# Review of Fourier Transform (FT)

1) Fourier Series (FS)
2) Fourier Transform (FT)
3) Discrete-Time Fourier Transform (DTFT)
4) Discrete Fourier Transform (DFT)
5) Sampling / Zero-Padding / Truncated Signals

# Review of Fourier Transform

| | |
|---|---|
| DTFT (Discrete Time Fourier Transform)<br>Time: Discrete<br>Frequency: Continuous | FT (Fourier Transform)<br>Time: Continuous<br>Frequency: Continuous |
| DFT (Discrete Fourier Transform)<br>Time: Discrete<br>Frequency: Discrete | FS (Fourier Series)<br>Time: Continuous<br>Frequency: Discrete |

| Time | | Frequency |
|---|---|---|
| Periodic | $\longrightarrow$ | Discrete |
| Discrete | $\longleftarrow$ | Periodic |

# Definition of Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, dt$$

## Properties of the Fourier Transform

| **Property** | **Aperiodic signal** | **Fourier transform** |
|---|---|---|
| Linearity | $ax(t) + by(t)$ | $aX(\omega) + bY(\omega)$ |
| **Time Shifting** | $x(t - t_0)$ | $e^{-j\omega t_0} X(\omega)$ |
| Frequency Shifting | $e^{j\omega_0 t} x(t)$ | $X(\omega - \omega_0)$ |
| Conjugation | $x^*(t)$ | $X^*(-\omega)$ |
| Time Reversal | $x(-t)$ | $X(-\omega)$ |
| Time and Frequency Scaling | $x(at)$ | $\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$ |
| Convolution | $x(t) * y(t)$ | $X(\omega)Y(\omega)$ |
| Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi} X(\omega) * Y(\omega)$ |
| Parseval's Relation | $\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt$ | $\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 d\omega$ |

# Transform Pair

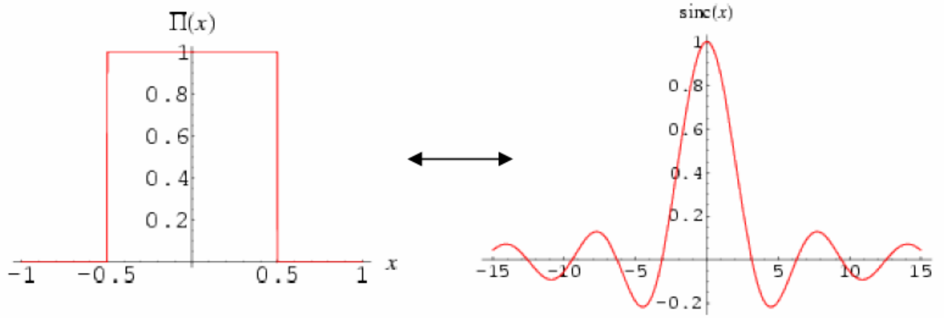



**TABLE 4.2 COMMON FOURIER TRANSFORM PAIRS**

$$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$$

$$-0.5 + u(t) \leftrightarrow \frac{1}{j\omega}$$

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - c) \leftrightarrow e^{-j\omega c}, \quad c \text{ any real number}$$

$$e^{-bt}u(t) \leftrightarrow \frac{1}{j\omega + b}, \quad b > 0$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0), \ \omega_0 \text{ any real number}$$

$$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$$

$$\tau \operatorname{sinc} \frac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$$

$$\left(1 - \frac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

$$\frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1 - \frac{2|\omega|}{\tau}\right)p_\tau(\omega)$$

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

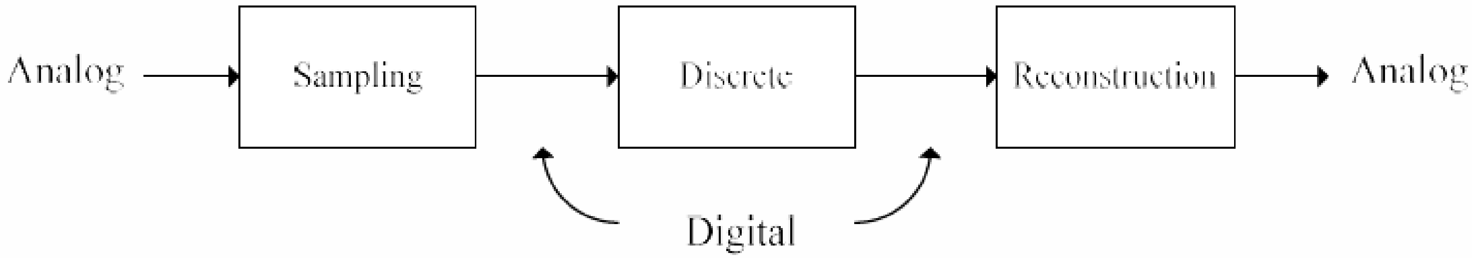
$$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$$

$$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$$

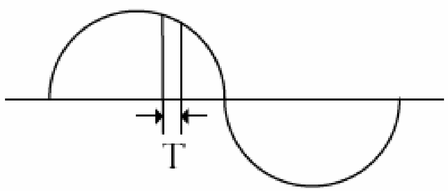
# Sampling and Reconstruction

## Meaning of sampling

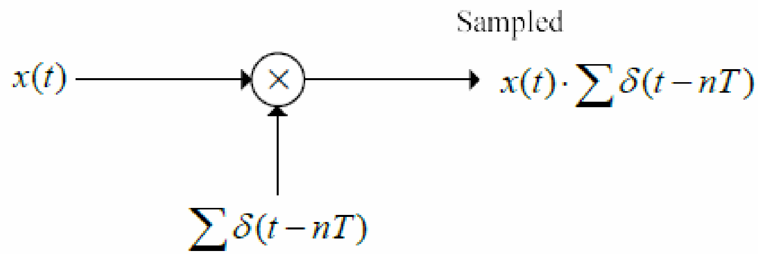



Ideal case of sampling:

$x[n] = x(nT)$




$\Rightarrow$




$$\begin{aligned} & x(t) \cdot \sum \delta(t - nT) \\ &= \sum x(t) \cdot \delta(t - nT) \\ &= \sum x(nT) \cdot \delta(t - nT) \end{aligned}$$

Note: $x(t)\delta(t - t_0)$
$= x(t_0)\delta(t - t_0)$

Relationship between Fourier transform and Discrete-time Fourier transform:

$$F[\sum x(nT)\cdot\delta(t-nT)]$$

$$=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(nT)\cdot\delta(t-nT)\cdot e^{-j\omega t}dt$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\cdot\int_{-\infty}^{\infty}\delta(t-nT)\cdot e^{-j\omega t}dt$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\cdot e^{-j\omega nT}=\sum_{n=-\infty}^{\infty}x[n]\cdot e^{-j\omega nT}$$

DTFT $\quad F[x[n]]=\sum_{-\infty}^{\infty}x[n]e^{-j\Omega n}$

Cont. FT $\quad F[x[n]]=\sum_{-\infty}^{\infty}x[n]e^{-j\omega Tn}$

From the above equations, we have the following relation:

$$\Omega=\omega T=\omega/f_s$$

$$\therefore \omega=\Omega f_s$$

$$\therefore f=\frac{\Omega}{2\pi}f_s$$

If we continue the Fourier transform calculation, we have

$$F[x(t) \cdot \sum \delta(t-nT)]$$

$$= \frac{1}{2\pi} F[x(t)] * F[\sum \delta(t-nT)]$$

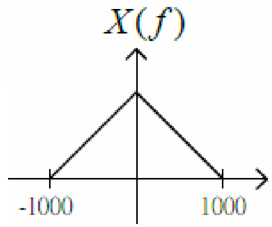
$$= \frac{1}{2\pi} X(\omega) * \left( \alpha \cdot \sum \delta(\omega - n\omega_0) \right)$$

$$= \frac{\alpha}{2\pi} \sum X(\omega) * \delta(\omega - n\omega_0)$$

$$= \frac{\alpha}{2\pi} \sum X(\omega - n\omega_0)$$

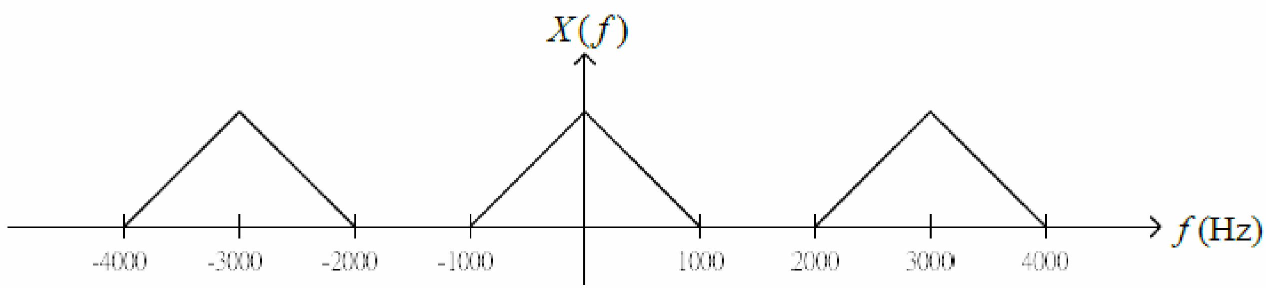
Note: $\alpha = 1/T$

*Ex:*

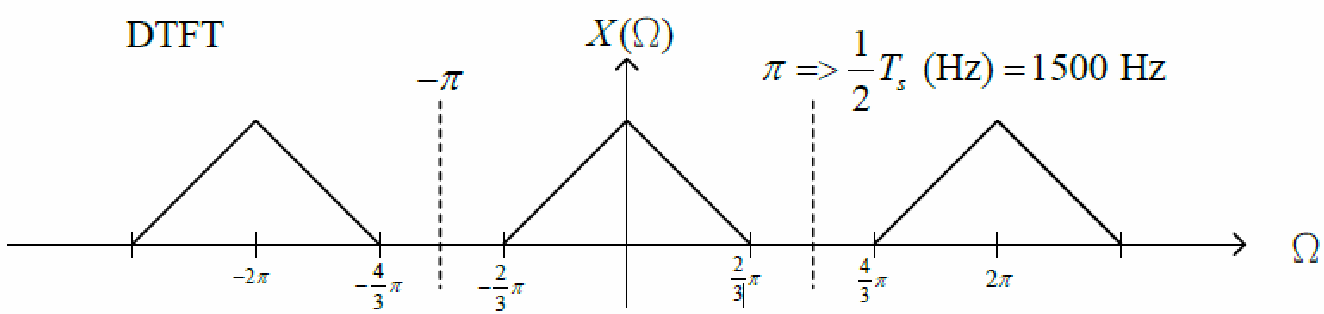



Sampling @ $T = \dfrac{1}{3000}$ s ($f_s = 3000$ Hz)

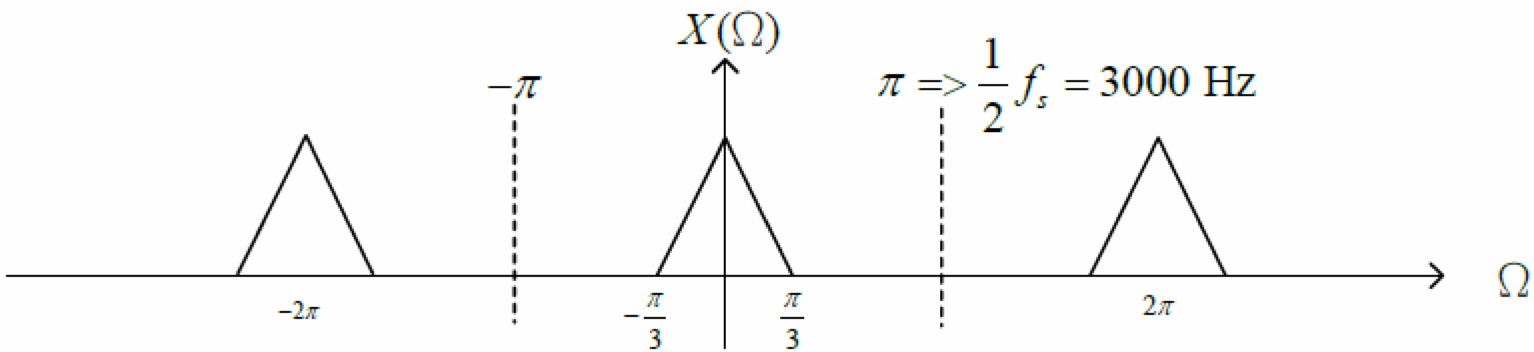
Continuous FT:




DTFT

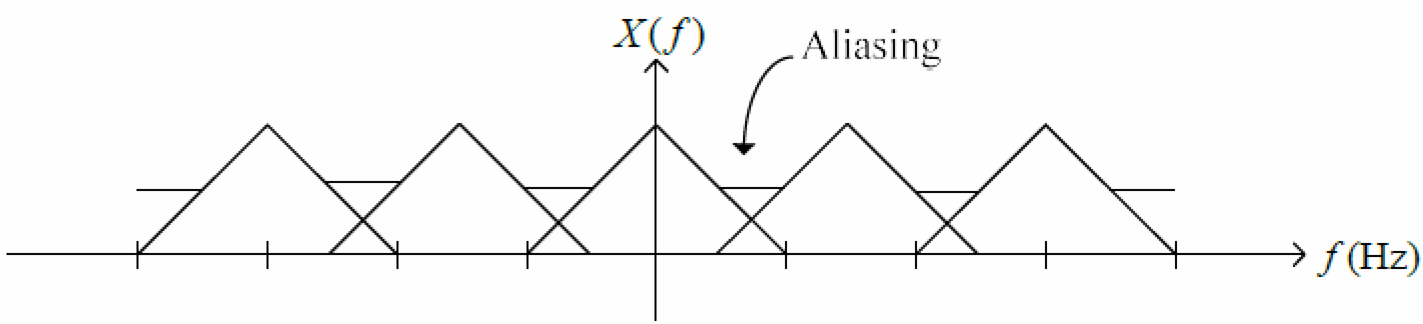


If $f_s = 6000$ Hz




Concept of spectrum aliasing.

