THE UNIVERSITY OF SOUTH ALABAMA
COLLEGE OF ARTS AND SCIENCES

Embedding Graphs on the Square Grid

BY

Daniel Hodgins

A Thesis

Submitted to the Graduate Faculty of the
University of South Alabama
in partial fulfillment of the
requirements for the degree of

Master of Science

in

Mathematics

May 2024

Approved:	**Date:**
Chair of Thesis Committee: Dr. Elena Pavelescu	
Committee Member: Dr. Steven Clontz	
Committee Member: Dr. Vasiliy Prokhorov	
Committee Member: Dr. Martin Frank	
Chair of Department: Dr. Madhuri Mulekar	
Director of Graduate Studies: Dr. Eric Loomis	
Dean of the Graduate School: Dr. J. Harold Pardue	

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ABSTRACT

Daniel Hodgins, M. S., University of South Alabama, May 2024. Embedding Graphs on the Square Grid. Chair of Committee: Elena Pavelescu, Ph.D.

Minimizing the lengths of connections in network systems can be useful in the real world when connections are cost prohibitive. In this paper, we define a function called the edge sum to evaluate embeddings of graphs on the square grid. The edge sum is determined by the lengths of the edges in the taxicab metric. In addition, our objective is to find bounds on the minimum edge sum for various classes of graphs. We prove what graphs have embeddings on the 2 dimensional grid. We describe operations that can be done to reduce the lengths of edges in the taxicab metric. We found results for: paths, cycles, wheels, triangulations, prisms, and trees in the 2 dimensional grid. We show results for complete graphs, wheel graphs, and trees in higher dimensional grids.

ACKNOWLEDGMENTS

I would like to thank Dr. Elena Pavelescu for her invaluable guidance as my thesis advisor.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

A graph is a structure composed of vertices (points) and edges connecting them. In [2], Erdos, Harary, and Tutte put forth the notion of graph dimension. The dimension of a graph G as the smallest positive integer n such that G can be drawn with its vertices being points of $\mathbb{R}^n$ where any two adjacent vertices are necessarily placed a unit-distance apart. A graph G is dimension-critical if every proper subgraph of G has dimension less than G. In [2], the authors establish the dimension of a few common families of graphs. In [3], Noble determines exactly which complete multipartite graphs are dimension-critical. In [4], Noble takes these ideas further by allowing edge lengths which are rational numbers. Due to allowed scaling, it is equivalent to ask for edge lengths which are integers. The Steiner's Problem as detailed in [5] states an interesting problem of what is the smallest tree which contains n given points on the grid. This problem allows other points to be included in the tree, but the n vertices are fixed points. Interestingly [6] showed that the problem detailed in [5] is NP-complete.

The current project explores the idea of edge sum for graphs whose vertices sit on the integer lattice, that are points in the plane with coordinates (a, b), $a, b \in \mathbb{Z}$. The edges of a graph are constrained to the square grid. The length of an edge is the arc length of the path describing the edge in the taxicab metric. The edge-sum of a graph on the square grid is the sum of all edge lengths in the graph. In Figure 2.1

are some examples of edge-sums of a few graph embeddings. This thesis focuses on two questions:

1. What graphs can be embedded on the grid?

2. What are the possible values for the edge-sums of these graphs? In particular, what is the minimum edge-sum for a particular class of graphs?

A starting upper bound for an arbitrary graph's minimum edge-sum is the edge sum of a random embedding of G. A starting lower bound for an arbitrary graph's minimum edge-sum is the number of edges in G.

In Chapter 2, we define many types of graphs and related terms with notation. In Chapter 3, we prove the existence theorem. In Chapter 4, we show a reduction move for single edges and a reduction move for multiple edges. In Chapter 5, we show bounds for the minimum edge sum for various graphs. In Chapter 6, we show results for higher dimensions. In Chapter 7, we discuss future extensions.

CHAPTER 2

DEFINITIONS AND NOTATION

Let G denote an abstract graph and $\tilde{G}$ represent an embedding of the graph G. Let V denote the set of vertices in G, and let E denote the set of edges in G. The order of graph G, is the number of vertices in G and is denoted by n. The size of graph G, is the number of edges in G is is denoted by m. The **degree** of a vertex v in a graph is the number of edges incident to v, denoted by $deg(v)$. The **maximum degree** of G, denoted as $\Delta(G)$, represents the maximum value among all $deg(v)$ for $v \in V$. In any graph that is embeddable into a square grid, the maximal degree is at most 4. This is due to each vertex lying at a grid intersection, and each grid intersection accommodating at most four incident edges.

A subgraph of a graph G is obtained by deleting edges and vertices of G. Deleting a vertex means deleting all edges incident to that vertex as well. Each edge is assigned a weight corresponding to the sum of the edge's arc lengths in the taxicab metric, also known as the rectilinear distance. For two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in the grid, the taxicab distance $d(p_1, p_2)$ is calculated as the sum of the absolute differences of their coordinates: $|x_1 - x_2| + |y_1 - y_2|$. This can be extended to higher dimensions by adding the absolute differences of each coordinate.

The **edge-sum** of $\tilde{G}$, denoted $es(\tilde{G})$, represents the sum of all the weights of the edges in $\tilde{G}$. The **minimum edge sum**, denoted as $mes(G)$, represents the smallest edge sum among all possible embeddings of the graph G. Figure 2.1 shows several graph embeddings and their edge sums.

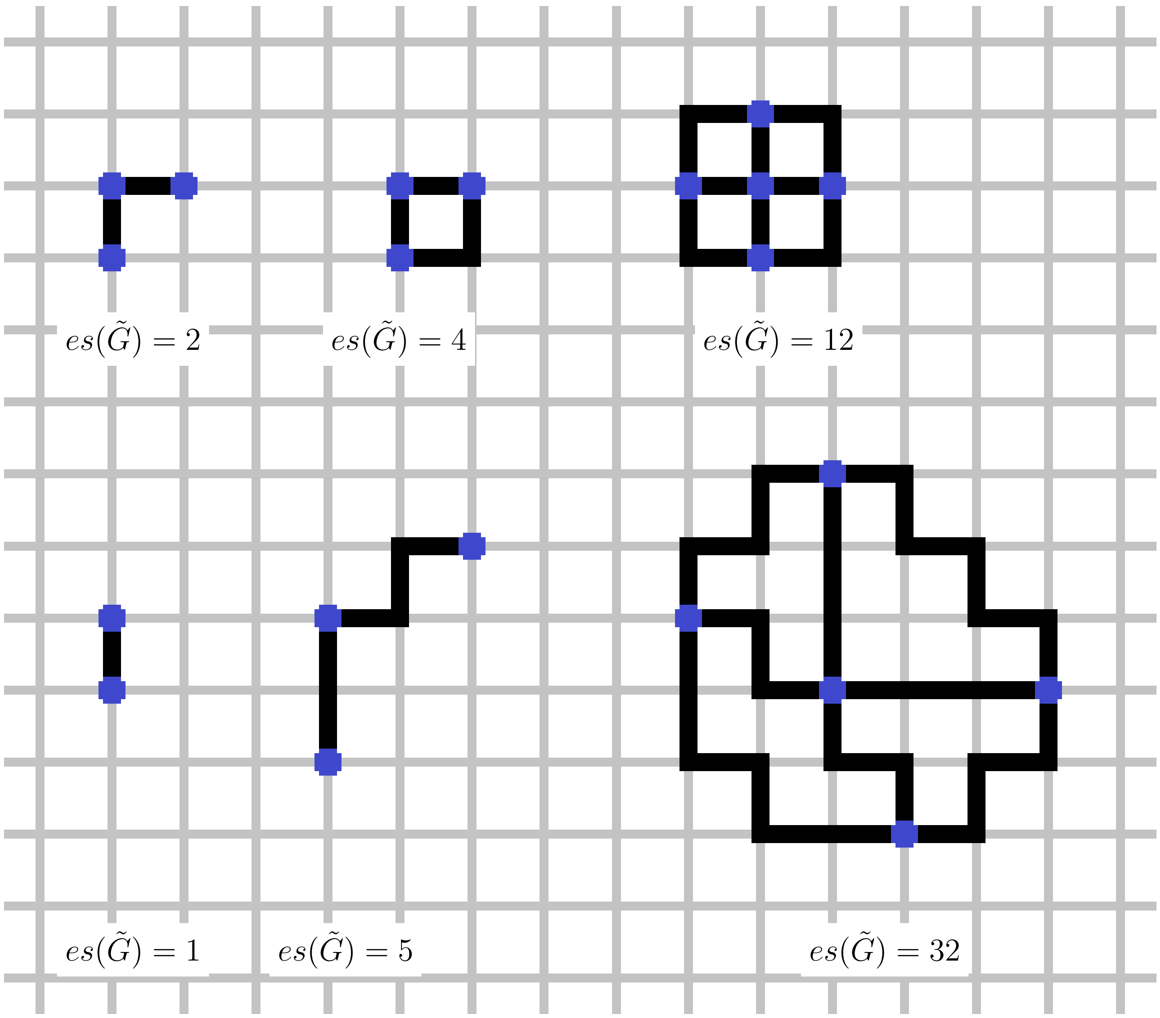

Figure 2.1: Examples of edge-sums.

Below are several graphs and their notation that will be examined in later sections. An **n-path**, denoted P_n, is a graph with n vertices, $v_1, v_2, \ldots, v_n$, and the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$.

