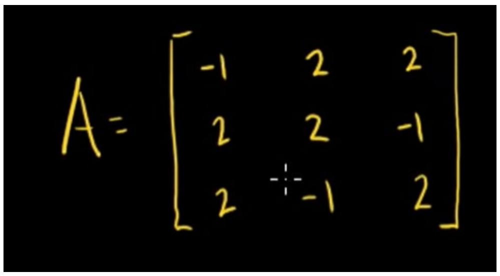
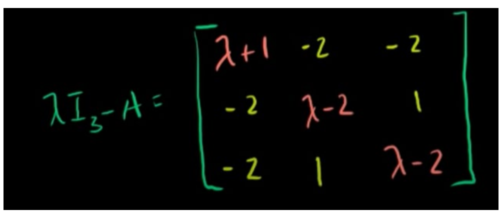
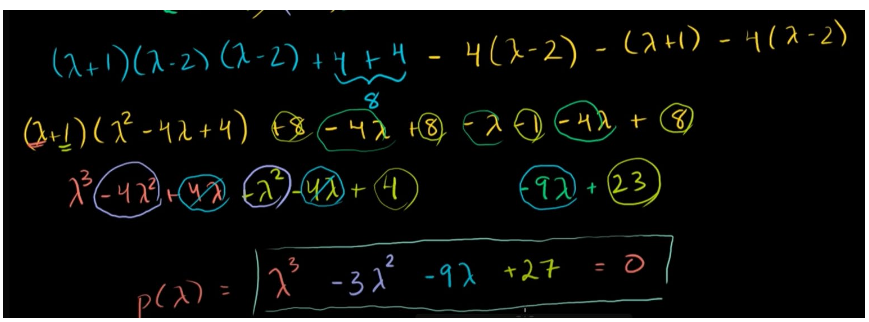
Linear Algebra: Unit II - Review of Main Topics

√ Characteristic Polynomial

- The characteristic polynomial $\det(A - \lambda I_n)$ of an $n \times n$ matrix, whose roots are the eigenvalues of A. Computation of eigenvectors, determining diagonalizability.
- **Definition:**
    1. This is the matrix that you get by subtracting λ from the diagonal of a matrix
- **Example:**
    1. 

    2. 

    3. 

- **Links:**
    1. [Khan Academy Eigen values of 3x3 matrix]
    2. [Eigenvalue and vector solver with steps]
    3. [Quiz 13, Q1 Example]

√ Coordinate Vectors

- **Definition:**
    1. Coordinate vectors $[\mathbf{v}]_S$ with respect to some ordered basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of a subspace of R^n.
    2. $$[\vec{w}]_S = \begin{bmatrix} w \cdot \vec{v}_1 \\ \vdots \\ \vec{w} \cdot \vec{v}_k \end{bmatrix} \quad \text{if} \quad S = \{\vec{v}_1, \ldots, \vec{v}_k\}$$
- **Links:**
    1. [Page 9 lecture notes]
    2. [Coordinate Vectors – Khan Academy]

√ Transition matrices

- **Definition:**
    1. Transition matrices: $P_{S \leftarrow T}\,[\mathbf{v}]_T = [\mathbf{v}]_S$. for all vectors $\mathbf{v}$ in the subspace. We have $P_{T\leftarrow S} = P^{-1}_{S\leftarrow T}$
- **Links:**
    1. [Transition Matrices – Khan Academy]v

√ Vector Lengths, distances, and angles

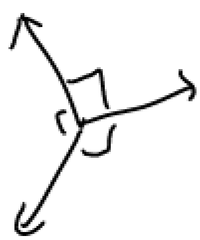
- Lengths, distances, and angles of vectors in terms of the dot product in R^n. Orthogonality of vectors and of subspaces; orthonormal = orthogonal and unit vectors.
- **Definition:**
    1. length $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
       distance $\|\vec{v} - \vec{w}\|$
       angle $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$
- **Links:**
    1. [Quiz 10 Answers and Submission for Angle and Distance between vectors]

√ Orthogonal projection onto a subspace (shorter Gram-Schmidt Alg.)