If $f_s = 1900$ Hz and the bandwidth of the signal remains 1000 Hz, then we observe the effect of alias. In general, alias is an undesirable situation and should be avoided.



## Sampling Theorem

For a bandlimited signal $x(t)$ with $X(f)=0$ for $f > f_m$, then $x(t)$ can be uniquely determined from its sampled version if

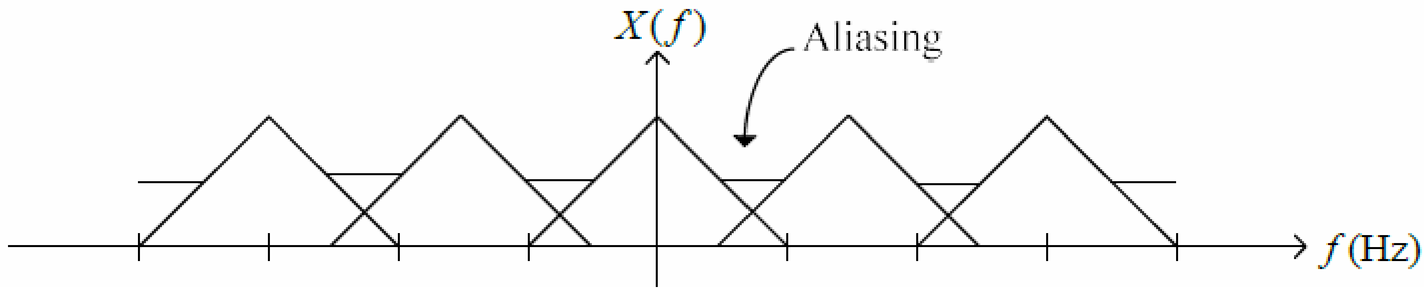
$$f_s\text{(Sampling freq.)} \geq 2f_m \text{ (Nyquist rate)}$$

What the sampling theorem states is that a sampled signal is as good as the original (continuous) signal if the constraint is met. Therefore, we may process and store the discrete version of the signal instead of the original one.

## Bandlimited signal

A signal is said to be bandlimited if

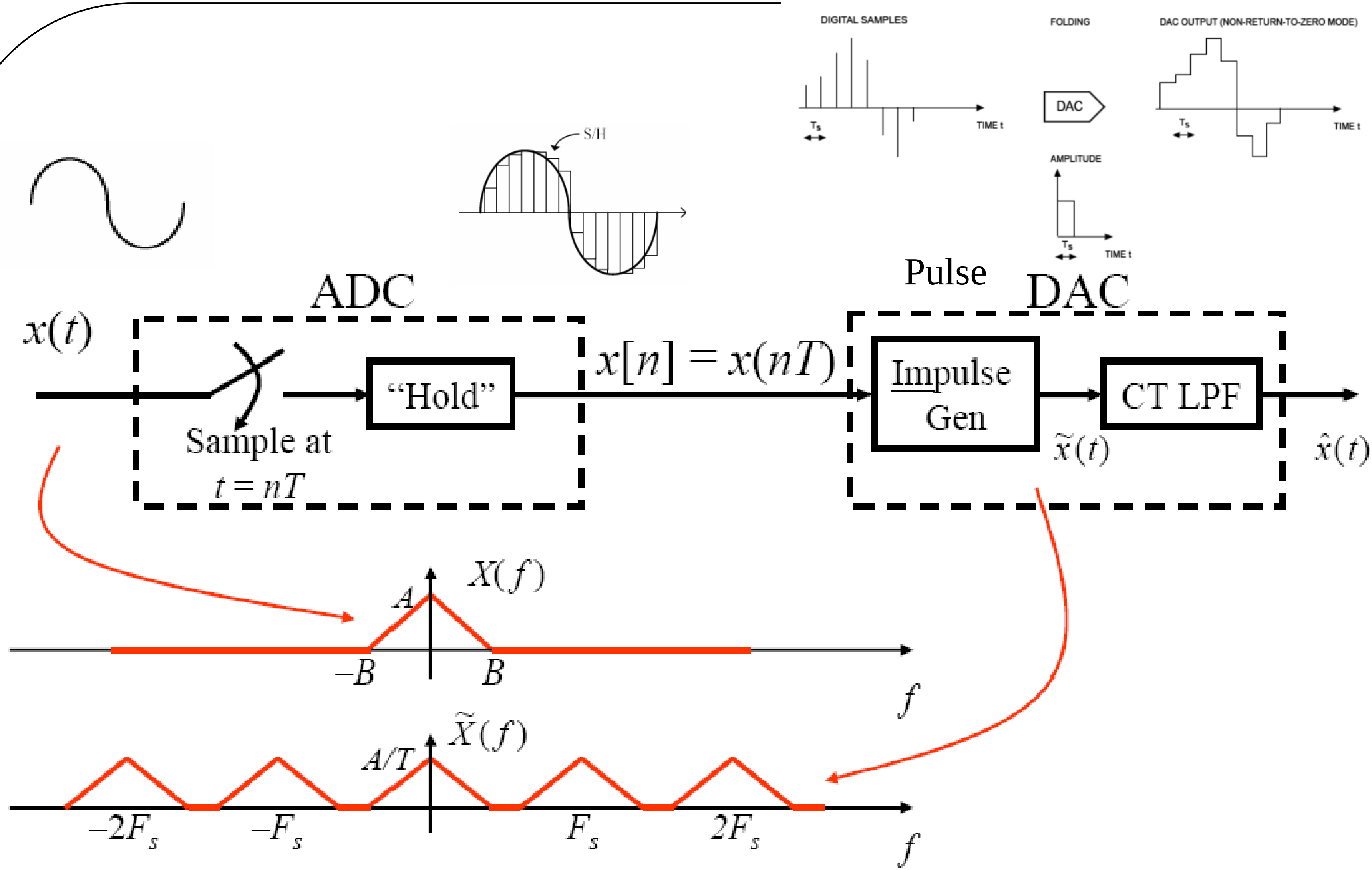
$$X(f)=0 \quad \text{for} \quad f > f_m \quad \text{or} \quad f < -f_m$$




## Aliasing

Aliasing = error / noise / distortion = N.G. Thereofire, we want to get rid of it.
A typical method to eliminate (or reduce) alias signal is to use a low-pass filter in front of the A/D converter. The filter is also referred to as the anti-aliasing filter. It is used to restrict the bandwidth of the input signal.

Note: In some cases, we deal with a signal that is practically bandlimited. In such cases, we still need the anti-aliasing filter to limit the bandwidth of the whole system. The filter is used to control the bandwidth of the input noise to prevent a high level of (aliased) noise due to its wide bandwidth.




As long as $F_s \geq 2B$ then we can clearly "see"…
a view of $X(f)$ in $\tilde{X}(f)$

# Quantization

The A/D and D/A have finite step sizes (resolutions). For example, a 12-bit A/D converts a continuous voltage to only $2^{12} = 4096$ different values. The difference between the exact value and the value after A/D is called quantization error. Quantization error may be treated as white noise if it is small. We can calculate S/N after quantization.

Rule of thumb: $6\ {}^{\text{dB}}\!/\!_{\text{bit}}$

This is the **Signal-to-Quantization Noise Ratio** (SQNR) and is given by

$$SQNR = \frac{S_0}{N_q} = \frac{\langle m^2(t) \rangle}{\left( m_p^2 / 3L^2 \right)} = 3L^2 \left( \frac{\langle m^2(t) \rangle}{m_p^2} \right)$$

It is usually expressed in decibels,

Note: $\left( \frac{\langle m^2(t) \rangle}{m_p^2} \right) \cong \frac{1}{2}$

$$SQNR_{dB} = 10 \cdot \log_{10}\left( \frac{S_0}{N_q} \right) \cong 10 \cdot \log_{10}\left( \frac{3L^2}{2} \right)$$

$$10 \cdot \log_{10}\left( \frac{S_0}{N_q} \right) \cong (1.76 + 6n)\ \text{dB}$$

| $L$ | $n$ | SNR |
|---|---|---|
| 32 | 5 | 31.8 dB |
| 64 | 6 | 37.8 dB |
| 128 | 7 | 43.8 dB |
| 256 | 8 | 49.8 dB |

# Take An Alternate Path to the DTFT!

$$\tilde{x}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = x(t)\delta_T(t)$$

**FS of $\delta_T(t)$**

$$\tilde{x}(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}x(t)e^{jk2\pi F_s t}$$

**FT & Mod. Prop**

$$\tilde{X}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X(\omega + k2\pi F_s)$$

Tells what $\tilde{X}(\omega)$ looks like!

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty}x[n]\delta(t-nT)$$

$$\tilde{X}(\omega) = \mathfrak{F}\left\{\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty}x[n]\mathfrak{F}\{\delta(t-nT)\}$$

$$\tilde{X}(\omega) = \sum_{n=-\infty}^{\infty}x[n]e^{-jn\omega T}$$

Tells how to compute $\tilde{X}(\omega)$!

Fourier Transform of a continuous-time signal

Definition: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

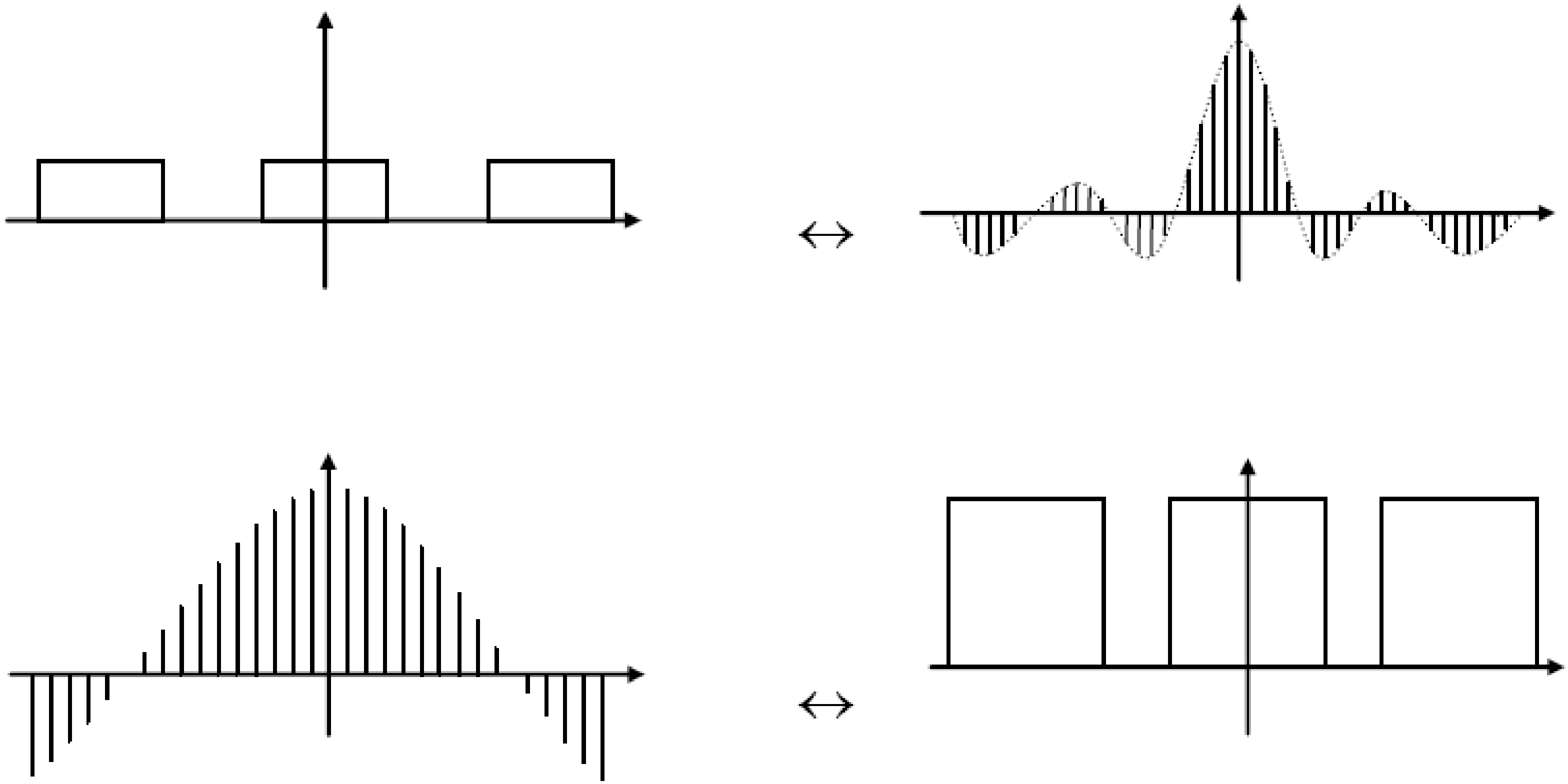
DTFT (Discrete-Time Fourier Transform) of x[n]

Definition: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \leftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega) e^{j\Omega n} d\Omega$

