

Large Cycles in Graphs Around Conjectures of Bondy and Jung - Modifications and Sharpness

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Abstract

A number of new sufficient conditions for generalized cycles (large cycles including Hamilton and dominating cycles as special cases) in an arbitrary κ -connected graph ($\kappa = 1, 2, \dots$) and new lower bounds for the circumference (the length of a longest cycle) are derived, inspiring a number of modifications of famous conjectures of Bondy (1980) and Jung (2001). All results are shown to be best possible in a sense and cannot be derived directly from Bondy's and Jung's conjectures as special cases.

Keywords: Hamilton cycle, Dominating cycle, Longest cycle, Large cycle.

MSC-class: 05C38 (primary), 05C45, 05C40 (secondary)

1 Introduction

Only finite undirected graphs without loops or multiple edges are considered. Notation and terminology not defined here follow that in [3]. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset $S \subset V(G)$, we denote by $G - S$ the maximum subgraph of G with vertex set $V(G) - S$. For a subgraph H of G , we use $G - H$ to denote $G - V(H)$.

We use δ and α to denote the minimum degree and the independence number of G , respectively. The minimum degree sum of any k independent vertices in a graph is denoted by σ_k , if $\alpha \geq k$. If $\alpha < k$, we set $\sigma_k = +\infty$. For a special case, we have $\sigma_1 = \delta$.

By the standard definition, a sequence $v_1 v_2 \dots v_t v_1$ of distinct vertices v_1, \dots, v_t in G is called a simple cycle (or just a cycle) of order t (the number of vertices) if $v_i v_{i+1} \in E(G)$ for each $i \in \{1, \dots, t\}$, where $v_{t+1} = v_1$. In particular, for $t = 1$, the cycle v_1 coincides with the vertex v_1 . So, each vertex and edge in a graph can be considered as a cycle of orders 1 and 2, respectively.

A cycle in a graph G is called a Hamilton cycle of G if it contains all the vertices of G . A graph G is called hamiltonian if G contains a Hamilton cycle.

A cycle Q of a graph G is called dominating cycle if every edge of G is incident with at least one vertex of Q .

In 1980, Bondy [4] introduced the first type of generalized cycles, including Hamilton and dominating cycles as special cases: for a positive integer λ , we call Q a D_λ -cycle if $|H| \leq \lambda - 1$ for every component H of $G - Q$. In other words, Q is a D_λ -cycle of G if and only if every connected subgraph of order λ of G has at least one vertex with Q in common. In fact, a D_λ -cycle dominates all connected subgraphs of order λ . According to this definition, Q is a Hamilton cycle if and only if Q is a D_1 -cycle; and Q is a dominating cycle if and only if Q is a D_2 -cycle.

In this paper, we consider another two types of generalized cycles including Hamilton and dominating cycles as special cases. For a positive integer λ , a cycle Q is called a PD_λ -cycle (PD - Path Dominating) if each path of order at least λ in G has at least one vertex with Q in common. Next, we call a cycle Q a CD_λ -cycle (CD - Cycle Dominating; introduced in [13]) if each cycle of order at least λ has at least one vertex with Q in common. Actually, a PD_λ -cycle dominates all paths of order λ in G ; and a CD_λ -cycle dominates all cycles of order λ in G . In terms of PD_λ and CD_λ -cycles, Q is a Hamilton cycle if and only if either Q is a PD_1 -cycle or a CD_1 -cycle. Further, Q is a dominating cycle if and only if either Q is a PD_2 -cycle or a CD_2 -cycle.

Throughout the paper, we consider a graph G on n vertices with minimum degree δ and connectivity κ . Further, let C be a longest cycle in G with $c = |C|$, and let \bar{p} and \bar{c} denote the orders of a longest path and

a longest cycle in $G - C$, respectively. In terms of \bar{p} and \bar{c} , C is a Hamilton cycle if and only if either $\bar{p} \leq 0$ or $\bar{c} \leq 0$. Similarly, C is a dominating cycle if and only if $\bar{p} \leq 1$ or $\bar{c} \leq 1$.

In 1980, Bondy [4] conjectured a common generalization of some well-known degree-sum conditions for PD_λ -cycles (called (σ, \bar{p}) -version) including Hamilton cycles (PD_1 -cycles) and dominating cycles (PD_2 -cycles) as special cases.

Conjecture A (Bondy [4], 1980): (σ, \bar{p}) -version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{p} \leq \lambda - 1$.

Parts of Conjecture A were proved by Ore [16] ($\lambda = 1$), Bondy [4] ($\lambda = 2$) and Zou [18] ($\lambda = 3$). For the general case, Conjecture A is still open.

