Introduction to Linear Algebra

Based on: Winston, W. L. 2004. Operations Research: Applications and Algorithms. 4th Edition.

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Vectors

Vector: A list of numbers or subscripts, where each subscript denotes the position of the value in the list.

147, 156, 179, 185, 164
w_1, w_2, w_3, w_4, w_5

W = $(w_1, w_2, w_3, w_4, w_5)$ is a linear array or vector.

Vectors

- Vector Addition

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

- Scalar Multiplication

The *scalar product* or, simply, *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Vectors

- Negative and Subtraction

Negatives and subtraction are defined in $\mathbf{R}^n$ as follows:

$$-u = (-1)u \qquad \text{and} \qquad u - v = u + (-v)$$

The vector $-u$ is called the *negative* of u, and $u - v$ is called the *difference* of u and v.

- Linear Combination

Now suppose we are given vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ and scalars $k_1, k_2, \ldots, k_m$ in $\mathbf{R}$. We can multiply the vectors by the corresponding scalars and then add the resultant scalar products to form the vector

$$v = k_1 u_1 + k_2 u_2 + k_3 u_3 + \cdots + k_m u_m$$

Such a vector v is called a *linear combination* of the vectors $u_1, u_2, \ldots, u_m$.

Matrices

A matrix A over a field K or, simply, a matrix A (when K is implicit) is a rectangular array of scalars usually presented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The rows of such a matrix A are the m horizontal lists of scalars:

$$(a_{11}, a_{12}, \dots, a_{1n}), \quad (a_{21}, a_{22}, \dots, a_{2n}), \quad \dots, \quad (a_{m1}, a_{m2}, \dots, a_{mn})$$

and the columns of A are the n vertical lists of scalars:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

Matrices

A matrix with *m* rows and *n* columns is called an *m* by *n* matrix. The pair of numbers *m* and *n* is called the size of the matrix.

- Matrix Addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices with the same size, say $m \times n$ matrices. The *sum* of A and B, written $A + B$, is the matrix obtained by adding corresponding elements from A and B. That is,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrices

- Scalar Multiplication

The *product* of the matrix A by a scalar k, written $k \cdot A$ or simply kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

- Negative of the Matrix and the Difference of Two Matrices

$$-A = (-1)A \qquad \text{and} \qquad A - B = A + (-B)$$

The matrix $-A$ is called the *negative* of the matrix A, and the matrix $A - B$ is called the *difference* of A and B. The sum of matrices with different sizes is not defined.

Matrices

- Matrix Multiplication

The product AB of a $\boxed{\text{row matrix } A} = [a_i]$ and a $\boxed{\text{column matrix } B} = [b_i]$ with the same number of elements is defined to be the scalar (or 1×1 matrix) obtained by multiplying corresponding entries and adding; that is,

$$AB = [a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{k=1}^{n} a_kb_k$$

AB is a scalar!

- Transpose of a Matrix

The *transpose* of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order, as rows. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad [1, -3, -5]^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

In other words, if $A = [a_{ij}]$ is an $m \times n$ matrix, then $A^T = [b_{ij}]$ is the $n \times m$ matrix where $b_{ij} = a_{ji}$.

System of Linear Equations

- Linear Equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Coefficient of x_k The constant term of the equation

- Solution

A list of values for the vector $u = (x_1,\ x_2,\ldots,x_n)$ in K^n that satisfies the equation above.

$$x_1 = k_1\,,\ x_2 = k_2\,,\ \ldots\,,\ x_n = k_n$$

- System of Linear Equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

System of Linear Equations

- System of Linear Equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- Solution:

A solution (or a particular solution) of the system is a list of values for the unknowns or, equivalently, a vector u in K^n , which is a solution of each of the equations in the system.

The set of all solutions of the system is called the solution set or the general solution of the system.

- Degenerate Linear Equation:

A linear equation is said to be degenerate if all the coefficients are zero—that is, if it has the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

  - If b$\neq$ 0, then the equation has no solution.
  - If b=0, then every vector $u = (k_1,\ k_2,\ldots,k_n)$ in K^n is a solution.

