LARGE DEVIATIONS AND FLUCTUATIONS OF REAL EIGENVALUES OF ELLIPTIC RANDOM MATRICES

SUNG-SOO BYUN, LESLIE MOLAG, AND NICK SIMM

ABSTRACT. We study real eigenvalues of $N \times N$ real elliptic Ginibre matrices indexed by a non-Hermiticity parameter $0 \le \tau < 1$, in both the strong and weak non-Hermiticity regime. Here N is assumed to be an even number. In both regimes, we prove a central limit theorem for the number of real eigenvalues. We also find the asymptotic behaviour of the probability $p_{N,k}^{(\tau)}$ that exactly k eigenvalues are real. In the strong non-Hermiticity regime, where τ is fixed, we find

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \log p_{N,k_N}^{(\tau)} = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\zeta(3/2)}{\sqrt{2\pi}}$$

for any sequence $(k_N)_N$ of even numbers such that $k_N = o(\frac{\sqrt{N}}{\log N})$ as $N \to \infty$, where ζ is the Riemann zeta function. In the weak non-Hermiticity regime, where $\tau = 1 - \frac{\alpha^2}{N}$, we obtain

$$\lim_{N \to \infty} \frac{1}{N} \log p_{N,k_N}^{(\tau)} \le \frac{2}{\pi} \int_0^1 \log \left(1 - e^{-\alpha^2 s^2} \right) \sqrt{1 - s^2} \, ds$$

for any sequence $(k_N)_N$ of even numbers such that $k_N = o(\frac{N}{\log N})$ as $n \to \infty$. This inequality is expected to be an equality.

1. Introduction and Main results

In 1965, Ginibre introduced three random matrix models that are essentially the unconstrained versions of the GOE, GUE and GSE, i.e. all entries are i.i.d. Gaussians without the requirement of Hermiticity [34]. Due to the lack of Hermiticity, the eigenvalues are not confined to the real line, and live on the full complex plane. These ensembles consist of $N \times N$ matrices M, with real ($\beta = 1$), complex ($\beta = 2$) or (real) quaternion ($\beta = 4$) entries, that are distributed according to the probability measure

$$\frac{1}{Z_N^{\beta}} e^{-\frac{1}{2}\beta \operatorname{Tr}(M^{\dagger}M)} dM_N^{\beta},$$

where Z_N^{β} is a normalisation constant, and dM_N^{β} is the standard Lebesgue measure on the corresponding spaces of matrices of real dimension βN^2 . Nowadays these models are called the real, complex and quaternion Ginibre ensembles (denoted as GinOE, GinUE and GinSE), and they are well-studied in the past half century. For a recent review on the Ginibre ensembles, we refer to the papers [8,9].

In the present paper we focus on $\beta = 1$. In fact, we consider a one-parameter deformation of the GinOE, called the real elliptic Ginibre ensemble (eGinOE). Likely inspired by Girko [35], the eGinOE was introduced in 1988 by Sommers, Crisanti, Sompolinsky and Stein [48]. The eGinOE with parameter $-1 < \tau < 1$ consists of $N \times N$ real matrices M, with centered Gaussian entries that satisfy

Date: May 5, 2023.

²⁰²⁰ Mathematics Subject Classification. Primary 60B20; Secondary 33C45.

Key words and phrases. Real elliptic Ginibre matrices, real eigenvalues, strong/weak non-Hermiticity, central limit theorem, large deviation.

the correlation structure

$$\mathbb{E}M_{ij}^2 = \frac{1}{N}, \qquad \mathbb{E}M_{ij}M_{ji} = \frac{\tau}{N}, \qquad \mathbb{E}M_{ii}^2 = \frac{1+\tau}{N}, \qquad i, j = 1, \dots, N \text{ and } i \neq j.$$

These are precisely the real random matrices M that are distributed according to the probability measure

$$\frac{1}{Z_N^{(\tau)}} e^{-\frac{1}{2(1-\tau^2)} \operatorname{Tr}(M^{\dagger} M - \tau M^2)} dM_N, \qquad dM_N = \prod_{i,j=1}^N dM_{ij},$$

where $Z_N^{(\tau)}$ is a normalisation constant. For $\tau=0$, we obtain the GinOE. On the other hand, in the limit $\tau\uparrow 1$, it is known that the eGinOE approaches the GOE. In the limit $\tau\downarrow -1$, the matrix M is real and anti-symmetric. This is equivalent (after multiplication by i) to what is known as the anti-symmetric GUE, see [13,28] and references therein for further details about this ensemble. One can define the eGinOE equivalently as the ensemble consisting of $N\times N$ matrices

(1.1)
$$M := \sqrt{\frac{1+\tau}{2}} \, S + \sqrt{\frac{1-\tau}{2}} \, A,$$

where S and A are matrices picked from the GOE and its anti-symmetric version. Nowadays, most authors require the parameter τ to be in (0,1) or [0,1) in the definition of the eGinOE.

The first occurrence of a GinOE matrix in an application was in a paper by May [41] in 1972, who investigated complex ecological systems. More precisely, May considered the stability of the solutions to

$$\vec{x}' = (-\mathbb{I} + \alpha G_N) \, \vec{x},$$

where G_N is a GinOE matrix. This allows to investigate general systems of differential equations $\vec{x}' = f(\vec{x})$ where f is unknown. More general systems of differential equations associated with the eGinOE were further investigated by Fyodorov and Khoruzhenko [29]. Over the years many other applications have been introduced, ranging from dynamics to random networks and cortical electric activity to quantum chromodynamics, see [36] for references.

For general fixed $\tau \in [0,1)$ and $N \to \infty$, it is well known that the eigenvalues are uniformly distributed on the ellipse

(1.2)
$$\mathscr{E}^{(\tau)} := \left\{ z \in \mathbb{C} : \left(\frac{\operatorname{Re} z}{1+\tau} \right)^2 + \left(\frac{\operatorname{Im} z}{1-\tau} \right)^2 \le 1 \right\},$$

see e.g. [35, 43, 48]. We also refer to [5] for the local law for elliptic random matrices.

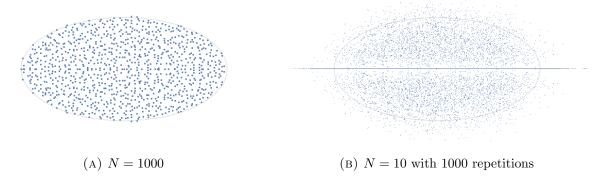


FIGURE 1. Eigenvalues of the eGinOE.

In his original paper, Ginibre only managed to derive the joint probability density function (JPDF) for the case that all N eigenvalues are real. It took about a quarter century longer before the JPDF in the general case was determined [39]. When N = 2n, and the number of real eigenvalues is 2k, the corresponding JPDF is given by

$$C_n^{(\tau)} \binom{n}{k} 2^{n-k} \prod_{j=1}^k \sqrt{\omega^{(\tau)}(\lambda_j)} \prod_{\ell=1}^{n-k} \omega^{(\tau)}(z_\ell) |\Delta\left(\lambda_1, \dots, \lambda_k, z_1, \overline{z}_1, \dots, z_{n-k}, \overline{z}_{n-k}\right)|,$$

where $C_n^{(\tau)}$ is an explicit constant,

$$\omega^{(\tau)}(z) = e^{-\frac{\operatorname{Re}(z^2)}{1+\tau}}\operatorname{erfc}\left(\sqrt{\frac{2}{1-\tau^2}}\operatorname{Im}z\right),\,$$

and Δ is a Vandermonde factor. We mention that the associated correlation functions exhibit a Pfaffian structure [3].

The difficulty in determining the JPDF for the GinOE and eGinOE, is due to the unique property, not seen in the complex and quaternion counterparts, that purely real eigenvalues occur with non-zero probability, see Figure 1. In particular, the probabilities

(1.3)
$$p_{N,k}^{(\tau)}, \qquad k = 0, 1, \dots, N$$

that a particular eigenvalue configuration of the eGinOE (and GinOE) has exactly k real eigenvalues are non-zero, when k has the same parity as N.

In the current paper, we shall focus on the case N is even as the odd N case requires separate treatment, see e.g. [24, 47]. We shall write

$$(1.4) N = 2n, n = 1, 2, \dots$$

henceforth.

1.1. Main result for fluctuations of the number of real eigenvalues. Let $\mathcal{N}_N^{(\tau)}$ be the number of real eigenvalues of M. For $\tau \in [0,1)$ fixed, it was shown by Forrester and Nagao [27] that the expected number of real eigenvalues is given by

(1.5)
$$\mathbb{E}\mathscr{N}_{N}^{(\tau)} = \sqrt{\frac{1+\tau}{1-\tau}}\sqrt{\frac{2N}{\pi}}(1+o(1)), \qquad (N\to\infty).$$

The formula (1.5) was first proved by Edelman, Kostlan, and Shub for the GinOE case ($\tau = 0$) [15]. In addition to the mean, the variance of the number of real eigenvalues satisfies the asymptotic behaviour

(1.6)
$$\operatorname{Var} \mathcal{N}_{N}^{(\tau)} = (2 - \sqrt{2}) \sqrt{\frac{1+\tau}{1-\tau}} \sqrt{\frac{2N}{\pi}} (1 + o(1)), \qquad (N \to \infty).$$

This was implicitly shown in [27], cf. [10, Remark 5.1].

It is obvious from (1.1) that for $\tau = 1$,

(1.7)
$$\mathbb{E}\mathscr{N}_{N}^{(1)} = N, \qquad \operatorname{Var}\mathscr{N}_{N}^{(1)} = 0.$$

From (1.5), (1.6) and (1.7), one can expect the occurrence of a non-trivial transition when $\tau \uparrow 1$. This occurs in the so-called weak non-Hermiticity regime, which was introduced in the pioneering work [30–32] of Fyodorov, Khoruzhenko, and Sommers. For the eGinOE, it corresponds to the regime

(1.8)
$$\tau = 1 - \frac{\alpha^2}{N}, \qquad \alpha \in (0, \infty) \text{ fixed.}$$

This regime is sometimes referred to as the weakly asymmetric regime in the case of real random matrices. Several interesting scaling limits arise in this regime, which interpolate between the GOE

and GinOE, see e.g. [4,27,33] and references therein. In this critical regime, it was shown in a recent work [10] that

(1.9)
$$\mathbb{E}\mathscr{N}_{N}^{(\tau)} = c(\alpha) N + O(1), \qquad (N \to \infty),$$

where

(1.10)
$$c(\alpha) := e^{-\alpha^2/2} \left[I_0\left(\frac{\alpha^2}{2}\right) + I_1\left(\frac{\alpha^2}{2}\right) \right].$$

Here

(1.11)
$$I_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)}$$

is the modified Bessel function of the first kind [44, Chapter 10]. See (5.2) for alternative representations of the function $c(\alpha)$. It was also shown in [10] that

(1.12)
$$\operatorname{Var} \mathcal{N}_{N}^{(\tau)} = 2\left(c(\alpha) - c(\sqrt{2}\alpha)\right)N + o(N), \qquad (N \to \infty).$$

(See [1] for analogous results for products of GinOE matrices.)

For the GinOE ($\tau = 0$), the central limit theorem for the number of real eigenvalues (or its linear statistics in general) was proved in [19,45]. In all other cases a central limit theorem was missing, and our first result is on this topic.

Theorem 1.1 (Central limit theorem for the number of real eigenvalues).

Let N be even. As $N \to \infty$, we have the convergence in distribution

(1.13)
$$\frac{\mathscr{N}_{N}^{(\tau)} - \mathbb{E}\mathscr{N}_{N}^{(\tau)}}{\sqrt{\mathbb{E}\mathscr{N}_{N}^{(\tau)}}} \to N(0, \sigma^{2}),$$

where $N(0, \sigma^2)$ denotes the normal distribution with mean 0 and variance

(1.14)
$$\sigma^2 = \begin{cases} 2 - \sqrt{2} & \text{if } \tau \in [0, 1) \text{ is fixed,} \\ 2 - 2\frac{c(\sqrt{2}\alpha)}{c(\alpha)} & \text{if } \tau = 1 - \frac{\alpha^2}{N} \text{ with fixed } \alpha \in (0, \infty). \end{cases}$$

The variance for the weak non-Hermiticity regime in (1.14) interpolates between the GOE ($\alpha \downarrow 0$) and GinOE ($\alpha \to \infty$). For fixed τ , it can be shown that the results are in fact valid for $-1 < \tau < 1$, see Corollary 1.7. Let us also mention that the full counting statistics of the GinUE and its generalisation were obtained in [7,11,16] with great precision.

1.2. Main results on large deviations for the number of real eigenvalues. We shall now discuss large deviations concerning the number of real eigenvalues of the GinOE and eGinOE. We know from [14, 27] that

(1.15)
$$p_{N,N}^{(\tau)} = \left(\frac{1+\tau}{2}\right)^{\frac{N(N-1)}{4}}.$$

The case of exactly one complex eigenvalue pair was studied in [3] for $\tau = 0$, which reads

$$\log p_{N,N-2}^{(0)} = -\frac{\log 2}{4} N^2 + \frac{\log(3\sqrt{2})}{2} N + o(N), \qquad (N \to \infty).$$

We also mention that the case $k \sim aN$, with 0 < a < 1 fixed, was studied in [12] using a Coulomb gas approach. In this paper however, we shall be interested in the case of a small number of real eigenvalues. It was shown in [37] that for the Ginibre case when $\tau = 0$,

(1.16)
$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \log p_{N,k_N}^{(0)} = -\frac{1}{\sqrt{2\pi}} \zeta(\frac{3}{2}),$$

whenever $(k_N)_N$ is a sequence of even numbers such that $k_N = o(\frac{\sqrt{N}}{\log N})$ as $n \to \infty$. We prove the analogous statement for the eGinOE.

Theorem 1.2 (Large deviations for real eigenvalues at strong non-Hermiticity).

Let $\tau \in [0,1)$ be fixed, and let N and k be even numbers. Then for any fixed k

(1.17)
$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \log p_{N,k}^{(\tau)} = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}} \zeta\left(\frac{3}{2}\right).$$

The limit holds with k replaced by any sequence $(k_N)_N$ of even numbers such that $k_N = o(\frac{\sqrt{N}}{\log N})$ as $N \to \infty$.

Theorem 1.2 in fact holds for $-1 < \tau < 1$, see Corollary 1.7.

We also state the analogue of Theorem 1.2 for the weak non-Hermiticity regime. Here, due to the lack of a uniform estimate as in Lemma 2.7, we are merely able to give an upper bound. We do believe that this upper bound is in fact sharp, i.e. the inequality in (1.18) is an equality. In any case, there are already some interesting conclusions that can be drawn from the upper bound, e.g. that the probability of having only a few real eigenvalues is of a much smaller order then in the fixed τ case.

Theorem 1.3 (Large deviations for real eigenvalues at weak non-Hermiticity).

Let N and k be even and let $\tau = 1 - \frac{\alpha^2}{N}$ with fixed $\alpha \in (0, \infty)$. Then

(1.18)
$$\lim_{N \to \infty} \frac{1}{N} \log p_{N,k}^{(\tau)} \le -d(\alpha),$$

where

(1.19)
$$d(\alpha) := \sum_{m=1}^{\infty} \frac{c(\sqrt{m}\,\alpha)}{2m} = -\frac{2}{\pi} \int_0^1 \log\left(1 - e^{-\alpha^2 s^2}\right) \sqrt{1 - s^2} \, ds.$$

Here $c(\alpha)$ is given by (1.10). Moreover, the inequality holds with k replaced by any sequence $(k_N)_N$ of even numbers such that $k_N = o(\frac{N}{\log N})$ as $N \to \infty$.

The second identity in (1.19) is shown in Lemma A.1 below.

Remark 1.4 (Interpolating property). Let us assume that the inequality in (1.18) is an equality, which is what we expect. We can then write Theorems 1.2 and 1.3 combined as

$$\lim_{N \to \infty} \frac{\log p_{N,k}^{(\tau)}}{\mathbb{E} \mathscr{N}_N^{(\tau)}} = - \begin{cases} \frac{1}{2} \zeta \left(\frac{3}{2}\right) & \text{if } \tau \in [0,1) \text{ is fixed,} \\ \frac{d(\alpha)}{c(\alpha)} & \text{if } \tau = 1 - \frac{\alpha^2}{N} \text{ with fixed } \alpha \in (0,\infty), \end{cases}$$

following directly from (1.5) and (1.9). Indeed, using (1.19), we have that

$$\frac{d(\alpha)}{c(\alpha)} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{c(\sqrt{m}\alpha)}{c(\alpha)} \to \begin{cases} \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} = \infty, & \alpha \to 0, \\ \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m\sqrt{m}} = \frac{1}{2} \zeta\left(\frac{3}{2}\right), & \alpha \to \infty, \end{cases}$$

which can straightforwardly be derived from the representation (5.2) for $c(\alpha)$. We thus observe an interpolation between the GinOE and the GOE $(p_{Nk}^{(1)} = 0 \text{ if } k < N)$.

In the form (1.20), the fixed τ limit does not depend on τ anymore. This might indicate a universality result. We can consider real matrices M, distributed by

$$\frac{1}{Z_N^{(V)}} e^{-\frac{1}{2}\operatorname{Tr} V(M)} dM_N,$$

for some external field V. It is an interesting question whether the probabilities $p_{N,k}^{(V)}$ of having k real eigenvalues satisfy

$$\lim_{n \to \infty} \frac{\log p_{N,k}^{(V)}}{\mathbb{E} \mathcal{N}_N^{(V)}} = -\frac{1}{2} \zeta \left(\frac{3}{2}\right)$$

for a general class of external fields V (when N and k have the same parity, and k growing sufficiently slowly with N).

