W3WI DS304.1 Applied Machine Learning Fundamentals

How to solve a homogeneous System of linear Equations (by Example)

Let the following system of linear equations be given:

$$\begin{aligned}
-1x_1 + 2x_2 \qquad\quad + 3x_4 &= 0 \\
-3x_2 \qquad\quad - 3x_4 &= 0 \\
+4x_1 - 4x_2 + 1x_3 - 5x_4 &= 0 \\
-4x_1 \qquad\quad - 2x_3 - 2x_4 &= 0
\end{aligned}$$

We begin by transforming the system of linear equations into the form $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{pmatrix} -1 & 2 & 0 & 3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix} \in \mathbb{R}^{4 \times 4}$$

is the **matrix of coefficients**, and

$$\boldsymbol{b} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$$

denotes the vector containing the right-hand side of the system. The solution of the system can be read off from the **reduced row echelon form** (German: *Treppennormalform*) of $\boldsymbol{A}$, which is often denoted by $\mathrm{rref}(\boldsymbol{A})$.

In the following we compute the reduced row echelon form of $\boldsymbol{A}$:

$$\begin{pmatrix} -1 & 2 & 0 & 3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix}$$

$[-1 \cdot \mathbf{I}]$

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & -3 & 0 & -3 \\ 4 & -4 & 1 & -5 \\ -4 & 0 & -2 & -2 \end{pmatrix}$$

$[-4 \cdot \mathbf{I} + \mathbf{III}$ **and** $4 \cdot \mathbf{I} + \mathbf{IV}]$

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \end{pmatrix}$$

$[-^1/_3 \cdot \mathbf{II}]$

$$\begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \end{pmatrix}$$

$[2 \cdot \mathbf{II} + \mathbf{I}$ **and** $-4 \cdot \mathbf{II} + \mathbf{III}$ **and** $8 \cdot \mathbf{II} + \mathbf{IV}]$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{pmatrix}$$

$[2 \cdot \mathbf{III} + \mathbf{IV}]$

$$\begin{pmatrix} \color{red}{1} & 0 & 0 & -1 \\ 0 & \color{red}{1} & 0 & 1 \\ 0 & 0 & \color{red}{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(By the way: The three red 1 entries in the reduced row echelon form are called ***pivot elements****. The number of pivots determines the rank of the matrix, i. e. the rank of* $\boldsymbol{A}$ *is equal to 3.)*

Subsequently we **delete all zero lines** from the reduced row echelon form of $\boldsymbol{A}$. Then we **add -1 entries** on the main diagonal in such a way that all pivots are located on the main diagonal and the resulting matrix is again quadratic (the remaining entries of the rows we insert consist of zeros). The result of this step is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \color{red}{-1} \\ 0 & 1 & 0 & \color{red}{1} \\ 0 & 0 & 1 & \color{red}{3} \\ 0 & 0 & 0 & \color{red}{-1} \end{pmatrix}.$$

The solution space is now given by **all linear combinations of the columns which contain an inserted -1 entry**. In our example the solution space is given by

$$L = \left\{ t \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, t \in \mathbb{R} \right\}.$$