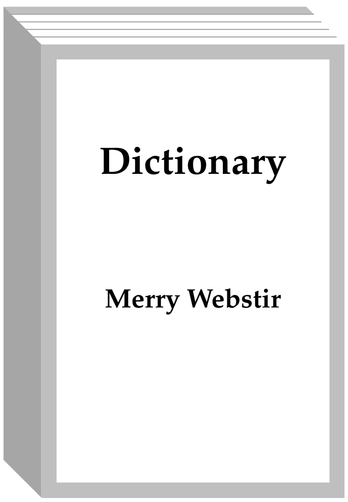
Outcomes: By the end of this lesson, we aim for you to be able to

- Understand what list coloring is
- Understand the basics of the proof for the choosability of planar graphs
- To know some applications of list coloring

Definitions

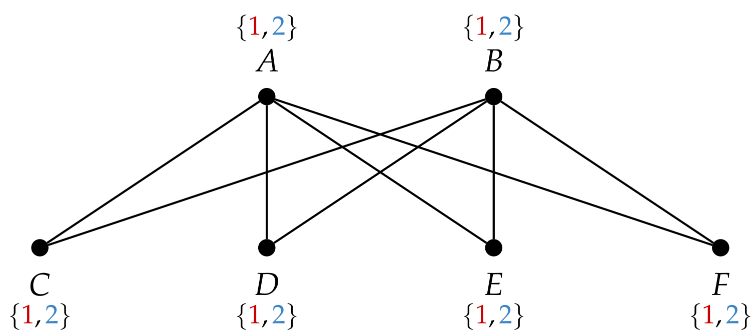


1. For each vertex v in the graph G, let $L(v)$ denote a set of available colors at vertex v. A **list coloring** is a proper coloring f of G such that $f(v) \in L(v)$ for all v in G.

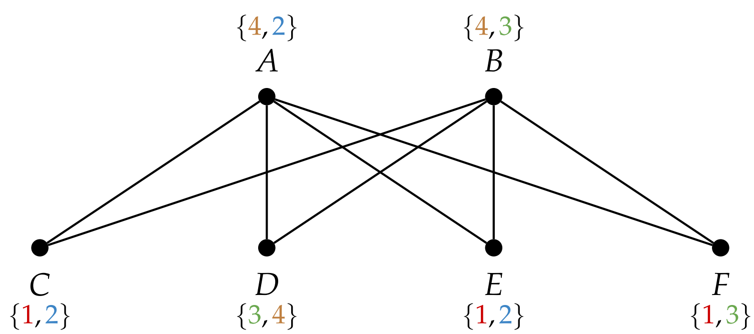
2. A graph G is ***k*-choosable** if every assignment of the k-element sets to the vertices allows a proper coloring.

3. The **list chromatic number**, denoted $\chi_l(G)$, is the minimum k s.t. G is k-choosable.

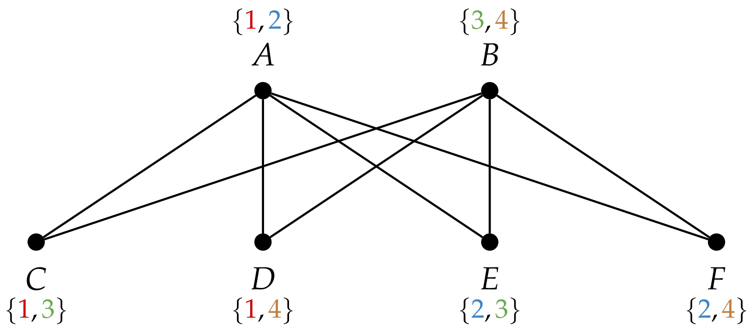
Example 1.



Example 2.



You try!



Remark:
What can we say about the relationship between traditional coloring $\chi(G)$ and list coloring $\chi_l(G)$?

Question:
How large can $\chi_l(G) - \chi(G)$ be?

