

# 1 Summary

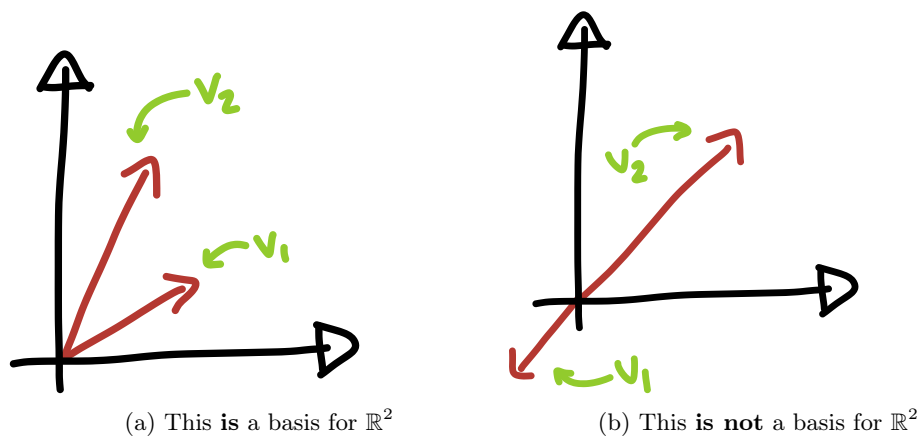
This is an informal overview of the "system och signaler" course.

## 2 Linear algebra precursor

To fully motivate the definition of the fourier transform, we'll need a bit of linear algebra first.

### 2.1 Basis

I'll begin with the concept of a *basis*. A list of vectors  $v_1, \dots, v_n$  can be thought of as a basis if every other vector can be represented by the basis by uniquely scaling and adding them.



#### 2.1.1 Example

For example the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a basis for  $\mathbb{R}^2$ .

#### 2.1.2 Question

Are the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

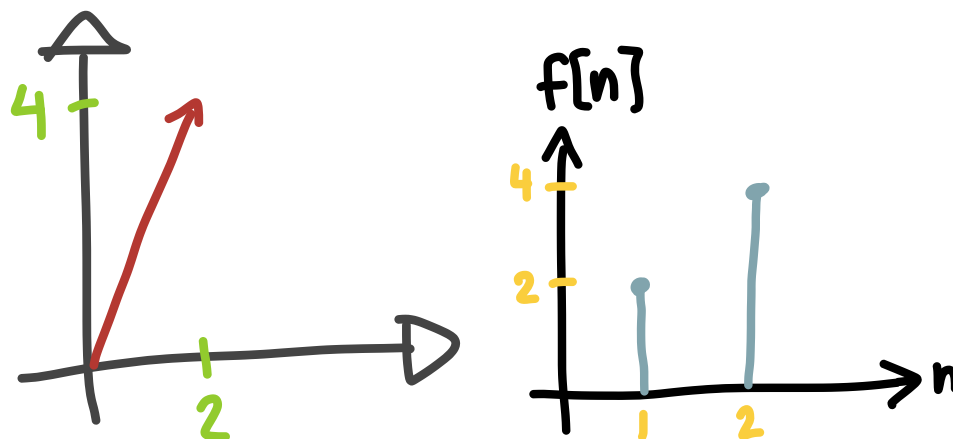
a basis for  $\mathbb{R}^2$ ? Why or why not?

## 2.2 Functions are vectors

From linear algebra we're used to thinking of vectors like

$$v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

as arrows, perhaps on the cartesian plane in the case of 2D vectors.

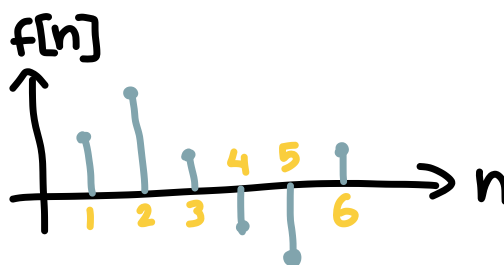


But an important insight we'll need on our journey into the fourier transform is that functions can actually be seen as vectors! To motivate this, let's just think about how we could represent the  $v$  above as a function  $f$ . Well the first entry has value 2, and the second entry has value 4, so

$$f[1] = 2, f[2] = 4$$

That certainly makes a lot of sense! And you could easily imagine how to convert arbitrary-dimensional vectors into functions.

Using this notation we can efficiently plot vectors which would otherwise be impossible, like 6-dimensional vectors!



The whole point of this has been to illustrate that functions are indeed also vectors, which means it makes sense to define the dot-product between two functions, the length (aka norm) of a function, etc. In the next section we'll see how the dot-product of two functions could be defined.