- **Definition:**
    1. Orthogonal *projection* onto a subspace W - is the closest vector in W to a given vector. Computed in terms of an orthogonal/orthonormal basis.
    2. Projection of *v onto w* below:

$$proj_w(v) = \Sigma_{j=1}\left(\frac{v \cdot w_j}{w_j \cdot w_j}\right) \cdot w_j$$

    3. Definition of **Orthogonality**:
        - if two vectors dotted together equal zero (0);
        - same is true for $v^T w$ dotted equal zero (0). This means that the two vectors meet at a *right angle*;
        - If v_i through v_j any v_i or v_j dotted equal zero are **pairwise orthogonal**
    4. Definition of **Orthonormal**
        - If v_i through v_j any v_i or v_j dotted equal zero AND v_i is a *unit vector*.
- **Example:**



    1.
- **Links:**
    1. Quiz 11 (Q1) Answers and Submission for *Orthogonal Projection*
    2. Page 16 Lecture Notes

√ Gram-Schmidt algorithm

- **Definition:**

1. $$\vec{V}_1 = \vec{W}_1, \quad \vec{V}_2 = \vec{W}_2 - \left(\frac{\vec{W}_2 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1}\right)\vec{V}_1$$
$$\vec{V}_3 = \vec{W}_3 - \left(\frac{\vec{W}_3 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1}\right)\vec{V}_1 - \left(\frac{\vec{W}_3 \cdot \vec{V}_2}{\vec{V}_2 \cdot \vec{V}_2}\right)\vec{V}_2$$

- **Example:**

Example Find an orthogonal basis for W,

The subspace of $\mathbb{R}^4$ spanned by

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Gram-Schmidt! $\vec{v}_1 = \vec{w}_1$

$$\vec{v}_2 = \vec{w}_2 - \left(\frac{w_2 \cdot v_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{-2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

note this is orthogonal to $\vec{v}_1 = \vec{w}_1$.

$$\vec{v}_3 = \vec{w}_3 - \left(\frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{-1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ -1 \end{bmatrix}$$

This is orth. to $\vec{v}_1, \vec{v}_2$.

- **Links:**

1. Page 12 Lecture Notes
2. Quiz 10 (Q2) Answers and Submission for Gramm-Schmidt Algorithm

√ Orthogonal complements

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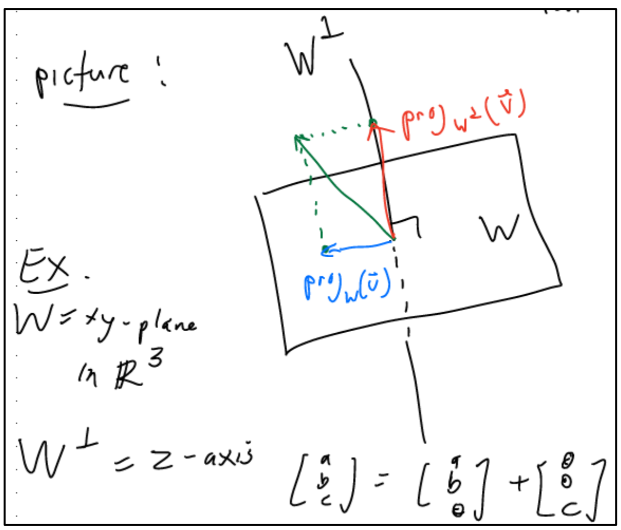
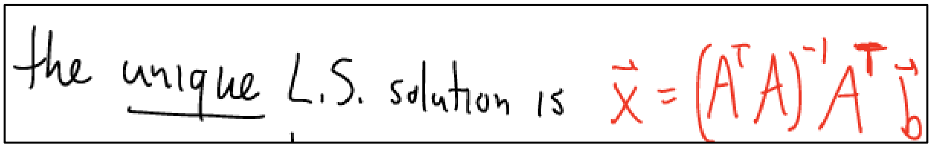
- **Definition:**
  1. **Formal Definition:**
     - $W^{\perp}$ is the subspace of all vectors that are orthogonal to W.
     - Every vector **v** in R^n can be decomposed into the sum of its projection to W and its projection to $W^{\perp}$
  2. **Simple Definition**
     - The orthonormal basis for $W^{\perp}$ is the *orthogonal complement*
     - if $\mathrm{v} \cdot \mathrm{w} = 0$, and w is a subspace that contains v, v is orthogonal to w.
     - $W^{\perp}$ is all of the vectors that are orthogonal to w, which is itself a *subspace*
- **Example:**
  1. 

