Mathematical-03

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Automaton

- in situ排列可以通过自动机实现

$$(\Sigma, \delta, q_0, q_f)$$
$$\delta : \Sigma \to [1..n]^3 \times \Sigma^n$$
$$\delta(q) = (i, j, k, q_1, \ldots, q_k)$$

Automaton-Σ

- 0: 对应第一行的循环
- 1: 验证它是否是cycle leader
- 2: 交换元素

1	**for** $j := 1$ **step** 1 **until** n **do**	1
2	**begin comment** the permutation has been	n
3	done on all cycles with leader $< j$;	n
4	$k := p(j)$;	n
5	**while** $k > j$ **do**	$n + a$
6	$k := p(k)$;	a
7	**if** $k = j$ **then**	n
8	**begin comment** j is a cycle leader;	b
9	$y := x[j]$; $l := p(k)$;	b
10	**while** $l \neq j$ **do**	$b + c$
11	**begin** $x[k] := x[l]$; $k := l$; $l := p(k)$ **end**;	c
12	$x[k] := y$;	b
13	**end** permutation on cycle;	b
14	**end** loop on j.	n

Automaton-δ

- 起始状态(0, 1), 终止状态(0, n + 1)

$$\delta(0, j) = (1, 1, j, (1, j, 1), \ldots, (1, j, n));$$

$$\delta(1, j, k) = \begin{cases} (1, 1, k, (1, j, 1), \ldots, (1, j, n)), & \text{if } k > j; \\ (1, 1, j, (2, j, j, 1), \ldots, (2, j, j, n)) & \text{if } k = j; \\ (1, 1, 1, (0, j+1), \ldots, (0, j+1)) & \text{if } k < j; \end{cases}$$

$$\delta(2, j, k, l) = \begin{cases} (k, l, l, (2, j, l, 1), \ldots, (2, j, l, n)), & \text{if } k \neq l; \\ (1, 1, 1, (0, j+1), \ldots, (0, j+1)), & \text{if } k = l; \end{cases}$$

ALGORITHM65 :FIND

- 类似快排的算法, 只对一部分排序

```
begin        integer I,J;
             if M < N then begin partition (A, M, N, I, J);
                                 if K≦I then find (A,M,I,K)
                                 else if J≦K then find (A,J,N,K)
                                 end
end          find
```

- ALGORITHM63: PARTITION

比较次数

- 类似快排的分析, 得到

$$C_{1,1} = 0;$$

$$C_{n,t} = n - 1 + \frac{1}{n}(A_{n,t} + B_{n,t}), \quad \text{for } 1 \le t \le n \text{ and } n \ge 2,$$

- 其中呢

$$A_{n,t} = C_{n-1,t-1} + C_{n-2,t-2} + \cdots + C_{n-t+1,1},$$

$$B_{n,t} = C_{t,t} + C_{t+1,t} + \cdots + C_{n-1,t}.$$

分别是要找的元素在枢轴量的右边和左边的情况.

解递推方程

$$\text{由定义得到这三个式子}\left\{\begin{aligned}
A_{n+1,t+1} - A_{n,t} &= C_{n,t} \\
B_{n+1,t} - B_{n,t} &= C_{n,t} \\
A_{n,t} + B_{n,t} &= nC_{n,t} - n(n-1)
\end{aligned}\right.$$

$$\begin{aligned}
A_{n+1,t+1} + B_{n+1,t+1} &= (n+1)C_{n+1,t+1} - n(n+1) \\
A_{n,t+1} + B_{n,t+1} &= nC_{n,t+1} - n(n-1) \\
B_{n+1,t+1} - B_{n,t+1} &= C_{n,t+1} \\
A_{n-1,t} + B_{n-1,t} &= (n-1)C_{n-1,t} - (n-2)(n-1)
\end{aligned}$$

解递推方程

$$\text{由定义得到这三个式子}\left\{\begin{aligned}
A_{n+1,t+1} - \cancel{A_{n,t}} &= C_{n,t} \\
B_{n+1,t} - \cancel{B_{n,t}} &= C_{n,t} \\
\cancel{A_{n,t}} + \cancel{B_{n,t}} &= nC_{n,t} - n(n-1)
\end{aligned}\right.$$

$$A_{n+1,t+1} + B_{n+1,t+1} = (n+1)C_{n+1,t+1} - n(n+1)$$

$$A_{n,t+1} + B_{n,t+1} = nC_{n,t+1} - n(n-1)$$

$$B_{n+1,t+1} - B_{n,t+1} = C_{n,t+1}$$

$$A_{n-1,t} + B_{n-1,t} = (n-1)C_{n-1,t} - (n-2)(n-1)$$

解递推方程

由定义得到这三个式子

$$\begin{aligned}
\cancel{A_{n+1,t+1}} - \cancel{A_{n,t}} &= C_{n,t} \\
B_{n+1,t} - \cancel{B_{n,t}} &= C_{n,t} \\
\cancel{A_{n,t}} + \cancel{B_{n,t}} &= nC_{n,t} - n(n-1)
\end{aligned}$$