System of Linear Equations

- Matrix representation

Using matrices can greatly simplify the statement and solution of a system of linear equations. To show how matrices can be used to compactly represent Equation (3), let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then (3) may be written as

$$A\mathbf{x} = \mathbf{b}$$

Gauss–Jordan Method

- The Gauss–Jordan Method for Solving Systems of Linear Equations:

Using the Gauss–Jordan method, we show that any system of linear equations must satisfy one of the following three cases:

  – Case 1: The system has no solution.
  – Case 2: The system has a unique solution.
  – Case 3: The system has an infinite number of solutions.

- Elementary Row Operations:
  1. B is obtained by multiplying any row of A by a nonzero scalar. Begin by multiplying any row of A (say, row i) by a nonzero scalar c.
  2. For some j i, let row j of B= c(row i of A) + row j of A, and let the other rows of B be the same as the rows of A.
  3. Interchange any two rows of A

Gauss–Jordan Method

The discussion in the previous section indicates that if the matrix $A'|\mathbf{b}'$ is obtained from $A|\mathbf{b}$ via an ERO, the systems $A\mathbf{x} = \mathbf{b}$ and $A'\mathbf{x} = \mathbf{b}'$ are equivalent. Thus, any sequence of EROs performed on the augmented matrix $A|\mathbf{b}$ corresponding to the system $A\mathbf{x} = \mathbf{b}$ will yield an equivalent linear system.

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 - x_2 + 2x_3 &= 6 \\ x_1 - x_2 + 2x_3 &= 5 \end{aligned}$$

$$A|\mathbf{b} = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

The augmented matrix representation.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The result obtained by performing an ERO on a system of equations

Gauss–Jordan Method (Example 1)

Linear System with No Solution

Find all solutions to the following linear system:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 4 \end{aligned} \tag{11}$$

We apply the Gauss–Jordan method to the matrix

$$A|\mathbf{b} = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 4 \end{array}\right]$$

We begin by replacing row 2 of $A|\mathbf{b}$ by -2(row 1 of $A|\mathbf{b}$) + row 2 of $A|\mathbf{b}$. The result of this Type 2 ERO is

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -2 \end{array}\right] \tag{12}$$

We would now like to transform the second column of (12) into

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but this is not possible. System (12) is equivalent to the following system of equations:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 + 0x_2 &= -2 \end{aligned} \tag{12$'$}$$

Whatever values we give to x_1 and x_2, the second equation in (12′) can never be satisfied. Thus, (12′) has no solution. Because (12′) was obtained from (11) by use of EROs, (11) also has no solution.

Gauss–Jordan Method (Example 2)

Linear System with Infinite Number of Solutions

Apply the Gauss–Jordan method to the following linear system:

$$\begin{aligned} x_1 + x_2 \quad\quad &= 1 \\ x_2 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \end{aligned} \tag{13}$$

The augmented matrix form of (13) is

$$A|\mathbf{b} = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

We begin by replacing row 3 (because the row 2, column 1 value is already 0) of $A|\mathbf{b}$ by -1(row 1 of $A|\mathbf{b}$) + row 3 of $A|\mathbf{b}$. The result of this Type 2 ERO is

$$A_1|\mathbf{b}_1 = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \tag{14}$$

Next we replace row 1 of $A_1|\mathbf{b}_1$ by -1(row 2 of $A_1|\mathbf{b}_1$) + row 1 of $A_1|\mathbf{b}_1$. The result of this Type 2 ERO is

$$A_2|\mathbf{b}_2 = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Gauss–Jordan Method (Example 2 - continued)

Now we replace row 3 of $A_2|\mathbf{b}_2$ by -1(row 2 of $A_2|\mathbf{b}_2$) + row 3 of $A_2|\mathbf{b}_2$. The result of this Type 2 ERO is

$$A_3|\mathbf{b}_3 = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We would now like to transform the third column of $A_3|\mathbf{b}_3$ into

