Advanced Linear Programming

Partially based on: Taha, H. A. 2017. Operations Research: An Introduction. 10th Edition. Boston, MA: Pearson & Gurobi Documentation

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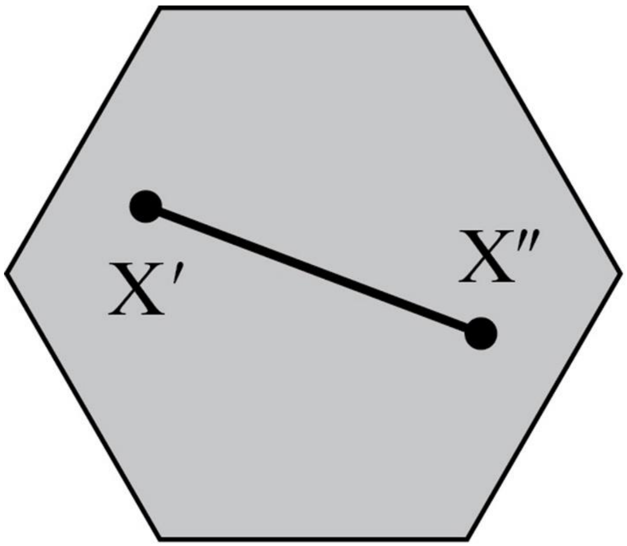
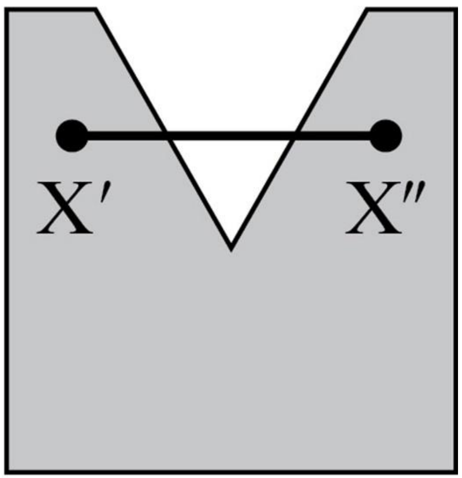
Convex sets and extreme points

- A solution space forms a convex set if the line segment joining any two distinct feasible points also falls in the set.
- An extreme point of a convex set is a feasible point that cannot lie on a line segment joining any two distinct feasible points in the set



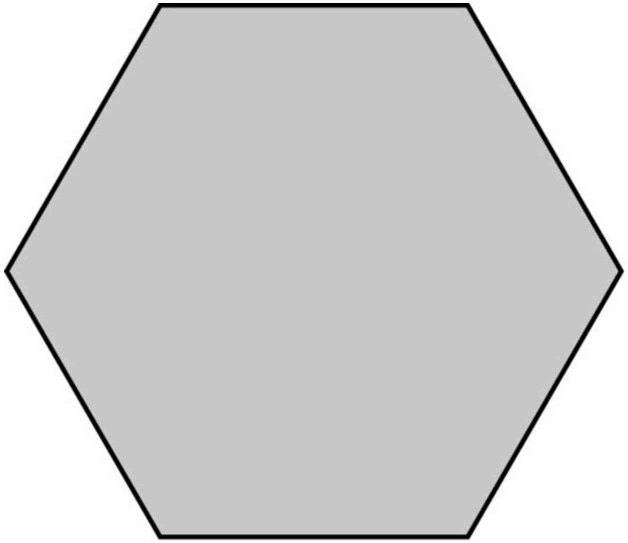
(a)



(b)

Convex sets and extreme points

- Every element in a convex set can be determined as a function of its extreme points



- The set of feasible solutions of an LP model is convex

From extreme points to Basic solutions

$$\min \boldsymbol{c}^T \boldsymbol{x}$$

s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

- Where $\boldsymbol{x}$ and $\boldsymbol{c}$ are vectors of size n
- $\boldsymbol{b}$ is a vector of size m
- $\boldsymbol{A}$ is a matrix of size $m \times n$

A basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is determined by setting $n - m$ variables equal to zero, and the solving the resulting m equations in the remaining m unknowns (provided that the resulting solution is unique)

From extreme points to Basic solutions

$$\min \boldsymbol{c}^T\boldsymbol{x}$$

s.t. $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

Note that $\boldsymbol{Ax} = \boldsymbol{b}$ can be written in vector form as:

$$\sum_{j=1}^{n} \boldsymbol{P}_j x_j = b$$

The vector P_j is the jth column of matrix A.

