figure 2: Multiplying the signal and the reference [4].

As said before, the lock-in measurements require a frequency reference. Typically, an experiment is excited at a fixed frequency (from an oscillator or function generator), and the lock-in detects the response from the experiment at the reference frequency. In the figure 3, the reference signal is a square wave at frequency $w_r$, if the sine output from the function generator is used to excite the experiment, the response might be the signal waveform shown in the figure. The signal is $V_{sig}\sin(w_r t + \theta_{sig})$ where $V_{sig}$ is the signal amplitude, $w_r$ is the signal frequency, and $\theta_{sig}$ is the signal phase [5].

Lock-in amplifiers generate their own internal reference signal usually by a phase-locked-loop locked to the external reference. In the figure 3, the external reference, the lock-in reference, and the signal are all shown. The internal reference is $V_L\sin(w_L t + \theta_{ref})$.

The lock-in amplifies the signal and then multiplies it by the lock-in reference using a PSD or multiplier. The output of the PSD is simply the product of two sine waves.

$$V_{psd} = V_{sig}V_L\sin(w_r t + \theta_{sig})\sin(w_L t + \theta_{ref}) \tag{1}$$

$$= \frac{1}{2}V_{sig}V_L[(\cos((w_r - w_L)t) + \theta_{sig} - \theta_{ref}) - (\cos((w_r + w_L)t) + \theta_{sig} + \theta_{ref})] \tag{2}$$

The PSD output is two AC signals, one at the difference frequency $(w_r - w_L)$ and the other at the sum frequency $(w_r + w_L)$. If the PSD output is passed through a low pass filter, the AC signals are removed. And what will be left is in the general case, nothing. However, if $w_r$ equals $w_L$, the difference frequency component will be a DC signal. In this case, the filtered PSD output will be:

$$V_{psd} = \frac{1}{2}V_{sig}V_{ref}\cos(\theta_{sig} - \theta_{ref}) \tag{3}$$

Suppose that instead of being a pure sine wave, the input is made up of signal plus noise. The PSD and low pass filter only detect signals whose frequencies are very close to the lock-in reference frequency. Noise signals, at frequencies far from the reference, are attenuated at the PSD output by the low pass filter.

Noise at frequencies very close to the reference frequency will result in very low frequency AC outputs from the PSD ($|w_{noise}w_{ref}|$ is small). Their attenuation depends upon the low pass filter bandwidth and roll off. A narrower bandwidth will remove noise sources very close to the reference frequency; a wider bandwidth allows these signals to pass. The low pass filter bandwidth determines the bandwidth of detection. Only the signal at the reference frequency will result in a true DC output and be unaffected by the low pass filter [5].

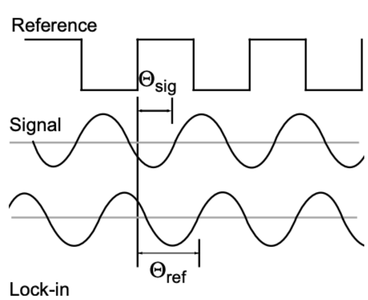


Figure 3: [5]

### 3.3.2 Magnitude and Phase

Remember that the PSD output is proportional to $V_{sig}\cos\theta$, where $\theta = (\theta_{sig} - \theta_{ref})$. $\theta$ is the phase difference between the signal and the lock-in reference oscillator. By adjusting $\theta_{ref}$ we can make $\theta$ equal to zero. In which case we can measure $V_{sig}(\cos\theta = 1)$. Conversely, if $\theta$ is 90°, there will be no output at all. A lock-in with a single PSD is called a single-phase lock-in and its output is $V_{sig}(\cos\theta)$ [5].

This phase dependency can be eliminated by adding a second PSD. If the second PSD multiplies the signal with the reference oscillator shifted by 90 degree, i.e. $V_L \sin(w_L t + \theta_{ref} + 90)$, its low pass filtered output will be:

$$V_{psd_2} = \frac{1}{2} V_{sig} V_{ref} \sin(\theta_{sig} - \theta_{ref}) \tag{4}$$

$$V_{psd_2} \approx V_{sig}\sin\theta \tag{5}$$

Now we have two outputs: one proportional to $\cos\theta$ and the other proportional to $\sin\theta$. If we call the first output X and the second Y,

$$X = V_{sig}\cos\theta \tag{6}$$

$$Y = V_{sig}\sin\theta \tag{7}$$

These two quantities represent the signal as a vector relative to the lock-in reference oscillator. X is called the "in-phase" component and Y the "quadrature" component. This is because when $\theta = 0$, X measures the signal while Y is zero. By computing the magnitude (R) of the signal vector, the phase dependency is removed.

$$R = (X^2 + Y^2)^{\frac{1}{2}} = V_{sig} \tag{8}$$

A dual-phase lock-in has two PSDs with reference oscillators 90 degree apart, and can measure X, Y and R directly. In addition,

$$\theta = \tan^{-1}(\frac{Y}{X}) \tag{9}$$

## 3.4 Low-pass Filter and Output Amplifier

As mentioned before, the purpose of the output filter is to remove the AC components from the desired DC output. There is usually also some form of output amplifier. The use of this amplifier, in conjunction with the input amplifier, allows the unit to handle a range of signal inputs. When there is little accompanying noise, the input amplifier can be operated at high gain without overloading the PSD, in which case little or even any gain is needed at the output. In the case of signals buried in very large noise voltages, the reverse is the case[4].

## 3.5 Output

The output from a lock-in amplifier was traditionally a DC voltage which was usually displayed on an analog panel meter. Nowadays, especially when the instruments are used under computer control, the output is more commonly a digital number although the analog DC voltage signal is usually provided as well[4].

