DETERMINISTIC SYSTEMS MODELS/SYSTEMS OPTIMIZATION
ISE 4623/5023
EXAM 1
Fall 2020

Last name: Perez

First name: Pepito

Student ID: 123456789

Section (mark with X): ______ ISE 4623
X ISE 5023

Pledge: *"On my honor, I have neither given nor received inappropriate assistance in the completion of this Exam."*

Student signature: Pepito Perez

Problem 1 - (20 points)

Pepita Perez is a famous woodworker that makes "rustic" large and "fancy" medium-sized dining tables. To make these tables, he uses only three materials: wood, paint, and glue. Table 1 provides the basic data of the problem:

Table 1. Raw material utilization and availability for production of tables, along with profit per type of table

	"Rustic" large dining tables (units of raw material per table)	"Fancy" medium dining tables (units of raw material per table)	Maximum daily availability (units of raw material)
Wood (2'x4'x16' studs)	10	5	90
Paint (ounces)	4	8	72
Glue (ounces)	2	2	22

Profit per table ($1000)	10	2

Given that Pepita Perez is so famous and respected, the demand for her tables is always very high, so she always sells all the tables she makes.

You want to determine the production plan that maximizes Pepita Perez's profit. To do this, first you decide to formulate this problem as an LP model. In particular:

a. (5 points) Define the decision variables for this model

X_1 : # of rustic tables made

X_2 : # of fancy tables made

b. (5 points) What is the objective function of this LP model? Do not forget to indicate if you are maximizing or minimizing it.

$$\text{Maximize} \quad 10X_1 + 2X_2$$

c. (5 points) What are the constraints for this problem?

$$10X_1 + 5X_2 \le 90$$
$$4X_1 + 8X_2 \le 72$$
$$2X_1 + 2X_2 \le 22$$
$$X_1 \ge 0$$
$$X_2 \ge 0$$

d. (5 points) If you were to generalize this problem for many types of tables and other wooden furniture, using many types of raw materials, what would be the associated general LP model? Include your sets, parameters, decision variables, objective function, and constraints.

Sets

T : types of tables

R : types of raw materials

Parameters

p_t : Profit of each type of table $t \in T$

a_r : Daily availability of raw material $r \in R$

$u_{r,t}$: amount of raw material $r \in R$ needed to produce one table of type $t \in T$

Variables

x_t : Amount of tables produced of type $t \in T$

Objective function

$$\text{Maximize} \quad \sum_{t \in T} p_t x_t$$

Constraints

$$\sum_{t \in T} u_{r,t} x_t \leq a_r \qquad \forall r \in R$$

$$x_t \in \mathbb{Z}^+ \cup \{0\} \qquad \forall t \in T$$

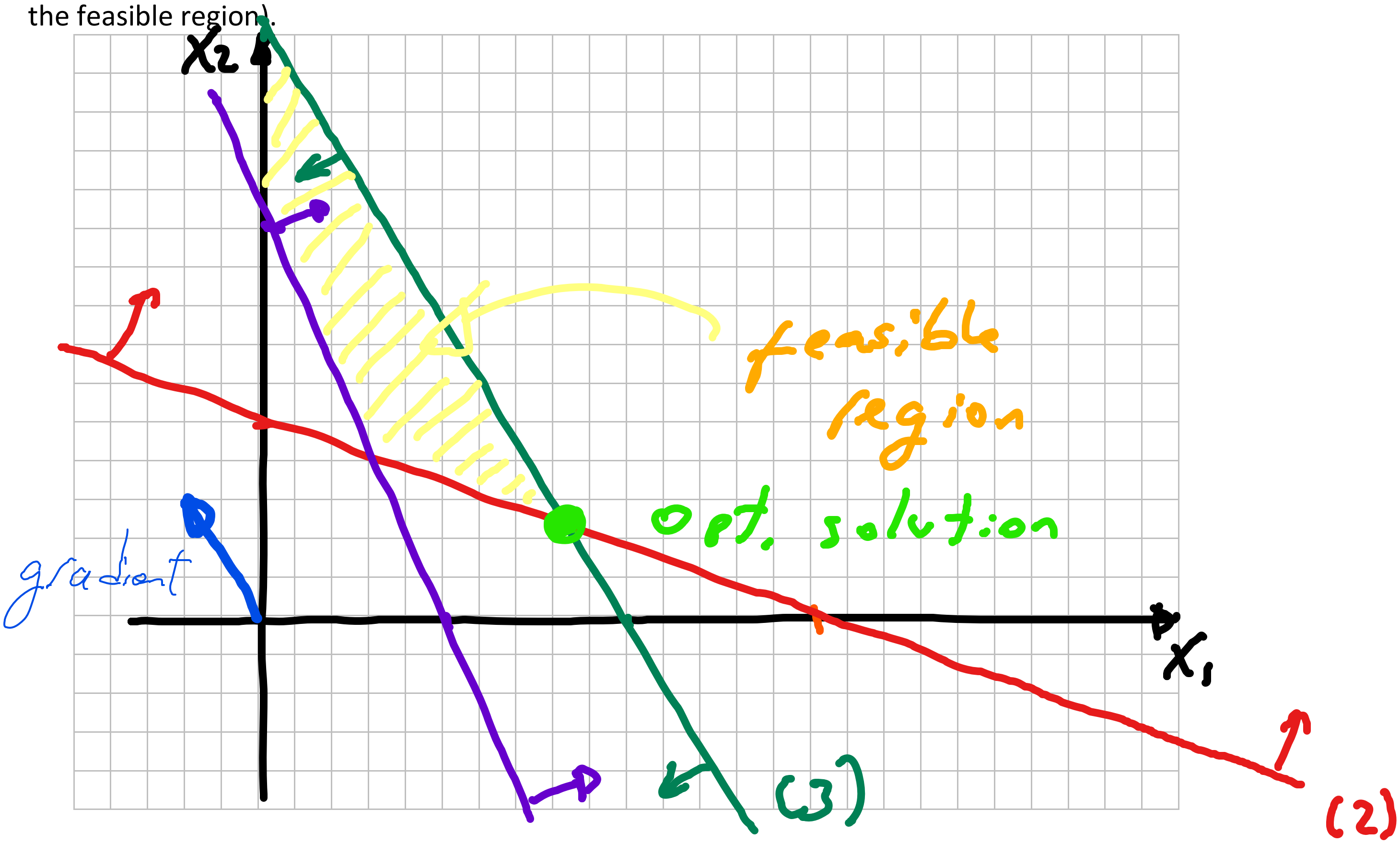
Problem 2 (25 points)
Suppose you have the following LP model

