Simplex Method and Sensitivity Analysis

Andrés D. González

Assistant Professor
School of Industrial and Systems Engineering, The University of Oklahoma

The University of Oklahoma, Norman, OK, USA

Converting inequalities into equations

- To convert a *less or equal* (≤) inequality to an equation, a nonnegative **slack variable** is **added** to the left-hand side of the constraint.
- Correspondingly, to convert a *greater or equal* (≥) inequality to an equation, a nonnegative **surplus variable** is **subtracted** to the left-hand side of the constraint.

  – For example, the $M1$ constraint of the Reddy Mikks model (Example 2.1-1), which was

$$6x_1 + 4x_2 \leq 24$$

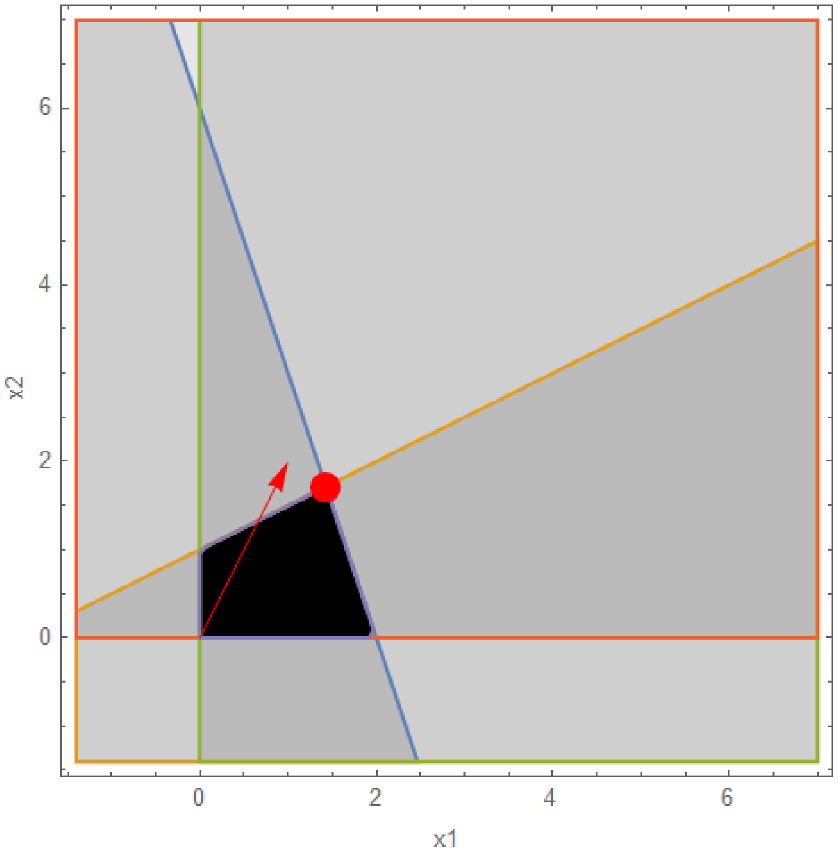
can be replaced by

$$6x_1 + 4x_2 + s1 = 24$$
$$s1 \geq 0$$

Example 1

- Suppose you have the following LP model

$$
\begin{aligned}
&\text{Maximize } z = x_1 + 2x_2 \\
&\text{subject to} \\
&\qquad 3x_1 + x_2 \le 6 \\
&\qquad 2x_1 - 4x_2 \ge -4 \\
&\qquad x_1, x_2 \ge 0
\end{aligned}
$$



Example 1

- Suppose you have the following LP model

Maximize $z = x_1 + 2x_2$
subject to

$$3x_1 + x_2 \leq 6$$
$$2x_1 - 4x_2 \geq -4$$
$$x_1, x_2 \geq 0$$

Replace it by an equivalent LP model with only equality constraints and non-negative right-hand sides

Example 1

- Suppose you have the following LP model

$$
\begin{aligned}
&\text{Maximize } z = x_1 + 2x_2\\
&\text{subject to}\\
&\qquad 3x_1 + x_2 \boxed{\leq} 6\\
&\qquad 2x_1 - 4x_2 \boxed{\geq} -4\\
&\qquad x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\text{Maximize } z = x_1 + 2x_2\\
&\text{subject to}\\
&\qquad 3x_1 + x_2 \boxed{+ s_1 \qquad =} 6\\
&\qquad 2x_1 - 4x_2 \quad \boxed{- s_2 =} -4\\
&\qquad x_1, x_2, s_1, s_2 \geq 0
\end{aligned}
$$

Replace it by an equivalent LP model with only equality constraints and non-negative right-hand sides

Example 1

- Suppose you have the following LP model

Maximize $z = x_1 + 2x_2$

subject to

$$3x_1 + x_2 \boxed{\leq} 6$$
$$2x_1 - 4x_2 \boxed{\geq} -4$$
$$x_1, x_2 \geq 0$$

Replace it by an equivalent LP model with only equality constraints and non-negative right-hand sides

Maximize $z = x_1 + 2x_2$

subject to

$$3x_1 + x_2 \boxed{+ s_1 \quad =} 6$$
$$2x_1 - 4x_2 \boxed{- s_2 =} \boxed{-4}$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Maximize $z = x_1 + 2x_2$

subject to

$$3x_1 + x_2 + s_1 = 6$$
$$\boxed{-2x_1 + 4x_2 + s_2 = 4}$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Example 1

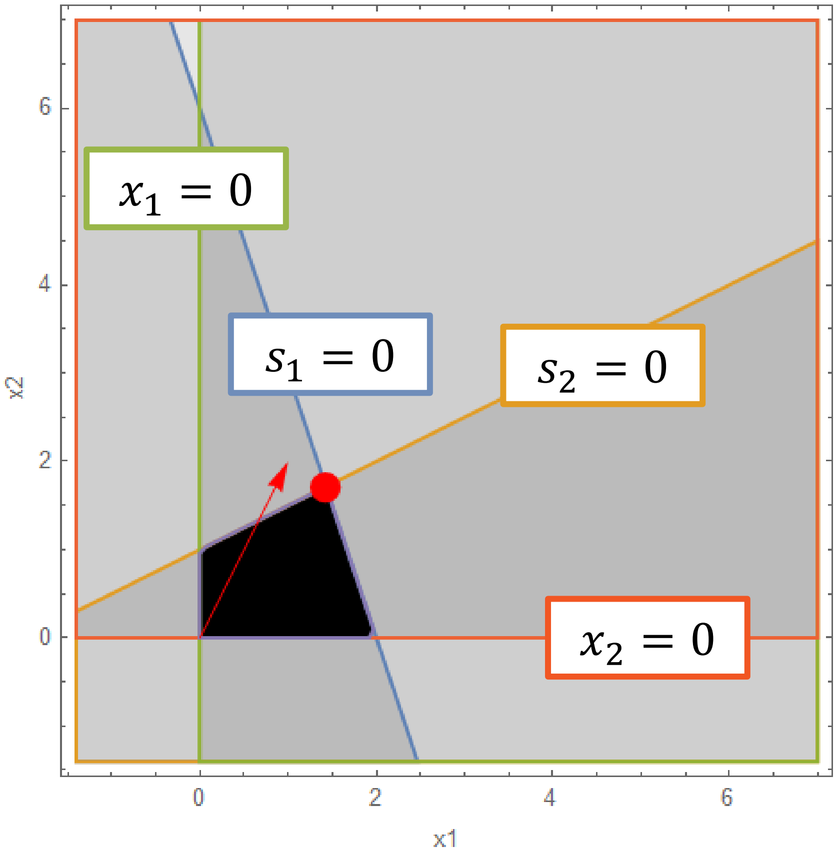
Maximize $z = x_1 + 2x_2$

subject to

$$3x_1 + x_2 + \boxed{s_1} = 6$$

$$-2x_1 + 4x_2 + \boxed{s_2} = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$



Dealing with non-positive and unrestricted variables

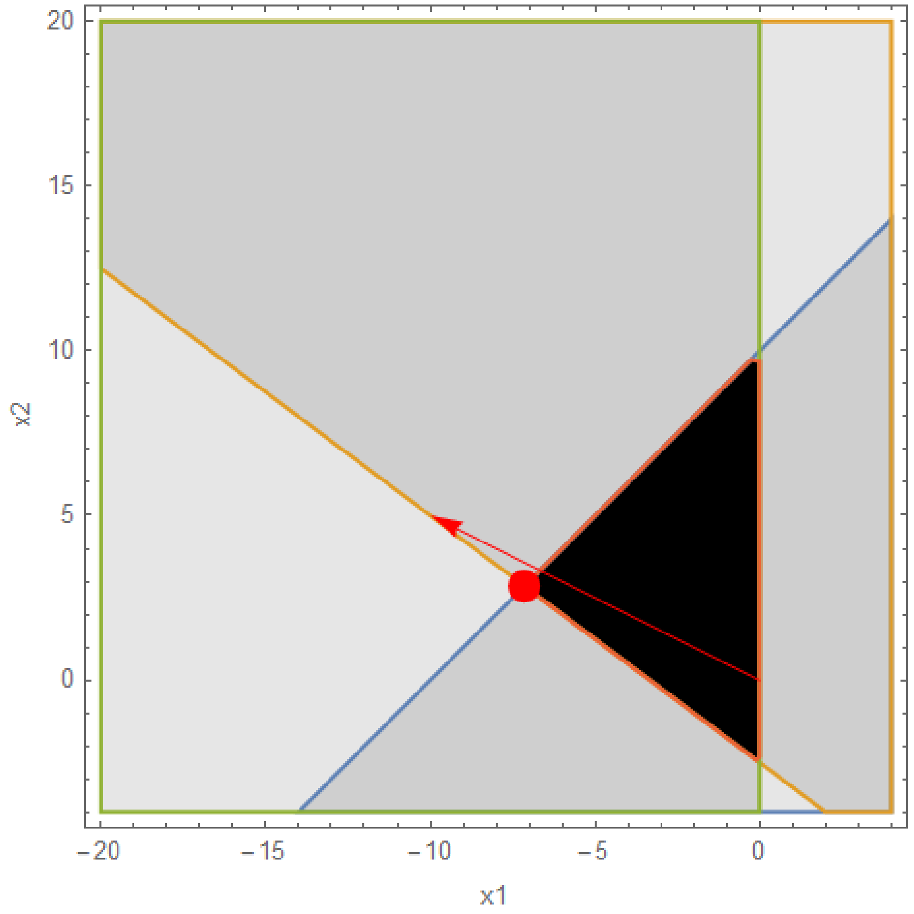
- Case 1: Suppose you have an LP model with a variable y that is non-positive. You could simply replace it (everywhere in the model) by $-\hat{y}$, where $\hat{y}$ is a non-negative variable
- Case 2: Suppose you have an LP model with a variable y that is free (or non-restricted). You could simply replace it (everywhere in the model) by $y^- - y^+$, where y^- and y^- are non-negative variables