1.3. Further results. We now present some further results. Let $\text{Li}_s(z)$ denote the polylog function of order s defined according to the Dirichlet series

(1.21)
$$\operatorname{Li}_{s}(z) = \sum_{k=1}^{\infty} \frac{z^{k}}{k^{s}},$$

which is an analytic function of z on the open unit disc |z| < 1. It can be analytically continued to $\mathbb{C} \setminus [1, \infty]$, and this analytic continuation can be continuously extended to $\mathbb{C} \setminus (1, \infty)$ when $\operatorname{Re} s > 1$. When z = 1 and $\operatorname{Re} s > 1$ it coincides with the Riemann zeta function: $\operatorname{Li}_s(1) = \zeta(s)$.

Theorem 1.5. Let N=2n be even, and let $\tau \in [0,1)$ be fixed. Then we have for all $x \in [0,2]$ that

(1.22)
$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \log \left(\sum_{k=0}^{n} p_{N,2k}^{(\tau)} x^k \right) = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}(1-x)}{\sqrt{2\pi}}.$$

The convergence is uniform on any compact subset of (0,2).

The uniform convergence cannot be extended to x = 0, since that would imply that the limit function is smooth at x = 0.

Remark 1.6. Expanding around x = 1, Theorem 1.5 gives us that

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \log \left(1 + \sum_{k=1}^{n} \sum_{j=k}^{n} {j \choose k} p_{N,2j}^{(\tau)} (x-1)^k \right) = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2} (1-x)}{\sqrt{2\pi}}$$

uniformly for x in compact subsets of (0,2). This implies that all the coefficients on the left-hand side converge to some limit as $N \to \infty$. We have

$$\begin{split} &\frac{1}{\sqrt{N}}\log\left(1+\sum_{k=1}^n\sum_{j=k}^n\binom{j}{k}p_{N,2j}^{(\tau)}(x-1)^k\right)\\ &=\frac{x-1}{\sqrt{N}}\sum_{j=1}^njp_{N,2j}^{(\tau)}-\frac{(x-1)^2}{\sqrt{N}}\bigg(\sum_{j=2}^n\frac{j(j-1)}{2}p_{N,2j}^{(\tau)}-\frac{1}{2}\sum_{j,\ell=1}^nj\ell p_{N,2j}^{(\tau)}p_{N,2\ell}^{(\tau)}\bigg)+\mathscr{O}((x-1)^3). \end{split}$$

We can use this as a generating function for various probabilistic expressions, by taking derivatives at x = 1. For example, the expected number of real eigenvalues is asymptotically given by

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \mathbb{E} \mathscr{N}_N^{(\tau)} = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k=0}^n 2j p_{N,2j}^{(\tau)} = 2\sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}'(0)}{\sqrt{2\pi}} = 2\sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}}.$$

Corollary 1.7. The implications of Theorem 1.1 and Theorem 1.2 are true for fixed $-1 < \tau < 1$.

Proof. The skew-orthogonal polynomials in (3.5) are also valid for $-1 < \tau < 0$. This implies in particular that the generating identity in Proposition 2.1 is also valid for $-1 < \tau < 0$. We may write

$$\frac{1}{\sqrt{N}}\log\left(\sum_{k=0}^{n} p_{N,2k}^{(\tau)} x^{k}\right) = \sum_{j=0}^{N} C_{N,j}(x)\tau^{j},$$

for some coefficients $C_{N,j}(x)$, for all $-1 < \tau < 1$. Theorem 1.5 means for the coefficients that we have

$$\lim_{N \to \infty} C_{N,j}(x) = -\frac{\text{Li}_{3/2}(1-x)}{\sqrt{2\pi}} \frac{1}{j!} \frac{d^j}{d\tau^j} \bigg|_{\tau=0} \sqrt{\frac{1+\tau}{1-\tau}},$$

and (1.22) is in particular then also true for $-1 < \tau < 0$ (at least for fixed x). Thus Theorem 1.2 follows directly (take x = 0), and along the lines of Remark 1.6, we obtain Theorem 1.1 as well.

Corollary 1.8. Let N and $(k_N)_N$ be even numbers. We have as $N \to \infty$ that

$$\frac{\log p_{N,k_N}^{(\tau)}}{\sqrt{N}} \le -\left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{\frac{1+\tau}{1-\tau}}\frac{\zeta(3/2)}{\sqrt{2\pi}} - \frac{k_N}{\sqrt{N}}\log 2 + o(1).$$

If $k = k_N^{(\tau)}$ is such that $p_{N,k}^{(\tau)}$ is maximal among $p_{N,0}^{(\tau)}, p_{N,2}^{(\tau)}, \dots, p_{N,N}^{(\tau)}$, then we have

$$\lim_{N \to \infty} \frac{k_N^{(\tau)}}{\sqrt{N}} \le \sqrt{\frac{1+\tau}{1-\tau}} \frac{\zeta(3/2)}{\sqrt{\pi \log 4}}.$$

Proof. Taking x = 2 in Theorem 1.5, we find that

$$\lim_{N \to \infty} \frac{\log \left(p_{N,k_N}^{(\tau)} 2^{k_N} \right)}{\sqrt{N}} \le \lim_{N \to \infty} \frac{1}{\sqrt{N}} \log \sum_{k=0}^n p_{N,2k}^{(\tau)} 2^k = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}(-1)}{\sqrt{2\pi}} = \left(1 - \frac{1}{\sqrt{2}}\right) \sqrt{\frac{1+\tau}{1-\tau}} \frac{\zeta(3/2)}{\sqrt{2\pi}},$$

from which the first assertion follows. This, combined with Theorem 1.2, yields the estimate for $k_N^{(\tau)}$ as $N \to \infty$.

The rest of this paper is organised as follows.

- In Section 2, we introduce key ingredients of our analysis and complete the proof of the main results. In Subsection 2.1, we present Propositions 2.1, 2.3, 2.4, 2.5 and Lemmas 2.7, 2.8, 2.9 some of which will be shown in the following sections. Combining all of these, Subsection 2.2 culminates in the proof of Theorems 1.1, 1.2 and 1.3.
- Section 3 is devoted to the analysis of the generating function of the number of real eigenvalues. In Subsections 3.1 and 3.2, we prove the finite-N result (Proposition 2.1) and provide some useful lemmas on the evaluations of the generating matrix. In Subsections 3.3 and 3.4, we show some preliminary estimates of the generating matrix and prove Lemmas 2.7 and 2.8.
- Sections 4 and 5 are devoted to the crucial asymptotic analysis of the generating matrix at strong and weak non-Hermiticity, respectively. In particular, we show Propositions 2.4 and 2.5, which complete the proofs of main results.

• This article contains two appendices. In Appendix A, we collect some auxiliary lemmas. In Appendix B, we introduce an equivalent determinantal formula of the generating function due to Forrester and Nagao, and compare it with Proposition 2.1.

Acknowledgements. SB is partially supported by Samsung Science and Technology Foundation (SSTF-BA1401-51), by a KIAS Individual Grant (SP083201) via the Center for Mathematical Challenges at Korea Institute for Advanced Study, by the National Research Foundation of Korea (NRF-2019R1A5A1028324), and by the POSCO TJ Park Foundation (POSCO Science Fellowship). LDM is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB 1283/2 2021–317210226 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications", and the Royal Society grant RF\ERE\210237. NS acknowledges financial support from the Royal Society grant URF\R1\180707.

2. Proofs of main results

In this section and later sections, we shall assume that N = 2n, where n is a positive integer. In the proceeding, we shall mostly express our results and proofs in terms of n rather than N.

For reader's convenience, we briefly explain the overall strategy of the proof.

- (i) We first derive a determinantal formula for the generating function of the number of real eigenvalues (Proposition 2.1) that holds for any τ and n. The generating matrix appearing in Proposition 2.1 can be implemented to express the probability that there is no real eigenvalue and cumulants of the number of real eigenvalues (Proposition 2.3).
- (ii) We then derive asymptotic behaviours of trace powers of the generating matrix both in the strong (Proposition 2.4) and weak (Proposition 2.5) non-Hermiticity. Together with Proposition 2.3 (ii), these lead to Theorem 1.1.
- (iii) To complete the proof of Theorems 1.2 and 1.3, we perform required error estimates (Lemmas 2.7 and 2.8).
- 2.1. **Key ingredients.** The first step of the proofs is a determinant formula for $p_{2n,2k}$. For this purpose, recall that the k-th Hermite polynomial H_k is given by

(2.1)
$$H_k(z) := (-1)^k e^{z^2} \frac{d^k}{dz^k} e^{-z^2} = k! \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-1)^m}{m!(k-2m)!} (2z)^{k-2m}.$$

and that the (regularised) hypergeometric function is defined by the Gauss series

(2.2)
$${}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!} z^s, \quad (|z|<1)$$

and by analytic continuation elsewhere.

Proposition 2.1 (Determinantal formula for the generating function). We have

(2.3)
$$\sum_{k=0}^{n} z^{k} p_{2n,2k}^{(\tau)} = \det \left[\delta_{j,k} + (z-1) M_{n}^{(\tau)}(j,k) \right]_{j,k=1}^{n},$$

where

$$(2.4) M_n^{(\tau)}(j,k) = \frac{1}{\sqrt{2\pi}} \frac{(\tau/2)^{j+k-2}}{\sqrt{\Gamma(2j-1)\Gamma(2k-1)}} \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} H_{2j-2}\left(\frac{x}{\sqrt{2\tau}}\right) H_{2k-2}\left(\frac{x}{\sqrt{2\tau}}\right) dx \\ = \frac{1}{\sqrt{2\pi}} \left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}} \frac{\Gamma(j+k-\frac{3}{2})_2 F_1(k-j+\frac{1}{2},j-k+\frac{1}{2};-j-k+\frac{5}{2};-\frac{\tau}{1-\tau})}{\sqrt{\Gamma(2j-1)\Gamma(2k-1)}}.$$

In particular, we have

(2.5)
$$p_{2n,0}^{(\tau)} = \det\left[I - M_n^{(\tau)}\right], \qquad M_n^{(\tau)} = \left[M_n^{(\tau)}(j,k)\right]_{j,k=1}^n.$$

For $\tau=0$, a similar formula was first derived by Kanzieper and Akemann [36]. We stress that the determinantal formula (2.3) is equivalent to Proposition B.1 due to Forrester and Nagao. In general, the determinantal formula for the generating function of number of real eigenvalues follows from the skew-orthogonal polynomial formalism of the generalised partition functions [21,26]. Let us also mention that similar formulas can be found in the context of induced Ginibre, spherical Ginibre, and truncated orthogonal matrices, see e.g. [17,18,20,23,25,38,40,46] and also [9, Section 4] for a comprehensive review.

Remark 2.2 (Extremal cases $\tau = 0, 1$). For $\tau = 0$, using that ${}_2F_1(a, b; c; 0) = 1$ and that $H_k(x)$ has leading coefficient 2^k we get

(2.6)
$$\left(\frac{\tau}{2}\right)^{k/2} H_k\left(\frac{x}{\sqrt{2\tau}}\right) \to x^k, \qquad \tau \to 0,$$

and consequently

$$(2.7) M_n^{(0)}(j,k) = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(j+k-\frac{3}{2})}{\sqrt{\Gamma(2j-1)\Gamma(2k-1)}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-x}}{x^{5/2}} \frac{x^j}{\sqrt{\Gamma(2j-1)}} \frac{x^k}{\sqrt{\Gamma(2k-1)}} dx.$$

Thus one can observe that for $\tau = 0$, Proposition 2.1 corresponds to [37, Lemma 2.1], see also [22, Proposition 1]. The second expression also agrees with the integral representation in [37, Eq.(A.19)]. We mention that for $\tau = 0$, this follows from exact formulas in [3, 26, 36], cf. see [6] for more on the Pfaffian integration theorem.

On the other hand, if $\tau = 1$, it follows from the orthogonality of the Hermite polynomials

(2.8)
$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \sqrt{\pi} \, 2^n \, n! \, \delta_{nm}$$

that $M_n^{(1)}(j,k) = \delta_{jk}$. Thus one can observe that

(2.9)
$$\sum_{k=0}^{n} z^{k} p_{2n,2k}(1) = \det[zI] = z^{n}, \quad \text{i.e.} \quad p_{2n,2k} = \begin{cases} 0 & \text{if } k = 0, 1, \dots, n-1, \\ 1 & \text{if } k = n \end{cases}$$

as expected.

Proposition 2.3 (Zero probabilities and cumulants in terms of the generating matrix). For any τ and n, we have the following.

(i) For any natural number $K_n \geq 1$, we have

(2.10)
$$\log p_{2n,0}^{(\tau)} = \operatorname{Tr} \log(I - M_n^{(\tau)}) = -\sum_{m=1}^{K_n} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m - R_n(K_n),$$

where

(2.11)
$$R_n(K) = \int_0^1 \text{Tr}\left(\frac{(M_n^{(\tau)})^{K+1}}{(1 - xM_n^{(\tau)})^{K+1}}\right) (1 - x)^K dx.$$

(ii) The l-th order cumulant κ_l of $\mathscr{N}_{2n}^{(\tau)}$ is given by

(2.12)
$$\kappa_l = 2^l \sum_{m=1}^l \frac{(-1)^{m+1}}{m} \sum_{\substack{\nu_1 + \dots + \nu_m = l \\ \nu_i \ge 1}} \frac{l!}{\nu_1! \dots \nu_m!} \operatorname{Tr}(M_n^{(\tau)})^m.$$

Proof. The first assertion is an immediate consequence of Proposition 2.1, whereas the second one was shown in [45, Lemma 2.4].

We need to show the following.

Proposition 2.4 (Asymptotics of trace powers at strong non-Hermiticity). For a fixed $\tau \in [0,1)$, and for any fixed integer m > 0,

(2.13)
$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \operatorname{Tr}(M_n^{(\tau)})^m = \sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{2\pi m}.$$

For $\tau = 0$, this proposition coincides with [37, Lemma 2.3].

Proposition 2.5 (Asymptotics of trace powers at weak non-Hermiticity). For $\tau = 1 - \alpha^2/(2n)$ with a fixed $\alpha \in (0, \infty)$, and for any fixed integer m > 0,

(2.14)
$$\lim_{n \to \infty} \frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^m = \frac{c(\sqrt{m} \alpha)}{2} = \frac{e^{-m\alpha^2/2}}{2} \left[I_0\left(\frac{m\alpha^2}{2}\right) + I_1\left(\frac{m\alpha^2}{2}\right) \right].$$

Example 2.6 (The first three cumulants). As a consequence of Proposition 2.3 (ii), for l = 1, 2, 3, we have

$$\mathbb{E}\mathscr{N}_{2n}^{(\tau)} = 2\operatorname{Tr} M_n^{(\tau)},$$

(2.16)
$$\operatorname{Var} \mathcal{N}_{2n}^{(\tau)} = 4 \Big(\operatorname{Tr} M_n^{(\tau)} - \operatorname{Tr} (M_n^{(\tau)})^2 \Big),$$

(2.17)
$$\mathbb{E}[(\mathscr{N}_{2n}^{(\tau)} - \mathbb{E}\mathscr{N}_{2n}^{(\tau)})^3] = 8\Big(\operatorname{Tr} M_n^{(\tau)} - 3\operatorname{Tr}(M_n^{(\tau)})^2 + 2\operatorname{Tr}(M_n^{(\tau)})^3\Big).$$

We emphasise that for m = 1, 2, Propositions 2.4 and 2.5 recover important results in [10, 27]. To be more precise, for m = 1, the asymptotic behaviours of the expected numbers (1.5) and (1.9) follow from (2.15), whereas for m = 2, the variance asymptotics (1.6) and (1.12) follow from (2.16).

To prove Theorem 1.2, we need a version of Proposition 2.4 that is uniform in m, rather than for fixed m. We shall need the following more refined bound.

Lemma 2.7. For any $\tau \in [0,1)$ and integers $n \ge 1$ and $m \ge \frac{1+\tau}{1-\tau}$ we have the inequality

(2.18)
$$\operatorname{Tr}(M_n^{(\tau)})^m \le \frac{1}{4} + \sqrt{\frac{1+\tau}{1-\tau}} \sqrt{\frac{n}{\pi m}} (1+2/n) + \frac{1}{8} \frac{1-\tau}{1+\tau} \sqrt{\frac{m}{\pi n}} (1+1/n).$$

Note that for $\tau = 0$, this lemma gives [37, Lemma 2.3].

Lemma 2.8 (Bounds for eigenvalues of $M_n^{(\tau)}$). Let $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ $(j = 1, 2, \ldots, n)$ be the eigenvalues of $M_n^{(\tau)}$. Then we have the following.