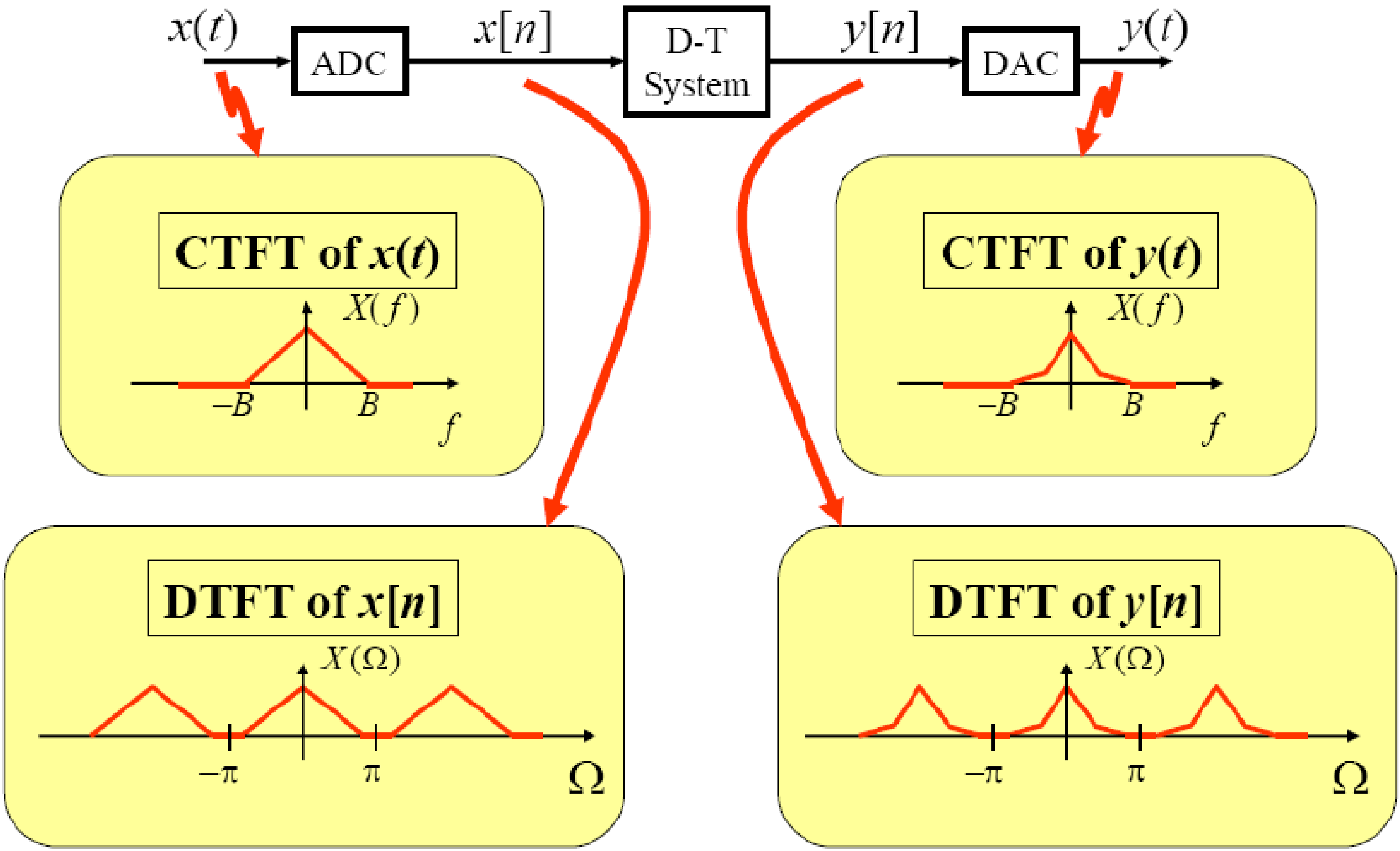
real variable (D-T frequency )

$\Omega = \omega T : (rad / \sec) \times (\sec / sample) = rad / sample$

# Discrete-Time Fourier Transformation

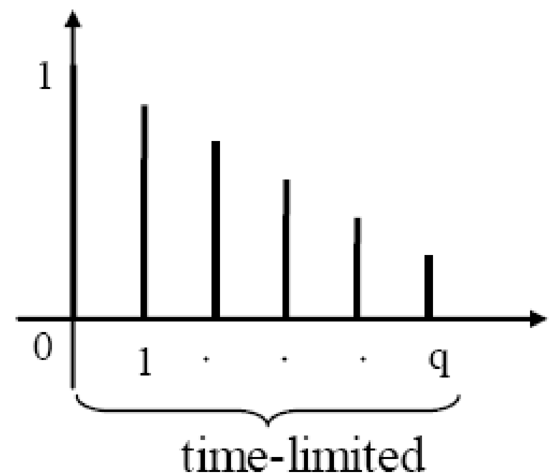


# Motivating D-T *System Analysis* using DTFT



existence of DTFT if $\underbrace{\sum_{n=-\infty}^{\infty}\left|x[n]\right| < \infty}_{\text{if time-limited } \to \text{ summable}}$ (absolutely summable)

$$Ex. \quad x[n] = \begin{cases} 0, & n < 0 \\ a^n, & 0 \le n \le q \\ 0, & n > q \end{cases}$$




$$\left( \sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1-r} \right)$$

✳✳Periodicity of $X(\Omega)$

$$X(\Omega+2\pi)=X(\Omega), \quad \textit{for all } \Omega, \quad -\infty<\Omega<\infty$$

pf/

$$\begin{aligned}
X(\Omega+2\pi) &= \sum_{n=-\infty}^{\infty} x[n]e^{-jn(\Omega+2\pi)} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}\underbrace{e^{-jn2\pi}}_{=\,1 \quad \forall n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} \\
&= X(\Omega)
\end{aligned}$$

✳✳Complex valued (in general) for $X(\Omega)$

- **$X(\Omega)$ in rectangular form**

$$X(\Omega) = R(\Omega) + jI(\Omega)$$

$$R(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\cos n\Omega \qquad R(-\Omega) = R(\Omega)$$

$$I(\Omega) = -\sum_{n=-\infty}^{\infty} x[n]\sin n\Omega \qquad I(-\Omega) = -I(\Omega)$$

$X(\Omega)$: frequency spectrum of $x[n]$

$|X(\Omega)|$: amplitude spectrum of $x[n]$

$\angle X(\Omega)$: phase spectrum of $x[n]$

◆ **$X(\Omega)$ in polar form**

$$X(\Omega) = |X(\Omega)| \exp[j\angle x(\Omega)]$$

$$|X(\Omega)| = \sqrt{R^2(\Omega) + I^2(\Omega)}$$

$$\Rightarrow |X(-\Omega)| = |X(\Omega)| \qquad \text{even function}$$

$$\angle X(\Omega) = \tan^{-1} \frac{I(\Omega)}{R(\Omega)}$$

$$\Rightarrow \angle X(-\Omega) = -\angle X(\Omega) \qquad \text{odd function}$$

## ∗∗Symmetry of $X(\Omega)$

$$Ex.\quad p[n]=\begin{cases}1, & n=-q,-q+1,\ldots,-1,0,1,\ldots q\\ 0, & other\ n\end{cases}$$

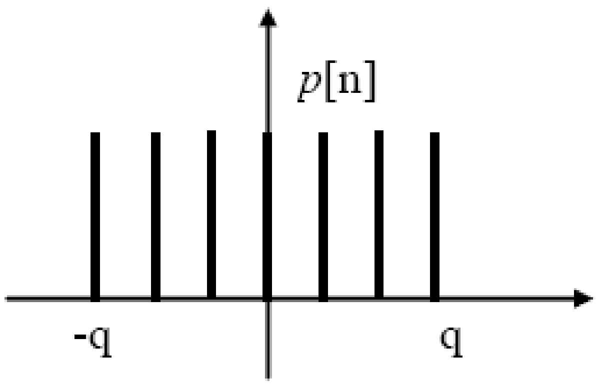
$$P(\Omega)=\sum_{n=-q}^{q}e^{-j\Omega n}$$

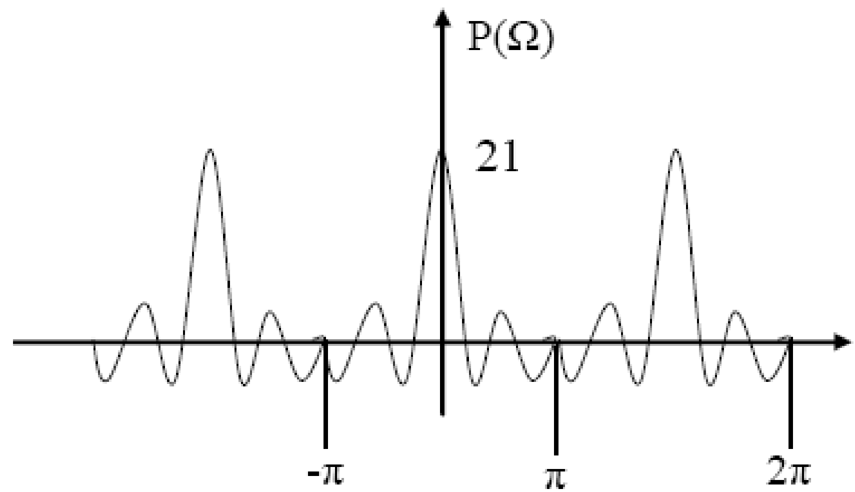
$$P(\Omega)=\frac{e^{j\Omega q}-e^{-j\Omega(q+1)}}{1-e^{-j\Omega}}\times\frac{e^{j\frac{\Omega}{2}}}{e^{j\frac{\Omega}{2}}}\quad=\frac{e^{j\Omega\left(q+\frac{1}{2}\right)}-e^{-j\Omega\left(q+\frac{1}{2}\right)}}{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}$$

$$=\frac{\sin\left[\left(q+\frac{1}{2}\right)\Omega\right]}{\sin\left(\frac{\Omega}{2}\right)}$$

when $q = 10$:

$$P(\Omega) = \left.\frac{\sin\left(\frac{21}{2}\Omega\right)}{\sin\frac{\Omega}{2}}\right|_{\Omega=0} = \frac{\frac{21}{2}}{\frac{1}{2}} \times \left.\frac{\cos\left(\frac{21}{2}\Omega\right)}{\cos\left(\frac{\Omega}{2}\right)}\right|_{\Omega=0} = 21$$



◆ **Signal with** $\begin{cases} \textbf{low frequency} \\ \textbf{high frequency} \end{cases}$

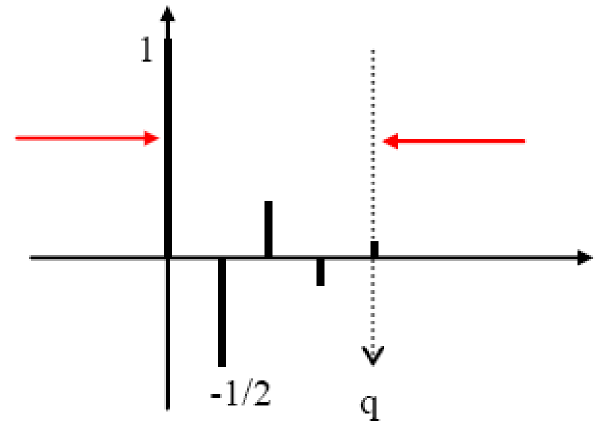
$$x[n] = (-0.5)^n u[n]$$

$$X(\Omega) = \sum_{n=0}^{\infty} (-0.5)^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(-0.5e^{-j\Omega}\right)^n$$

$$= \frac{1 - \left(-0.5e^{-j\Omega}\right)^{q+1}\Big|_{q\to\infty}}{1 + 0.5e^{-j\Omega}} \quad = 0$$

$$= \frac{1}{1 + 0.5e^{-j\Omega}}$$




⇩

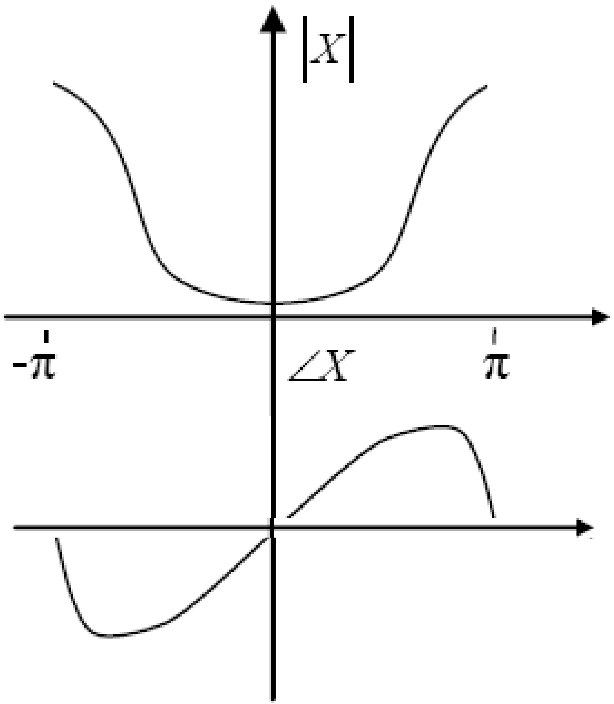
high-pass functions

$$|X(\Omega)| = \frac{1}{\sqrt{(1+0.5\cos\Omega)^2 + (0.5\sin\Omega)^2}}$$

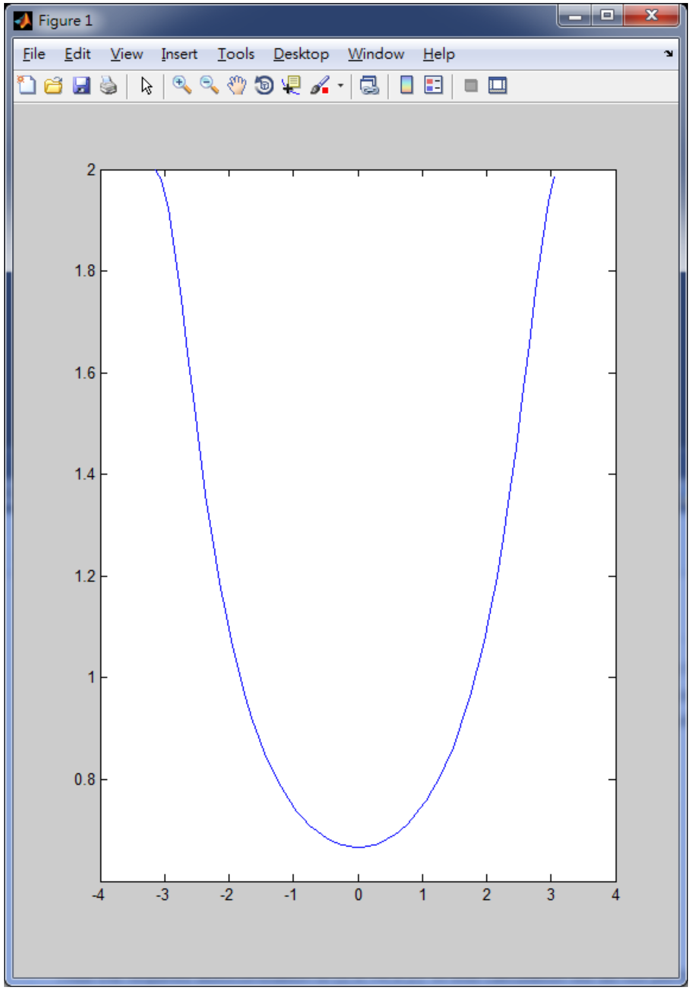
$$= \frac{1}{\sqrt{1+2\cdot 0.5\cos\Omega + 0.25\cos^2\Omega + 0.25\sin^2\Omega}}$$