(So far we've been working with discrete functions, but using similar arguments it's also possible to see that continuous functions are vectors.)

## 2.3 Dot-product ("skalärprodukt")

From linear algebra we learned that a dot-product between two real vectors (i.e. with all elements real valued)

$$v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, w = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

is defined as

$$v \cdot w = a_1 b_1 + \cdots + a_n b_n$$

Note that  $v \cdot v = a_1^2 + \cdots + a_n^2 = \|v\|^2$ , since  $\|v\|$  is defined as  $\|v\| = \sqrt{a_1^2 + \cdots + a_n^2}$ .

Above we restricted ourselves to vectors with real-valued entries, but what about vectors with complex-valued entries? What happens if we use the above definition without modification? Well suppose we want to find out the length of  $u = \begin{bmatrix} i \\ 0 \end{bmatrix}$ . Intuitively you'd think the length of this would be 1, i.e.  $\|u\| = 1$ . But look what happens if we try to calculate  $u$  using the fact that  $\|u\| = \sqrt{u \cdot u}$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{i^2 + 0^2} = i$$

So we get a complex length! Now that's not very intuitive!

Instead of using this definition, we'll modify it slightly to ensure the norm is always a positive real. Recall that if  $z$  is a complex number, then  $z\bar{z}$  is always a positive real. Using this as inspiration we define the dot-product of two complex vectors (i.e. with complex-valued entries)

$$v' = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, w' = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

to be

$$v' \cdot w' = z_1 \bar{h}_1 + \cdots + z_n \bar{h}_n$$

Notice that when  $v'$  and  $w'$  are real, this turns into the normal (real) dot-product (since  $a = \bar{a}$ , for real  $a$ ). So we've really just extended the notion of the dot-product, not replaced it.

### 2.3.1 Dot-products of functions

So far we've only looked at the dot-product of vectors in  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , but it's actually possible to also define the dot-product of functions.

To motivate the definition, let's recall that for two complex vectors

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

the following is their dot-product

$$v \cdot w = v_1 \bar{w}_1 + \cdots + v_n \bar{w}_n$$

note that this can also be written as

$$v \cdot w = \sum_{k=1}^n v_k \overline{w_k}$$

So the natural definition for a couple of discrete functions  $t[x], h[x]$ , where  $x$  goes from 1 to  $n$ , is just

$$t[x] \cdot h[x] = \sum_{k=1}^n t[k] \overline{h[k]}$$

What about continuous functions? We'll begin with periodic functions. When we go from the discrete to the continuous, series turn into integrals, thus it's pretty natural that the definition should be something like

$$p(x) \cdot q(x) = \int f(x) \overline{g(x)} dx$$

where  $p(x)$  and  $q(x)$  are periodic with period  $T_0$ . Notice that I didn't write the integration limits. What should these be? Well since an integral during a period of a periodic function really contains all the information about the area we need, it'd be reasonable to have the limits be one of the periods (e.g. from 0 to  $T_0$ )

$$p(x) \cdot q(x) = \int_{T_0} f(x) \overline{g(x)} dx$$

Note that the integral will be equivalent no matter which period we choose.

What about aperiodic functions  $f(x)$  and  $g(x)$ ? These can be seen as periodic functions but with an infinite period, so we just extend the above to

$$f(x) \cdot g(x) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

Sometimes the dot-product between two functions is denoted  $\langle f, g \rangle$ , to avoid confusion from normal multiplication. I'll use this notation in the remainder of this document.

## 2.4 Orthogonality

Orthogonal is just a fancy word for perpendicular.

So the definition of two vectors  $v$  and  $w$  being orthogonal is just that

$$v \cdot w = 0$$

or, using the notation for the dot-product between function two functions  $f$  and  $g$ ,

$$\langle f, g \rangle = 0$$

We're especially interested in orthogonal bases, i.e. where all vectors in the basis are pairwise orthogonal. Why do we care about these types of bases? To motivate this, let's investigate what happens when we have a basis which **isn't** pairwise orthogonal, like

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Since  $v_1 \cdot v_2 = 4 \neq 0$ , this is not an orthogonal basis.

Now suppose we're given some vector

$$v = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

and we want to find the scalars  $\alpha_1, \alpha_n$  such that

$$v = \alpha_1 v_1 + \alpha_2 v_2$$

In other words, such that

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 2\alpha_1 \end{bmatrix} + \begin{bmatrix} 2\alpha_2 \\ \alpha_2 \end{bmatrix}$$

So we need to solve the following system of equations

$$\begin{cases} \alpha_1 + 2\alpha_2 = 4 \\ 2\alpha_1 + \alpha_2 = 0 \end{cases}$$

How do we do this? You guessed it! Matrices!

$$\left[ \begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 0 \end{array} \right]$$

I'll skip the details, but eventually we get that

$$\alpha_1 = \alpha_2 = \frac{4}{3}$$

Which you should verify is correct.

