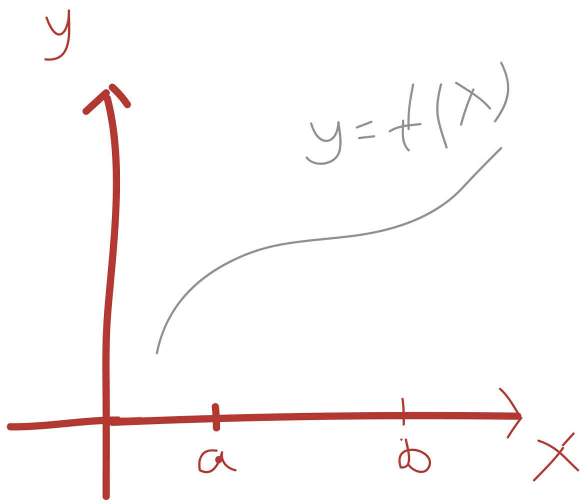
1 1. Plan area

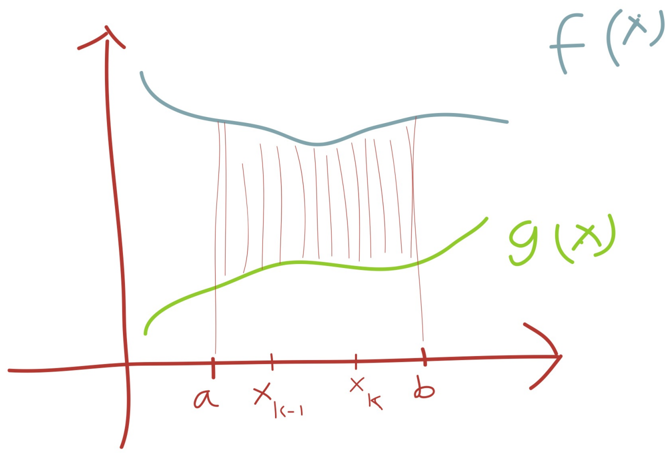


Repetition

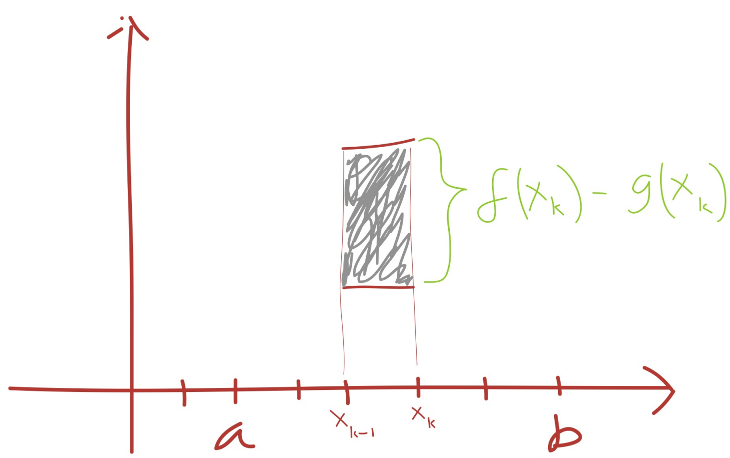
$$\int_a^b f(x)\ dx = \lim \sum_k f(x_k)\Delta x_k$$

$$D = \{(x,y) : a \leq x \leq b, g(x) \leq y \leq f(x)\}$$

där g, f är kontinuerliga.



$$A(D) = \int_a^b (f(x) - g(x))\, dx$$



$$\Delta x_k = x_k - x_{k-1}$$

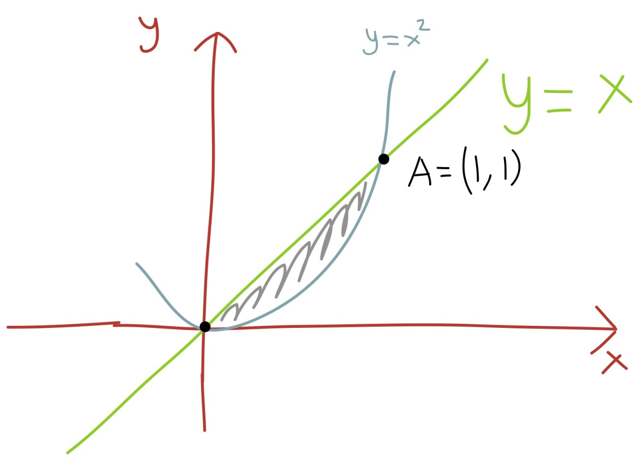
$$\Delta A_k \approx (f(x_k) - g(x_k))\Delta x_k$$

