1 Q1

$\mathcal{C}_1 = \phi$ is open, close, bounded and compact, its interior, closure, boundary and accumulation point set is ϕ.

$\mathcal{C}_2 = \mathcal{R}^n$ is open, close, not bounded and thus not compact, its interior, closure, boundary and accumulation points set is $\mathcal{R}^n$.

$\mathcal{C}_3 = [0,1) \cup [2,3] \cup (4,5]$ is not open, not close, is bounded, but not compact, its interior is $(0,1) \cup (2,3) \cup (4,5)$, its closure is $[0,1] \cup [2,3] \cup [4,5]$, its boundary is $\{0,1,2,3,4,5\}$, and its accumulation points are $[0,1] \cup [2,3] \cup [4,5]$.

$\mathcal{C}_4 = \{(x,y)^T |\ x \geq 0, y > 0\}$ is not open, not close, not bounded, not compact, and its interior is $\{(x,y)|\ x > 0, y > 0\}$, its closure is $\{(x,y)|\ x \geq 0, y \geq 0\}$, its boundary is $\{(0,y)|y \geq 0\} \cup \{(x,0)|\ x \geq 0\}$, and is accumulation points is $\{(x,y)|\ x \geq 0, y \geq 0\}$.

$\mathcal{C}_5 = \{k|\ k \in \mathcal{Z}\}$ is not open, but is closed, is not bounded, and not compact, its interior is ϕ, its closure and boundary is itself, $\{k|\ k \in \mathcal{Z}\}$, and its accumulation points are ϕ.

$\mathcal{C}_6 = \{k^{-1}|\ k \in \mathcal{Z}\}$ is not open, not closed, but is bounded, and is not compact, its interior is ϕ, its closure and boundary is $\{k^{-1}|\ k \in \mathcal{Z}\} \cup \{0\}$, its accumulation point is $\{0\}$.

$\mathcal{C}_7 = \{(1/k, \sin k^T|\ k \in \mathcal{Z})\}$ is not open, not closed, but is bounded, and is not compact, its interior is ϕ, its closure and boundary are $\{(1/k, \sin k^T|\ k \in \mathcal{Z})\} \cup \{(0,y)|\ -1 \leq y \leq 1\}$, and its accumulation points are $\{(0,y)|\ -1 \leq y \leq 1\}$.

2 Q2

1. suppose $\mathcal{C}$ is closed, if there exists x^* which is the limit of one convergent sequence in $\mathcal{C}$, but $x^* \notin \mathcal{C}$, thus $x^* \in \mathcal{C}^c$, which is an open set, so we have

$$\exists \epsilon\ s.t.\ (\cup(x^*, \epsilon)) \cap \mathcal{C} = \phi \tag{1}$$

for we have $\cup(x^*, \epsilon) \subseteq \mathcal{C}^c$. but there exists $\{x_k\}_1^\infty \subseteq \mathcal{C}$ s.t. $\lim_{k\to\infty} x_k = x^*$, that is to say,

$$\forall \epsilon \quad (\cup(x^*, \epsilon)) \cap \mathcal{C} \neq \phi \tag{2}$$

contradiction! so for all x^* which is the limit of one convergent sequence in $\mathcal{C}$, we have $x^* \in \mathcal{C}$

2. suppose $\mathcal{C}$ is not closed, $i.e.$ $\mathcal{C}^c$ is not open, that is to say,

$$\exists x^* \in \mathcal{C}^c\ \forall\ \epsilon > 0\ (\cup(x^*, \epsilon)) \cap \mathcal{C} \neq \phi \tag{3}$$

for we have $(\mathcal{C}^c)^c = \mathcal{C}$. so we choose a sequence of $\epsilon_k \to 0$ and find $x_k \in (\cup(x^*, \epsilon)) \cap \mathcal{C}$, then $lim_{k\to\infty} x_k = x^*$, so $x^* \in \mathcal{C}$, contradiction! so $\mathcal{C}$ must be closed.

3. from 1 and 2, we have a set $\mathcal{C} \subseteq \mathcal{R}^n$ is closed iff it contains the limit point of every convergent sequence in it.

3 Q3

$x \in \partial\mathcal{C} = \bar{\mathcal{C}} \setminus \mathcal{C}^o = ((\mathcal{C}^c)^o)^c \setminus \mathcal{C}^o = ((\mathcal{C}^c)^o)^c \cap (\mathcal{C}^o)^c$

lemma 1 by definition, $x \in \mathcal{C}^o \iff \exists\epsilon > 0\ \cup(x, \epsilon) \subseteq \mathcal{C}$, thus we have

$$x \in (\mathcal{C}^o)^c \iff \epsilon > 0 \exists z \notin \mathcal{C} \tag{4}$$