Example 2

- Suppose you have the following LP model

Maximize $z = -10x_1 + 5x_2$
subject to
$-x_1 + x_2 \leq 10$
$-3x_1 - 4x_2 \leq 10$
$x_1 \leq 0$
$x_2 \text{ is free}$

Replace it by an equivalent LP model with only equality constraints, non-negative right-hand sides, and non-negative variables



Example 2

- Suppose you have the following LP model

$$\begin{aligned}&\text{Maximize } z = -10x_1 + 5x_2\\&\text{subject to}\\&\quad -x_1 + x_2 \le 10\\&\quad -3x_1 - 4x_2 \le 10\\&\quad x_1 \le 0\\&\quad \boxed{x_2\ is\ free}\end{aligned}$$

Replace it by an equivalent LP model with only equality constraints, non-negative right hand sides, and non-negative variables

$$\begin{aligned}&\text{Maximize } z = -10x_1 + 5\boxed{(x_2^- - x_2^+)}\\&\text{subject to}\\&\quad -x_1 + \boxed{(x_2^- - x_2^+)} + s_1 = 10\\&\quad -3x_1 - 4\boxed{(x_2^- - x_2^+)} + s_2 = 10\\&\quad \boxed{x_1 \le 0}\\&\quad \boxed{x_2^-, x_2^+ \ge 0}\end{aligned}$$

$$\begin{aligned}&\text{Maximize } z = -10\boxed{(-\hat{x}_1)} + 5(x_2^- - x_2^+)\\&\text{subject to}\\&\quad -\boxed{(-\hat{x}_1)} + (x_2^- - x_2^+) + s_1 = 10\\&\quad -3\boxed{(-\hat{x}_1)} - 4(x_2^- - x_2^+) + s_2 = 10\\&\quad \boxed{\hat{x}_1 \ge 0}\\&\quad x_2^-, x_2^+ \ge 0\end{aligned}$$

Example 2

Maximize $z = -10(-\hat{x}_1) + 5(x_2^- - x_2^+)$

subject to

$$\begin{aligned} -(-\hat{x}_1) + (x_2^- - x_2^+) + s_1 \quad &= 10 \\ -3(-\hat{x}_1) - 4(x_2^- - x_2^+) \quad + s_2 &= 10 \\ \hat{x}_1 &\geq 0 \\ x_2^-, x_2^+ &\geq 0 \end{aligned}$$

Rearranging and simplifying

Maximize $z = 10\hat{x}_1 + 5x_2^- - 5x_2^+$

subject to

$$\begin{aligned} \hat{x}_1 + x_2^- - x_2^+ + s_1 \quad &= 10 \\ 3\hat{x}_1 - 4x_2^- + 4x_2^+ \quad + s_2 &= 10 \\ \hat{x}_1, x_2^-, x_2^+ &\geq 0 \end{aligned}$$

Example 3

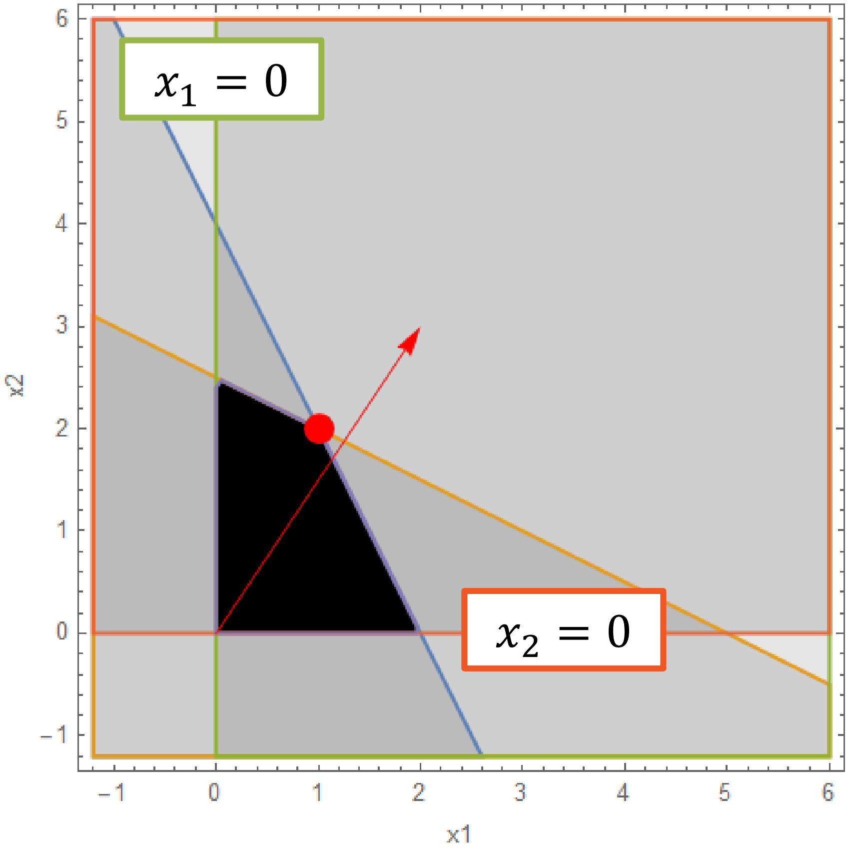
- Consider the following LP with two variables

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 \le 4$$
$$x_1 + 2x_2 \le 5$$
$$x_1, x_2 \ge 0$$

- Let us solve it using the graphical method



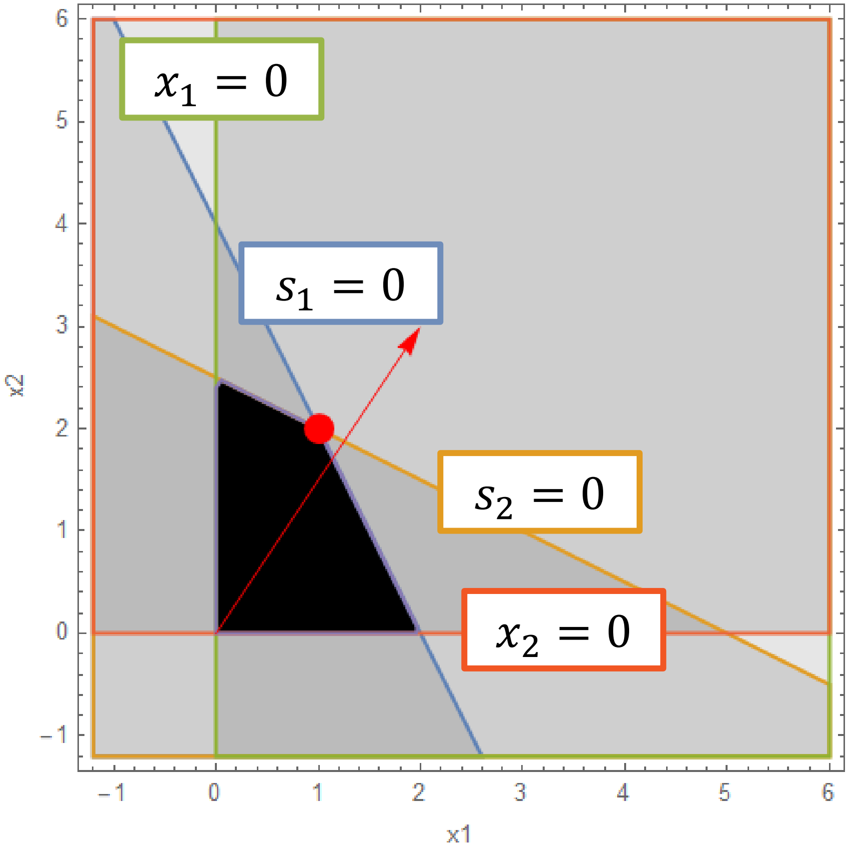
Example 3

- What is the equivalent LP with:
  - Only equality constraints
  - Only non-negative right-hand side
  - Only non-negative variables

Maximize $z = 2x_1 + 3x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 + s_1 \quad\quad &= 4 \\ x_1 + 2x_2 \quad\quad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$



