Duality and Post-Optimal Analysis

Partially based on: Taha, H. A. 2017. Operations Research: An Introduction. 10th Edition. Boston, MA: Pearson
Gurobi Documentation

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Sensitivity analysis

– Suppose you have the problem

$$\begin{aligned}
&\text{Maximize} \quad z = \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\qquad\qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
&\qquad\qquad\ \ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}$$

– There are some sensitivity analyses of interest:
  - Changes/perturbations in $\boldsymbol{c}$: vector of costs
  - Changes/perturbations in $\boldsymbol{A}$: matrix of coefficients
  - Changes/perturbations in $\boldsymbol{b}$: right-hand side vector

Example 3 (from Simplex slides)

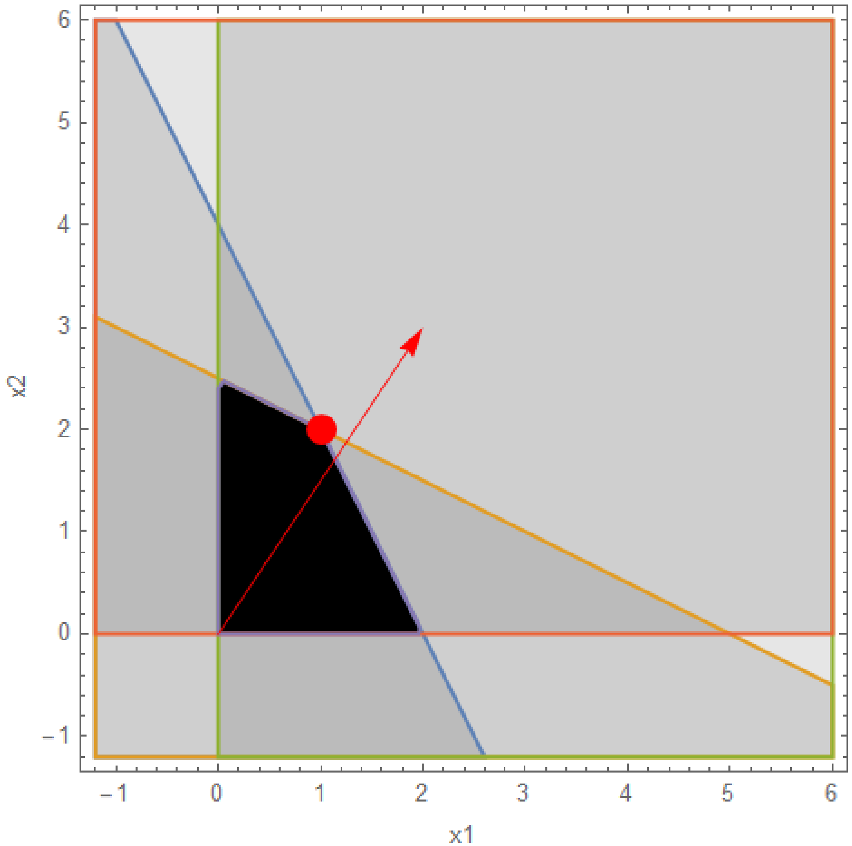
- Consider the following LP with two variables

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

- Imagine that the right-hand side coefficients represent limited resources. How much "should I pay" for "one extra unit" of each resource?



The gradient is: (2,3)
The optimum solution is: $\{x_1 \to 1, x_2 \to 2\}$
The optimal value of the objective function is: 8

Example 3 (from Simplex slides)