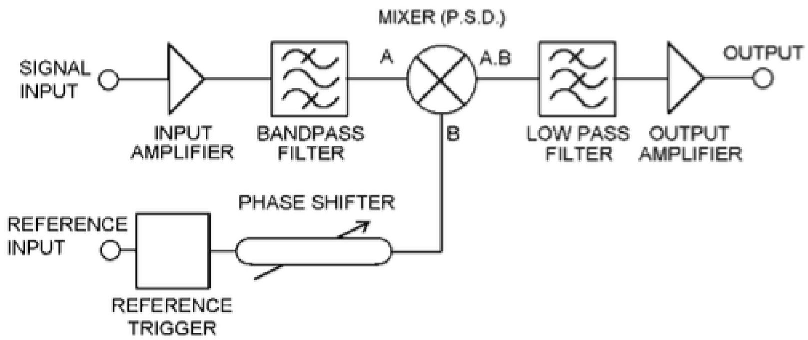


Figure 4: block diagram of a typical lock-in amplifier [4]

# 4 Multiplication and Filtering

The core of a lock-in amplifier, consists of a multiplier and a low-pass filter. The signal (plus noise) and the reference signal are multiplied, and the resulting signal is then filtered to block off contributions from higher frequencies.

Figure 5 shows the generic scheme involved in taking a measurement using a lock-in amplifier. The form of the periodic signal and of the reference can vary. The periodic signal from the oscillator forms the input (excitation) $e$ of the experiment. This gives rise to an output $s$ of the experiment that is superimposed by noise $n$.

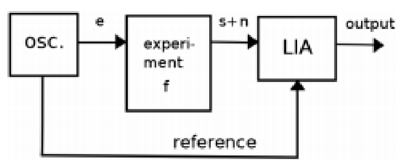


Figure 5: Measurement taken with a lock-in amplifier. The reference signal provides information on the frequency of the signal.

There are different cases in which the form of the periodic signal and of the reference can vary, I will go through mathematics proof of these cases in the following [3].

## 4.1 Sinusoidal Signal and Reference

The simplest case: both the signal and the reference are sinusoidal. The response that is fed to the lock-in amplifier is a superposition of a sinusoidal signal $s$ and a noise contribution $n$. To gain information on the signal $s$, a reference signal $r$ is derived from the oscillator that is used as a second input signal. The reference signal contains information on the frequency of the excitation signal.

The reference signal is:

$$r = A \sin \omega t \tag{10}$$

Where $\omega = 2\pi f$, and $A$ is the amplitude of the reference signal. If the signal $s$ features a phase shift with respect to the reference signal, then $s$ can be written as:

$$s = a_1 \sin(\omega t + \phi) \tag{11}$$

Where $a_1$is an indication of the reference signal. The input of the lock-in amplifier is a superposition of the signal and noise. In the multiplication-and-filter unit, the superposition of the signal and noise is combined with the reference and the low-pass filter corresponds to an averaging process. We can therefore describe the output $u$ of the lock-in amplifier by:

$$u = \frac{1}{\tau} \int_0^\tau (s + n).rdt \tag{12}$$

Since the low-pass filter corresponds to an averaging process and $\tau$ is the time constant of the low-pass filter, after multiplication, the noise is suppressed by the low-pass filter, so:

$$n = \frac{1}{\tau} \int_0^\tau n.rdt = 0 \tag{13}$$

The product of the signal and reference is:

$$s.r = a_1 A \sin(\omega t + \phi) \sin \omega t = \frac{a_1 A}{2} [\cos \phi - \cos(2\omega t + \phi)] \tag{14}$$

Therefore:

$$u = \frac{a_1 A}{2\tau} \int_0^\tau \cos \phi dt - \frac{a_1 A}{2\tau} \int_0^\tau \cos(2\omega t + \phi) dt \tag{15}$$

Note: If the integration time is sufficiently long, the positive and negative contributions in the second integral will cancel out because the integrand is a periodic function. It must be stated that there is an idealization in this mathematical derivation. The response of the low-pass filter is described as a rectangular function even though in reality it has a more complicated form.

The integrand in the first integral, on the other hand, is constant:

$$u = \frac{a_1 A}{2\tau} \int_0^\tau \cos \phi dt = \frac{a_1 A}{2} \cos \phi \tag{16}$$

In this way, an output voltage is generated that is proportional to the amplitude $a_1$ of the signal and to the cosine of the phase shift between the reference and the signal. Equation 4.1 reflects the situation in which the frequencies of the signal and the reference are the same. Now we can consider the situation in which the reference and the signal have different frequencies. For this purpose, we introduce the symbol $\Omega$ for the reference:

$$r = A \sin \Omega t \tag{17}$$

while the definition of the signal is kept unchanged:

$$s = a_1 \sin(\omega t + \phi) \tag{18}$$

As we know:

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \tag{19}$$

$$\sin(x) \sin(y) = \frac{1}{2} (\cos(x - y) - \cos(x + y)) \tag{20}$$

$$\sin(x) \cos(y) = \frac{1}{2} (\sin(x - y) + \sin(x + y)) \tag{21}$$

Then:

$$s.r = a_1 A \sin(\omega t + \phi) \sin(\Omega t) = a_1 A (\sin(\omega t). \cos(\phi). \sin(\Omega t) + \cos(\omega t). \sin(\phi). \sin(\Omega t)) \tag{22}$$

The periodic terms cancel out during this integration process. Only a nonzero DC part in the sum will contribute to the output voltage. This is the case where the argument of the cosine function is zero, i.e., where $\omega = \Omega$. Therefore, we obtain:

$$u = \frac{a_1 A}{2} \cos(\phi) \tag{23}$$

Even though we considered the more general situation this time, we obtained the same result as when the frequency of the signal and the reference were the same.

## 4.2 Square-wave signal and reference

Lock-in amplifiers can also be operated using square waves as the signal and reference.