Example 3

Maximize $z = 2x_1 + 3x_2$
subject to

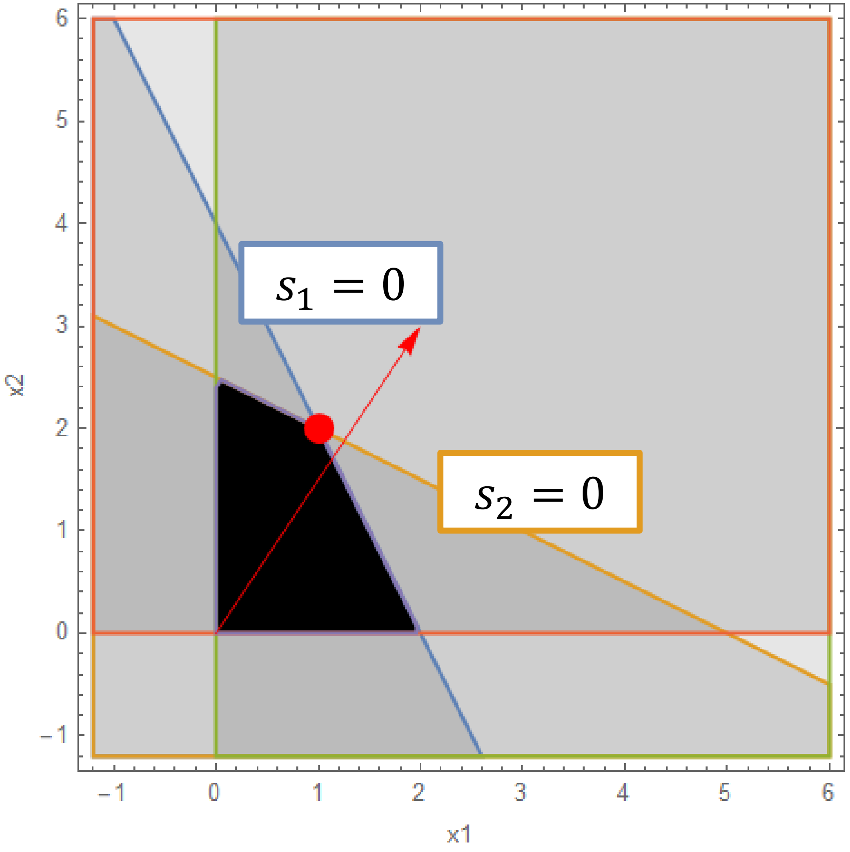
$$\begin{aligned} 2x_1 + x_2 + s_1 \quad &= 4 \\ x_1 + 2x_2 \quad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

- For this problem, in the **optimal** solution s_1 and s_2 are *zero*
  - Thus, the optimal values of the other variables can be determined by solving

$$\boxed{\begin{aligned} 2x_1 + x_2 + 0 \quad &= 4 \\ x_1 + 2x_2 \quad + 0 &= 5 \end{aligned}}$$

  - Then, using Gauss-Jordan, it is easy to see that there is a unique solution:

$$x_1 = 1, x_2 = 2$$



How about the other corner points?

Maximize $z = 2x_1 + 3x_2$
subject to

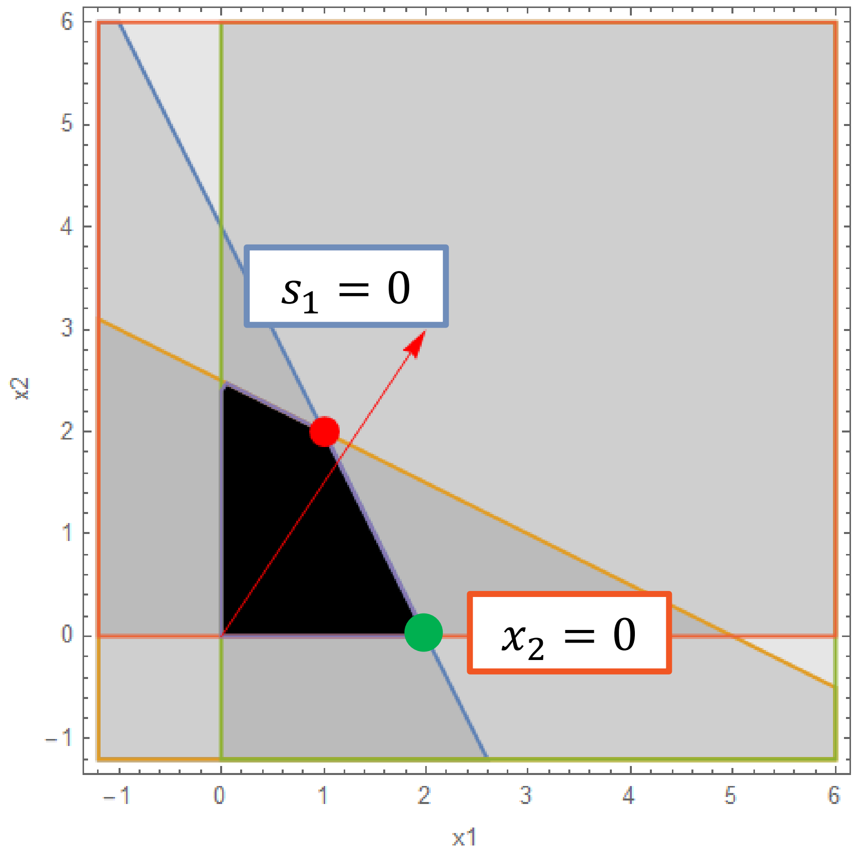
$$\begin{aligned} 2x_1 + x_2 + s_1 \quad\quad &= 4 \\ x_1 + 2x_2 \quad\quad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

- For this *feasible* solution, s_1 and x_2 are *zero*
  - Thus, the values of the other variables can be determined by solving the system of equations:

$$\boxed{\begin{aligned} 2x_1 + 0 + 0 \quad\quad &= 4 \\ x_1 + 0 \quad\quad + s_2 &= 5 \end{aligned}}$$

  - Then, using Gauss-Jordan, it is easy to see that there is a unique solution to this system of eqns:

$$x_1 = 2, s_2 = 3$$



How about the other corner points?

Maximize $z = 2x_1 + 3x_2$
subject to

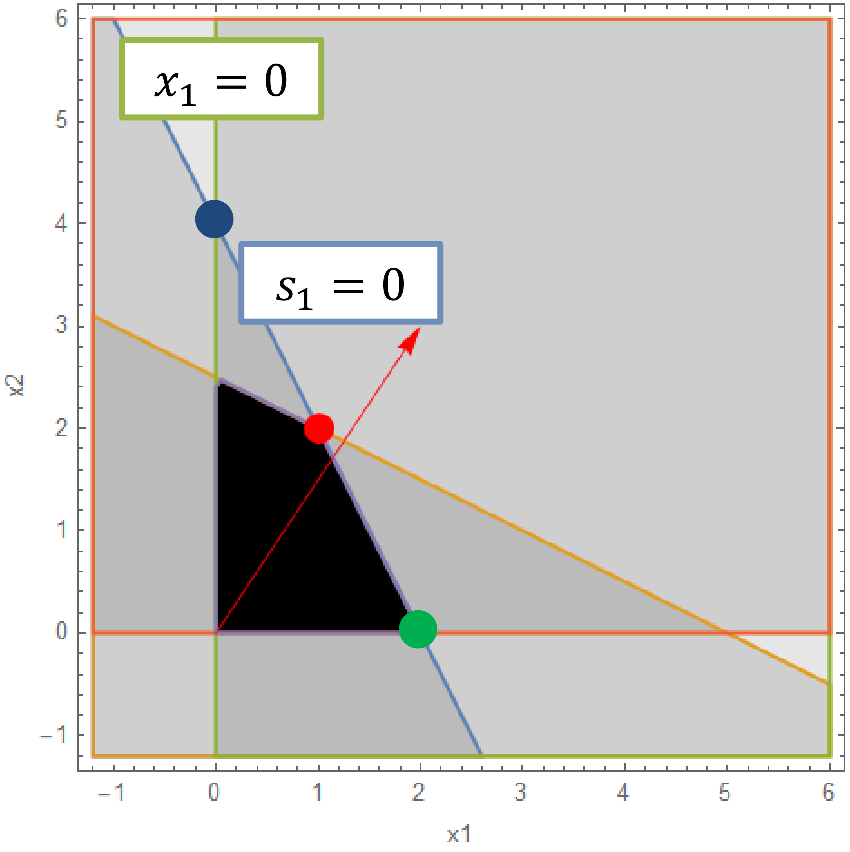
$$\begin{aligned} 2x_1 + x_2 + s_1 \quad &= 4 \\ x_1 + 2x_2 \quad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

- For this *infeasible* solution, x_1 and s_1 are *zero*
  - Thus, the values of the other variables can be determined by solving the system of equations:

$$\boxed{\begin{aligned} 0 + x_2 + 0 \quad &= 4 \\ 0 + 2x_2 \quad + s_2 &= 5 \end{aligned}}$$

  - Then, using Gauss-Jordan, it is easy to see that there is a unique solution to this system of eqns:

$$x_2 = 4, s_2 = -3$$



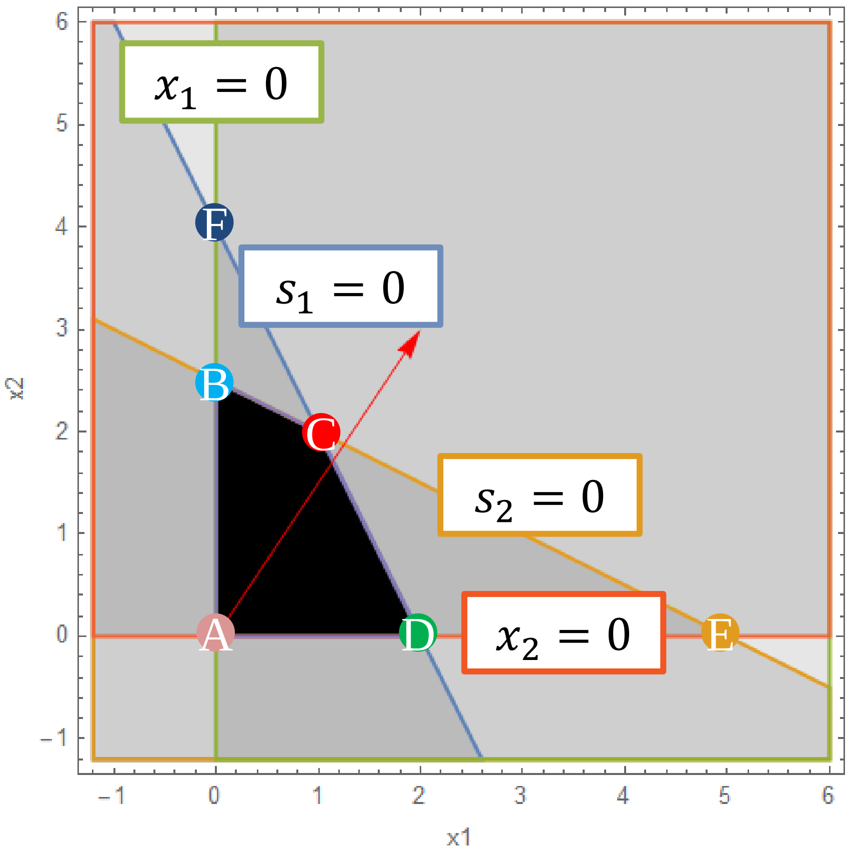
How about the other corner points?

