1 Q1

$\mathcal{C}_1 = \phi$ is open, close, bounded and compact, its interior, closure, boundary and accumulation point set is ϕ.

$\mathcal{C}_2 = \mathcal{R}^n$ is open, close, not bounded and thus not compact, its interior, closure, boundary and accumulation points set is $\mathcal{R}^n$.

$\mathcal{C}_3 = [0,1) \cup [2,3] \cup (4,5]$ is not open, not close, is bounded, but not compact, its interior is $(0,1) \cup (2,3) \cup (4,5)$, its closure is $[0,1] \cup [2,3] \cup [4,5]$, its boundary is $\{0,1,2,3,4,5\}$, and its accumulation points are $[0,1] \cup [2,3] \cup [4,5]$.

$\mathcal{C}_4 = \{(x,y)^T |\ x \geq 0, y > 0\}$ is not open, not close, not bounded, not compact, and its interior is $\{(x,y)|\ x > 0, y > 0\}$, its closure is $\{(x,y)|\ x \geq 0, y \geq 0\}$, its boundary is $\{(0,y)|y \geq 0\} \cup \{(x,0)|\ x \geq 0\}$, and is accumulation points is $\{(x,y)|\ x \geq 0, y \geq 0\}$.

$\mathcal{C}_5 = \{k|\ k \in \mathcal{Z}\}$ is not open, but is closed, is not bounded, and not compact, its interior is ϕ, its closure and boundary is itself, $\{k|\ k \in \mathcal{Z}\}$, and its accumulation points are ϕ.

$\mathcal{C}_6 = \{k^{-1}|\ k \in \mathcal{Z}\}$ is not open, not closed, but is bounded, and is not compact, its interior is ϕ, its closure and boundary is $\{k^{-1}|\ k \in \mathcal{Z}\} \cup \{0\}$, its accumulation point is $\{0\}$.

$\mathcal{C}_7 = \{(1/k, \sin k^T |\ k \in \mathcal{Z})\}$ is not open, not closed, but is bounded, and is not compact, its interior is ϕ, its closure and boundary are $\{(1/k, \sin k^T |\ k \in \mathcal{Z})\} \cup \{(0,y)|\ -1 \leq y \leq 1\}$, and its accumulation points are $\{(0,y)|\ -1 \leq y \leq 1\}$.

2 Q2

1. suppose $\mathcal{C}$ is closed, if there exists x^* which is the limit of one convergent sequence in $\mathcal{C}$, but $x^* \notin \mathcal{C}$, thus $x^* \in \mathcal{C}^c$, which is an open set, so we have

$$\exists \epsilon \ s.t. \ (\cup(x^*, \epsilon)) \cap \mathcal{C} = \phi \tag{1}$$

for we have $\cup(x^*, \epsilon) \subseteq \mathcal{C}^c$. but there exists $\{x_k\}_1^\infty \subseteq \mathcal{C}$ s.t. $\lim_{k\to\infty} x_k = x^*$, that is to say,

$$\forall \epsilon \ \ (\cup(x^*, \epsilon)) \cap \mathcal{C} \neq \phi \tag{2}$$

contradiction! so for all x^* which is the limit of one convergent sequence in $\mathcal{C}$, we have $x^* \in \mathcal{C}$

2. if $\mathcal{C}$ contains the limit point of every convergent sequence in it, suppose $\mathcal{C}$ is not closed, *i.e.* $\mathcal{C}^c$ is not open, that is to say,

$$\exists x^* \in \mathcal{C}^c \ \forall \ \epsilon > 0 \ (\cup(x^*, \epsilon)) \cap \mathcal{C} \neq \phi \tag{3}$$

for we have $(\mathcal{C}^c)^c = \mathcal{C}$. so we choose a sequence of $\epsilon_k \to 0$ and find $x_k \in (\cup(x^*, \epsilon)) \cap \mathcal{C}$, then $lim_{k\to\infty} x_k = x^*$, so $x^* \in \mathcal{C}$, contradiction! so $\mathcal{C}$ must be closed.