**Mark-space square wave** A mark-space square wave can be generated by repeatedly switching a DC signal on and off, fig 6. (The term mark-space square wave is used in the literature to distinguish it from a bipolar square wave.)

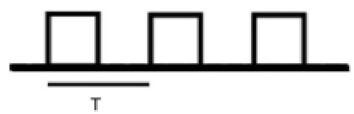


Figure 6: Mark-space square wave.

A divided definition is necessary to describe such a function:

$$r = A \begin{bmatrix} +1 & [0, \frac{T}{2}] \\ 0 & [\frac{T}{2}, T] \end{bmatrix} \tag{24}$$

This pattern is repeated periodically. The pattern of the signal is similar. In the special case where the patterns coincide completely, the signal is denoted as:

$$s = a1 \begin{bmatrix} +1 & [0, \frac{T}{2}] \\ 0 & [\frac{T}{2}, T] \end{bmatrix} \tag{25}$$

We now consider the more general case where there is a time difference $p$ between both pulses. To determine the lock-in amplifier output:

$$u = \frac{1}{T} \int_0^T srdt$$

It is convenient to distinguish between the two cases. In the first case, the time difference $p$ of the rising edge of the square wave of the reference and the signal is smaller than $\frac{T}{2}$. This corresponds to the situation depicted in Fig 7. Because we are dealing with squares, the result of the integration can be directly concluded from Fig 7:

$$u = \frac{1}{T} a_1 A(\frac{T}{2} - p) \quad for \quad p \in [0, \frac{T}{2}]$$

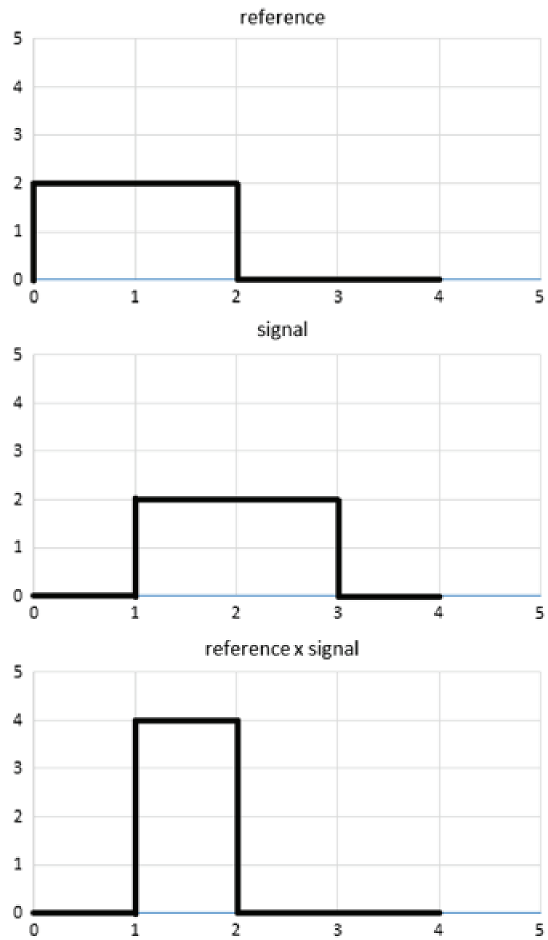


Figure 7: Multiplying mark-space square waves: first case.

In the second case, the time difference $p$ of the rising edge of the square wave of the reference and the signal is greater than T/2, as illustrated in Fig 8. So, with an analogous consideration, we obtain the following result for the other interval, where $p \in [\frac{T}{2}, T]$:

$$u = \frac{1}{T} a_1 A (p - \frac{T}{2}) \quad for \quad p \in [\frac{T}{2}, T]$$

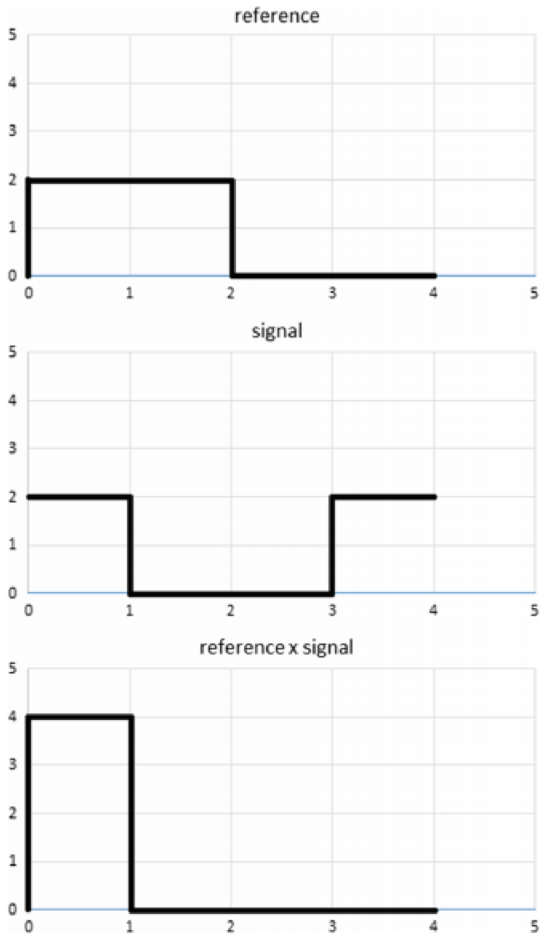


Figure 8: Multiplying mark-space square waves: second case.