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 + s_1 \qquad = 4$$
$$x_1 + 2x_2 \qquad + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4,5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	$(4, -3)$	F	No	12
(x_1, s_2)	(x_2, s_1)	(2.5,1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2,3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	$(5, -6)$	E	No	10
(s_1, s_2)	(x_1, x_2)	(1,2)	C	Yes	8



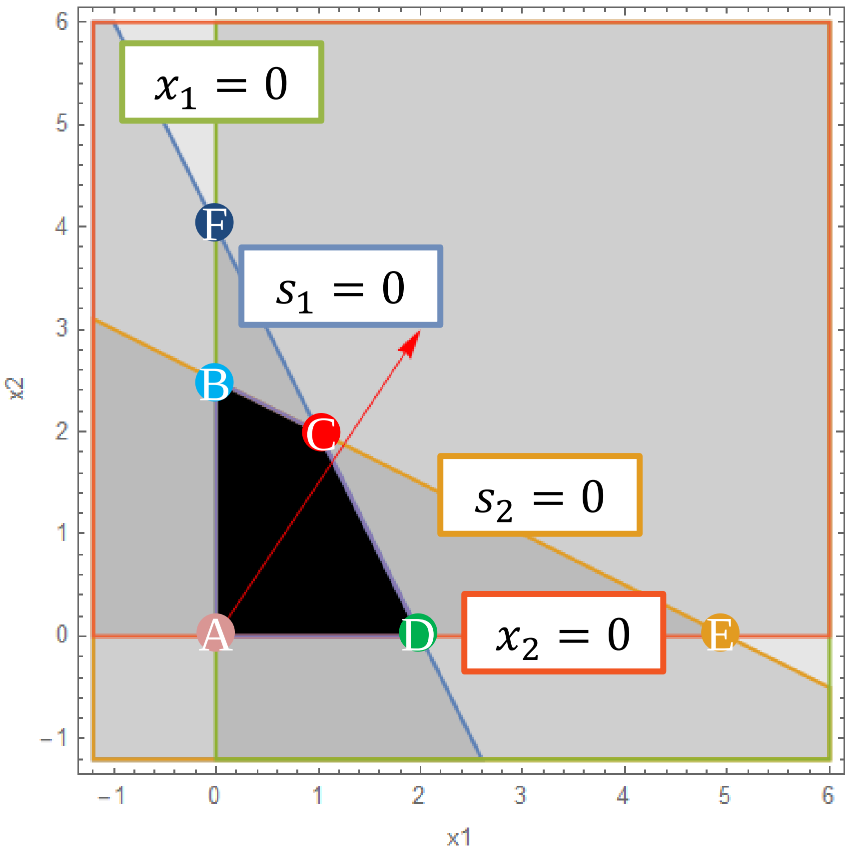
How about the other corner points?

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4,5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	$(4, -3)$	F	No	12
(x_1, s_2)	(x_2, s_1)	(2.5,1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2,3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	$(5, -6)$	E	No	10
(s_1, s_2)	**(x_1, x_2)**	**(1,2)**	**C**	**Yes**	**8**



Transition from graphical to algebraic solution

Graphical Method	Algebraic Method
Graph all constraints, including nonnegativity restrictions	Represent the solution space by m equations in n variables and restrict all variables to nonnegative values, $m < n$
Solution space consists of infinity of **feasible** points	The system has infinity of **feasible** solutions
↓	↓
Identify **feasible corner points** of the solution space	Determine the **feasible basic solutions** of the equations
Candidates for the optimum solution are given by a *finite* number of **corner points**	Candidates for the optimum solution are given by a *finite* number of **basic feasible solutions**
↓	↓
Use the objective function to determine the **optimum corner point** from among all the candidates	Use the objective function to determine the **optimum basic feasible solution** from among all the candidates

The Simplex method

- In general, for a problem with non-empty set of feasible solutions, we have $\boldsymbol{m}$ **linearly independent equations** and $\boldsymbol{n}$ **variables (including slack and surplus variables)** (where $\boldsymbol{m} \leq \boldsymbol{n}$)
  - The maximum number of *basic solutions* (which correspond to *corner points* in the graphical solution space) is

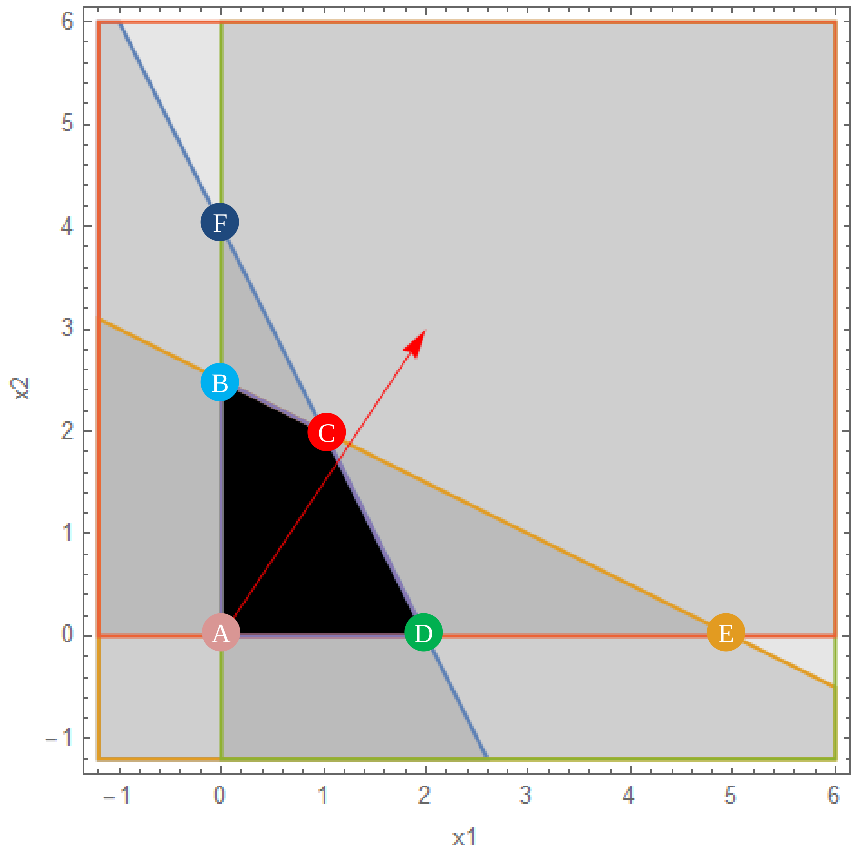
$$C_m^n = \frac{n!}{m!\,(n-m)!}$$

This number is VERY large. For example, this for a problem with 20 variables and 10 constraints (a very small problem in most realistic applications) you'll have 184,756 basic solutions!

- We need a (much) more efficient algorithm: SIMPLEX

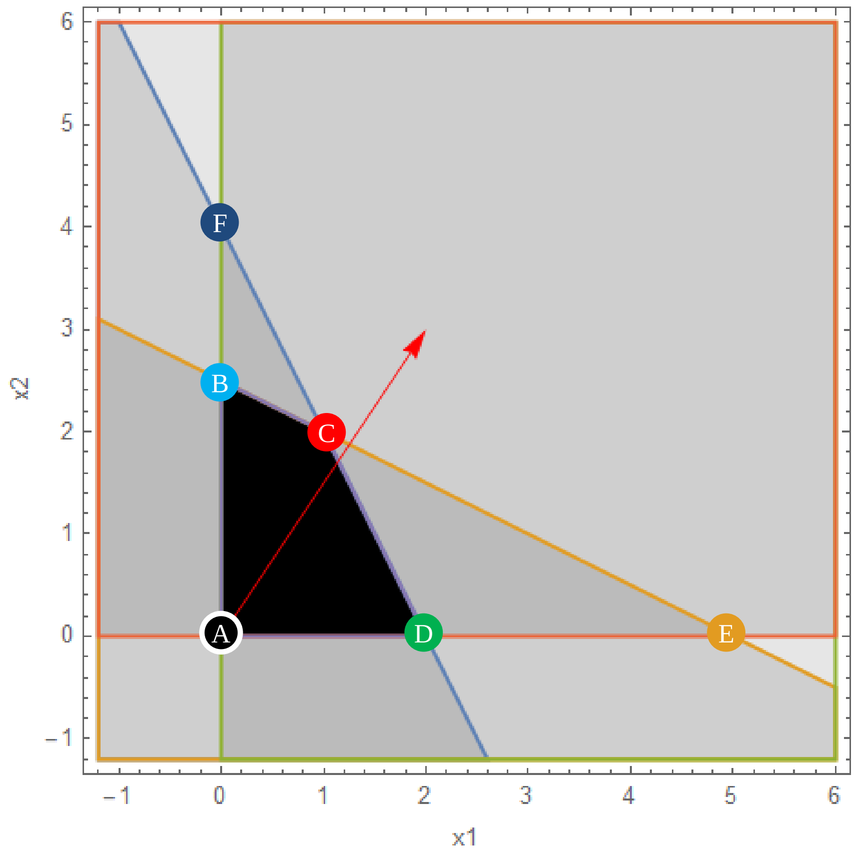
Example 3 – revisited (Simplex)

- Let's revisit Example 3…



Example 3 – revisited (Simplex)