- (i) The matrix $M_n^{(\tau)}$ is positive definite, i.e. $\lambda_n > 0$.
- (ii) There exists $\mu > 0$ such that for sufficiently large n,

(2.19)
$$\lambda_1 \le 1 - \frac{\mu}{n^a}, \qquad a = \begin{cases} 1 & \text{if } \tau \text{ is fixed,} \\ 2 & \text{if } \tau = 1 - \frac{\alpha^2}{2n}. \end{cases}$$

Lemma 2.9. Let a be given in (2.19). Then we have

(2.20)
$$\frac{1}{(2n)^{a/2}} |R_n(K)| \le \frac{1}{\sqrt{2^a \mu}} \frac{1}{\sqrt{K+1}} \operatorname{Tr} \left((M_n^{(\tau)})^{K+1} \right).$$

Proof. By (2.11) and Lemma 2.8, it suffices to show that for $0 < \lambda < 1$,

(2.21)
$$\lambda^{K+1} \int_0^1 \frac{(1-x)^K}{(1-\lambda x)^{K+1}} dx = -\log(1-\lambda) - \sum_{j=1}^K \frac{\lambda^j}{j} \le \frac{\lambda^{K+1}}{\sqrt{2K+1}} \frac{1}{\sqrt{1-\lambda}}.$$

For this, note that

$$\lambda^{K+1} \int_0^1 \frac{(1-x)^K}{(1-\lambda x)^{K+1}} dx = \int_0^\lambda \frac{(\lambda-x)^K}{(1-x)^{K+1}} dx = \int_{1-\lambda}^1 \frac{(\lambda-1+x)^K}{x^{K+1}} dx$$

$$= \int_{1-\lambda}^1 \sum_{j=0}^K \binom{K}{j} (\lambda-1)^j x^{-j-1} dx$$

$$= -\log(1-\lambda) - \sum_{j=1}^K \frac{1}{j} \binom{K}{j} \left((\lambda-1)^j - (-1)^j \right) = -\log(1-\lambda) - \sum_{j=1}^K \frac{\lambda^j}{j}.$$

Here, the identity

$$\sum_{j=1}^{K} \frac{1}{j} {K \choose j} \left((\lambda - 1)^{j} - (-1)^{j} \right) = \sum_{j=1}^{K} \frac{\lambda^{j}}{j}$$

follows from

$$\sum_{j=1}^{K} {K \choose j} (\lambda - 1)^{j-1} = \frac{(1 + \lambda - 1)^K - 1}{\lambda - 1} = \sum_{j=1}^{K} \lambda^{j-1}.$$

The inequality follows by Cauchy-Schwarz, as follows.

$$-\log(1-\lambda) - \sum_{j=1}^{K} \frac{\lambda^{j}}{j} = \sum_{j=K+1}^{\infty} \frac{\lambda^{j}}{j} = \int_{0}^{\lambda} \frac{x^{K}}{1-x} dx$$

$$\leq \sqrt{\int_{0}^{\lambda} x^{2K} dx \int_{0}^{\lambda} \frac{dx}{(1-x)^{2}}} = \frac{\lambda^{K+1/2}}{\sqrt{2K+1}} \frac{\sqrt{\lambda}}{\sqrt{1-\lambda}}.$$

2.2. **Proofs of Theorems 1.1, 1.2, 1.3 and 1.5.** This section culminates in the proofs of our main results.

Proof of Theorem 1.1. As a consequence of Propositions 2.4 and 2.5, for any fixed m, we have that $\text{Tr}(M_n^{(\tau)})^m$ is of order $n^{a/2}$, where a is given by (2.19). Then it follows from Proposition 2.3 (ii) that all the cumulants have order $n^{a/2}$. This shows the desired central limit theorem.

Proof of Theorem 1.2. It is enough to show Theorem 1.2 for k=0:

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \log p_{2n,0}^{(\tau)} = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}} \zeta\left(\frac{3}{2}\right).$$

The statement for general k can be proved entirely analogously to [37, Section 2.2], which only needs the result of Lemma 2.8 and Proposition 2.4.

Let $m_{\tau} \geq \frac{1+\tau}{1-\tau}$ be an integer. For any positive integer $K > m_{\tau}$, we can use the inequality from Lemma 2.7 to show that

$$\frac{1}{\sqrt{2n}} \sum_{m=m_{\tau}}^{K} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m \le \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+2/n}{\sqrt{2\pi}} \sum_{m=m_{\tau}}^{K} \frac{1}{m^{3/2}} + \frac{\log K}{4\sqrt{2n}} + \frac{\sqrt{K}}{8n\sqrt{2\pi}} \frac{1-\tau}{1+\tau}.$$

Now we take $K = K_n$ such that both $n^{-1/2}K_n \to \infty$ and $K_n = \mathcal{O}(n\log^2 n)$ as $n \to \infty$. Then we obtain

$$\frac{1}{\sqrt{2n}} \sum_{m=m_{\tau}}^{K_n} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m \le \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+2/n}{\sqrt{2\pi}} \sum_{m=m_{\tau}}^{K} \frac{1}{m^{3/2}} + C \frac{\log n}{\sqrt{n}}$$

for some constant C > 0. By Lemma 2.9 the remainder satisfies

$$\frac{1}{\sqrt{2n}}|R_n(K_n)| \le c\frac{\sqrt{n}}{K_n}$$

for some constant c > 0. We infer that

$$-\frac{1}{\sqrt{2n}}\log p_{2n,0}^{(\tau)} \le \sum_{m=1}^{m_{\tau}-1} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m + \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+2/n}{\sqrt{2\pi}} \sum_{m=m_{\tau}}^{K_n} \frac{1}{m^{3/2}} + C\frac{\log n}{\sqrt{n}} + c\frac{\sqrt{n}}{K_n}.$$

For any fixed number K, we have

$$-\frac{1}{\sqrt{2n}}\log p_{2n,0}^{(\tau)} \ge \frac{1}{\sqrt{2n}} \sum_{m=1}^{K} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m.$$

Taking the limit $n \to \infty$, and using Proposition 2.4, we find that

$$\sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}} \sum_{m=1}^{K} \frac{1}{m^{3/2}} \le -\lim_{n\to\infty} \frac{1}{\sqrt{2n}} \log p_{2n,0}^{(\tau)} \le \sqrt{\frac{1+\tau}{1-\tau}} \frac{\zeta(3/2)}{\sqrt{2\pi}}.$$

Since this is true for any positive integer K, we conclude that

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \log p_{2n,0}^{(\tau)} = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\zeta(3/2)}{\sqrt{2\pi}}.$$

Proof of Theorem 1.3. For k=0, this immediately follows from Lemma 2.8 and Proposition 2.5. Furthermore, the statement for general k can be again proved along the same lines to [37, Section 2.2], which only needs the result of Lemma 2.8 and Proposition 2.5.

Proof of Theorem 1.5. We already proved the statement for x=0 in Theorem 1.2. By Lemma 2.7, for every $x \in (0,1]$, we have for $m_{\tau} \geq \frac{1+\tau}{1-\tau}$ that

$$\frac{1}{\sqrt{2n}} \sum_{m=m_{\tau}}^{\infty} \text{Tr}(M_{n}^{(\tau)})^{m} \frac{(1-x)^{m}}{m}
(2.22) \qquad \leq \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+2/n}{\sqrt{2\pi}} \sum_{m=m_{\tau}}^{\infty} \frac{(1-x)^{m}}{m^{3/2}} + \frac{1}{4\sqrt{2n}} \sum_{m=1}^{\infty} \frac{(1-x)^{m}}{m} + \frac{1-\tau}{1+\tau} \frac{1+1/n}{8n\sqrt{2\pi}} \sum_{m=1}^{\infty} \frac{(1-x)^{m}}{\sqrt{m}}
= \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+2/n}{\sqrt{2\pi}} \sum_{m=m_{\tau}}^{\infty} \frac{(1-x)^{m}}{m^{3/2}} + \frac{\log 1/x}{4\sqrt{2n}} + \sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{1/2}(1-x)}{8n\sqrt{2\pi}} (1+1/n).$$

Hence for every integer K > 0 we have

$$\sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}} \sum_{m=1}^K \frac{(1-x)^m}{m^{3/2}} \le -\lim_{n\to\infty} \log \left(\sum_{k=0}^{2n} p_{2n,2k}^{(\tau)} x^{2k} \right) \le \sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}(1-x)}{\sqrt{2\pi}}.$$

Since this is true for any K, we conclude that for any $x \in (0,1]$

$$\lim_{n \to \infty} \log \left(\sum_{k=0}^{2n} p_{2n,2k}^{(\tau)} x^{2k} \right) = -\sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}(1-x)}{\sqrt{2\pi}}.$$

Furthermore, we notice that for all $x \in [0, 1]$

$$\sum_{m=K+1}^{\infty} \frac{(1-x)^m}{m^{3/2}} \le \int_K^{\infty} \frac{(1-x)^m}{m^{3/2}} dm = \frac{(1-x)^K}{\sqrt{K}} + \sqrt{\pi} \sqrt{-\log(1-x)} \operatorname{erfc}(\sqrt{-K\log(1-x)}),$$

where we read the expression on the right-hand side as a limit (which is 0) when x = 1. This, combined with (2.22), gives a uniform bound for all $x \in [r, 1]$ for any fixed r > 0.

Let us now turn to the case $x \in [1, 2]$. We can write

$$\begin{split} \sum_{m=1}^{\infty} \mathrm{Tr}(M_n^{(\tau)})^m \frac{(1-x)^m}{m} &= \sum_{m=1}^{\infty} (-1)^m \, \mathrm{Tr}(M_n^{(\tau)})^m \frac{|1-x|^m}{m} \\ &= \sum_{m=1}^{\infty} \mathrm{Tr}(M_n^{(\tau)})^m \frac{|1-x|^m}{m} - 2 \sum_{m=1}^{\infty} \mathrm{Tr}(M_n^{(\tau)})^{2m} \frac{|1-x|^{2m}}{2m}. \end{split}$$

Both series in the last line can be treated with the same arguments that we used for the $x \in (0,1]$ case, yielding

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \sum_{m=1}^{\infty} \text{Tr}(M_n^{(\tau)})^m \frac{(1-x)^m}{m} = \sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi}} \left(\text{Li}_{3/2}(|1-x|) - \frac{1}{\sqrt{2}} \text{Li}_{3/2}(|1-x|^2) \right)$$
$$= \sqrt{\frac{1+\tau}{1-\tau}} \frac{\text{Li}_{3/2}(1-x)}{\sqrt{2\pi}}.$$

The uniform convergence follows along similar lines.

3. Analysis of the generating function

The focus of this section is on examining the generating function for the number of real eigenvalues.

3.1. **Proof of Proposition 2.1.** We first prove Proposition 2.1.

Proof of Proposition 2.1. Let us write

$$\langle f, g \rangle := \langle f, g \rangle_{\mathbb{R}} + \langle f, g \rangle_{\mathbb{C}},$$

where

$$(3.2) \quad \langle f, g \rangle_{\mathbb{R}} := \frac{1}{2} \int_{\mathbb{R}^2} dx \, dy \, e^{-\frac{x^2 + y^2}{2(1+\tau)}} \operatorname{sgn}(y - x) f(x) g(y),$$

$$(3.3) \quad \langle f, g \rangle_{\mathbb{C}} := i \int_{\mathbb{R}} dx \int_{0}^{\infty} dy \, e^{\frac{y^{2} - x^{2}}{1 + \tau}} \operatorname{erfc}\left(\sqrt{\frac{2}{1 - \tau^{2}}y}\right) [f(x + iy)g(x - iy) - g(x + iy)f(x - iy)].$$

In terms of the scaled monic Hermite polynomials

(3.4)
$$C_k(z) := \left(\frac{\tau}{2}\right)^{k/2} H_k\left(\frac{z}{\sqrt{2\tau}}\right),$$

we define

(3.5)
$$q_{2j}(x) := C_{2j}(x), \qquad q_{2j+1}(x) := C_{2j+1}(x) - 2jC_{2j-1}(x).$$

Then by [27, Theorem 1], $\{q_j\}$ forms a family of monic skew-orthogonal polynomials with respect to (3.1). Here, the skew-norm r_j is given by

(3.6)
$$r_j := \langle q_{2j}, q_{2j+1} \rangle = \sqrt{2\pi} (1+\tau) \Gamma(2j+1).$$

We also write

$$\mathbf{A}_{j,k} = \langle q_{j-1}, q_{k-1} \rangle_{\mathbb{R}}.$$

Let

(3.8)
$$g(z) \equiv g_{2n}^{(\tau)}(z) := \sum_{k=0}^{n} z^{k} p_{2n,2k}^{(\tau)}.$$

Then along the lines of the proof of [37, Lemma 2.1], it follows that

(3.9)
$$g_{2n}^{(\tau)}(z) = \det \left[\delta_{jk} + \frac{z-1}{\sqrt{r_{j-1}r_{k-1}}} \mathbf{A}_{2j-1,2k} \right]_{j,k=1}^{n}.$$

Therefore it suffices to evaluate (3.7). For this purpose, let

(3.10)
$$I_{j,k} = \int_{\mathbb{R}^2} dx \, dy \, e^{-\frac{x^2 + y^2}{2(1+\tau)}} C_{2j+1}(x) C_{2k}(y) \operatorname{sgn}(y-x).$$

Then by (3.5), $\mathbf{A}_{2i-1,2k}$ can be written in terms of $I_{i,k}$ as

(3.11)
$$\mathbf{A}_{2j-1,2k} = -\frac{1}{2} \Big(I_{k-1,j-1} - 2(k-1)I_{k-2,j-1} \Big).$$

On the other hand, by [27, Eq.(5.9)], $I_{j,k}$ satisfies the recurrence relation

$$(3.12) I_{j+1,k} = (2j+2)I_{j,k} - 2\xi_{j,k}, \xi_{j,k} = (1+\tau) \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} C_{2j+2}(x) C_{2k}(x) dx.$$

Combining (3.11) and (3.12), we have

$$\mathbf{A}_{2j-1,2k} = \xi_{k-2,j-1}.$$

Now it remains to evaluate $I_{j,k}$. For this, we use the following integration formula that can be found in the proof of [10, Lemma 5.2]: for j + k even,

(3.14)
$$\int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} H_j\left(\frac{x}{\sqrt{2\tau}}\right) H_k\left(\frac{x}{\sqrt{2\tau}}\right) dx$$

$$= \left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}} \left(\frac{\tau}{2}\right)^{-\frac{j+k}{2}} \Gamma\left(\frac{j+k+1}{2}\right)_2 F_1\left(\frac{j-k+1}{2}, \frac{k-j+1}{2}; \frac{1-j-k}{2}; -\frac{\tau}{1-\tau}\right).$$

Then by (3.4), we have

(3.15)
$$\xi_{j,k} = (1+\tau) \left(\frac{\tau}{2}\right)^{j+k+1} \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} H_{2j+2} \left(\frac{x}{\sqrt{2\tau}}\right) H_{2k} \left(\frac{x}{\sqrt{2\tau}}\right) dx \\ = (1+\tau) \left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}} \Gamma\left(j+k+\frac{3}{2}\right) {}_{2}F_{1} \left(j-k+\frac{3}{2},k-j-\frac{1}{2};-j-k-\frac{1}{2};-\frac{\tau}{1-\tau}\right).$$

Then it follows from (3.13) and (3.15) that

(3.16)
$$\mathbf{A}_{2j-1,2k} = (1+\tau) \left(\frac{\tau}{2}\right)^{j+k-2} \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} H_{2j-2} \left(\frac{x}{\sqrt{2\tau}}\right) H_{2k-2} \left(\frac{x}{\sqrt{2\tau}}\right) dx \\ = (1+\tau) \left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}} \Gamma\left(j+k-\frac{3}{2}\right)_2 F_1 \left(k-j+\frac{1}{2},j-k+\frac{1}{2};-j-k+\frac{5}{2};-\frac{\tau}{1-\tau}\right).$$

Combining (3.9), (3.6) and (3.16), we obtain the desired identity (2.4), where the second expression follows from the reflection formula of the Gamma function

(3.17)
$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z).$$

3.2. Evaluations of trace powers. In this subsection, we derive two different expressions of $\text{Tr}(M_n^{(\tau)})^m$ in Lemmas 3.1 and 3.5.

Lemma 3.1. We have

(3.18)
$$\operatorname{Tr}(M_n^{(\tau)})^m = \int_{\mathbb{R}^m} K_n^{(\tau)}(x_1, x_2) K_n^{(\tau)}(x_2, x_3) \cdots K_n^{(\tau)}(x_m, x_1) dx_1 \cdots dx_m,$$

where

(3.19)
$$K_n^{(\tau)}(x,y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 + y^2}{2(1+\tau)}} \sum_{j=0}^{n-1} \frac{(\tau/2)^{2j}}{(2j)!} H_{2j}\left(\frac{x}{\sqrt{2\tau}}\right) H_{2j}\left(\frac{y}{\sqrt{2\tau}}\right).$$

Proof. By definition,

(3.20)
$$\operatorname{Tr}(M_n^{(\tau)})^m = \sum_{j_1, j_2, \dots, j_m = 1}^n M_n^{(\tau)}(j_1, j_2) M_n^{(\tau)}(j_2, j_3) \cdots M_n^{(\tau)}(j_{m-1}, j_m) M_n^{(\tau)}(j_m, j_1).$$

Then the expression (3.19) follows by plugging in the first line of (2.4).

Remark 3.2. The kernel $K_n^{(\tau)}$ can also be written in terms of the Laguerre polynomials

(3.21)
$$L_j^{\nu}(z) := \sum_{k=0}^{J} \frac{\Gamma(j+\nu+1)}{(j-k)! \Gamma(\nu+k+1)} \frac{(-z)^k}{k!}$$

as

(3.22)
$$K_n^{(\tau)}(x,y) = \frac{1}{\sqrt{2}} e^{-\frac{x^2 + y^2}{2(1+\tau)}} \sum_{j=0}^{n-1} \frac{\tau^{2j} j!}{\Gamma(j+\frac{1}{2})} L_j^{-1/2} \left(\frac{x^2}{2\tau}\right) L_j^{-1/2} \left(\frac{y^2}{2\tau}\right).$$

This follows from the relation

(3.23)
$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2)$$

and the duplication formula of the gamma function:

(3.24)
$$\Gamma(2z+1) = \frac{1}{\sqrt{\pi}} 2^{2z} \Gamma(z+\frac{1}{2}) \Gamma(z+1).$$

Remark 3.3. Let $T_n^{(\tau)}$ be defined as the operator

$$f \mapsto T_n^{(\tau)}(f) := \int_{\mathbb{R}} f(y) K_n^{(\tau)}(x, y) dy.$$

Then we have

$$-\log p_{2n,0}^{(\tau)} = \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{Tr}(M_n^{(\tau)})^m = \sum_{m=1}^{\infty} \frac{1}{m} \int_{\mathbb{R}^m} K_n^{(\tau)}(x_1, x_2) \cdots K_n^{(\tau)}(x_m, x_1) \prod_{k=1}^m dx_k$$
$$= \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{Tr}\left((T_n^{(\tau)})^m\right) = -\log \det(1 - T_n^{(\tau)}),$$

where the determinant in the last line is the Fredholm determinant,

$$\det(1 - T_n^{(\tau)}) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \det\left(K_n^{(\tau)}(x_j, x_k)\right)_{1 \le j, k \le m} \prod_{k=1}^m dx_k.$$

Example 3.4. Note that by (2.6),

(3.25)
$$K_n^{(0)}(x,y) = \frac{e^{-\frac{x^2+y^2}{2}}}{\sqrt{2\pi}} \sum_{j=0}^{n-1} \frac{(xy)^{2j}}{(2j)!} = \frac{e^{-\frac{x^2+y^2}{2}}}{\sqrt{2\pi}} \cosh_{n-1}(xy),$$

where $\cosh_n(x) = \sum_{j=0}^n x^{2j}/(2j)!$. Therefore

$$\operatorname{Tr}(M_n^{(\tau)})^m = \int_{\mathbb{R}^m} \frac{e^{-x_1^2 - x_2^2 - \dots - x_m^2}}{(2\pi)^{m/2}} \cosh_{n-1}(x_1 x_2) \cosh_{n-1}(x_2 x_3) \dots \cosh_{n-1}(x_m x_1) dx_1 \dots dx_m$$

$$= \int_0^\infty \frac{dx_1}{\sqrt{2\pi x_1}} \int_0^\infty \frac{dx_2}{\sqrt{2\pi x_2}} \dots \int_0^\infty \frac{dx_m}{\sqrt{2\pi x_m}} e^{-x_1 - \dots - x_m} \cosh_{n-1}(\sqrt{x_1 x_2}) \dots \cosh_{n-1}(\sqrt{x_m x_1}),$$

which corresponds to the integral representation in [37, Eq.(A.23)].

