Example control problems

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November 28, 2018

This document contains some toy examples of control problems demonstrating that the prototype code in this project is working as intended. These examples are chosen due to their correspondence with control problems I've seen in the literature and in experiments, though I emphasize again that they are *toy examples* for the purposes of code validation, and I am not in any way claiming they are actually useful. Nevertheless, the optimized sequences do satisfy properties designed into the objective of the search, and also have interesting shapes.

The general setup of the control problems is that we have a matrix differential equation of the form

$$\dot{U}(t) = (G_0 + \sum_k a_k(t) G_k) U(t) \tag{1}$$

with $U(0) = \mathbb{1}_n$ (the identity), where $\dot{U}(t)$ represents the derivative of $U(t)$ with respect to time, and $U(t), G_0$, and the G_k are all in M_n. The $a_k(t)$ represent real valued controllable parameters, and our goal is to choose the $a_k(t)$ so that the trajectory $U(t)$ satisfies certain properties. The G_0 and G_k matrices are called *generators*, and the whole matrix $G(t) = G_0 + \sum_k a_k(t) G_k$ is called the generator of the system.

Everything we will consider is within the context of quantum mechanics, in which case $G_0 = -iH_0$ and $G_k = -iH_k$, where H_0 and H_k are Hermitian matrices, and hence $U(t)$ will be a unitary matrix for all t. We will also assume that the time interval over which we perform our search $[0, T]$ is discretized into N steps, and that the $a_k(t)$ are piecewise constant over those steps.

We will search over the $a_k(t)$ so that the trajectory $U(t)$ satisfies two different types of criteria:

1. $U(T)$ takes a particular value. That is, the final unitary matrix implements a particular quantum gate.

2. $U(T)$ is *robust* to certain variations in the generator. That is, if we change the generator a little by adding a small change in a particular direction, then $U(T)$ doesn't change much.

In numerical searches, the first criteria is commonly quantified in an objective via the *fidelity*

$$|\langle U_{\text{target}}, U(T) \rangle|^2, \tag{2}$$

where U_{target} is the desired transformation. The optimal value is n, the dimension of the matrices, and it is also common to normalize the above function by this factor.

The second criteria can be encoded in various ways and mean many different things, depending on if we want the system to be robust to variations of the internal Hamiltonian of the system, or if we additionally want it to be robust against couplings to other systems. For these small examples we will only worry about the first possibility. As it is the whole purpose of the code, we will quantify the robustness of $U(T)$ with respect to a (constant) variation G_{v} in the generator $G(t)$ by the size of the directional derivative

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{T}\exp\left(\int_0^T dt_1 (G(t_1) + \epsilon G_{\mathrm{v}})\right) = U(T)\int_0^T dt_1 U^*(t_1) G_{\mathrm{v}} U(t_1). \tag{3}$$

In particular, to find a sequence that is robust against a variation G_{v}, we will include the function

$$\left\| U(T)\int_0^T dt_1 U^*(t_1) G_{\mathrm{v}} U(t_1) \right\|_2^2 \tag{4}$$

in our objective, where $\|A\|_2 = \sqrt{\mathrm{Tr}(A^* A)}$ is the Hilbert-Schmidt norm.[1]

In general, we can quantify robustness in this way with respect to higher derivatives as well (and the code can handle this), but for now we will only worry about the first derivative. Hence, in these examples, we will find control sequences implementing a certain gate, and that are robust with respect to particular variations in the generator, in the sense that the derivative of the function

$$\left| \left\langle U_{\mathrm{target}}, \mathcal{T}\exp\left(\int_0^T dt_1 (G(t_1) + \epsilon G_{\mathrm{v}})\right) \right\rangle \right|^2, \tag{5}$$

with respect to ϵ at $\epsilon = 0$ will be 0.

Both examples will be qubit examples, so our systems will be in M_2, and we will use the notation

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

for the Pauli matrices. When summing over Pauli matrices, we will also use the notation $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$.

1 Example 1: With only X control, implement an X gate that is robust variations in Z

This problem is implemented within the function `x_sys_dec_z` in `control_problems`. The system generator terms are given by:

$$G_0 = 0, \text{ and } G_1 = -i\pi X, \tag{7}$$

i.e. there is no drift and we only have X control.