3. from 1 and 2, we have a set $\mathcal{C} \subseteq \mathcal{R}^n$ is closed iff it contains the limit point of every convergent sequence in it.

3 Q3

$x \in \partial\mathcal{C} = \bar{\mathcal{C}} \setminus \mathcal{C}^o = ((\mathcal{C}^c)^o)^c \setminus \mathcal{C}^o = ((\mathcal{C}^c)^o)^c \cap (\mathcal{C}^o)^c$

by definition, $x \in \mathcal{C}^o \iff \exists \epsilon > 0 \ \cup(x, \epsilon) \subseteq \mathcal{C}$, thus we have

$$x \in (\mathcal{C}^o)^c \iff \forall \epsilon > 0 \ \exists z \notin \mathcal{C} \ |z - x|_2 < \epsilon \tag{4}$$

and if we change $\mathcal{C}^o$ in equation (4) into $(\mathcal{C}^c)^o$, we have

$$x \in ((\mathcal{C}^c)^o)^c \iff \forall \epsilon > 0 \ \exists y \in \mathcal{C} \ |y - x|_2 < \epsilon \tag{5}$$

so $x \in \partial\mathcal{C} = ((\mathcal{C}^c)^o)^c \cap (\mathcal{C}^o)^c \iff \forall \epsilon > 0 \ \exists y \in \mathcal{C} \ |y - x|_2 < \epsilon \ \exists z \notin \mathcal{C} \ |z - x|_2 < \epsilon$

4 Q4

1.1 if $\mathcal{C}$ is closed, then for all x^* in $\partial\mathcal{C}$, from $Q3$, we have $\forall \epsilon > 0 \ \exists y \in \mathcal{C} \ |y - x^*|_2 < \epsilon$, then x^* must be a limit of one convergent sequence in $\mathcal{C}$, then from $Q2$, we have $x^* \in \mathcal{C}$, so $\mathcal{C} \supseteq \partial\mathcal{C}$.

1.2 if $\mathcal{C} \supseteq \partial\mathcal{C}$, suppose $\mathcal{C}$ is not closed, from $Q2$, there exists one point x^* which is a limit of one convergent sequence in $\mathcal{C}$, but $x^* \notin \mathcal{C}$, then we have

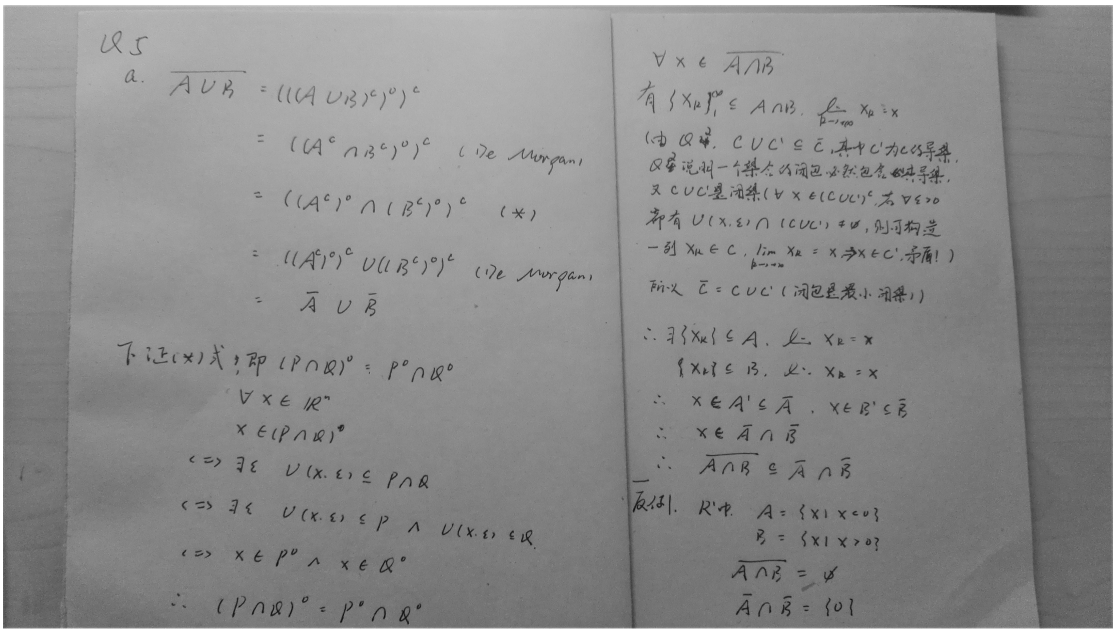
$$\forall \epsilon > 0 \ \exists y \in \mathcal{C} \ |y - x^*|_2 < \epsilon \ \exists z \notin \mathcal{C} \ |z - x^*|_2 < \epsilon, \tag{6}$$