$$A(D) \approx \sum \Delta A = \sum f(x_k) - g(x_k)\Delta x_k$$

$$\Delta x_k \to 0 \implies A(D) = \int_a^b (f(x) - g(x))\, dx$$

1.1 Exempel 1

Bestäm arean av det område som begränsas av $y = x$ och $y = x^2$

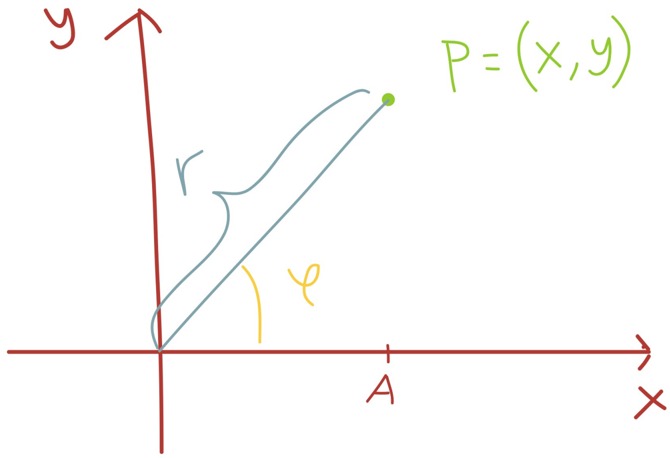


$$D = \{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq x\}$$

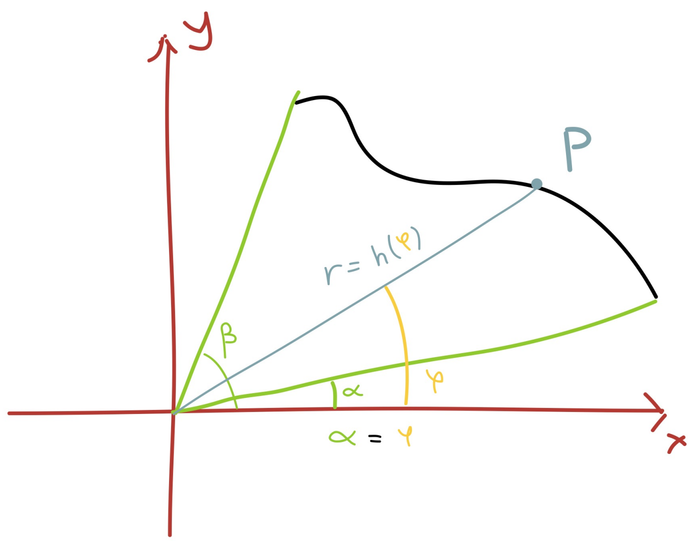
$A(D) = \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right] = \frac{1}{6}$

2 Arean på polär form

Polära koordinater (ϕ, r)



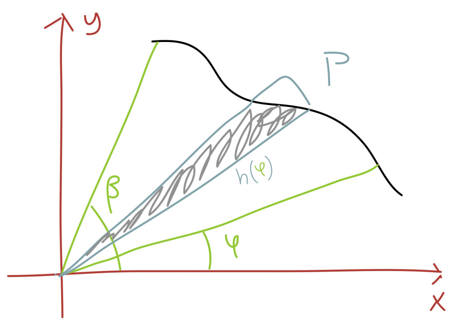
$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

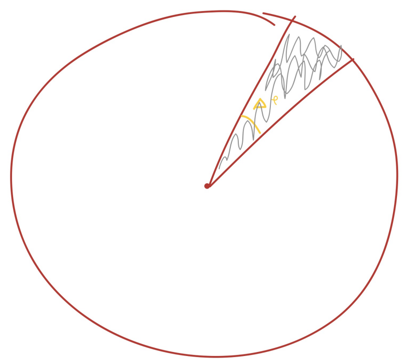


$$D = \{(x, y) : \alpha \leq \phi \leq \beta, 0 \leq r \leq h(\phi)\}$$

Om h är kontinuerlig då kan vi beräkna arean av D som:

$$A(D) = \frac{1}{2}\int_{\alpha}^{\beta} h^2(\phi)\ d\phi$$





$\Delta\phi$ är litet

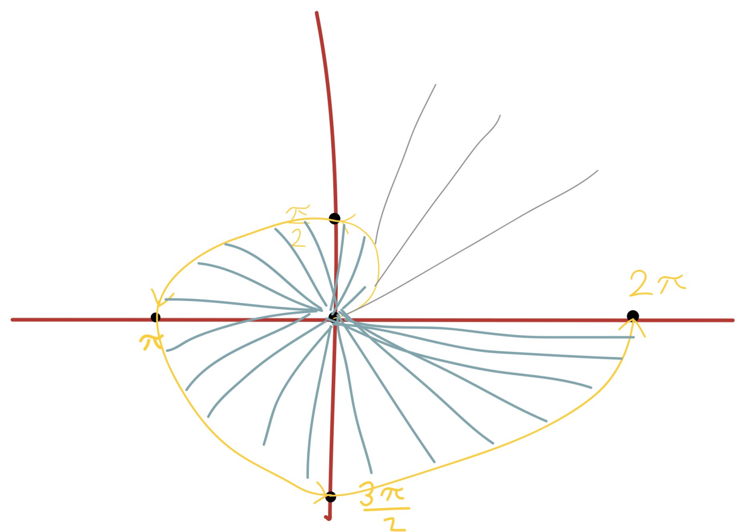
$$\Delta A = \frac{\pi h^2(\phi)}{2\pi} * \Delta\phi$$

$$A(D) = \sum \Delta A = \sum \frac{1}{2} h^2(\phi)\Delta\phi$$

$$\Delta\phi \to 0 \implies A(D) = \frac{1}{2}\int_{\alpha}^{\beta} h^2(\phi)\ d\phi$$

2.1 Exempel 2

En kurva ges i polära koordinater av $r = \phi, 0 \leq \phi \leq 2\pi$ Bestäm arean av område som innesluts av kurvan.



$$\begin{cases} \phi = 0 \implies r = 0 \\ \phi = \frac{\pi}{2} \implies r = \frac{\pi}{2} \\ \phi = \pi \implies r = \pi \\ \phi = \frac{3\pi}{2} \implies r = \frac{3\pi}{2} \end{cases}$$

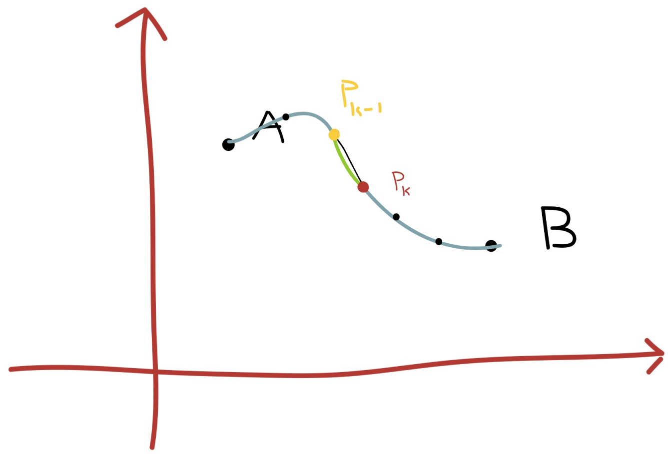
$$A(D) = \frac{1}{2}\int_0^{2\pi} \phi^2 \ d\phi = \frac{1}{2}\left[\frac{1}{3}\phi^3\right]_0^{2\pi} = \frac{4}{3}\pi^3$$