1) Start from a feasible basic solution

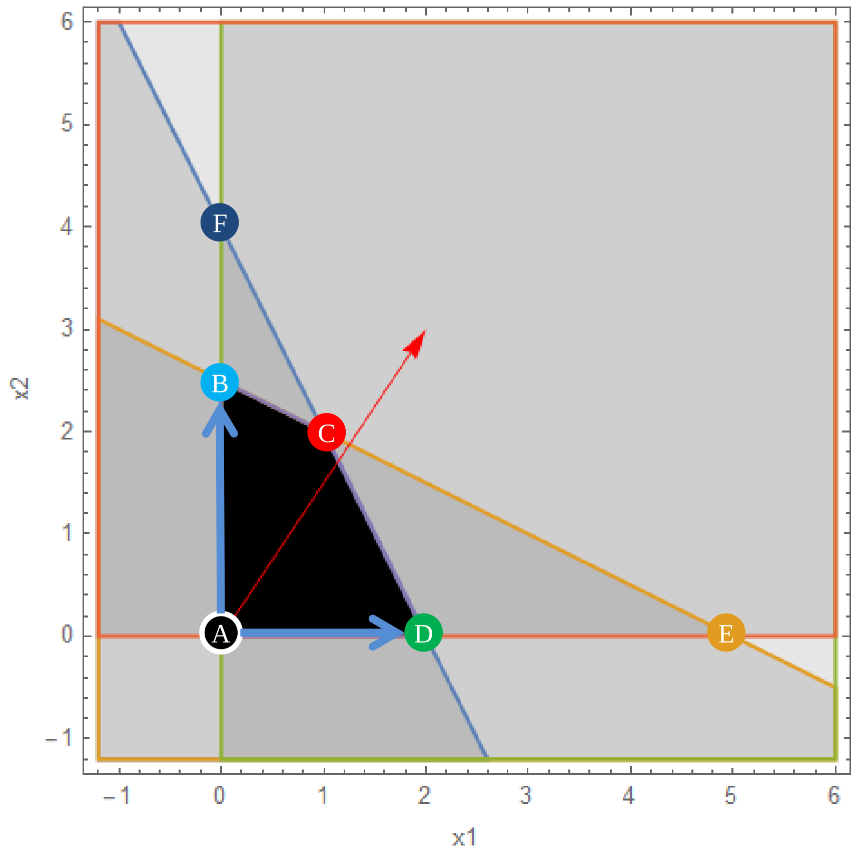


Example 3 – revisited (Simplex)

1) Start from a feasible basic solution

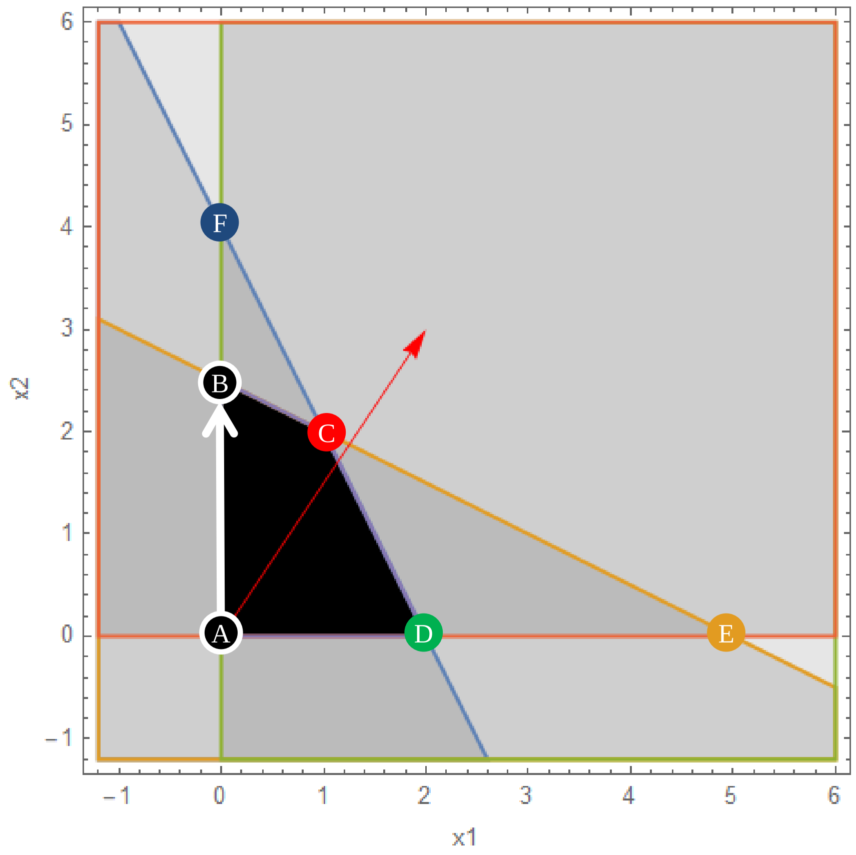
2) Check if there is any "promising" direction to move

- It needs to be along a "constraint"
- You cannot move beyond the first "constraint" you find in your path



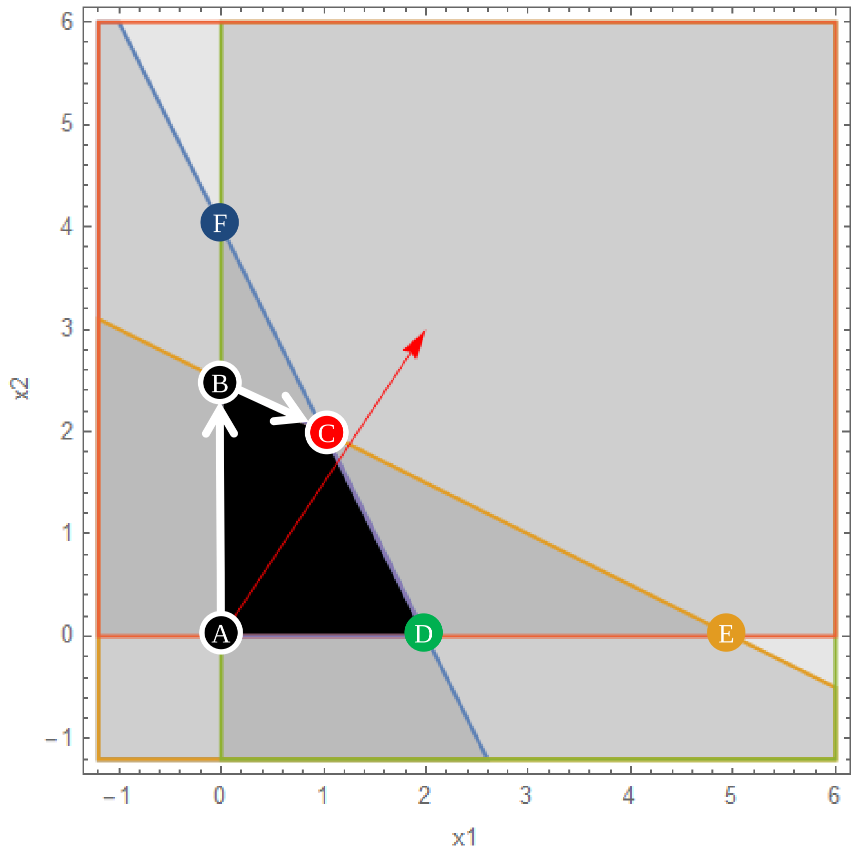
Example 3 – revisited (Simplex)

1) Start from a feasible basic solution
2) Check if there is any "promising" direction to move
   - It needs to be along a "constraint"
   - You cannot move beyond the first "constraint" you find in your path
3) From all the "promising" directions, move in the "most promising" direction, and update basic and non-basic variables
   - If there are no "promising" directions, STOP, since you are in the OPTIMAL SOLUTION



Example 3 – revisited (Simplex)

1) Start from a feasible basic solution
2) Check if there is any "promising" direction to move
   - It needs to be along a "constraint"
   - You cannot move beyond the first "constraint" you find in your path
3) From all the "promising" directions, move in the "most promising" direction, and update basic and non-basic variables
   - If there are no "promising" directions, STOP, since you are in the OPTIMAL SOLUTION
4) Repeat 2) and 3)



Let's follow the same idea… but algebraically

Maximize $z = 2x_1 + 3x_2$

subject to

$$
\begin{aligned}
2x_1 + x_2 &\leq 4 \\
x_1 + 2x_2 &\leq 5 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

- **Step 1: Transform to LP model with only equality constraints, non-negative right-hand side, and non-negative variables**

Maximize $z = 2x_1 + 3x_2$

subject to

$$
\begin{aligned}
2x_1 + x_2 + s_1 \quad\quad &= 4 \\
x_1 + 2x_2 \quad\quad + s_2 &= 5 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

Let's follow the same idea… but algebraically

Maximize $z = 2x_1 + 3x_2$

subject to

$$
\begin{aligned}
2x_1 + x_2 + s_1 \quad\quad &= 4 \\
x_1 + 2x_2 \quad\quad + s_2 &= 5 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

- **<u>Step 2: Find an initial feasible basic solution:</u>**

In this case, $x_1 = x_2 = 0$ (the trivial solution) is feasible! Thus, a feasible basic solution would be:

$s_1 = 4, s_2 = 5 \qquad\qquad \rightarrow BASIC\ VARIABLES$

$x_1 = x_2 = 0 \qquad\qquad \rightarrow NONBASIC\ VARIABLES$

Note: what if the trivial solution is not feasible? Then we need to use an initialization method (e.g., *big-M method, two-phase method*)

Let's start working in a table form (a.k.a. Simplex Tableau)

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 + s_1 \qquad = 4$$
$$x_1 + 2x_2 \qquad + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

$s_1 = 4, s_2 = 5 \quad \rightarrow BASIC\ VARIABLES$
$x_1 = x_2 = 0 \quad \rightarrow NONBASIC\ VARIABLES$

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
s_2	0	1	2	0	1	5	s_2 row

Simplex Tableau

Maximize $z = 2x_1 + 3x_2$ → $\boxed{z - 2x_1 - 3x_2} = 0$

subject to

$$\boxed{\begin{aligned} 2x_1 + x_2 + s_1 \\ x_1 + 2x_2 \quad + s_2 \end{aligned}} \begin{aligned} &= 4 \\ &= 5 \end{aligned}$$

$$x_1, x_2, s_1, s_2 \geq 0$$

$s_1 = 4, s_2 = 5 \quad \rightarrow BASIC\ VARIABLES$

$x_1 = x_2 = 0 \quad \rightarrow NONBASIC\ VARIABLES$

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
s_2	0	1	2	0	1	5	s_2 row

Simplex Tableau

- **<u>Step 3: Find the "most promising" direction to move:</u>**

In other words, determine which non-basic variable (i.e., is zero) would improve the objective function the most if it became basic (i.e., if it became non-zero)