we can just choose $z = x^*$, then from $Q3$, $x^* \in \partial\mathcal{C} \subseteq \mathcal{C}$, contradiction! so $\mathcal{C}$ nust be closed.

2.1 if $\mathcal{C}$ is open, suppose there exists $x^* \in \mathcal{C} \cap \partial\mathcal{C}$, then $\exists \epsilon > 0 \ \cup(x^*, \epsilon) \subseteq \mathcal{C}$ and $\forall \epsilon > 0 \ \exists z \notin \mathcal{C} \ |z - x^*|_2 < \epsilon$, contradiction! so $\mathcal{C} \cap \partial\mathcal{C} = \phi$.

2.2 if $\mathcal{C} \cap \partial\mathcal{C} = \phi$, suppose $\mathcal{C}$ is not open, *i.e.* $\exists x^* \in \mathcal{C} \ \forall \epsilon > 0 \ \exists z \notin \mathcal{C} \ |z - x^*|_2 < \epsilon$, then from $Q3$, $x^* \in \partial\mathcal{C}$, contradiction! so $\mathcal{C}$ is open.

5 Q5, 6, 7的答案(图片)



Q5

a. $\overline{A \cup B} = (((A \cup B)^c)^o)^c$

$= ((A^c \cap B^c)^o)^c$ (De Morgan)

$= ((A^c)^o \cap (B^c)^o)^c$ (*)

$= ((A^c)^o)^c \cup ((B^c)^o)^c$ (De Morgan)

$= \bar{A} \cup \bar{B}$

下证(*)式，即 $(P \cap Q)^o = P^o \cap Q^o$

$\forall x \in \mathbb{R}^n$

$x \in (P \cap Q)^o$

$\iff \exists \varepsilon \ U(x, \varepsilon) \subseteq P \cap Q$

$\iff \exists \varepsilon \ U(x, \varepsilon) \subseteq P \land U(x, \varepsilon) \subseteq Q$

$\iff x \in P^o \land x \in Q^o$

$\therefore (P \cap Q)^o = P^o \cap Q^o$

$\forall x \in \overline{A \cap B}$

有 $\{x_k\}_1^\infty \subseteq A \cap B$, $\lim_{k \to +\infty} x_k = x$

(由 Q2, $C \cup C' \subseteq \bar{C}$, 其中 C' 为 C 的导集, Q2 说明一个集合的闭包必然包含其导集, 又 $C \cup C'$ 是闭集 ($\forall x \in (C \cup C')^c$, 若 $\forall \varepsilon > 0$ 都有 $U(x, \varepsilon) \cap (C \cup C') \neq \emptyset$, 则可构造一列 $x_k \in C$, $\lim_{k \to +\infty} x_k = x \Rightarrow x \in C'$, 矛盾!)