3 Kurvlängd

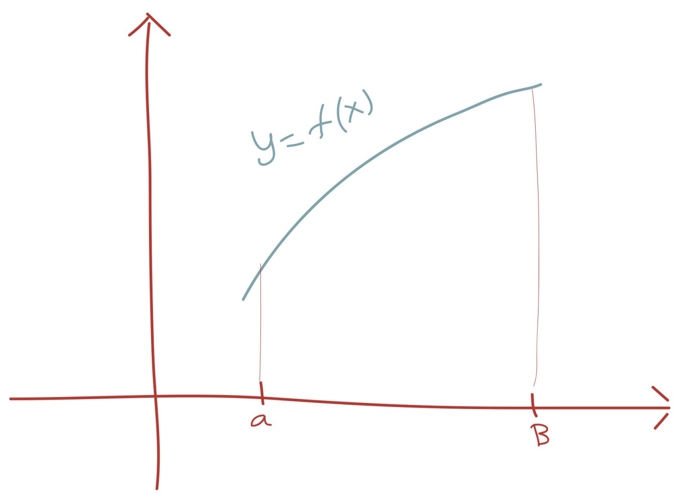
3.1 Kurvor på parameterform

$$x(t), y(t)$$

- kontinuerligt deriverbara



$$a \leq t \leq b$$
$$t \longmapsto (x(t), y(t))$$



$$t = a \implies (x(a), y(a)) = A$$

$$t = b \implies (x(b), y(b)) = B$$

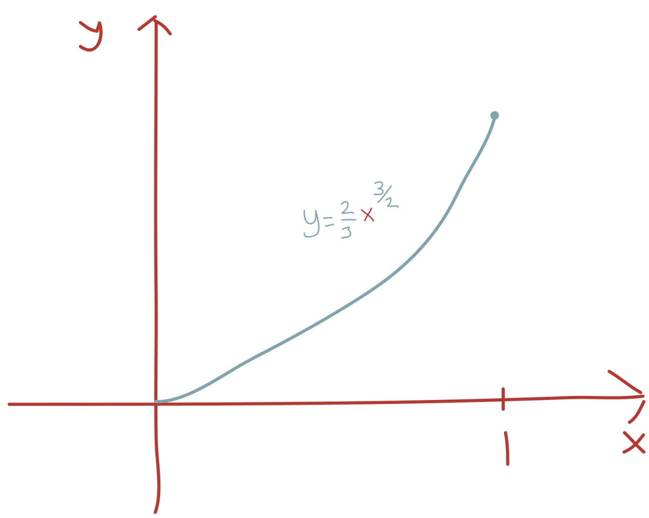
$$\Delta t_k = t_k - t_{k-1}$$

$$\Delta s_k = \sqrt{(x(t_k)) - x(t_{k-1}))^2 + (y(t_k)) - y(t_{k-1}))^2} =$$
$$\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2} * \Delta t_k}{\Delta t_k} =$$
$$\sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{\Delta t_k^2}} * \Delta t_k =$$
$$\sqrt{\left(\frac{\Delta x}{\Delta t_k}\right)^2 + \left(\frac{\Delta y}{\Delta t_k}\right)^2} * \Delta t_k \to \sqrt{x'^2 + y'^2}\ dt, \Delta t_k \to 0$$