The long cycles analogue (so called reverse version) of Bondy's conjecture (Conjecture A) can be formulated as follows.

Conjecture B : (reverse, σ, \bar{p})-version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G . If $\bar{p} \geq \lambda - 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Parts of Conjecture B were proved by Dirac [6] ($\lambda = 1$), Bondy [2], Bermond [1], Linial [11] ($\lambda = 2$), Fraisse, Yung [8] ($\lambda = 3$) and Chiba, Tsugaki, Yamashita [5] ($\lambda = 4$).

The initial motivations of Conjecture A and Conjecture B come from their minimal degree versions - the most popular and much studied versions, which also remain unsolved.

Conjecture C (Bondy [4], 1980): (δ, \bar{p}) -version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{p} \leq \lambda - 1$.

Parts of Conjecture C were proved by Dirac [6] ($\lambda = 1$), Nash-Williams [12] ($\lambda = 2$) and Fan [7] ($\lambda = 3$).

Conjecture D : (Jung [10], 2001): (reverse, δ, \bar{p})-version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G . If $\bar{p} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Parts of Conjecture D were proved by Dirac [6] ($\lambda = 1$), Dirac [6] ($\lambda = 2$), Voss, Zuluaga [17] ($\lambda = 3$) and Jung [9] ($\lambda = 4$).

Note that CD_λ -cycles are more suitable for research than PD_λ -cycles since cycles in $G - C$ are more symmetrical than paths in view of the connections between $G - C$ and CD_λ -cycles. This is the main reason why some minimum degree versions of Conjectures C and D have been solved just for CD_λ -cycles.

According to above arguments, it is natural to consider the exact analogues of Bondy's generalized conjecture (Conjecture A) and its reverse version (Conjecture B) for CD_λ -cycles which we call (σ, \bar{c}) and (reverse, σ, \bar{c})-versions, respectively.

Conjecture E : (σ, \bar{c}) -version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \lambda - 1$.

Conjecture F : (reverse, σ, \bar{c})-version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \lambda - 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

In 2009, the author proved [14] the validity of minimum degree versions of Conjectures E and F.

Theorem A ([14], 2009): (δ, \bar{c}) -version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \lambda - 1$.

Theorem B ([14], 2009): (reverse, δ, \bar{c})-version

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Actually, in [14] it was proved a significantly stronger result than Theorem A by showing that the conclusion $\bar{c} \leq \lambda - 1$ in Theorem A can be strengthened to $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$, called \bar{c} -improvement.

Theorem C ([14], 2009): (δ, \bar{c}) -version, \bar{c} -improvement

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Recently, further improvements of Theorems A and C are presented [15] inspiring new conjectures in forms of improvements of initial generalized conjecture of Bondy.

Theorem D ([15], 2022): (δ, \bar{c}) -version, κ -improvement

Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \lambda - 1$.

Theorem E ([15], 2022): (δ, \bar{c}) -version, (\bar{c}, κ) -improvement

Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Analogously, the condition $\bar{c} \geq \lambda - 1$ in Theorem B was weakened [14] to $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$.

Theorem F ([14], 2009): (reverse, δ, \bar{c})-version, \bar{c} -improvement

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G . If $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

Furthermore, it was proved [15] that the connectivity condition $\kappa \geq \lambda$ in Theorem B can be weakened to $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$.

Theorem G ([15], 2022): (reverse, δ, \bar{c})-version, κ -improvement

Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\bar{c} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Finally, it was proved [15] that the connectivity condition $\kappa \geq \lambda$ in Theorem F can be weakened to $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$.

Theorem H ([15], 2022): (reverse, δ, \bar{c})-version, (\bar{c}, κ) -improvement

Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

In this paper we present a number of new sufficient conditions for large cycles and new lower bounds for the circumference around conjectures of Bondy and Jung which cannot be derived directly from these conjectures as special cases. They can be characterized as modifications around conjectures of Bondy and Jung. Each of these modifications is shown to be best possible in a sense. Consider Theorem A to determine some kinds of sharpness. We say that Theorem A is \bar{c} -sharp if the conclusion $\bar{c} \leq \lambda - 1$ in Theorem A cannot be strengthened to $\bar{c} \leq \lambda - 2$, that is, the upper bound $\lambda - 1$ is available. Further, Theorem A is δ -sharp if the minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ cannot be weakened to $\delta \geq \frac{n+1}{\lambda+1} + \lambda - 2$. Finally, Theorem A is κ -sharp if the connectivity condition $\kappa \geq \lambda$ cannot be weakened to $\kappa \geq \lambda - 1$. The c -sharpness, κ -sharpness and \bar{c} -sharpness for long cycle versions (reverse versions), say for theorem B, can be defined analogously.

