DETERMINISTIC SYSTEMS MODELS/SYSTEMS OPTIMIZATION
ISE 4623/5023
EXAM 2
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Section (mark with X): ______ ISE 4623
X ISE 5023

Pledge: "On my honor, I have neither given nor received inappropriate assistance in the completion of this Exam."

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Problem 1 (40 points)
Suppose you have the following LP model

$$\text{Minimize } z = 2x_1 - 4x_2 + 8x_3$$
s.t.
$$\begin{aligned} 2x_1 + x_2 &\leq 20 \\ x_1 + 4x_2 + 8x_3 &\leq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

You would like to solve it using Simplex Tableau. For this:

a. (5 points) Construct the associated "Simplex ready" LP model. In other words, construct an LP model that is equivalent to the one shown above, but that contains only non-negative right-hand side, non-negative variables, and equality constraints. Indicate clearly the variables, objective function, the associated constraints, and the nature of the variables (if they are non-negative, non-positive, etc.) for this new "Simplex ready" LP model.

$$\text{Min } 2X_1 - 4X_2 + 8X_3$$
s.t.
$$\begin{aligned} 2X_1 + X_2 \qquad\qquad + S_1 \qquad &= 20 \\ X_1 + 4X_2 + 8X_3 \qquad + S_2 &= 40 \\ X_1, X_2, X_3, S_1, S_2 &\geq 0 \end{aligned}$$

b. (10 points) Using the table below, write the "Simplex ready" model in Tableau form. Is this Tableau well-posed (i.e., does it comply with all the required conditions to start your Simplex iterations)? Explain your answer in detail. If is not well-posed, fix it.

It is well-posed: the basic variables (S_1, S_2) have reduced cost $\emptyset$, their columns form the identity matrix, and their solution is non-negative.

Basic	z	x_1	x_2	X_3	S_1	S_2	Solution
z	1	-2	4	-8	0	0	0
S_1	0	2	1	0	1	0	20
S_2	0	1	4	8	0	1	40

c. (5 points) Using the Tableau form from part b, determine the current basic and non-basic variables (and their values) and the current value of the objective function. Is this solution feasible? Is this solution optimal? Explain your answers in detail.

Since the Tableau was well-posed, these values can be taken directly from there.

Current value of objective function
0

Basic variables	Value
s_1	20
s_2	40

Non-basic variables	Value
x_1	0
x_2	0
x_3	0

d. (20 points) Starting from the Tableau from part b, use Simplex method **to solve this problem to optimality**. In the tables below, write your final Tableau (the one containing the optimal solution after all your iterations), the basic and non-basic variables (and their values) and the value of the objective function associated with the optimal solution.

The largest positive reduced cost is 4 (associated with X_2) so we choose X_2 to enter the basis.
Using the minimum ratio test, since $\frac{40}{4} < \frac{20}{1}$, we select s_2 to leave the basis.

$$\Rightarrow R_{x_2} \leftarrow R_{s_2}/4$$

$$R_z \leftarrow R_z - 4R_{x_2}$$

$$R_{s_1} \leftarrow R_{s_1} - R_{x_2}$$

After the update, we see that all reduced cost are non-positive, so we are in the optimal solution

Basic	z	x_1	x_2	x_3	s_1	s_2	Solution
z	1	−3	0	−16	0	−1	−40
s_1	0	1.75	0	−2	1	−0.25	10
x_2	0	0.25	1	2	0	0.25	10

Optimal value of objective function
−40

Basic variables	Value
s_1	10
x_2	10

Non-basic variables	Value
x_1	0
x_3	0
s_2	0

Problem 2 (40 points)
Suppose you have the following LP model

$$\text{Minimize } z = 2x_1 - 4x_2 + 8x_3$$
s.t.
$$2x_1 + x_2 \leq 20$$
$$x_1 + 4x_2 + 8x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

a. (10 points) Construct its associated dual problem

$$\text{Max} \quad 20w_1 + 40w_2$$
$$2w_1 + 1w_2 \leq 2$$
$$1w_1 + 4w_2 \leq -4$$
$$0w_1 + 8w_2 \leq 8$$
$$w_1, w_2 \leq 0$$

b. (5 points) Transform the dual problem into a mathematical formulation with a format suitable for the standard Tableau (i.e, change the dual problem into a formulation with only equality constraints, non-negative right-hand side, and non-negative variables)