Next, we show the following.

Lemma 3.5. Let $j_1, j_2, \ldots, j_m = j_0 \in \mathbb{N}$. Then for any $n, m \in \mathbb{N}$, we have

$$(3.26) \qquad \operatorname{Tr}(M_n^{(\tau)})^m = \sum_{j_1,\dots,j_m=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{\frac{m}{2}+2\sum_{k=1}^m j_k} \prod_{k=1}^m \left(\sum_{l=0}^{j_{k-1}} \frac{(1-\tau)^{2l}(1+\tau)^{-2l}(2j_k)!}{2^{2l}l!(l+j_k-j_{k-1})!(2j_{k-1}-2l)!}\right).$$

Remark 3.6. For given l in the inner summation, it suffices to consider the summands with

$$(3.27) j_k \ge l, j_k \ge j_{k-1} - l, k = 1, 2, \dots, m.$$

In particular, if l = 0, this gives $j_1 = j_2 = \cdots = j_m$.

Remark 3.7. Recall that for $\tau = 1$, we have $M_n^{(1)} = I$. Thus $\text{Tr}(M_n^{(1)})^m = n$. This can be checked using the identity (3.26); namely, for $\tau = 1$, it reads

$$(3.28) \operatorname{Tr}(M_n^{(1)})^m = \sum_{j_1,\dots,j_m=0}^{n-1} \prod_{k=1}^m \frac{(2j_k)!}{(j_k - j_{k-1})!(2j_{k-1})!} = \sum_{j_1,\dots,j_m=0}^{n-1} \prod_{k=1}^m \frac{1}{(j_k - j_{k-1})!} = \sum_{j=0}^{n-1} \prod_{k=1}^n \frac{1}{0!} = n.$$

Proof of Lemma 3.5. Using the contour integral representation

(3.29)
$$H_k(x) = \frac{k!}{2\pi i} \oint \frac{e^{2\zeta x - \zeta^2}}{\zeta^{k+1}} d\zeta,$$

and (3.19), we have

(3.30)
$$K_n^{(\tau)}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 + y^2}{2(1+\tau)}} \sum_{j=0}^{n-1} (2j)! \left(\frac{\tau}{2}\right)^{2j} \underset{\zeta,\eta=0}{\text{Res}} \left[\frac{e^{2\zeta \frac{x}{\sqrt{2\tau}} + 2\eta \frac{y}{\sqrt{2\tau}} - \zeta^2 - \eta^2}}{(\zeta\eta)^{2j+1}} \right].$$

Here and in the sequel, we shall use the shorthand notation

$$\operatorname{Res}_{\zeta_1,\dots,\zeta_k=0}[f(\zeta,\eta)] := \operatorname{Res}_{\zeta_1=0}[\cdots[\operatorname{Res}_{\zeta_k=0}[f(\zeta_1,\dots,\zeta_k)]].$$

Thus we obtain

$$K_n^{(\tau)}(x_1, x_2) K_n^{(\tau)}(x_2, x_3) \cdots K_n^{(\tau)}(x_m, x_1)$$

$$= \frac{1}{(2\pi)^{m/2}} e^{-\frac{x_1^2 + \dots x_m^2}{1 + \tau}} \sum_{j_1, \dots, j_m = 0}^{n-1} (2j_1)! \dots (2j_m)! \left(\frac{\tau}{2}\right)^{2(j_1 + \dots + j_m)}$$

$$\times \operatorname{Res}_{\substack{\zeta_k, \eta_k = 0 \\ k = 1, \dots, m}} \left[\prod_{k=1}^m \exp\left(\sqrt{\frac{2}{\tau}} (\zeta_k + \eta_k) x_k - \zeta_k^2 - \eta_k^2\right) \frac{1}{\zeta_k^{2j_{k-1} + 1} \eta_k^{2j_k + 1}} \right].$$

Since

(3.31)
$$\int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau} + \sqrt{\frac{2}{\tau}}(\zeta + \eta)x} dx = \sqrt{\pi (1+\tau)} \exp\left(\frac{1+\tau}{2\tau} (\zeta + \eta)^2\right),$$

it follows from Lemma 3.1 that

(3.32)
$$\operatorname{Tr}(M_n^{(\tau)})^m = \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} (2j_1)! \dots (2j_m)! (\tau/2)^{2(j_1+\dots+j_m)} \times \operatorname{Res}_{\substack{\zeta_k,\eta_k=0\\k=1,\dots,m}} \left[\prod_{k=1}^m \exp\left(\frac{1+\tau}{2\tau}(\zeta_k+\eta_k)^2 - \zeta_k^2 - \eta_k^2\right) \frac{1}{\zeta_k^{2j_{k-1}+1}\eta_k^{2j_k+1}}\right].$$

Note that

$$\operatorname{Res}_{\substack{\zeta_k, \eta_k = 0 \\ k = 1, \dots, m}} \left[\prod_{k=1}^m \exp\left(\frac{1+\tau}{2\tau} (\zeta_k + \eta_k)^2 - \zeta_k^2 - \eta_k^2\right) \frac{1}{\zeta_k^{2j_{k-1}+1} \eta_k^{2j_k+1}} \right]$$

$$= \prod_{k=1}^m \operatorname{Res}_{\zeta_k, \eta_k = 0} \left[\exp\left(\frac{1+\tau}{2\tau} (\zeta_k + \eta_k)^2 - \zeta_k^2 - \eta_k^2\right) \frac{1}{\zeta_k^{2j_{k-1}+1} \eta_k^{2j_k+1}} \right].$$

Since

$$\exp\left(\frac{1+\tau}{2\tau}(\zeta+\eta)^{2}-\zeta^{2}-\eta^{2}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1-\tau}{2\tau}(\zeta^{2}+\eta^{2}) + \frac{1+\tau}{2\tau}2\zeta\eta\right)^{k}$$

we have, applying Newton's binomial formula twice, that (only k = p + q can survive)

$$\operatorname{Res}_{\zeta,\eta=0} \left[\exp\left(\frac{1+\tau}{2\tau}(\zeta+\eta)^2 - \zeta^2 - \eta^2\right) \frac{1}{\zeta^{2p+1}\eta^{2q+1}} \right] \\
= \frac{1}{(p+q)!} \sum_{l=0}^{p} \left(\frac{1-\tau}{2\tau}\right)^{2l+q-p} \left(\frac{1+\tau}{2\tau}\right)^{2p-2l} 2^{2p-2l} \binom{p+q}{2l-p+q} \binom{2l-p+q}{l} \\
= \frac{1}{(p+q)!} \sum_{l=0}^{p} \left(\frac{1-\tau}{2\tau}\right)^{2l+q-p} \left(\frac{1+\tau}{2\tau}\right)^{2p-2l} \frac{2^{2p-2l}(p+q)!}{l!(l+q-p)!(2p-2l)!} \\
= \frac{2^{p-q}}{\tau^{p+q}} \sum_{l=0}^{p} \frac{(1-\tau)^{2l+q-p}(1+\tau)^{2p-2l}}{2^{2l}l!(l+q-p)!(2p-2l)!} =: f(p,q).$$

This gives

$$\operatorname{Res}_{\substack{\zeta_k, \eta_k = 0 \\ k = 1, \dots, m}} \left[\prod_{k=1}^m \exp\left(\frac{1+\tau}{2\tau} (\zeta_k + \eta_k)^2 - \zeta_k^2 - \eta_k^2\right) \frac{1}{\zeta_k^{2j_{k-1}+1} \eta_k^{2j_k+1}} \right] = \prod_{k=1}^m f(j_{k-1}, j_k)$$

Therefore we obtain

$$\operatorname{Tr}(M_n^{(\tau)})^m = \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} (2j_1)! \dots (2j_m)! \left(\frac{\tau}{2}\right)^{2(j_1+\dots+j_m)} \prod_{k=1}^m f(j_{k-1},j_k)$$

$$= \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} (2j_1)! \dots (2j_m)! \left(\frac{1}{2}\right)^{2(j_1+\dots+j_m)} \prod_{k=1}^m \left(\sum_{l=0}^{j_{k-1}} \frac{(1-\tau)^{2l+j_k-j_{k-1}}(1+\tau)^{2j_{k-1}-2l}}{2^{2l}l!(l+j_k-j_{k-1})!(2j_{k-1}-2l)!}\right),$$

which gives the lemma.

3.3. Estimates of trace powers.

Lemma 3.8. We have

$$(3.33) \operatorname{Tr}(M_n^{(\tau)})^m \leq \frac{1}{(2\pi)^{\frac{m-1}{2}}} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(x_1^2 + x_m^2)} \mathscr{K}_n^{(\tau)}(x_m, x_1) e^{-\sum_{j=2}^{m-1} x_j^2} \prod_{j=2}^m \cosh\left(x_{j-1} x_j\right) dx_1 \cdots dx_m,$$

where

(3.34)
$$\mathscr{K}_{n}^{(\tau)}(x,y) := \sqrt{1-\tau^{2}}K_{n}^{(\tau)}(\sqrt{1-\tau^{2}}\,x,\sqrt{1-\tau^{2}}\,y).$$

Here, $K_n^{(\tau)}$ is given by (3.19).

Proof. Recall the multiplication theorem for Hermite polynomials

(3.35)
$$H_{2n}(\lambda x) = \sum_{k=0}^{n} \frac{(2n)!}{(2k)!(n-k)!} \lambda^{2k} (\lambda^2 - 1)^{n-k} H_{2k}(x),$$

see e.g. [44, Eq.(18.18.13)]. Letting

$$e_{\tau} = \frac{1+\tau}{\sqrt{1-\tau^2}},$$

we have

$$H_{2j}\left(\sqrt{\frac{1-\tau^2}{2\tau}}x\right) = H_{2j}\left(\frac{e_{\tau}}{\sqrt{2\tau}}(1-\tau)x\right) = \sum_{k=0}^{j} \left(\frac{e_{\tau}}{\sqrt{\tau}}\right)^{2k} \left(\frac{e_{\tau}^2}{\tau}-1\right)^{j-k} \frac{(2j)!}{(2k)!(j-k)!} H_{2k}\left(\frac{1-\tau}{\sqrt{2}}x\right).$$

Note that

$$\sum_{m=0}^{n-1} \frac{(\tau/2)^{2m}}{(2m)!} H_{2m} \left(\sqrt{\frac{1-\tau^2}{2\tau}} x \right) H_{2m} \left(\sqrt{\frac{1-\tau^2}{2\tau}} y \right)
= \sum_{m=0}^{n-1} \sum_{j,k=0}^{m} \left(\frac{1}{1-\tau e_{\tau}^{-2}} \right)^{j+k} \left(\frac{e_{\tau}^2}{\tau} - 1 \right)^{2m} \frac{(\tau/2)^{2m} (2m)!}{(2k)!(m-k)!(2j)!(m-j)!} H_{2k} \left(\frac{1-\tau}{\sqrt{2}} x \right) H_{2j} \left(\frac{1-\tau}{\sqrt{2}} y \right).$$

We can thus write the kernel as a combination of Hermite functions

$$\mathcal{K}_{n}^{(\tau)}(x,y) = \sqrt{\frac{1-\tau^{2}}{2\pi}}e^{-\frac{1-\tau}{2}(x^{2}+y^{2})} \sum_{j=0}^{n-1} \frac{(\tau/2)^{2j}}{(2j)!} H_{2j}\left(\sqrt{\frac{1-\tau^{2}}{2\tau}}x\right) H_{2j}\left(\sqrt{\frac{1-\tau^{2}}{2\tau}}y\right) \\
= \sum_{j,k=0}^{n-1} e^{-\frac{1-\tau}{2}x^{2}} H_{2j}\left(\frac{1-\tau}{\sqrt{2}}x\right) \hat{M}_{n,jk} e^{-\frac{1-\tau}{2}y^{2}} H_{2k}\left(\frac{1-\tau}{\sqrt{2}}y\right)$$

for some matrix \hat{M}_n that has positive elements. In fact, we see that

$$(3.37) \qquad \hat{M}_{n,jk} = \sqrt{\frac{1-\tau^2}{2\pi}} \left(\frac{1}{1-\tau e_{\tau}^{-2}}\right)^{j+k} \sum_{m=\max(j,k)}^{n-1} \frac{(\tau/2)^{2m}(2m)!}{(2k)!(m-k)!(2j)!(m-j)!} \left(\frac{e_{\tau}^2}{\tau} - 1\right)^{2m}.$$

By Lemma 3.1 and the change of variables, we have

(3.38)
$$\operatorname{Tr}(M_n^{(\tau)})^m = \int_{\mathbb{R}^m} \mathscr{K}_n^{(\tau)}(x_1, x_2) \mathscr{K}_n^{(\tau)}(x_2, x_3) \cdots \mathscr{K}_n^{(\tau)}(x_m, x_1) dx_1 \cdots dx_m,$$

Since the Hermite functions are orthonormal, we infer that $\text{Tr}(M_n^{(\tau)})^m$ is a sum of products of elements of \hat{M}_n (which are positive).

Now suppose that we add an extra term

$$\frac{(\tau/2)^{2n}}{(2n)!}H_{2n}\left(\sqrt{\frac{1-\tau^2}{2\tau}}x_1\right)H_{2n}\left(\sqrt{\frac{1-\tau^2}{2\tau}}x_2\right)$$

to $\mathscr{K}_{n}^{(\tau)}(x_{1},x_{2})$. The corresponding matrix \hat{M}_{n+1} then has elements that are greater than or equal to their counterparts of \hat{M}_{n} . Furthermore, it has 2n-1 elements more, which are all positive. Hence the sum over products of elements of \hat{M}_{n+1} and \hat{M}_{n} has increased overall. That is, $\text{Tr}(M_{n}^{(\tau)})^{m}$ has increased. We can repeat this argument inductively and extend the summation in the kernel (3.19) over all non-negative integers j. Recall here that the Mehler kernel formula is given by

(3.39)
$$\sum_{j=0}^{\infty} \frac{(\tau/2)^j}{j!} H_j(x) H_j(y) = \frac{1}{\sqrt{1-\tau^2}} \exp\left(\frac{2\tau}{1-\tau^2} xy - \frac{\tau^2}{1-\tau^2} (x^2+y^2)\right),$$

which leads to

$$\sum_{j=0}^{\infty} \frac{(\tau/2)^{2j}}{(2j)!} H_{2j}(x) H_{2j}(y) = \frac{1}{2} \Big(\sum_{j=0}^{\infty} \frac{(\tau/2)^j}{j!} H_j(x) H_j(y) + \sum_{j=0}^{\infty} \frac{(\tau/2)^j}{j!} H_j(-x) H_j(y) \Big)$$
$$= \frac{1}{\sqrt{1-\tau^2}} e^{-\frac{\tau^2}{1-\tau^2}(x^2+y^2)} \cosh\left(\frac{2\tau}{1-\tau^2}xy\right).$$

Using this, the infinite sum evaluates to

$$\mathscr{K}_{\infty}^{(\tau)}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 + y^2)} \cosh(xy).$$

Replacing m-1 of the m kernels by this expression, we arrive at the result.

Lemma 3.9. We have

$$(3.40) \qquad \operatorname{Tr}(M_n^{(\tau)})^m \le \frac{\sqrt{1+\tau}}{2\pi} \oint_{\gamma} \frac{s^{-2n+1}}{\sqrt{(1-s)((m-1-(m+1)\tau)s+(m+1)-(m-1)\tau)}} \frac{ds}{1-s^2},$$

where γ is a small loop around 0 with positive direction.

Remark 3.10. For the case $\tau = 0$, the inequality (3.40) agrees with [37, Eq.(A.31)].