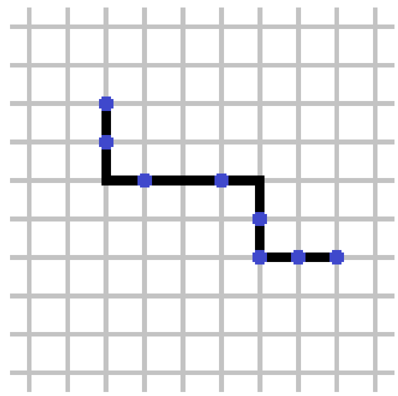

Figure 2.2: A 8-path graph embedded in a square grid.

A **n-cycle**, denoted C_n, is a graph with n vertices, $v_1, v_2, \ldots, v_n$, and contains the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$.

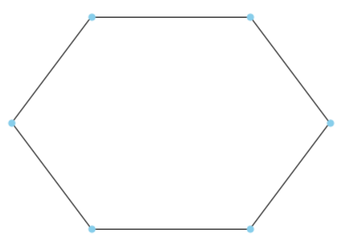

Figure 2.3: A 6-cycle graph.

A **wheel** with n vertices, denoted W_n, is a graph created from connecting a single vertex by edges to every vertex of an $(n-1)$-cycle.

A graph G is **planar** if there exist an drawing $\tilde{G}$ in the plane such that the edges only intersect at their endpoints. A graph is a **maximal planar graph** if it is planar and would become non-planar if an edge was added between any two vertices.

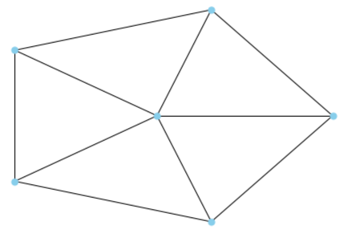

Figure 2.4: A wheel graph with 6 vertices.

Another term for a maximal planar graph is a **triangulation** since the planar regions determined by the embedding are triangles.

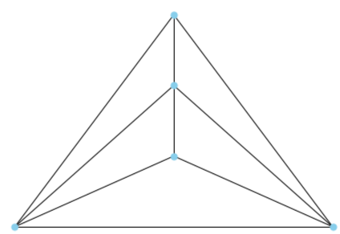

Figure 2.5: A maximal planar graph with 5 vertices.

A **leaf** is a vertex with a degree of one. A **connected graph** is a graph where there exists a path between any two vertices. A **tree**, is a connected graph that contains no cycles as subgraphs. The vertices in a tree that are not leaves are called **internal vertices**. For each $v \in V$, $u \in V$, let P_{vu} be a shortest path from v to u. The **radius** of G is

$$radius(G) = \min_{v \in V} \max_{u \in V}(|E(P_{vu})|)$$

A **central vertex** is a internal vertex that is the v in the definition of radius which realizes the minimum.

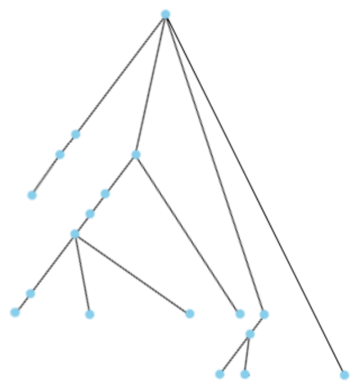

Figure 2.6: A tree graph with 8 leaves.

A **caterpillar graph** is a tree where the deletion of the leaves results in a path graph. We follow the terminology in [7].

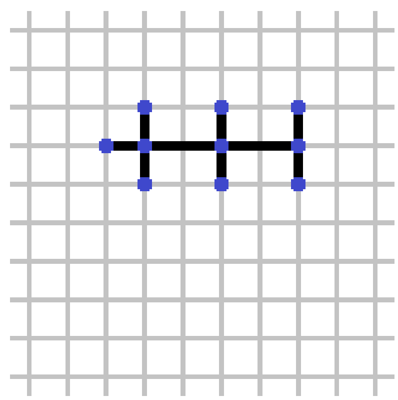

Figure 2.7: A caterpillar graph with seven leaves embedded on a grid.

A **lobster graph** is a tree in which the deletion of the leaves results in a caterpillar graph. We follow the terminology in [7].

A **complete graph**, denoted K_n, is a graph with n vertices where each vertex is connected to every other vertex by an edge.

A **Halin graph**, denoted H_n, is a type of planar graph with n vertices constructed by connecting the leaves of a tree to form a cycle. The tree must have a minimum of four vertices, and none of its vertices have exactly two neighboring vertices.

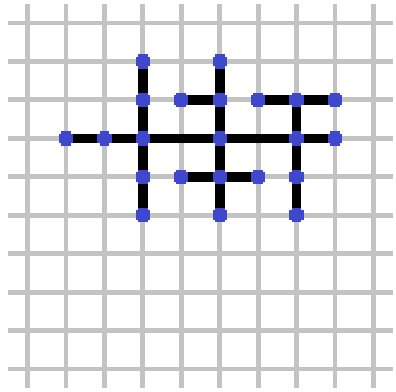

Figure 2.8: A lobster graph with 12 leaves embedded on a grid.

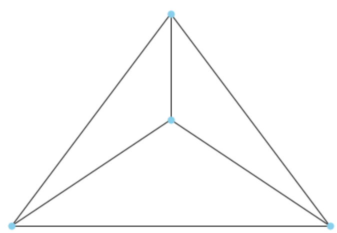

Figure 2.9: The complete graph K_4.

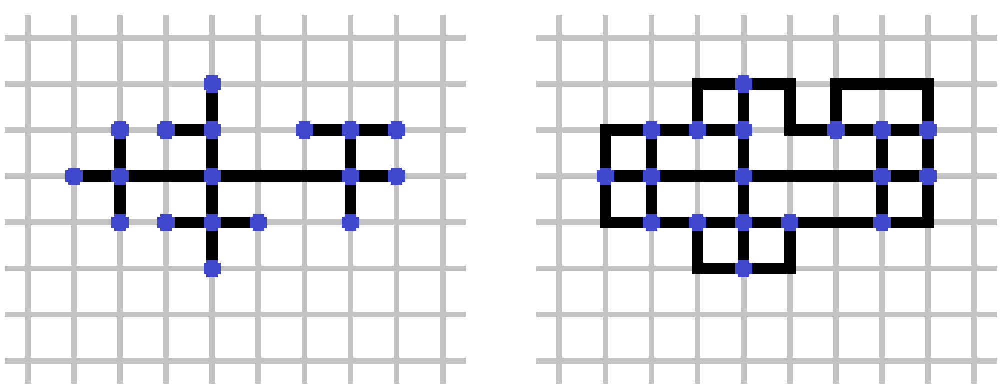

Figure 2.10: A Halin graph embedded on a grid with its tree subgraph on the left.

A **prism graph** with $2n$ vertices is a graph that contains two cycles. The vertices of the cycles are denoted as $c_1, c_2, \ldots, c_n$ and $b_1, b_2, \ldots, b_n$. These cycles are connected to each other with edges $c_1b_1, c_2b_2, \ldots, c_nb_n$.

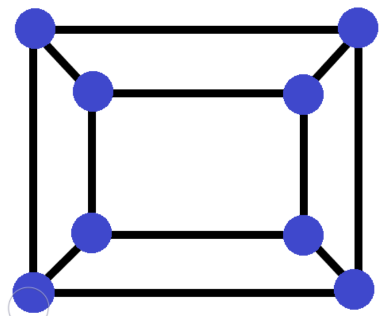

Figure 2.11: A prism graph with 8 vertices.

CHAPTER 3

THE EXISTENCE THEOREM

We consider embeddings that are piecewise linear embeddings where each edge is a finite union of line segments. Recall that a graph G with $\Delta(G) > 4$ can not be embedded in a 2-dimensional grid since each grid intersection only has four accommodating places for edges to be embedded.

Theorem 1. *Every plane graph embedding $\tilde{G}$ with maximum degree of 4 can be embedded into a square grid given fine adjustments to the positioning of the vertices. By 'fine adjustments' we mean that we can shift any point with irrational coordinates to a nearby point with rational coordinates such that the distance between the two points is arbitrarily small.*

Proof. Take an arbitrary plane embedding of the graph G with n vertices. Our goal is to show a process to find a isomorphic embedding of the plane embedding onto the square grid. First we shall show how vertices can be positioned on the grid, then we will construct neighborhoods of those vertices, and then we will construct neighborhoods of the edges.

Vertices in the graph may initially have coordinates represented by irrational values; however, we can shift each vertex's position slightly within an arbitrarily small neighborhood to a point with rational coordinates. This is possible since the rationals are dense within the set $\mathbb{R}$. The new point for each vertex will be represented as $(\frac{a_i}{b_i}, \frac{c_i}{d_i})$, where $gcd(a_i, b_i) = 1$ and $gcd(c_i, d_i) = 1$, $i = 1, 2, \ldots, n$. Let

$l = lcm(b_1, b_2, \ldots, b_n, d_1, d_2, \ldots, d_n)$. We then draw horizontal and vertical grid lines starting from any vertex with a distance of $\frac{1}{l}$ between every two consecutive grid lines. This ensures that each vertex lies precisely on a grid intersection.