2 Modifications

The first modification can be obtained from Theorem A by replacing the conclusion $\bar{c} \leq \lambda - 1$ with $\bar{c} \leq \delta - \lambda$, called \bar{c} -modification.

Theorem 1: (δ, \bar{c}) -version, \bar{c} -modification

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \delta - \lambda$.

The next modification can be obtained by replacing the connectivity condition $\kappa \geq \lambda$ in Theorem A with $\kappa \geq \delta - \lambda + 1$, called κ -modification.

Theorem 2: (δ, \bar{c}) -version, κ -modification

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \lambda - 1$.

Another modification can be obtained by replacing the conclusion $\bar{c} \leq \lambda - 1$ in Theorem 2 with $\bar{c} \leq \delta - \lambda$, we obtain a new modification, called (\bar{c}, κ) -modification.

Theorem 3: (δ, \bar{c}) -version, (\bar{c}, κ) -modification

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \delta - \lambda$.

Further, Theorem 2 can be improved by strengthening the conclusion $\bar{c} \leq \lambda - 1$ to $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$, called κ -modification with \bar{c} -improvement.

Theorem 4: (δ, \bar{c}) -version, κ -modification, \bar{c} -improvement

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Finally, Theorem 3 can be improved by relaxing the connectivity condition $\kappa \geq \delta - \lambda + 1$ to $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$, called \bar{c} -modification with κ -improvement.

Theorem 5: (δ, \bar{c}) -version, \bar{c} -modification, κ -improvement

Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\bar{c} \leq \delta - \lambda$.

The reverse versions of Theorems 1-5 are generated from Theorem B using different combinations of conditions $\kappa \geq \lambda$, $\kappa \geq \delta - \lambda + 2$, $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\bar{c} \geq \lambda - 1$, $\bar{c} \geq \delta - \lambda + 1$, $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$.

The first result can be considered as \bar{c} -modification of Theorem B.

Theorem 6: (reverse, δ , \bar{c})-version, \bar{c} -modification

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

The κ -modification of Theorem B can be formulated as follows.

Theorem 7: (reverse, δ , \bar{c})-version, κ -modification

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem B after \bar{c} -modification and κ -modification, can be formulated as follows.

Theorem 8: (reverse, δ , \bar{c})-version, (\bar{c}, κ) -modification

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

The next result can be obtained from Theorem 8 by \bar{c} -improvement.

Theorem 9: (reverse, δ , \bar{c})-version, κ -modification, \bar{c} -improvement

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

Finally, κ -improvement of Theorem 8 implies the following.

Theorem 10: (reverse, δ , \bar{c})-version, κ -improvement, \bar{c} -modification

Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Observe that none of theorems 1-10 follows from Conjectures of Bondy and Jung as a special case.

3 Generalized modifications

In this section, motivated by Theorems 1-10 (minimum degree versions), we propose their generalized versions in terms of degree sums $((\sigma, \bar{c})$ -versions) as generalized modifications around Conjectures of Bondy and Jung.

The first conjecture is a (σ, \bar{c}) -generalization of Theorem 1.

Conjecture 1: (σ, \bar{c}) -version, \bar{c} -modification

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \delta - \lambda$.

The (σ, \bar{c}) -version of Theorem 2 can be formulated as follows.

Conjecture 2: (σ, \bar{c}) -version, κ -modification

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \lambda - 1$.

After (σ, \bar{c}) -generalization, Theorem 3 implies the following.

Conjecture 3: (σ, \bar{c}) -version, κ -modification, \bar{c} -modification

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \delta - \lambda$.

Theorem 4, after (σ, \bar{c}) -generalization, implies the following,

Conjecture 4: (σ, \bar{c}) -version, κ -modification, \bar{c} -improvement

Let C be a longest cycle in a $(\delta - \lambda + 1)$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Finally, Theorem 5 implies the next conjecture after (σ, \bar{c}) -generalization.

Conjecture 5: (σ, \bar{c}) -version, \bar{c} -modification, κ -improvement

Let C be a longest cycle in a $\min\{\lambda, \delta - \lambda + 1\}$ -connected ($1 \leq \lambda \leq \delta$) graph G of order n . If $\sigma_{\lambda+1} \geq n + \lambda(\lambda - 1)$, then $\bar{c} \leq \delta - \lambda$.