- Let's use this rule (for now):
  - If maximization problem: select "most negative" in z-row
  - If minimization problem: select "most positive" in z-row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	$\boxed{\mathbf{-3}}$	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
s_2	0	1	**2**	0	1	5	s_2 row

Note: if there is no "promising" variable (i.e., if no non-basic variable would improve the objective function by becoming basic), you are in the OPTIMAL SOLUTION

Simplex Tableau

- **Step 4: Move in the "most promising" direction as much as you can:**

In other words, determine which basic variable would become zero (i.e., become non-basic) first

- You cannot move beyond that point, otherwise your variable would become negative (i.e., infeasible)

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	**-3**	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
s_2	0	1	**2**	0	1	5	s_2 row

Basic	Entering (x_2)	Solution	Ratio (or intercept)	
s_1	1	4	$x_2 = 4/1 = 4$	
s_2	2	5	$x_2 = 5/2 = 2.5$	Minimum

Conclusion: x_2 **enters** (becomes basic) and s_2 **leaves** (becomes non-basic)

Simplex Tableau

- **Step 5: Update table:**

x_2 **enters** (pivot column) and s_2 **leaves** (pivot row). The intersection is called pivot element

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	**-3**	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
s_2	0	1	**2**	0	1	5	s_2 row

You need to update the table using Gauss-Jordan row operations

1. Pivot row
   a) Replace the leaving variable in the Basic column with the entering variable
   b) New pivot row = Current pivot row ÷ Pivot element
2. All other rows, including z
   New row = (Current row) – (Pivot column coefficient) x (New pivot row)

Simplex Tableau

- **Step 5: Update table:**

1. Pivot row
   a) Replace the leaving variable in the Basic column with the entering variable

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
s_2	0	1	2	0	1	5	s_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	1	2	0	1	5	x_2 row

Simplex Tableau

- **Step 5: Update table:**

1. Pivot row
   a) Replace the leaving variable in the Basic column with the entering variable
   b) New pivot row = Current pivot row ÷ Pivot element

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	**-3**	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
x_2	0	1	**2**	0	1	5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	**-3**	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
x_2	0/2	1/2	**2/2**	0/2	1/2	5/2	x_2 row

Simplex Tableau

- **Step 5: Update table:**

1. Pivot row
   a) Replace the leaving variable in the Basic column with the entering variable
   b) New pivot row = Current pivot row ÷ Pivot element

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	1	2	0	1	5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	**-3**	0	0	0	z-row
s_1	0	2	**1**	1	0	4	s_1 row
x_2	0	0.5	**1**	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table:**

2. All other rows, including z

   New row = (Current row) – (Pivot column coefficient) x (New pivot row)

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table:**

2. All other rows, including z

   New row = (Current row) – (Pivot column coefficient) x (New pivot row)

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1-((-3)x0)	-2-((-3)x0.5)	-3-((-3)x1)	0-((-3)x0)	0-((-3)x0.5)	0-((-3)x2.5)	z-row
s_1	0-(1x0)	2-(1x0.5)	1-(1x1)	1-(1x0)	0-(1x0.5)	4-(1x2.5)	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table:**

2. All other rows, including z

   New row = (Current row) – (Pivot column coefficient) x (New pivot row)

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1-((-3)x0)	-2-((-3)x0.5)	-3-((-3)x1)	0-((-3)x0)	0-((-3)x0.5)	0-((-3)x2.5)	z-row
s_1	0-(1x0)	2-(1x0.5)	1-(1x1)	1-(1x0)	0-(1x0.5)	4-(1x2.5)	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table:**

2. All other rows, including z

New row = (Current row) – (Pivot column coefficient) x (New pivot row)

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-2	-3	0	0	0	z-row
s_1	0	2	1	1	0	4	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 6: Go back to Step 3**

Basic	z	x_1	x_2	s_1	s_2	Solution	
$\boldsymbol{z}$	1	-0.5	0	0	1.5	7.5	z-row
$\boldsymbol{s_1}$	0	1.5	0	1	-0.5	1.5	s_1 row
$\boldsymbol{x_2}$	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 3: Find the "most promising" direction to move:**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 4: Move in the "most promising" direction as much as you can**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
s_1	0	1.5	0	1	-0.5	1.5	s_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
x_1	0/1.5	1.5/1.5	0/1.5	1/1.5	-0.5/1.5	1.5/1.5	x_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Simplex Tableau

- **Step 5: Update table**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1-((-0.5)x0)	-0.5 -((-0.5)x1)	0 -((-0.5)x0)	0-((-0.5) x(2/3))	1.5-((-0.5) x(-1/3))	7.5-((-0.5) x1)	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0-(0.5x0)	0.5-(0.5x1)	1-(0.5x0)	0(0.5 x(2/3))	0.5-(0.5 x(-1/3))	2.5-(0.5x1)	x_2 row

Simplex Tableau

- **Step 5: Update table**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	-0.5	0	0	1.5	7.5	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0.5	1	0	0.5	2.5	x_2 row

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	0	0	1/3	4/3	8	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0	1	-1/3	2/3	2	x_2 row

Simplex Tableau

- **Step 6: Go back to Step 3**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	0	0	1/3	4/3	8	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0	1	-1/3	2/3	2	x_2 row

Simplex Tableau

- **Step 3: Find the "most promising" direction to move:**

Basic	z	x_1	x_2	s_1	s_2	Solution	
z	1	0	0	1/3	4/3	8	z-row
x_1	0	1	0	2/3	-1/3	1	x_1 row
x_2	0	0	1	-1/3	2/3	2	x_2 row

There is no "promising" direction, thereby we are in the OPTIMAL SOLUTION

$s_1 = 0, s_2 = 0$ → *Non-basic variables*

$x_1 = 1, x_2 = 2$ → *Basic variables*

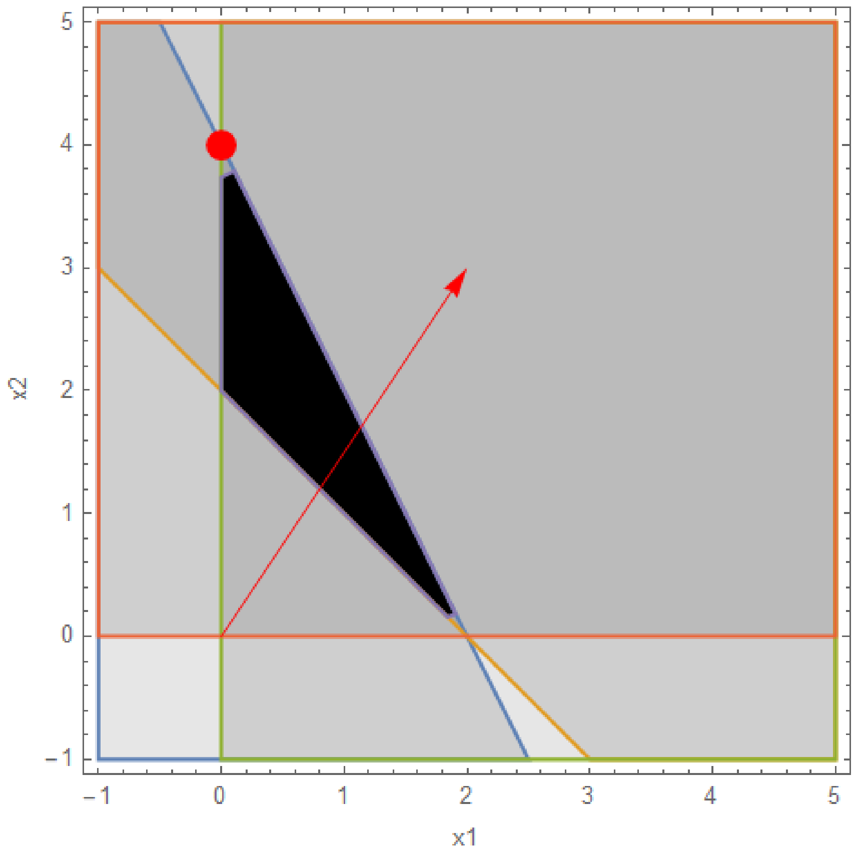
$z = 8$ → *Objective function*

What if the "trivial solution" is not feasible?

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$



- The point $x_1 = x_2 = 0$ is not feasible
  - How can we find a feasible basic solution (to initialize our algorithm)?

Big M method & Two-phase method

$$\begin{aligned} \text{Maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} \quad & \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- M method (a.k.a. Big M method)

$$\begin{aligned} \text{Maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{M}\boldsymbol{r} \\ \text{subject to} \quad & \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{r} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{r} \geq \boldsymbol{0} \end{aligned}$$

Note: in minimization problems, the penalization should be $+\boldsymbol{M}\boldsymbol{r}$

- Two-phase method
  - Phase 1: find initial feasible solution

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{1}^T\boldsymbol{r} \\ \text{subject to} \quad & \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{r} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{r} \geq \boldsymbol{0} \end{aligned}$$

  - Phase 2: If you found a feasible solution, use it to initialize your original problem
    - If you are using Tableau, don't forget to use the updated coefficients