The two previous equations can be combined to obtain a result for the output signal generated by multiplying and integrating two mark-space square waves:

$$u = \frac{a_1 A}{T} \begin{bmatrix} \frac{T}{2} - p & [0, \frac{T}{2}] \\ p - \frac{T}{2} & [\frac{T}{2}.T] \end{bmatrix}$$

**Bipolar square wave** A bipolar square wave is generated by repeatedly changing the polarity of a DC signal, Fig 9. Again, a divided definition of the function on the interval [0, T] is used:

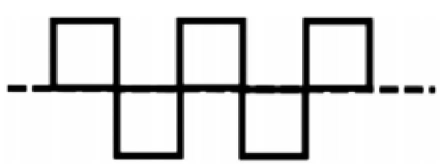

Figure 9: Bipolar square wave.

$$r = A\begin{bmatrix} +1 \ [0,\frac{T}{2}] \\ -1 \ \ [\frac{T}{2},T] \end{bmatrix}$$

This interval is the part of the square wave that is repeated periodically. A bipolar square wave might be considered as a mark-space square wave with twice the amplitude plus a negative DC offset. As before, we consider two cases and then according to Fig 10, simply collect the contributions of the different rectangles; it is important to the take the correct sign into account:

$$u = \frac{1}{T}[(-a_1A)p + a_1A(\frac{T}{2} - P) + (-a_1A)p + a_1A(\frac{T}{2} - P)] \quad p \in [0,\frac{T}{2}]$$

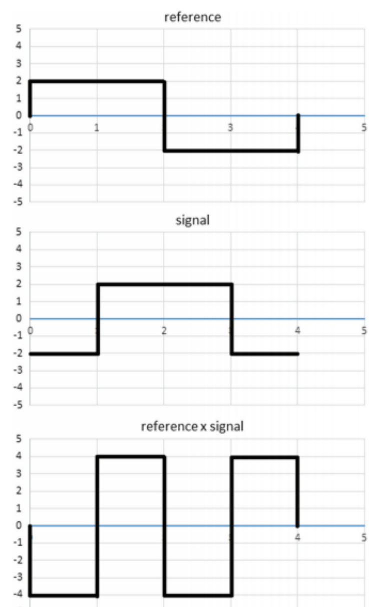


Figure 10: Multiplying bipolar square waves: first case.

Finally,

$$u = \frac{1}{T}a_1A(T - 4P) \quad p \in [0,\frac{T}{2}]$$

And for the second case where the time difference p of the rising edge of the square wave of the reference and of the signal is greater than T/2. Fig 11 illustrates this situation.

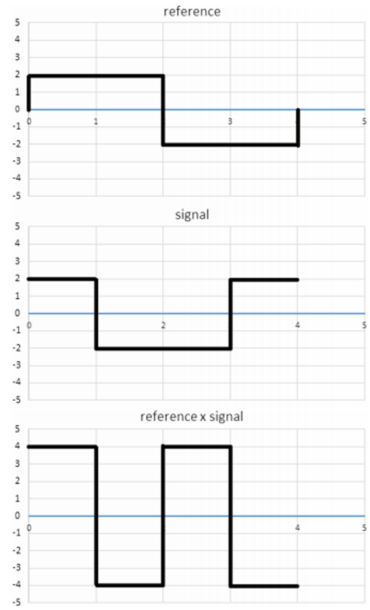


Figure 11: Multiplying bipolar square waves: second case.

$$u = \frac{1}{T} a_1 A \left[ P - \tfrac{T}{2} - (T - P) + P - \tfrac{T}{2} - (T - P) \right] \quad p \in [\frac{T}{2}, T]$$

So,

$$u = \frac{1}{T} a_1 A (4P - 3T) \quad p \in [\frac{T}{2}, T]$$

And again, the overall result is:

$$u = \frac{a_1 A}{T} \begin{bmatrix} T - 4P & [0, \frac{T}{2}] \\ 4P - 3T & [\frac{T}{2}.T] \end{bmatrix}$$

**Mark-space square wave and bipolar square wave** For completeness, the result obtained from a mark-space square wave as a signal, and a bipolar square wave as a reference, is also provided. This can be derived in the same way as before by adding the rectangles of the product function and taking the correct sign into account.

$$u = \frac{a_1 A}{T} \begin{bmatrix} \frac{T}{2} - 2P & [0, \frac{T}{2}] \\ 2P - \frac{3T}{2} & [\frac{T}{2}.T] \end{bmatrix}$$

## 4.3 Square-wave reference signal and comb-filter effect

The situation where the reference signal is a square wave and the signal is a sinusoidal function, as is encountered in some spectroscopic experiments.

The square wave can also be a bipolar function:

$$r = A \begin{bmatrix} +1 & [0, \frac{T}{2}] \\ -1 & [\frac{T}{2}, T] \end{bmatrix}$$

Or a mark square wave:

$$r = A \begin{bmatrix} +1 \ [0,\frac{T}{2}] \\ 0 \ \ [\frac{T}{2},T] \end{bmatrix}$$

We consider A=1 and also do the calculation for the first case and then refer to the previous result for the second case!

**CALCULATIONS**: Finding the Fourier series of Bipolar square wave and Mark square wave according to the reference [6].

As we know the Fourier series is equal to:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx\ \pi/L) + \sum_{n=1}^{\infty} b_n \sin(nx\ \pi/L) \tag{26}$$

Where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{27}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx\ \pi/L)dx \tag{28}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(nx\ \pi/L)dx \tag{29}$$

For a Bipolar square wave, according to the calculations in the link mentioned:

$$a_0 = 0,\ all\ a_n = 0\ ,\ b_n = 0\ when\ n\ is\ even\ and\ b_n = \frac{4h}{n\pi}\ when\ n\ is\ odd.$$

Finally a Bipolar square wave function is:

$$f(x) = \frac{4h}{\pi}(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + ...) = \frac{4h}{\pi}(\frac{\sin((2n+1)x)}{2n+1}) \tag{30}$$

For a Mark square wave, according to the calculations in the reference:

$$a_0 = \frac{1}{2},\ all\ a_n = 0\ ,\ b_n = 0\ when\ n\ is\ even\ and\ b_n = \frac{2h}{n\pi}\ when\ n\ is\ odd.$$