The next five conjectures can be proposed by replacing the conclusion $c \geq \lambda(\delta - \lambda + 2)$ in Theorems 6-10 with $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Conjecture 6: (reverse, σ, \bar{c})-version, \bar{c} -modification

Let C be a longest cycle in a λ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Conjecture 7: (reverse, σ, \bar{c})-version, κ -modification

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \lambda - 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Conjecture 8: (reverse, σ, \bar{c})-version, κ -modification, \bar{c} -modification

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Conjecture 9: (reverse, δ, \bar{c})-version, κ -modification, \bar{c} -improvement

Let C be a longest cycle in a $(\delta - \lambda + 2)$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

Conjecture 10: (reverse, δ, \bar{c})-version, \bar{c} -modification, κ -improvement

Let C be a longest cycle in a $\min\{\lambda, \delta - \lambda + 2\}$ -connected ($1 \leq \lambda \leq \delta$) graph. If $\bar{c} \geq \delta - \lambda + 1$, then $c \geq \sigma_\lambda - \lambda(\lambda - 2)$.

The (σ, \bar{p}) -versions of Conjectures 1-10 can be proposed by a similar way.

4 Proofs

The proofs are based on several implications with the following two schemes:

$$\begin{aligned}
\text{Theorem E} &\Rightarrow \begin{cases} \text{Theorem C} \Rightarrow \text{Theorem A,} \\ \text{Theorem D} \Rightarrow \text{Theorem A,} \\ \text{Theorem 4} \Rightarrow \text{Theorem 2 and Theorem 3.} \\ \text{Theorem 5} \Rightarrow \text{Theorem 1 and Theorem 3,} \end{cases} \\
\text{Theorem H} &\Rightarrow \begin{cases} \text{Theorem F} \Rightarrow \text{Theorem B,} \\ \text{Theorem G} \Rightarrow \text{Theorem B,} \\ \text{Theorem 9} \Rightarrow \text{Theorem 7 and Theorem 8,} \\ \text{Theorem 10} \Rightarrow \text{Theorem 6 and Theorem 8.} \end{cases}
\end{aligned}$$

5 On sharpness

To show the $c, \bar{c}, \kappa, \delta$ -sharpness of all modifications (Theorems 1-10), we need analogous observations around original Theorems A, B and for all improvements - Theorems C, D, E, F, G and H.

For Theorem A, take $\delta - \lambda + 3$ disjoint copies of the complete graph $K_{\lambda-1}$ and join each vertex in their union to every vertex of a disjoint complete graph $K_{\delta-\lambda+2}$, denoted by $G = (\delta - \lambda + 3)K_{\lambda-1} + K_{\delta-\lambda+2}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \lambda - 1$ and $n(G) = \lambda(\delta - \lambda + 3) - 1$. The connectivity condition $\kappa(G) = \delta - \lambda + 2 \geq \lambda$ holds whenever $\lambda \leq \frac{\delta+1}{2}$. Further, the minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ is guaranteed whenever $\lambda \leq \frac{\delta+1}{2}$. So, Theorem A is \bar{c} -sharp for each $\lambda \leq \frac{\delta+1}{2}$.

Now consider the κ -sharpness of Theorem A. Let $G = \lambda K_{\delta-\lambda+2} + K_{\lambda-1}$. Then $\delta(G) = \delta$, $\kappa(G) = \lambda - 1$ and $n(G) = \lambda(\delta - \lambda + 3) - 1$. If $\lambda \leq \frac{\delta+1}{2}$, then it is easy to check that $\bar{c}(G) = \delta - \lambda + 2 \geq \lambda$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$. This graph example shows that the connectivity condition $\kappa \geq \lambda$ cannot be relaxed to $\kappa \geq \lambda - 1$. So, Theorem A is κ -sharp for each $\lambda \leq \frac{\delta+1}{2}$.

As for δ -sharpness of Theorem A, we consider the graph example $G = (\delta - \lambda + 2)K_{\lambda} + K_{\delta-\lambda+1}$ with $\delta(G) = \delta$, $\bar{c}(G) = \lambda$ and $n(G) = (\lambda + 1)(\delta - \lambda + 2) - 1$. Then

$$\delta = \frac{(\lambda + 1)(\delta - \lambda + 2)}{\lambda + 1} + \lambda - 2 = \frac{n + 1}{\lambda + 1} + \lambda - 2.$$

Since the connectivity condition $\kappa(G) = \delta - \lambda + 1 \geq \lambda$ holds for each $\lambda \leq \frac{\delta+1}{2}$, we conclude that Theorem A is δ -sharp for each $\lambda \leq \frac{\delta+1}{2}$.

Combining above observations, we obtain the following.