A subset of m vectors forms a basis, $\boldsymbol{B}$, if, and only if, the selected m vectors are linearly independent. In this case, $\boldsymbol{B}$ is nonsingular. Defining X_B as a vector of size m of the basic variables, then

$$\boldsymbol{Bx_B} = \boldsymbol{b}$$

Thus, $\boldsymbol{B^{-1}Bx_B} = \boldsymbol{B^{-1}b} = \boldsymbol{x_B}$

Then, if $\boldsymbol{B^{-1}b} \geq \boldsymbol{0}$, then $\boldsymbol{x_B}$ is feasible

Example 7.1-2

- Determine all the basic (feasible and infeasible) solutions of the following system of equations

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Generalized Simplex Tableau in Matrix Form

Consider the problem

$$\max \boldsymbol{c}^T \boldsymbol{x}$$

s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

Equivalently, the problem can be written as

$$\begin{pmatrix} 1 & -\boldsymbol{c}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} z \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix}$$

Suppose that $\boldsymbol{B}$ is a feasible basis of the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$, and let $\boldsymbol{x_B}$ be the corresponding vector of basic variables and $\boldsymbol{c_B}$ its associated objective vector. Given all the nonbasic variables are zero, the solution is then computed as

$$\begin{pmatrix} z \\ \boldsymbol{x_B} \end{pmatrix} = \begin{pmatrix} 1 & -\boldsymbol{c}_B^T \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \\ \boldsymbol{0} & \boldsymbol{B}^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b} \\ \boldsymbol{B}^{-1} \boldsymbol{b} \end{pmatrix}$$

Generalized Simplex Tableau in Matrix Form

The complete simplex tableau in matrix form can be derived from the original equations

$$\begin{pmatrix} 1 & \boldsymbol{c}_B^T\boldsymbol{B}^{-1} \\ \mathbf{0} & \boldsymbol{B}^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{c}_B^T\boldsymbol{B}^{-1} \\ \mathbf{0} & \boldsymbol{B}^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\boldsymbol{c}_B^T \\ \mathbf{0} & \boldsymbol{A} \end{pmatrix}\begin{pmatrix} z \\ \boldsymbol{x} \end{pmatrix}$$

Matrix manipulations then yield the following equations

$$\begin{pmatrix} 1 & \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A} - \boldsymbol{c}^T \\ \mathbf{0} & \boldsymbol{B}^{-1}\boldsymbol{A} \end{pmatrix}\begin{pmatrix} z \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_B^T\boldsymbol{B}^{-1} \\ \boldsymbol{B}^{-1}\boldsymbol{b} \end{pmatrix}$$

Basic	z	$\boldsymbol{x}$	Solution
z	1	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$
$\boldsymbol{x}_B$	$\mathbf{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Generalized Simplex Tableau in Matrix Form

Basic	z	$\boldsymbol{x}$	Solution
z	1	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}-\boldsymbol{c}^T$	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Given that jth vector P_j of A, the simplex tableau can be written as:

Basic	z	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	…	$\boldsymbol{x}_j$	…	$\boldsymbol{x}_n$	Solution
z	1	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{P}_1-\boldsymbol{c}_1$	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{P}_2-\boldsymbol{c}_2$		$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{P}_j-\boldsymbol{c}_j$		$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{P}_n-\boldsymbol{c}_n$	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1}\boldsymbol{P}_1$	$\boldsymbol{B}^{-1}\boldsymbol{P}_2$		$\boldsymbol{B}^{-1}\boldsymbol{P}_j$		$\boldsymbol{B}^{-1}\boldsymbol{P}_n$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Example 7.1-3