And Mark square wave function is:

$$f(x) = \frac{1}{2} + \frac{2h}{\pi}(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + ...) = \frac{1}{2} + \frac{2h}{\pi}(\frac{\sin((2n+1)x)}{2n+1}) \tag{31}$$

**Bipolar square wave as a reference** To state the square-wave function as a superposition of harmonic functions: The function is expanded into a Fourier series [3]:

$$r = \begin{bmatrix} +1 & [0, \frac{T}{2}] \\ -1 & [\frac{T}{2}, T] \end{bmatrix} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\Omega t]}{2k+1} \tag{32}$$

and

$$s.r = \frac{4a_1}{\pi} \sin(\omega t + \phi) \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\Omega t]}{2k+1} \tag{33}$$

Therefore, related to formula for sin and cos:

$$s.r = \frac{4a_1}{\pi} \sin(\omega t) \cos\phi \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\Omega t]}{2k+1} + \frac{4a_1}{\pi} \cos(\omega t) \sin\phi \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\Omega t]}{2k+1} \tag{34}$$

We assume that the noise part in the lock-in amplifier input is rejected by the multiplication-and-filtering process. Now the product of the two functions is integrated to obtain the output voltage of the lock-in amplifier:

$$u = \int_0^{\tau} s.r dt \tag{35}$$

And,

$$u = \frac{2a_1}{\pi\tau} \cos\phi \int_0^{\tau} dt \sum_{k=0}^{\infty} \frac{\cos[\omega - (2k+1)\Omega]t - \cos[\omega + (2k+1)\Omega]t}{2k+1} \tag{36}$$

$$+ \frac{2a_1}{\pi\tau} \sin\phi \int_0^{\tau} dt \sum_{k=0}^{\infty} \frac{\sin[-\omega + (2k+1)\Omega]t + \sin[\omega + (2k+1)\Omega]t}{2k+1} \tag{37}$$

The periodic components of the sum vanish during integration. Only those components that are constant and nonzero contribute to the output voltage. This condition holds for the cosine term with a vanishing argument, i.e.,

$$\omega - (2k+1)\Omega = 0 \tag{38}$$

Therefore:

$$u = \frac{2a_1}{\pi} \cos\phi \int_0^{\tau} \frac{1}{2k+1} dt = \frac{2a_1}{\pi} \cos\phi \frac{1}{2k+1} \tag{39}$$

This equation implies that we will measure:

$$u = \frac{2a_1}{\pi} \cos\phi \ at \ frequency \ \omega = \Omega \tag{40}$$

$$u = \frac{2a_1}{3\pi}\cos\phi \ at \ frequency \ \omega = 3\Omega$$
$$u = \frac{2a_1}{5\pi}\cos\phi \ at \ frequency \ \omega = 5\Omega$$

These lines correspond to the cases where k = 0, 1, 2 and can be continued in an analogous way.

**Mark-space square wave as a reference** According to previous section and calculations, the output voltage of the lock-in amplifier when a Mark-space square wave is as a reference is equal to [3]:

$$u = \frac{a_1}{\pi}\cos\phi\frac{1}{2k+1} \qquad k = 1,2,3,...$$

As was the case when both the signal and the reference were periodic, the locking amplifier outputs a voltage that is proportional to the cosine of the phase difference.

And finally, the results obtained for the different combinations of signal and reference and are represented succinctly in the Fig 12.

| Reference $r$ | Signal $s$ | Output $u$ |
|---|---|---|
| Sine wave | Sine wave | $u = \frac{a_1 A}{2}\cos\phi$ |
| Mark-space square wave | Mark-space square wave | $u = \frac{a_1 A}{T}\left\{\begin{matrix} \frac{T}{2} - p & \left[0, \frac{T}{2}\right] \\ p - \frac{T}{2} & \left[\frac{T}{2}, T\right] \end{matrix}\right\}$ |
| Bipolar square wave | Bipolar square wave | $u = \frac{a_1 A}{T}\left\{\begin{matrix} T - 4p & \left[0, \frac{T}{2}\right] \\ 4p - 3T & \left[\frac{T}{2}, T\right] \end{matrix}\right\}$ |
| Bipolar square wave | Mark-space square wave | $u = \frac{a_1 A}{T}\left\{\begin{matrix} \frac{T}{2} - 2p & \left[0, \frac{T}{2}\right] \\ 2p - \frac{3T}{2} & \left[\frac{T}{2}, T\right] \end{matrix}\right\}$ |
| Bipolar square wave | Sine wave | $u = \frac{2a_1 A}{\pi}\cos\phi\frac{1}{2k+1}, \ k = 0, 1, 2, \ldots$ |
| Mark-space square wave | Sine wave | $u = \frac{a_1 A}{\pi}\cos\phi\frac{1}{2k+1}, \ k = 0, 1, 2, \ldots$ |

Figure 12: Outputs obtained for different combinations of reference and signal [3].

## 4.4 Nonlinearities and Higher Harmonics

A transfer function that features nonlinearities gives rise to higher harmonics. There are a variety of physical effects in materials that feature a quadratic dependence. We start with a consideration of a quadratic nonlinearity at 2f, i.e., the double frequency, and then generalize this reasoning to higher-order nonlinearities [3].

#### Detection at multiples of the frequency of the first harmonic

**Detection at 2f** In the experimental arrangement depicted in Fig 13, the transfer function of the experiment is now nonlinear; i.e., it is a quadratic function.

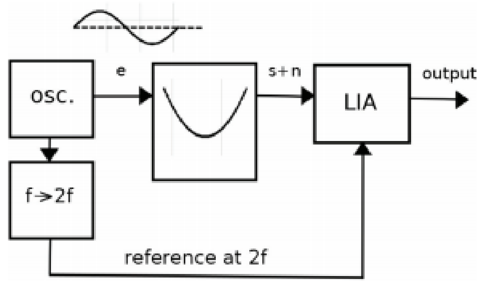


Figure 13: Experiment featuring a quadratic transfer function and detection at 2f [3].