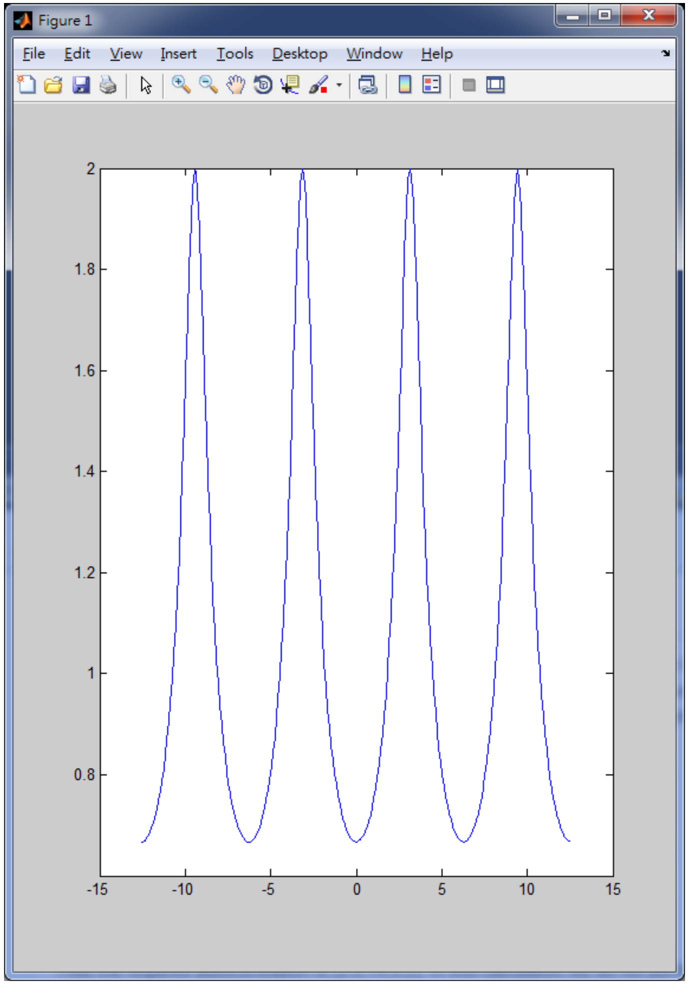
$$= \frac{1}{\sqrt{1.25+\cos\Omega}}$$

$$\angle X(\Omega) = -\tan^{-1}\frac{-0.5\sin\Omega}{1+0.5\cos\Omega}$$

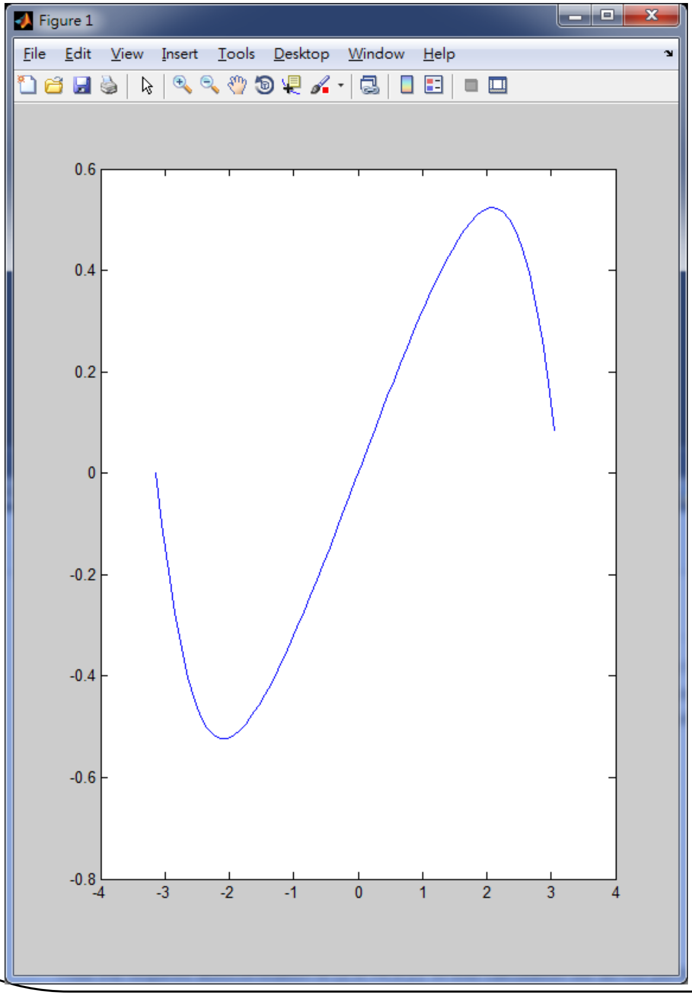


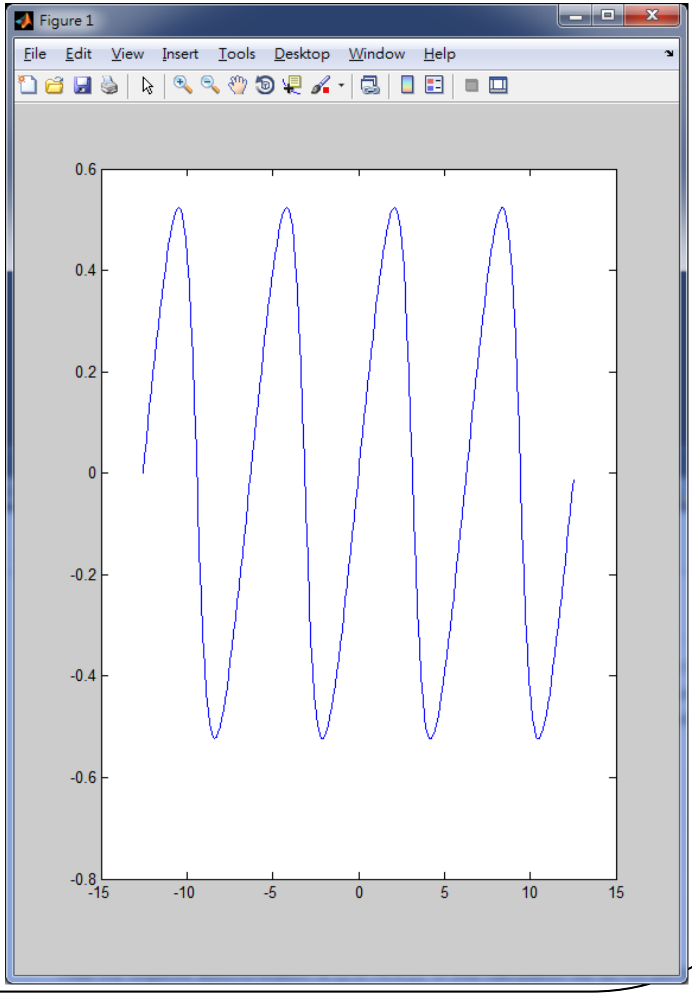
```
>> w=-pi:0.1:pi;
>> X_amp=1./((1.25+cos(w)).^0.5);
>> plot(w,X_amp)
```



```
>> w=-pi:0.1:pi;
>> X_ang=-atan(-0.5*sin(w)./(1+0.5*cos(w)));
>> plot(w,X_ang)
```



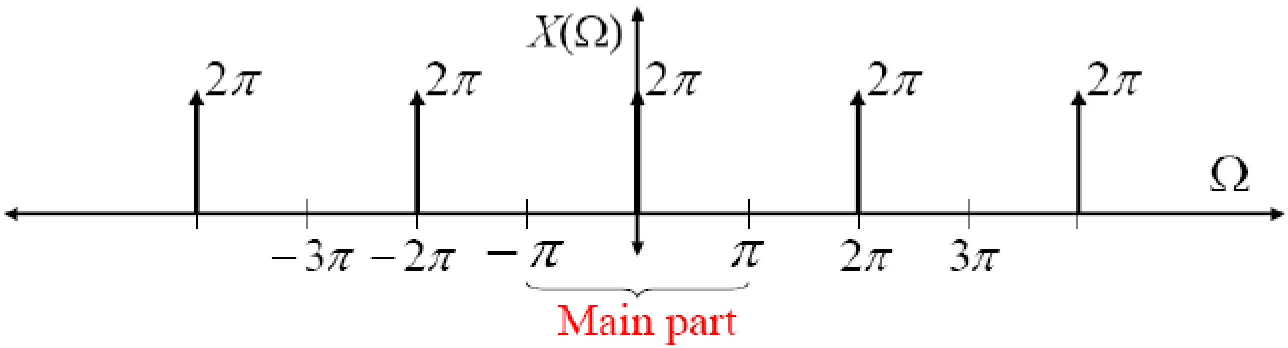



## Generalized DTFT

Periodic D-T signals have DTFT's that contain delta functions

Example: $x[n] = 1, \ \forall n \quad \leftrightarrow \quad X(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ periodic, & elsewhere \end{cases}$

With a period of $2\pi$




Another way of writing this is:

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$$

How do we derive the result? Work backwards!

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{jn\Omega}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega)e^{jn\Omega}d\Omega$$

Sifting property

$$= e^{jn\cdot 0}$$

$$= 1$$

## Table 3.2 COMMON FOURIER TRANSFORM PAIRS

$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$

$-0.5 + u(t) \leftrightarrow \frac{1}{j\omega}$

$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$\delta(t - c) \leftrightarrow e^{-j\omega c}$, $c$ any real number

$e^{-bt}u(t) \leftrightarrow \frac{1}{j\omega + b}$, $b > 0$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, $\omega_0$ any real number

$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$

$\tau \operatorname{sinc} \frac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1 - \frac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$

$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1 - \frac{2|\omega|}{\tau}\right)p_\tau(\omega)$

$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$

$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

## Table 4.1 COMMON DTFT PAIRS

$1, \text{ all } n \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi k)$

$\operatorname{sgn}[n] \leftrightarrow \frac{2}{1 - e^{-j\Omega}}$, where $\operatorname{sgn}[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ -1, & n = -1, -2, \ldots \end{cases}$

$u[n] \leftrightarrow \frac{1}{1 - e^{-j\Omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$

$\delta[n] \leftrightarrow 1$

$\delta[n - N] \leftrightarrow e^{-jN\Omega}$, $N = \pm 1, \pm 2, \ldots$

$a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\Omega}}$, $|a| < 1$

$e^{j\Omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi k)$

$p_q[n] \leftrightarrow \frac{\sin[(q + \frac{1}{2})\Omega]}{\sin(\Omega/2)}$

$\frac{B}{\pi}\operatorname{sinc}\left(\frac{B}{\pi}n\right) \leftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$

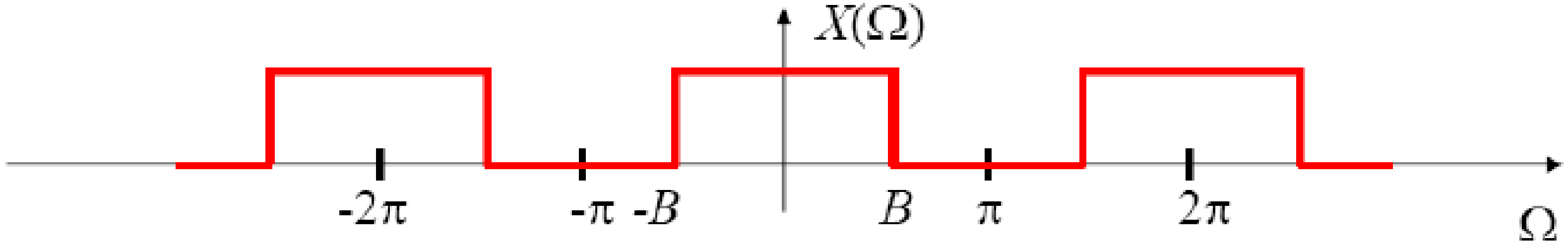
$\cos \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)]$

$\sin \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$

$\cos\left(\Omega_0 n + \theta\right) \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$

Careful here the book's table doesn't have this subscript… see next slide.