- **Links:**
  1. Page 13 Lecture Notes
  2. Quiz 11 (Q2) Answers for computing the Orthogonal Complement

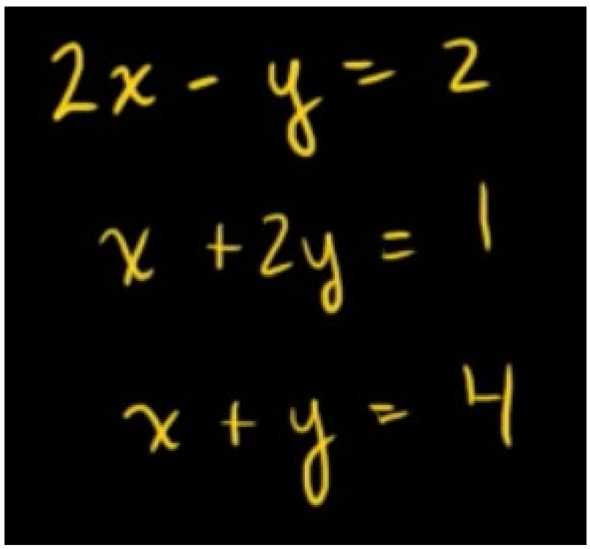
√ Least Squares

- **Definition:**
  1. If $A\mathbf{x} = \mathbf{b}$ is an *inconsistent system*, solving $A^T A\mathbf{x} = A^T\mathbf{b}$ minimizes the length through $|| A\mathbf{x} - \mathbf{b} ||$.
  2. "Sum of Squares"

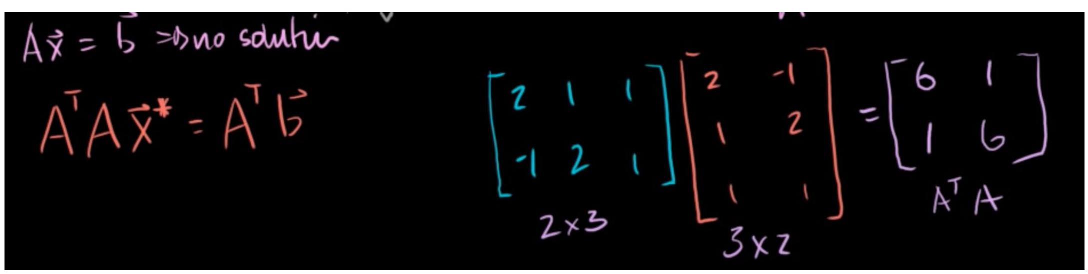


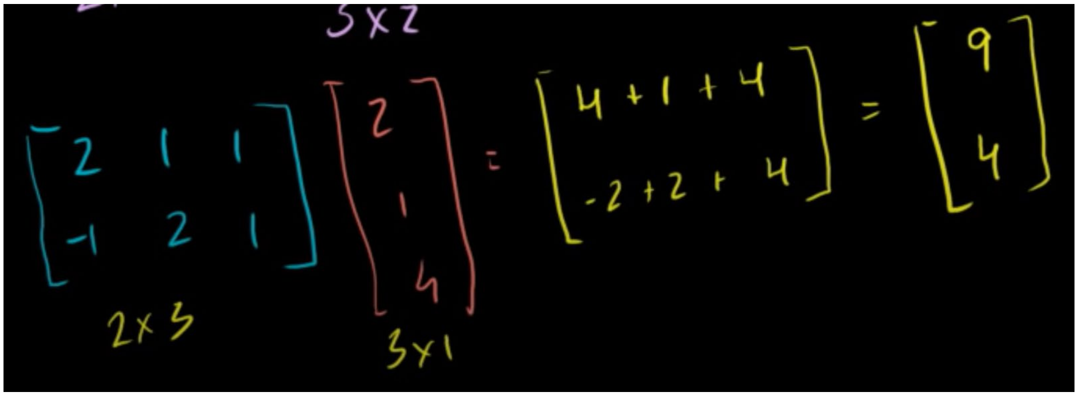
This matrix represents a projection onto the column space of A