A sine wave is used as an input signal (excitation) for the experiment:

$$e = e_\omega \cos(\omega t) \tag{41}$$

The transfer function is quadratic where the constant that precedes the square of the excitation is labelled $d$

$$s = d_2 e^2 \tag{42}$$

As we know:

$$\cos 2x = 2\cos^2 x - 1 \tag{43}$$

So

$$s = \frac{1}{2} d_2 e_w^2 + \frac{1}{2} d_2 e_w^2 \cos(2\omega t) \tag{44}$$

The first part of the equation is a $DC$ signal, while the second part is a periodic signal at frequency 2f. The nonlinearity described by the constant $d_2$ can therefore be quantified using a harmonic excitation signal and lock-in detection at 2f [3].

**Detection at higher-order harmonics** In a more general case:

$$s = \sum_{k=1}^{4} d_k e^k = d_1 e^1 + d_2 e^2 + d_3 e^3 + d_4 e^4 \tag{45}$$

To obtain expressions of the system response in terms of higher frequencies, we apply the following trigonometric identities:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{46}$$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x) \tag{47}$$

$$\cos^4 x = \frac{1}{8}(3 + 4\cos 2x + \cos 4x) \tag{48}$$

Finally, the system response is [3]:

$$s = \frac{1}{2}d_2.w_e^2 + \frac{3}{8}d_4.w_e^4 + \cos(\omega t).(d_1.e_w + \frac{3}{4}d_3.e_w^3)\ for\ detection\ at\ f. \tag{49}$$
$$+\cos(2\omega t).(\frac{1}{2}d_2.e_w^2 + \frac{1}{2}d_4.e_w^4)\ for\ detection\ at\ 2f.$$
$$+\cos(3\omega t).(\frac{1}{4}d_3.e_w^3)\ for\ detection\ at\ 3f.$$
$$+\cos(4\omega t).(\frac{1}{8}d_4.e_w^4)\ for\ detection\ at\ 4f.$$

# 5 Practical Example: Optical Detection using Lock-in Amplifier

In this section different approaches to detect a signal are provided. It then discusses different experimental approaches using lock-in amplifiers [5].

## 5.1 The DC Approach

In the simplest form of light measurement, a suitable current meter measures the DC current generated in an optical detector. Such a detection system has its use in applications such as camera light meters but much lower light levels, errors will begin to appear. One important problem of DC approach is that there is no way of separating the output signal caused by the wanted input signal from noises come from stray light entering the detector [7].

## 5.2 The Modulated Light Approach

The most widely used method of measuring a low level optical signal is to apply a modulation to the light source and then recover the signal at the modulation frequency [7]. The modulation can be of any periodic form, but sinusoidal and square waves are most commonly used. The AC signal generated at the detector output allows the experimenter to use any one of a variety of AC measurement techniques and reduces some of the problems which plague the DC method [7].

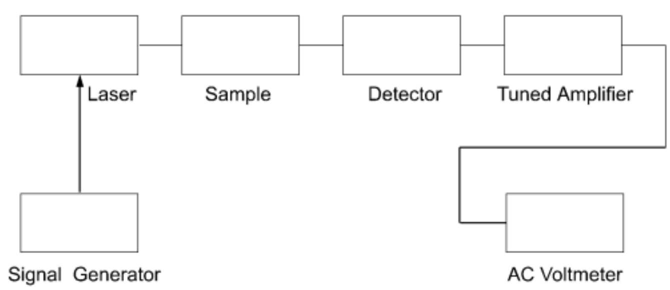


Figure 14: AC Measurement System using Tuned Amplifier [7].

In figure 14, the optical signal generated by a laser is modulated at the frequency output by the signal generator. The output of the detector is therefore now the unwanted DC signal caused by the thermal leakage and an AC signal at the same frequency as the modulation. The signal then passes to a tuned amplifier, which consists of a signal filter and amplifier stage. The filter is set to a bandpass mode, which limits the bandwidth of the measuring system to those frequencies close to the modulation frequency, and its output is then measured using an AC voltmeter. The lower limit of light detection using this method represents a improvement when compared to the DC system. Still, there are some shortcomings in this method. The minimum signal that can be detected is primarily determined by the selectivity of the detection system, which in this case is set by the Q-factor of the filter. And another problem is that tuned amplifiers are not the best instruments to use in the "front end" of an experiment. By the way, the AC filter method may still be preferred over more sophisticated techniques [7].

## 5.3 The Lock-In Amplifier Method

A better approach to the AC detection method is to use a lock-in amplifier to measure the AC signal from the detector. The lock-in amplifier uses a frequency-selective technique. lock-in amplifier measures only the amplitude of signals at a frequency equal to the applied reference frequency. Figure 15 illustrates a optical detection setup using a mechanical light chopper and a lock-in amplifier. Since the rotation of the chopper blade causes the optical signal path to be interrupted, the light source that stimulates the experiment is in the form of an AC excitation [7].

In addition to modulating the light source, the chopper also provides a synchronous reference signal capable of driving the reference channel of a lock-in amplifier. Any discrete frequencies or noise voltages not equal to the reference frequency will be rejected by the lock-in amplifier.