$$\text{Max} \quad -20\hat{w}_1 - 40\hat{w}_2$$
s.t.
$$-2\hat{w}_1 - \hat{w}_2 + y_1 = 2$$
$$\hat{w}_1 + 4\hat{w}_2 - y_2 = 4$$
$$-8\hat{w}_2 + y_3 = 8$$
$$\hat{w}_1, \hat{w}_2, y_1, y_2, y_3 \geq 0$$

c. (5 points) Looking at the formulation from part b., you notice that the "trivial solution" (all the non-slack variables being zero) is not a feasible solution to your problem. Thus, you need to use an initialization method for your problem. If you wanted to use the big M method, what would be the associated big M optimization problem to be solved?

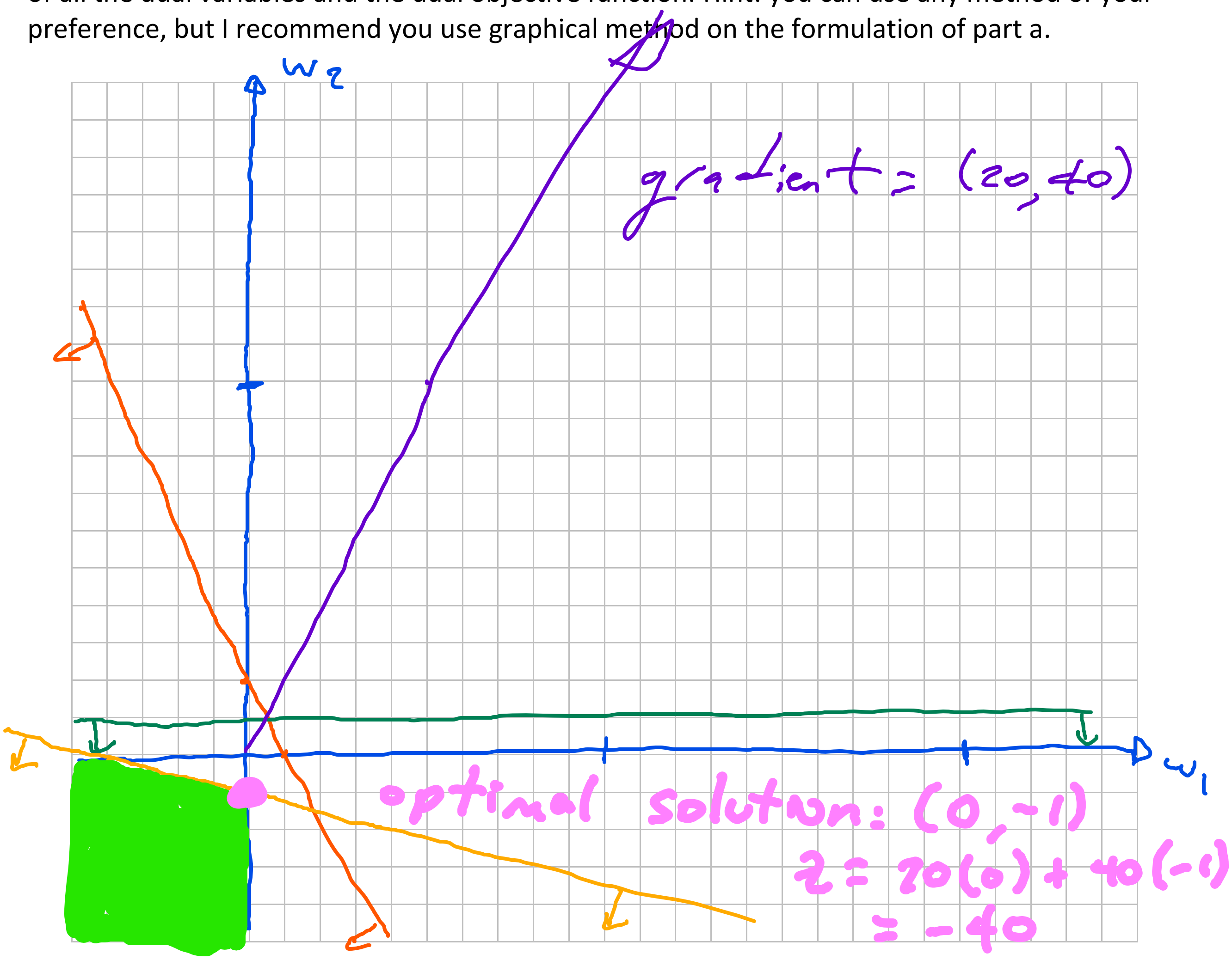
$$\text{Max} \quad -20\hat{w}_1 - 40\hat{w}_2 - Mr$$
s.t.
$$-\hat{w}_1 - \hat{w}_2 + y_1 = 2$$
$$\hat{w}_1 + 4\hat{w}_2 - y_2 + r = 4$$
$$-8\hat{w}_2 + y_3 = 8$$
$$\hat{w}_1, \hat{w}_2, y_1, y_2, y_3, r \geq 0$$