Next, we show a construction to align the edges with the grid lines. Near each vertex, the incident edges e_1, e_2, e_3, and e_4 need to connect to the vertex in different cardinal directions. However, in the arbitrary embedding, vertices may have multiple edges connecting in the same quadrant. The worst-case scenario is when four edges exit the same quadrant of the neighborhood. We shall assume all edges are positioned at the bottom right quadrant without loss of generality. We create a staircase pattern with the edges, as illustrated in Figure 3.1, which also preserves the edge ordering. The space required for this staircase pattern around the vertex must have a neighborhood with a radius of 5. This refinement will make our grid lines ten times closer. At this step, vertices and their neighborhoods are embedded onto the grid.

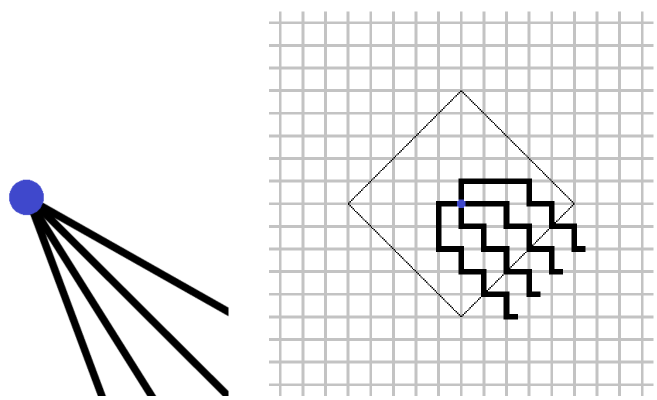

Figure 3.1: Modifying the neighborhood of a vertex. The ordering of the edges e_1, e_2, e_3, and e_4 is preserved.

The last step is to embed the arcs of edges between their incident neighborhoods. Note that intersecting edges are not an issue, since outside of the neighborhoods of the vertices, edges have mutually disjoint neighborhoods in the Euclidean metric. Now we shall create mutually disjoint neighborhoods for the arcs of the edges in the grid. While edges also require neighborhoods, they only need to be of width 3. This provides enough space for the arcs of edges to be positioned and to change directions without intersecting another arc. This involves adding an arbitrary but finite number of grid lines, depending on the original embedding; however, after adding a sufficient number of grid lines, the embedding will now lie on the gridlines.

□

From now on, when we refer to an embedding of the graph G, we will mean that the embedding $\tilde{G}$ is an embedding on the grid.

CHAPTER 4

REDUCTION MOVES

Multi-edge Reduction:

Let $\tilde{G}$ be an embedding of G such that there exists a grid line that intersects $\tilde{G}$ at finitely many points along the edges and does not intersect the vertices. Then there exists an embedding of the graph where each edge intersecting the grid line is shorten by one unit. This is done by removing one unit of length to the right (up) of each intersection point with the grid line and translating the connected component to the right (up) of the grid line by one unit to the left (down). This will reduce the edge sum by the number of edge intersections with the grid line. This process will be refered to as a **multi-edge reduction**, which can be performed along both vertical and horizontal lines. If there exist multiple grid lines which meet the condition, then the multi-edge reduction can be performed in any order for each grid line. Refer to Figure 4.1 for illustration.

Single-edge Reduction:

Let $\tilde{G}$ be an embedding. Let an edge e of $\tilde{G}$, with a and b being grid points on e but not necessarily the endpoints of e, such that a and b lie on the same grid line. The arc of e between a and b can be replaced by the straight line segment from a to b, resulting in a shorter edge, provided that the straight line segment from a to b does not intersect any parts of the graph. This operation reduces the edge sum by at

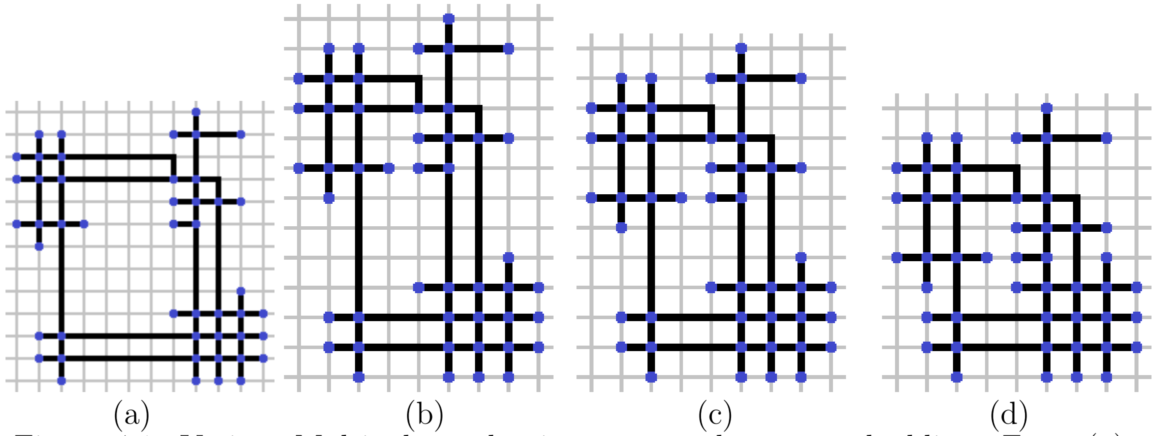

Figure 4.1: Various Multi-edge reductions operated on an embedding. From (a) to (b) three multi-edge reductions are done. From (b) to (c) one multi-edge reduction is done. From (c) to (d) the edges are reduced, but this operation would not be considered a multi-edge reduction.

least 2. This process will be refered to as a **single-edge reduction**. Refer to Figure 4.2 for an illustration.

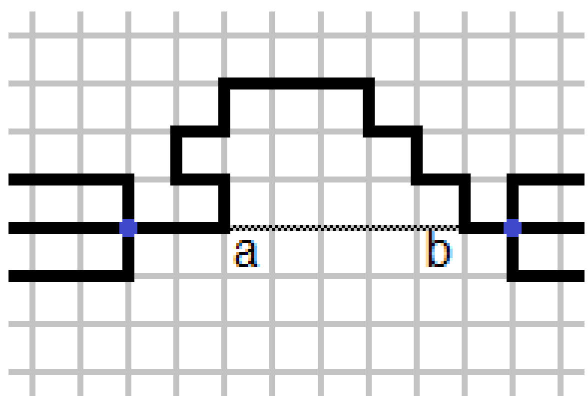

Figure 4.2: A single-edge reduction operated on a embedding of a graph. The bold path, indicating the edge, can be replaced with the shaded line segment from point a to b in order to reduce the edge sum.

If an embedding of a graph $\tilde{G}$ already has a minimum edge sum, neither single-edge reductions nor multi-edge reductions can be performed on the embedding. Therefore, single-edge reductions and multi-edge reductions on embeddings are useful for identifying potential candidates for embeddings with the minimum edge sum.

CHAPTER 5

MINIMAL EDGE-SUM VALUES FOR SEVERAL CLASSES OF GRAPHS

In this section we examine the following classes of graphs: paths, cycles, wheels, trees, prisms, and triangulations.

5.1 Paths, Cycles, and Wheels

Proposition 1. *Let P_n be a path with n vertices. Then $mes(P_n) = n - 1$.*

Proof. Let P_n be a path with n vertices. The vertices $v_1, v_2, \ldots, v_n$ of P_n can be drawn at the points $(0,0), (1,0), \ldots, (n-1,0)$, where vertex v_k is embedded at $(k-1,0)$ for $k \in \{1,2,\ldots,n-1\}$. The edges v_kv_{k+1} can be drawn as segments with length 1. Since this is the smallest possible length for each edge, $mes(P_n) = n - 1$.

□

Proposition 2. *For a cycle C_n, $n \geq 3$, any embedding of C_n in the grid has even length.*

Proof. Without loss of generality place a vertex v at the point $(0,0)$. A grid point on the cycle which is k units away from $(0,0)$ along the cycle has coordinates (a,b) where $a+b$ and k has the same parity. Since the cycle closes at $(0,0)$, its length has the same parity as 0, that is even. □

Proposition 3. *Let G be a cycle with $n \geq 3$ vertices. If n is even, then mes(G)=n. If n is odd, then mes(G)=$n+1$.*

Proof. Case 1: n is even.

Consider that a rectangle on the grid whose perimeter equals n. Place a vertex of C_n at each grid intersection. This is an embedding of the n-cycle with edge lengths of 1 as shown in Figure 5.1.

