Distribution of Angle between Two Linked Triangles

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1 One Triangle

Suppose we have a triangle composed of unit length edges with one edge designated the 'axis'. Fix the midpoint of this axis at the origin. Then, the endpoints of the axis lie at opposite ends of the sphere with radius 1/2. We use sphedica coordinates to parameterize the axis $[-u(\phi,\omega), u(\phi,\omega)]$.

$$u(\phi,\omega) = \left(\frac{1}{2}\cos(\phi), \frac{1}{2}\sin(\phi)\cos(\omega), \frac{1}{2}\sin(\phi)\sin(\omega)\right) \tag{1}$$

Now for a particular choice of u we wish to parameterize the loci $v : |v-u| = |v+u| = 1$ which correspond to the set of possible points that (along with u and $-u$) would form a unit equilateral triangle. Let v^0 be such a point. Then, the remaining locus points can be parameterized as $v = R(u,\theta)v^0$ where R is the rotation matrix that rotates points along the axis u by θ.

$$R = \begin{bmatrix} \cos\theta + 4u_x^2(1-\cos\theta) & 4u_xu_y(1-\cos\theta) - 2u_z\sin\theta & 4u_xu_z(1-\cos\theta) + 2u_y\sin\theta \\ 4u_yu_x(1-\cos\theta) + 2u_z\sin\theta & \cos\theta + 4u_y^2(1-\cos\theta) & 4u_yu_z(1-\cos\theta) - 2u_x\sin\theta \\ 4u_zu_x(1-\cos\theta) - 2u_y\sin\theta & 4u_zu_y(1-\cos\theta) + 2u_x\sin\theta & \cos\theta + 4u_z^2(1-\cos\theta) \end{bmatrix} \tag{2}$$

Parameterization

$$f(\phi,\omega,\theta) = \begin{bmatrix} u \\ -u \\ v \end{bmatrix} \tag{3}$$

$$Df(\phi,\omega,\theta) = \begin{bmatrix} Du \\ -Du \\ Dv \end{bmatrix} \tag{4}$$

$$(Df)^T\, Df = 2\,(Du)^T\, Du + (Dv)^T\, Dv \tag{5}$$

$$Du(\phi,\omega,\theta) = \begin{bmatrix} -\frac{1}{2}\sin\phi & 0 & 0 \\ \frac{1}{2}\cos\phi\cos\omega & -\frac{1}{2}\sin\phi\sin\omega & 0 \\ \frac{1}{2}\cos\phi\sin\omega & \frac{1}{2}\sin\phi\cos\omega & 0 \end{bmatrix} \tag{6}$$

$$(Du)^T\, Du = \begin{bmatrix} \frac{1}{4}\sin^2\phi + \frac{1}{4}\cos^2\phi\cos^2\omega + \frac{1}{4}\cos^2\phi\sin^2\omega & \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega - \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega & 0 \\ \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega - \frac{1}{4}\cos\phi\cos\omega\sin\phi\sin\omega & \frac{1}{4}\sin^2\phi\sin^2\omega + \frac{1}{4}\sin^2\phi\cos^2\omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$(Du)^T\, Du = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4}\sin^2\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$Dv(\phi,\omega,\theta) = \begin{bmatrix} \frac{\partial R}{\partial\phi}v^0 & \frac{\partial R}{\partial\omega}v^0 & \frac{\partial R}{\partial\theta}v^0 \end{bmatrix} \tag{9}$$

$$(Dv)^T\, Dv = \begin{bmatrix} (v^0)^T(\frac{\partial R}{\partial\phi})^T\frac{\partial R}{\partial\phi}v^0 & (v^0)^T(\frac{\partial R}{\partial\phi})^T\frac{\partial R}{\partial\omega}v^0 & (v^0)^T(\frac{\partial R}{\partial\phi})^T\frac{\partial R}{\partial\theta}v^0 \\ (v^0)^T(\frac{\partial R}{\partial\omega})^T\frac{\partial R}{\partial\phi}v^0 & (v^0)^T(\frac{\partial R}{\partial\omega})^T\frac{\partial R}{\partial\omega}v^0 & (v^0)^T(\frac{\partial R}{\partial\omega})^T\frac{\partial R}{\partial\theta}v^0 \\ (v^0)^T(\frac{\partial R}{\partial\theta})^T\frac{\partial R}{\partial\phi}v^0 & (v^0)^T(\frac{\partial R}{\partial\theta})^T\frac{\partial R}{\partial\omega}v^0 & (v^0)^T(\frac{\partial R}{\partial\theta})^T\frac{\partial R}{\partial\theta}v^0 \end{bmatrix} \tag{10}$$

$$R = \cos\theta I + 4(1-\cos\theta)uu^T + 2\sin\theta[u]_\times \tag{11}$$

$$\frac{\partial R}{\partial\phi} = 4(1-\cos\theta)(u\frac{\partial u^T}{\partial\phi} + \frac{\partial u}{\partial\phi}u^T) + 2\sin\theta\frac{\partial[u]_\times}{\partial\phi} \tag{12}$$