- Consider the following LP

$$\max z = x_1 + 4x_2 + 7x_3 + 5x_4$$

s.t.
$$2x_1 + x_2 + 2x_3 + 4x_4 = 10$$
$$3x_1 - x_2 - 2x_3 + 6x_4 = 5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Generate the simplex tableau associated with the basis $\boldsymbol{B} = (\boldsymbol{P_1}, \boldsymbol{P_2})$

Revised Simplex

- From our Tableau:

Basic	z	x_1	x_2	x_j	x_n	Solution
z	1	$\boldsymbol{c_B^T B^{-1} P_1 - c_1}$	$\boldsymbol{c_B^T B^{-1} P_2 - c_2}$	$\boldsymbol{c_B^T B^{-1} P_j - c_j}$	$\boldsymbol{c_B^T B^{-1} P_n - c_n}$	$\boldsymbol{c_B^T B^{-1} b}$
$\boldsymbol{x}_B$	$\mathbf{0}$	$\boldsymbol{B^{-1} P_1}$	$\boldsymbol{B^{-1} P_2}$	$\boldsymbol{B^{-1} P_j}$	$\boldsymbol{B^{-1} P_n}$	$\boldsymbol{B^{-1} b}$

we know that

$$\begin{pmatrix} 1 & \boldsymbol{c_B^T B^{-1} P_1 - c_1} & \boldsymbol{c_B^T B^{-1} P_2 - c_2} & \dots & \boldsymbol{c_B^T B^{-1} P_j - c_j} & \dots & \boldsymbol{c_B^T B^{-1} P_n - c_n} \\ 0 & \boldsymbol{B^{-1} P_1} & \boldsymbol{B^{-1} P_2} & \dots & \boldsymbol{B^{-1} P_j} & \dots & \boldsymbol{B^{-1} P_n} \end{pmatrix} \begin{pmatrix} z \\ \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \dots \\ \boldsymbol{x_j} \\ \dots \\ \boldsymbol{x_n} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c_B^T B^{-1} b} \\ \boldsymbol{B^{-1} b} \end{pmatrix}$$

Thus, defining $z_j = \boldsymbol{c_B^T B^{-1} P_j}$, we have the following equations:

$$z + \sum_{j=1}^{n} (z_j - c_j) x_j = \boldsymbol{c_B^T B^{-1} b}$$

$$(x_B)_i + \sum_{j=1}^{n} \left(\boldsymbol{B^{-1} P_j}\right)_i x_j = \left(\boldsymbol{B^{-1} b}\right)_{\boldsymbol{i}} \qquad \forall \boldsymbol{i} \in \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{m}\}$$

Revised Simplex

The **reduced cost** of x_j is computed as

$$z_j - c_j = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{P}_j - c_j$$

Optimality condition

From equation $z + \sum_{j=1}^{n} (z_j - c_j) x_j = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b}$ we can see that

- In the case of maximization, an increase in nonbasic x_j above its current zero value can improve the value of z (relative to its current value, $\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b}$) only if $z_j - c_j \leq 0$.
- For minimization, the condition is $z_j - c_j \geq 0$
- The entering vector is selected as the nonbasic vector $z_j - c_j$ with the most negative value, if maximizing (or most positive value, if minimizing)

Revised Simplex

Feasibility condition

Given the entering vector $\boldsymbol{P_j}$ as determined by the optimality condition, the equations $(x_B)_i + \sum_{j=1}^{n} \left(\boldsymbol{B^{-1}P_j}\right)_i x_j = \left(\boldsymbol{B^{-1}b}\right)_{\boldsymbol{i}} \quad \forall i \in \{1,2,\dots,m\}$ reduce to

$$(x_B)_i + \left(\boldsymbol{B^{-1}P_j}\right)_i x_j = \left(\boldsymbol{B^{-1}b}\right)_{\boldsymbol{i}}$$

Since all the remaining $n-1$ nonbasic variables are zero.