Case 2: n is odd. From Proposition 2, for an odd cycle, the lower bound for the minimum edge sum is the smallest even number larger than n. We can construct a rectangle where $es(C_n) = n + 1$, thus $mes(C_n) = n + 1$, as shown in Figure 5.2. $\square$

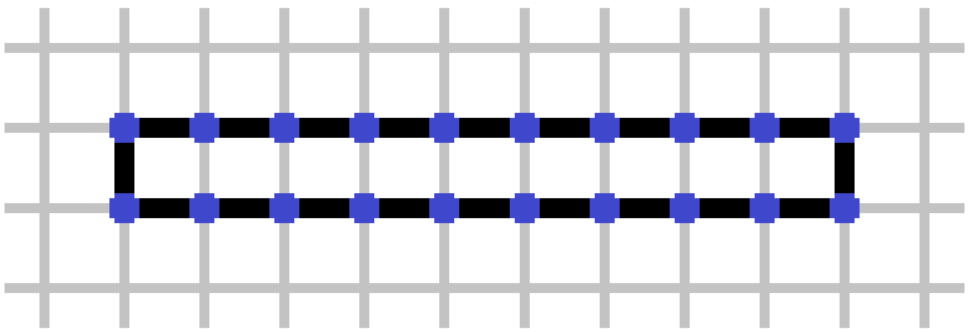

Figure 5.1: An even cycle with 20 vertices with minimum edge sum

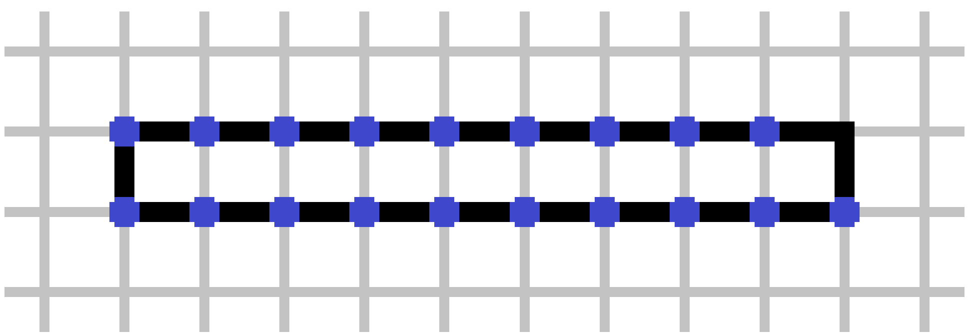

Figure 5.2: An odd cycle with 19 vertices with minimum edge sum

Corollary 1. *Let G be a maximal planar graph with n vertices and maximum degree $\Delta(G) \leq 4$. Then $4n - 8 \leq mes(G)$.*

Proof. Since every face in a maximal planar graph is a triangle, the edges and vertices in each face form a 3-cycle. Thus, we know that $mes(G) \geq \lceil \frac{4}{3}m \rceil$. For a graph with n vertices, the maximum number of edges is $3n - 6$. Therefore, we obtain $mes(G) \geq \lceil \frac{4}{3}m \rceil = \lceil \frac{4}{3}(3n-6) \rceil = 4(n-2) = 4n - 8$. □

Proposition 4. *Let G be a prism graph with $2n$ vertices.*

If $n > 3$ is even, then $3n \leq mes(G) \leq 3n + 8$.

If $n > 3$ is odd, then $3n + 2 \leq mes(G) \leq 3n + 10$.

If $n = 3$, then $mes(G) = 15$

Proof. For arbitrary n, consider a cycle of length n together with a cycle of length $n + 8$ surrounding it. Then add the n edges connecting the cycles as segments of length 1. Case 1: Let $n > 3$ be even. The number of edges is $3n$. Thus $mes(G) \geq 3n$. We have a construction of an embedding for $n = 6$ illustrated in Figure 5.3, which give $es(G) \leq 3n + 8$.

Case 2: Let $n > 3$ be odd. The number of edges is 3n. Thus $mes(G) \geq 3n$. The cycles are odd, thus we have an additional two edge lengths according to Proposition 3. Thus $3n + 2 \leq mes(G)$.

We have a construction of an embedding illustrated in Figure 5.4, which gives $mes(G) \leq 3n + 10$

Case 3: Let $n = 3$. We have a construction of an embedding illustrated in Figure 5.5, which gives $es(G) \leq 15$. □

Proposition 5. *Consider an embedding $\tilde{G}$ of G in the grid. Let v be a vertex in G with $deg(v) = 3$. Then there exist a face containing v whose area is larger than a unit square where the embedding of G is in the grid.*

Proof. For a vertex with degree 3, there will be three incident edge arcs at the vertex that form an "T" shape. Since one of the faces has a vertex on a straight line then

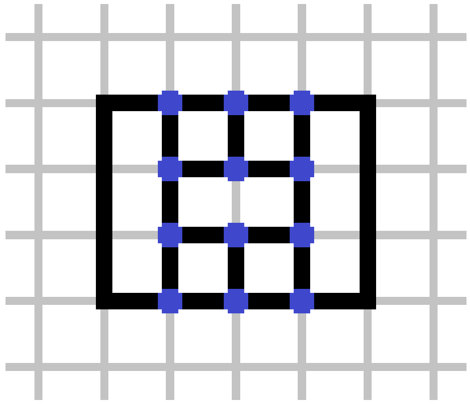

Figure 5.3: A prism graph with 12 vertices that has a minimum edge sum embedding.

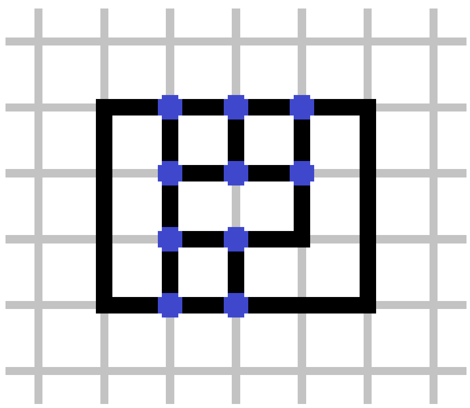

Figure 5.4: A prism graph with 10 vertices that has a minimum edge sum embedding.

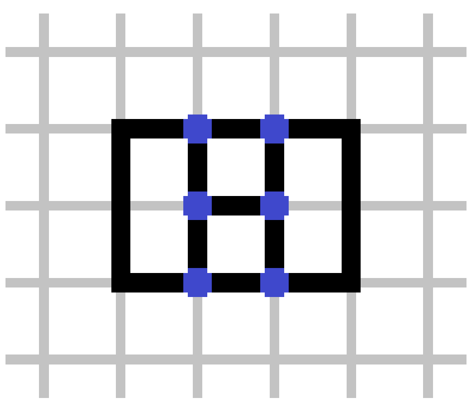

Figure 5.5: A prism graph with 6 vertices that has a minimum edge sum embedding.

that face must not be a unit square. □

Proposition 6. *There are only two wheel graphs with maximum degree $\Delta(G) \leq 4$, W_4 and W_5. The minimum edge sum for these graphs are $mes(W_4) = 11$ and $mes(W_5) =$* 12.

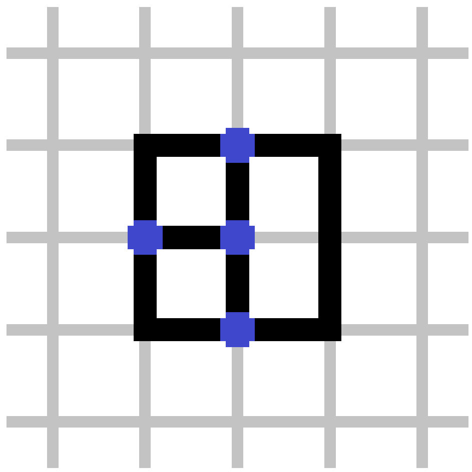

Figure 5.6: A minimum edge sum embedding of the wheel with 4 vertices.

Proof. Proof for W_4: To establish a lower bound for the minimum edge sum of W_4, we can apply Proposition 5. Therefore, we know that an embedding of W_4 with minimum edge sum has a face that is not a unit square at each vertex. Thus, the embedding shown in Figure 5.6 realize the minimum edge sum for W_4. We have $mes(W_4) = 12$.
Proof for W_5:

To establish the minimum edge sum of W_5, we consider the minimum edge sum embedding of W_4 as illustrated in 5.6. Note that adding an edge to a graph adds one to the lower bound of the minimum edge sum. Since $mes(W_4) = 11$ and W_5 has only 1 more edge, then $12 \leq mes(W_5)$. However, the construction in Figure 5.7 has $es(\tilde{W_5}) = 12$, thus $mes(W_5) = 12$.

□

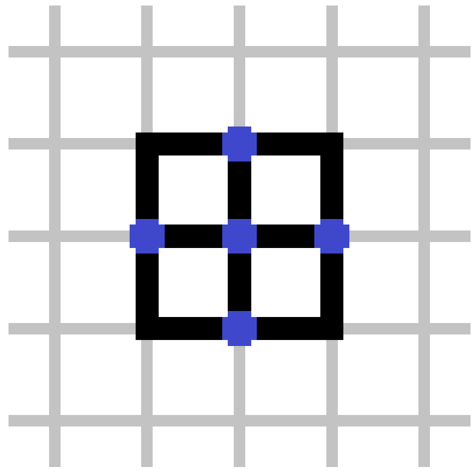

Figure 5.7: A minimum edge sum embedding of the wheel with 5 vertices .

5.2 Triangulations

In this subsection, we consider maximal planar graphs with n vertices, where $n \geq 3$.

Theorem 2. ***(Euler's Formula)*** *Let G be a connected plane graph with n vertices, m edges, and f faces, then $n - m + f = 2$.*

Corollary 2. *In a maximal planar graph with n vertices there are $3n - 6$ edges.*

Proof. A proof for both Euler's Formula and the corollary can be found in Diestel [1]. □

Proposition 7. *Every maximal planar graph embedded into the square grid has at most six vertices.*

Proof. Recall that for a graph G embedded into the square grid, $deg(v) \leq 4$ for all $v \in G$. Therefore, by the handshake lemma, we have $2m = \sum deg(v) \leq 4n$, where m is the number of edges and n is the number of vertices.

For a maximal planar graph, the number of edges can be found using Euler's formula: $3n-6=m$. Combining the two equations, we obtain $2(3n-6) \leq 4n$, which simplifies to $n \leq 6$. Therefore, every maximal planar graph that can be embedded into the square grid has at most six vertices. $\square$

Triangulations with $n = 3$, 4, or 5 vertices are unique up to isomorphism. We denote these as Tr_3, Tr_4, and Tr_5. The minimum edge sum for triangulations with $n = 3$ and $n = 4$ vertices are already known:

- $mes(Tr_3) = 4$ because Tr_3 is a cycle graph C_3.
- $mes(Tr_4) = 11$ because Tr_4 is a wheel graph W_4, which is also the complete graph K_4.