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

but this is not possible. The linear system corresponding to $A_3|\mathbf{b}_3$ is

$$x_1 \quad - x_3 = -2 \tag{14.1}$$

$$x_2 + x_3 = 3 \tag{14.2}$$

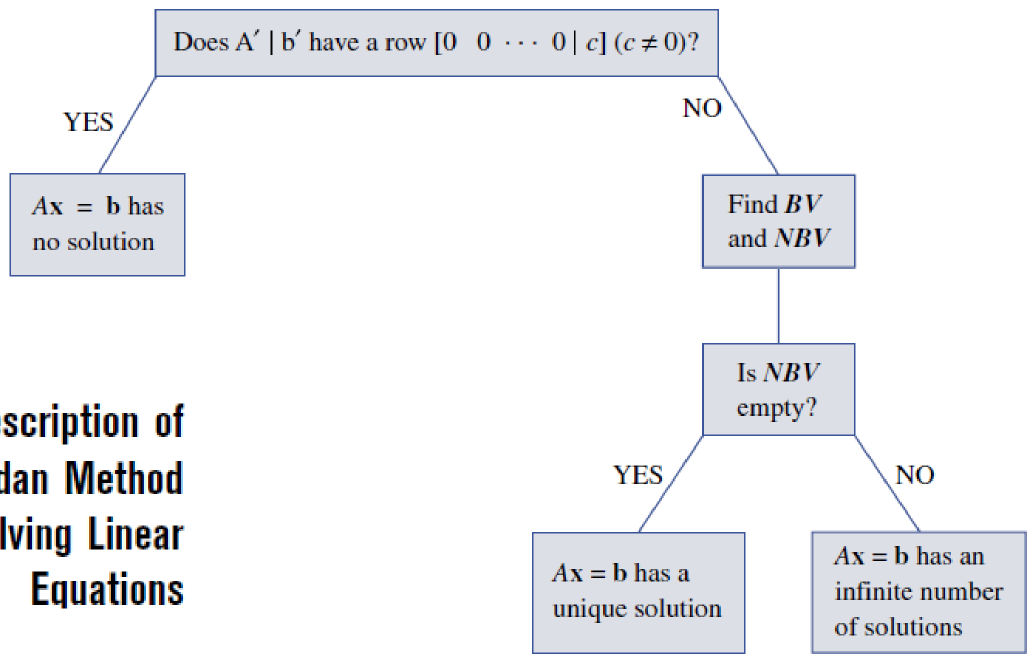
$$0x_1 + 0x_2 + 0x_3 = 0 \tag{14.3}$$

Suppose we assign an arbitrary value k to x_3. Then (14.1) will be satisfied if $x_1 - k = -2$, or $x_1 = k - 2$. Similarly, (14.2) will be satisfied if $x_2 + k = 3$, or $x_2 = 3 - k$. Of course, (14.3) will be satisfied for any values of x_1, x_2, and x_3. Thus, for any number k, $x_1 = k - 2$, $x_2 = 3 - k$, $x_3 = k$ is a solution to (14). Thus, (14) has an infinite number of solutions (one for each number k). Because (14) was obtained from (13) via EROs, (13) also has an infinite number of solutions. A more formal characterization of linear systems that have an infinite number of solutions will be given after the following summary of the Gauss–Jordan method.

Gauss–Jordan Method

After the Gauss–Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable** (BV). ■

Any variable that is not a basic variable is called a **nonbasic variable** (NBV). ■



Description of Gauss–Jordan Method for Solving Linear Equations

Linear Combination of Vectors

A **linear combination** of the vectors in V is any vector of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ $\cdots + c_k\mathbf{v}_k$, where $c_1, c_2, \ldots, c_k$ are arbitrary scalars. ■

For example, if $V = \{[1 \quad 2], [2 \quad 1]\}$, then

$$\begin{aligned} 2\mathbf{v}_1 - \mathbf{v}_2 &= 2([1 \quad 2]) - [2 \quad 1] = [0 \quad 3] \\ \mathbf{v}_1 + 3\mathbf{v}_2 &= [1 \quad 2] + 3([2 \quad 1]) = [7 \quad 5] \\ 0\mathbf{v}_1 + 3\mathbf{v}_2 &= [0 \quad 0] + 3([2 \quad 1]) = [6 \quad 3] \end{aligned}$$

are linear combinations of vectors in V. The foregoing definition may also be applied to a set of column vectors.

Linear Dependence of Vectors

The vectors $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of them is a linear combination of the others.