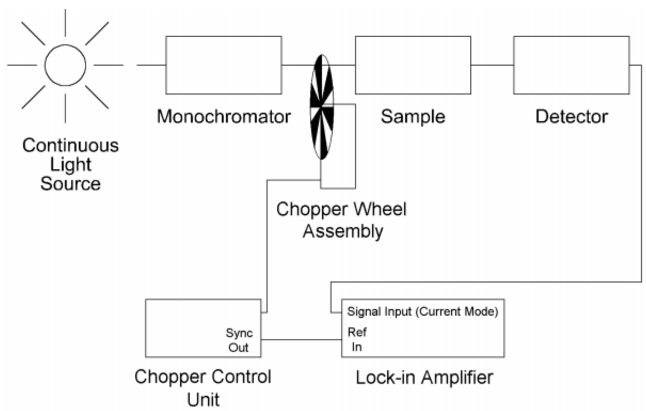


Figure 15: AC Measurement System using a Lock-in Amplifier and Mechanical Chopper [7].

## 5.4 Source Compensation - Ratiometric Spectroscopy

Although the use of a lock-in amplifier enhances the ability to measure a signal buried in noise, there can be sources of measurement errors other than noise and background voltages. In optical measurements, one source of error can be attributed to variations in light source intensity. Moreover, the efficiency of a scanning monochromator may vary as a function of wavelength. These are common problem that can only be addressed by introducing a second detection path which measures the optical output of the excitation source and by using a ratiometric technique to normalize for source fluctuations [7].

In figure 16, the optical output from the monochromator is split off and sent to a separate detector and preamplifier (not shown). This generates a DC voltage, the magnitude of which is determined by the intensity of the source as well as the relative efficiency of the monochromator. It was mentioned earlier that the lock-in generates a DC voltage at its output as part of the detection process. In this configuration, a second DC voltage is now available which represents only the optical signal from the monochromator. By calculating the ratio of the DC output of the lock-in amplifier to the DC voltage generated as a result of the normalizing beam, a third DC voltage is generated which is proportional to only those changes due to properties in the sample path. The block labeled "Ratiometer" may be an analog circuit, or more likely a digital system that calculates the ratio of the two DC voltages and provides an output in some digital form. The neutral density filter is used to adjust the level of the normalizing beam for the appropriate nominal input voltage to the "B" input of the ratiometer. Again referring to figure 16, note that it is essential not to allow stray light to fall on the detector in the normalizing path.

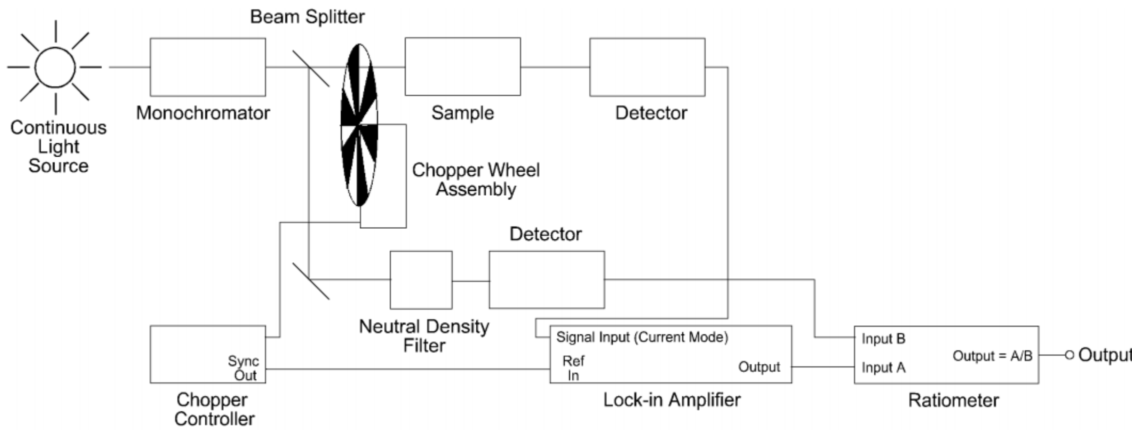


Figure 16: AC Measurement System using a Lock-in Amplifier, Mechanical Chopper and DC Source Compensation [7].

## 5.5 Source Compensation Using Two Lock-Ins

An improved version of the ratiometer approach is shown in figure 17. Basically, the difference is that both the normalizing and signal beams are chopped, and the two beams are recombined into a single detector. This eliminates any error due to mismatching of optical detectors.

As shown in Figure 17, a second lock-in amplifier, number 2, is used to detect the normalizing beam. The second lockin's output is used for the denominator of the ratio calculation.

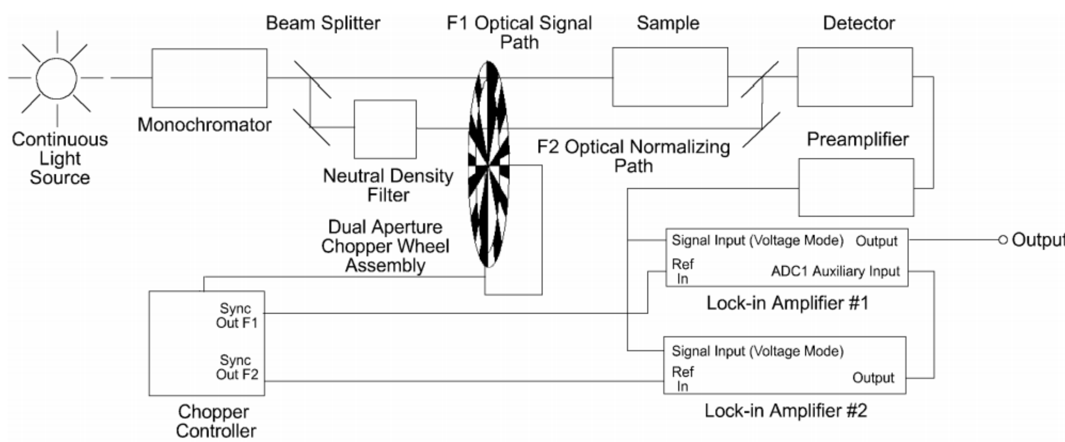


Figure 17: AC Measurement System using two Lock-in Amplifiers, Dual-Beam Mechanical Chopper and AC Source Compensation [7].