Up to isomorphism, there are two triangulations with $n = 6$ vertices, but only one has $\Delta(G) \leq 4$; we denote this graph as Tr_6.

Proposition 8. *Let Tr_5 be the triangulation of the plane with five vertices. Then $mes(Tr_5) = 22$.*

Proof. In Tr_5, there exists a 3-cycle where each vertex has degree four. Consequently, any region containing Tr_5 must be at least four grid lines wide in one direction and five grid lines wide in the other direction. This is due to the 3-cycle requiring two edges not on the cycle coming from each vertex on the cycle, and two grid lines for the cycle itself for a total minimum width of four. The width of five in the other direction is due to the 3-cycle having three vertices of degree four that need at least a length of 4 to be embedded. The 3-cycle contributes at least 12 to the edge sum since it is at least four edge lengths tall and two edge lengths wide to avoid the vertices not on the cycle. The edges branching from the cycle contribute at least 6 more to the

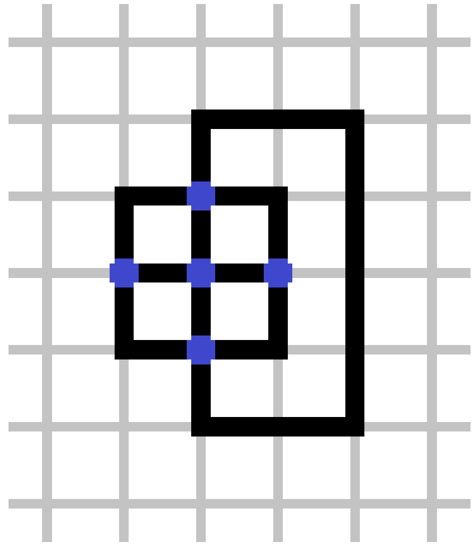

Figure 5.8: An embedding of Tr_5 with minimum edge sum.

edge sum since they are 6 edges. Additionally, the two vertices branching from the 3-cycle connect to every vertex on the cycle, each contributing at least 2 more to the edge sum since they form odd cycles. Therefore, we have $mes(Tr_5) \geq 12+6+4 = 22$. The embedding shown in Figure 5.8 shows an edge sum $es(\tilde{Tr_5}) = 22$, resulting in $mes(Tr_5) = 22$. □

Proposition 9. *Let Tr_6 be the triangulation of the plane with six vertices. Then $mes(Tr_6) \leq 36$.*

Proof. The embedding shown in Figure 5.9 shows an edge sum $es(\tilde{Tr_6}) = 36$, thus resulting in $mes(Tr_6) \leq 36$. □

5.3 Trees

If G is a tree with $\Delta(G) \leq 2$, then G is a path, and the result in Proposition 1 applies. Now, assume G is a tree with $\Delta(G) \geq 3$.

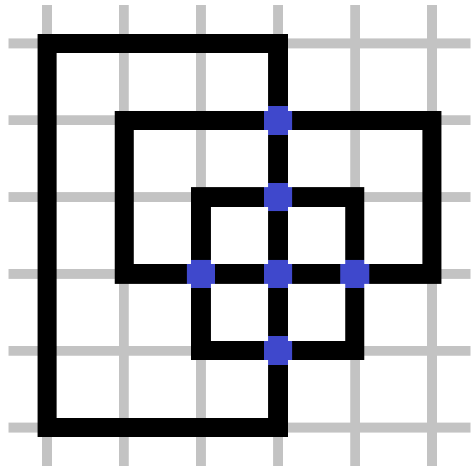

Figure 5.9: An embedding of Tr_6 with minimum edge sum.

There are a couple of trees with $\Delta(G) \geq 3$ that can be constructed from paths: caterpillars and lobsters.

Proposition 10. *Caterpillar graphs with* n *vertices have minimum edge sum of* $n-1$.

Proof. Caterpillar graphs are a type of tree where all leaves are adjacent to a path. This characteristic allows each edge in a caterpillar graph to be represented as a segment of length 1. □

Proposition 11. *Let* L_n *be a lobster graph with* n *vertices. Then* $mes(L_n) \leq \frac{11}{9}(n-8)+7$.

Proof. In a lobster graph, the maximum number of vertices on a sub-tree branching from the central path is four. We assume that every vertex on the central path, formed by removing leaves twice from the lobster, has two maximum branches of four vertices. Each vertex on this path forms a module with an edge sum of 11 and 9 vertices, except for the endpoints, which consist of an additional four vertices branch with one unit each. We have $(n-8)$ vertices in modules, and seven edges not in the modules: 4 edges on one end of the central path and 3 edges on the other end of

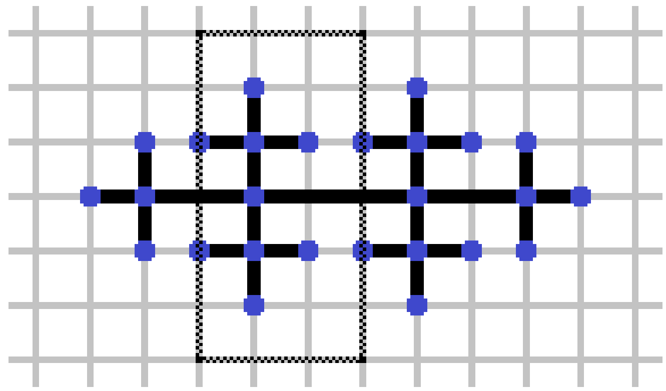

Figure 5.10: A minimum embedding for a lobster graph. The module of vertices is in a box.

the central path. Thus we have $mes(G) \leq \frac{11}{9}(n-8)+7$ Refer to Figure 5.10 for an illustration of these modules. $\square$

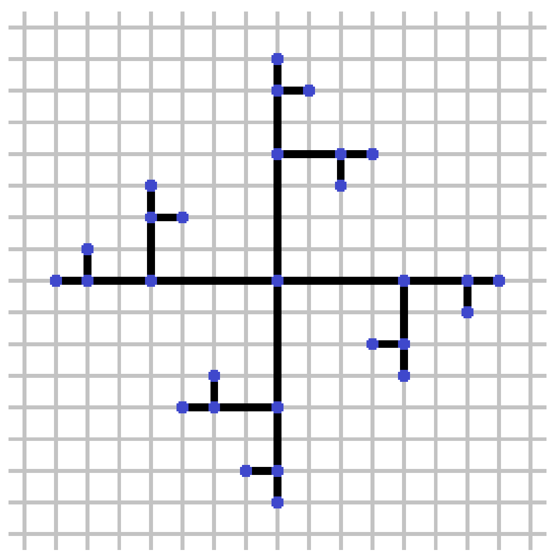

Figure 5.11: A tree of radius 3 with non-central internal vertices of degree 3.

Proposition 12. *Consider G a tree of radius(G) $= r$ whose internal vertices all have degree 3, except the central vertex which has degree b, where $2 \leq b \leq 4$. Then the* $mes(G) \leq b * r * 2^{r-1}$.

Proof. Let r represents the radius of a branch. Then in the tree, each branch connected to the central vertex has $2^{r-1}-1$ edges since going toward the leaves we double the vertices starting from the central vertex. At each level, starting from the leaves and working inward toward the central vertex, we double the lengths of the edges. Consequently, each branch will have an edge sum of

$$\sum_{i=1}^{r}(2^{r-i})(2^{i-1}) = \sum_{i=1}^{r} 2^{r-1} = r2^{r-1}.$$

Therefore, the minimum edge sum of the entire binary tree T is given by $mes(T) \leq br2^{r-1}$ for $0 \leq b \leq 4$. Refer to Figure 5.11 for a visual representation of a tree with radius 3.

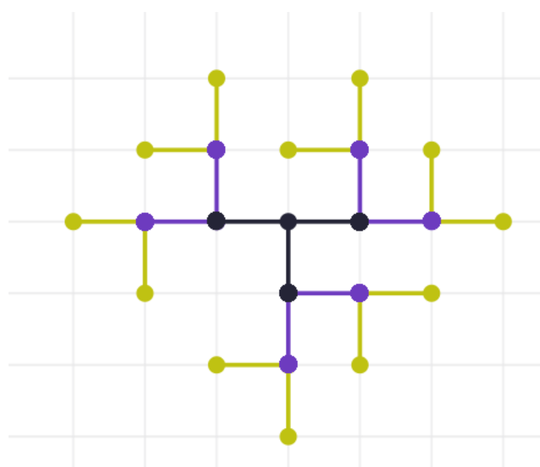

Figure 5.12: Tree with radius 3 and all internal nodes of degree 3.

$\square$

Proposition 13. *Consider a tree T_n with radius r where all internal vertices have degree 4. Such a tree has $2 \cdot (3^r - 1)$ edges and minimum edge sum of $mes(T_n) \leq 4(3^r - 2^r)$.*

Proof. As we move away from the central vertex, the number of vertices on each branch triples from the previous level, starting with one vertex adjacent to the central vertex on each branch. We can suspect that we can half the edge length, when we triple the vertices at each step from the central vertex. Specifically, for a branch at radius s from the central vertex, we have $(2^{s-1})^2$ spots for vertices, accommodating 3^{s-1} vertices. So we have that $(2^{s-1})^2 = 4^{s-1} > 3^{s-1}$, this relationship implies that the available space grows exponentially faster than the number of vertices, providing ample room for embedding the vertices.