Maximize $z = 2x_1 + 3x_2$
subject to

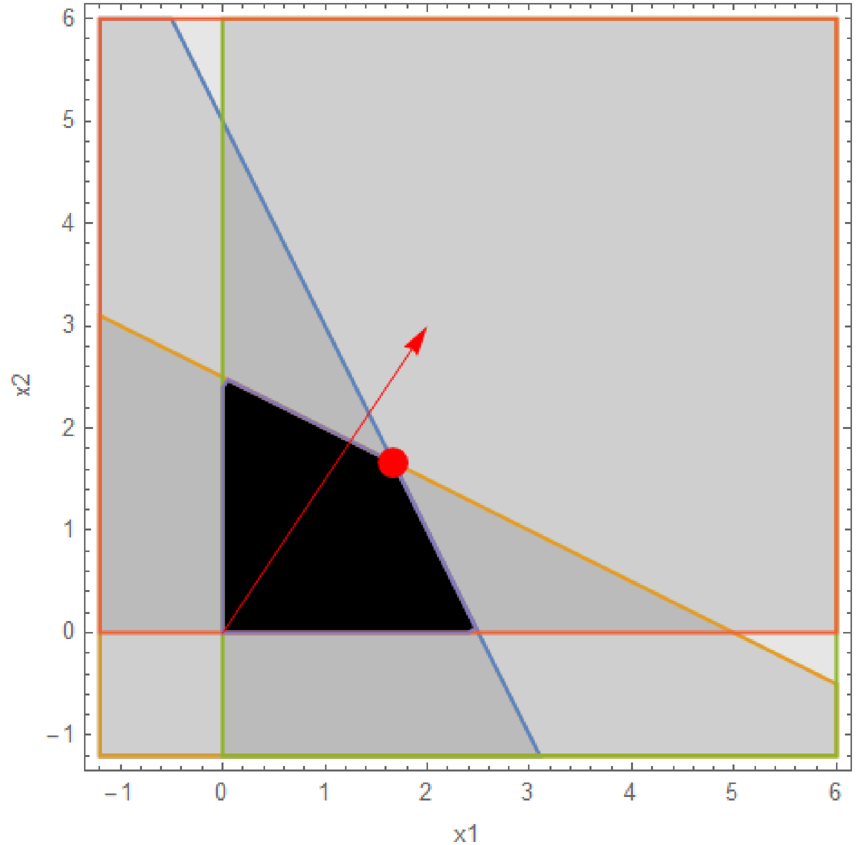
$$2x_1 + x_2 \leq 4 + \mathbf{1}$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

- If we increase in one unit (i.e., from 4 to 5) the "first resource" (right-hand side of the first constraint) our optimal objective value from 8 to $25/3$
  - Then, the associated rate of revenue change is:

$$\frac{\left(\frac{25}{3} - 8\right)}{5 - 4} = \frac{\left(\frac{1}{3}\right)}{1} = 1/3$$

This mean that you "should not pay" more than 1/3 for an additional unit of "resource 1"

  - This value is often called **the dual (or shadow) price** associated with that resource/constraint



The gradient is: (2,3)

The optimum solution is: $\{x_1 \rightarrow \frac{5}{3}, x_2 \rightarrow \frac{5}{3}\}$

The optimal value of the objective function is: $\frac{25}{3}$

Example 3 (from Simplex slides)

Maximize $z = 2x_1 + 3x_2$
subject to

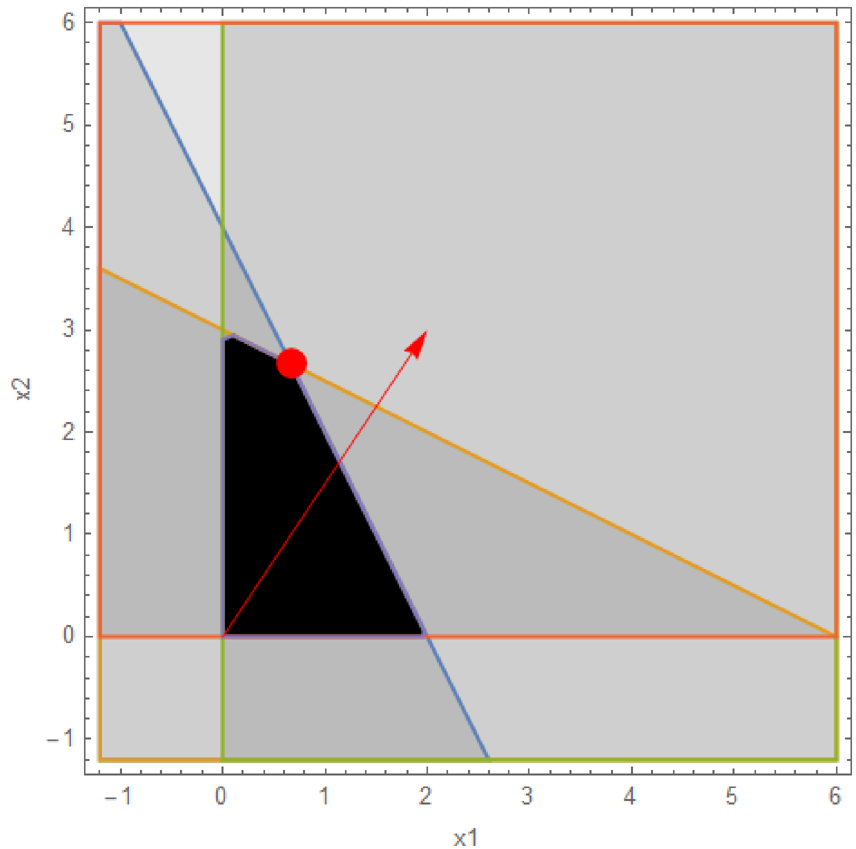
$2x_1 + x_2 \leq 4$
$x_1 + 2x_2 \leq 5\ +\mathbf{1}$
$x_1, x_2 \geq 0$