d. (5 points) If you wanted to initialize the problem of part b. with the two-phase method, what would be the associated phase-1 optimization problem to be solved?

$$\text{Min} \quad r$$

$$\text{s.t.} \quad \begin{aligned} -\hat{w}_1 - \hat{w}_2 + y_1 &= 2 \\ \hat{w}_1 + 4\hat{w}_2 - y_2 + r &= 4 \\ -8\hat{w}_2 + y_3 &= 8 \\ \hat{w}_1, \hat{w}_2, y_1, y_2, y_3, r &\geq 0 \end{aligned}$$

e. (10 points) What is the optimal solution to the dual problem? You need to indicate the optimal values of all the dual variables and the dual objective function. Hint: you can use any method of your preference, but I recommend you use graphical method on the formulation of part a.



f. (5 points) How can you interpret the solutions obtained for the dual variables? *Hint: remember the concept of "shadow" prices*

This means that incrementing a unit in the right-hand side (RHS) of constraint 1 won't have any effect on improving the objective function; on the other hand, the objective function has the potential to improve in one unit (-1) per unit of increase in the RHS of constraint 2.

Problem 3 (10 points)
Suppose you have the following LP model

$$\begin{aligned} &\text{Minimize } z = 2x_1 - 4x_2 + 8x_3 \\ &\text{s.t.} \\ &\qquad 2x_1 + x_2 \leq 20 \\ &\qquad x_1 + 4x_2 + 8x_3 \leq 40 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Assume that you are studying a basic solution with Basis B = (P2, P4), where P2 and P4 are the columns of A (the matrix of coefficients) associated with the variables x2 and s1, respectively.

a. (5 points) Calculate the values of all basic and non-basic variables. Use the equation $x_B = B^{-1}b$. Is this basic solution feasible?

Help: Remember that the inverse of a matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is $M^{-1} = \frac{1}{M_{11}M_{22} - M_{12}M_{21}} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$

$$B = (P_2, P_4) = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} \Rightarrow B^{-1} = \frac{1}{(1)(0)-(1)(4)} \begin{pmatrix} 0 & -1 \\ -4 & 1 \end{pmatrix}$$

$$= \frac{1}{-4} \begin{pmatrix} 0 & -1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1/4 \\ 1 & -1/4 \end{pmatrix}$$

$$\Rightarrow x_B = B^{-1}b = \begin{pmatrix} 0 & 1/4 \\ 1 & -1/4 \end{pmatrix} \begin{pmatrix} 20 \\ 40 \end{pmatrix} = \begin{pmatrix} 0+10 \\ 20-10 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

Basic variables	Value
x_2	10
S_1	10

Non-basic variables	Value
x_1	0
x_3	0
S_2	0

b. (5 points) Calculate the value of the objective function associated with this basic solution. Use the equation $z = c^T x = c_B^T x_B$

$$c_B = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \Rightarrow z = c_B^T x_B = (-4 \quad 0) \begin{pmatrix} 10 \\ 10 \end{pmatrix} = -40$$

Note that this is exactly the solution shown in Problem 1d.