$$\begin{aligned}
\text{Minimize } & z = -4x_1 + 6x_2 \\
\text{s.t.} & \\
& 4x_1 + 2x_2 \geq 40 \quad (1) \\
& 2x_1 + 6x_2 \geq 60 \quad (2) \\
& 6x_1 + 4x_2 \leq 120 \quad (3) \\
& x_1, x_2 \geq 0
\end{aligned}$$

a. (5 points) What is the gradient of this problem?

$(-4, 6)$

b. (15 points) Plot the gradient and the feasible region (clearly indicating all the constraints and "shading" the feasible region).



c. (5 points) What is the optimal solution for this problem? Indicate the values of all the variables and the objective function.

$2X_1 + 6X_2 = 60 \Rightarrow X_1 = (60 - 6X_2)/2$

$6X_1 + 4X_2 = 120 \Rightarrow 6(30 - 3X_2) + 4X_2 = 120$

$\Rightarrow 180 - 18X_2 + 4X_2 = 120$

$\Rightarrow 180 - 120 = 14X_2 \Rightarrow X_2 = \frac{60}{14} = \boxed{\frac{30}{7}} \Rightarrow X_1 = \left[60 - 6\left(\frac{30}{7}\right)\right]/2$

$= \left(\frac{420 - 180}{14}\right) = \boxed{\frac{120}{7}}$

$\Rightarrow \text{O.F.} = -4X_1 + 6X_2 = -4\left(\frac{120}{7}\right) + 6\left(\frac{30}{7}\right) = \boxed{-300/7}$

Problem 3 - (30 points)

Mr. Buythen has been very successful with his business in the last year and was able to save $10 million ($3,000,000 million in cash, $4,500,000 million in real estate, and $2,500,000 in mutual funds). He has decided to use his assets to invest in a portfolio composed by up to five stock groups (A, B, C, D, and E). However, Mr. Buythen is not an expert in finance, so he has asked you to advise him about coming up with a good investment strategy, following the criteria below:

1. After the investment, Mr. Buythen wants to keep at least twice as much in real state as he keeps in mutual funds
2. Mr. Buythen wants to pay no more than $100,000 in financial charges (the financial institution will charge Mr. Buythen 5% if he acquires stocks using cash, 4% if he uses mutual funds, and 10% if he uses real estate).
3. Mr. Buythen wants to invest in group C stocks at least four times what he invests in group E stocks.
4. Mr. Buythen wants no more than 20% of the stock portfolio invested in group B stocks.
5. Mr. Buythen wants to guarantee that his aggregate investment in stock groups A and B represents at least 50% of his entire investment portfolio

The stock groups and the estimated rate of return for each investment is provided in Table 2.

Table 2. Rate of expected annual return for different stock groups

Stock groups	Expected annual rate of return
Group A	0.10
Group B	0.15
Group C	0.10
Group D	0.10
Group E	0.25

The goal is to come up with an investment strategy that maximizes total Mr. Buythen total profit (including investment returns and financial charges). To do this, first you need to mathematically formulate the problem as a linear model, using a generalized format. In particular:

a. (3 points) What are the "Sets" to be used in your model?

S : Set of sources $(1, 2, 3) \rightarrow$ (cash, real state, mutual funds)

G : Set of stock groups $(1, 2, 3, 4, 5)$
$\rightarrow (A, B, C, D, E)$

b. (3 points) What are the "Parameters" to be used in your model?

r_j : Expected rate of return for stock group $j \in G$

f_i : Financial charge for source $i \in S$

C_i : Current amount of source $i \in S$

M : Max financial charges to be paid

c. (3 points) What are the "Decision Variables" to be used in your model?