**Example   Finding a DTFT pair from a CTFT pair**

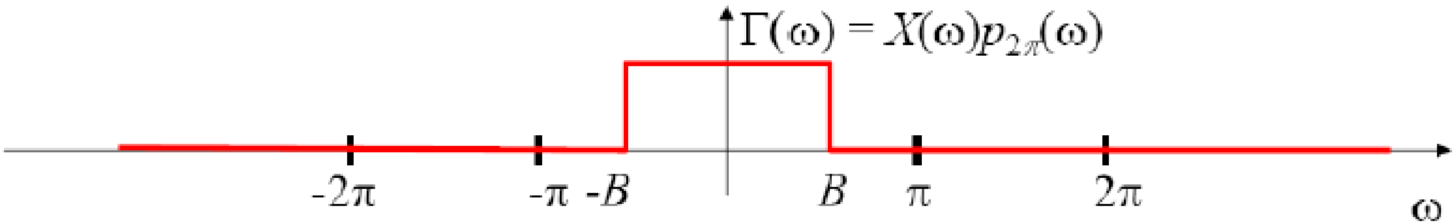



Say we are given this DTFT and want to invert it…

The four steps for using "Relationship to Inverse CTFT" property are:

1. Truncate the DTFT $X(\Omega)$ to the $-\pi$ to $\pi$ range and set it to zero elsewhere
2. Then treat the resulting function as a function of $\omega$… call this $\Gamma(\omega)$

$$\boxed{\Gamma(\omega) = X(\omega)p_{2\pi}(\omega)}$$




3. Find the inverse CTFT of $\Gamma(\omega)$ from a CTFT table, call it $\gamma(t)$

From CTFT table:

$$\gamma(t) = \frac{B}{\pi}\operatorname{sinc}\left(\frac{B}{\pi}t\right)$$

4. Get the $x[n]$ by replacing $t$ by $n$ in $\gamma(t)$

$$\boxed{x[n] = \gamma(t)\big|_{t=n} = \frac{B}{\pi}\operatorname{sinc}\left(\frac{B}{\pi}n\right)}$$

## Example of DTFT of sinusoid

$$x[n] = \cos(\Omega_0 n) \quad \leftrightarrow \quad X(\Omega) = ?$$

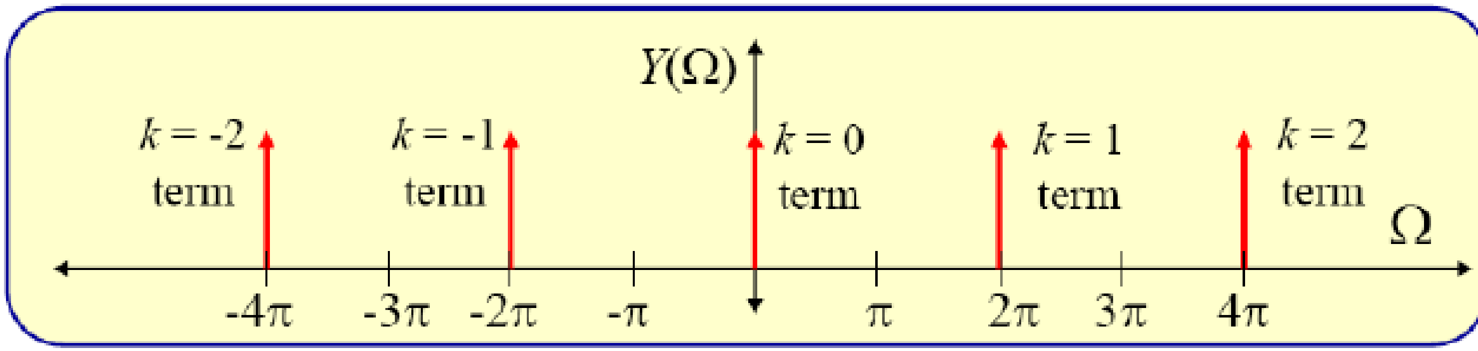
Note that: $x[n] = 1 \times \cos(\Omega_0 n)$

So… use the "mult. by sinusoid" property

$$\triangleq y[n] = 1$$

From DTFT Table

$$\boxed{Y(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)}$$




Another way of writing this:

$$\boxed{Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} & \end{cases}}$$

Recall: $x[n] = 1 \times \cos(\Omega_0 n)$ so we can use the "mult. by sinusoid" result

$$\Rightarrow X(\Omega) = \frac{1}{2}\left[Y(\Omega + \Omega_0) + Y(\Omega - \Omega_0)\right]$$

"mult. by sinusoid" property says we shift up & down by $\Omega_0$

Using the second form for $Y(\Omega)$ gives:

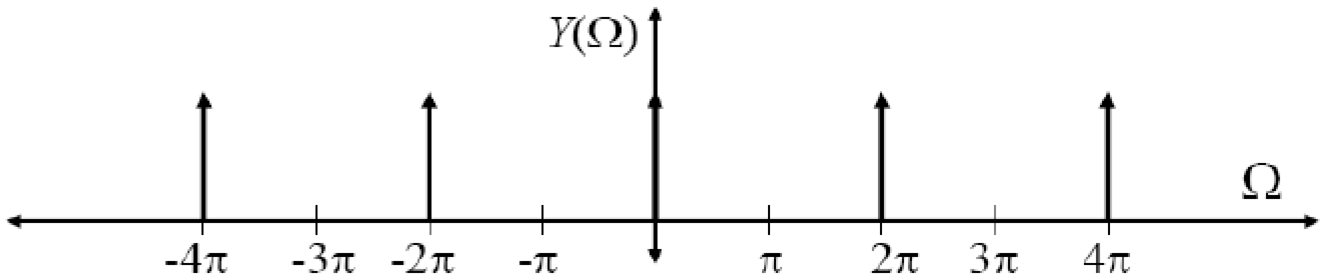
$$\boxed{X(\Omega) = \begin{cases} \pi\left[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right], & -\pi < \Omega < \pi \\ 2\pi - periodic\ elsewhere \end{cases}}$$

Or…using the first form for $Y(\Omega)$ gives:

$$\boxed{Y(\Omega) = \pi \sum_{k=-\infty}^{\infty}\left[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)\right]}$$

To see this graphically:

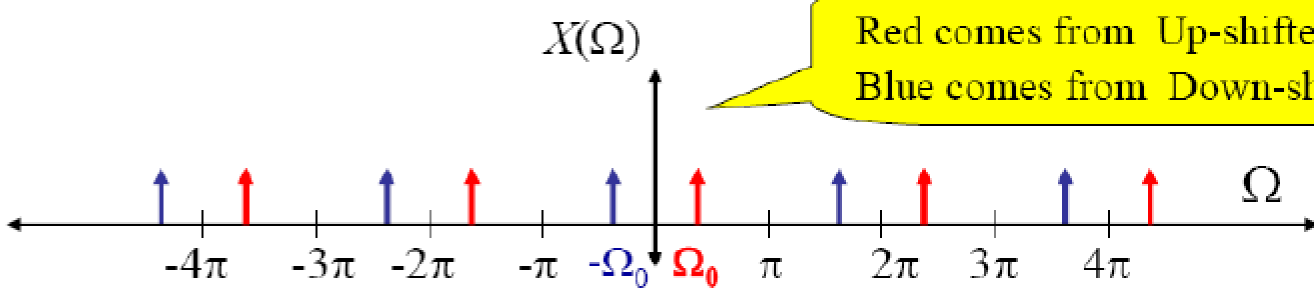
$$Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$




$$X(\Omega) = \begin{cases} \pi\left[\delta(\Omega+\Omega_0) + \delta(\Omega-\Omega_0)\right], & -\pi < \Omega < \pi \\ 2\pi - \textit{periodic elsewhere} \end{cases}$$

Red comes from Up-shifted $Y(\Omega)$

Blue comes from Down-shifted $Y(\Omega)$



◆**DTFT**

Transform Pair

$$\cos\omega_0 t \leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

$$\Leftrightarrow \cos\Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty}\pi\left[\delta(\Omega+\Omega_0-2\pi k)+\delta(\Omega-\Omega_0-2\pi k)\right]$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0) \Leftrightarrow e^{j\Omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty}2\pi\delta(\Omega-\Omega_0-2\pi k)$$

## § DFT (Discrete Fourier Transform)

$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}\quad \text{(DTFT)}$$ can be computed analytically

(at least in principle) when we have an equation model for $x[n]$

Q: **Well… why can't we use a computer to compute the DTFT from Data?**

A: There are two reasons why we can't!!

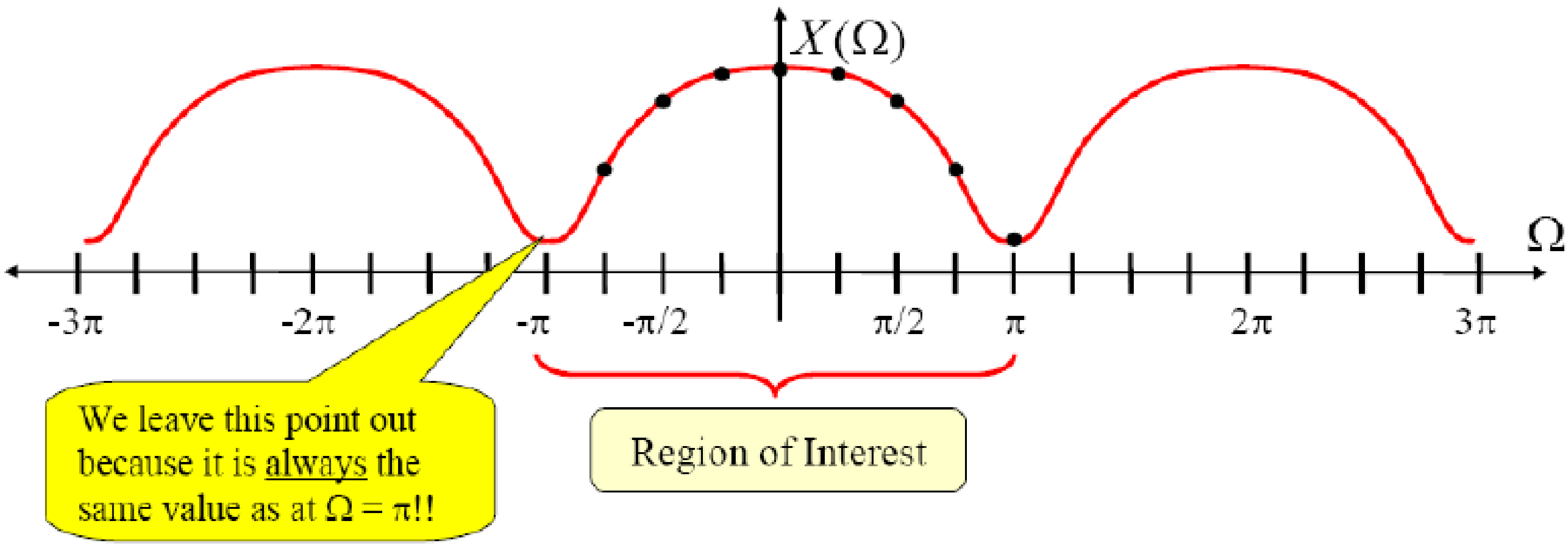
1. The DTFT requires an infinite number of terms to be summed over $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
2. The DTFT must be evaluated at an infinite number of points over the interval $\Omega\in(-\pi, \pi]$

-The first one ("infinite # of terms")… isn't a problem if $x[n]$ has "finite duration"

-The second one ("infinitely many points")… is always a problem!!

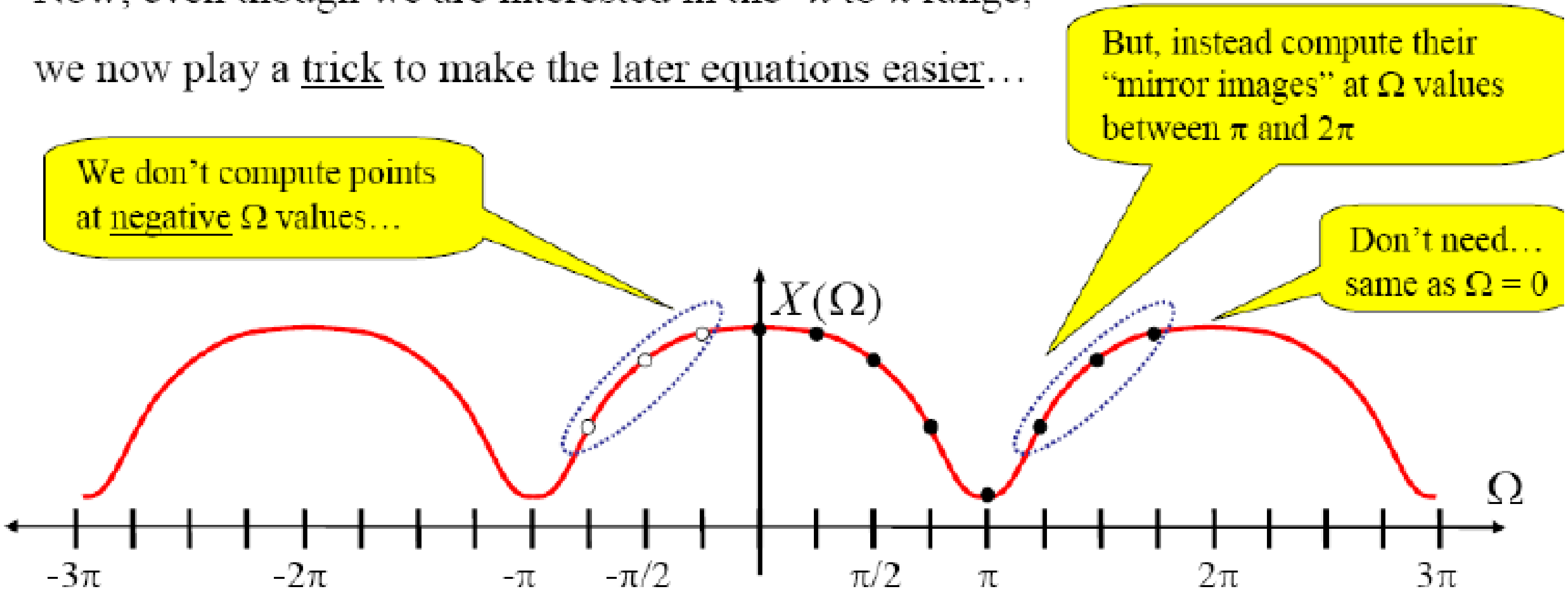
Now suppose we take the numerical data $x[n]$ for $n = 0, \ldots, N\text{-}1$

and just compute this DTFT at a <u>finite number of $\Omega$ values</u> (8 points here)…