消去$A_{n+1,t+1}$

$$\begin{aligned}
\cancel{A_{n+1,t+1}} + B_{n+1,t+1} &= (n+1)C_{n+1,t+1} - n(n+1) \\
A_{n,t+1} + B_{n,t+1} &= nC_{n,t+1} - n(n-1) \\
B_{n+1,t+1} - B_{n,t+1} &= C_{n,t+1} \\
A_{n-1,t} + B_{n-1,t} &= (n-1)C_{n-1,t} - (n-2)(n-1)
\end{aligned}$$

解递推方程

由定义得到这三个式子

$$
\begin{aligned}
\cancel{A_{n+1,t+1}} - \cancel{A_{n,t}} &= C_{n,t} \\
B_{n+1,t} - \cancel{B_{n,t}} &= C_{n,t} \\
\cancel{A_{n,t}} + \cancel{B_{n,t}} &= nC_{n,t} - n(n-1)
\end{aligned}
$$

消去$A_{n+1,t+1}$和$B_{n+1,t+1}$

$$
\begin{aligned}
\cancel{A_{n+1,t+1}} + \cancel{B_{n+1,t+1}} &= (n+1)C_{n+1,t+1} - n(n+1) \\
A_{n,t+1} + \cancel{B_{n,t+1}} &= nC_{n,t+1} - n(n-1) \\
\cancel{B_{n+1,t+1}} - \cancel{B_{n,t+1}} &= C_{n,t+1} \\
A_{n-1,t} + B_{n-1,t} &= (n-1)C_{n-1,t} - (n-2)(n-1)
\end{aligned}
$$

解递推方程

由定义得到这三个式子

$$\cancel{A_{n+1,t+1}} - \cancel{A_{n,t}} = C_{n,t}$$

$$B_{n+1,t} - \cancel{B_{n,t}} = C_{n,t}$$

$$\cancel{A_{n,t}} + \cancel{B_{n,t}} = nC_{n,t} - n(n-1)$$

消去$A_{n+1,t+1}$和$B_{n+1,t+1}$

$$\cancel{A_{n+1,t+1}} + \cancel{B_{n+1,t+1}} = (n+1)C_{n+1,t+1} - n(n+1)$$

$$A_{n,t+1} + \cancel{B_{n,t+1}} = nC_{n,t+1} - n(n-1)$$

$$\cancel{B_{n+1,t+1}} - \cancel{B_{n,t+1}} = C_{n,t+1}$$

消去最后两项

$$A_{n-1,t} + B_{n-1,t} = (n-1)C_{n-1,t} - (n-2)(n-1)$$

解递推方程

- 所以只需

$$\alpha C_{n+1,t+1}+\beta C_{n,t+1}+\gamma C_{n,t}+\theta C_{n-1,t}$$

就可以得到和AB无关的式子

$$C_{n+1,t+1}-C_{n,t+1}-C_{n,t}+C_{n-1,t}=\frac{2}{n+1}$$

边界条件

- 类似对快排时间复杂度的分析

$$C_{n,1} = n - 1 + \frac{1}{n}(C_{1,1} + \cdots + C_{n-1,1})$$

$$n(C_{n,1} - (n-1)) = \sum_{i=1}^{n-1} C_{i,1}$$

$$(n+1)(C_{n+1,1} - n) = \sum_{i=1}^{n} C_{i,1}$$

$$C_{n+1,1} - C_{n,1} = 2 - \frac{2}{n+1}$$

边界条件

- 类似对快排时间复杂度的分析

$$
\begin{aligned}
C_{n,1} &= 2 - \frac{2}{n} + C_{n-1,1} \\
&= \dots \\
&= 2n - 2\sum_{i=1}^{n} \frac{1}{i} \\
&= 2n - 2H_n
\end{aligned}
$$

解递推方程

- 这样就可以求解刚才的方程了

$$
\begin{aligned}
C_{n,t} - C_{n-1,t-1} &= \frac{2}{n} + C_{n-1,t} - C_{n-2,t-1} \\
&= 2\sum_{i=t+1}^{n} \frac{1}{i} + C_{t,t} - C_{t-1,t-1} \\
&= 2(H_n - H_t) + 2 - \frac{2}{t}
\end{aligned}
$$