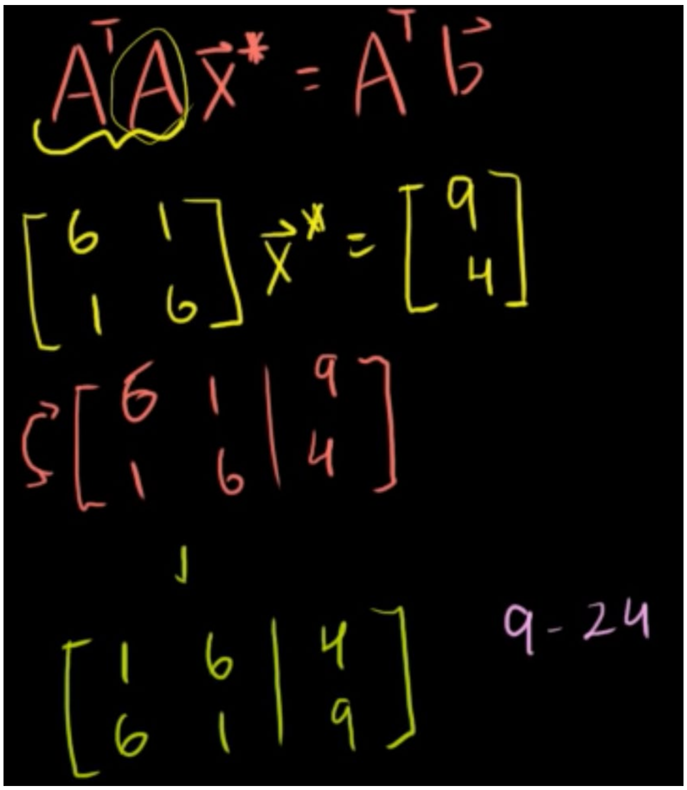
- **Example:**
  1. Equations

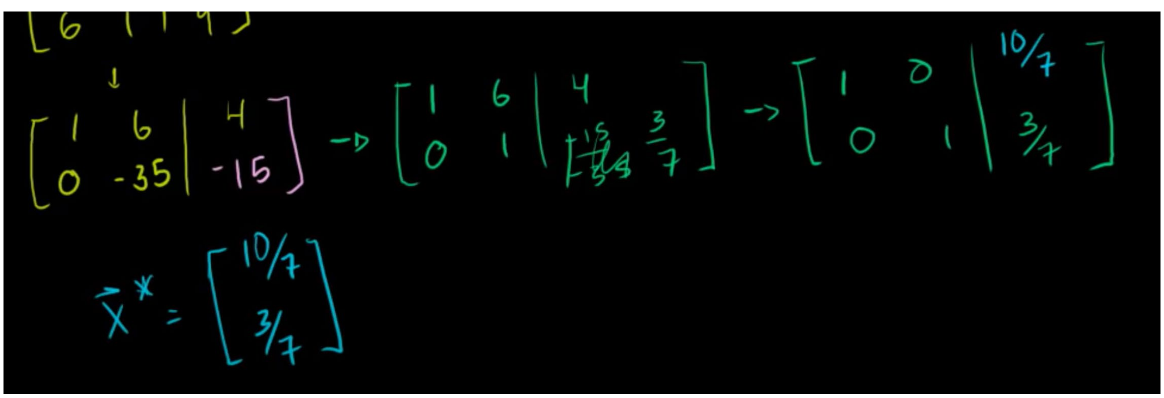


  2. Least Squares



- 

3. Find Solution
   - 

   - 

- **Links:**
  1. [Page 31 Lecture Notes]()
  2. [Least Squares Khan Academy – Minute 5:00]()