The function performs a search for a sequence that implements an X gate that is robust to variations in Z. The objective function (to be minimized) is thus chosen to be:

$$f(U(t)) = -|\langle X, U(T)\rangle|^2 + \left\| U(T)\int_0^T dt_1 U^*(t_1) Z U(t_1) \right\|_2^2, \tag{8}$$

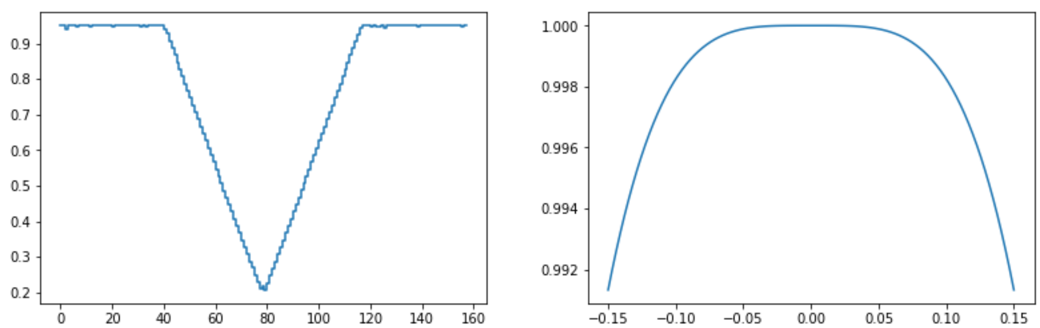
[1]We do note that the benefit of defining robustness in this way is that even though there is no mention of other quantum systems, if a trajectory $U(t)$ sets the integral in Equation (3) to 0, then $U(T)$ will also be robust (with respect to this definition) to couplings to other systems of the form $G_{\mathrm{v}} \otimes K$.

where first term is to drive $U(T)$ to be an X gate, and the second term is to drive $U(T)$ to be robust against variations in Z. Note that the best possible value for this objective is -4.

The function `x_sys_dec_z` is set up to perform the search with the following parameters:

- Number of time steps: $N = 157$
- Time step size: $\Delta t = 0.0125$
- The objective has also been modified with "penalties" to restrict the shape of the control sequence. The penalties enforce:
  - The control amplitude is always within $[-1, 1]$
  - Between time steps, the control amplitude changes by no more than 0.025

With these parameters, the search finds a sequence whose objective value is -3.9999999999929416, which is within 10^{-11} of the optimal value -4. The resulting pulse and a plot of the fidelity of $U(T)$ with respect to variations of the form $\epsilon(-i\pi Z)$, as a function of ϵ, can be found in Figure 1.



(a) The control sequence amplitude as a function of time step.

(b) $|\langle X, U(T)\rangle|^2/4$ as a function of ϵ, with respect to variations in the generator $\epsilon(-i\pi Z)$

Figure 1: Plots for the control sequence that implements an X gate while being robust against Z.

The shape of the pulse is quite interesting, and observe that as the fidelity of the final unitary with respect to the variation $\epsilon(-i\pi Z)$, when plotted as a function of ϵ, is flat at $\epsilon = 0$, which is ensured by the form of the objective.

Note that this particular length of pulse was found with a little bit of trial and error. If the amount of time is made much longer, then the objective will be optimized well, but the pulse shape will have less structure.

2 Example 2: With X and Y control, implement an identity gate that is robust against *all* constant variations

This problem is implemented within the function `universal_id_xy` in `control_problems`. The system generator terms are given by:

$$G_0 = 0, G_1 = -i\pi X, \text{ and } G_2 = -i\pi Y \tag{9}$$

i.e. there is no drift and we have X and Y control.

The function performs a search so that $U(T) = \mathbb{1}_2$, and $U(T)$ is robust with respect to *all* possible variations in the generator (to first order). That is, we are searching for an identity gate that performs "universal first-order decoupling". The objective (to be minimized) is thus

$$f(U(t)) = -|\langle \mathbb{1}_2, U(T)\rangle|^2 + \sum_{i=1}^{3} \left\| U(T) \int_0^T dt_1 U^*(t_1)\sigma_i U(t_1) \right\|_2^2, \tag{10}$$

where, again, the first term is to ensure $U(T) = \mathbb{1}_2$, and the second term ensures robustness against all Pauli matrices. For this objective, the optimal value is -4.

The function `universal_id_xy` is set up to perform the search with the following parameters:

- Number of time steps: $N = 80$
- Time step size: $\Delta t = 0.05$
- The objective has also been modified with "penalties" to restrict the shape of the control sequence. The penalties enforce:
  - Both control amplitudes are always within $[-1, 1]$
  - Between time steps, the control amplitudes change by no more than 0.05

With these parameters, the search finds a sequence achieving objective value -3.9999999995477555, which is with 10^{-9} of the best possible value -4. The pulse shape may be found in Figure 2, and the robustness as a function of variations in the generator is displayed in Figure 3.

