Chapter 1 Bayesian Network

Theorem 1.1

Let $\{X_i\}_{i=1}^{n}$ are random variables, then

$$P(\cap_{i=1}^{n} X_i) = \prod_{j=1}^{n} P(X_j | \cap_{i=1}^{j-1} X_i)$$

consider the following probability directed acyclic graph:

$$\begin{aligned} P(X_1, X_2, X_3, X_4) = & \quad P(X_1|X_2, X_3, X_4) \times P(X_2|X_3, X_4) \times P(X_3|X_4) \times P(X_4) \\ = & \quad P(X_1|X_3, X_4) \times P(X_2|X_3) \times P(X_3) \times P(X_4) \end{aligned}$$

Theorem 1.2

Let $G =< V, E >$ be a probability directed acyclic graph. where $V = \{X\}_{i=1}^{n}$.

$$P(\cap_{i=1}^{n} X_i) = \prod_{i=1}^{n} P(X_i | parent(X_i))$$

Proof. let's try a constructive proof,at first,we have known that $P(\cap_{i\in[1,n]\cap\mathbb{N}} X_i) = P(\cap_{i\in permuate([1,n])} X_i)$

$$P(\cap_{i=1}^{n} X_i) = \quad P(X_1|X_2, X_3, \ldots, X_n) \times P(X_2, X_3, \ldots, X_n)$$

therefore $Parent(X_1) \subseteq \{X_i\}_{i=2}^{n}$ hold.

then consider $P(X_2, X_3, \ldots, X_n) = P(X_2|X_3, \ldots, X_n) \times P(X_3, \ldots, X_n)$,Since G is a directed acyclic graph, there are only three cases to decompose it into the one we want:

Case1 $X_1 \notin Parent(X_2)$ and $X_2 \notin Parent(X_1)$

In this case,it's trival

Case2 $X_1 \notin Parent(X_2)$ and $X_2 \in Parent(X_1)$

All right,we can continue to decomposition $P(X_2, \ldots, X_n)$ to $P(X_2|X_3, \ldots, X_n)$, because G is a DAG (directed acyclic graph.

Case3 $X_1 \in Parent(X_2)$ and $X_2 \notin Parent(X_1)$

Back to previous step and replace $P(X_2, X_1, X_3, \ldots, X_n)$ with $P(X_1, X_2, X_3, \ldots, X_n)$

Continue with the above process until the end of the decomposition and we have got

$$P(\cap_{i=1}^{n} X_i) = \prod_{i=1}^{n} P(X_i | parent(X_i))$$

■