W3WI DS304.1 Applied Machine Learning Fundamentals

Exercise Sheet # 3 - Linear Regression

Question 1 (Matrix operations)

Let $\boldsymbol{X} \in \mathbb{R}^{n \times (m+1)}$, $\boldsymbol{y} \in \mathbb{R}^n$, and $\boldsymbol{\theta} \in \mathbb{R}^{m+1}$ be given. Do the following operations result in a matrix, a vector, or a scalar? What are the respective dimensions?

- $\boldsymbol{X}^\intercal \boldsymbol{X}$
- $\boldsymbol{X}^\intercal \boldsymbol{y}$
- $\boldsymbol{y}^\intercal \boldsymbol{y}$
- $\|\boldsymbol{X}\boldsymbol{\theta} - \boldsymbol{y}\|^2$

Question 2 EX 2022 (Normal equation)

Let the training dataset

$$\mathcal{D}_{\text{train}} = \Big\{(1,2), (2,1), (3,3)\Big\}$$

be given. Each training example is a tuple of the form (x, y), where $x \in \mathbb{R}$ is the only feature, and $y \in \mathbb{R}$ the corresponding label. Please work through the following tasks:

1. Compute the optimal model parameters $\boldsymbol{\theta}^\star$ using the **normal equation**

   $$\boldsymbol{\theta}^\star := \left(\boldsymbol{X}^\intercal \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\intercal \boldsymbol{y}.$$

   Do not apply any regularization. **Hint:** Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be an invertible matrix. Its inverse is given by $\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\det \boldsymbol{A} = ad - bc$ is the determinant of $\boldsymbol{A}$.

2. Figure 1 below plots the training dataset $\mathcal{D}_{\text{train}}$. Add the regression function produced by your model to the plot.

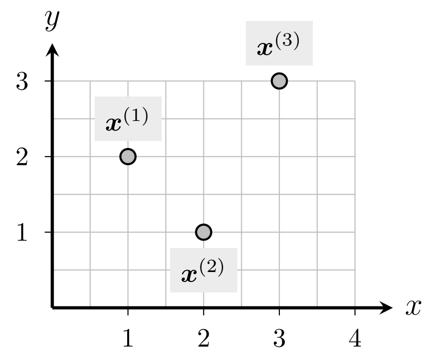

Figure 1: Plot of the training data set $\mathcal{D}_{\text{train}}$.

Question 3 [EX 2022] (Least squares error)

A colleague of yours provides you with a dedicated test dataset $\mathcal{D}_{\text{test}}$ so that you can thoroughly validate the model you have trained in question 2:

$$\mathcal{D}_{\text{test}} = \Big\{(0.5, 3), (1, 1.5), (2.5, 1), (3, 1.5)\Big\}.$$

Each tuple has the same form as in question 2. Please work through the following tasks:

1. Calculate the least squares error of your model from question 2 on the training dataset $\mathcal{D}_{\text{train}}$. **Hint:** Use $\boldsymbol{\theta}^\star = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$ in your calculations.

2. Compute the least squares error on $\mathcal{D}_{\text{test}}$! **Hint:** Use $\boldsymbol{\theta}^\star = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$ in your calculations.

3. Figure 2 visualizes both datasets, $\mathcal{D}_{\text{train}}$ and $\mathcal{D}_{\text{test}}$. How do you rate the model performance given figure 2 as well as the least squares errors you have computed on both datasets? What can you do to improve the model?

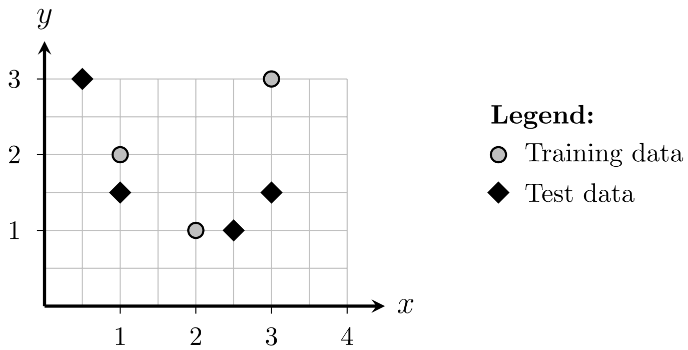

Figure 2: Plot of the complete dataset comprising train and test data.