- Basis

$$v_j = b_j^{-1} b_1 v_1 - \cdots - b_j^{-1} b_{j-1} v_{j-1} - b_j^{-1} b_{j+1} v_{j+1} - \cdots - b_j^{-1} b_m v_m$$

A set S = $\{v_1, v_2, \ldots, v_n\}$ of vectors is a basis of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

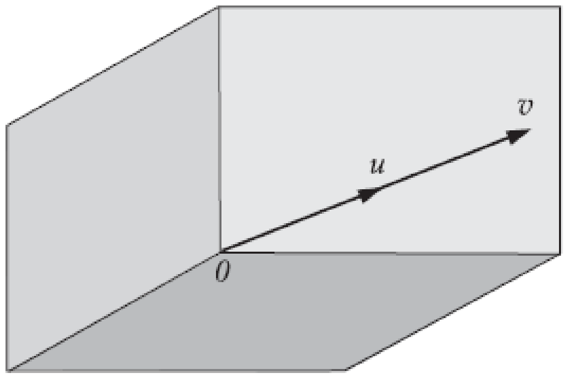
Linear Dependence of Vectors

Linear dependence in the vector space $V = \mathbf{R}^3$ can be described geometrically as follows:

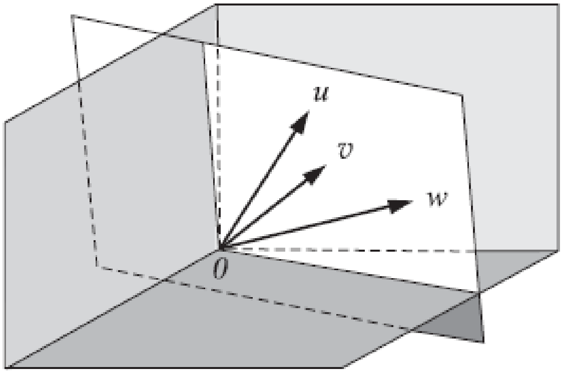
(a) Any two vectors u and v in $\mathbf{R}^3$ are linearly dependent if and only if they lie on the same line through the origin O, as shown in Fig. 4-3(a).

(b) Any three vectors u, v, w in $\mathbf{R}^3$ are linearly dependent if and only if they lie on the same plane through the origin O, as shown in Fig. 4-3(b).

Later, we will be able to show that any four or more vectors in $\mathbf{R}^3$ are automatically linearly dependent.



(a) u and v are linearly dependent.



(b) u, v, and w are linearly dependent.

Figure 4-3

The Inverse of a Matrix

A **square matrix** is any matrix that has an equal number of rows and columns. ■

The **diagonal elements** of a square matrix are those elements a_{ij} such that $i = j$. ■

A square matrix for which all diagonal elements are equal to 1 and all nondiagonal elements are equal to 0 is called an **identity matrix.** ■

For a given $m \times m$ matrix A, the $m \times m$ matrix B is the **inverse** of A if

$$BA = AB = I_m \quad (16)$$

(It can be shown that if $BA = I_m$ or $AB = I_m$, then the other quantity will also equal I_m.) ■

- Why the inverse of matrix important?

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

or

$$I_m\mathbf{x} = A^{-1}\mathbf{b}$$

or

$$\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$$

The Inverse of a Matrix

Inverting Matrices with Excel

The Excel =MINVERSE command makes it easy to invert a matrix. See Figure 8 and file Minverse.xls. Suppose we want to invert the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Simply enter the matrix in E3:G5 and select the range (we chose E7:G9) where you want A^{-1} to be computed. In the upper left-hand corner of the range E7:G9 (cell E7), we enter the formula

= MINVERSE(E3:G5)

and select **Control Shift Enter.** This enters an array function that computes A^{-1} in the range E7:G9. You cannot edit part of an array function, so if you want to delete A^{-1}, you must delete the entire range where A^{-1} is present.

Determinant of Matrix

If A is an $m \times m$ matrix, then for any values of i and j, the ijth **minor** of A (written A_{ij}) is the $(m-1) \times (m-1)$ submatrix of A obtained by deleting row i and column j of A. ■

For example,

$$\text{if} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \quad \text{and} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Let A be any $m \times m$ matrix. We may write A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

To compute det A, pick any value of i $(i = 1, 2, \ldots, m)$ and compute det A:

$$\det A = (-1)^{i+1} a_{i1}(\det A_{i1}) + (-1)^{i+2} a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m} a_{im}(\det A_{im}) \quad \textbf{(23)}$$

THANK YOU

QUESTIONS?