Linear Quadratic regulators

LQR + LQI:

Assume that we have a regular state space model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The LQR control law

$$u = -L\,x + Li\,x_i$$

with integral action LQI

$$\dot{x}_i = r - y = r - Cx - Du$$
$$x_i = \int \dot{x}_i = \int (r - Cx - Du)$$

This will create the augmented state space model.

Also added precompensator vector BK_r where $K_r \geq 0, \in \Re^m, B \in \Re^{nxm}$. That vector need to be specified from the user.

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} [u] + \begin{bmatrix} BK_r \\ 1 \end{bmatrix} [r]$$
$$y = Cx + Du$$

With the LQR and LQI control law. The augmented state space model will then be:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} (A - BL) & BLi \\ (DL - C) & -DLi \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} BK_r \\ 1 \end{bmatrix} [r]$$
$$[y] = \begin{bmatrix} (C - DL) & DLi \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix}$$

This is what Matacecontrol is using.

LQR + LQE: Also called LQG controller

Assume a state space gaussian model

$$\dot{x} = Ax + Bu + B_d d$$
$$y = Cx + Du + B_n n$$

And a state state space gaussian observer

$$\dot{\hat{x}} = A\,\hat{x} + Bu + Ke$$
$$\hat{y} = C\,\hat{x} + Du$$
$$e = y - \hat{y} = Cx + Du + B_n n - C\,\hat{x} - Du = Cx + B_n n - C\,\hat{x}$$

The LQR control law

$$u = -L\,x + r$$

One important step is to create one error state of diffrence between real state and estimated state

$$\tilde{x}=x-\hat{x}$$

This will change the control law to a new control law

$$u=-L\hat{x}+r=L\tilde{x}-Lx+r$$

Because

$$\tilde{x}-x=-\hat{x}$$

Insert the new control law into the state space gaussian model

$$\dot{x}=Ax+B(L\tilde{x}-Lx+r)+B_d d$$
$$\dot{x}=(A-BL)x+BL\tilde{x}+Br+B_d d$$

Insert the new control law to the output of the state space gaussian model

$$y=Cx+D(L\tilde{x}-Lx+r)+B_n n$$
$$y=(C-DL)x+DL\tilde{x}+Dr+B_n n$$

Insert the new control law into the state state space gaussian observer

$$\dot{\hat{x}}=A\hat{x}+Bu+Ke$$
$$\dot{\hat{x}}=A\hat{x}+B(L\tilde{x}-Lx+r)+K(Cx+B_n n-C\hat{x})$$
$$\dot{\hat{x}}=A\hat{x}+BL\tilde{x}-BLx+Br+KCx+KB_n n-KC\hat{x}$$

Assume now

$$\dot{\tilde{x}}=\dot{x}-\dot{\hat{x}}$$
$$\dot{\tilde{x}}=[(A-BL)x+BL\tilde{x}+r+B_d d]-[A\hat{x}+BL\tilde{x}-BLx+r+KCx+KB_n n-KC\hat{x}]$$
$$\dot{\tilde{x}}=Ax-BLx+BL\tilde{x}+r+B_d d-A\hat{x}-BL\tilde{x}+BLx-r-KCx-KB_n n+KC\hat{x}$$
$$\dot{\tilde{x}}=Ax+B_d d-A\hat{x}-KCx-KB_n n+KC\hat{x}$$

Remember

$$\tilde{x}=x-\hat{x}$$

Which results:

$$\dot{\tilde{x}}=Ax+B_d d-A\hat{x}-KCx-KB_n n+KC\hat{x}$$
$$\dot{\tilde{x}}=A(x-\hat{x})+B_d d-KC(x-\hat{x})-KB_n n$$
$$\dot{\tilde{x}}=(A-KC)\tilde{x}+B_d d-KB_n n$$

The whole state space model will then be: Also added precompensator factor K_r where $K_r\geq 0,\in\Re^m, B\in\Re^{nxm}$. That factor is computed automaticly.

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} (A-BL) & BL \\ 0 & (A-KC) \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BK_r & B_d & 0 \\ 0 & B_d & -KB_n \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

$$y = \begin{bmatrix} (C-DL) & DL \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} D & 0 & B_n \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Because the most important equations are:

$$\dot{x} = (A-BL)x + BL\tilde{x} + Br + B_d d$$
$$\dot{\tilde{x}} = (A-KC)\tilde{x} + B_d d - KB_n n$$
$$y = (C-DL)x + DL\tilde{x} + Dr + B_n n$$

That was the deep theory!

But the state space model in Matavecontrol is:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} (A-BL) & BL \\ 0 & (A-KC) \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BK_r & B_d & 0 \\ 0 & B_d & -KB_n \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

$$\begin{bmatrix} y \\ y_m \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ C & 0 \\ -L & L \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & B_n \\ K_r & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Where y_m is the measured signal and y is the filtered output.

