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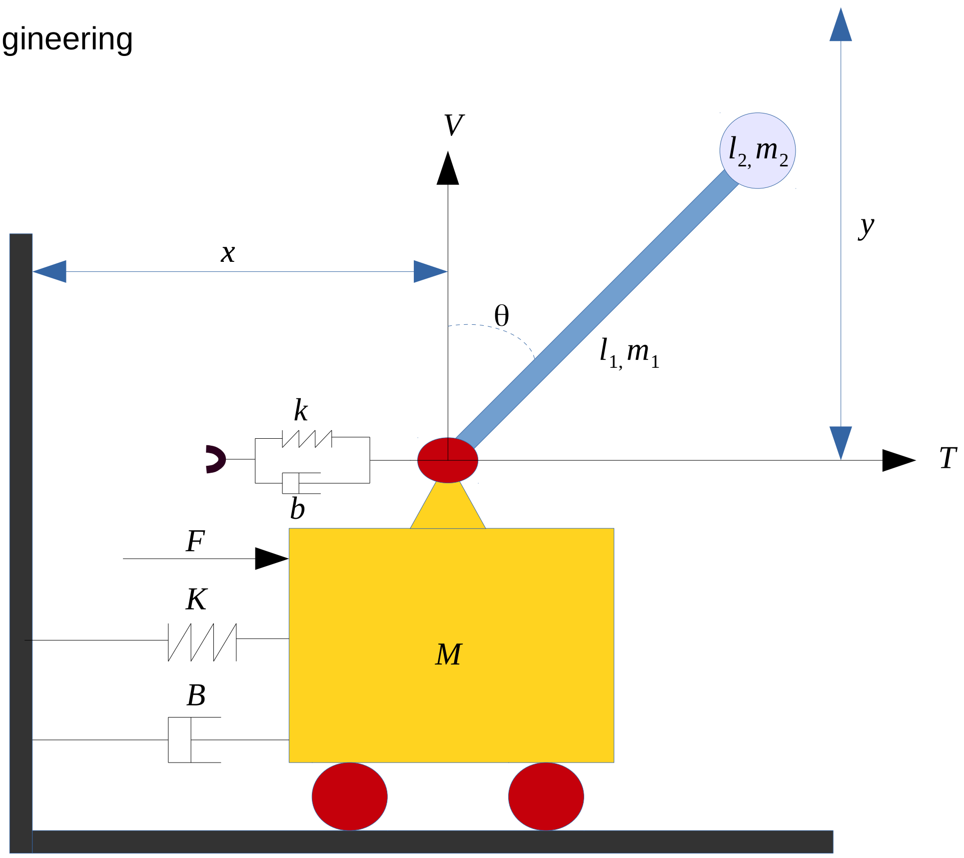

Langrage potential energy:

$$V = m_2 g l_2 \cos\theta + m_1 g l_1 \frac{1}{2} \cos\theta$$

Langrage kinect energy:

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_2 (\dot{x} + l_2 \dot{\theta} \cos\theta)^2 + \frac{1}{2} m_2 (-l_2 \dot{\theta} \sin\theta)^2 + \frac{1}{2} m_1 (\dot{x} + l_1 \dot{\theta} \cos\theta)^2 + \frac{1}{2} m_1 (-l_1 \dot{\theta} \sin\theta)^2$$

Lagrangian:

$$L = T - V$$

$$L = \frac{1}{2} M \dot{x}^2 - \frac{1}{2} g l_1 m_1 \cos\theta - g l_2 m_2 \cos\theta + \frac{1}{2} l_1^2 m_1 \sin^2\theta\, \dot{\theta}^2 + \frac{1}{2} l_2^2 m_2 \sin^2\theta\, \dot{\theta}^2 + \frac{1}{2} m_1 (l_1 \dot{\theta} \cos\theta + \dot{x})^2 + \frac{1}{2} m_2 (l_2 \dot{\theta} \cos\theta + \dot{x})^2$$

Langrage equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad Q_i = \begin{bmatrix} F - Kx - B\dot{x} \\ -k\theta - b\dot{\theta} \end{bmatrix}, q_1 = x, q_2 = \theta, \dot{q}_1 = \dot{x}, \dot{q}_2 = \dot{\theta}$$