- If we increase in one unit (i.e., from 5 to 6) the "first resource" (right-hand side of the first constraint) our optimal objective value from 8 to $28/3$
  - Then, the associated rate of revenue change is:

$$\frac{\left(\frac{28}{3} - 8\right)}{6-5} = \frac{\left(\frac{4}{3}\right)}{1} = 4/3$$

This mean that you "should not pay" more than $4/3$ for an additional unit of "resource 2"

  - This value is often called **the dual (or shadow) price** associated with that resource/constraint



The gradient is: (2,3)
The optimum solution is: $\{x_1 \to \frac{2}{3}, x_2 \to \frac{8}{3}\}$
The optimal value of the objective function is: $\frac{28}{3}$

Example 3 (from Simplex slides)

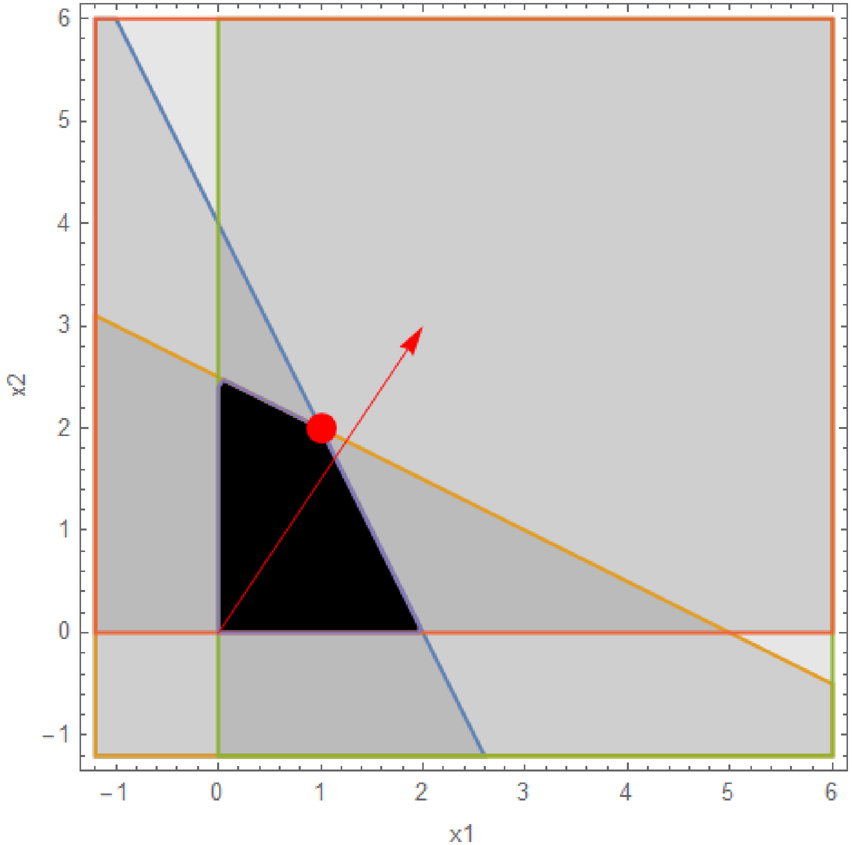
- We can make it general by adding a "dual" variable to each constraint