Proof of Lemma 3.9. We start with the inequality from Lemma 3.8. We use Lemma A.2 for the integrations over x_2, \ldots, x_{m-1} . This yields

$$\operatorname{Tr}(M_n^{(\tau)})^m \leq \frac{1}{\sqrt{2\pi(m-1)}} \int_{\mathbb{R}^2} \mathscr{K}_n^{(\tau)}(x_m, x_1) e^{-\frac{1}{2(m-1)}(x_1^2 + x_m^2)} \cosh\left(\frac{x_1 x_m}{m-1}\right) dx_1 dx_m$$

$$= \frac{1}{\sqrt{2\pi(m-1)}} \int_{\mathbb{R}^2} \mathscr{K}_n^{(\tau)}(x_m, x_1) e^{-\frac{1}{2(m-1)}(x_1 - x_m)^2} dx_1 dx_m.$$

Here, we have used the symmetry $K_n^{(\tau)}(x_m, x_1) = K_n^{(\tau)}(-x_m, x_1)$. Then we plug in the single integral representation for the kernel from [2, Eq.(26)], which, adapted to only yield even indexed terms, takes the form

$$(3.42) \quad \mathscr{K}_{n}^{(\tau)}(x,y) = \sqrt{\frac{1-\tau^{2}}{2\pi}}e^{-\frac{1-\tau}{2}(x^{2}+y^{2})}\frac{1}{2\pi i}\oint_{\gamma}e^{\frac{1-\tau^{2}}{4\tau}\frac{s}{1+s}(x+y)^{2}-\frac{1-\tau^{2}}{4\tau}\frac{s}{1-s}(x-y)^{2}}\frac{(\tau/s)^{2n}}{(\tau/s)^{2}-1}\frac{ds}{s\sqrt{1-s^{2}}}.$$

Interchanging the order of integration, we have

$$\frac{1}{\sqrt{2\pi(m-1)}} \int_{\mathbb{R}^2} \mathscr{K}_n^{(\tau)}(x_m, x_1) e^{-\frac{1}{2(m-1)}(x_1 - x_m)^2} dx_1 dx_m$$

$$= \sqrt{\frac{1-\tau^2}{m-1}} \frac{1}{4\pi^2 i} \oint_{\gamma} \int_{\mathbb{R}^2} e^{-\frac{1-\tau}{2}(x^2 + y^2) + \frac{1-\tau^2}{2\tau} \frac{s}{1+s} x^2 - (\frac{1-\tau^2}{2\tau} \frac{s}{1-s} + \frac{1}{m-1})y^2} dx dy \frac{(\tau/s)^{2n}}{(\tau/s)^2 - 1} \frac{ds}{s\sqrt{1-s^2}},$$

where now we use the light-cone coordinates:

$$x = \frac{x_1 + x_m}{\sqrt{2}}, \qquad y = \frac{x_1 - x_m}{\sqrt{2}}.$$

Since

$$\begin{split} & \int_{\mathbb{R}^2} e^{-\frac{1-\tau}{2}(x^2+y^2) + \frac{1-\tau^2}{2\tau} \frac{s}{1+s} x^2 - (\frac{1-\tau^2}{2\tau} \frac{s}{1-s} + \frac{1}{m-1}) y^2} \, dx \, dy \\ &= \pi \bigg(-\frac{1-\tau}{2} + \frac{1-\tau^2}{2\tau} \frac{s}{1+s} \bigg)^{-\frac{1}{2}} \bigg(-\frac{1-\tau}{2} - \frac{1-\tau^2}{2\tau} \frac{s}{1-s} - \frac{1}{m-1} \bigg)^{-\frac{1}{2}} \\ &= 2\pi \sqrt{\frac{1-s^2}{1-\tau} (m-1)} \bigg(\frac{1}{s/\tau - 1} \frac{1}{(1+m+\tau - m\tau) - s(1-m+\tau m+\tau)/\tau} \bigg)^{\frac{1}{2}}, \end{split}$$

we have

$$\begin{split} &\sqrt{\frac{1-\tau^2}{m-1}}\frac{1}{4\pi^2i}\int_{\mathbb{R}^2}e^{-\frac{1-\tau}{2}(x^2+y^2)+\frac{1-\tau^2}{2\tau}\frac{s}{1+s}x^2-(\frac{1-\tau^2}{2\tau}\frac{s}{1-s}+\frac{1}{m-1})y^2}\,dx\,dy\frac{(\tau/s)^{2n}}{(\tau/s)^2-1}\frac{1}{s\sqrt{1-s^2}}\\ &=\frac{\sqrt{1+\tau}}{2\pi i}\Big(\frac{1}{s/\tau-1}\frac{1}{m+1-\tau(m-1)-s(m+1-(m-1)\tau^{-1})}\Big)^{\frac{1}{2}}\frac{(\tau/s)^{2n}s}{\tau^2-s^2}. \end{split}$$

This yields

$$\operatorname{Tr}(M_n^{(\tau)})^m \leq \frac{\sqrt{1+\tau}}{2\pi} \oint_{\mathbb{R}^n} \frac{1}{\sqrt{(1-s/\tau)(m+1-\tau(m-1)-(m+1-\tau^{-1}(m-1))s)}} \frac{(\tau/s)^{2n}s}{\tau^2-s^2} \, ds.$$

We arrive at the result after a substitution $s \to \tau s$.

3.4. **Proofs of Lemmas 2.7 and 2.8.** We now prove Lemma 2.7. Our proof is similar to the argument given in [37, Appendix A.3].

Proof of Lemma 2.7. Our starting point is the estimate (3.40). Recall that γ in (3.40) is a small loop around 0 with positive direction. The integrand on the RHS in (3.40) has three singularities, 1, -1 and a singularity that we will denote by

$$a = -\frac{m+1 - (m-1)\tau}{m-1 - (m+1)\tau}.$$

First, we consider the case that $\frac{m-1}{m+1} > \tau$. In that case a < -1. Then we deform γ to a band L_- around $(-\infty, a)$, a band L_+ around $(1, \infty)$, connected by two (almost) semicircles (and a small circle

around the pole s = -1). The integrals over the semicircles tend to 0 as we increase the radius to ∞ . The residue at s = -1 gives a contribution 1/4. The integral over L_- gives (without the prefactor)

$$\begin{vmatrix}
2 \int_{-\infty}^{a} \frac{s^{-2n+1}}{\sqrt{(1-s)((m-1-(m+1)\tau)s+(m+1)-(m-1)\tau)}} \frac{ds}{s^{2}-1} \\
\leq \frac{2|a|}{(1+|a|)^{3/2}(|a|-1)} \frac{1}{\sqrt{m-1-(m+1)\tau}} \left| \int_{-\infty}^{a} \frac{s^{-2n}}{\sqrt{s-a}} ds \right| \\
= \frac{2|a|^{-2n}}{|a|-1} \left(\frac{|a|}{1+|a|} \right)^{3/2} \frac{1}{\sqrt{m-1-(m+1)\tau}} \sqrt{\frac{\pi}{n}} \int_{0}^{\infty} \frac{(1+\frac{s}{2n})^{-2n}}{\sqrt{\pi s}} ds.$$

We shall use the following elementary inequality

(3.43)
$$\int_0^\infty \frac{(1+\frac{s}{2n})^{-2n}}{\sqrt{\pi s}} ds \le 1 + 1/n,$$

which can be found in [37, Eq.(A.33)]. Using this, have

$$\frac{\sqrt{1+\tau}}{2\pi} \int_{L_{-}} \frac{s^{-2n+1}}{\sqrt{(1-s)((m-1-(m+1)\tau)s+(m+1)-(m-1)\tau)}} \frac{ds}{1-s^2} \\ \leq \sqrt{\frac{1+\tau}{2\pi n}} \frac{1}{|a|-1} \frac{1+1/n}{\sqrt{m-1-(m+1)\tau}} = \frac{1}{\sqrt{2\pi n(1+\tau)}} \sqrt{m-1-(m+1)\tau} \frac{1+1/n}{2(1+\tau)}.$$

Lastly, we estimate the integral over L_+ . Since the integrand has a factor $(1-s)^{-3/2}$ we first perform a partial integration. Then we may take the bandwidth to 0. The dominant part (without prefactor) is given by

$$4n \int_{1}^{\infty} \frac{s^{-2n}}{\sqrt{(1-s)((m-1-(m+1)\tau)s+(m+1)-(m-1)\tau)}} \frac{ds}{1+s}$$

$$\leq \frac{4n}{2\sqrt{2m(1-\tau)}} \int_{1}^{\infty} \frac{s^{-2n}}{\sqrt{s-1}} ds = \frac{2\sqrt{n}}{\sqrt{2m(1-\tau)}} \sqrt{\pi} \int_{0}^{\infty} \frac{(1+\frac{s}{2n})^{-2n}}{\sqrt{\pi s}} ds.$$

Then we can again use the estimate (3.43). The part that is not dominant can be estimated using

$$\int_{1}^{\infty} \frac{s^{-2n}}{\sqrt{(1-s)(s-a)}} \left(\frac{1}{s-a} + \frac{2}{(s+1)^2}\right) ds \leq \frac{1}{\sqrt{1-a}} \int_{1}^{\infty} \frac{s^{-2n}}{\sqrt{1-s}} ds \leq \sqrt{\frac{\pi}{1-a}} (1+1/n).$$

Combining all of the above, we conclude that

$$\operatorname{Tr}(M_n^{(\tau)})^m \le \frac{1}{4} + \sqrt{\frac{1+\tau}{1-\tau}} \sqrt{\frac{n}{\pi m}} (1+1/n) + \frac{1}{2} \sqrt{\frac{1+\tau}{1-\tau}} \frac{1+1/n}{\sqrt{2\pi m n}} + \frac{1}{8} \sqrt{\frac{m-1-(m+1)\tau}{(1+\tau)\pi n}} (1+1/n).$$

This is also true when $m = \frac{1+\tau}{1-\tau}$. This case is similar, but easier, since there is no band L_{-} in this case. Some easy inequalities finish the proof.

Proof of Lemma 2.8. Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n \setminus \{0\}$. Then by (2.4), it follows that

$$\langle v, M_n^{(\tau)} v \rangle = \sum_{j,k=1}^n M_n^{(\tau)}(j,k) v_j v_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} \left[\sum_{j=1}^n \frac{(\tau/2)^{j-1}}{\sqrt{\Gamma(2j-1)}} H_{2j-2} \left(\frac{x}{\sqrt{2\tau}} \right) v_j \right]^2 dx > 0,$$

which gives rise to the first assertion.

Next, we show the second assertion. Let m be the smallest integer such that

$$m > \frac{1+\tau}{1-\tau}n.$$

Then we have for n large enough

$$\frac{1}{8} \frac{1-\tau}{1+\tau} \sqrt{\frac{m}{\pi n}} (1+1/n) \le \frac{1}{8} \frac{1-\tau}{1+\tau} \sqrt{\frac{\frac{1+\tau}{1-\tau}n+1}{\pi n}} (1+1/n)
\le \frac{1}{8\sqrt{\pi}} \sqrt{\frac{1-\tau}{1+\tau}} \sqrt{1+\frac{1-\tau}{1+\tau} \frac{1}{n}} (1+2/n) \le \frac{1}{8\sqrt{\pi}}.$$

Hence, by Lemma 2.7, we have

$$\operatorname{Tr}(M_n^{(\tau)})^m \le \frac{1}{4} + \frac{1+1/n}{\sqrt{\pi}} + \frac{1}{8\sqrt{\pi}} \le 1 - a$$

for some constant $a \in (0,1)$ when n is big enough. The rest of the proof is similar (but with a different power) to [37] yielding

$$\lambda_{\max} \le (1-a)^{1/m} \le 1 - \sqrt{\frac{1-\tau}{1+\tau}} \frac{a}{n}.$$

The proof for weak non-Hermiticity is analogous, here one takes m to be a multiple of n^2 (indeed, then $m \ge \frac{1+\tau}{1-\tau}$).

4. Asymptotic analysis at strong non-Hermiticity

4.1. **Asymptotics of the kernel.** Since the Hermite polynomials are odd (resp., even) for j odd (resp., even), by symmetry, one can rewrite (3.19) as

(4.1)
$$K_n^{(\tau)}(x,y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2 + y^2}{2(1+\tau)}} \sum_{j=0}^{2n-1} \frac{(\tau/2)^j}{j!} H_j\left(\frac{x}{\sqrt{2\tau}}\right) H_j\left(\frac{y}{\sqrt{2\tau}}\right) + \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2 + y^2}{2(1+\tau)}} \sum_{j=0}^{2n-1} \frac{(\tau/2)^j}{j!} H_j\left(\frac{-x}{\sqrt{2\tau}}\right) H_j\left(\frac{y}{\sqrt{2\tau}}\right).$$

Asymptotics for strong non-Hermiticity can be directly extracted from [2,42]. It will be convenient to define the rescaled kernel

(4.2)
$$\widehat{\mathscr{K}}_{n}^{(\tau)}(x,y) = \sqrt{2n}\,\mathscr{K}_{n}^{(\tau)}(\sqrt{2n}x,\sqrt{2n}y).$$

where we recall that $\mathscr{K}_n^{(\tau)}$ is given by (3.34).

We also define the edge and focal point

(4.3)
$$e_{\tau} = \frac{1+\tau}{\sqrt{1-\tau^2}}, \qquad f_{\tau} = \frac{2\sqrt{\tau}}{\sqrt{1-\tau^2}}.$$

It will also be convenient to define

(4.4)
$$\xi_x = \begin{cases} 0, & |x| \le f_\tau, \\ \cosh^{-1}(x/f_\tau), & |x| > f_\tau. \end{cases}$$

In what follows H will be the Heaviside function, i.e. H(x) = 1 for $x \ge 0$ and H(x) = 0 for x < 0.

Lemma 4.1. Let $0 < \tau < 1$ and $\frac{1}{3} < \mu < \frac{1}{2}$ be fixed. Then there exist constants c, C > 0 such that

$$H(2\xi_{e_{\tau}} - n^{-\mu} - \xi_{x} - \xi_{y}) \left(\sqrt{\frac{n}{\pi}} e^{-n(x^{2} + y^{2})} \cosh(2nxy) - e^{-cn^{1-2\mu}} \right) - Cn^{\mu} e^{-cn(|x| - e_{\tau})^{2}} e^{-cn(|y| - e_{\tau})^{2}}$$

$$\leq \hat{\mathcal{K}}_{n}^{(\tau)}(x, y)$$

$$\leq H(2\xi_{e_{\tau}} + n^{-\mu} - \xi_x - \xi_y) \left(\sqrt{\frac{n}{\pi}} e^{-n(x^2 + y^2)} \cosh(2nxy) + e^{-cn^{1-2\mu}} \right) + Cn^{\mu} e^{-cn(|x| - e_{\tau})^2} e^{-cn(|y| - e_{\tau})^2}$$

uniformly for all $(x, y) \in \mathbb{R}^2$.

Proof. Without loss of generality we assume that $x, y \ge 0$. We shall use [2] in what follows. This paper treats the kernel

$$\kappa_n(z,w) = \frac{n}{\pi\sqrt{1-\tau^2}}\sqrt{\omega(z)\omega(w)}\sum_{j=0}^{n-1}H_j\left(\frac{\sqrt{n}\,z}{\sqrt{2\tau}}\right)H_j\left(\frac{\sqrt{n}\,\overline{w}}{\sqrt{2\tau}}\right),$$

where the weight is given by $\omega(z) = \exp\left(-n\frac{(\operatorname{Re}z)^2}{1+\tau} - n\frac{(\operatorname{Im}z)^2}{1-\tau}\right)$. Our kernel (4.2) can be expressed as

$$\widehat{\mathscr{K}}_n^{(\tau)}(x,y) = \frac{\sqrt{\pi n}}{2} \sqrt{1-\tau^2} \left(\kappa_{2n} \left(\sqrt{1-\tau^2} x, \sqrt{1-\tau^2} y \right) + \kappa_{2n} \left(-\sqrt{1-\tau^2} x, \sqrt{1-\tau^2} y \right) \right).$$

For $\xi_x + \xi_y > 2\xi_{e_\tau} + n^{-\mu}$, it follows from [2, Theorem I.1 and Remark I.2], and some easy estimations, that

$$\left|\widehat{\mathscr{K}}_{n}^{(\tau)}(x,y)\right| \leq C n^{\mu} e^{-nh(x)} e^{-nh(y)},$$

for some constant C > 0, where h is the continuous function

$$(4.6) h(x) = \begin{cases} 2\xi_{e_{\tau}} - \frac{(1-\tau)^2}{2\tau} x^2, & x \in [0, f_{\tau}], \\ 2\xi_{e_{\tau}} - 1 + (1-\tau)x^2 - \frac{f_{\tau}^2}{(x+\sqrt{x^2-f_{\tau}^2})^2} + \log\left(\frac{f_{\tau}^2}{(x+\sqrt{x^2-f_{\tau}^2})^2}\right), & x > f_{\tau}, \end{cases}$$

see [2, Eq.(80)] with $x = f_{\tau} \cos \eta$ or $x = f_{\tau} \cosh \xi$. We remark that one can alternatively use (3.42) as a starting point, and then follow the approach in [2] to reach the same conclusion. As shown in [2], h has a double zero in $x = e_{\tau}$ and is positive for all $x \neq e_{\tau}$. In fact, we can show that

$$\frac{h(x)}{(x - e_{\tau})^2} \ge \lim_{x \to \infty} \frac{h(x)}{(x - e_{\tau})^2} = 1 - \tau$$

for all $x > f_{\tau}$. We conclude that

$$Cn^{\mu}e^{-nh(x)}e^{-nh(y)} < Cn^{\mu}e^{-cn(x-e_{\tau})^{2}}e^{-cn(y-e_{\tau})^{2}}$$

where c>0 is some constant. For $\xi_x+\xi_y<2\xi_{e_\tau}-n^{-\mu}$ we have by [2, Theorem III.5] that

$$\left|\widehat{\mathscr{K}}_n^{(\tau)}(x,y) - e^{-n(x^2 + y^2)}\cosh(2nxy)\right| \le e^{-cn^{1-2\mu}},$$

where we possibly redefine the constant c. We used here that at least one of x, y is $\leq e_{\tau} - an^{-\mu}$ for some constant a > 0. Lastly, we look at the region $|\xi_x + \xi_y - 2\xi_{e_{\tau}}| \leq n^{-\mu}$. Then we have by [42, Proposition V.1] that

$$\widehat{\mathcal{K}}_{n}^{(\tau)}(x,y) = \frac{1}{4} \sqrt{\frac{n}{\pi}} e^{-n(x-y)^{2}} \operatorname{erfc}\left(\sqrt{n}(x+y-2e_{\tau})\right) + e^{-n(x-e_{\tau})^{2}} e^{-n(y-e_{\tau})^{2}} \mathscr{O}(n^{1-2\mu}) + H(2\xi_{e_{\tau}} - \xi_{x} - \xi_{y}) \frac{1}{2} \sqrt{\frac{n}{\pi}} e^{-n(x+y)^{2}} + \mathscr{O}(e^{-nh(x)} e^{-nh(y)}).$$

Since the complementary error function takes values in (0,2) for real arguments, we infer that

$$\widehat{\mathscr{K}}_{n}^{(\tau)}(x,y) \leq \sqrt{\frac{n}{\pi}} e^{-n(x^{2}+y^{2})} \cosh(2nxy) + Cn^{1-2\mu} e^{-cn(x-e_{\tau})^{2}} e^{-cn(y-e_{\tau})^{2}}$$

$$\leq \sqrt{\frac{n}{\pi}} e^{-n(x^{2}+y^{2})} \cosh(2nxy) + Cn^{\mu} e^{-cn(x-e_{\tau})^{2}} e^{-cn(y-e_{\tau})^{2}},$$

for some constant C > 0. For the lower bound we have trivially

$$\widehat{\mathscr{K}}_n^{(\tau)}(x,y) \ge -Cn^{\mu}e^{-cn(x-e_{\tau})^2}e^{-cn(y-e_{\tau})^2}.$$

Putting it all together, we obtain the result.