Now, even though we are interested in the $-\pi$ to $\pi$ range,

we now play a trick to make the later equations easier…




So say we want to compute the DTFT at $M$ points, then choose

$$\Omega_k = k\boxed{\frac{2\pi}{M}}, \quad for \quad k = 0,1,2,\ldots, M-1$$

Spacing between computed Ω values

In otherwords:

$$\boxed{\Omega_0 = 0,} \quad \boxed{\Omega_1 = \frac{2\pi}{M},} \quad \boxed{\Omega_2 = 2\frac{2\pi}{M},} \quad \ldots \quad ,\boxed{\Omega_{M-1} = (M-1)\frac{2\pi}{M}}$$

Thus… mathematically what we have computed for our finite-duration signal is:

$$X(\Omega_k) = \sum_{n=0}^{N-1} x[n]e^{-jn\Omega_k} = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{M}}, \quad for \quad k = 0, 1, 2, \ldots, M-1$$

There is just one last step needed to define the **D**iscrete **F**ourier **T**ransform **(DFT):**

We must set $M = N$…

Done for a few mathematical reasons… later we'll learn a trick called "zero-padding" to get around this!

In other words: Compute as many "frequency points" as "signal points"

So… Given $N$ signal data points $x[n]$ for $n = 0, \ldots, N$-1
Compute $N$ DFT points using:

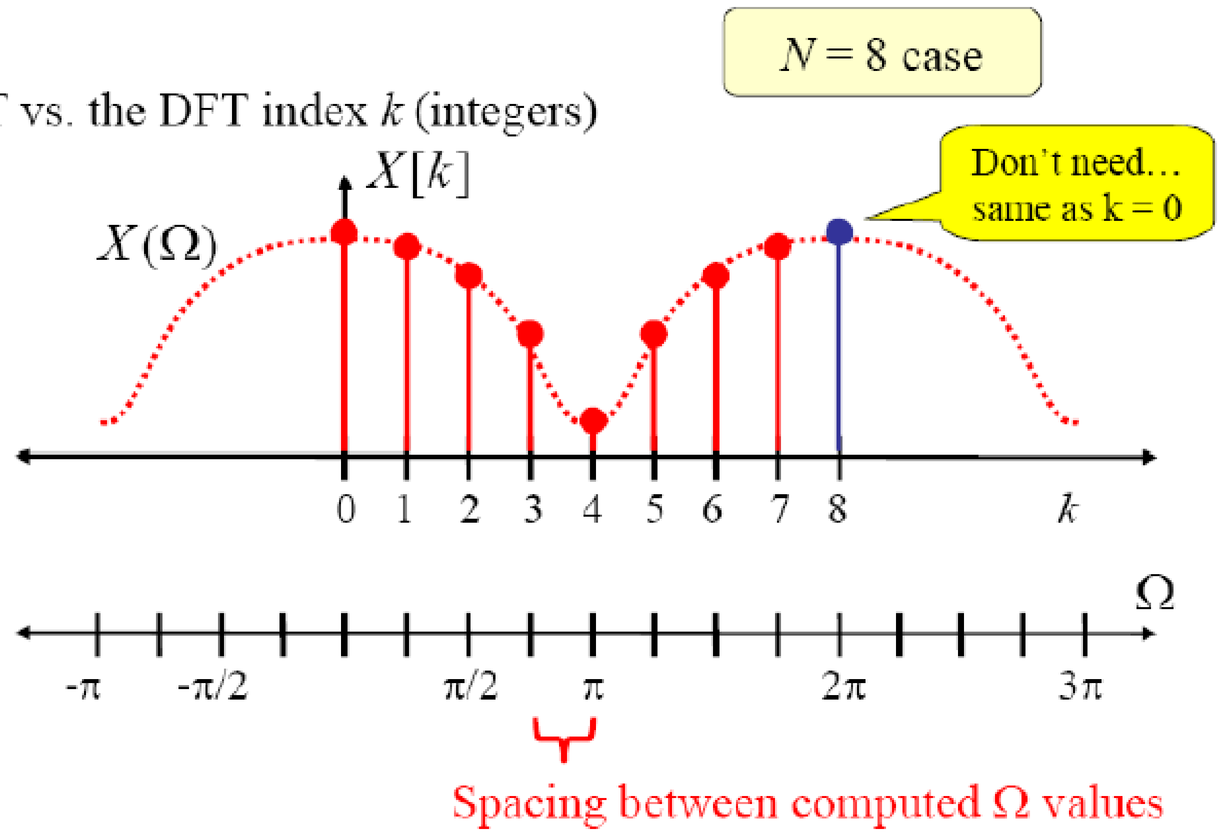
$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k = 0,1,2,\ldots,N-1}$$

Book uses $X_k$ notation

**Definition of the DFT**

## Plotting the DFT

We often plot the DFT vs. the DFT index $k$ (integers)




**But… we know that these points can be tied back to the true D-T frequency Ω:**

$$\frac{2\pi}{N} \Rightarrow \frac{2\pi}{8} = \frac{\pi}{4}$$

# Properties of the DFT

1. **Symmetry of the DFT**

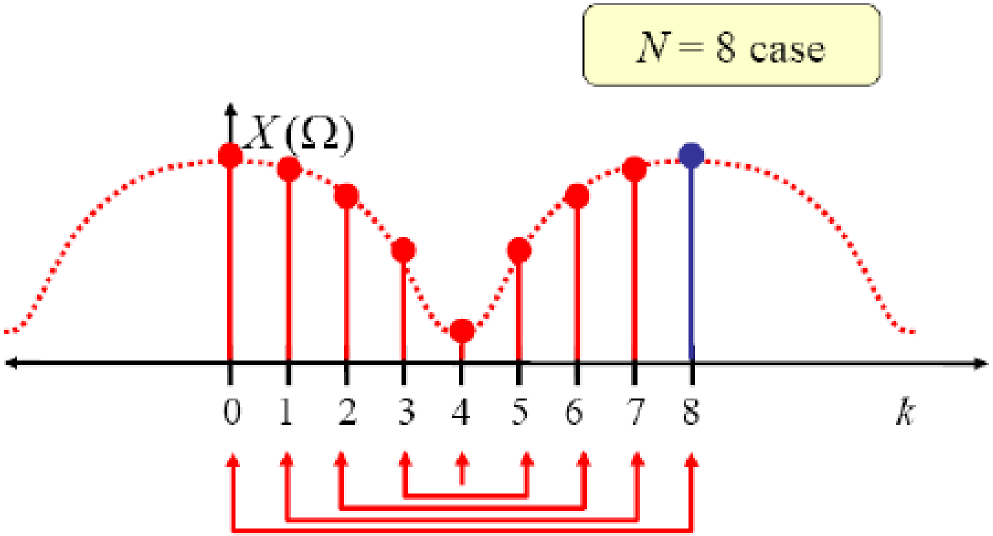
We arrived at the DFT via the DTFT so it should be no surprise that the DFT inherits some sort of symmetry from the DTFT.

$$\boxed{X[N-k] = \overline{X}[k], \quad k = 0,1,2,\ldots,N-1}$$

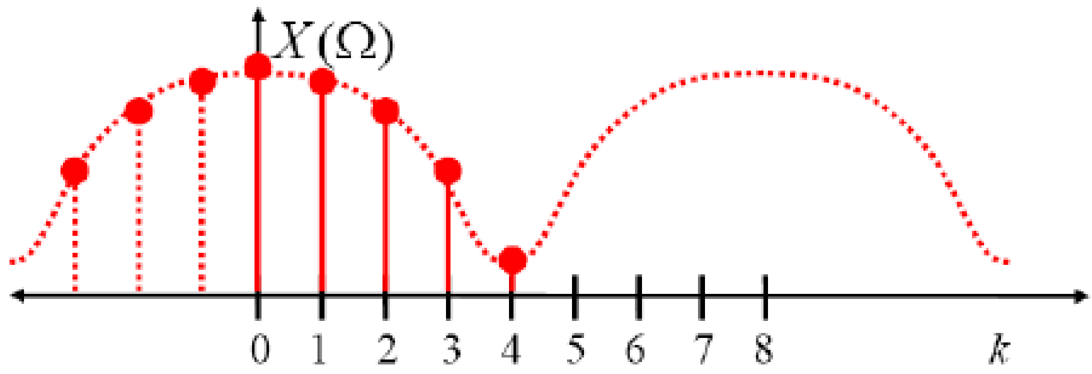
**Illustration of DFT Symmetry**

$$\boxed{X[N-k] = \overline{X}[k], \quad k = 0,1,2,\ldots,N-1}$$

In this example we don't see the effect of the conjugate because we made the DFT real-valued for ease



Because the "upper" DFT points are just like the "negative index" DFT points… this DFT symmetry property is exactly the same as the DTFT symmetry around the origin:



## 2. Inverse DFT

Recall that the DTFT can be inverted… given $X(\Omega)$ you can find the signal $x[n]$

Because we arrived at the DFT via the DTFT… it should be no surprise that the DFT inherits an inverse property from the DTFT.

Actually, we needed to force $M = N$ to enable the DFT inverse property to hold!!

So… Given $N$ DFT points $X[k]$ for $k = 0, \ldots, N$-1
Compute $N$ signal data points using:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1$$

**Inverse DFT (IDFT)**

Compare to the DFT… a remarkably similar structure:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1$$

**DFT**

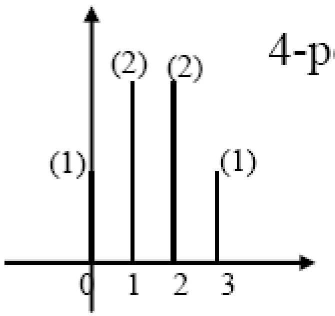
$k$

Ex. $x[0]=1, \quad x[1]=2, \quad x[2]=2, \quad x[3]=1, \quad x[n]=0, elsewhere$

DFT=? $$X_k = \sum_{n=0}^{3} x[n] e^{-j2\pi kn/4}, \quad k = 0,1,2,3$$

$$= x[0] + x[1] e^{-j\frac{1}{2}\pi k} + x[2] e^{-j\pi k} + x[3] e^{-j\frac{3}{2}\pi k}$$

$$= 1 + 2e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + e^{-j\frac{3\pi k}{2}}$$




4-point

## DFT & DTFT: Finite Duration Case

If $x[n] = 0$ for $n < 0$ and $n \geq N$ then the DTFT is:

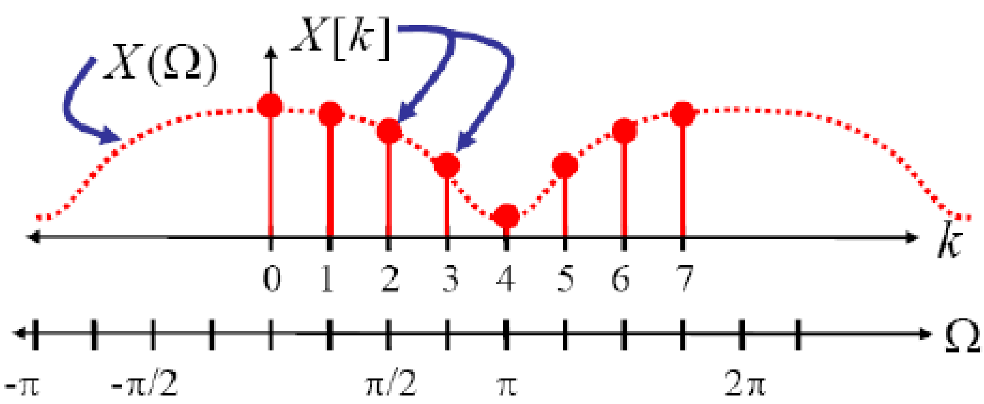
$$\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}}$$

we can leave out terms that are zero

Now… if we take these $N$ samples and compute the DFT (using the FFT, perhaps) we get:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k = 0,1,2,\ldots,N-1}$$

Comparing these we see that for the finite-duration signal case: $\boxed{X[k] = X\left(k\frac{2\pi}{N}\right)}$




**DTFT & DFT :**

DFT points lie exactly on the finite-duration signal's DTFT!!!

## Summary of DFT & DTFT for a ***finite*** duration $x[n]$

$$x[n] \xrightarrow{\text{DTFT}} X(\Omega)$$

$$x[n] \xrightarrow{\text{DFT}} X[k] = X\left(k\frac{2\pi}{N}\right)$$

**Points of DFT are "samples" of DTFT of $x[n]$**

The number of samples $N$ sets how closely spaced these "samples" are on the DTFT… seems to be a limitation.