Proposition 1.

$$\text{Theorem A is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}. \end{cases}$$

For Theorem 3, assume first that $G = (\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$. Then $\delta(G) = \delta$ and $n(G) = (\lambda + 2)(\delta - \lambda + 1) - 1$. It is easy to see that the connectivity condition $\kappa(G) = \lambda + 1 \geq \delta - \lambda + 1$ and the minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ are guaranteed whenever $\lambda \geq \frac{\delta+1}{2}$. Since $\bar{c}(G) = \delta - \lambda$, Theorem 3 is \bar{c} -sharp for each $\lambda \geq \frac{\delta+1}{2}$.

Now let $G = (\delta - \lambda + 1)K_{\lambda+1} + K_{\delta-\lambda}$. We have $\delta(G) = \delta$ and $n(G) = (\lambda + 2)(\delta - \lambda + 1) - 1$. The inequality $\bar{c}(G) = \lambda + 1 \geq \delta - \lambda + 1$ and the minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ are guaranteed whenever $\lambda \geq \frac{\delta+1}{2}$. Observing that $\kappa(G) = \delta - \lambda$, we conclude that Theorem 3 is κ -sharp for each $\lambda \geq \frac{\delta+1}{2}$.

For δ -sharpness, we let $G = (\delta - \lambda + 2)K_{\lambda} + K_{\delta-\lambda+1}$. Then $\delta(G) = \delta$, $\kappa(G) = \delta - \lambda + 1$ and $n(G) = (\lambda + 1)(\delta - \lambda + 2) - 1$. Therefore,

$$\delta = \frac{(\lambda + 1)(\delta - \lambda + 2)}{\lambda + 1} + \lambda - 2 = \frac{n + 1}{\lambda + 1} + \lambda - 2.$$

Since the inequality $\bar{c}(G) = \lambda \geq \delta - \lambda + 1$ holds whenever $\lambda \geq \frac{\delta+1}{2}$, we conclude that Theorem 3 is δ -sharp for each $\lambda \geq \frac{\delta+1}{2}$.

Thus, we have the following.

Proposition 2.

$$\text{Theorem 3 is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}. \end{cases}$$

Consider the sharpness of Theorem 1. Let $G = (\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$. Then $\delta(G) = \delta$, $\kappa(G) = \lambda + 1 > \lambda$ and $n(G) = (\lambda + 2)(\delta - \lambda + 1) - 1$. It is easy to see that the minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ is guaranteed whenever $\lambda \geq \frac{\delta+1}{2}$. Since $\bar{c}(G) = \delta - \lambda$, Theorem 1 is \bar{c} -sharp for each $\lambda \geq \frac{\delta+1}{2}$.

Consider the κ -sharpness of Theorem 1. Let $G = \lambda K_{\delta-\lambda+2} + K_{\lambda-1}$. Then $\delta(G) = \delta$, $\kappa(G) = \lambda - 1$, $\bar{c}(G) = \delta - \lambda + 2 > \delta - \lambda + 1$ and $n(G) = \lambda(\delta - \lambda + 3) - 1$. If $\lambda \leq \frac{\delta+1}{2}$, then it is easy to check that $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$. This graph example shows that the connectivity condition $\kappa \geq \lambda$ cannot be relaxed to $\kappa \geq \lambda - 1$. So, Theorem 1 is κ -sharp for each $\lambda \leq \frac{\delta+1}{2}$.

Let $G = (\lambda + 1)K_{\delta-\lambda+1} + K_{\lambda}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \delta - \lambda + 1$, $\kappa(G) = \lambda$ and $n(G) = (\lambda + 1)(\delta - \lambda + 2) - 1$. Observing also that

$$\delta = \frac{(\lambda + 1)(\delta - \lambda + 2)}{\lambda + 1} = \frac{n + 1}{\lambda + 1} + \lambda - 2,$$

we conclude that Theorem 1 is δ -sharp for each $1 \leq \lambda \leq \delta$.

Thus, we have the following.

Proposition 3.

$$\text{Theorem 1 is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

For Theorem 2, take $G = (\delta - \lambda + 3)K_{\lambda-1} + K_{\delta-\lambda+2}$. Then $\delta(G) = \delta$, $n(G) = \lambda(\delta - \lambda + 3) - 1$ and $\kappa(G) = \delta - \lambda + 2 > \delta - \lambda + 1$. The minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ is guaranteed whenever $\lambda \leq \frac{\delta+1}{2}$. Recalling that $\bar{c}(G) = \lambda - 1$, we conclude that Theorem 2 is \bar{c} -sharp for each $\lambda \leq \frac{\delta+1}{2}$.