LQR + LQE + LQI: (Not included in Matavecontrol)

Assume a state space gaussian model

$$\dot{x} = Ax + Bu + B_d d$$
$$y = Cx + Du + B_n n$$

And a state state space gaussian observer

$$\dot{\hat{x}} = A\hat{x} + Bu + Ke$$
$$\hat{y} = C\hat{x} + Du$$
$$e = y - \hat{y} = Cx + Du + B_n n - C\hat{x} - Du = Cx + B_n n - C\hat{x}$$

The LQR control law

$$u = -L\hat{x} + Li\, x_i$$

with integral action LQI

$$\dot{x}_i = r - y = r - Cx - Du - B_n n$$
$$x_i = \int \dot{x}_i = \int (r - Cx - Du - B_n n)$$

One important step is to create one error state of diffrence between real state and estimated state

$$\tilde{x} = x - \hat{x}$$

This will change the control law to a new control law

$$u=-L\hat{x}+Li\,x_i=L\tilde{x}-L\,x+Li\,x_i$$

Because

$$\tilde{x}-x=-\hat{x}$$

Insert the new control law into the integral state

$$\dot{x}_i=r-y=r-Cx-D(L\tilde{x}-L\,x+Li\,x_i)-B_n n$$
$$\dot{x}_i=r-y=r-Cx-DL\tilde{x}+DL\,x-DLi\,x_i-B_n n=r+(DL-C)\,x-DL\tilde{x}-DLi\,x_i-B_n n$$

Insert the new control law into the state space gaussian model

$$\dot{x}=Ax+B(L\tilde{x}-L\,x+Li\,x_i)+B_d d$$
$$\dot{x}=(A-BL)\,x+BL\tilde{x}+BLi\,x_i+B_d d$$

Insert the new control law to the output of the state space gaussian model

$$y=Cx+D(L\tilde{x}-L\,x+Li\,x_i)+B_n n$$
$$y=(C-DL)\,x+DL\tilde{x}+DLi\,x_i+B_n n$$

Insert the new control law into the state state space gaussian observer

$$\dot{\hat{x}}=A\hat{x}+Bu+Ke$$
$$\dot{\hat{x}}=A\hat{x}+B(L\tilde{x}-L\,x+Li\,x_i)+K(Cx+B_n n-C\hat{x})$$
$$\dot{\hat{x}}=A\hat{x}+BL\tilde{x}-BL\,x+BLi\,x_i+KCx+KB_n n-KC\hat{x}$$

Assume now

$$\dot{\tilde{x}}=\dot{x}-\dot{\hat{x}}$$
$$\dot{\tilde{x}}=[(A-BL)\,x+BL\tilde{x}+BLi\,x_i+B_d d]-[A\hat{x}+BL\tilde{x}-BL\,x+BLi\,x_i+KCx+KB_n n-KC\hat{x}]$$
$$\dot{\tilde{x}}=Ax-BLx+BL\tilde{x}+BLi\,x_i+B_d d-A\hat{x}-BL\tilde{x}+BL\,x-BLi\,x_i-KCx-KB_n n+KC\hat{x}$$
$$\dot{\tilde{x}}=Ax+B_d d-A\hat{x}-KCx-KB_n n+KC\hat{x}$$

Remember

$$\tilde{x}=x-\hat{x}$$

Which results

$$\dot{\tilde{x}}=Ax+B_d d-A\hat{x}-KCx-KB_n n+KC\hat{x}$$
$$\dot{\tilde{x}}=A(x-\hat{x})+B_d d-KC(x-\hat{x})-KB_n n$$
$$\dot{\tilde{x}}=(A-KC)\tilde{x}+B_d d-KB_n n$$

The whole state space model will then be:

$$\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\\ \dot{x}_i\end{bmatrix}=\begin{bmatrix}(A-BL) & BL & BLi\\ 0 & (A-KC) & 0\\ (DL-C) & -DL & -DLi\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\\ x_i\end{bmatrix}+\begin{bmatrix}0 & B_d & 0\\ 0 & B_d & -KB_n\\ 1 & 0 & -B_n\end{bmatrix}\begin{bmatrix}r\\ d\\ n\end{bmatrix}$$
$$y=\begin{bmatrix}(C-DL) & DL & DLi\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\\ x_i\end{bmatrix}+\begin{bmatrix}0 & 0 & B_n\end{bmatrix}\begin{bmatrix}r\\ d\\ n\end{bmatrix}$$

Because the most important equations are:

$$\dot{x} = (A - BL)x + BL\tilde{x} + BLi\,x_i + B_d d$$
$$\dot{\tilde{x}} = (A - KC)\tilde{x} + B_d d - KB_n n$$
$$\dot{x}_i = r + (DL - C)x - DL\tilde{x} - DLi\,x_i - B_n n$$
$$y = (C - DL)x + DL\tilde{x} + DLi\,x_i + B_n n$$