Artificial Intelligence and Machine Learning

How to solve a homogeneous System of linear Equations (by Example)

Let the following system of linear equations be given:

$$\begin{aligned} -1x_1 + 2x_2 \qquad + 3x_4 &= 0 \\ -3x_2 \qquad - 3x_4 &= 0 \\ +4x_1 - 4x_2 + 1x_3 - 5x_4 &= 0 \\ -4x_1 \qquad - 2x_3 - 2x_4 &= 0 \end{aligned}$$

We begin by transforming this system into the standard form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{A} := \begin{pmatrix} -1 & 2 & 0 & 3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

is the **matrix of coefficients**, and

$$\boldsymbol{b} := \boldsymbol{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$$

the vector containing the right-hand side of the system. Its solution can be read off from the **reduced row echelon form** (German: *Treppennormalform*) of $\boldsymbol{A}$, which is often denoted by $\text{rref}(\boldsymbol{A})$.

In the following we compute the reduced row echelon form of $\boldsymbol{A}$:

$$\begin{pmatrix} -1 & 2 & 0 & 3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix}$$

[$-1 \cdot$ I]

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix}$$

[$-4 \cdot$ I + III and $4 \cdot$ I + IV]

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \end{pmatrix}$$

[$-1/3 \cdot$ II]

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \end{pmatrix}$$

[$2 \cdot$ II + I and $-4 \cdot$ II + III and $8 \cdot$ II + IV]

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{pmatrix}$$

[$2 \cdot$ III + IV]

$$\begin{pmatrix} \color{red}{1} & 0 & 0 & -1 \\ 0 & \color{red}{1} & 0 & 1 \\ 0 & 0 & \color{red}{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

*(The three red 1 entries in the reduced row echelon form are called **pivot elements**. The number of pivots determines the rank of the matrix, i. e. the rank of $\boldsymbol{A}$ is equal to 3.)*

We continue by deleting all zero lines from the reduced row echelon form of $\boldsymbol{A}$. Then we add -1 entries on the main diagonal in such a way that all pivots are located on the main diagonal and the resulting matrix is again quadratic (the remaining entries of the rows we insert consist of zeros). The result of this step is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \color{red}{-1} \\ 0 & 1 & 0 & \color{red}{1} \\ 0 & 0 & 1 & \color{red}{3} \\ 0 & 0 & 0 & \color{red}{-1} \end{pmatrix}.$$

The solution space is now given by *all linear combinations of the columns which contain an inserted -1 entry*. In our example the solution space is given by

$$L = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$