Then, $(\boldsymbol{x_B})_i = \left(\boldsymbol{B^{-1}b}\right)_{\boldsymbol{i}} - \left(\boldsymbol{B^{-1}P_j}\right)_i x_j$

The idea is to increase x_j as much as possible, while guaranteeing that $(\boldsymbol{x_B})_i \geq 0$. Then,

$$x_j = \min_i \left\{ \frac{\left(\boldsymbol{B^{-1}b}\right)_i}{\left(\boldsymbol{B^{-1}P_j}\right)_i} \mid \left(\boldsymbol{B^{-1}P_j}\right)_i > 0 \right\}$$

Suppose that $(\boldsymbol{x_B})_k$ is the basic variable that corresponds to the minimum ratio. Then the leaving vector is $\boldsymbol{P_k}$, and its associated (basic) variable must become nonbasic.

Revised Simplex Algorithm

- Step 0. Construct initial basic feasible solution, and let B and c_B be the associated basis and objective coefficients vector
- Step 1. Compute B^{-1}
- Step 2. For each nonbasic vector P_j compute the reduced cost

$$z_j - c_j = C_B B^{-1} P_j - c_j$$

If all reduced costs are greater than or equal to zero (less than or equal to zero) in maximization (minimization), stop. The optimal solution is $x_B = B^{-1}b$, and $z = c_B x_B$

Else, determine the entering vector P_j having the most negative (positive) reduced cost in case of maximization (minimization)

Revised Simplex Algorithm

- Step 3. Compute $B^{-1}P_j$.

If all elements of $B^{-1}P_j$ are negative or zero, stop, as the solution is unbounded.

Else, use the ratio test to determine the leaving vector P_i

- Step 4. For the next basis by replacing the leaving vector P_i with the entering vector P_j in the current basis. Go to step 1 to start a new iteration.

Duality

- Suppose you have a primal problem in equation form with m constraints and n variables, defined as

$$\max \boldsymbol{z} = \boldsymbol{c}^T \boldsymbol{x}$$

s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

- Let the vector $y = (y_1, y_2, \dots, y_m)$ define the dual variables. Then the associated dual problem is

$$\min \boldsymbol{w} = \boldsymbol{b}^T \boldsymbol{y}$$

s.t. $\boldsymbol{A}^T \boldsymbol{y} \geq \boldsymbol{c}$

$\boldsymbol{y}$ unrestricted

Weak duality theorem

- For any pair of feasible primal and dual solutions, (x, y), the value of the objective function in the minimization problem sets an upper bound on the value of the objective function in the maximization problem.

Proof

The feasible pair (x, y) satisfies all the constraints in both problems

Since $Ax = b$, then $y^T(Ax) = y^T(b) = b^T y = w$

Also, since $A^T y \geq c$ and $x \geq 0$, then $(A^T y)^T x \geq (c)^T x = z$

Then, $w \geq z$

Strong duality theorem

- Let $\boldsymbol{x}$ be the optimal solution to the primal problem, and let $\boldsymbol{B}$ and $\boldsymbol{c_B}$ be the associated basis and objective coefficients vector. Then, the optimal solution to the dual problem is $\boldsymbol{y^T} = \boldsymbol{c_B^T B^{-1}}$, and both objective function values are equal (i.e., $z = w$)

Proof

Since x is optimal, we know that $\boldsymbol{c_B^T B^{-1} P_j} - c_j \geq 0$ for all (basic and nonbasic) variables. In other words:

$$\boldsymbol{c_B^T B^{-1} A - c^T \geq 0}$$

Transposing, we obtain $\left(\boldsymbol{c_B^T B^{-1} A}\right)^{\boldsymbol{T}} \geq \boldsymbol{c}$

Which can be written as $(\boldsymbol{A})^{\boldsymbol{T}}\left(\boldsymbol{c_B^T B^{-1}}\right)^{\boldsymbol{T}} \geq \boldsymbol{c}$

Thus, $\boldsymbol{y} = \left(\boldsymbol{c_B^T B^{-1}}\right)^{\mathbf{T}}$ is a <u>feasible solution</u> to the dual problem

Thus, the associated dual objective function is $w = \boldsymbol{b^T y} = \boldsymbol{y^T b} = \boldsymbol{c_B^T B^{-1} b}$

But we know that $z = \boldsymbol{c^T}(\boldsymbol{x}) = \boldsymbol{c_B^T x_B} = \boldsymbol{c_B^T}(\boldsymbol{B^{-1} b})$

Then, $z = w$