# Module 4. LP models for common structures

Including (but not limited to):

- Understanding common problems and their LP models/structures
  - Data Envelopment Analysis (DEA)
    - DMUs, inputs, outputs
    - Efficiency, reference set, etc.
    - Generalized network models
      - Transportation problem
      - Assignment problem
      - Minimum cost flow problem
        - » Connection with transportation problem, assignment problem, etc.
        - » Shortest path problem
        - » Max flow problem
- Knowing how to adapt/modify/extend common structures to fit particular contexts
  - Handle unbalanced networks (total demand is different from total supply capacity)
  - Handle possible unmet demand or excess production
  - Adding capacities to nodes (e.g., maximum flow that can go through some nodes)
  - Adding/adapting flow constraints for different types of capacities (number of units, weight, size, etc.)

# Data Envelopment Analysis (DEA) Efficiency Model

- Sets:

$N$ = set of DMUs

$I$ = set of *inputs*

$O$ = set of *outputs*

- Parameters:

$y_{oj}$ = amount of output $o \in O$ produced by DMU $j \in N$

$x_{ij}$ = amount of input $i \in I$ produced by DMU $j \in N$

$p$ = Individual DMU under study

- Decision variables:

$\lambda_j$ = fraction of the $j$th DMU ($j \in N$) used to achieve efficiency (i.e., get to the efficiency frontier)

$\phi_p$ = Proportional increase (growth rate) of DMU under study ($p$)

- Objective function:

$$\max \ \phi_p$$

- Constraints:

$$\sum_{j \in N} y_{oj}\lambda_j \geq y_{op}\phi_p \qquad , \forall o \in O$$

Sum of output * weights must be more than the person of comparison

$$\sum_{j \in N} x_{ij}\lambda_j \geq x_{io} \qquad , \forall i \in I$$

Sum of inputs * weights for others in sample must be less than the person of comparison? Check notes

$$\lambda_j \geq 0 \qquad , \forall j \in N$$

$$\phi_p \ \ unrestricted$$

# Transportation Problem

- Sets:

$S$ = set of *sources* $\{1,2,\ldots,m\}$

$D$ = set of *destinations* $\{1,2,\ldots,n\}$

- Parameters:

$a_i$ = supply of source node $i \in S$

$b_j$ = demand of destination node $j \in D$

$c_{ij}$ = unit cost of flow through arc $(i,j)$, where $i \in S$ and $j \in D$

- Decision variables:

$x_{ij}$ = flow through arc $(i,j)$, where $i \in S$ and $j \in D$

- Objective function:

$$\min \mathrm{z} = \sum_{i \in S} \sum_{j \in D} c_{ij} x_{ij}$$

Flows meet the supply or demand

Change to <= if unmet demand

- Constraints:

$$\sum_{i \in S} x_{ij} = b_j \qquad ,\forall j \in D$$

flow equals demand

$$\sum_{j \in D} x_{ij} = a_i \qquad ,\forall i \in S$$

flow equals supply

$$x_{ij} \geq 0 \qquad ,\forall i \in S, \forall j \in D$$

# Assignment Problem

- Sets:

$W$: Set of workers $\{1,2,3,\dots,n\}$

$J$: Set of jobs $\{1,2,3,\dots,n\}$

- Parameters:

$c_{ij}$: cost of assigning worker $i \in W$ to job $j \in J$

- Variables:

$x_{ij}$: binary variable that is 1 if worker $i \in W$ is assigned to job $j \in J$, and is 0 otherwise.

- Objective function:

$$\min z = \sum_{i \in W} \sum_{j \in J} c_{ij} x_{ij}$$

- Constraints:

$$\sum_{i \in W} x_{ij} = 1 \qquad , \forall j \in J$$

Each worker must be assigned a job

$$\sum_{j \in J} x_{ij} = 1 \qquad , \forall i \in W$$

$$x_{ij} \in \{0,1\} \qquad , \forall i \in W, \ \forall j \in J$$

# Minimum Cost Flow Problem (MCFP)

- Sets:

$N$: Set of nodes $\{1,2,3,\dots,n\}$

$A$: Set of arcs

- Parameters:

$c_{ij}$: unit cost of sending a commodity through arc $(i,j) \in A$

$b_i$: demand/supply of commodities in node $i \in N$

$u_{ij}$: maximum flow through arc $(i,j) \in A$

$l_{ij}$: minimum flow through arc $(i,j) \in A$

- Variables:

$x_{ij}$: flow through arc $(i,j) \in A$

- Objective function:

$$\min z = \sum_{(i,j) \in A} c_{ij} x_{ij}$$

- Constraints:

$$\sum_{j:(i,j)\epsilon A} x_{ij} - \sum_{j:(j,i)\epsilon A} x_{ji} = b_i \qquad , \forall i \in N$$

$$x_{ij} \leq u_{ij} \qquad , \forall (i,j) \in A$$

Within upper bound

$$x_{ij} \geq l_{ij} \qquad , \forall (i,j) \in A$$

Within Lower bound

Flows meet the supply or demand

Change to <= if unmet demand

# Multicommodity Minimum Cost Flow Problem (MMCFP)

- Sets:

$N$: Set of nodes $\{1,2,3,\dots,n\}$

$A$: Set of arcs

$K$: Set of commodities

- Parameters:

$c_{ijk}$: unit cost of sending a commodity $k \in K$ through arc $(i,j) \in A$

$b_{ik}$: demand/supply of commodity $k \in K$ in node $i \in N$

$u_{ij}$: maximum total flow through arc $(i,j) \in A$

$l_{ij}$: minimum total flow through arc $(i,j) \in A$

- Variables:

$x_{ijl}$: flow of commodity $l \in L$ through arc $(i,j) \in A$

- Objective function:

$$\min z = \sum_{k \in K} \sum_{(i,j) \in A} c_{ijk} x_{ijk}$$

- Constraints:

Flows meet the supply / Demand (change <= for unmet demand)

$$\sum_{j:(i,j) \epsilon A} x_{ijk} - \sum_{j:(j,i) \epsilon A} x_{jik} = b_{ik} \quad , \forall i \in N, \forall k \in K$$

$$\sum_{k \in K} x_{ijk} \leq u_{ij} \quad , \forall (i,j) \in A$$

Within upper bound

$$\sum_{k \in K} x_{ijk} \geq l_{ij} \quad , \forall (i,j) \in A$$

Within Lower bound