For κ -sharpness, let $G = (\delta - \lambda + 1)K_{\lambda+1} + K_{\delta-\lambda}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \lambda + 1 > \lambda$ and $n(G) = (\lambda + 2)(\delta - \lambda + 1) - 1$. The minimum degree condition $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$ is guaranteed whenever $\lambda \geq \frac{\delta+1}{2}$. Observing also that $\kappa(G) = \delta - \lambda$, we conclude that Theorem 2 is κ -sharp for each $\lambda \geq \frac{\delta+1}{2}$.

For δ -sharpness, let $G = (\delta - \lambda + 2)K_{\lambda} + K_{\delta-\lambda+1}$. Then $\delta(G) = \delta$, $\kappa(G) = \delta - \lambda + 1$, $\bar{c}(G) = \lambda$ and $n(G) = (\lambda + 1)(\delta - \lambda + 2) - 1$. Therefore,

$$\delta = \frac{(\lambda + 1)(\delta - \lambda + 2)}{\lambda + 1} + \lambda - 2 = \frac{n + 1}{\lambda + 1} + \lambda - 2.$$

Hence, Theorem 2 is δ -sharp for each $1 \leq \lambda \leq \delta$.

Thus, we have the following.

Proposition 4.

$$\text{Theorem 2 is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem C. If $\lambda \leq \frac{\delta+1}{2}$, then we have Theorem A. By Proposition 1, Theorem C is $(\bar{c}, \kappa, \delta)$ -sharp. If $\lambda \geq \frac{\delta+1}{2}$, then we have Theorem 1. By Proposition 3, Theorem C is (\bar{c}, δ) -sharp. So, we have the following.

Proposition 5.

$$\text{Theorem C is } \begin{cases} \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem D. If $\lambda \leq \frac{\delta+1}{2}$, then we have Theorem A. By Proposition 1, Theorem D is $(\bar{c}, \kappa, \delta)$ -sharp. If $\lambda \geq \frac{\delta+1}{2}$, then we have Theorem 2. By Proposition 4, Theorem D is (κ, δ) -sharp. So, we have the following.

Proposition 6.

$$\text{Theorem D is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem 5. If $\lambda \leq \frac{\delta+1}{2}$, then we have Theorem 1. By Proposition 3, Theorem 5 is (κ, δ) -sharp. If $\lambda \geq \frac{\delta+1}{2}$, then we have Theorem 3. By Proposition 2, Theorem 5 is $(\bar{c}, \kappa, \delta)$ -sharp. So, we have the following.

Proposition 7.

$$\text{Theorem 5 is } \begin{cases} \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem 4. If $\lambda \leq \frac{\delta+1}{2}$, then we have Theorem 2. By Proposition 4, Theorem 4 is (\bar{c}, δ) -sharp. If $\lambda \geq \frac{\delta+1}{2}$, then we have Theorem 3. By Proposition 2, Theorem 4 is $(\bar{c}, \kappa, \delta)$ -sharp. So, we have the following.

Proposition 8.

$$\text{Theorem 4 is } \begin{cases} \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+1}{2}, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem E. If $\lambda \leq \frac{\delta+1}{2}$, then we have Theorem A. By Proposition 1, Theorem E is $(\bar{c}, \kappa, \delta)$ -sharp. If $\lambda \geq \frac{\delta+1}{2}$, then we have Theorem 3. By Proposition 2, Theorem E is $(\bar{c}, \kappa, \delta)$ -sharp as well. So, we have the following.

Proposition 9.

$$\text{Theorem E is } \begin{cases} \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \delta\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Now we turn to the sharpness observations around Jung's conjecture.

Consider the sharpness of Theorem B. Let $G = (\delta - \lambda + 3)K_{\lambda-1} + K_{\delta-\lambda+2}$. Then $\delta(G) = \delta$ and $\bar{c}(G) = \lambda - 1$. The connectivity condition $\kappa(G) = \delta - \lambda + 2 \geq \lambda$ holds whenever $\lambda \leq \frac{\delta+2}{2}$. Observing also that $c(G) = \lambda(\delta - \lambda + 2)$, we conclude that Theorem B is c -sharp for each $\lambda \leq \frac{\delta+2}{2}$.

Consider the κ -sharpness of Theorem B. Let $G = \lambda K_{\delta-\lambda+2} + K_{\lambda-1}$. Then $\delta(G) = \delta$ and $c(G) = (\lambda - 1)(\delta - \lambda + 3)$. The inequalities $\bar{c}(G) = \delta - \lambda + 2 \geq \lambda - 1$ and $c(G) < \lambda(\delta - \lambda + 2)$ are satisfied whenever $\lambda \leq \frac{\delta+2}{2}$. Observing also that $\kappa(G) = \lambda - 1$, we conclude that Theorem B is κ -sharp for each $\lambda \leq \frac{\delta+2}{2}$.