**<u>"Zero-Padding Trick"</u>**

After we collect our $N$ samples, we tack on some additional zeros at the end to trick the "DFT Processing" into thinking there are really more samples.

(Since these are <u>zeros</u> tacked on they don't change the values in the DFT sums)

If we now have a total of $N_Z$ "samples" (including the tacked on zeros), then the spacing between DFT points is $2\pi/N_Z$ which is smaller than $2\pi/N$

## Example:

$$x[n] = \begin{cases} 1, & n = 0,1,2,\ldots 2q \\ 0, & otherwise \end{cases} \qquad \text{Recall}: \; p_q[n] = \begin{cases} 1, & n = -q, \ldots, -1, 0, 1, \ldots, q \\ 0, & otherwise \end{cases}$$

Then… $x[n] = p_q[n-q]$

Note: we'll need the delay property for DTFT

From DTFT Table: $\boxed{p_q[n] \leftrightarrow P_q(\Omega) = \dfrac{\sin[(q+0.5)\Omega]}{\sin[\Omega/2]}}$

From DTFT Property Table

(Delay Property): $\boxed{X(\Omega) = \dfrac{\sin[(q+0.5)\Omega]}{\sin[\Omega/2]} e^{-jq\Omega}}$

Since $x[n]$ is a finite-duration signal then the DFT of the $N = 2q+1$ non-zero samples is just samples of the DTFT: $\boxed{X[k] = X\left(k\dfrac{2\pi}{N}\right)}$

$$\boxed{X[k] = \frac{\sin[(q+.5)2\pi k/N]}{\sin[\pi k/N]} e^{-jq2\pi k/N}}$$

for DFT. $X_k$ for $k = 0,1,2,\ldots,N-1$

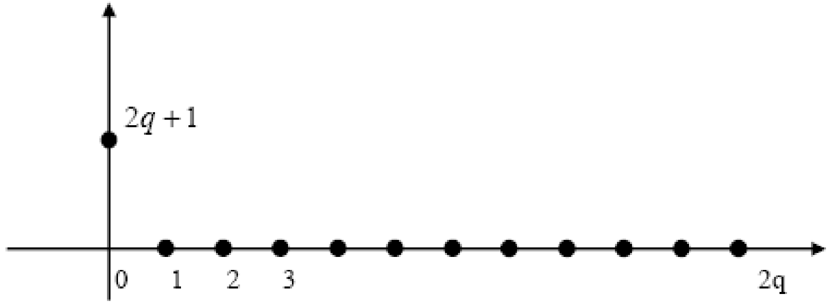
if $N = 2q+1$

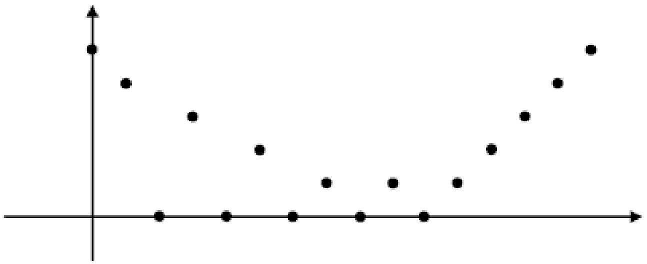
$$|X_k| = |X(\Omega)|_{\Omega=\frac{2\pi k}{N}} = \frac{\left|\sin\left(q+\frac{1}{2}\right)\left(\frac{2\pi k}{N}\right)\right|}{\left|\sin\left(\frac{2\pi k}{2N}\right)\right|}$$

let $N = 2q+1 \Rightarrow |X_k| = \dfrac{|\sin(\pi k)|}{\left|\sin\left(\frac{\pi k}{2q+1}\right)\right|}, \quad k = 0,1,\ldots,2q$

$$\Rightarrow |X_k| = \begin{cases} 2q+1, & k = 0 \\ 0, & k = 1,2,\ldots,2q \end{cases}$$

By L'Hopital's Rule



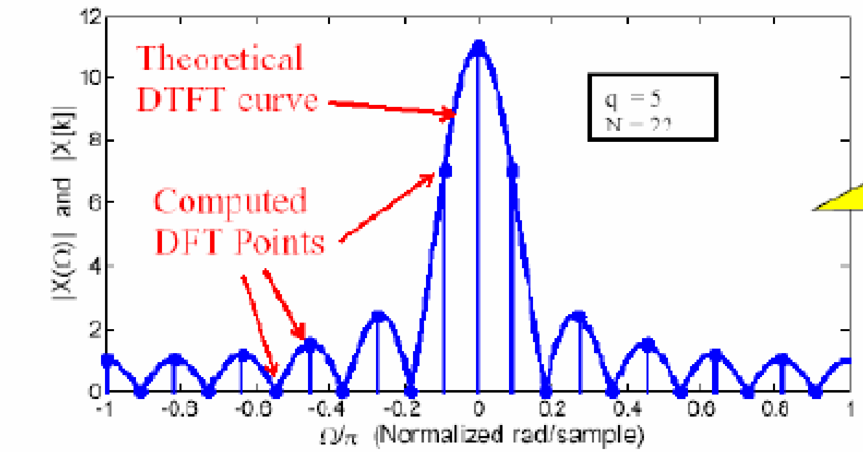
$N = 2(2q+1) = 22$

let $q = 5$

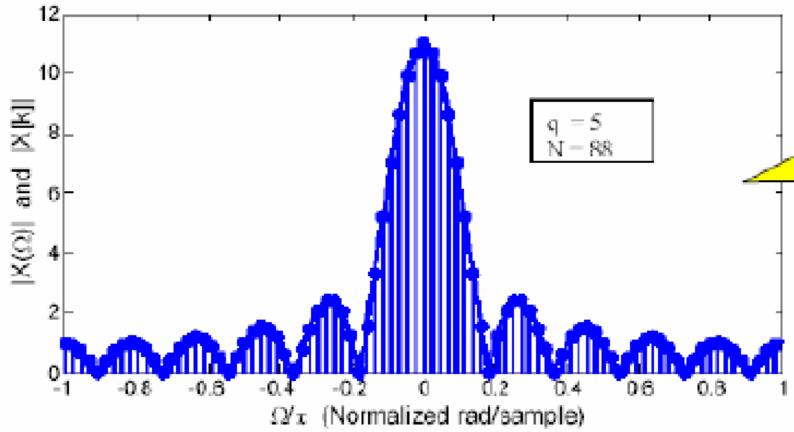
Note that if we don't zero pad, then all but the $k = 0$ DFT values are zero!!!

That doesn't show what the DTFT looks like! So we need to use zero-padding.

Here are two numerically computed examples, both for the case of $q = 5$:




For the case of zero-padding 11 zeros onto the end of the signal… the DFT points still don't really show what the DTFT looks like!




For the case of zero-padding 77 zeros onto the end of the signal… **NOW** the DFT points really show what the DTFT looks like!

## Important Points for *Finite-Duration* Signal Case

- DFT points lie on the DTFT curve… perfect view of the DTFT
  - But… only if the DFT points are spaced closely enough
- Zero-Padding doesn't change the shape of the DFT…
- It just gives a denser set of DFT points… all of which lie on the true DTFT
  - Zero-padding provides a better view of this "perfect" view of the DTFT

## DFT & DTFT: Infinite Duration Case

As we said… in a computer we cannot deal with an infinite number of signal samples.

So say there is some signal that "goes on forever" (or at least continues on for longer than we can or are willing to grab samples)

$x[n] \quad n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

We **only grab $N$ samples**: $x[n]$, $n = 0, \ldots, N-1$ **We've lost some information!**

We can define an "imagined" finite-duration signal:

$$x_N[n] = \begin{cases} x[n], & n = 0, 1, 2, \ldots, N-1 \\ 0, & elsewhere \end{cases}$$

We can compute the DFT of the $N$ collected samples:

$$X_N[k] = \sum_{n=0}^{N-1} x_N[n] e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1$$

Q: How does this DFT of the "truncated signal" relate to the "true" DTFT of the full-duration $x[n]$? …which is what we really want to see!!

$$\text{"True" DTFT}: X_\infty(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

What we *want* to see

$$\text{DTFT of truncated signal}: X_N(\Omega) = \sum_{n=-\infty}^{\infty} x_N[n]e^{-j\Omega n}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

A *distorted* version of what we want to see

$$\text{DFT of collected signal data}: X_N[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

What we *can* see

$$\text{DFT gives samples of } X_N(\Omega)$$

So… DFT of collected data gives "samples" of $\underbrace{\text{DTFT of } \textit{truncated} \text{ signal}}_{\neq \text{"True" DTFT}}$

⇒DFT of collected data does not perfectly show DTFT of complete signal.

Instead, the DFT of the data shows the DTFT of the *truncated* signal…

So **our goal** is to understand what kinds of "errors" are in the "truncated" DTFT

…then we'll know what "errors" are in the computed DFT of the data

To see what the DFT <u>does</u> show we need to understand how

$X_N(\Omega)$ relates to $X_\infty(\Omega)$

First, we note that:

$$x_N[n] = x[n]\underbrace{p_q[n-q]}_{} \xrightarrow{\text{DTFT}} P_q(\Omega) = \frac{\sin[N\Omega/2]}{\sin[\Omega/2]} e^{-j(N-1)\Omega/2}$$

with $N=2q+1$

From "mult. in time domain" property in DTFT Property Table:

$$\boxed{X_N(\Omega) = X_\infty(\Omega) * P_q(\Omega)}$$

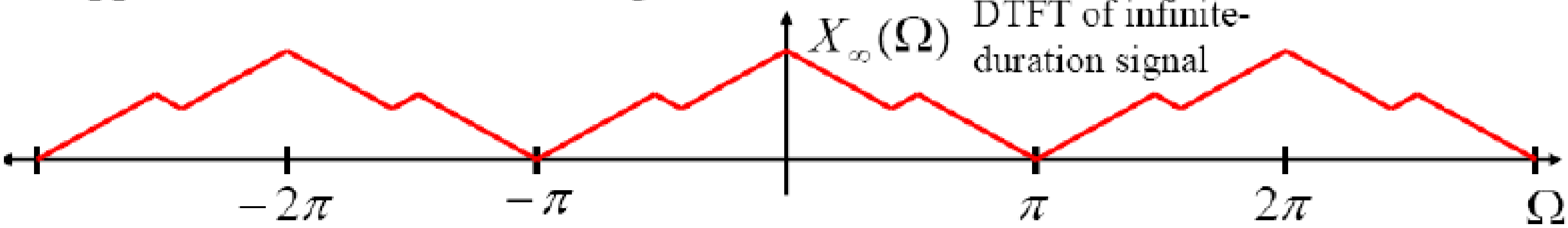
Convolution causes "smearing" of $X_\infty(\Omega)$

$\Rightarrow$ So… $X_N(\Omega)$ …which we can see via the DFT $X_N[k]$ …

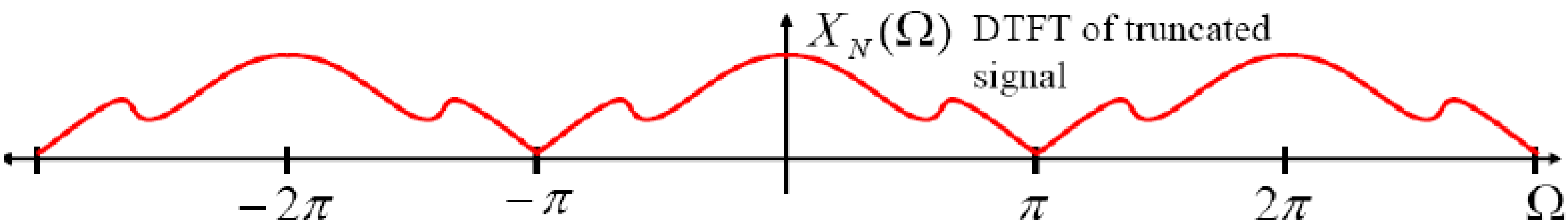
is a "smeared" version of $X_\infty(\Omega)$

**"Fact": The more data you collect, the less smearing**
**… because $P_q(\Omega)$ becomes more like $\delta(\Omega)$**

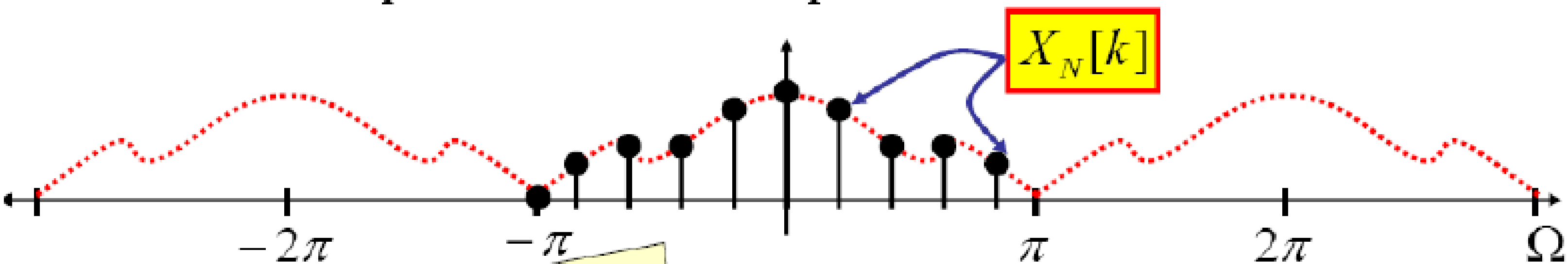
**Suppose the infinite-duration signal's DTFT is:**




**Then it gets smeared into something that might look like this:**




**Then the DFT computed from the *N* data points is:**




The DFT points are shown *after* "upper" points are moved (e.g., by matlab's "fftshift"

## Example: Infinite-Duration Complex Sinusoid & DFT

Suppose we have the signal $x[n] = e^{-j\Omega_o n}$ $\quad n = \ldots, -3, -2, -1, 0, 1, 2, \ldots$

and we want to compute the DFT of $N$ collected samples ($n$ = 0, 1, 2, …, $N$-1).