Maximize $z = 2x_1 + 3x_2 + f(w_1, w_2)$

subject to

$$
\begin{aligned}
2x_1 + x_2 &\leq 4 + w_1 \\
x_1 + 2x_2 &\leq 5 \qquad + w_2 \\
x_1, x_2, w_1, w_2 &\geq 0
\end{aligned}
$$

- How did we determine the "nature" (e.g., non-negativity, non-positivity) of the dual variables?



The gradient is: (2,3)
The optimum solution is: $\{x_1 \to 1, x_2 \to 2\}$
The optimal value of the objective function is: 8

Adding the dual variables…

Basic	z	x_1	x_2	s_1	s_2	Solution
z	1	-2	-3	0	0	0
s_1	0	2	1	1	0	4
s_2	0	1	2	0	1	5

Basic	z	x_1	x_2	s_1	s_2	Solution	w_1	w_2
z	1	-2	-3	0	0	0	0	0
s_1	0	2	1	1	0	4	1	0
s_2	0	1	2	0	1	5	0	1

Adding the dual variables…

Basic	z	x_1	x_2	s_1	s_2	Solution
z	1	-2	-3	0	0	0
s_1	0	2	1	1	0	4
s_2	0	1	2	0	1	5

Basic	z	x_1	x_2	s_1	s_2	Solution	w_1	w_2
z	1	-2	-3	0	0	0	0	0
s_1	0	2	1	1	0	4	1	0
s_2	0	1	2	0	1	5	0	1

How about other Simplex iterations?

Basic	z	x_1	x_2	s_1	s_2	Solution
z	1	0	0	1/3	4/3	8
x_1	0	1	0	2/3	-1/3	1
x_2	0	0	1	-1/3	2/3	2

Basic	z	x_1	x_2	s_1	s_2	Solution	w_1	w_2
z	1	0	0	1/3	4/3	8	1/3	4/3
x_1	0	1	0	2/3	-1/3	1	2/3	-1/3
x_2	0	0	1	-1/3	2/3	2	-1/3	2/3

How about other Simplex iterations?

Basic	z	x_1	x_2	s_1	s_2	Solution
z	1	0	0	1/3	4/3	8
x_1	0	1	0	2/3	-1/3	1
x_2	0	0	1	-1/3	2/3	2

↓

Basic	z	x_1	x_2	s_1	s_2	Solution	w_1	w_2
z	1	0	0	1/3	4/3	8	1/3	4/3
x_1	0	1	0	2/3	-1/3	1	2/3	-1/3
x_2	0	0	1	-1/3	2/3	2	-1/3	2/3

Duality

Primal problem

Maximize $\boldsymbol{c}^T\boldsymbol{x}$
s.t.
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual problem

Minimize $\boldsymbol{b}^T\boldsymbol{x}$
s.t.
$$\boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{c}$$
$$\boldsymbol{w} \geq \boldsymbol{0}$$

Maximize $\boldsymbol{c}^T\boldsymbol{x}$
s.t.
$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Minimize $\boldsymbol{b}^T\boldsymbol{x}$
s.t.
$$\boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c}$$
$\boldsymbol{w}$ **is unrestricted**