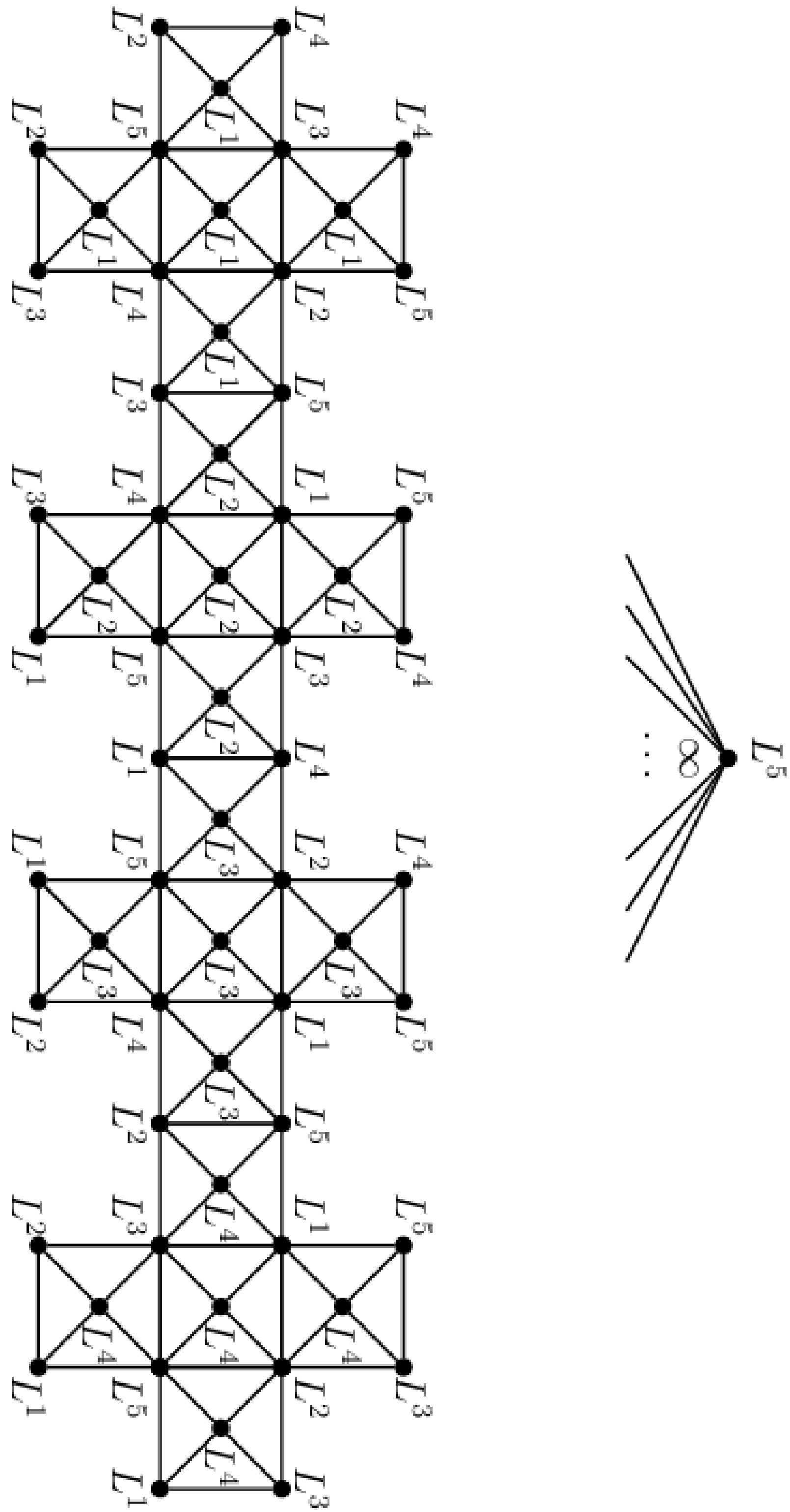
You Try!
Prove that $\chi_l(G) \leq 1 + \Delta(G)$

Choosability of Planar Graphs

Background:
Theorem Planar graphs have chromatic number at most 4, Vizing [1976].

Conjecture Erdős-Rubin-Taylor [1979] The maximum list chromatic number for planar graphs is 5.

Lemma There are some planar graphs that are not 4-choosable, Voigt[1993] and Mirzakhani [1996].



Remark Some 3 colorable graphs are not 4-choosable, Gutner [1996], Voigt-Wirth [1997].

Theorem 8.4.32 For planar graphs, $\chi_l(G) \leq 5$, Thomassen [1994b].

Proof. The addition of edges obviously do not decrease $\chi_l(G)$, thus we shall consider plane graphs where the outer face is a cycle and every bounded face is a triangle.
We shall prove a stronger result: a coloring can be choosen even when two adjacent external vertices have distinct list of size 1 and the other external vertices have size 3, (internal vertices have list of size 5).
Now we shall use induction on the number of vertices.
Base step: $n = 3$
This is clear since this is a triangle, which means the third vertex always has an usable color.
Now consider $n > 3$. Let v_p and v_1 be the vertices with singleton lists. Let the external face have cycle C and $v_1, v_2, \ldots, v_p$ be in $V(C)$ in a negative direction around the cycle.

Case 1: C has a chord $v_i v_j$ s.t. $1 \leq i \leq j-2 \leq p-2$. We apply the induction hypothesis to the graph consisting of the cycle $v_1, \ldots, v_j, \ldots, v_p$ and the interior. This process reduces v_i and v_j to have forced singleton colors then we apply induction to the cycle $v_i, v_{i+1}, \ldots, v_j$ and its interior to obtain the list coloring of G.

Case 2: C has no chord. Let v_1, u_2, v_3 be the neighbors of v_2 in order. Because the interior is triangulated, G contains a path P with vertices $v_1, u_1, \ldots, u_m, v_3$, the neighbors of v_2. Since C is chordless $u_1, \ldots, u_m$ are interior vertices, and the outer face of $G' = G \setminus v_2$ is bounded by a cycle C' where v_2 is replaced by P.

Let c be the color assigned to v_1. Since $|L(v_2)| \geq 3$, we may choose distinct colors $x, y \in L(v_2) \setminus \{c\}$. Thus, we are disallowing x, y from $u_1, \ldots, u_m$ such that we can color v_2 with x or y. Since $|L(u_i)| \geq 5$, we have $|L(u_i) \setminus \{x, y\} \geq 3$. Thus, inductively $u_1, \ldots, u_m$ have list of size at least 3 with other vertices' lists being unchanged. We extend this to G by choosing a color for v_2 in the set $\{x, y\}$ that does not appear on v_3, this is sufficient since v_2 is the only additional vertex in G.

$\square$