$$ds = \sqrt{x'^2 + y'^2}\ dt$$

$$s = \int_a^b ds = \int_a^b \sqrt{x'^2(t) + y'^2(t)}\ dt$$

3.2 Funktionskurvor



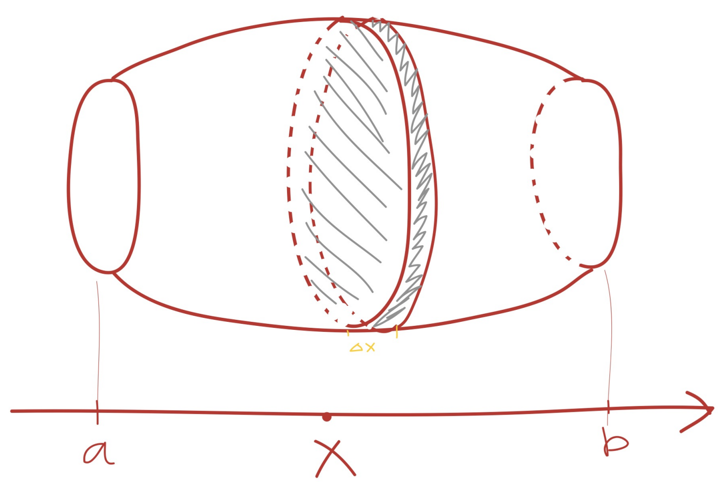
$$y = f(x), a \leq x \leq b$$

$$\begin{cases} x = t, a \leq t \leq b \\ y = f(t) \end{cases}$$

$$x'(t) = 1$$
$$y'(t) = f'(t)$$

$$s = \int_a^b \sqrt{1 + f'^2(t)}\ dt$$
$$s = \int_a^b \sqrt{1 + f'^2(x)}\ dx$$

3.3 Exempel 3



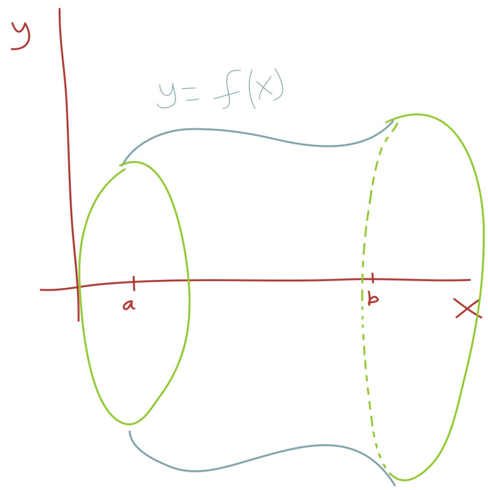
Bestäm längden av $y = \frac{2}{3}x^{\frac{3}{2}}, 0 \leq x \leq 1$

$$f(x) = \frac{2}{3}x^{\frac{3}{2}}$$

$$f'(x) = x^{\frac{1}{2}}$$

$$s = \int_0^1 1 + x\ dx = \left[\frac{1 + x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

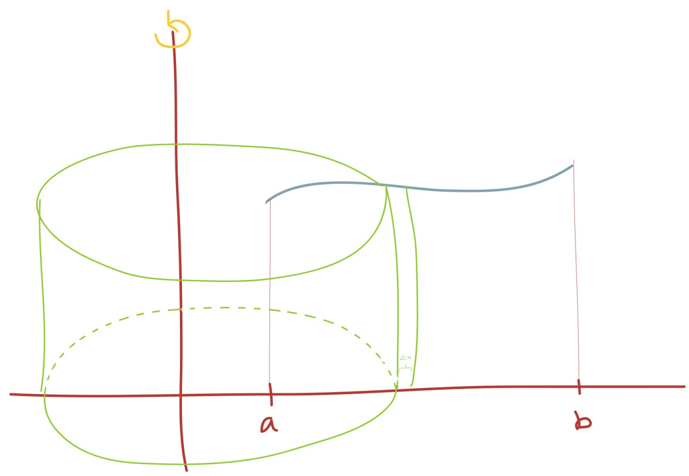
4 Volym



$$dV = A(x)\Delta x$$

$$V = \int_a^b A(x)\ dx$$

4.1 Rotationsvolym, skivformeln



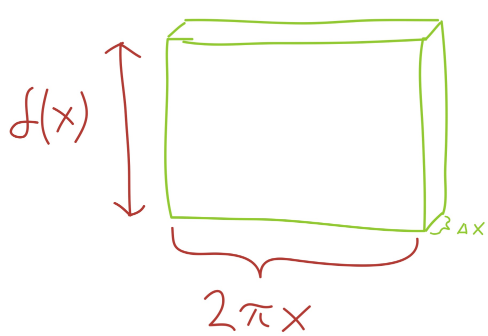
$$D = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$$

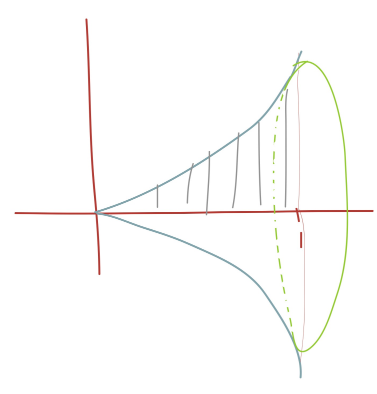
$$A(x) = \pi f^2(x)$$

Skivformeln:

$$V = \pi \int_a^b f^2(x)\ dx$$

4.2 Rotationsvolym, rörformeln





$$dV = 2\pi x f(x) \Delta x$$

Rörformeln:

$$V = 2\pi \int_a^b x f(x)\ dx$$

4.3 Exempel 4

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$$

Roteras ett varv kring x-axeln.

$$dV = \pi (x^2)^2\ dx$$

$$V = \pi \int_0^1 x^4\ dx = \left[\pi \frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

Om D roteras ett varv kring y-axeln.

$$V = 2\pi \int_0^1 x(x^2)\ dx = 2\pi \int_0^1 x^3\ dx = 2\pi \left[\frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}$$