Duality

Primal problem

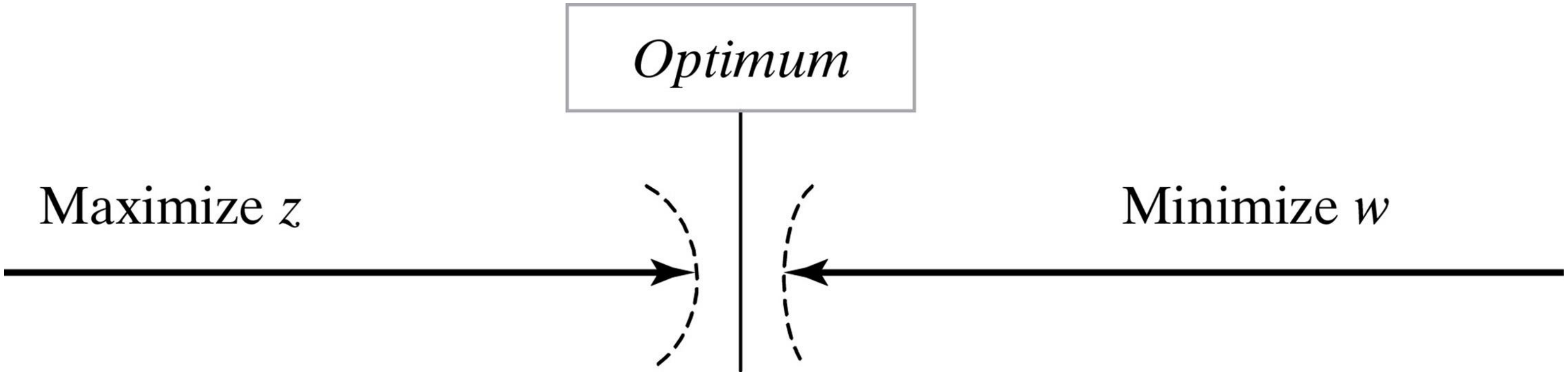
Maximize $\boldsymbol{c}^T\boldsymbol{x}$
subject to

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual problem

Minimize $\boldsymbol{b}^T\boldsymbol{x}$
subject to

$$\boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{c}$$
$$\boldsymbol{w} \geq \boldsymbol{0}$$



General rules for constructing the dual problem

Maximization problem		Minimization problem
Constraints		***Variables***
$\geq$	$\Leftrightarrow$	$\leq\ 0$
$\leq$	$\Leftrightarrow$	$\geq\ 0$
$=$	$\Leftrightarrow$	Unrestricted
Variables		***Constraints***
$\geq\ 0$	$\Leftrightarrow$	$\geq$
$\leq\ 0$	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Example

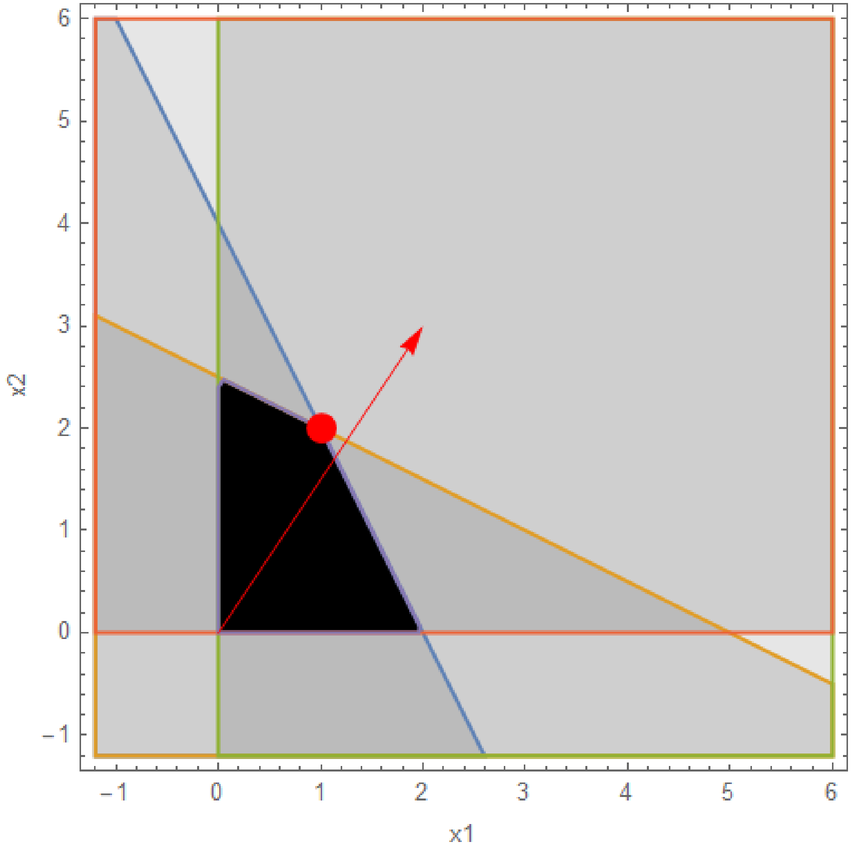
- Consider the following LP with two variables

Maximize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

- What is the corresponding Dual Problem?



