DETERMINISTIC SYSTEMS MODELS/SYSTEMS OPTIMIZATION
ISE 4623/5023
EXAM 1
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Section (mark with X): _____ ISE 4623
_X_ ISE 5023

Pledge: "On my honor, I have neither given nor received inappropriate assistance in the completion of this Exam."

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**Problem 1 - (20 points)**

One of your best friends wants to be certified as a CrossFit Level 1 athlete (certification that provides an introductory education on the fundamental principles and movements of CrossFit). To do so, your friend must spend time training next week in both cardiovascular and weightlifting exercises. Your friend does not like cardio that much, but your friend must train a minimum of two hours of cardio sessions per the week. Nonetheless, since excessive cardio can have a negative impact in muscle growth, the maximum time that your friend is willing to spend doing cardio is six hours per week. Additionally, given that proper rest must be guaranteed to maximize muscle and tissue recovery, your friend should not exceed eight hours of weekly exercises in total (considering both cardio and weightlifting training). Finally, your friend's gym charges $1 dollar per hour for cardio training and $2 dollars per hour for weightlifting, and your friend has an available weekly budget of $13 dollars.

Your friend has estimated that their final score in the certification course is proportional to 0.6 points per cardio hour and 1 point per weightlifting hour trained during the week. Your friend recognizes that this is an optimization problem, and knowing that you are an expert in optimization and linear programming, asks for your help to determine how many cardio and weightlifting training hours next week would maximize the number of points scored in the certification, while satisfying their time and budget requirements.

To do this, first you decide to formulate this problem as an LP model. In particular:

a. (4 points) Define the decision variables of this LP model

$x_1$ → # of hours of cardio trained

$x_2$ → # of hours of weightlifting trained

b. (4 points) What is the objective function of this LP model? Do not forget to indicate if you are maximizing or minimizing it.

$$\text{Max } z = 0.6x_1 + x_2$$

c. (12 points) What are the constraints of this LP model?

$$x_1 \geq 2 \quad (1)$$
$$x_1 \leq 6 \quad (2)$$
$$x_1 + x_2 \leq 8 \quad (3)$$
$$x_1 + 2x_2 \leq 13 \quad (4)$$
$$x_1 \geq 0 \quad (5)$$
$$x_2 \geq 0 \quad (6)$$

**Problem 2 (25 points)**
Suppose you have the following LP model

$$\text{Minimize } z = 5x_1 + 2x_2$$

s.t.

$$\begin{aligned} 2x_1 + 8x_2 &\geq 8 \\ 2x_1 + 2x_2 &\leq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$(0,1)$, $(4,0)$
$(0,3)$, $(3,0)$
$(0,4)$, $(2,0)$

a. (5 points) What is the gradient of this problem?

$$\nabla z = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

b. (15 points) Plot the gradient and the feasible region (clearly indicating all the constraints and "shading" the feasible region).

c. (5 points) What is the optimal solution for this problem? Indicate the values of all the variables (including the slacks) and the objective function.

$\boxed{x_1 = 0}$, $\boxed{x_2 = 1}$ $\Rightarrow$ $2(0) + 8(1) - s_1 = 8 \Rightarrow \boxed{s_1 = 0}$

$2(0) + 2(1) + s_2 = 6 \Rightarrow \boxed{s_2 = 4}$

$4(0) + 2(1) + s_3 = 8 \Rightarrow \boxed{s_3 = 6}$

$z = 5(0) + 2(1) = \boxed{2}$

**Problem 3 (35 points)**
Suppose you have the following LP model

$$\text{Maximize } z = -2x_1 - x_2$$
s.t.
$$\begin{aligned} -x_1 - 4\,x_2 \quad\quad &\leq -4 \\ 2x_1 + \quad x_2 - \quad x_3 &= 4 \\ x_1, x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

You would like to solve it using Simplex Tableau. For this:

a. (5 points) Write the problem in standard form (so that it contains only non-negative right-hand side, non-negative variables, and equality constraints). Indicate the variables, objective function, the associated constraints, and the nature of the variables (if they are non-negative, non-positive, etc.).

$$\begin{aligned} \text{Max} \quad & z = -2x_1 - x_2 \rightarrow z + 2x_1 + x_2 = 0 \\ \text{s.t.} \quad & x_1 + 4x_2 - s_1 = 4 \\ & 2x_1 + x_2 + \hat{x}_3 = 4 \\ & x_1, x_2, \hat{x}_3 \geq 0 \end{aligned}$$

b. (5 points) If you want to use the big M method to initialize and solve the problem from part a, what would be the associated big M optimization problem to be solved?

$$\begin{aligned} \text{Max} \quad & z = -2x_1 - x_2 - Mr \\ \text{s.t.} \quad & x_1 + 4x_2 - s_1 + r = 4 \\ & 2x_1 + x_2 + \hat{x}_3 = 4 \\ & x_1, x_2, \hat{x}_3, s_1, r \geq 0 \end{aligned}$$

c. (5 points) If you want to initialize and solve the problem from part a with the two-phase method, what would be the associated phase-1 optimization problem to be solved?

$$\begin{aligned} \text{min} \quad & z = r \rightarrow z - r = 0 \\ \text{s.t.} \quad & x_1 + 4x_2 - s_1 + r = 4 \\ & 2x_1 + x_2 + \hat{x}_3 = 4 \\ & x_1, x_2, \hat{x}_3, s_1, r \geq 0 \end{aligned}$$

d. (10 points) Using Simplex Tableau solve the associated phase-1 optimization problem and complete the table below with the optimal basic solution for the phase-1 problem. Don't forget to write all the steps/iterations made, including the row operations used.