解递推方程

- 这同样是一个递推式

$$
\begin{aligned}
C_{n,t} &= 2(H_n - H_t) + 2 - \frac{2}{t} + C_{n-1,t-1} \\
&= 2(H_n - H_t) + 2 - \frac{2}{t} + 2(H_{n-1} - H_{t-1}) + 2 - \frac{2}{t-1} + C_{n-2,t-2} \\
&= \dots \\
&= 2\sum_{2\le k\le t}(H_{n-t+k} - H_k + 1 - \frac{1}{k}) + C_{n+1-t,1}
\end{aligned}
$$

解递推方程

- 归纳

$$C_{n,t} = 2\sum_{2\le k\le t}(H_{n-t+k} - H_k + 1 - \frac{1}{k}) + C_{n+1-t,1}$$

$$= 2((n+1)H_n - (n+3-t)H_{n+1-t} - (t+2)H_t + n + 3) \quad \text{why?}$$

$$\delta(\text{left}) = 2(H_{m+1} - H_{m-t+2} + 1 - \frac{1}{m+2-t})$$

$$\delta(\text{right}) = 2(H_{m+1} - H_{m+2-t} - \frac{m+3-t}{m+2-t} + 1)$$

求中位数

- 将n = 2t - 1带入

$$C_{2t+1,t} = 4t(H_{2t-1} - H_t) + 4t - 8H_t + 4$$
$$= (4 + 4\ln 2)t - 8 + 1 - 8\gamma + O(t^{-1})$$

$$H_n = \sum_{k=1}^{n} = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \dots.$$

Type B?

- 比较次数的界在哪里?
  - t = 1 ==> n - 1
  - t = 2 ==> n - 2 + [$\log_2$ n]
  - t = 3 ==> ???

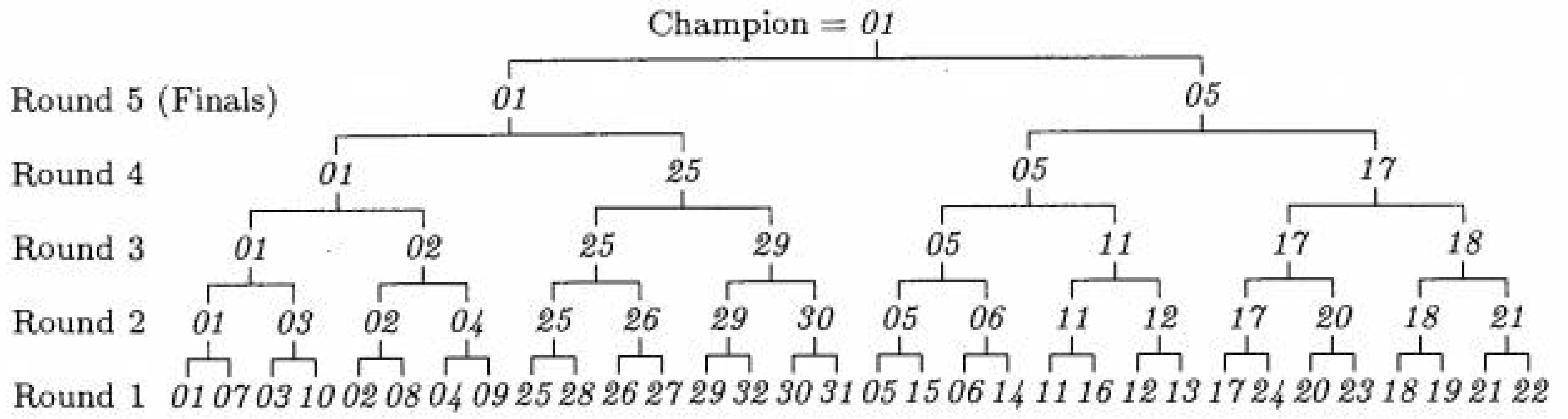

Fig. 39. A knockout tournament with 32 players.

Type B?

- 对任意的n, t, 比较次数不超过 5.2n
- 平均次数不超过

  $n + \min(t, n + 1 - t) + o(n)$
- 中位数可以做到

  $1.5n + O(n^{2/3} \log n)$

  平均$1.25n + o(n)$

Table 1

VALUES OF $V_t(n)$ FOR SMALL n

n	$V_1(n)$	$V_2(n)$	$V_3(n)$	$V_4(n)$	$V_5(n)$	$V_6(n)$	$V_7(n)$	$V_8(n)$	$V_9(n)$	$V_{10}(n)$
1	0									
2	1	1								
3	2	3	2							
4	3	4	4	3						
5	4	6	6	6	4					
6	5	7	8	8	7	5				
7	6	8	10	10*	10	8	6			
8	7	9	11	12	12	11	9	7		
9	8	11	12	14	14*	14	12	11	8	
10	9	12	14*	15	16**	16**	15	14*	12	9

* Exercises 10–12 give constructions that improve on Eq. (11) in these cases.

** See K. Noshita, *Trans. of the IECE of Japan* **E59**, 12 (December 1976), 17–18.