Nonlinear state space model:

$$M\ddot{x} - l_1 m_1 \dot{\theta}^2 \sin\theta + l_1 m_1 \ddot{\theta} \cos\theta - l_2 m_2 \dot{\theta}^2 \sin\theta + l_2 m_2 \ddot{\theta} \cos\theta + m_1 \ddot{x} + m_2 \ddot{x} = F - Kx - B\dot{x}$$

$$-\frac{1}{2} g l_1 m_1 \sin\theta - g l_2 m_2 \sin\theta + l_1^2 m_1 \ddot{\theta} + l_1 m_1 \ddot{x} \cos\theta + l_2^2 m_2 \ddot{\theta} + l_2 m_2 \ddot{x} \cos\theta = -k\theta - b\dot{\theta}$$

Linear state space:

Lineareize points:

$$\sin\theta=\theta \qquad M\ddot{x}+l_1 m_1\ddot{\theta}+l_2 m_2\ddot{\theta}+m_1\ddot{x}+m_2\ddot{x}=F-Kx-B\dot{x}$$

$$\cos\theta=1 \qquad -\frac{1}{2}g\,l_1 m_1\theta-g\,l_2 m_2\theta+l_1{}^2 m_1\ddot{\theta}+l_1 m_1\ddot{x}+l_2{}^2 m_2\ddot{\theta}+l_2 m_2\ddot{x}=-k\theta-b\dot{\theta}$$

$$\dot{\theta}^2=0$$

Simplifying second order system:

$$\begin{bmatrix} M+m_1 m_2 & l_1 m_1+l_2 m_2 \\ l_1 m_1+l_2 m_2 & l_1{}^2 m_1+l_2{}^2 m_2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}=\begin{bmatrix} -K & -B & 0 & 0 \\ 0 & 0 & \left(\frac{1}{2}g\,l_1 m_1+g\,l_2 m_2-k\right) & -b \end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}F$$

Transformation:

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}=\begin{bmatrix} M+m_1 m_2 & l_1 m_1+m_2+m_2 \\ l_1 m_1+m_2 & l_1{}^2 m_1+l_2{}^2 m_2 \end{bmatrix}^{-1}\begin{bmatrix} -K & -B & 0 & 0 \\ 0 & 0 & \left(\frac{1}{2}g\,l_1 m_1+g\,l_2 m_2-k\right) & -b \end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix}+\begin{bmatrix} M+m_1 m_2 & l_1 m_1+m_2+m_2 \\ l_1 m_1+m_2 & l_1{}^2 m_1+l_2{}^2 m_2 \end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}F$$

$$\dot{x}=M^{-1}Ax+M^{-1}Bu$$

$$\Omega\leftarrow M^{-1}A\leftarrow\begin{bmatrix} \frac{K(l_1^2m_1+l_2^2m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} & \frac{B(l_1^2m_1+l_2^2m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} & \frac{(l_1m_1+l_2m_2)(gl_1m_1+2gl_2m_2-2k)}{2\left((l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)\right)} & -\frac{b(l_1m_1+l_2m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} \\ -\frac{K(l_1m_1+l_2m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} & -\frac{B(l_1m_1+l_2m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} & -\frac{(M+m_1+m_2)(gl_1m_1+2gl_2m_2-2k)}{2(l_1m_1+l_2m_2)^2-2(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} & \frac{b(M+m_1+m_2)}{(l_1m_1+l_2m_2)^2-(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} \end{bmatrix}$$

$$\Gamma\leftarrow M^{-1}B\leftarrow\begin{bmatrix} \frac{-(-l_1m_1-l_2m_2)(l_1m_1+l_2m_2)-(l_1m_1+l_2m_2)^2+(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)}{\left(-(l_1m_1+l_2m_2)^2+(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)\right)(M+m_1+m_2)} \\ \frac{-l_1m_1-l_2m_2}{-(l_1m_1+l_2m_2)^2+(l_1^2m_1+l_2^2m_2)(M+m_1+m_2)} \end{bmatrix}$$

First order system state space representation:

$$\begin{bmatrix}\dot{r}_1\\ \dot{r}_2\\ \dot{r}_3\\ \dot{r}_4\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \Omega(1,1) & \Omega(1,2) & \Omega(1,3) & \Omega(1,4) \\ 0 & 0 & 0 & 1 \\ \Omega(2,1) & \Omega(2,2) & \Omega(2,3) & \Omega(2,4) \end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}+\begin{bmatrix}0\\ \Gamma(1,1)\\ 0\\ \Gamma(2,1)\end{bmatrix}F \qquad r_1=x,r_2=\dot{x},r_3=\theta=r_4=\dot{\theta}$$

$$\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}F$$