Consider the \bar{c} -sharpness of Theorem B. Let $G = (\delta - \lambda + 4)K_{\lambda-2} + K_{\delta-\lambda+3}$. Then $\delta(G) = \delta$ and $c(G) = (\lambda - 1)(\delta - \lambda + 3)$. The connectivity condition $\kappa(G) = \delta - \lambda + 3 \geq \lambda$ and the inequality $c(G) < \lambda(\delta - \lambda + 2)$ are satisfied whenever $\lambda \leq \frac{\delta+2}{2}$. Recalling also that $\bar{c}(G) = \lambda - 2$, we conclude that Theorem B is \bar{c} -sharp for each $\lambda \leq \frac{\delta+2}{2}$.

Combining these observations, we have the following.

Proposition 10.

$$\text{Theorem B is } \begin{cases} c\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem 8. Let $G = (\delta - \lambda + 3)K_{\lambda-1} + K_{\delta-\lambda+2}$ with $\delta(G) = \delta$, $\kappa(G) = \delta - \lambda + 2$ and $\bar{c}(G) = \lambda - 1$. The inequality $\bar{c}(G) = \lambda - 1 \geq \delta - \lambda + 1$ holds whenever $\lambda \geq \frac{\delta+2}{2}$. Observing also that $c(G) = \lambda(\delta - \lambda + 2)$, we conclude that Theorem 8 is c -sharp for each $\lambda \geq \frac{\delta+2}{2}$.

Consider the κ -sharpness of Theorem 8. Let $G = (\delta - \lambda + 2)K_{\lambda} + K_{\delta-\lambda+1}$. Then $\delta(G) = \delta$, $c(G) = (\lambda + 1)(\delta - \lambda + 1)$. The inequalities $\bar{c}(G) = \lambda \geq \delta - \lambda + 1$ and $c(G) < \lambda(\delta - \lambda + 2)$ are satisfied whenever $\lambda \geq \frac{\delta+2}{2}$. Since $\kappa(G) = \delta - \lambda + 1$, Theorem 8 is κ -sharp for each $\lambda \geq \frac{\delta+2}{2}$.

Consider the \bar{c} -sharpness of Theorem 8. Let $G = (\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$ with $\delta(G) = \delta$ and $c(G) = (\lambda + 1)(\delta - \lambda + 1)$. The inequalities $\kappa(G) = \lambda + 1 \geq \delta - \lambda + 2$ and $c(G) < \lambda(\delta - \lambda + 2)$ are satisfied whenever $\lambda \geq \frac{\delta+2}{2}$. Observing also that $\bar{c}(G) = \delta - \lambda$, we conclude that Theorem 8 is \bar{c} -sharp for each $\lambda \geq \frac{\delta+2}{2}$.

Combining these observations, we have the following.

Proposition 11.

$$\text{Theorem 8 is } \begin{cases} c\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem 6. Let $G = (\lambda + 1)K_{\delta-\lambda+1} + K_{\lambda}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \delta - \lambda + 1$ and $\kappa(G) = \lambda$. Since $c(G) = \lambda(\delta - \lambda + 2)$, Theorem 6 is c -sharp for each $1 \leq \lambda \leq \delta$.

Consider the κ -sharpness of Theorem 6. Let $G = \lambda K_{\delta-\lambda+2} + K_{\lambda-1}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \delta - \lambda + 2 > \delta - \lambda + 1$ and $c(G) = (\lambda - 1)(\delta - \lambda + 3)$. The inequality $c(G) < \lambda(\delta - \lambda + 2)$ is satisfied whenever $\lambda \leq \frac{\delta+2}{2}$. Since $\kappa(G) = \lambda - 1$, Theorem B is κ -sharp for each $\lambda \leq \frac{\delta+2}{2}$.

Let $G = (\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$ with $\delta(G) = \delta$, $\kappa(G) = \lambda + 1 > \lambda$ and $c(G) = (\lambda + 1)(\delta - \lambda + 1)$. The inequality $c(G) < \lambda(\delta - \lambda + 2)$ is satisfied whenever $\lambda \geq \frac{\delta+2}{2}$. Since $\bar{c}(G) = \delta - \lambda$, Theorem 8 is \bar{c} -sharp for each $\lambda \geq \frac{\delta+2}{2}$.

Combining these observations, we have the following.

Proposition 12.

$$\text{Theorem 6 is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem 7. Let $G = (\delta - \lambda + 3)K_{\lambda-1} + K_{\delta-\lambda+2}$. Then $\delta(G) = \delta$, $\bar{c}(G) = \lambda - 1$ and $\kappa(G) = \delta - \lambda + 2$. Since $c(G) = \lambda(\delta - \lambda + 2)$, Theorem 7 is c -sharp for each $1 \leq \lambda \leq \delta$.