$$\frac{\partial R}{\partial\omega} = 4(1-\cos\theta)(u\frac{\partial u^T}{\partial\omega} + \frac{\partial u}{\partial\omega}u^T) + 2\sin\theta\frac{\partial[u]_\times}{\partial\omega} \tag{13}$$

$$\frac{\partial R}{\partial\theta} = -\sin\theta I + 4\sin\theta uu^T + 2\cos\theta[u]_\times \tag{14}$$

$$\tag{15}$$

$$\frac{\partial u}{\partial \phi}u^T = \frac{1}{4}\begin{bmatrix} -\cos\phi\sin\phi & \cos^2\phi\cos\omega & \cos^2\phi\sin\omega \\ -\sin^2\phi\cos\omega & \sin\phi\cos\phi\cos^2\omega & \sin\phi\cos\phi\sin\omega\cos\omega \\ -\sin^2\phi\sin\omega & \sin\phi\cos\phi\sin\omega\cos\omega & \sin\phi\cos\phi\sin^2\omega \end{bmatrix} \tag{16}$$

$$u\frac{\partial u^T}{\partial \phi} + \frac{\partial u}{\partial \phi}u^T = \frac{1}{4}\begin{bmatrix} -\sin 2\phi & \cos 2\phi\cos\omega & \cos 2\phi\sin\omega \\ \cos 2\phi\omega & \sin 2\phi\cos^2\omega & \sin 2\phi\sin 2\omega \\ \cos 2\phi\sin\omega & \sin 2\phi\sin 2\omega & \sin 2\phi\sin^2\omega \end{bmatrix} \tag{17}$$

$$\frac{\partial u}{\partial \omega}u^T = \frac{1}{4}\begin{bmatrix} 0 & -\sin\phi\cos\phi\sin\omega & \sin\phi\cos\phi\cos\omega \\ 0 & -\sin^2\phi\sin\omega\cos\omega & \sin^2\phi\cos^2\omega \\ 0 & -\sin^2\phi\sin^2\omega & \sin^2\phi\sin\omega\cos\omega \end{bmatrix} \tag{18}$$

$$u\frac{\partial u^T}{\partial \omega} + \frac{\partial u}{\partial \omega}u^T = \frac{1}{4}\begin{bmatrix} 0 & -\sin\phi\cos\phi\sin\omega & \sin\phi\cos\phi\cos\omega \\ -\sin\phi\cos\phi\sin\omega & -\sin^2\phi\sin 2\omega & \sin^2\phi\cos 2\omega \\ \sin\phi\cos\phi\cos\omega & \sin^2\phi\cos 2\omega & \sin^2\phi\sin 2\omega \end{bmatrix} \tag{19}$$

$$uu^T = \frac{1}{4}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi\cos\omega & \sin\phi\cos\phi\sin\omega \\ \sin\phi\cos\phi\cos\omega & \sin^2\phi\cos^2\omega & \sin^2\phi\sin\omega\cos\omega \\ \sin\phi\cos\phi\sin\omega & \sin^2\phi\sin\omega\cos\omega & \sin^2\phi\sin^2\omega \end{bmatrix} \tag{20}$$

$$[u]_\times = \frac{1}{2}\begin{bmatrix} 0 & -\sin\phi\sin\omega & \sin\phi\cos\omega \\ \sin\phi\sin\omega & 0 & -\cos\phi \\ -\sin\phi\cos\omega & \cos\phi & 0 \end{bmatrix} \tag{21}$$

$$\frac{\partial [u]_\times}{\partial \phi} = \frac{1}{2}\begin{bmatrix} 0 & -\cos\phi\sin\omega & \cos\phi\cos\omega \\ \cos\phi\sin\omega & 0 & \sin\phi \\ -\cos\phi\cos\omega & -\sin\phi & 0 \end{bmatrix} \tag{22}$$

$$\frac{\partial [u]_\times}{\partial \omega} = \frac{1}{2}\begin{bmatrix} 0 & -\sin\phi\cos\omega & -\sin\phi\sin\omega \\ \sin\phi\cos\omega & 0 & 0 \\ \sin\phi\sin\omega & 0 & 0 \end{bmatrix} \tag{23}$$

$$(\frac{\partial R}{\partial \theta})^T \frac{\partial R}{\partial \theta} = (-\sin\theta I + 4\sin\theta u u^T + 2\cos\theta [u]_\times)^T (-\sin\theta I + 4\sin\theta u u^T + 2\cos\theta [u]_\times) \tag{24}$$

$$= \sin^2\theta I - 4\sin^2\theta u u^T - 2\sin\theta\cos\theta [u]_\times - 4\sin^2\theta u u^T + 16\sin^2 u u^T u u^T + 8\sin\theta\cos\theta u u \tag{25}$$