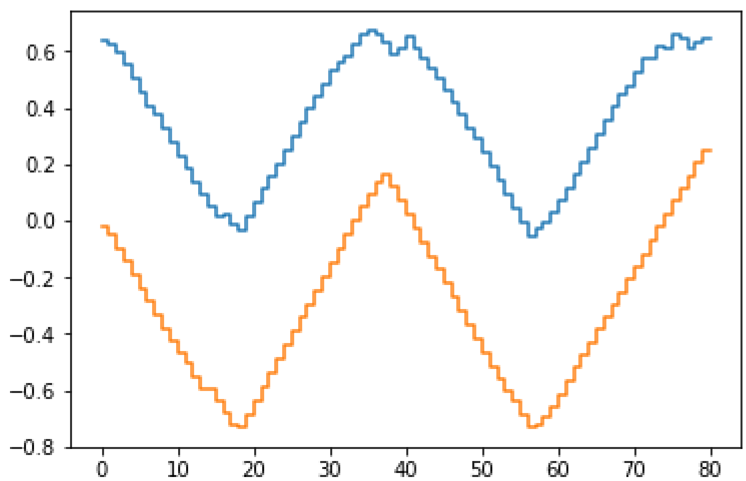
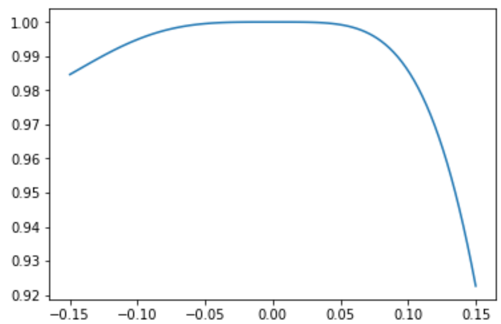
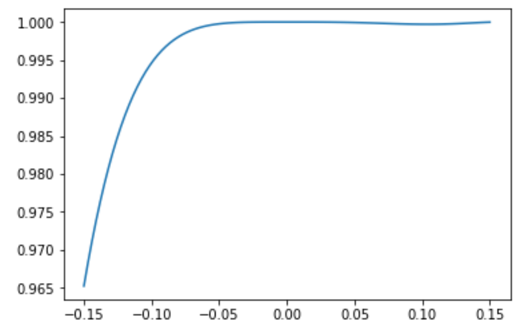


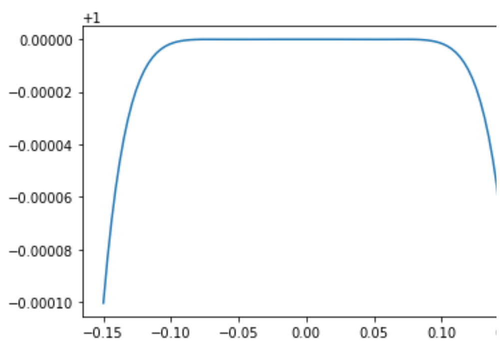
Figure 2: The x (blue) and y (orange) control amplitudes over the period of the pulse, as a function of time step. Note that the pulse seems to be switching between driving along two axes, where each axis is driven along twice (note that it ends and begins with a "half rotation" about the same axis). Hence, this is somewhat like a continuous time version of XY4 (though the axes of rotation are not actually orthogonal).



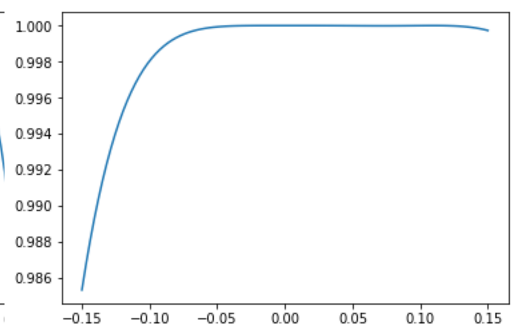
(a) Fidelity as a function of ϵ for a generator with variation $\epsilon(-i\pi X)$



(b) Fidelity as a function of ϵ for a generator with variation $\epsilon(-i\pi Y)$



(c) Fidelity as a function of ϵ for a generator with variation $\epsilon(-i\pi Z)$



(d) Fidelity as a function of ϵ for a generator with variation $\epsilon\left(-i\pi\frac{Y+Z}{\sqrt{2}}\right)$

Figure 3: Each plot displays the fidelity with respect to generator variations in different directions. Observe that, at $\epsilon = 0$, the fidelity is flat, as is ensured by our search criterion.