This give the construction where we begin at the leaves, where edges have a length of 1 unit. As we move towards the central vertex, each successive edge in the path doubles in length according to the taxicab metric.

Because at each level from the central vertex we have 2^{r-1-i} units of length for 3^i edges, the total sum of edge lengths can be expressed as:

$$\begin{aligned} mes(T_n) &\leq 4\sum_{i=0}^{r-1} 2^{r-1-i}3^i \\ &= 4 \cdot 2^{r-1} \sum_{i=0}^{r-1} \left(\frac{3}{2}\right)^i \\ &= 4 \cdot 2^{r-1} \cdot 2\left(\left(\frac{3}{2}\right)^r - 1\right) \\ &= 4(3^r - 2^r) \end{aligned}$$

□

Now we describe a way to reduce the edge sum for trees with internal vertices of degree 4.

Proposition 14. *The bound in Proposition 13 can be further improved to* $mes(T_n) \leq 16(3^{r-2} - 2^{r-2}) + 4 * 3^{r-1} + 2(2^{r-1} + 2^{r-2})$.

Proof. In the construction of the tree, we are able to halve the length of any edge that is perpendicular to a longer straight series of edges. This is because, in the original construction, at each branch that turns direction, the edges can be halved a second time. The edge that could overlap is a quarter of the length of the branching edge. Therefore, we obtain the sum:

$$mes(T_n) \leq 4\sum_{i=0}^{r-3}(2^{r-3-i}\frac{2}{3}+\frac{2^{r-2-i}}{3})3^{i+1}+4*3^{r-1}+2(2^{r-1}+2^{r-2})$$

$$=4*2^{r-1}\sum_{i=0}^{r-3}(\frac{3}{2})^{i}+4*3^{r-1}+2(2^{r-1}+2^{r-2})$$

$$=4[3^{r-2}4-2^{r}]4*3^{r-1}+2(2^{r-1}+2^{r-2})$$

. □

We have developed code to compute the edge sum for trees believed to be minimal. In what follows, a tree of radius r is called the **level r** tree. The code and the tiles used can be found at https://github.com/DanielHodgins/Edge_Sum/tree/main .

The code performs a few tasks before examining an image of a graph: opening gray scale versions of reference tiles, converting gray scale tiles into numpy arrays, rotating reference tiles to create rotationally symmetric reference arrays.

A function named the "compare region" function, which iterates through reference tiles, is used to find the tile that is the least different from a region in the image of the graph. It assigns scores based on tile types: 0 for empty tiles, 0.5 for leaf tiles, 1 for degree two vertex tiles, 1.5 for degree three vertex tiles as well as the bent and straight edge tiles, 2 for degree four vertex tiles. The values for vertex tiles are halved due to the handshake lemma. The handshake lemma also applies to the non vertex tiles, since the graph is just a subgraph of the grid which is a graph where each grid intersection can be thought of as a vertex.

Another function called the "edge sum" examines the graph image by checking its width and height. It then iterates through each 16 pixel by 16 pixel region. For each region, it looks up a score using the "compare region" function and adds it to a running tally. Finally, it outputs the tally which is the total edge sum of the embedding represented by the image.

As a result, we have obtained a list of upper bound values for trees with a radius between one and nine. The edge sums for each level, denoted as es(level), are as follows: es(level1) = 4, es(level2) = 18, es(level3) = 62, es(level4) = 204, es(level5) = 640, es(level6) = 1976, es(level7) = 6026, es(level8) = 18268, and es(level9) = 55160. Here, "level" refers to the number of edges on a path connecting the central vertex to any leaf. The construction of the level 6 tree can be found in Figure 5.15.

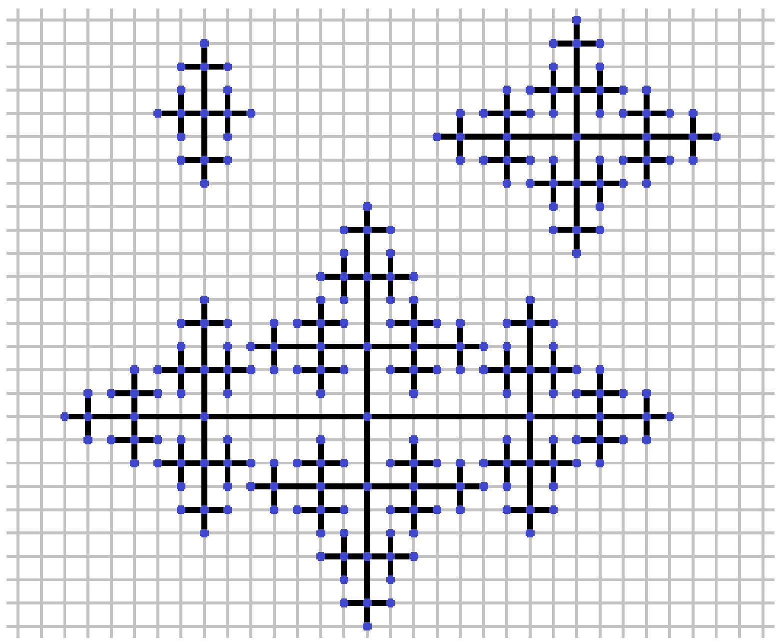

Figure 5.13: Trees of radius 2, 3, and 4 that are believed to have a minimum edge sum.

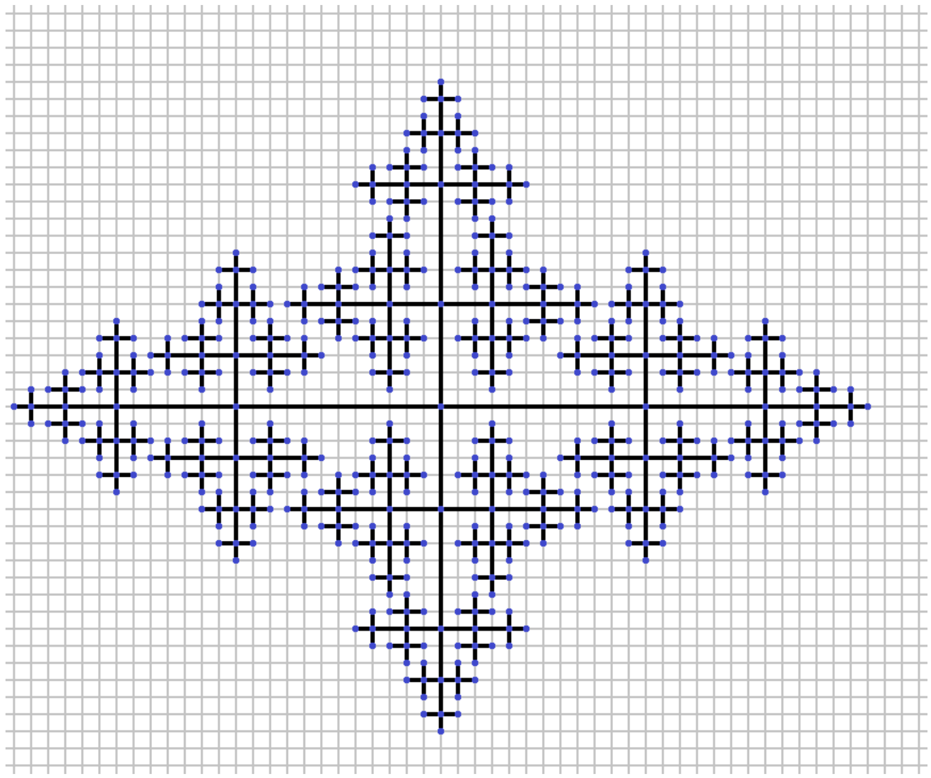

Figure 5.14: A tree of radius 5 that is believed to have a minimum edge sum.

5.4 Other extensions

In a more general context, for planar graphs with a vertex v with $deg(v) > 4$, we can replace v with a d-cycle to allow the graph to be embedded into the square grid. This replacement substitutes the single vertex v with degree d with d vertices, each with a degree of 3. As a result we could give graphs that would normally not have an edge sum a way of evaluating the lengths of its edges. However in the next Chapter we shall take a different approach and instead embed graphs into grids of higher dimensions.