X_{ij} : Money from source $i \in S$ invested in stock group $j \in G$

d. (3 points) If you want to maximize the total amount of money that Mr. Buythen has after his investment (including the expected returns), what would be the "Objective Function" in your model?

$$\text{Max} \quad \sum_{i \in S} \sum_{j \in G} X_{ij} (1 + r_j - t_i)$$

e. (3 points) What is(are) the constraint(s) that guarantee(s) "criteria" 1?

$$C_2 - \sum_{j \in G} X_{2j} \geq 2 \left(C_3 - \sum_{j \in G} X_{3j} \right)$$

f. (3 points) What is(are) the constraint(s) that guarantee(s) "criteria" 2?

$$\sum_{i \in S} t_i \left[\sum_{j \in G} X_{ij} \right] \leq M$$

g. (3 points) What is(are) the constraint(s) that guarantee(s) "criteria" 3?

$$\sum_{i \in S} X_{i,c} \geq 4 \sum_{i \in S} X_{i,E}$$

h. (3 points) What is(are) the constraint(s) that guarantee(s) “criteria” 4?

$$\sum_{i \in S} X_{i,B} \leq 0.2 \left(\sum_{i \in S} \sum_{j \in G} X_{ij} \right)$$

i. (3 points) What is(are) the constraint(s) that guarantee(s) “criteria” 5?

$$\sum_{i \in S} \left(X_{i,A} + X_{i,B} \right) \geq 0.5 \left(\sum_{i \in S} \sum_{j \in G} X_{ij} \right)$$

j. (3 points) What are the constraints associated with the nature of the variables?

$$X_{ij} \geq 0 \qquad \forall i \in S, \forall j \in G$$

Problem 4 (25 points)

Suppose you have the following LP model, associated with a capacitated transportation problem

- Sets:

S = set of *sources* $\{1,2,\ldots,m\}$

D = set of *destinations* $\{1,2,\ldots,n\}$

- Parameters:

a_i = supply of source node $i \in S$

b_j = demand of destination node $j \in D$

c_{ij} = unit cost of flow through arc (i,j), where $i \in S$ and $j \in D$

u_{ij} = flow capacity for arc (i,j), where $i \in S$ and $j \in D$

- Decision variables:

x_{ij} = flow through arc (i,j), where $i \in S$ and $j \in D$

- Objective function:

$$\min z = \sum_{i\in S}\sum_{j\in D} c_{ij}x_{ij} \qquad \text{(o1)}$$

- Constraints:

$$\sum_{i\in S} x_{ij} = b_j \qquad ,\forall j \in D \qquad \text{(c1)}$$

$$\sum_{j\in D} x_{ij} = a_i \qquad ,\forall i \in S \qquad \text{(c2)}$$

$$x_{ij} \le u_{ij} \qquad ,\forall i \in S, \forall j \in D \qquad \text{(c3)}$$

$$x_{ij} \ge 0 \qquad ,\forall i \in S, \forall j \in D \qquad \text{(c4)}$$

You were given a particular instance of this problem to be solved in Gurobi/Python. The code to initialize the sets and parameters has been given to you (below).

```
## SETS
# S -> set of sources
# D -> set of destinations

S=["s1","s2""]
D=["d1","d2","d3"]

## Parameters
# c_{ij}, u_{ij}, a{i}, b{j}

SD, c, u = multidict({
    ("s1","d1"):[1, 10],
    ("s1","d2"):[2, 20],
    ("s1","d3"):[3, 15],
    ("s2","d1"):[2, 10],
    ("s2","d2"):[1, 30],
    ("s2","d3"):[2, 40]})

a={("s1"):30,("s2"):20}
b={("d1"):15,("d2"):20,("d3"):15}
```

Now you want to write and solve the associated LP model. In particular, you must:

a. (3 points) Write the code in Python/Gurobi to import "Gurobipy"

from gurobipy import *

b. (3 points) Write the code in Python/Gurobi to define/create a Gurobi model named "Exam1_Problem4"

```
model = Model ("Exam1_Problem4")
```

c. (3 points) Write the code in Python/Gurobi to add the decision variables to the Gurobi model. Do not forget to indicate their respective coefficients (for the objective function) when defining the variables.

```
X = model.addVars(S, D, obj: C, name = "x")
```

d. (3 points) Write the code in Python/Gurobi to set the "sense" of the Gurobi model to "Minimize"

```
model.modelSense = GRB.Minimize
```

e. (10 points) Write the code in Python/Gurobi to add the constraints (c1, c2, and c3) to the Gurobi model

```
model.addConstrs((X.sum('*', j) == b[j]) for j in D, "c1")
model.addConstrs((X.sum(i, '*') == a[i]) for i in S, "c2")
model.addConstrs((X[i, j] ≤ U[i, j]) for i in S for j
                                                 in D, "c3")
```

f. (3 points) Write the code in Python/Gurobi to solve (optimize) the Gurobi model

```
model.optimize()
```