| Basis | $z$ | $x_1$ | $x_2$ | $\hat{x}_3$ | $s_1$ | $r$ | sol |
|---|---|---|---|---|---|---|---|
| | 1 | 0 | 0 | 0 | 0 | −1 | 0 |
| | 0 | 1 | 4 | 0 | −1 | 1 | 4 |
| | 0 | 2 | 1 | 1 | 0 | 0 | 4 |

$R_1 \leftarrow R_1 - (-1) R_2$

| Basis | $z$ | $x_1$ | $x_2$ | $\hat{x}_3$ | $s_1$ | $r$ | sol | |
|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 4 | 0 | −1 | 0 | 4 | |
| $r$ | 0 | 1 | 4 | 0 | −1 | 1 | 4 | 4/4 |
| $\hat{x}_3$ | 0 | 2 | 1 | 1 | 0 | 0 | 4 | 4/1 |

$R_2 \leftarrow R_2/4 \Rightarrow$ 0 1/4 1 0 −1/4 1/4 1

After updating $R_2$ $\Big\} \rightarrow$ $R_1 \leftarrow R_1 - (4 R_2)$
$R_3 \leftarrow R_3 - (1 R_2)$

| Basic | $z$ | $x_1$ | $x_2$ | $\hat{x}_3$ | $s_1$ | $r$ | Solution |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 0 | 0 | 0 | −1 | 0 |
| $x_2$ | 0 | 1/4 | 1 | 0 | −1/4 | 1/4 | 1 |
| $\hat{x}_3$ | 0 | 7/4 | 0 | 1 | 1/4 | −1/4 | 3 |

| Optimal value of objective function of phase-1 problem |
|---|
| 0 |

| Variable | Value | Basic or non-basic? |
|---|---|---|
| $x_1$ | 0 | Non-basic |
| $x_2$ | 1 | Basic |
| $\hat{x}_3$ | 3 | Basic |
| $s_1$ | 0 | Non-basic |
| $r$ | 0 | Non-basic |

e. (10 points) Initialize the phase-2 problem (using the solution from part d) and solve it to optimality using Simplex Tableau. Complete the table below with the associated basic solution. Don't forget to write all the steps/iterations made, including the row operations used. Is this solution feasible for the problem in part a? Is this solution optimal for the problem in part a? Explain your answers in detail.

Phase-2 → Max $z = -2x_1 - x_2$ → $z + 2x_1 + x_2 = 0$

| Basic | $z$ | $x_1$ | $x_2$ | $\hat{x}_3$ | $s_1$ | sol |
|---|---|---|---|---|---|---|
| | 1 | 2 | 1 | 0 | 0 | 0 |
| | 0 | 1/4 | 1 | 0 | -1/4 | 1 |
| | 0 | 7/4 | 0 | 1 | 1/4 | 3 |

$R_1 \leftarrow R_1 - (1\ R_2)$

| Basic | $z$ | $x_1$ | $x_2$ | $\hat{x}_3$ | $s_1$ | Solution |
|---|---|---|---|---|---|---|
| $z$ | 1 | 7/4 | 0 | 0 | 1/4 | -1 |
| $x_2$ | 0 | 1/4 | 1 | 0 | -1/4 | 1 |
| $\hat{x}_3$ | 0 | 7/4 | 0 | 1 | 1/4 | 3 |

| Optimal value of objective function of phase-2 problem |
|---|
| -1 |

| Variable | Value | Basic or non-basic? |
|---|---|---|
| $x_1$ | 0 | Non-basic |
| $x_2$ | 1 | Basic |
| $\hat{x}_3$ | 3 | Basic |
| $s_1$ | 0 | Non-basic |

Is this basic solution feasible (with respect to the problem in part a)? Yes / No

Explanation: This basic solution comes from the optimal of phase-1, where 1 was non-basic. Also, from this tableau, "sol" column has only positive values.

Is this basic solution optimal (with respect to the problem in part a)? Yes / No

Explanation: There is no further promising direction to keep iterating

**Problem 4 (20 points)**

Suppose that you have the following Python/Gurobi code associated with an LP model

```
from gurobipy import *
model=Model("PepitaPerez_Company")

x1={}                                          (6)
x1=model.addVar(vtype=GRB.CONTINUOUS, lb=2, ub=GRB.INFINITY)

x2={}
x2=model.addVar(vtype=GRB.CONTINUOUS, lb=0, ub=6)

z=2*x1+4*x2                                    (5) (4)
model.setObjective(z)
model.modelSense=GRB.MAXIMIZE
model.update()

model.addConstr(3*x1+2*x2<=12)  → (1)
model.addConstr(x1+2*x2<=6)     → (2)
model.addConstr((-1)*x1+x2<=1)  → (3)

model.update()
model.optimize()

if model.status==GRB.OPTIMAL:
    print ("\n Optimal value (profit in USD thousands):", model.objVal)
    print ("--- Production quantities---")
    print ("x1", x1.x)
    print ("x2", x2.x)
```

a. (4 points) What are the decision variables associated with this LP model?

$x_1$, $x_2$

b. (4 points) What is the objective function of this LP model? Do not forget to indicate if you are maximizing or minimizing it.

Max $z = 2x_1 + 4x_2$

c. (12 points) What are the constraints associated with this LP model?

$$3x_1 + 2x_2 \le 12 \quad (1)$$
$$x_1 + 2x_2 \le 6 \quad (2)$$
$$-x_1 + x_2 \le 1 \quad (3)$$
$$x_2 \le 6 \quad (4)$$
$$x_2 \ge 0 \quad (5)$$
$$x_1 \ge 2 \quad (6)$$