所以 $\bar{C} = C \cup C'$ (闭包是最小闭集))

$\therefore \exists \{x_k\} \subseteq A$, $\lim x_k = x$

$\{x_k\} \subseteq B$, $\lim x_k = x$

$\therefore x \in A' \subseteq \bar{A}$, $x \in B' \subseteq \bar{B}$

$\therefore x \in \bar{A} \cap \bar{B}$

$\therefore \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

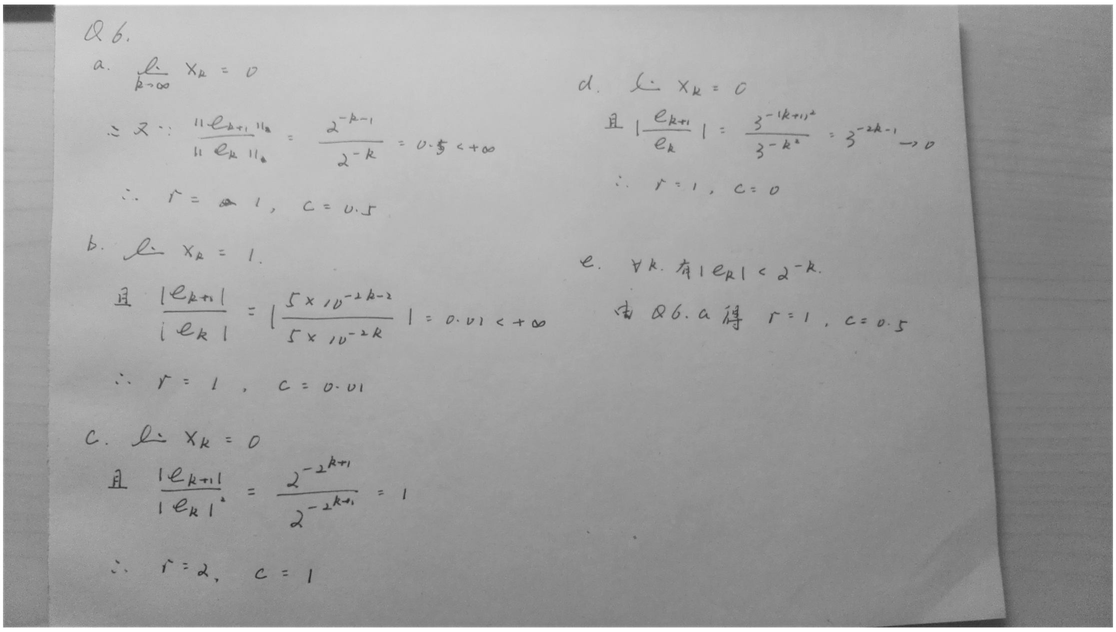
反例. R^1 中 $A = \{x \mid x < 0\}$

$B = \{x \mid x > 0\}$

$\overline{A \cap B} = \emptyset$

$\bar{A} \cap \bar{B} = \{0\}$

Figure 1: Q5a



Q6.

a. $\lim_{k\to\infty} x_k = 0$

又∵ $\frac{\|e_{k+1}\|_*}{\|e_k\|_*} = \frac{2^{-k-1}}{2^{-k}} = 0.5 < +\infty$

∴ $r = 1$, $c = 0.5$

b. $\lim x_k = 1$.

且 $\frac{|e_{k+1}|}{|e_k|} = \left|\frac{5\times10^{-2k-2}}{5\times10^{-2k}}\right| = 0.01 < +\infty$