4.2. Proof of Proposition 2.4.

Proof of Proposition 2.4. In what follows, we shall make the identification $x_{m+1} = x_1$. By Lemma 4.1, we infer that, for fixed m

$$\int_{\mathbb{R}^m} \prod_{j=1}^m \widehat{\mathscr{H}}_n^{(\tau)}(x_j, x_{j+1}) dx_j
\geq 2^m \left(\frac{n}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m_+} e^{-2n\sum_{j=1}^m x_j^2} \prod_{j=1}^m H(2\xi_{e_{\tau}} - n^{-\mu} - \xi_{x_j} - \xi_x) \cosh(2nx_j x_{j+1}) dx_j - c_m n^{-\frac{1}{2} + \mu},$$

for some constant $c_m > 0$. There exists a constant a > 0 such that

$$2^{m} \int_{\mathbb{R}_{+}^{m}} e^{-2n \sum_{j=1}^{m} x_{j}^{2}} \prod_{j=1}^{m} H(2\xi_{e_{\tau}} - n^{-\mu} - \xi_{x_{j}} - \xi_{x}) \cosh(2nx_{j}x_{j+1}) dx_{j}$$

$$\geq \int_{[-e_{\tau} + an^{-\mu}, e_{\tau} - an^{-\mu}]^{m}} e^{-2n \sum_{j=1}^{m} x_{j}^{2}} \prod_{j=1}^{m} \cosh(2nx_{j}x_{j+1}) dx_{j}$$

$$\geq \frac{1}{2} \int_{[-e_{\tau} + an^{-\mu}, e_{\tau} - an^{-\mu}]^{m}} e^{-2n \sum_{j=1}^{m} (x_{j}^{2} - x_{j}x_{j+1})} \prod_{j=1}^{m} dx_{j}.$$

In the last step we took only the combination of exponentials $e^{\pm 2nx_jx_{j+1}}$ such that after substitutions $x_j \to \pm x_j$ we obtain $e^{n\sum_{j=1}^m x_j}$ in the integrand, which is half of all the combinations. Now we make a substitution $y_m = x_1 + \ldots + x_m$ and $y_j = x_{j+1} - x_j$ for $j = 1, \ldots, m-1$. Then we have

$$\int_{[-e_{\tau}+an^{-\mu},e_{\tau}-an^{-\mu}]^{m}} e^{-2n\sum_{j=1}^{m}(x_{j}^{2}+2x_{j}x_{j+1})} \prod_{j=1}^{m} dx_{j}$$

$$= \frac{1}{m} \int_{-m(e_{\tau}-an^{-\mu})}^{m(e_{\tau}-an^{-\mu})} \int_{[-e_{\tau}+n^{-\mu},e_{\tau}-n^{-\mu}]^{m-1}} e^{-n(\sum_{j=1}^{m-1}y_{j}^{2}+(\sum_{j=1}^{m-1}y_{j})^{2})} \prod_{j=1}^{m} dy_{j}.$$

This we can write as

$$(e_{\tau} - an^{-\mu}) \int_{[-e_{\tau} + n^{-\mu}, e_{\tau} - n^{-\mu}]^{m-1}} \int_{-\infty}^{\infty} e^{-n\lambda^{2} + 2n\lambda \sum_{j=1}^{m-1} y_{j}} e^{-n\sum_{j=1}^{m-1} y_{j}^{2}} d\lambda \prod_{j=1}^{m-1} dy_{j}$$

$$= (e_{\tau} - an^{-\mu}) \int_{-\infty}^{\infty} e^{-\frac{n}{2}\lambda^{2}} \left(\int_{-e_{\tau} + n^{-\mu}}^{e_{\tau} - n^{-\mu}} e^{-nx^{2} + 2n\lambda x} \right)^{m-1} d\lambda$$

$$\geq (e_{\tau} - an^{-\mu}) \int_{-e_{\tau} + \varepsilon}^{e_{\tau} - \varepsilon} e^{-n\lambda^{2}} \left(\int_{-e_{\tau} + n^{-\mu}}^{e_{\tau} - n^{-\mu}} e^{-nx^{2} + 2n\lambda x} \right)^{m-1} d\lambda$$

for any $\varepsilon > 0$ (for n big enough). We have

$$\int_{-e_{\tau}+\varepsilon}^{e_{\tau}-\varepsilon} e^{-n\lambda^{2}} \left(\int_{-e_{\tau}+n^{-\mu}}^{e_{\tau}-n^{-\mu}} e^{-nx^{2}+2n\lambda x} \right)^{m-1} d\lambda = \int_{-e_{\tau}+\varepsilon}^{e_{\tau}-\varepsilon} e^{-n\lambda^{2}} \left(\frac{\pi}{n} \right)^{\frac{m-1}{2}} e^{-(m-1)n\lambda^{2}} (1 + \mathcal{O}(1/n)) d\lambda$$
$$= 2(e_{\tau}-\varepsilon) \left(\frac{\pi}{n} \right)^{\frac{m}{2}} \frac{1}{\sqrt{m}} (1 + \mathcal{O}(1/n)).$$

We conclude that

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \operatorname{Tr}(M_n^{(\tau)})^m \ge (e_{\tau} - \varepsilon) \frac{1}{\sqrt{2\pi m}}.$$

Since this is true for arbitrary $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \operatorname{Tr}(M_n^{(\tau)})^m \ge e_{\tau} \frac{1}{\sqrt{2\pi m}} = \sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{2\pi m}}.$$

For $m \ge \frac{1+\tau}{1-\tau}$ we have an upper bound already, but we need one for the remaining m. We start with (3.41) and plug in the result from Lemma 4.1 for the remaining kernel. This yields

$$\operatorname{Tr}(M_n^{(\tau)})^m \le \frac{n}{\pi} \frac{1}{\sqrt{m-1}} \int_{\mathbb{R}^2_+} H(2\xi_{e_\tau} + n^{-\mu} - \xi_x - \xi_y) 4e^{-n\frac{m}{m-1}(x^2 + y^2)} \cosh\left(\frac{2nxy}{m-1}\right) \cosh(nxy) \, dx \, dy.$$

We can write

$$\begin{split} &4e^{-n\frac{m}{m-1}(x^2+y^2)}\cosh\left(\frac{2nxy}{m-1}\right)\cosh(nxy)\\ &=e^{-n\frac{m}{m-1}(x^2+y^2)}\left(e^{2n\frac{m}{m-1}xy}+e^{-2n\frac{m}{m-1}xy}+e^{2n\frac{m-2}{m-1}xy}+e^{-2n\frac{m-2}{m-1}xy}\right)\\ &=e^{-n\frac{m}{m-1}(x-y)^2}+e^{-n\frac{m}{m-1}(x+y)^2}+e^{-\frac{4n}{m}x^2}\left(e^{-n\frac{m}{m-1}(y-\frac{m-2}{m}x)^2}+e^{-n\frac{m}{m-1}(y+\frac{m-2}{m}x)^2}\right). \end{split}$$

Note that we may replace the integration domain \mathbb{R}^2_+ by $[0, g_\tau + bn^{-\mu}]^2$ for some constant b > 0, and

$$g_{\tau} = f_{\tau} \cosh(2\xi_{e_{\tau}}) = \frac{1+\tau^2}{2\tau} f_{\tau}.$$

This is allowed because the Heaviside function vanishes outside this region. Now we divide this region into $[f_{\tau}, g_{\tau} + bn^{-\mu}]$, and the remaining region. For the latter, we may bound the Heaviside function by 1 (boundary contribution is of small order), and it follows straightforwardly by steepest descent arguments that the corresponding integral equals

$$\frac{n}{\pi} \frac{1}{\sqrt{m-1}} \int_0^{f_\tau} \sqrt{\pi \frac{m-1}{mn}} dx = \sqrt{\frac{n}{\pi m}} f_\tau.$$

up to leading order. For the first region we find that the corresponding integral equals

$$\frac{1}{2}\sqrt{\frac{n}{\pi m}}\int_{f_{\tau}}^{g_{\tau}}\left(\operatorname{erf}\left(\sqrt{\frac{mn}{m-1}}x\right)+\operatorname{erf}\left(\sqrt{\frac{mn}{m-1}}\left(f_{\tau}\cosh(2\xi_{e_{\tau}}-\xi_{x})-x\right)\right)\right)dx$$

to leading order. For $x > 1 + \tau$, we have

$$f_{\tau} \cosh(2\xi_{e_{\tau}} - \xi_x) - x < f_{\tau} \cosh(\xi_{e_{\tau}}) - e_{\tau} = 0,$$

and the two error functions cancel in the limit $n \to \infty$. What remains is

$$\frac{1}{2}\sqrt{\frac{n}{\pi m}} \int_{f_{\tau}}^{g_{\tau}} 2 \, dx = \sqrt{\frac{n}{\pi m}} (e_{\tau} - f_{\tau}).$$

Together with the contribution from $[0, f_{\tau}]^2$, this gives

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} \operatorname{Tr}(M_n^{(\tau)})^m \le \frac{1}{\sqrt{2\pi m}} (f_{\tau} + e_{\tau} - f_{\tau}) = \sqrt{\frac{1+\tau}{1-\tau}} \frac{1}{\sqrt{\pi m}}.$$

5. Asymptotic analysis at weak non-Hermiticity

In this section, we show Proposition 2.5, i.e. for any fixed m > 0,

(5.1)
$$\lim_{n \to \infty} \frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^m = \frac{c(\sqrt{m} \alpha)}{2} = \frac{e^{-m\alpha^2/2}}{2} \left[I_0\left(\frac{m\alpha^2}{2}\right) + I_1\left(\frac{m\alpha^2}{2}\right) \right].$$

Note that $c(\alpha)$ in (1.10) can be written as

(5.2)
$$c(\alpha) = \frac{2}{\alpha\sqrt{\pi}} \int_0^1 \operatorname{erf}(\alpha\sqrt{1-s^2}) \, ds = \sum_{k=0}^\infty \frac{(2k-1)!!}{2^k \, k! \, (k+1)!} (-1)^k \alpha^{2k},$$

see e.g. [10, Remark 2.7]. Therefore, the right-hand side of (5.1) can be written as

(5.3)
$$\frac{c(\sqrt{m}\,\alpha)}{2} = \sum_{k=0}^{\infty} \frac{m^k}{2^{2k+1}(k+1)!} {2k \choose k} (-\alpha^2)^k.$$

Recall that by Lemma 3.5, $Tr(M_n^{(\tau)})^m$ is evaluated as

(5.4)
$$\operatorname{Tr}(M_n^{(\tau)})^m = \sum_{j_1, j_2 = 0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2\sum_{k=1}^m j_k + \frac{m}{2}} \prod_{k=1}^m \left(\sum_{l=0}^{n-1} \frac{(1-\tau)^{2l}(1+\tau)^{-2l}(2j_k)!}{2^{2l}l!(l+j_k-j_{k-1})!(2j_{k-1}-2l)!}\right).$$

Here and in the sequel, we use the convention $j_0 = j_m$.

Remark 5.1. We first discuss the contribution from l = 0 in the expression (5.4). If l = 0, the right-hand side of (5.4) is given by

$$\left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} \prod_{k=1}^m \left(\frac{1+\tau}{2}\right)^{2j_k} \frac{(2j_k)!}{(j_k-j_{k-1})!(2j_{k-1})!}$$

$$= \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} \prod_{k=1}^m \left(\frac{1+\tau}{2}\right)^{2j_k} \frac{1}{(j_k-j_{k-1})!}$$

$$= \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j=0}^{n-1} \prod_{k=1}^m \left(\frac{1+\tau}{2}\right)^{2j} = \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2jm} = \left(\frac{1+\tau}{2}\right)^{m/2} \frac{1-\left(\frac{1+\tau}{2}\right)^{2mn}}{1-\left(\frac{1+\tau}{2}\right)^{2m}}.$$

(From the second to the third line, see Remark 3.6.) This gives that for $\tau = 1 - \alpha^2/(2n)$, as $n \to \infty$,

(5.5)
$$\frac{1}{2n} \left(\frac{1+\tau}{2}\right)^{m/2} \sum_{j_1,\dots,j_m=0}^{n-1} \prod_{k=1}^m \left(\frac{1+\tau}{2}\right)^{2j_k} \frac{(2j_k)!}{(j_k-j_{k-1})!(2j_{k-1})!} = \frac{1-e^{-\frac{\alpha^2m}{2}}}{\alpha^2m} + \mathcal{O}\left(\frac{1}{n}\right).$$

- 5.1. Expectation and variance revisited. It is instructive to first consider the cases m = 1, 2 before dealing with the general m. By Proposition 2.3 (ii), the analysis for m = 1, 2 below provides an alternative and more unified proof of [10, Theorems 2.1 and 2.3].
- 5.1.1. The case m=1. We first consider the simplest case m=1. Then by (5.4), we have

(5.6)
$$\operatorname{Tr}(M_n^{(\tau)}) = \sum_{l=0}^{n-1} \frac{1}{(l!)^2} \frac{(1-\tau)^{2l}(1+\tau)^{-2l}}{2^{2l}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!}.$$

Note that for $\tau = 1 - \alpha^2/(2n)$, we have

$$\frac{1}{2n} \frac{(1-\tau)^{2l}(1+\tau)^{-2l}}{2^{2l}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!} \sim \frac{(\alpha/2)^{4l}}{(2n)^{2l+1}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!}.$$

For l = o(n), the Riemann sum approximation gives

$$\frac{1}{(2n)^{2l+1}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!} = \frac{1}{2n} \sum_{j=0}^{n-1} \left(1 - \frac{\alpha^2}{4n}\right)^{2j} \frac{2j}{2n} \frac{2j-1}{2n} \dots \frac{2j-2l+1}{2n}$$
$$\sim \frac{1}{2} \int_0^1 e^{-\frac{\alpha^2 x}{2}} x^{2l} dx = \frac{2^{2l}}{\alpha^{4l+2}} \gamma \left(2l+1, \frac{\alpha^2}{2}\right).$$

Note here that for l = 0, it matches with (5.5) since $\gamma(1, x) = 1 - e^{-x}$. Combining the above, for l = o(n),

(5.7)
$$\frac{1}{2n} \frac{(1-\tau)^{2l}(1+\tau)^{-2l}}{2^{2l}(l!)^2} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!} \sim \frac{1}{\alpha^2} \frac{1}{2^{2l}(l!)^2} \gamma \left(2l+1, \frac{\alpha^2}{2}\right).$$

This gives that for a sufficiently large L > 0,

$$\frac{1}{2n}\operatorname{Tr}(M_n^{(\tau)}) \sim \frac{1}{\alpha^2} \sum_{l=0}^{L} \frac{1}{2^{2l}(l!)^2} \gamma \left(2l+1, \frac{\alpha^2}{2}\right) + \sum_{l=L+1}^{n-1} \frac{1}{(l!)^2} \frac{(\alpha/2)^{4l}}{(2n)^{2l+1}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!}.$$

Note here that for any l = 0, ..., n - 1,

$$(5.8) \qquad \frac{1}{(2n)^{2l+1}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!} = \frac{1}{2n} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{2j}{2n} \frac{2j-1}{2n} \dots \frac{2j-2l+1}{2n} < \frac{1}{2}.$$

Thus as $L \to \infty$ keeping L = o(N), we have

$$\sum_{l=L+1}^{n-1} \frac{1}{(l!)^2} \frac{(\alpha/2)^{4l}}{(2n)^{2l+1}} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2j+\frac{1}{2}} \frac{(2j)!}{(2j-2l)!} \le \sum_{l=L+1}^{n-1} \frac{(\alpha/2)^{4l}}{2(l!)^2} \to 0.$$

Therefore we obtain

(5.9)
$$\lim_{n \to \infty} \frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)}) = \frac{1}{\alpha^2} \sum_{l=0}^{\infty} \frac{1}{2^{2l} (l!)^2} \gamma \left(2l+1, \frac{\alpha^2}{2}\right).$$

Now Proposition 2.5 for m=1 follows from the following lemma.