Special cases in the Simplex method

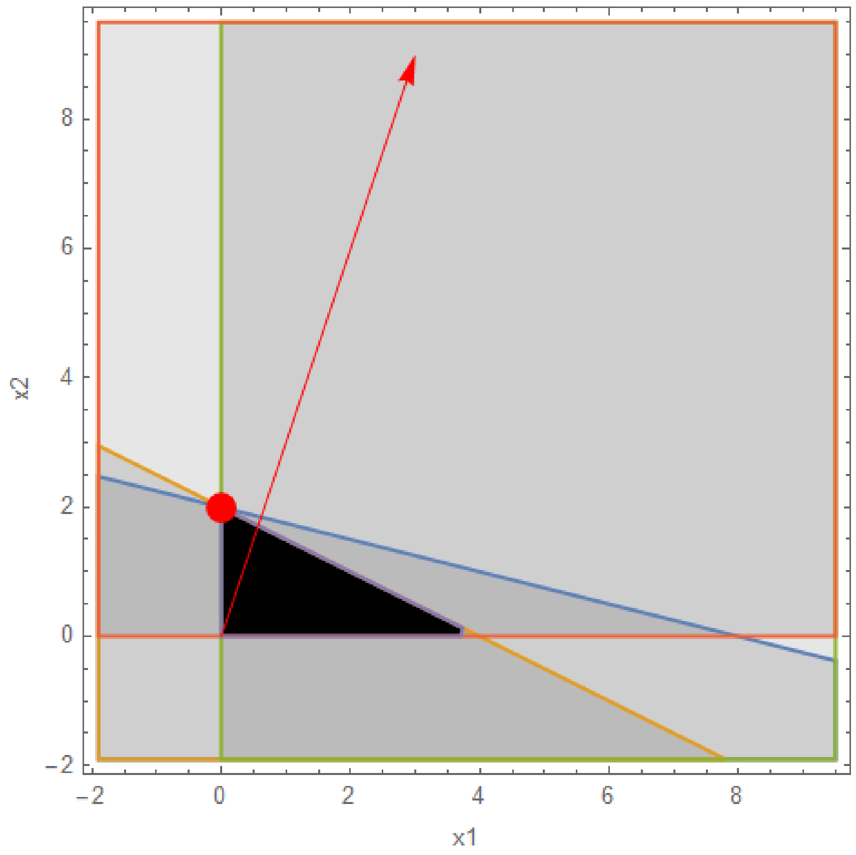
- There are four special cases that arise in the use of the simple method
  - Degeneracy
  - Alternative optima
  - Unbounded solutions
  - Non-existing (or infeasible) solutions

Degeneracy

Maximize $z = 3x_1 + 9x_2$

subject to

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



- There is a basic solution that has more than one constraint going through it
  - This means that (in addition to all non-basic variables) at least one basic variable has to be zero
  - This can lead to **cycling**

Degeneracy

Maximize $z = 3x_1 + 9x_2$

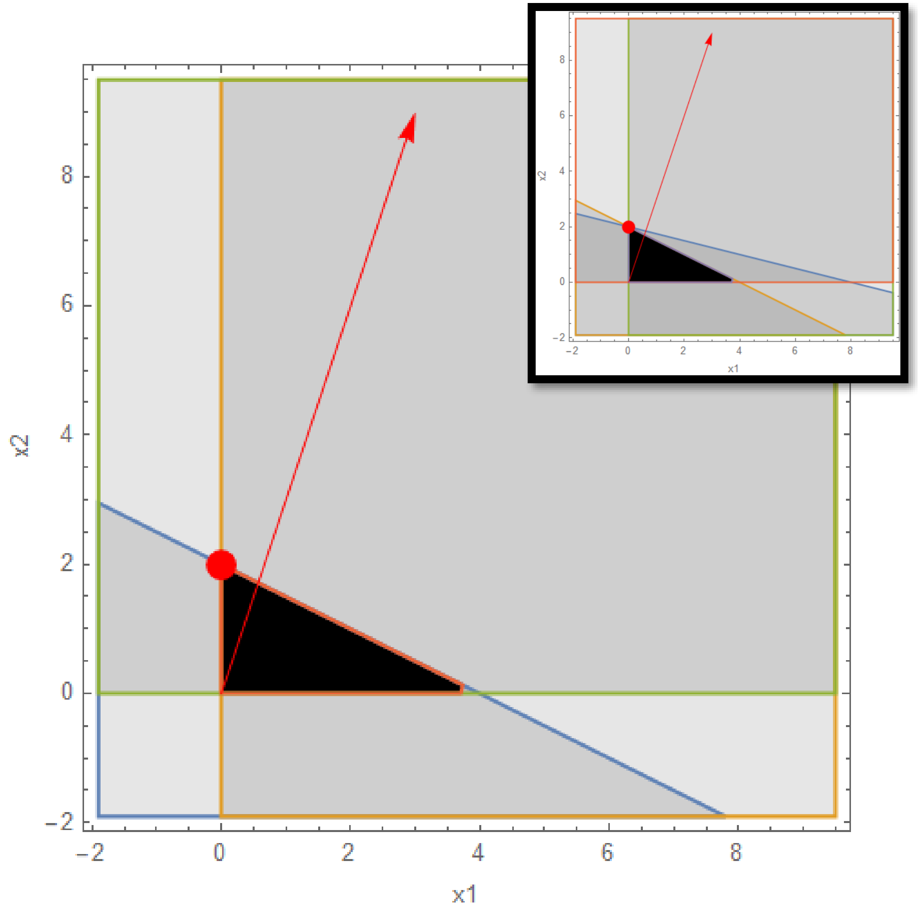
subject to

$x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

- There is a basic solution that has more than one constraint going through it
  - There is a redundant constraint (that you should try to eliminate)



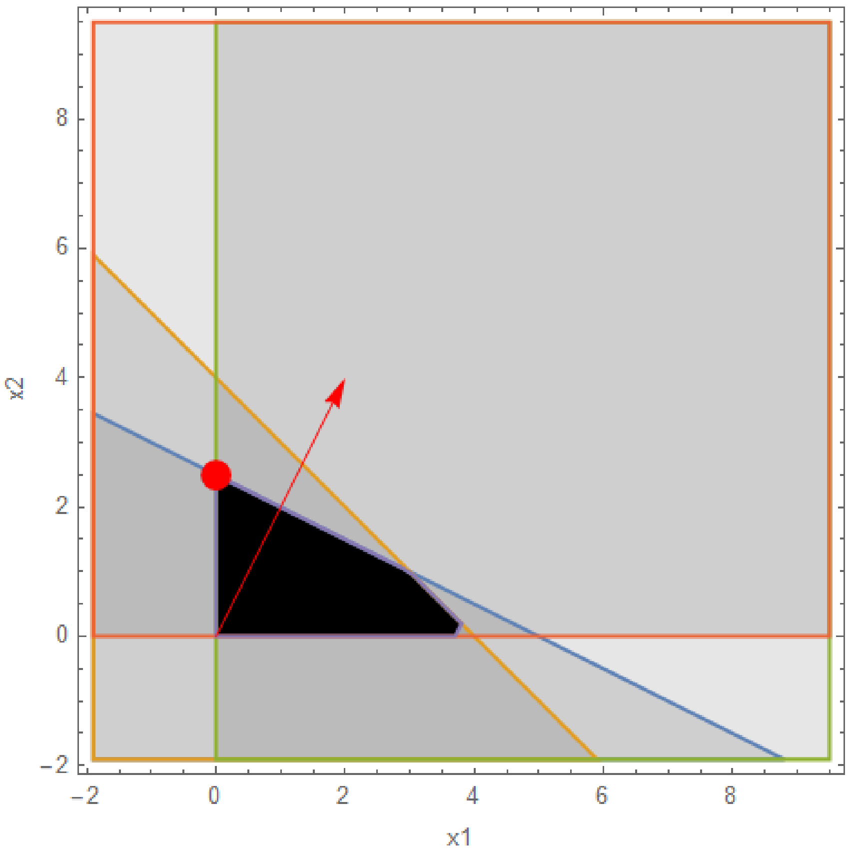
Alternative optima

Maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

- When solving this example with Simplex, you will converge to an optimal solution
  - However…

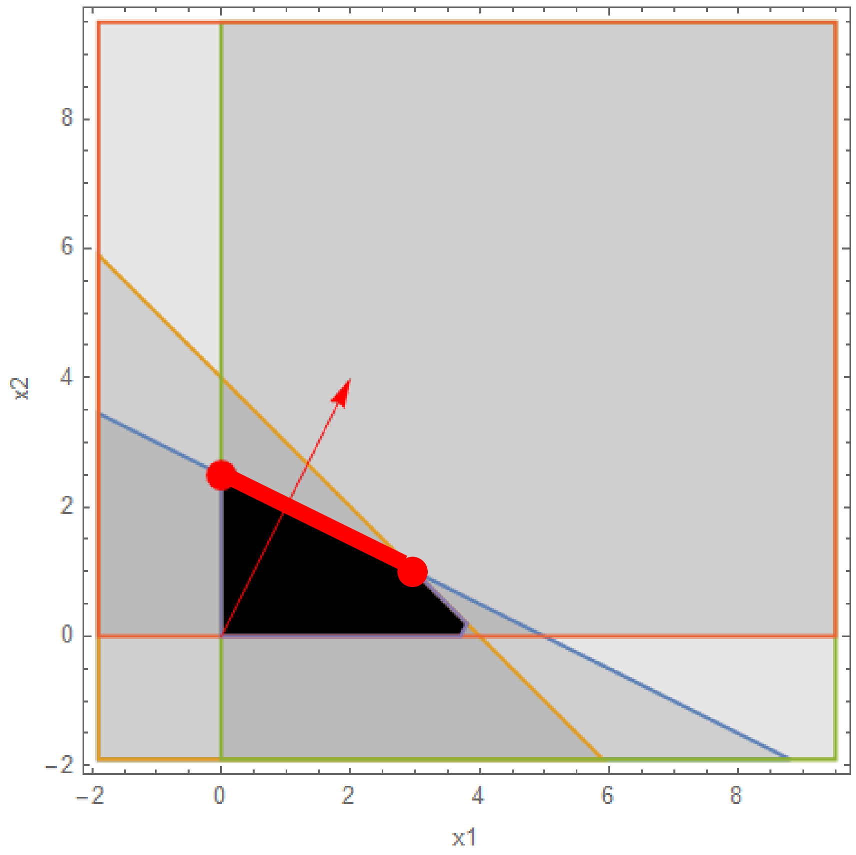


Alternative optima

Maximize $z = 2x_1 + 4x_2$
subject to

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

- When solving this example with Simplex, you will converge to an optimal solution
  - However there are multiple solutions that are optimal