### 5.6 Dual Reference Lock-In Amplifiers

As it is shown in Figure 18, a lock-in amplifier which has a dual reference capability can detect two signals simultaneously, and as a result, only one lock-in is needed for a ratiometric experiment. As in the case of the two lock-in approach, the output from the monochromator is split off and a dual Optical Chopper is used to chop the light source at the two chopping frequencies. The chopper also generates a reference signal. This signal is fed back to the reference input connector of the lock-in which has the two required reference signals, one corresponding to the signal channel path and one relating to the optical normalizing path.Finally the lock-in can measure and display the amplitude of both signals appearing at its input connector.

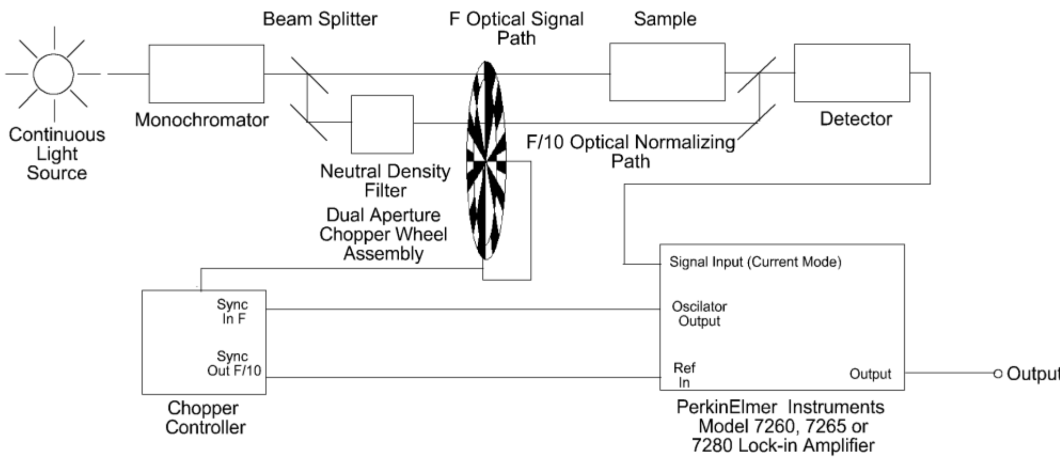


Figure 18: AC Measurement System using Dual-Reference Lock-in Amplifier, Dual-Beam Mechanical Chopper and AC Source Compensation [7].

## 6 Mechanical Light Choppers

Knowing light chopper specifications is crucial to decide which one is best suited for a particular experiment. In this section, some of the more important features will be explained.

### 6.1 Frequency Range

The chopping frequency is variable to allow the user the ability to both select a frequency which is optimum for the detector as well as avoid troublesome frequencies. It's usually a good idea to chop at frequencies above the 1/f noise level (typically 100 Hz). In addition, a chopping frequency near the power line frequency or any harmonic of it should be avoided. For choppers using a dual aperture blade one has the ability to chop at two different frequencies.

### 6.2 External Frequency Control

In addition to a frequency control on the chopper itself, the frequency of most choppers can be controlled externally. In some choppers this is done by the application of a DC voltage. Other choppers are controlled by applying an external reference frequency signal. Most modern lock-in amplifiers incorporate an internal oscillator, the output of which can be connected to the chopper's reference frequency input. If this is done, then changing the oscillator frequency also changes the frequency of the modulation generated by the chopper. If the lock-in is operated under computer control then the oscillator frequency can be set by the program, allowing for example, with suitable software, automatic selection of an operating frequency where any interfering signals are smallest. Another use of this technique is to prolong the chopper's motor life by reducing its speed whenever measurements are not being taken.

## 6.3 Jitter

Jitter is the short term variation in the period of one chopper cycle to the next. Its effect is to add noise to a measurement. The source of jitter is twofold; one is the mechanical imperfections in the chopper blade, the other is from the speed control electronics and motor combination.

Figure 19 illustrates how jitter manifests itself. Jitter can be expressed in either degree rms values or peak-to-peak units as a percent.

A difficulty may arise when comparing two chopper jitter specifications where the two different values are specified. For example, one might be specified to exhibit 0.5 percent peak-to-peak jitter while one might be specified as of 0.5 degree rms under similar operating conditions.

Naturally, a comparison must be made in the same mathematical units by converting the numbers. The 0.5 percent peak-to-peak specification refers to the percentage of a complete wheel rotation or 360 degrees. In this case, the peak-to-peak jitter is (.005 X 360) or 1.8 degrees. The calculated 1.8 degrees is still in peak-to-peak units, so it is necessary to convert to rms values. Peak-to-peak values are 2.8 times larger than rms values. In this case, it is necessary to divide the calculated 1.8 degrees by 2.8 in order to arrive at a rms value. In this case, a conversion to rms will yield $1.8/2.8 = 0.64$ degrees rms. In this case, the two choppers have a very similar jitter specification.

Other factors to consider when choosing a chopper are mechanical configuration, beam size to be chopped, and how one wishes to externally control the speed.

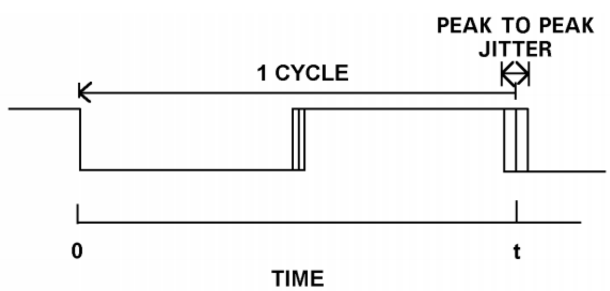


Figure 19: [7].

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