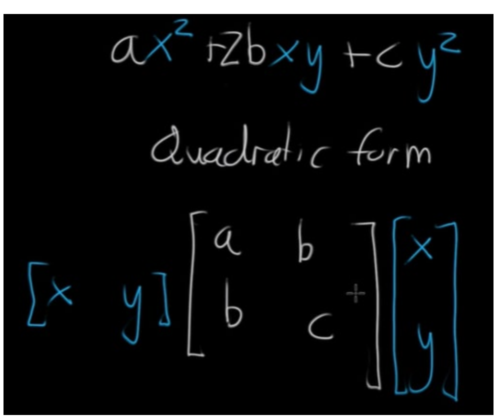
√ QR Factorization

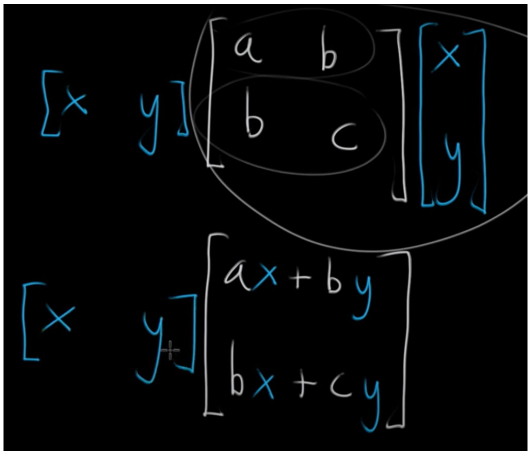
- **Definition:**
    1. Want least squares solution having orthonormal columns
- **Links:**
    1. Page 51 Lecture Notes
    2. Quiz 12 (Q1) Answers for QR Factorization Computation

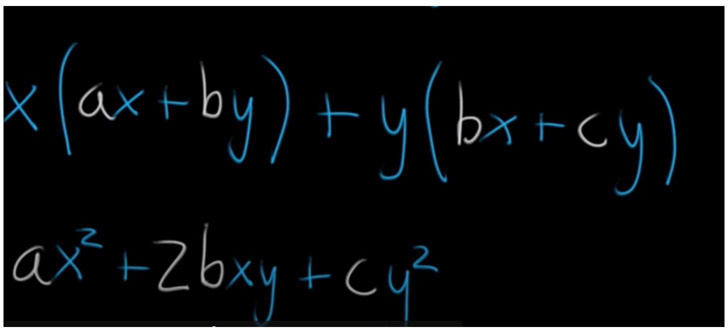
√ Spectral Decomposition: $P^T AP$

- **Definition:**
- If $A = A^T$, then A is diagonalizable and has real eigenvalues. The eigenspaces corresponding to *distinct* eigenvalues are automatically orthogonal. We can find a matrix P so that $P^T = P^{-1}$ and $P^T AP$ is diagonal. Moreover, if $\{v_1, \ldots, v_n\}$ form an orthonormal basis for $\mathbb{R}^n$ composed of eigenvectors of A, then $A = \sum_{i=1}^{n} \lambda_i(v_i v_i^T)$; this is a spectral decomposition of A
- **Links:**
    1. Quiz 12 (Q2) Answers for Spectral Decomposition Computation