∴ $r = 1$, $c = 0.01$

c. $\lim x_k = 0$

且 $\frac{|e_{k+1}|}{|e_k|^2} = \frac{2^{-2^{k+1}}}{2^{-2^{k+1}}} = 1$

∴ $r = 2$, $c = 1$

d. $\lim x_k = 0$

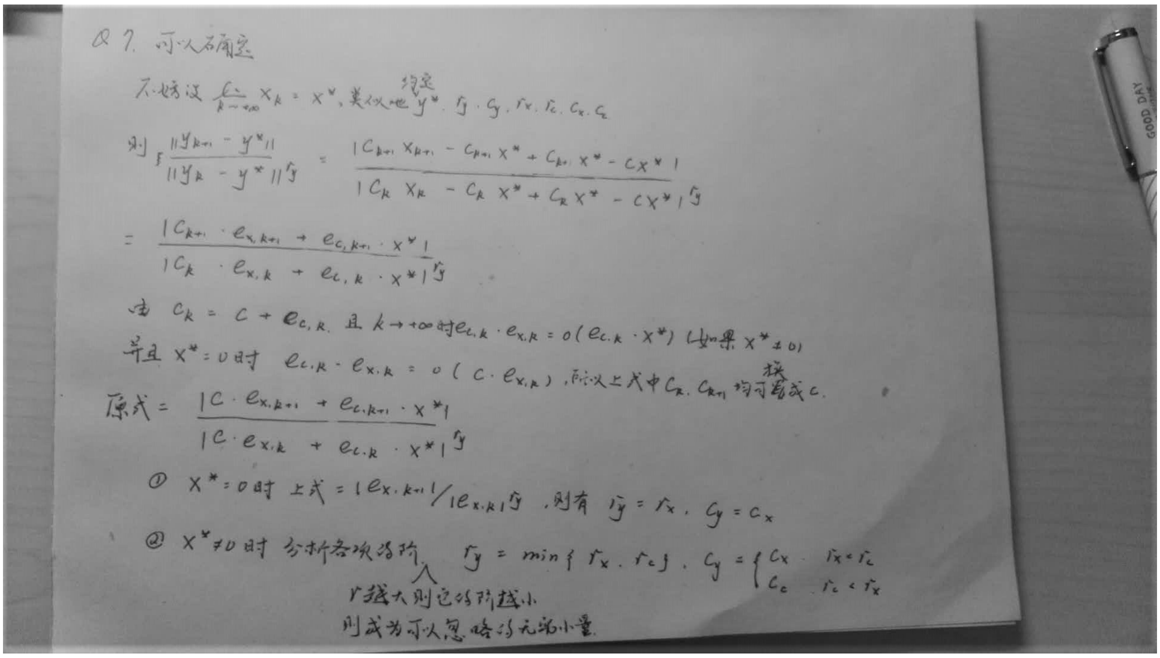
且 $\left|\frac{e_{k+1}}{e_k}\right| = \frac{3^{-(k+1)^2}}{3^{-k^2}} = 3^{-2k-1} \to 0$

∴ $r = 1$, $c = 0$

e. $\forall k$. 有 $|e_k| < 2^{-k}$.

由 Q6.a 得 $r = 1$, $c = 0.5$

Figure 2: Q6



Q7. 可以确定

不妨设 $\lim_{k\to+\infty} x_k = x^*$，类似地约定 y^*. r_y. c_y. r_x. r_c. c_x. c_c

则 $\frac{\|y_{k+1} - y^*\|}{\|y_k - y^*\|^{r_y}} = \frac{|c_{k+1} x_{k+1} - c_{k+1} x^* + c_{k+1} x^* - c x^*|}{|c_k x_k - c_k x^* + c_k x^* - c x^*|^{r_y}}$

$= \frac{|c_{k+1}\cdot e_{x,k+1} + e_{c,k+1}\cdot x^*|}{|c_k\cdot e_{x,k} + e_{c,k}\cdot x^*|^{r_y}}$

由 $c_k = c + e_{c,k}$ 且 $k\to+\infty$ 时 $e_{c,k}\cdot e_{x,k} = o(e_{c,k}\cdot x^*)$ (如果 $x^*\neq 0$)

并且 $x^* = 0$ 时 $e_{c,k}\cdot e_{x,k} = o(c\cdot e_{x,k})$，所以上式中 c_k. c_{k+1} 均可替换成 c.

