Distribution of Angle between Two Linked Triangles

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1 One Dimensional Constrsaint Manifold

First we consider the case in which one triangle is fixed with the coordinates $(0, 1/2, 0), (0, -1/2, 0), (\sqrt{3}/2, 0, 0)$. The second face will share the vertivies $(0, 1/2, 0)$ and $(0, -1/2, 0)$ and its third vertex v is allowed to vary subject to the constraints of being unit distance from the other vertices. The corresponding constraint space can be described by parameterizing v by θ, the dihedral angle between faces.

$$v(\theta) = \left[\sqrt{3}/2 \cos\theta, 0, \sqrt{3}/2 \sin\theta \right]^T \tag{1}$$

Diaconis et al. outline a method for sampling uniformly from a manifold $\mathcal{M}$. If the manifold is of m dimensions sitting in an n dimensional ambient space, and a parameterization $f : \mathbb{R}^m \to \mathbb{R}^n$ the sampling distribution on the parameters that corresponds to a uniform sampling on $\mathcal{M}$ is given by $h(\omega, \theta, \phi) \propto \left(\det \left[(Df)^T Df \right] \right)^{1/2}$.

In this example, we have the following parameterization.

$$f(\theta) = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ -1/2 \\ 0 \\ \sqrt{3}/2 \\ 0 \\ 0 \\ \sqrt{3}/2\cos\theta \\ 0 \\ \sqrt{3}/2\sin\theta \end{bmatrix} \tag{2}$$

$$Df(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\sqrt{3}/2\sin\theta \\ 0 \\ \sqrt{3}/2\cos\theta \end{bmatrix} \tag{3}$$

Thus the sampling distribution on θ is uniform.

$$h(\theta) \propto \sqrt{\det[3/4]} \tag{4}$$
$$\propto 1 \tag{5}$$

2 Three Dimensional Constrsaint Manifold

Now we let the common edge rotate about the x-axis while its center remains fixed at the origin. More formally, define the point $u = \left[0, \frac{1}{2}\cos\omega, \frac{1}{2}\sin\omega\right]^T$ in the y-z plane. Then, we parameterize the shared edge by $\omega \in [0, 2\pi)$ with

its endpoints being $u(\omega)$ and $-u(\omega)$. Then we use θ and ϕ to parameterize the remaining two verticies v_1 and v_2 as follows.

$$v_1 = \left[\frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta \sin \omega, -\frac{\sqrt{3}}{2} \sin \theta \cos \omega \right]^T \tag{6}$$

$$v_2 = \left[\frac{\sqrt{3}}{2} \cos(\theta + \phi), \frac{\sqrt{3}}{2} \sin(\theta + \phi) \sin \omega, -\frac{\sqrt{3}}{2} \sin(\theta + \phi) \cos \omega \right]^T \tag{7}$$

With these paramerizations, it is trivial to verify that the constrains $|u - (-u)| = |v_1 - u| = |v_1 + u| = |v_2 - u| = |v_2 + u| = 1$ are all satisfied for any choice of (ω, θ, ϕ). Additionally, the dihedral angle between the two triangles is given by ϕ.

We proceed in deriving the sampling distributions of the 3 parameters that corresponds to uniform sampling of the 3 dimensional constraint manifold embedded in $\mathbb{R}^{12}$ using the method outlined in Diaconis et al. First we concattenate our parameterizations into a function $f : \mathbb{R}^3 \to \mathbb{R}^{12}$ That maps

$(\omega, \theta, \phi) \in [0, 2\pi)^3$ onto the constraint manifold.

$$f(\omega, \theta, \phi) = \begin{bmatrix} 0 \\ \frac{1}{2}\cos\omega \\ \frac{1}{2}\sin\omega \\ 0 \\ -\frac{1}{2}\cos\omega \\ -\frac{1}{2}\sin\omega \\ \frac{\sqrt{3}}{2}\cos\theta \\ \frac{\sqrt{3}}{2}\sin\theta\sin\omega \\ -\frac{\sqrt{3}}{2}\sin\theta\cos\omega \\ \frac{\sqrt{3}}{2}\cos(\theta+\phi) \\ \frac{\sqrt{3}}{2}\sin(\theta+\phi)\sin\omega \\ -\frac{\sqrt{3}}{2}\sin(\theta+\phi)\cos\omega \end{bmatrix} \tag{8}$$