Lemma 5.2. We have

(5.10)
$$\frac{1}{\alpha^2} \sum_{l=0}^{\infty} \frac{1}{2^{2l} (l!)^2} \gamma \left(2l + 1, \frac{\alpha^2}{2} \right) = \frac{c(\alpha)}{2}.$$

Proof. Note that

(5.11)
$$\gamma(j,x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{j+s}}{s!(j+s)},$$

see [44, Eq.(8.7.1)]. Thus we have

(5.12)
$$\frac{1}{\alpha^2} \frac{1}{2^{2l}(l!)^2} \gamma \left(2l+1, \frac{\alpha^2}{2}\right) = \frac{1}{2^{4l+1}(l!)^2} \sum_{s=0}^{\infty} \frac{1}{2^s s! (2l+1+s)} (-\alpha^2)^{s+2l}.$$

This gives that

$$\begin{split} \frac{1}{\alpha^2} \sum_{l=0}^{\infty} \frac{1}{2^{2l}(l!)^2} \gamma \Big(2l+1, \frac{\alpha^2}{2} \Big) &= \sum_{k=0}^{\infty} \bigg(\sum_{\substack{s,l=0\\s+2l=k}}^{\infty} \frac{1}{2^{4l+s+1}(l!)^2 s! (2l+1+s)} \bigg) (-\alpha^2)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \bigg(\sum_{\substack{s,l=0\\s+2l=k}}^{\infty} \frac{2^{s-2k-1}}{(l!)^2 s!} \bigg) (-\alpha^2)^k. \end{split}$$

Then by (5.3), it suffices to show that

(5.13)
$$\sum_{\substack{s,l=0\\s+2l=k}}^{\infty} 2^{s} \frac{k!}{(l!)^{2} s!} = {2k \choose k}.$$

This combinatorial identity follows by comparing the coefficient of the $(xy)^k$ term in

$$(5.14) (x^2 + 2xy + y^2)^k = (x+y)^{2k},$$

which completes the proof.

5.1.2. The case m = 2. For m = 2, by (5.4), we have

(5.15)
$$\operatorname{Tr}(M_n^{(\tau)})^2 = \sum_{j_1, j_2 = 0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2(j_1 + j_2) + 1} \times \sum_{l_1, l_2 = 0}^{n-1} \frac{(1-\tau)^{2(l_1 + l_2)} (1+\tau)^{-2(l_1 + l_2)} (2j_1)! (2j_2)!}{2^{2(l_1 + l_2)} l_1! l_2! (l_1 + j_1 - j_2)! (l_2 + j_2 - j_1)! (2j_2 - 2l_1)! (2j_1 - 2l_2)!}.$$

Here, the summand does not vanish only for the set of indices

$$(5.16) j_1 - j_2 \ge -l_1, j_2 - j_1 \ge -l_2, j_2 \ge l_1, j_1 \ge l_2.$$

As before, by the rapid decay of the factor $1/(l_1!l_2!)$, it suffices to consider the case l_1 and l_2 are finite. Furthermore, due to the term

$$\frac{1}{(l_1+j_1-j_2)!(l_2+j_2-j_1)!},$$

it is enough to consider the case $j_1 - j_2$ is finite. By letting $M = j_1 - j_2$, it follows that

(5.17)
$$\frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^2 \sim \sum_{M=-\infty}^{\infty} \sum_{l_1, l_2=0}^{\infty} \frac{1}{l_1! l_2! (l_1+M)! (l_2-M)!} \times \frac{1}{2n} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{4j} \frac{(1-\tau)^{2(l_1+l_2)} (1+\tau)^{-2(l_1+l_2)} (2j+M)! (2j-M)!}{2^{2(l_1+l_2)} (2j-2M-2l_1)! (2j-2l_2)!}.$$

Here, by the Riemann sum approximation, we have

(5.18)
$$\frac{1}{2n} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{4j} \frac{(1-\tau)^{2(l_1+l_2)}(1+\tau)^{-2(l_1+l_2)}(2j+M)!(2j-M)!}{2^{2(l_1+l_2)}(2j-2M-2l_1)!(2j-2l_2)!} \sim \frac{(\alpha/2)^{4l}}{2} \int_0^1 e^{-\alpha^2 x} x^{2l} dx = \frac{1}{\alpha^2 2^{4l+1}} \gamma \left(2l+1,\alpha^2\right),$$

where we write $l = l_1 + l_2$. Therefore we obtain

(5.19)
$$\lim_{n \to \infty} \frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^2 = \frac{1}{\alpha^2} \sum_{M = -\infty}^{\infty} \sum_{\substack{l_1, l_2 = 0 \\ l_1 + l_2 = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{l_1! l_2! (l_1 + M)! (l_2 - M)!} \gamma \left(2l + 1, \alpha^2\right).$$

Now it suffices to show the following.

Lemma 5.3. We have

(5.20)
$$\frac{1}{\alpha^2} \sum_{M=-\infty}^{\infty} \sum_{\substack{l_1,l_2=0\\l_1+l_2=l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{l_1! l_2! (l_1+M)! (l_2-M)!} \gamma \left(2l+1,\alpha^2\right) = \frac{c(\sqrt{2}\alpha)}{2}.$$

Proof. Using (5.11), we have

$$\frac{1}{\alpha^2} \sum_{\substack{l_1, l_2 = 0 \\ l_1 + l_2 = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{(l_1!)^2 (l_2!)^2} \gamma \left(2l+1, \alpha^2\right) = \sum_{\substack{l_1, l_2 = 0 \\ l_1 + l_2 = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{(l_1)!^2 (l_2)!^2} \sum_{s=0}^{\infty} \frac{(-\alpha^2)^{s+2l}}{s!(2l+1+s)}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{\substack{s, l = 0 \\ s+2l=k}}^{\infty} \sum_{\substack{l_1, l_2 = 0 \\ l_1 + l_2 = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{(l_1)!^2 (l_2)!^2} \frac{1}{s!(2l+1+s)}\right) (-\alpha^2)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\sum_{\substack{s, l = 0 \\ s+2l-k}}^{\infty} \sum_{\substack{l_1, l_2 = 0 \\ l_1 + l_2 = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{(l_1)!^2 (l_2)!^2} \frac{1}{s!}\right) (-\alpha^2)^k.$$

Then by (5.3), it suffices to show that

(5.21)
$$\sum_{M=-\infty}^{\infty} \sum_{\substack{s,l=0\\s+2l-k}}^{\infty} \sum_{\substack{l_1,l_2=0\\k+l_1+l_2-l}}^{\infty} 2^{s-2l} \frac{k!}{l_1! l_2! (l_1+M)! (l_2-M)! s!} = \binom{2k}{k}.$$

Note that for any k, there are only finite numbers of non-trivial summands in the left-hand side of this identity. To prove this combinatorial identity, we can compare the coefficient of the $(xy)^k$ term in the expansion of

(5.22)
$$\left(\frac{x^2}{2} + \frac{x^2}{2} + 2xy + \frac{y^2}{2} + \frac{y^2}{2}\right)^k = (x+y)^{2k}.$$

To be more precise, the left-hand side of (5.21) can be obtained as the coefficient of the $(xy)^k$ term in the expansion of the left-hand side of (5.22). For this, we choose l_1 and l_2 instances of the first two terms $x^2/2$ and $x^2/2$, respectively, s instances of the term 2xy, and $l_1 + M$ and $l_2 - M$ instances of the last two terms $y^2/2$ and $y^2/2$, respectively. We then collect all possible combinations of l_1 , l_2 , and M such that

$$l_1 + l_2 + s + (l_1 + M) + (l_2 - M) = s + 2(l_1 + l_2) = s + 2l = k$$

This completes the proof.

5.2. **Proof of Proposition 2.5.** We now consider the general case with $m \ge 1$. Let us first rewrite (5.4) as

$$\operatorname{Tr}(M_n^{(\tau)})^m = \sum_{j_1, \dots, j_m = 0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2\sum_{k=1}^m j_k + \frac{m}{2}} \sum_{l_1, \dots, l_m = 0}^{n-1} \frac{(1-\tau)^{2(l_1 + \dots + l_m)} (1+\tau)^{-2(l_1 + \dots + l_m)}}{2^{2(l_1 + \dots + l_m)} l_1! \dots l_m!} \times \frac{(2j_1)! \dots (2j_m)!}{(l_1 + M_1)! \dots (l_m + M_m)! (2j_m - 2l_1)! (2j_1 - 2l_2)! \dots (2j_{m-1} - 2l_m)!},$$

where $M_k = j_k - j_{k-1}$ (k = 1, ..., m). Again, it suffices to consider the case that the l_j 's are finite. As a consequence, due to the terms

$$\frac{1}{(l_1+M_1)!\cdots(l_m+M_m)!}$$

in (5.4), it is enough to take the case $j_k - j_{k-1}$ (k = 1, ..., m) finite into account. We write

$$\widetilde{M}_k = \sum_{p=2}^k M_p = j_k - j_1.$$

Then by combining the above, we obtain

$$\frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^m \sim \sum_{\substack{M_1, \dots, M_m = -\infty \\ M_1 + \dots + M_m = 0}}^{\infty} \sum_{\substack{l_1, \dots l_m = 0 \\ l_1 + \dots + l_m = l}}^{\infty} \frac{1}{l_1! \dots l_m! (l_1 + M_1)! \dots (l_m + M_m)!} \times \frac{1}{2n} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2mj + \frac{m}{2}} \frac{(1-\tau)^{2l}}{(2(1+\tau))^{2l}} \frac{(2j)! (2j+2\widetilde{M}_2) \dots (2j+2\widetilde{M}_m)!}{(2j+2\widetilde{M}_m - 2l_1)! (2j-2l_2)! \dots (2j+2\widetilde{M}_{m-1} - 2l_m)!},$$

where $l = l_1 + \cdots + l_m$. Note that

$$(1-\tau)^{2l} \frac{(2j)!(2j+2\widetilde{M}_2)\dots(2j+2\widetilde{M}_m)!}{(2j+2\widetilde{M}_m-2l_1)!(2j-2l_2)!\dots(2j+2\widetilde{M}_{m-1}-2l_m)!}$$

$$=\alpha^{4l} \left(\frac{2j+2\widetilde{M}_m-2l_1+1}{2n}\dots\frac{2j}{2n}\right)\dots\left(\frac{2j+2\widetilde{M}_{m-1}-2l_m+1}{2n}\dots\frac{2j+2\widetilde{M}_m}{2n}\right).$$

Then it follows from the Riemann sum approximation that

(5.23)
$$\frac{1}{2n} \sum_{j=0}^{n-1} \left(\frac{1+\tau}{2}\right)^{2mj+\frac{m}{2}} \frac{(1-\tau)^{2l}}{(2(1+\tau))^{2l}} \frac{(2j)!(2j+2\widetilde{M}_2)\dots(2j+2\widetilde{M}_m)!}{(2j+2\widetilde{M}_m-2l_1)!(2j-2l_2)!\dots(2j+2\widetilde{M}_{m-1}-2l_m)!} \sim \frac{(\alpha/2)^{4l}}{2} \int_0^1 e^{-\frac{m\alpha^2}{2}x} x^{2l} dx = \frac{1}{\alpha^2} \frac{1}{2^{2l}m^{2l+1}} \gamma \left(2l+1, \frac{\alpha^2 m}{2}\right).$$

Therefore we obtain

$$\lim_{n \to \infty} \frac{1}{2n} \operatorname{Tr}(M_n^{(\tau)})^m$$

$$= \frac{1}{\alpha^2} \sum_{\substack{M_1, \dots, M_m = -\infty \\ M_1 + \dots + M_m = 0}}^{\infty} \sum_{\substack{l_1, \dots l_m = 0 \\ l_1 + \dots + l_m = l}}^{\infty} \frac{1}{l_1! \dots l_m! (l_1 + M_1)! \dots (l_m + M_m)!} \frac{1}{2^{2l} m^{2l+1}} \gamma \left(2l + 1, \frac{\alpha^2 m}{2}\right).$$

Then the following lemma completes the proof of Proposition 2.5.

Lemma 5.4. We have

$$\frac{1}{\alpha^2} \sum_{\substack{M_1, \dots, M_m = -\infty \\ M_1 + \dots + M_m = 0}}^{\infty} \sum_{\substack{l_1, \dots l_m = 0 \\ l_1 + \dots + l_m = l}}^{\infty} \frac{1}{l_1! \dots l_m! (l_1 + M_1)! \dots (l_m + M_m)!} \frac{1}{2^{2l} m^{2l+1}} \gamma \left(2l + 1, \frac{\alpha^2 m}{2}\right) = \frac{c(\sqrt{m}\alpha)}{2}.$$

Proof. Using (5.11), we have

$$\frac{1}{\alpha^{2}} \sum_{\substack{l_{1}, \dots l_{m} = 0 \\ l_{1} + \dots + l_{m} = l}}^{\infty} \frac{1}{l_{1}! \dots l_{m}! (l_{1} + M_{1})! \dots (l_{m} + M_{m})!} \frac{1}{2^{2l} m^{2l+1}} \gamma \left(2l + 1, \frac{\alpha^{2} m}{2}\right)$$

$$= \frac{1}{\alpha^{2}} \sum_{\substack{l_{1}, \dots l_{m} = 0 \\ l_{1} + \dots + l_{m} = l}}^{\infty} \frac{1}{l_{1}! \dots l_{m}! (l_{1} + M_{1})! \dots (l_{m} + M_{m})!} \frac{1}{2^{2l} m^{2l+1}} \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s! (s + 2l + 1)} \left(\frac{\alpha^{2} m}{2}\right)^{s+2l+1}$$

$$= \sum_{\substack{l_{1}, \dots l_{m} = 0 \\ l_{1} + \dots + l_{m} = l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{l_{1}! \dots l_{m}! (l_{1} + M_{1})! \dots (l_{m} + M_{m})!} \sum_{s=0}^{\infty} \frac{m^{s}}{2^{s} s! (s + 2l + 1)} (-\alpha^{2})^{s+2l}.$$

This can be rewritten as

$$\sum_{\substack{l_1,\dots l_m=0\\l_1+\dots+l_m=l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{l_1!\dots l_m!(l_1+M_1)!\dots(l_m+M_m)!} \sum_{s=0}^{\infty} \frac{m^s}{2^s s!(s+2l+1)} (-\alpha^2)^{s+2l}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{s,l=0\\s+2l=k}}^{\infty} \sum_{\substack{l_1,\dots l_m=0\\l_1+\dots+l_m=l}}^{\infty} \frac{1}{2^{4l+1}} \frac{1}{l_1!\dots l_m!(l_1+M_1)!\dots(l_m+M_m)!} \frac{m^s}{2^s s!(s+2l+1)} (-\alpha^2)^k.$$

By (5.3), all we need to show is

(5.25)
$$\sum_{\substack{M_1,\dots,M_m=-\infty\\M_1+\dots+M_m=0}}^{\infty} \sum_{\substack{s,l=0\\s+2l=k}}^{\infty} \sum_{\substack{l_1,\dots l_m=0\\l_1+\dots+l_m=l}}^{\infty} \frac{2^s \, m^{s-k} \, k!}{l_1!\dots l_m!(l_1+M_1)!\dots(l_m+M_m)! \, s!} = \binom{2k}{k}.$$

As before, this identity follows by comparing the coefficient of $(xy)^k$ term in

(5.26)
$$\left(\frac{x^2}{m} + \dots + \frac{x^2}{m} + 2xy + \frac{y^2}{m} + \dots + \frac{y^2}{m}\right)^k = (x+y)^{2k}.$$

Namely, we choose l_1, l_2, \ldots, l_m instances of the first m terms $x^2/m, \ldots, x^2/m$, respectively, s instances of the term 2xy, and l_1+M_1, \ldots, l_m+M_m instances of the last m terms $y^2/m, \ldots, y^2/m$, respectively. We then collect all possible combinations of l_1, \ldots, l_m , and M_1, \ldots, M_m such that $M_1 + \ldots M_m = 0$ and

$$(l_1 + \dots + l_m) + s + ((l_1 + M_1) + \dots + (l_m + M_m)) = s + 2(l_1 + \dots + l_m) = s + 2l = k.$$

Notice here that the exponent of m is

$$-(l_1 + \dots + l_m) - ((l_1 + M_1) + \dots + (l_m + M_m)) = -2l = s - k.$$

This completes the proof.