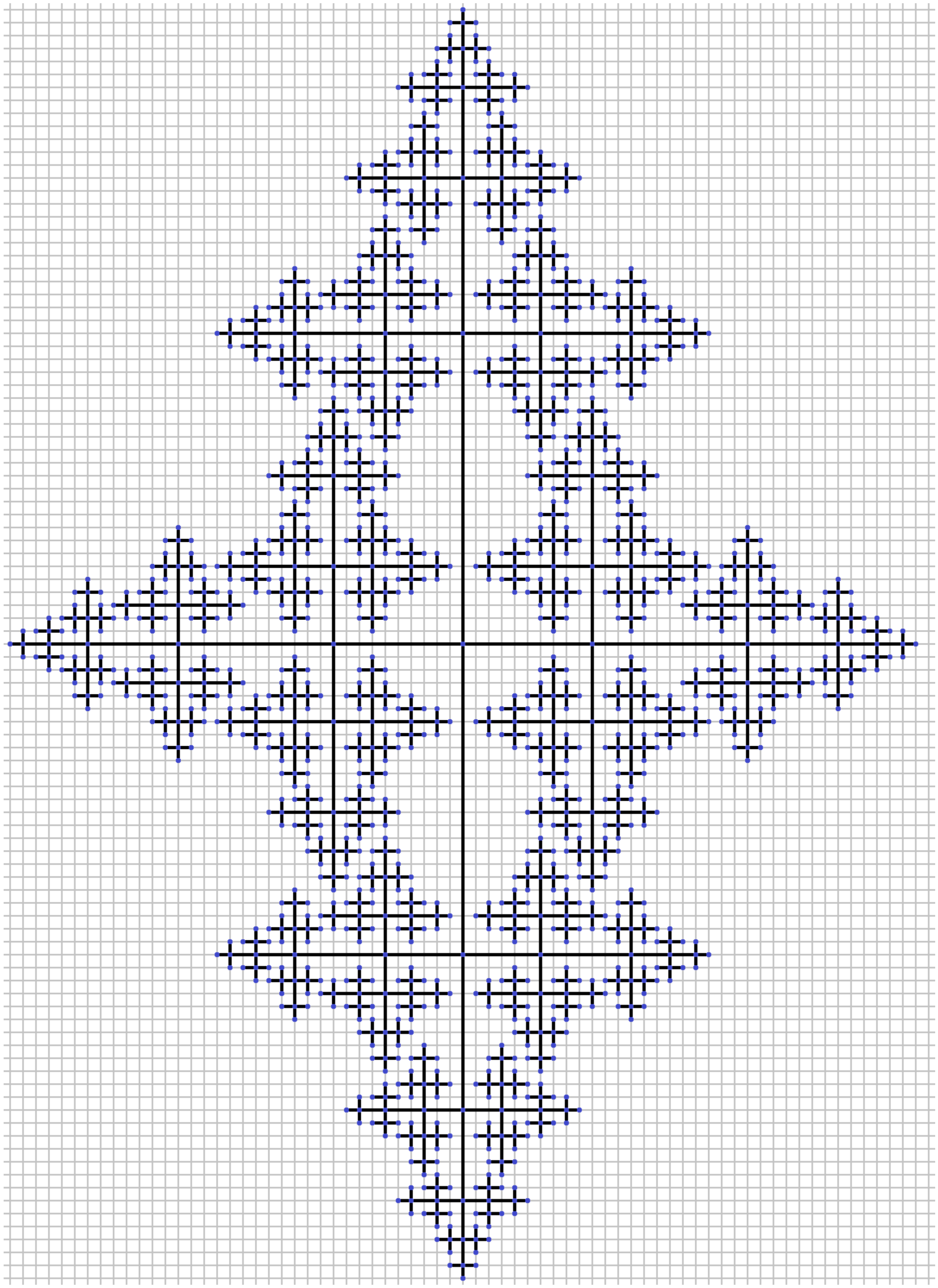

Figure 5.15: A tree of radius 6 that is believed to have a minimum edge sum.

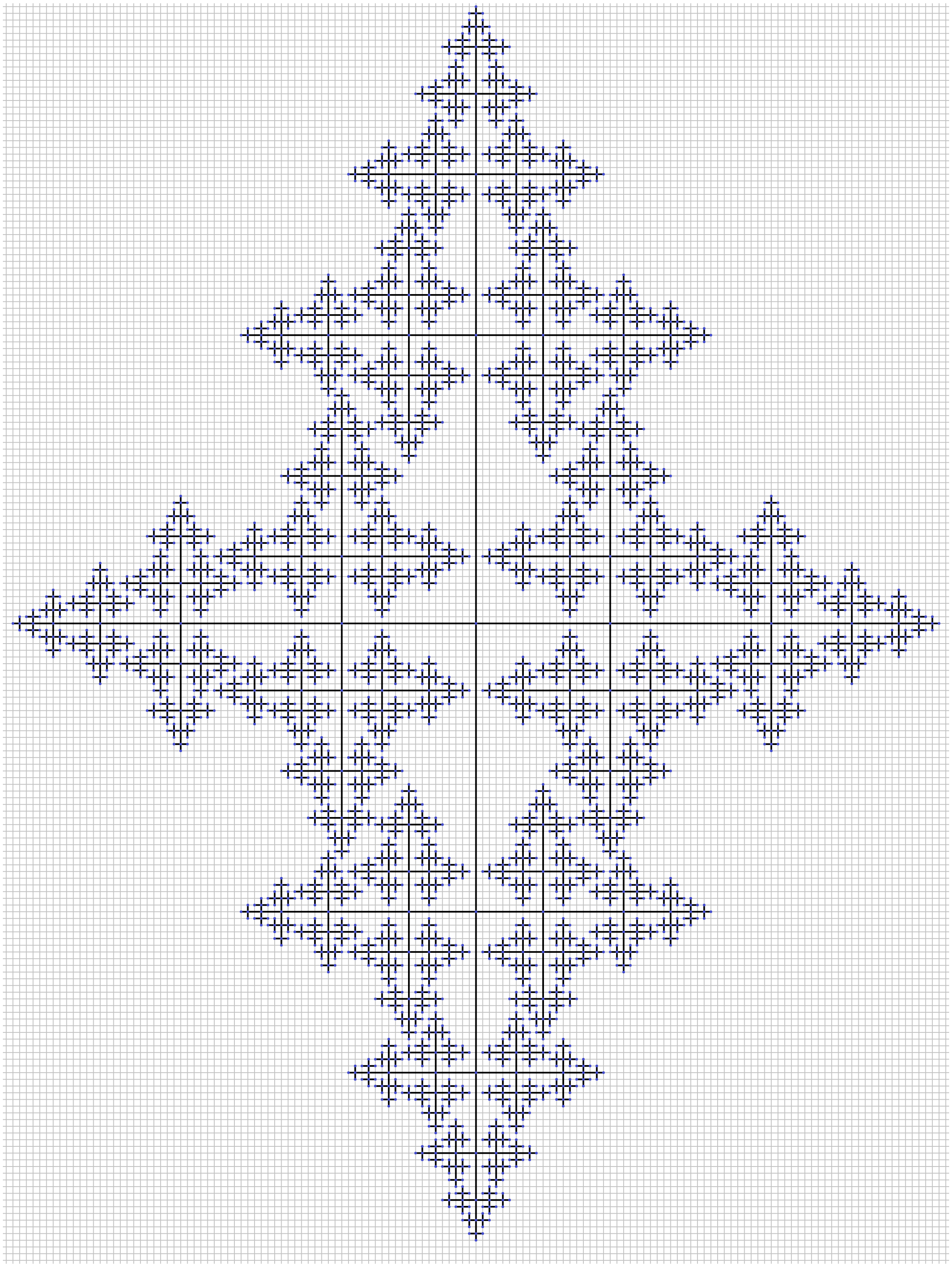

Figure 5.16: A tree of radius 7 that is believed to have a minimum edge sum.

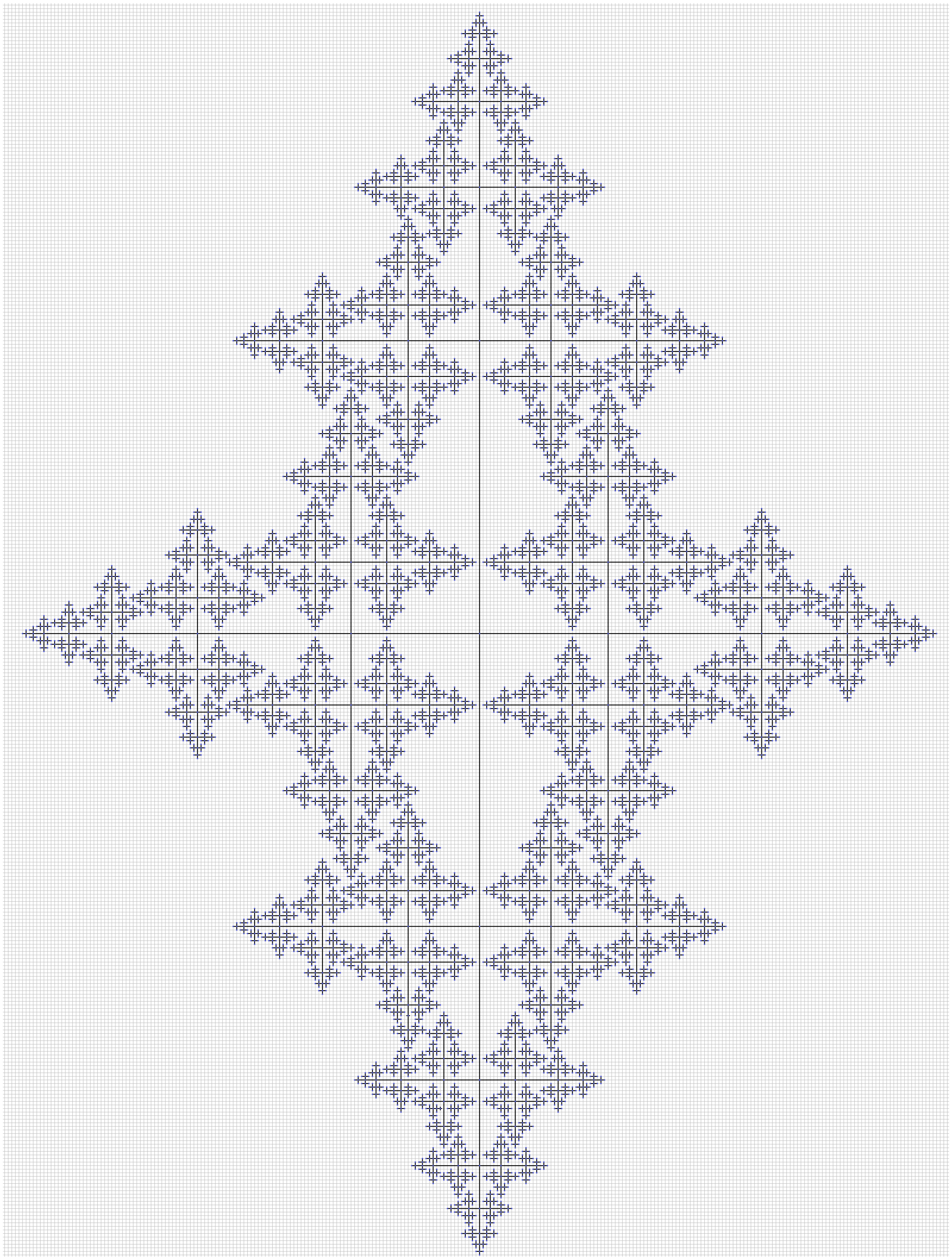

Figure 5.17: A tree of radius 8 that is believed to have a minimum edge sum.