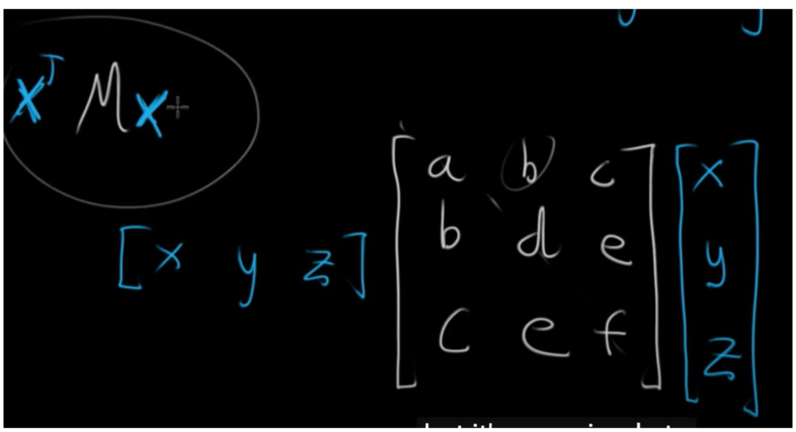
√ Quadratic Forms

- Quadratic forms = functions of the form $q(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$ for some symmetric matrix A. Positive definite/semidefinite quadratic forms (or symmetric matrices) = those associated to A with all eigenvalues positive/nonnegative. Quadrics = equations of the form $q(\mathbf{v}) = c$ for some real number c. These can be studied effectively by diagonalizing A.
- **Definition:**
    1. Form Matrix:
        - $A_{ij} = \frac{b_{ij}+b_{ji}}{2}$
    2. Every variable in the equation is a quadratic expression, no constants etc
- **Example:**
    1. Notation of Quadratic Forms:
        - Setup
        - 

- Multiply right hand two matrices
- 

- Multiply left hand two
- 

- **Key:** You end up with the ***same equation that you started with***

2. Example 3 x 3 Matrix
   - 

3. In Class Example

- Exercise

  In general, let $a_{ij} = \frac{b_{ij} + b_{ji}}{2}$ and set

  $$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ then $A = A^T$ and

  $$\boxed{q(\vec{v}) = \vec{v}^T A \vec{v}}$$

  Ex $q(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 - 5 x_1 x_3 =$

  if $A = \begin{bmatrix} 0 & 1/2 & -5/2 \\ 1/2 & 0 & 1/2 \\ -5/2 & 1/2 & 0 \end{bmatrix}$ $\quad [x_1\ x_2\ x_3] A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- **Links:**
  1. Khan Academy – Keys to Expressing quadratic form via Matrix Notation
  2. Eigenvalue and eigenvector calculator

√ Singular Value Decomposition (SVD)

- **Definition:**

    1. Singular Value Decompositions: if A is an $m \ x \ n$ matrix, then $A = V \Sigma U^T$ where U is an nxn matrix with orthonormal columns which are eigenvectors of $A^T A$. The nonzero entries in Σ occur along its "diagonal" and are the square roots of the eigenvalues of $A^T A$; these are the (nonzero) singular values of A. The first $r = \dim C(A)$ columns of V are determined by the conditions $A\mathbf{u}_i = \sigma_i \mathbf{v}_i$. The remaining $m - r$ columns are any orthonormal basis of $N(A^T)$.

    2. $\boxed{\text{Formula for SVD: } A = V \Sigma U^T}$

- **Example:**

    1. Goal: Compute $A = V \Sigma U^T$

    2. Steps (See Pseudoinverses/In Class Examples)

        - **Solve for "U"**

            i. $A^T A$

            ii. Get Eigenvalues

            iii. Get Eigenvectors

            iv. Put eigenvectors in matrix form [v_1, v_2, v_3. . .v_k] and multiply by $\frac{1}{\sqrt{contents^2}}$

        - **Solve for "Σ"**

            i. Put square root of eigenvalues in RREF form, e.g. have the eigenvalues be the leading ones of identity

                - $\Sigma = \begin{bmatrix} \sqrt{eigan_1} & 0 \\ 1 & \sqrt{eigan_2} \end{bmatrix}$

        - **Solve for "V"**

            i. Eigenvector 1, the rest of the matrix are 0's because they will be killed off anyways

- **Links:**

    1. In class Example – SVD and Pseudoinverse – start at beginning of video

    2. Quiz 13 Solutions

    3. Also see video lecture (*not posted yet*) for Quiz 13 Solutions

√ Pseudoinverses

- **Definition:**
  1. Pseudoinverses: if $A = V\Sigma U^T$ is an SVD of A, then its pseudoinverse has SVD $\tilde{A} = U\tilde{\Sigma}V^T$, where $\tilde{\Sigma}$ is the transpose of Σ with each (positive) σ_i replaced by $1/\sigma_i$.
  2. Formula of Pseudoinverse: $\tilde{A} = U\tilde{\Sigma}V^T$
- **Uses:**
  1. Compute a "best fit" (least squares) solution to a system of linear equations that *lacks a unique solution*
- **Example:**
  1. **In Class Examples**

Ex Compute pseudoinverse of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

1st Need SVD of A.

