Longest Narrow Path Polyominoes

Auburn REU Seminar Talk

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Outline

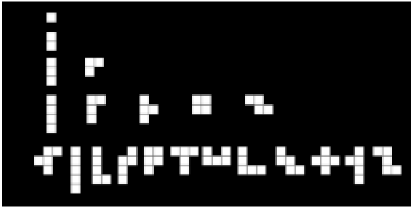
Polyominoes

Definition (Solomon Golomb 1966)

An n-omino is a set of n rook-wise connected unit squares.

(Alternative) Definition

A shape formed by a set of unit squares such that each unit square is orthogonal to another unit square.



Problem Statement

- For some $n \in \mathbb{N}$, using a set of one of each of the n-ominoes what is the longest possible "path", interior hole, which is only one unit square wide at any position and is solidly surrounded (diagonals are filled in)?

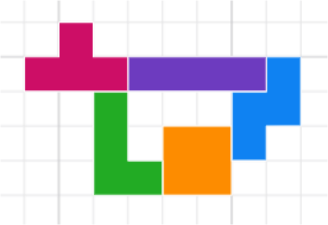


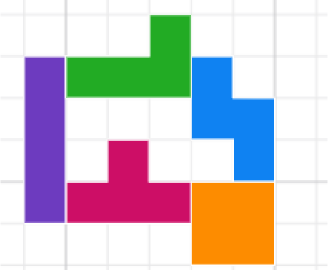
Figure: A length 4 path where $n = 4$.

Known Lower Bounds

- monominoes, dominoes, and triominoes are impossible.
- Kadon Enterprises asked about hexomminoes, Aad van de Wetering gave a construction for 146 in 2004.
- Aad van de Wetering 2004, Eric Harshbarger 2005, and Dan Edlefsen 2006 each found a unique 36 path length for pentominoes

Tetrominoes Lower Bound

- There are no apparent results online, probably due to the "simplicity" of this version.
- Here is a construction for length of 6, which is probably the best you can do.
- You can probably solve this one using a computer search (relatively small search space).



Main Result

- Tetrominoes: lower bound 6, upper bound 8
- Pentominoes: lower bound 36, upper bound 38
- Hexominoes: lower bound 146, upper bound 162

Hexomino Lower Bound

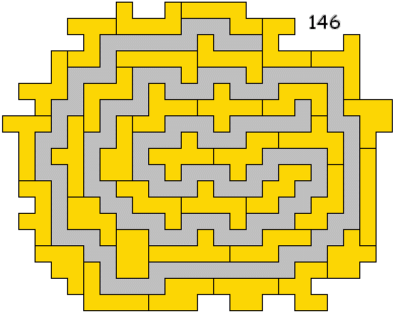


Figure: Aad van de Wetering 146 length hexomino construction

Main Result

- Tetrominoes with Pentominoes: lower bound 44, upper bound 54
- Tetrominoes with Hexominoes*:lower bound 146, upper bound 180
- Pentominoes with Hexominoes*:lower bound 146, upper bound 218
- all three*: lower bound 146, 236

Proof

Trivial proof outline:

Step 1

Find the total perimeter of the desired set of polyominoes.

Step 2

Consider the average number of polyominoes a polyomino will border: 2.

Subtract these border segments twice (Handshake Lemma) from the perimeter to obtain T

Trivial upper bound: $\frac{T-2}{2}$

Proof

Proof outline:
Step 1
Note the area of the polyominoes and add that to the know lower bound length to obtain the minimum area A of the longest narrow path configuration.
Step 2
By the Isoperimetric Theorem for rectangles with fixed integer perimeters, the square and m by $m+1$ rectangle are the shapes with maximum area.

Proof

Proof continue:
Thus, we find the smallest m such that A fits in a m by m or m by $m+1$ rectangle. Thus, the minimum exterior perimeter of the configuration of the longest narrow path for by the polyominos is $4m$ or $4m+4$.
Step 3
Upper bound: $\frac{T-4m\pm 2}{2}$

Limitations of the Proof

- The proof works for any set of polyominoes.
- With improved lower bound, the upper bound can improve
- Once the lower and upper bound become close, like for pentominoes, the proof can not narrow down the gap any more
- the proof does not address any constraints due to the shape of the polyominos beyond area and perimeter

Open Problems

1. Q: Using pentominoes, can you construct a path such that the resulting configuration fits in a 10 by 10 square with no more than one pair of pentominoes having a border of length 2 with the rest of the pairs having borders of at most 1?
2. Q: Using pentominoes can you construct a path that fits in a 10 by 11 rectangle with every pair of pentominoes sharing at most a border of length 1?
3. Q: What are lower bounds for multiple sets of polyominoes? (Only two sets of pentominoes have a nontrivial lower bound that is known)

Pentomino Lower Bound

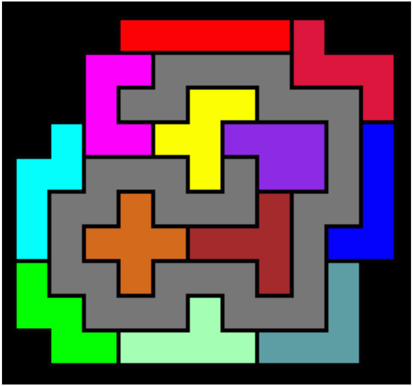


Figure: Aad van de Wetering 36 length pentomino construction

Near Miss Pentomino Lower Bound

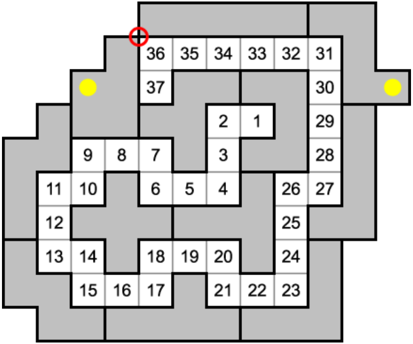


Figure: Erich Harsbarger in 2018 found a "near miss" length 37

Variations

1. V: Polyominoes use squares, what about using other regular polygons?
2. V: The problem for $n > 8$ polyominoes is ill defined, and heptominoes are also possibly ill define due to having holes. How would you generalize polyominoes with holes?
3. V: What about volume paths of polycubes, or 3D paths of polyominoids?
4. V: Allow touching diagonally or not require the path to be border at corners. (this variation for pentominoes have shown up in *The Journal of Recreational Mathematics*)

Questions?

Thank you for your time.

And thanks to Eric Harshbarger for maintaining much of the previously known results and introducing the problem to me.