Let $G = (\delta - \lambda + 2)K_{\lambda} + K_{\delta-\lambda+1}$ with $\delta(G) = \delta$, $\bar{c}(G) = \lambda > \lambda - 1$ and $c(G) = (\lambda + 1)(\delta - \lambda + 1)$. The inequality $c(G) < \lambda(\delta - \lambda + 2)$ holds whenever $\lambda \geq \frac{\delta+2}{2}$. Since $\kappa(G) = \delta - \lambda + 1$, Theorem 7 is κ -sharp for each $\lambda \geq \frac{\delta+2}{2}$.

Let $G = (\delta - \lambda + 4)K_{\lambda-2} + K_{\delta-\lambda+3}$. Then $\delta(G) = \delta$, $\kappa(G) = \delta - \lambda + 3 > \delta - \lambda + 2$ and $c(G) = (\lambda - 1)(\delta - \lambda + 3)$. The inequality $c(G) < \lambda(\delta - \lambda + 2)$ is satisfied whenever $\lambda \leq \frac{\delta+2}{2}$. Since $\bar{c} = \lambda - 2$, Theorem 7 is \bar{c} -sharp for each $\lambda \leq \frac{\delta+2}{2}$.

Combining these observations, we have the following.

Proposition 13.

$$\text{Theorem 7 is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem F. If $\lambda \leq \frac{\delta+2}{2}$, then we have Theorem B. By Proposition 10, Theorem F is (c, κ, \bar{c}) -sharp for each $\lambda \leq \frac{\delta+2}{2}$. If $\lambda \geq \frac{\delta+2}{2}$, then we have Theorem 6. By Proposition 12, Theorem F

is (c, \bar{c}) -sharp for each $\lambda \geq \frac{\delta+2}{2}$. Combining these observations, we obtain the following.

Proposition 14.

$$\text{Theorem F is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem G. If $\lambda \leq \frac{\delta+2}{2}$, then we have Theorem B. By Proposition 10, Theorem G is (c, κ, \bar{c}) -sharp for each $\lambda \leq \frac{\delta+2}{2}$. If $\lambda \geq \frac{\delta+2}{2}$, then we have Theorem 7. By Proposition 13, Theorem G is (c, κ) -sharp for each $\lambda \geq \frac{\delta+2}{2}$. Combining these observations, we obtain the following.

Proposition 15.

$$\text{Theorem G is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \bar{c}\text{-sharp for each } \lambda \leq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem 10. If $\lambda \leq \frac{\delta+2}{2}$, then we have Theorem 6. By Proposition 12, Theorem 10 is (c, κ) -sharp for each $\lambda \leq \frac{\delta+2}{2}$. If $\lambda \geq \frac{\delta+2}{2}$, then we have Theorem 8. By Proposition 11, Theorem 10 is (c, κ, \bar{c}) -sharp for each $\lambda \geq \frac{\delta+2}{2}$. Combining these observations, we obtain the following.

Proposition 16.

$$\text{Theorem 10 is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \bar{c}\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}. \end{cases}$$

Consider the sharpness of Theorem 9. If $\lambda \leq \frac{\delta+2}{2}$, then we have Theorem 7. By Proposition 13, Theorem 9 is (c, \bar{c}) -sharp for each $\lambda \leq \frac{\delta+2}{2}$. If $\lambda \geq \frac{\delta+2}{2}$, then we have Theorem 8. By Proposition 11, Theorem 9 is (c, κ, \bar{c}) -sharp for each $\lambda \geq \frac{\delta+2}{2}$. Combining these observations, we obtain the following.

Proposition 17.

$$\text{Theorem 9 is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } \lambda \geq \frac{\delta+2}{2}, \\ \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

Consider the sharpness of Theorem H. If $\lambda \leq \frac{\delta+2}{2}$, then we have Theorem B. By Proposition 10, Theorem H is (c, κ, \bar{c}) -sharp for each $\lambda \leq \frac{\delta+2}{2}$. If $\lambda \geq \frac{\delta+2}{2}$, then we have Theorem 8. By Proposition 11, Theorem H is (c, κ, \bar{c}) -sharp for each $\lambda \geq \frac{\delta+2}{2}$. Combining these observations, we obtain the following.

Proposition 18.

$$\text{Theorem H is } \begin{cases} c\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \kappa\text{-sharp for each } 1 \leq \lambda \leq \delta, \\ \bar{c}\text{-sharp for each } 1 \leq \lambda \leq \delta. \end{cases}$$

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