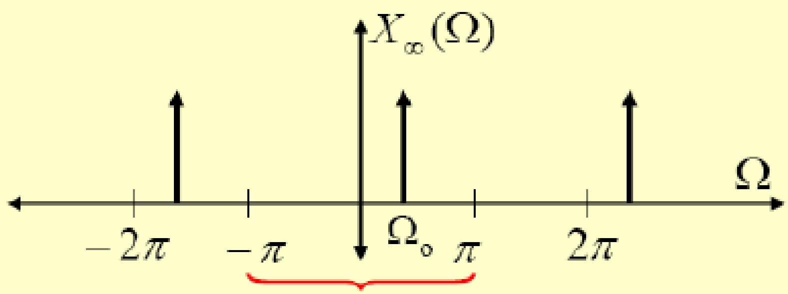
This is an important example because in practice we often have signals that consists of a few significant sinusoids among some other signals (e.g. radar and sonar).

In practice we just get the $N$ samples and we compute the DFT… but before we do that we need to understand what the DFT of the $N$ samples will show.

So we first need to theoretically find the DTFT of the infinite-duration signal.

From DTFT Table we have:

$$X_\infty(\Omega) = \begin{cases} \delta(\Omega - \Omega_0), & -\pi < \Omega < \pi \\ \textit{periodic elsewhere} & \end{cases}$$



From our previous results we know that the DTFT of the collected data is:

$P_q(\Omega)$

$$X_N(\Omega) = X_\infty(\Omega) * \left[ \frac{\sin[N\Omega/2]}{\sin[\Omega/2]} e^{-j(N-1)\Omega/2} \right]$$

Just Delta's in here $\Rightarrow$ Use Sifting Property!!

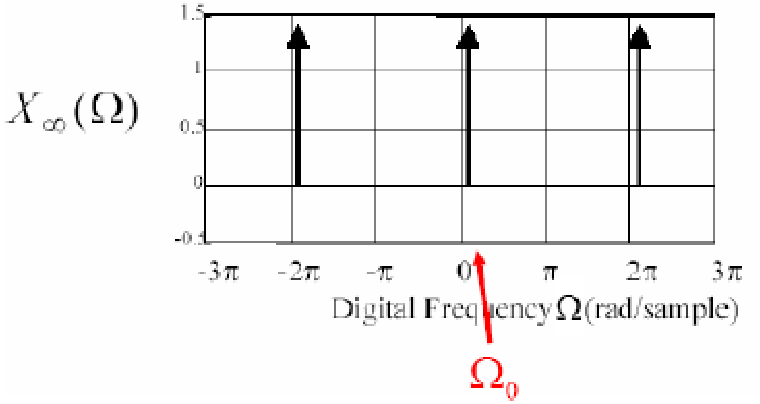
$$X_\infty(\Omega) = \begin{cases} \delta(\Omega - \Omega_0), & -\pi < \Omega < \pi \\ \textit{periodic elsewhere} \end{cases}$$

Just a shifted version of $P_q(\Omega)$

$$X_N(\Omega) = \begin{cases} \dfrac{\sin\left[\dfrac{N(\Omega - \Omega_0)}{2}\right]}{\sin\left[\dfrac{(\Omega - \Omega_0)}{2}\right]} e^{-j(N-1)(\Omega-\Omega_0)/2}, & -\pi < \Omega < \pi \\ \textit{periodic elsewhere} \end{cases}$$

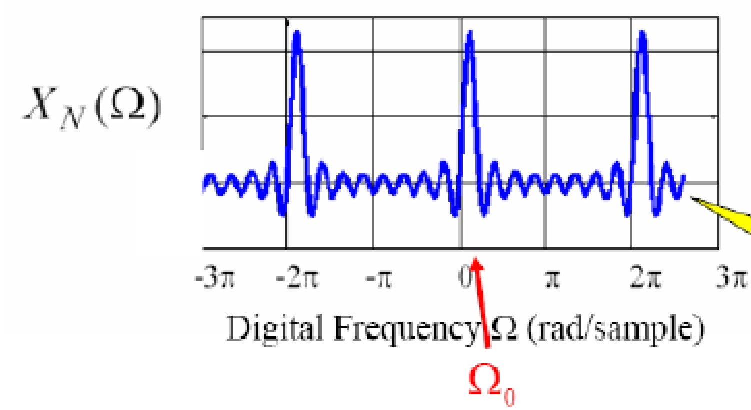
This is the DTFT on which our data-computed DFT points will lie… so looking at this DTFT shows us what we can expect from our DFT processing!!!

## True DTFT of Infinite Duration Complex Sinusoid




$$X_\infty(\Omega) = \begin{cases} \delta(\Omega - \Omega_0), & -\pi < \Omega < \pi \\ \text{periodic elsewhere} \end{cases}$$

## DTFT of Finite Number of Samples of a Complex Sinusoid

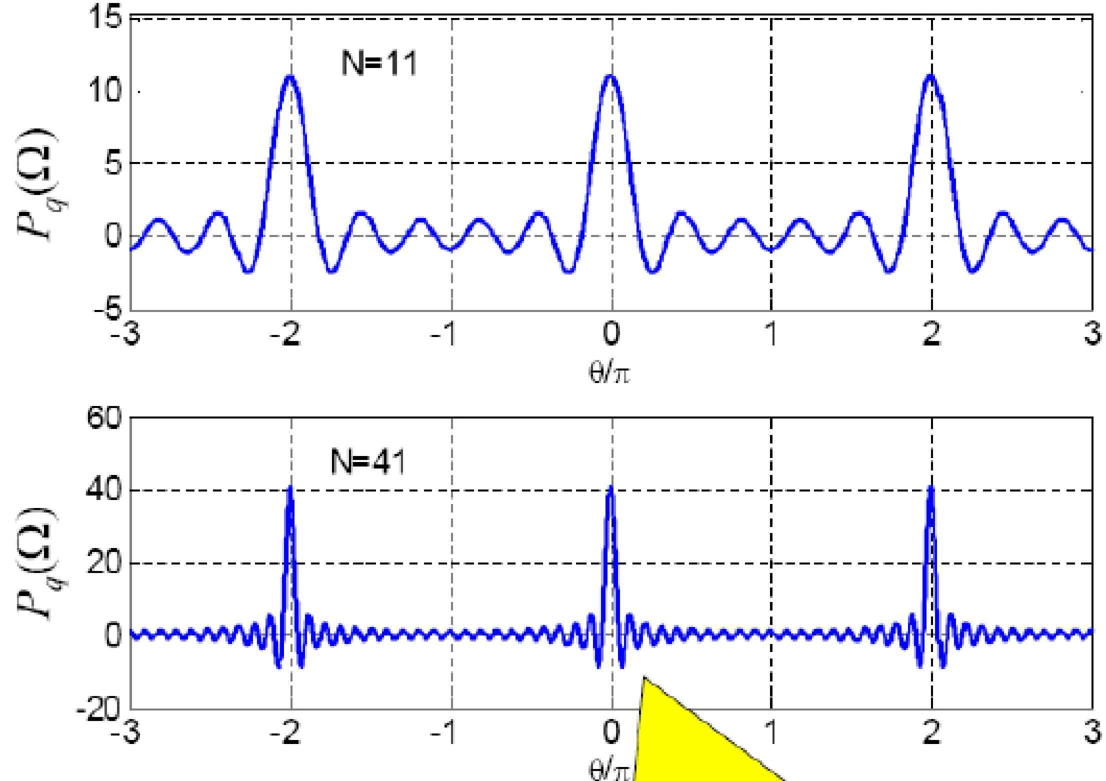



$$X_N(\Omega) = \begin{cases} \dfrac{\sin\left[\dfrac{N(\Omega - \Omega_0)}{2}\right]}{\sin\left[\dfrac{(\Omega - \Omega_0)}{2}\right]} e^{-j(N-1)(\Omega-\Omega_0)/2}, & -\pi < \Omega < \pi \\ \text{periodic elsewhere} \end{cases}$$

The computed DFT would give points on this curve… the spacing of points is controlled through "zero padding"

**So… what effect does our choice of $N$ have???**

To answer that we can simply look at $P_q(\Omega)$ for different values of $N = 2q+1$

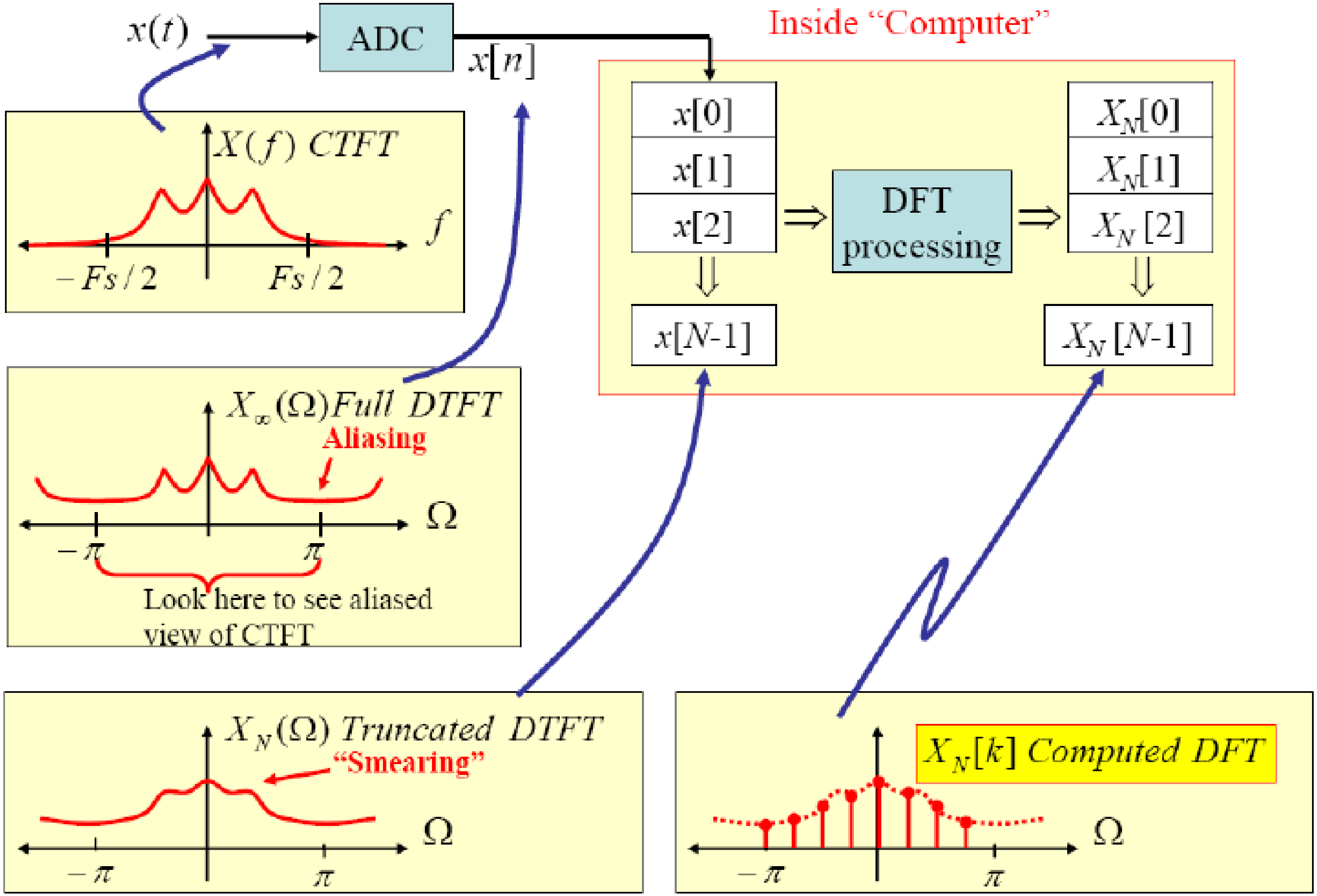



As $N$ grows… looks more like a delta!!
So… less smearing of $X_N(\Omega)$!!

## Important points for Infinite-Duration Signal Case

1. DTFT of finite collected data is a "smeared" version of the DTFT of the infinite-duration data
2. The computed DFT points lie on the "smeared" DTFT curve… not the "true" DTFT
   a. This gives an imperfect view of the true DTFT!
3. "Zero-padding" gives denser set of DFT points… a better view of this imperfect view of the desired DTFT!!!

# Connections between the CTFT, DTFT, & DFT



# Errors in a Computed DFT

$$\text{CTFT} \rightarrow \text{DTFT}_{\infty} \rightarrow \text{DTFT}_N \rightarrow \text{DFT}$$

**Aliasing Error** – control through $F_s$ choice
(i.e. through proper sampling)

See DSP course

**"Smearing" Error** – control through $\begin{cases} N \text{ choice} \\ \text{"window" choice} \end{cases}$

**"Grid" Error** – control through $\begin{cases} N \text{ choice} \\ \text{"zero padding"} \end{cases}$

**This is the only thing we can compute from data… and it has all these "errors" in it!! The theory covered here allows an engineer to understand how to control the amount of those errors!!!**

Zero padding trick

Collect $N$ samples → defines $X_N(\Omega)$

Tack $M$ zeros on at the end of the samples

Take $(N + M)$pt. DFT → gives points on $X_N(\Omega)$ spaced by $2\pi/(N+M)$
(rather than $2\pi/N$)