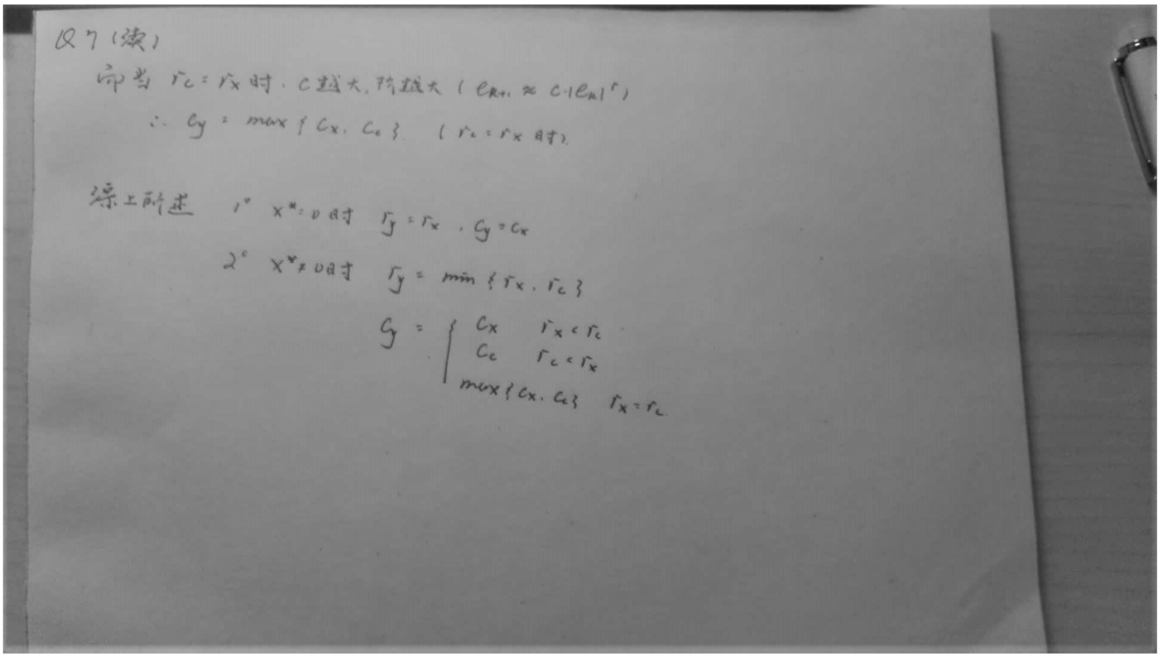
原式 $= \frac{|c\cdot e_{x,k+1} + e_{c,k+1}\cdot x^*|}{|c\cdot e_{x,k} + e_{c,k}\cdot x^*|^{r_y}}$

① $x^* = 0$ 时 上式 $= |e_{x,k+1}|/|e_{x,k}|^{r_y}$，则有 $r_y = r_x$, $c_y = c_x$

② $x^*\neq 0$ 时 分析各项的阶 $r_y = \min\{r_x, r_c\}$, $c_y = \begin{cases} c_x & r_x < r_c \\ c_c & r_c < r_x \end{cases}$

r越大则它的阶越小 则成为可以忽略的无穷小量.

Figure 3: Q7



Q7（续）

而当 $r_c = r_x$ 时，C越大，阶越大（$e_{k+1} \approx C \cdot |e_k|^r$）

$\therefore C_y = \max\{C_x, C_c\}$（$r_c = r_x$ 时）

综上所述 1° $x^* = 0$ 时 $r_y = r_x$，$C_y = C_x$

2° $x^* \neq 0$ 时 $r_y = \min\{r_x, r_c\}$

$$C_y = \begin{cases} C_x & r_x < r_c \\ C_c & r_c < r_x \\ \max\{C_x, C_c\} & r_x = r_c \end{cases}$$

Figure 4: Q7cont