Artificial Intelligence and Machine Learning

Lagrange Optimization (Example)

Task: Optimize the following function subject to the given constraint:

Maximize

$$f(x_1, x_2) = 1 - x_1^2 - x_2^2$$

subject to

$$h(x_1, x_2) = x_1 + x_2 - 1 = 0$$

Step 1) Compute the Lagrange function $\mathfrak{L}$. It is given by:

$$\mathfrak{L}(x_1, x_2, \lambda) := 1 - x_1^2 - x_2^2 + \lambda(x_1 + x_2 - 1)$$

Step 2) Compute the partial derivatives of $\mathfrak{L}$ and set them to zero:

Equation I

$$\frac{\partial}{\partial x_1}\mathfrak{L}(x_1, x_2, \lambda) = -2x_1 + \lambda \quad \stackrel{!}{=} 0$$

Equation II

$$\frac{\partial}{\partial x_2}\mathfrak{L}(x_1, x_2, \lambda) = -2x_2 + \lambda \quad \stackrel{!}{=} 0$$

Equation III

$$\frac{\partial}{\partial \lambda}\mathfrak{L}(x_1, x_2, \lambda) = x_1 + x_2 - 1 \stackrel{!}{=} 0$$

Step 3) Solve the system of equations:

From I:

$$x_1 = \frac{1}{2}\lambda$$

From II:

$$x_2 = \frac{1}{2}\lambda$$

Now substitute these results into equation III:

$$\frac{1}{2}\lambda + \frac{1}{2}\lambda = 1 \iff \lambda = 1$$

Substituting $\lambda = 1$ into the equations I and II yields the solution: $x_1^\star = x_2^\star = \frac{1}{2}$.

Remark: The point $(x_1^\star, x_2^\star)$ is obviously a maximum of the function *(see the function definition!)*.