(Note if columns or rows of A are lin. ind., then there's a much faster way to find $\tilde{A}$).

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$$

$\lambda = 15$ $\quad \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}}$

$\lambda = 0$ $\quad \vec{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\frac{1}{\sqrt{5}}$ $\qquad U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$$A\vec{u}_1 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}\frac{1}{\sqrt{5}} = \frac{\sqrt{75}}{\sqrt{5}}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}\frac{1}{\sqrt{75}} \qquad \Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \sqrt{15}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}\frac{1}{\sqrt{75}} = \sigma_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}} = \sigma_1\vec{v}_1 \qquad A\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ where $\vec{v}_2, \vec{v}_3$ are orthogonal unit vectors, orthogonal to $\vec{v}_1$.

(won't actually matter to $\tilde{A}$, as we'll see).

So $A = V\Sigma U^T$, in fact $A = \underbrace{\frac{1}{\sqrt{75}}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}}_{\vec{v}_1}\underbrace{\sqrt{15}}_{\sigma_1}\cdot\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}\frac{1}{\sqrt{5}}}_{u_1^T}$

and $\tilde{A} = U\tilde{\Sigma}V^T$ where $\tilde{\Sigma} = \begin{bmatrix} 1/\sqrt{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ & \vec{v}_2^T & \\ & \vec{v}_3^T & \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{45} & 1/\sqrt{45} & 1/\sqrt{45} \\ 0 & 0 & 0 \end{bmatrix} \qquad 5\cdot 45 = 15^2$$

$$= \boxed{\begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{2}{15} \end{bmatrix}} = \frac{1}{15}A^T$$

($\tilde{A}$ is not always a multiple of A^T)

Note $A\tilde{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\frac{1}{15} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\frac{1}{3} = \text{proj}_{C(A)}$

i.e. if $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, the closest vector to it in $C(A)$ is $A\tilde{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}(x+y+z)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- **Links:**
  1. [In class Example – SVD and Pseudoinverse – start at beginning of video]
  2. [Quiz 13 Solutions]
  3. Also see video lecture (*not posted yet*) for Quiz 13 Solutions
  4. [Pseudoinverse Calculator]

√ Generalized Eigenvectors

- **Definition:**
  1. Take the determinant of a matrix to get the (A – lambda)
  2. The result from raising to exponent ***m*** is equal to 0
  3. However, the result from raising to exponent ***m - 1*** is ***not*** equal to 0
- **Uses:**
  1. Calculating exponential matrices
- **Example:**
  1.
- **Links:**
  1. Page 115 Lecture Notes
  2. Quiz 14 Answers (Page 147 Lecture Notes)

√ Exponential Matrices

- **Definition:**
  1. e^A
- **Uses:**
  1. Infinite series, like geometric series [1/(1 + r)]
  2. Differential Equations
- **Example:**
  1. Basic Examples (Page 121 Lecture Notes)
- **Links:**
  1. Page 119 Lecture Notes

√ Jordan Canonical Form (aka Jordan Normal Form)

- **Definition:**
  1. The Jordan form of a matrix generalizes diagonalization and If A is not diagonalizable "As Jordan form is "as close as possible" to being diagonal.
  2. The idea will be to find a good basis of generalized eigenvector so if p is the matrix with them as columns then $P^{-1}AP$ will be the Jordan form.
- **Uses:**
  1. Page 143 Lecture Notes
- **Example:**
  1. Compute Jordan Form without computing (knowing) P (Page 149 Lecture Notes)
- **Links:**
  1. Page 135 Lecture Notes

Relevant/Useful/Important Theorems:

Diagonalizable Properties

- If A is a square matrix with n distinct eigenvalues, it is diagonalizable (the converse is false).

Eigenvectors and Linear Independence

- The eigenvectors of a matrix A associated to different eigenvalues are automatically linearly independent (and if A is symmetric then they are orthogonal).

Characteristic Polynomial and Similar Matrices

- if A and B are similar matrices (meaning $B = P^{-1}AP$ for some nonsingular matrix P), then they have the same characteristic polynomial (the converse is false).