$$Df(\omega, \theta, \phi) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2}\sin\omega & 0 & 0 \\ \frac{1}{2}\cos\omega & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}\sin\omega & 0 & 0 \\ -\frac{1}{2}\cos\omega & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2}\sin\theta & 0 \\ \frac{\sqrt{3}}{2}\sin\theta\cos\omega & \frac{\sqrt{3}}{2}\cos\theta\sin\omega & 0 \\ \frac{\sqrt{3}}{2}\sin\theta\sin\omega & -\frac{\sqrt{3}}{2}\cos\theta\cos\omega & 0 \\ 0 & -\frac{\sqrt{3}}{2}\sin(\theta+\phi) & -\frac{\sqrt{3}}{2}\sin(\theta+\phi) \\ \frac{\sqrt{3}}{2}\sin(\theta+\phi)\cos\omega & \frac{\sqrt{3}}{2}\cos(\theta+\phi)\sin\omega & \frac{\sqrt{3}}{2}\cos(\theta+\phi)\sin\omega \\ \frac{\sqrt{3}}{2}\sin(\theta+\phi)\sin\omega & -\frac{\sqrt{3}}{2}\cos(\theta+\phi)\cos\omega & -\frac{\sqrt{3}}{2}\cos(\theta+\phi)\cos\omega \end{bmatrix} \tag{9}$$

$$(Df)^T Df = \begin{bmatrix} \frac{1}{2} + \frac{3}{4}\sin^2\theta + \frac{3}{4}\sin^2(\theta+\phi) & 0 & 0 \\ 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{3}{4} \end{bmatrix} \tag{10}$$

Thus the sampling distribution is computed.

$$h(\omega, \theta, \phi) \propto \sqrt{1 + \frac{3}{2}\sin^2\theta + \frac{3}{2}\sin^2(\theta+\phi)} \tag{11}$$

3 Four Dimensional Constraint Manifold

Suppose we relax the constraints of the previous calculation. Fix the midpoint of the shared edge at the origin, but allow it the freedom to rotate in any direction. Then, the endpoints of the shared edge lie at opposite ends of the sphere with radius $1/2$. We use sphedical coordinates to parameterize the axis $[-u(\phi, \omega), u(\phi, \omega)]$.

$$u(\phi, \omega) = \left(\frac{1}{2}\cos(\phi), \frac{1}{2}\sin(\phi)\cos(\omega), \frac{1}{2}\sin(\phi)\sin(\omega) \right) \tag{12}$$

Now we parameterize the remaining two vertices as $v^1 = R(2u, \theta)v^0$ and $v^2 = R(2u, \theta+\xi)v^0$ where v^0 is any fixed point satisfying $|v^0 - u| = |v^0 + u| = 1$ and $R(\hat{u}, \hat{\theta})$ is the rotation matrix with unit axis of rotation $\hat{u}$ and angle of rotation $\hat{\theta}$.

$$R(2u, \theta) = \begin{bmatrix} \cos\theta + 4u_x^2(1-\cos\theta) & 4u_x u_y(1-\cos\theta) - 2u_z\sin\theta & 4u_x u_z(1-\cos\theta) + 2u_y\sin\theta \\ 4u_y u_x(1-\cos\theta) + 2u_z\sin\theta & \cos\theta + 4u_y^2(1-\cos\theta) & 4u_y u_z(1-\cos\theta) - 2u_x\sin\theta \\ 4u_z u_x(1-\cos\theta) - 2u_y\sin\theta & 4u_z u_y(1-\cos\theta) + 2u_x\sin\theta & \cos\theta + 4u_z^2(1-\cos\theta) \end{bmatrix} \tag{13}$$

$$= \cos\theta I + 4(1-\cos\theta)uu^T + 2\sin\theta[u]_\times \tag{14}$$

$$\tag{15}$$

Thus, we compute as follows.

$$f(\phi, \omega, \theta, \xi) = \begin{bmatrix} u \\ -u \\ v^1 \\ v^2 \end{bmatrix} \tag{16}$$

$$Df(\phi, \omega, \theta, \xi) = \begin{bmatrix} Du \\ -Du \\ Dv^1 \\ Dv^2 \end{bmatrix} \tag{17}$$

$$(Df)^T Df = 2(Du)^T Du + \left(Dv^1\right)^T Dv^1 + \left(Dv^2\right)^T Dv^2 \tag{18}$$

$$Du(\phi, \omega, \theta) = \begin{bmatrix} -\frac{1}{2}\sin\phi & 0 & 0 & 0 \\ \frac{1}{2}\cos\phi\cos\omega & -\frac{1}{2}\sin\phi\sin\omega & 0 & 0 \\ \frac{1}{2}\cos\phi\sin\omega & \frac{1}{2}\sin\phi\cos\omega & 0 & 0 \end{bmatrix} \tag{19}$$

$$(Du)^T Du = \begin{bmatrix} \frac{1}{4}\sin^2\phi + \frac{1}{4}\cos^2\phi\cos^2\omega + \frac{1}{4}\cos^2\phi\sin^2\omega & \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega - \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega & 0 & 0 \\ \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega - \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega & \frac{1}{4}\sin^2\phi\sin^2\omega + \frac{1}{4}\sin^2\phi\cos^2\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4}\sin^2\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$