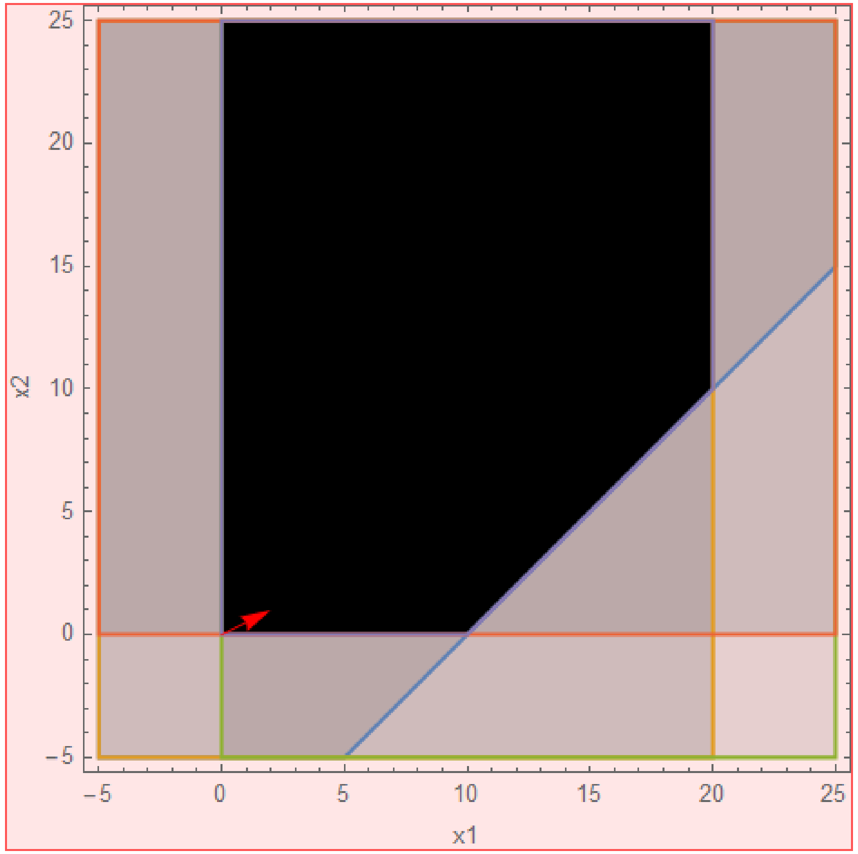


Unbounded solution space

Maximize $z = 2x_1 + x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\le 10 \\ 2x_1 \quad &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

- The feasible solution space is not bounded
  - For this example, you could move in a direction that increases the objective function indefinitely
  - However…

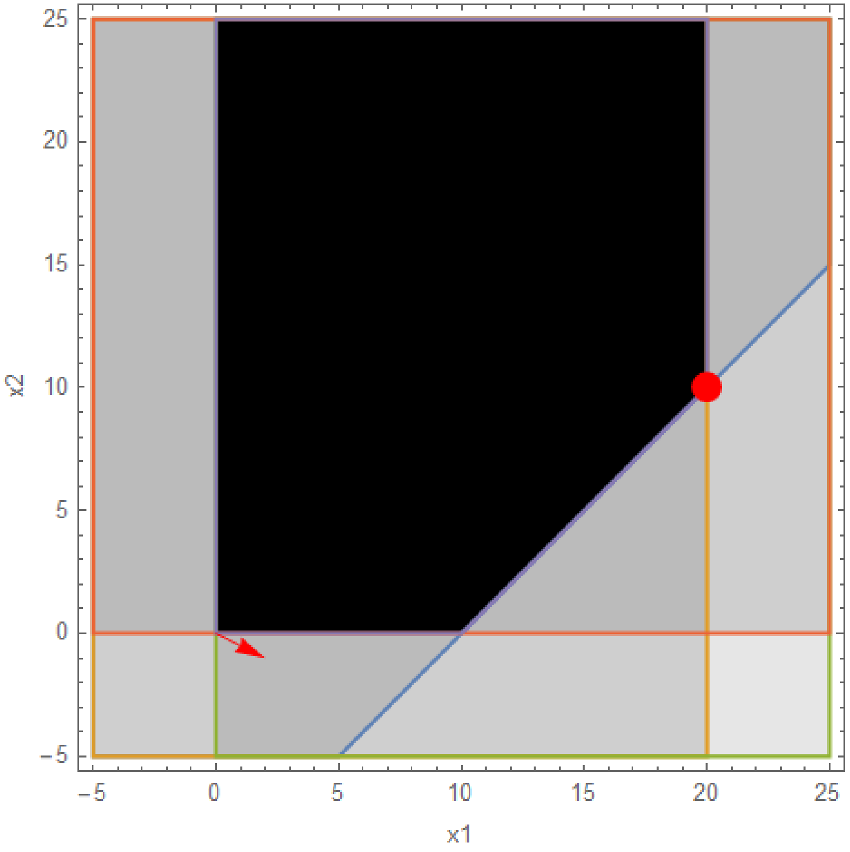


Unbounded solution space

Maximize $z = 2x_1 + x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\leq 10 \\ 2x_1 \quad &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- The feasible solution space is not bounded
  - For this example, you could move in a direction that increases the objective function indefinitely
  - However, depending on the objective function (or its maximization or minimization) you may have a finite optimal solution

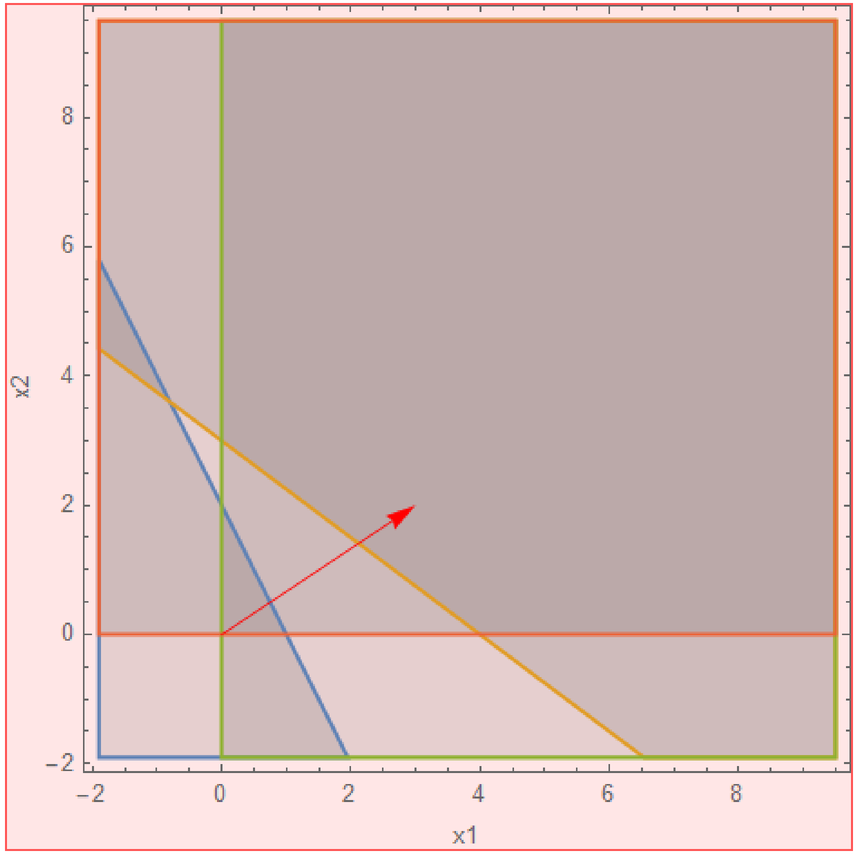


Infeasible solution space

Maximize $z = 3x_1 + 2x_2$

subject to

$$\begin{aligned} x_1 + \ \ x_2 &\leq 2 \\ 3x_1 + 4x_2 &\geq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- There is no feasible solution
  - Thereby, there is no optimal solution

Computational issues in linear programming

- An LP code is deemed robust if it satisfies two fundamental requirements:
  - Speed
  - Accuracy
- Key aspects to consider
  - Simplex entering variable (pivot) rule – How do we determine the "most promising" direction?
    - Classical
    - Most improvement
    - Steepest edge
  - Primal vs. dual simplex algorithm
  - Revised simplex vs. tableau simplex
  - Barrier (interior point) algorithm vs. simplex algorithm
  - Degeneracy
  - Input model conditioning (pre-solving)
  - Advances in computers

Sensitivity analysis

– Suppose you have the problem

$$
\begin{aligned}
&\text{Maximize} \quad z = \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
&\qquad \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

– There are some sensitivity analyses of interest:
  - Changes/perturbations in $\boldsymbol{c}$: vector of costs
  - Changes/perturbations in $\boldsymbol{A}$: matrix of coefficients
  - Changes/perturbations in $\boldsymbol{b}$: right-hand side vector

Sensitivity analysis

- Suppose you have the problem

$$
\begin{aligned}
&\text{Maximize} \quad z = \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
&\qquad \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

- There are some sensitivity analyses of interest:
  - Changes/perturbations in $\boldsymbol{c}$: vector of costs
  - Changes/perturbations in $\boldsymbol{A}$: matrix of coefficients
  - Changes/perturbations in $\boldsymbol{b}$: right-hand side vector

MORE ON THIS LATER…

In-class Exercise

Solve the "Reddy Mikks" paint production problem (Example 2.1-1) using Simplex

Maximize $z = 5x_1 + 4x_2$

subject to

$$
\begin{aligned}
6x_1 + 4x_2 &\leq 24 \\
x_1 + 2x_2 &\leq 6 \\
-x_1 + x_2 &\leq 1 \\
x_2 &\leq 2 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

LP model in equation form

- Simplex is algorithm to solve an LP model
  - If it has an optimal solution, simplex will reach in a finite amount of steps
  - In addition to require an LP formulation, it also requires:
    - All the constraints should be equations with non-negative right-hand side
    - All variables are non-negative
  - That means that, you should be able to write your LP model in the following (matrix) form

$$\begin{aligned} &\text{Maximize} \quad z = \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to} \\ &\qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ &\qquad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- $\boldsymbol{c}$: vector of costs
- $\boldsymbol{A}$: matrix of coefficients
- $\boldsymbol{b}$: right-hand side vector
- $\boldsymbol{x}$: vector of decision variables

THANK YOU

QUESTIONS?

Andrés D. González | andres.gonzalez@ou.edu