APPENDIX A. AUXILIARY LEMMAS

Lemma A.1. For fixed $\alpha > 0$, we have

(A.1)
$$-\sum_{m=1}^{\infty} \frac{c(\sqrt{m}\,\alpha)}{2m} = \frac{2}{\pi} \int_0^1 \log\left(1 - e^{-\alpha^2 s^2}\right) \sqrt{1 - s^2} \, ds.$$

Proof. We notice that

$$\frac{1}{\alpha} \int_0^1 \operatorname{erf}(\alpha \sqrt{1 - s^2}) \, ds = \frac{1}{\alpha} \int_0^1 \operatorname{erf}(\alpha s) \frac{s \, ds}{\sqrt{1 - s^2}}$$

$$= \left[-\frac{1}{\alpha} \operatorname{erf}(\alpha s) \sqrt{1 - s^2} \right]_0^1 + \frac{2}{\sqrt{\pi}} \int_0^1 e^{-(\alpha s)^2} \sqrt{1 - s^2} \, ds = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-\alpha^2 s^2} \sqrt{1 - s^2} \, ds.$$

We conclude, using (1.10) and (5.2), that

$$-\sum_{m=1}^{\infty} \frac{c(\sqrt{m}\,\alpha)}{2m} = -\frac{2}{\pi} \sum_{m=1}^{\infty} \int_0^1 \frac{1}{m} e^{-m\alpha^2 s^2} \sqrt{1-s^2} \, ds = \frac{2}{\pi} \int_0^1 \log\left(1-e^{-\alpha^2 s^2}\right) \sqrt{1-s^2} \, ds.$$

We need to prove that we may indeed interchange the order of summation and integration. We start with the observation, based on steepest descent arguments, that there exists an integer M > 0 such that m > M implies that

$$\left| \int_0^1 e^{-m\alpha^2 s^2} \sqrt{1 - s^2} ds - \frac{1}{2\alpha} \sqrt{\frac{\pi}{m}} \right| \le C \frac{1}{m\sqrt{m}}$$

for some uniform constant C (depending only on α). Now let $\varepsilon > 0$ satisfy $\varepsilon < M^{-2}$. Let M_{ε} be the integer in $(\varepsilon^{-1/2}, 1 + \varepsilon^{-1/2}]$. Then we have $M_{\varepsilon} > M$, and thus

$$\begin{split} \sum_{m=1}^{\infty} \frac{1}{m} \int_{0}^{\varepsilon} e^{-m\alpha^{2}s^{2}} \sqrt{1-s^{2}} ds &= \sum_{m=1}^{M_{\varepsilon}} \frac{1}{m} \int_{0}^{\varepsilon} e^{-m\alpha^{2}s^{2}} \sqrt{1-s^{2}} ds + \sum_{m=M_{\varepsilon}+1}^{\infty} \frac{1}{m} \int_{0}^{\varepsilon} e^{-m\alpha^{2}s^{2}} \sqrt{1-s^{2}} ds \\ &\leq \sum_{m=1}^{M_{\varepsilon}} \frac{\varepsilon}{m} + \sum_{m=M_{\varepsilon}+1}^{\infty} \frac{1}{m} \Big(\frac{1}{2\alpha} \sqrt{\frac{\pi}{m}} + \frac{C}{m\sqrt{m}} \Big) \\ &\leq \varepsilon (1 + \log M_{\varepsilon}) + \frac{\sqrt{\pi}}{\alpha} \frac{1}{\sqrt{M_{\varepsilon}}} + \frac{2}{3} \frac{C}{M_{\varepsilon} \sqrt{M_{\varepsilon}}}. \end{split}$$

This tends to 0 as $\varepsilon \to 0$, and we are done.

Lemma A.2. Let k > 1 be an integer and let $x_0, x_k \in \mathbb{R}$. Then we have

$$\int_{\mathbb{R}^{k-1}_+} e^{-\sum_{j=1}^{k-1} x_j^2} \prod_{j=1}^k \cosh(x_{j-1}x_j) \, dx_1 \cdots dx_{k-1} = \frac{1}{\sqrt{k}} \left(\frac{\pi}{2}\right)^{\frac{k-1}{2}} e^{\frac{k-1}{k} \frac{x_0^2 + x_k^2}{2}} \cosh\left(\frac{x_0 x_k}{k}\right).$$

Proof. First, we verify that the statement is true for k=2.

$$\int_{\mathbb{R}_{+}} e^{-x_{1}^{2}} \cosh(x_{0}x_{1}) \cosh(x_{1}x_{2}) dx_{1}$$

$$= \frac{1}{4} \int_{\mathbb{R}_{+}} e^{-x_{1}^{2}} \left(e^{x_{1}(x_{0}+x_{2})} + e^{-x_{1}(x_{0}+x_{2})} + e^{x_{1}(x_{0}-x_{2})} + e^{-x_{1}(x_{0}-x_{2})} \right) dx_{1}$$

$$= \frac{1}{4} \int_{\mathbb{R}} e^{-x_{1}^{2}} \left(e^{x_{1}(x_{0}+x_{2})} + e^{-x_{1}(x_{0}+x_{2})} \right) dx_{1}$$

$$= \frac{\sqrt{\pi}}{4} e^{\frac{(x_{0}+x_{2})^{2}}{4}} + \frac{\sqrt{\pi}}{4} e^{\frac{(x_{0}-x_{2})^{2}}{4}} = \frac{\sqrt{\pi}}{2} e^{\frac{x_{0}^{2}+x_{2}^{2}}{4}} \cosh\left(\frac{x_{0}x_{2}}{2}\right).$$

Now we use the induction argument and suppose that the statement is true for k. Then we have

$$\int_{\mathbb{R}_{+}^{k}} e^{-\sum_{j=1}^{k} x_{j}^{2}} \prod_{j=1}^{k+1} \cosh(x_{j-1}x_{j}) dx_{1} \cdots dx_{k}$$

$$= \frac{1}{\sqrt{k}} \left(\frac{\pi}{2}\right)^{\frac{k-1}{2}} e^{\frac{k-1}{k} \frac{x_{0}^{2}}{2}} \int_{\mathbb{R}_{+}} e^{-\frac{k+1}{2k} x_{k}^{2}} \cosh\left(\frac{x_{0}x_{k}}{k}\right) \cosh(x_{k}x_{k+1}) dx_{k}$$

$$= \frac{1}{\sqrt{k}} \left(\frac{\pi}{2}\right)^{\frac{k-1}{2}} e^{\frac{k-1}{k} \frac{x_{0}^{2}}{2}} \sqrt{\frac{2k}{k+1}} \int_{\mathbb{R}_{+}} e^{-x_{k}^{2}} \cosh\left(\frac{1}{k} \sqrt{\frac{2k}{k+1}} x_{0}x_{k}\right) \cosh\left(\sqrt{\frac{2k}{k+1}} x_{k}x_{k+1}\right) dx_{k}$$

$$= \frac{\sqrt{2}}{\sqrt{k+1}} \left(\frac{\pi}{2}\right)^{\frac{k-1}{2}} e^{\frac{k-1}{k} \frac{x_{0}^{2}}{2}} \sqrt{\pi} e^{\frac{1}{4} \left(\frac{2}{k(k+1)} x_{0}^{2} + \frac{2k}{k+1} x_{k+1}^{2}\right)} \cosh\left(\frac{x_{0}x_{k+1}}{k+1}\right),$$

and the result follows after noting that $\frac{k-1}{k} + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

APPENDIX B. AN EQUIVALENT DETERMINANTAL FORMULA

The following proposition is given in [27, Section 3].

Proposition B.1 (Cf. Section 3 in [27]). We have

(B.1)
$$p_{N,k}^{(\tau)} = \left(\frac{1+\tau}{2}\right)^{N(N-1)/4} \frac{1}{2^{N/2} \prod_{l=1}^{N} \Gamma(l/2)} [z^{k/2}] \det \left[z \, \alpha_{2j-1,2m} + \beta_{2j-1,2m}[1]\right]_{j,m=1}^{N/2},$$

where

(B.2)
$$\alpha_{2j-1,2m}[1] = 2^m (m-1)! \sum_{p=1}^m \frac{\Gamma(j+p-3/2)}{2^{p-1}(p-1)!},$$

(B.3)
$$\beta_{2j-1,2m}[1] = -4 \operatorname{Im} \int_{\mathbb{R}} dx \int_{0}^{\infty} dy \, e^{y^{2}-x^{2}} \operatorname{erfc}\left(\sqrt{\frac{2}{1-\tau}}y\right) (x+iy)^{2j-2} (x-iy)^{2m-1}.$$

In particular, we have

(B.4)
$$p_{2n,0}^{(\tau)} = \left(\frac{1+\tau}{2}\right)^{n(2n-1)/2} \frac{1}{2^n \prod_{l=1}^{2n} \Gamma(l/2)} \det\left[\beta_{2j-1,2m}[1]\right]_{j,m=1}^n.$$

Example B.2. By (2.4), we have

(B.5)
$$M_2^{(\tau)} = \begin{bmatrix} \frac{\sqrt{2(1+\tau)}}{2} & \frac{(1-\tau)\sqrt{1+\tau}}{4} \\ \frac{(1-\tau)\sqrt{1+\tau}}{4} & \frac{\sqrt{2(1+\tau)(3+2t+3t^2)}}{16} \end{bmatrix}$$

Then one can check that the formulas (2.5) for n = 1, 2 give rise to

(B.6)
$$p_{2,0}^{(\tau)} = 1 - \frac{\sqrt{2(1+\tau)}}{2}, \qquad p_{4,0}^{(\tau)} = \frac{9+3\tau+3\tau^2+\tau^3}{8} - \frac{\sqrt{2(1+\tau)}(11+2\tau+3\tau^2)}{16}.$$

In particular, for $\tau = 0$, we have

(B.7)
$$p_{2,0}^{(0)} = 1 - \frac{\sqrt{2}}{2}, \qquad p_{4,0}^{(0)} = \frac{18 - 11\sqrt{2}}{16}.$$

The formula (B.7) also appeared in [14]. It is obvious that in the symmetric case when $\tau = 1$, $p_{2.0}^{(1)} = p_{4.0}^{(1)} = 0$.

On the other hand, by direct computations using (B.3), we have

$$\beta_{1,2}[1] = 2\sqrt{\pi} \frac{\sqrt{2} - s}{s}, \qquad \beta_{3,4}[1] = \sqrt{\pi} \frac{12\sqrt{2} - 16\sqrt{2}s + 12\sqrt{2}s^4 - 7s^5}{2s^5},$$
$$\beta_{3,2}[1] = -\sqrt{\pi} \frac{2\sqrt{2} - 2\sqrt{2}s^2 + s^3}{s^3}, \qquad \beta_{1,4}[1] = -\sqrt{\pi} \frac{2\sqrt{2} - 6\sqrt{2}s^2 + 5s^3}{s^3},$$

where $s = \sqrt{1+\tau}$. Then by (B.4), we have (B.6).

References

- [1] G. Akemann and S.-S. Byun. The product of m real $N \times N$ Ginibre matrices: Real eigenvalues in the critical regime m = O(N). Constr. Approx. (online). arXiv:2201.07668, 2022.
- [2] G. Akemann, M. Duits, and L. D. Molag. The elliptic Ginibre ensemble: A unifying approach to local and global statistics for higher dimensions. *J. Math. Phys.*, 64:023503, 2023.
- [3] G. Akemann and E. Kanzieper. Integrable structure of Ginibre's ensemble of real random matrices and a Pfaffian integration theorem. J. Stat. Phys., 129(5-6):1159–1231, 2007.
- [4] G. Akemann and M. J. Phillips. The interpolating Airy kernels for the $\beta = 1$ and $\beta = 4$ elliptic Ginibre ensembles. J. Stat. Phys., 155(3):421–465, 2014.
- [5] J. Alt and T. Krüger. Local elliptic law. Bernoulli, 28(2):886-909, 2022.
- [6] A. Borodin and E. Kanzieper. A note on the Pfaffian integration theorem. J. Phys. A, 40(36):F849–F855, 2007.
- [7] S.-S. Byun and C. Charlier. On the characteristic polynomial of the eigenvalue moduli of random normal matrices. preprint arXiv:2205.04298, 2022.
- [8] S.-S. Byun and P. J. Forrester. Progress on the study of the Ginibre ensembles I: GinUE. preprint arXiv:2211.16223, 2022.
- [9] S.-S. Byun and P. J. Forrester. Progress on the study of the Ginibre ensembles II: GinOE and GinSE. preprint arXiv:2301.05022, 2023.
- [10] S.-S. Byun, N.-G. Kang, J. O. Lee, and J. Lee. Real eigenvalues of elliptic random matrices. Int. Math. Res. Not., (3):2243–2280, 2023.
- [11] C. Charlier. Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles. *Adv. Math.*, 408(part A):Paper No. 108600, 36, 2022.
- [12] L. C. G. del Molino, K. Pakdaman, J. Touboul, and G. Wainrib. The real Ginibre ensemble with k = O(n) real eigenvalues. J. Stat. Phys., 163(2):303–323, 2016.
- [13] I. Dumitriu and P. J. Forrester. Tridiagonal realization of the antisymmetric Gaussian β -ensemble. J. Math. Phys., 51(9):093302, 25, 2010.
- [14] A. Edelman. The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law. J. $Multivariate\ Anal.$, 60(2):203-232, 1997.
- [15] A. Edelman, E. Kostlan, and M. Shub. How many eigenvalues of a random matrix are real? J. Amer. Math. Soc., 7(1):247–267, 1994.
- [16] M. Fenzl and G. Lambert. Precise deviations for disk counting statistics of invariant determinantal processes. Int. Math. Res. Not., 2022(10):7420-7494, 2022.
- [17] J. Fischmann, W. Bruzda, B. A. Khoruzhenko, H.-J. Sommers, and K. Życzkowski. Induced Ginibre ensemble of random matrices and quantum operations. J. Phys. A, 45(7):075203, 2012.
- [18] J. Fischmann and P. J. Forrester. One-component plasma on a spherical annulus and a random matrix ensemble. J. Stat. Mech.: Theor. Exp., 2011(10):P10003, 2011.

- [19] W. FitzGerald and N. Simm. Fluctuations and correlations for products of real asymmetric random matrices. Ann. Inst. Henri Poincare (B) Probab. Stat (to appear) arXiv:2109.00322, 2021.
- [20] P. J. Forrester. The limiting Kac random polynomial and truncated random orthogonal matrices. J. Stat. Mech.: Theor. Exp., 2010(12):P12018, 2010.
- [21] P. J. Forrester. Skew orthogonal polynomials for the real and quaternion real ginibre ensembles and generalizations. J. Phys. A, 46(24):245203, 2013.
- [22] P. J. Forrester. Diffusion processes and the asymptotic bulk gap probability for the real Ginibre ensemble. *J. Phys.* A, 48(32):324001, 2015.
- [23] P. J. Forrester, J. R. Ipsen, and S. Kumar. How many eigenvalues of a product of truncated orthogonal matrices are real? Exp. Math., 29(3):276–290, 2020.
- [24] P. J. Forrester and A. Mays. A method to calculate correlation functions for $\beta = 1$ random matrices of odd size. *J. Stat. Phys.*, 134(3):443–462, 2009.
- [25] P. J. Forrester and A. Mays. Pfaffian point process for the Gaussian real generalised eigenvalue problem. Probab. Theory Related Fields, 154(1-2):1–47, 2012.
- [26] P. J. Forrester and T. Nagao. Eigenvalue statistics of the real Ginibre ensemble. Phys. Rev. Lett., 99(5):050603, 2007.
- [27] P. J. Forrester and T. Nagao. Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble. J. Phys.~A,~41(37):375003,~19,~2008.
- [28] P. J. Forrester and E. Nordenstam. The anti-symmetric GUE minor process. Mosc. Math. J., 9(4):749-774, 934, 2009.
- [29] Y. V. Fyodorov and B. A. Khoruzhenko. Nonlinear analogue of the May-Wigner instability transition. Proc. Natl. Acad. Sci. USA, 113(25):6827–6832, 2016.
- [30] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers. Almost Hermitian random matrices: crossover from Wigner-Dyson to Ginibre eigenvalue statistics. *Phys. Rev. Lett.*, 79(4):557–560, 1997.
- [31] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers. Almost-Hermitian random matrices: eigenvalue density in the complex plane. Phys. Lett. A, 226(1-2):46-52, 1997.
- [32] Y. V. Fyodorov, H.-J. Sommers, and B. A. Khoruzhenko. Universality in the random matrix spectra in the regime of weak non-Hermiticity. Ann. Inst. H. Poincaré Phys. Théor., 68(4):449–489, 1998.
- [33] Y. V. Fyodorov and W. Tarnowski. Condition numbers for real eigenvalues in the real elliptic Gaussian ensemble. Ann. Henri Poincaré, 22(1):309–330, 2021.
- [34] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. J. Math. Phys., 6(3):440–449, 1965.
- [35] V. L. Girko. Elliptic law. Theory Probab. Appl., 30(4):677–690, 1986.
- [36] E. Kanzieper and G. Akemann. Statistics of real eigenvalues in Ginibre's ensemble of random real matrices. *Phys. Rev. Lett.*, 95(23):230201, 4, 2005.
- [37] E. Kanzieper, M. Poplavskyi, C. Timm, R. Tribe, and O. Zaboronski. What is the probability that a large random matrix has no real eigenvalues? *Ann. Appl. Probab.*, 26(5):2733–2753, 2016.
- [38] B. A. Khoruzhenko, H.-J. Sommers, and K. Życzkowski. Truncations of random orthogonal matrices. *Phys. Rev. E*, 82(4):040106, 2010.
- [39] N. Lehmann and H.-J. Sommers. Eigenvalue statistics of random real matrices. *Phys. Rev. Lett.*, 67(8):941–944, 1991.
- [40] A. Little, F. Mezzadri, and N. Simm. On the number of real eigenvalues of a product of truncated orthogonal random matrices. *Electron. J. Probab.*, 27:Paper No. 5, 32, 2022.
- [41] R. M. May. Will a large complex system be stable? Nature, 238:413-414, 1972.
- [42] L. D. Molag. Edge universality of random normal matrices generalizing to higher dimensions. preprint arXiv:2208.12676, 2022.
- [43] H. H. Nguyen and S. O'Rourke. The elliptic law. Int. Math. Res. Not., 2015(17):7620-7689, 2015.
- [44] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Editors). NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge, 2010.
- [45] N. Simm. Central limit theorems for the real eigenvalues of large Gaussian random matrices. Random Matrices Theory Appl., 6(1):1750002, 18, 2017.
- [46] N. Simm. On the real spectrum of a product of Gaussian matrices. Electron. Commun. Probab., 22:Paper No. 41, 11, 2017.
- [47] C. D. Sinclair. Averages over Ginibre's ensemble of random real matrices. *Int. Math. Res. Not.*, (5):Art. ID rnm015, 15, 2007.
- [48] H.-J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein. Spectrum of large random asymmetric matrices. Phys. Rev. Lett., 60(19):1895–1898, 1988.

Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemungu, Seoul 02455, Republic of Korea

 $Email\ address: {\tt sungsoobyun@kias.re.kr}$

Department of Mathematics, University of Sussex, Brighton, BN1 9RH, United Kingdom $\it Email\ address$: L.D.Molag@sussex.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SUSSEX, BRIGHTON, BN1 9RH, UNITED KINGDOM

 $Email\ address{:}\ {\tt n.j.simm@sussex.ac.uk}$