Problem 4 (10 points)
Suppose you have the following LP model

$$\begin{aligned} &\text{Minimize } z = 2x_1 - 4x_2 \\ &\text{s.t.} \\ &\quad 2x_1 + x_2 \leq 20 \\ &\quad x_1 + 4x_2 \leq 40 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

Fill in the empty boxes below (there are 10 empty boxes in total) to complete the Gurobi/Python code associated with this problem:

```
#Import gurobi and name model
from gurobipy import *
m=Model("Problem3_Exam2")

#Define decision variables
x1={}
x1=m.addVar (vtype=GRB.CONTINUOUS , lb=0, ub=GRB.INFINITY)

x2={}
x2=m.addVar (vtype=GRB.CONTINUOUS , lb=0, ub=GRB.INFINITY)

#Define objective function
z=2*x1-4*x2
m.setObjective(z)
m.modelSense=GRB.MINIMIZE
m.update()

#Add constraints
m.addConstr (2*x1+x2<=20)
m.addConstr (x1+4*x2<=40)

m.update()

#Solve the model
m.optimize()

#printing outputs
if m.status==GRB.OPTIMAL:
    print ("\n Optimal objective function value:", m.objVal )
    print ("--- Optimal solution---")
    print ("x1", x1.x)
    print ("x2", x2.x)
```

Boxes 1,2, and 3

Boxes 4,5, and 6

Box 7

Box 8

Box 9

Box 10

Problem 5 (10 points) – Graduate students only (or extra credit for undergraduate students)

Suppose you have the following DEA formulation to study your relative efficiency to other students with respect to their performance in Exam 2. For this, assume that you have a list of all the students in the class, and their respective number of hours studied for Exam 2, and the score they obtained. Thus, for your DEA formulation, you have only one input (the number of hours studied) and one output (the score obtained). Also, assume that you are the student "1" in the list.

Given this, the associated DEA model is the following:

- Sets:

P: Set of students

- Parameters:

x_p: number of hours that student $p \in P$ studied for Exam 2
y_p: score obtained by student $p \in P$ in Exam 2

- Variables:

λ_p: multiplier coefficient for benchmark student $p \in P$
ϕ: multiplier for percentual increase in output (from current level to the output level if 100% efficient)

- Objective function:

$$\max \phi$$

- Constraints:

$$\sum_{p \in P} x_p \lambda_p \leq x_1$$
$$\sum_{p \in P} y_p \lambda_p \geq y_1 \phi$$
$$\sum_{p \in P} \lambda_p = 1$$
$$\lambda_p \geq 0, \ \forall p \in P$$
$$\phi \ is \ free$$

However, now you are interested in gaining better insights about the "shadow prices" associated with the input (hours studied) and output (exam score) of your model, so you decide to construct the dual problem:

a. (5 points) Assume that the class has only three people. In that case, the associated primal DEA model would be:

max $0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 1\phi$
s.t

$$x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 + 0\phi \leq x_1$$
$$y_1\lambda_1 + y_2\lambda_2 + y_3\lambda_3 - y_1\phi \geq 0$$
$$1\lambda_1 + 1\lambda_2 + 1\lambda_3 + 0\phi = 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$
$$\phi \text{ is free}$$

Construct the associated dual problem for this case. **Help:** remember that the variables are $\lambda_1, \lambda_2, \lambda_3$, and ϕ, and all the others values (i.e., $x_1, x_2, x_3, y_1, y_2, y_3$) are constants.

$$\min \quad x_1 w_1 + 0 w_2 + 1 w_3$$
s.t.
$$x_1 w_1 + y_1 w_2 + 1 w_3 \geq 0$$
$$x_2 w_1 + y_2 w_2 + 1 w_3 \geq 0$$
$$x_3 w_1 + y_3 w_2 + 1 w_3 \geq 0$$
$$0 w_1 - y_1 w_2 + 0 w_3 = 1$$
$$w_1 \geq 0$$
$$w_2 \leq 0$$
$$w_3 \text{ is free}$$

y1w1 values vary from the online calculator? Be careful

b. (5 points) Now assume again that you are studying the general case (when you have a set of students P). Construct the associated dual problem for this general case. **Help:** note that the number of constraints in the primal problem does not change, since the number of constraints does not depend on the number of people in the class. Thereby, the *number of dual variables does not depend on the number of students in the class*.

$$\min \quad x_1 w_1 + w_3$$
s.t.
$$x_p w_1 + y_p w_2 + w_3 \geq 0 \quad , \forall p \in P$$
$$- y_1 w_2 = 1$$
$$w_1 \geq 0$$
$$w_2 \leq 0$$
$$w_3 \text{ is free}$$

Goal here is to get rid of all constants (x1, x2, x3 in this case) and replace condense to one line and replace the constant subscripts with p's