The gradient is: (2,3)
The optimum solution is: $\{x_1 \to 1, x_2 \to 2\}$
The optimal value of the objective function is: 8

Solution

Maximize $z = 2x_1 + 3x_2$

subject to

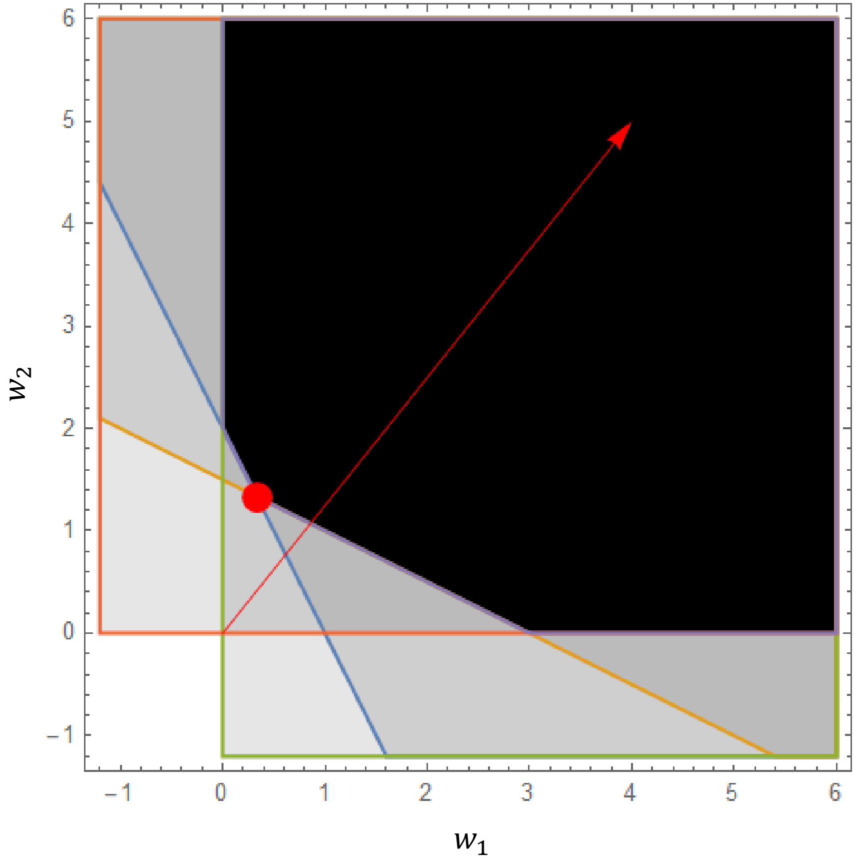
$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

$\Updownarrow$

Minimize $4w_1 + 5w_2$

subject to

$$2w_1 + w_2 \geq 2$$
$$w_1 + 2w_2 \geq 3$$
$$w_1, w_2 \geq 0$$



The gradient is: (4,5)

The optimum solution is: $\{w_1 \to 1/3 \; w_2 \to 4/3\}$

The optimal value of the objective function is: 8

Post-Optimal Analysis

- Some of the considerations if there are changes to the model after it has been solved:
  - Changes affecting feasibility
    - Changes in the right-hand side
    - Addition of a new constraint
  - Changes affecting optimality
    - Changes in the objective function coefficients
    - Addition of a new activity

In-class Exercise

Find the Dual problem associated with the "Reddy Mikks" paint production problem (Example 2.1-1) and solve it. What was the solution? How does it connect with the primal problem?

Maximize $z = 5x_1 + 4x_2$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$