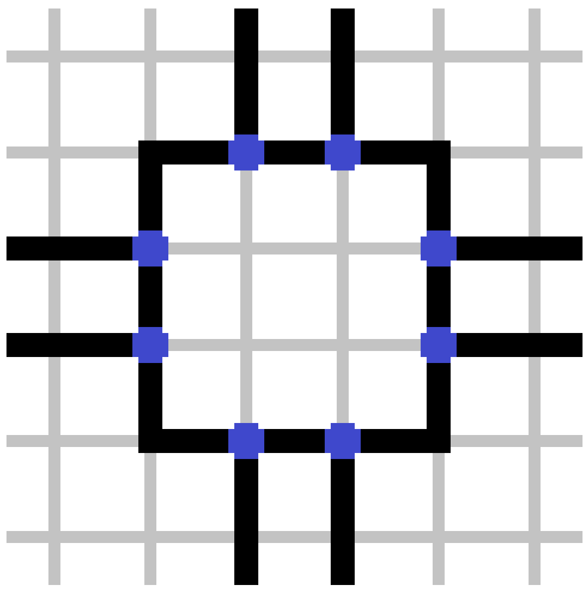

Figure 5.18: The result of a transformation of a vertex with degree 8 into a 8-cycle embedded on the grid.

CHAPTER 6

GRAPHS OF HIGHER DIMENSIONS

In this Chapter, we discuss results regarding graphs embedded in grids of dimensions higher than 2. To better explain the results we will modify the notation for minimum edge sum for a graph G to $mes_d(G)$ where d denotes the dimension of the embedding grid.

Graphs in higher dimensions are more challenging to visualize. To simplify the explanations of proofs for graphs in this section, we will adopt the following notation. Let each vertex in a graph G be notated as $v = (a_1, a_2, \ldots, a_d)$ where d notates the number of dimensions and a_k for $k \in \{1, 2, \ldots, d\}$ represents the value of the k^{th} coordinate. This notation will be referred to as **tuple notation**.

6.1 Wheels W_n

Proposition 15. *Let W_n be a wheel with n vertices. Then $mes_d(W_n) = 3n - 3$ where the dimension is $d = \lceil \frac{(n-1)}{2} \rceil$.*

Proof. The bound $mes(G) \geq 3n - 3$ discussed earlier in the section on planar wheels with $\Delta(G) \leq 4$ now becomes very useful in higher dimensions. We can obtain an embedding where each triangular face has an edge sum of 4 by the following process:

1. Place the center vertex of the wheel at $(0, 0, \ldots, 0)$, then place all other vertices at a distance of 1 from $(0, 0, \ldots, 0)$.

2. A total of $\lceil\frac{n-1}{2}\rceil$ vertices can be placed at points with a single non-zero coordinate that equals 1. The remaining vertices can be place at points with a single non-zero coordinate that equals -1.

3. Continue placing vertices until all coordinates are exhausted. The last vertex will be adjacent to the vertex at $(0, 0, \ldots, -1)$ if n is odd, or at $(0, 0, \ldots, -1, 0)$ if n is even.

4. Make the last vertex added adjacent to the second vertex added.

The resulting graph will achieve the minimum edge sum given by the equation:

$$mes_{\lceil\frac{(n-1)}{2}\rceil}(W_n) \geq 3n - 3$$

. □

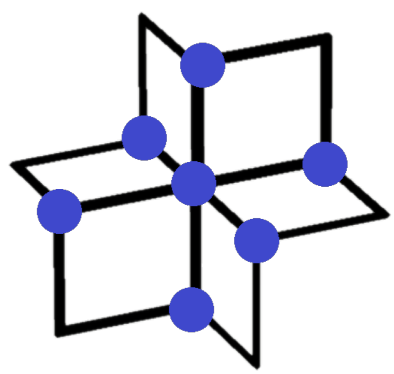

Figure 6.1: A isomorphic projection of the wheel W_7 embedded in the 3 dimensional grid with minimum edge sum for 3 dimensions.

6.2 Complete Graphs K_n

Proposition 16. *Let K_n be the complete graph with n vertices embedded into n dimensions. The minimum edge sum is $mes_n(K_n) = n^2 - 2n + 1$.*

Proof. Following the same idea as in the proofs for wheels, place one vertex at $(0, 0, \ldots, 0)$ and the other $(n-1)$ vertices at

$$(0, 1, 0, \ldots, 0, 0), (0, 0, 1, \ldots, 0, 0), \ldots, (0, 1, 0, \ldots, 1, 0), (0, 1, 0, \ldots, 0, 1)$$

. This way, every face is a minimum 3-cycle, which has a length of 4: two edges have length 1 and one edge has length 2.

All $(n-1)$ edges incident to the vertex at $(0, 0, \ldots, 0)$ have length 1. The remaining edges have length 2. There are $\binom{n-1}{2}$ edges of length 2. This gives a total edge sum of

$$(n-1) + 2\frac{(n-1)(n-2)}{2}$$

$$= (n-1)^2$$

$$= n^2 - 2n + 1$$

.

$\square$

6.3 Trees

The propositions from Chapter 5 for trees can be use to find the edge sums of trees with vertices of large degree when embedded in higher dimensional grids. Let T_r be a tree of radius r whose internal vertices have degree $2d$. This tree can be embedded on the d dimensional grid. We have the following corollary.

Corollary 3. *For T_r as described above, $mes_d(T_r) \leq 2d\frac{(2d-1)^r - 2^r}{2d-3}$ where d denotes the number of dimensions.*

Proof. Three is the number of available connections in Proposition 13, so we can replace this with $2d - 1$. The constant 4 in Proposition 13 simply counts the number of branches, and likewise can be replaced by $2d$. The rest of the proof for trees still holds.

$\square$

CHAPTER 7

FUTURE EXTENSIONS

7.1 Other Grid Tilings

Exploring alternative grid tilings could be interesting as to address the limitations imposed by vertex degree and face lengths for those tilings. For example, a hexagonal grid would feature faces with a minimum length of six, while maintaining a maximum vertex degree of three. Additionally, investigating tilings with mismatched shapes presents an interesting opportunity, as "grid lines" in such configurations may not be subject to the same maximum degree restriction and may result in location and orientation dependent edge sums.

7.2 Halin Graphs

While trees have proven to be quite cumbersome, with modifications to some of the suboptimal boundaries, a generalized Halin formula may be found.

7.3 Limit of Edge Sum

While most of the bounds have been described in terms of the number of vertices, it may prove more useful to extend this to include the number of edges, or other specific features. Doing so could provide non trivial bounds that are closer to the minimum edge sum.

7.4 Further Questions

The following are questions that are of interest.

1. When is $\frac{mes(G)}{|E|} = 1$?

2. Is there a constant C such that $\frac{mes(G)}{|E|} \leq C$ for all graphs G? If so, what is C? If not, why not?

3. For a graph G with n vertices what is the upper bound for $mes(G)$?

   We suspect that exploring what happens when taking an embedding from a higher dimension to a one lower dimension may be insightful.

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BIOGRAPHICAL SKETCH

Name of Author: Daniel Hodgins

Graduate and Undergraduate Schools Attended:

University of South Alabama, Mobile, Alabama

Degrees Awarded:

Master of Science in Mathematics, 2024, Mobile, Alabama

Bachelor of Science in Mechanical Engineering, 2023, Mobile, Alabama

Awards and Honors:

"Chris Nash Special Award (Highest Score) " Mobile, Alabama Spring 2023

"Pi Mu Epsilon Math Honor Society" Mobile, Alabama Spring 2022

"Mortar Board Senior Honor Society" Mobile, Alabama Spring 2022

"Department of Mechanical Engineering Scholarship" Mobile, Alabama Spring 2022

"Chris Nash Scholarship (Second Highest Score) " Mobile, Alabama Spring 2022

"Department of Mathematics and Statistics" Mobile, Alabama Spring 2022

"Chris Nash Scholarship (Highest Score) " Mobile, Alabama Spring 2021

"Department of Mathematics and Statistics" Mobile, Alabama Spring 2021

"Eagle Scout" Mobile, Alabama Spring 2018