Question 4 [EX 2022] (Regularization)

Tick the correct statements concerning the regularization of linear regression models (two statements are correct)!

- ☐ Regularization mitigates the danger of underfitting.
- ☐ Regularization mitigates the danger of overfitting.
- ☐ The regularization parameter λ should be chosen as large as possible.
- ☐ The formula for ridge regression is given by $\boldsymbol{\theta}^\star := \left(\boldsymbol{X}^\intercal \boldsymbol{X} + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}^\intercal \boldsymbol{y}$.
- ☐ It is not possible to use regularization and basis functions simultaneously.

Question 5 EX 2020, modified (Basis functions)

Name some examples of basis functions. Also do some research online: What other types exist? What is the purpose of basis functions? Can any function serve as a basis function?

Question 6 (Polynomial basis functions)

You are given the following dataset consisting of the two features x_1 and x_2:

Row	x_1	x_2	y
1	3	2	5
2	1	1	4
3	4	3	2
4	1	2	3
5	6	1	1

Please compute the transformations $\boldsymbol{\varphi}(\boldsymbol{x}^{(i)})$ of the data points $\boldsymbol{x}^{(i)}$ using the following polynomial basis functions:

$$\boldsymbol{\varphi} : \mathbb{R}^2 \to \mathbb{R}^6, \qquad \boldsymbol{\varphi}(\boldsymbol{x}) = \begin{pmatrix} \varphi_0(\boldsymbol{x}) \\ \varphi_1(\boldsymbol{x}) \\ \varphi_2(\boldsymbol{x}) \\ \varphi_3(\boldsymbol{x}) \\ \varphi_4(\boldsymbol{x}) \\ \varphi_5(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ x_2^3 \\ x_2^5 \end{pmatrix} \in \mathbb{R}^6.$$

Question 7 (Normal equation, polynomial basis functions, and regularization)

The following training dataset is presented to you (see figure 3):

$$\mathcal{D}_{\text{train}} = \left\{(1, 1/2), (2, 1), (3, 1/2), (4, 2)\right\}.$$

Your task is to train a linear regression model on $\mathcal{D}_{\text{train}}$. Due to the fact that the data is slightly non linear, you decide to use **polynomial basis functions** to transform the features in a non-linear fashion. You choose the feature functions

$$\boldsymbol{\varphi}(x) := \begin{pmatrix} 1 \\ x \\ x^3 \end{pmatrix},$$

so that the model function will take the form $h_{\boldsymbol{\theta}}(x) = \theta_0 + \theta_1 x + \theta_2 x^3$. In order to prevent overfitting you decide to regularize the model using the regularization parameter $\lambda := 1/2$.

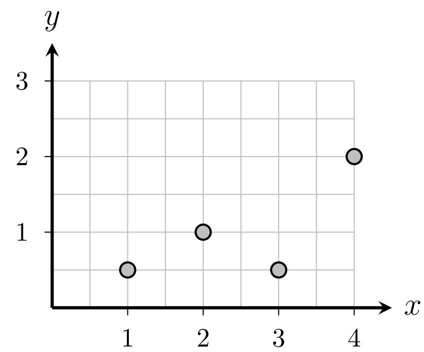

Figure 3: Plot of the training dataset $\mathcal{D}_{\text{train}}$.

Please answer the following questions:

1. Compute the optimal model parameters $\boldsymbol{\theta}^\star$ using the normal equation
$$\boldsymbol{\theta}^\star := \left(\boldsymbol{\Phi}^\intercal \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^\intercal \boldsymbol{y}.$$
(It is a valuable exercise to compute it by hand, but you may also use software.)

2. What is the prediction for the unknown data point $x_q = 1/2$?

Question 8 (Implement linear regression)

Use the following Python snippet to generate a regression problem:

```
1  # imports
   from sklearn.datasets import make_regression
3
   # create data set
5  X, y = make_regression(
       100,
7      n_features=1,
       noise=35.0,
9      random_state=45)

11 # uncomment this for task 3!
   #y = y**2 / 10000
```

1. Solve this regression task **analytically** and plot the results.

2. Now implement the **gradient descent algorithm** to solve the same task. Plot the results and compare them to your analytic solution from task 1.

3. Uncomment the non-linear transformation of the target labels in line 14 and adapt your code to solve this new task, i. e. implement suitable **basis functions**.

4. Compare the results of your implementation to the results obtained using `sklearn`.