Why did we do all this? To illustrate the point that arbitrary bases are annoying. What we really want, it turns out, are orthogonal bases!

### 2.4.1 Orthogonal bases

When we have an orthogonal basis it turns out there exists a really nice formula for each of the scalars. To motivate this, recall that the formula for projecting a vector  $v$  onto another vector  $u$  is denoted by  $proj_u(v)$  and is given by

$$proj_u(v) = \frac{v \cdot u}{\|u\|^2} u$$

If  $u$  is normalized (i.e.  $\|u\| = 1$ ) this turns just into

$$proj_u(v) = (v \cdot u)u$$

So with this in mind, let's return to the original problem. Suppose we have some vector  $v$  and we want to find the scalars  $\alpha_1, \dots, \alpha_n$  such that

$$v = \alpha_1 e_1 + \dots + \alpha_n e_n$$

where  $e_1, \dots, e_n$  is an orthonormal (i.e. pairwise orthogonal and normalized) basis. Then we can sort of imagine "projecting"  $v$  onto all basis vectors. Thus the scalars should be

$$v = (v \cdot e_1)e_1 + \dots + (v \cdot e_n)e_n$$

and indeed this is the correct representation of  $v$  in terms of the basis!

## 3 Fourier series

Finally we come to what the course is actually about. Luckily this step will be super simple thanks to the background we've provided.

### 3.1 Searching for an orthonormal basis to the periodic functions

So we've motivated why orthonormal bases are important/nice. And we've seen some examples of orthonormal bases for  $\mathbb{R}^2$ . We'd now like to find an orthonormal basis for the set of periodic functions.

#### 3.1.1 A first attempt (sinusoids of the same phase)

So suppose we have some periodic function  $x_{T_0}(t)$  with period  $T_0$ . What might be an appropriate choice of building blocks to represent this? Well some of the most famously periodic functions are the sinusoids, so it seems reasonable that summing a bunch of these will eventually construct our function

$$x_{T_0}(t) \stackrel{?}{=} \sum_{n=0}^{\infty} \alpha_n \sin(n\omega_0 t)$$

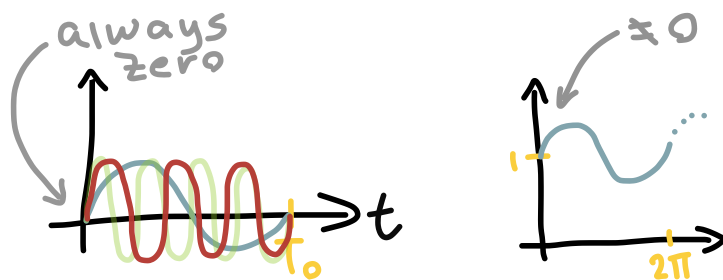
Here  $\omega_0$  denotes the angular frequency, which is given by  $\omega_0 = \frac{2\pi}{T_0}$  (since  $\omega_0 = 2\pi f_0$  and  $T_0 = \frac{1}{f_0}$ ).

Notice here that all of these sinusoids have the same phase (they all start at zero and end at zero). Is there any problems with this? Are there periodic functions which cannot be represented as sinusoids, all of the same phase? It turns out yes.

Consider the simple sine, shifted up by 1

$$s(t) = 1 + \sin(t)$$

Of all periodic function, surely you'd expect this to be representable by a series of sines! But consider what happens at  $t = 0$ . Here  $s(0) = 1$  but our sinusoids all start at zero! Adding an infinite amount of zeroes will never yeild 1, so this function is not representable by our series.



You might think; "well if we need the  $x(0)$  to be 1, perhaps that's a bad choice, switch to a series of cosines instead!" But the problem remains here, for a series of cosines (all of the same phase) always start at 1, so how do we represent the most basic function of all;  $x(t) = 0$ ?

### 3.1.2 A second attempt (sinusoids of different phases)

So sinusoids of the same phase won't work. What about if we also allow the phase to vary?

$$x_{T_0}(t) \stackrel{?}{=} \sum_{n=0}^{\infty} \alpha_n \sin(n\omega_0 t + \phi_n)$$

Making  $\omega_0 = 1, \phi_0 = \pi, \alpha_0 = 1, \phi_1 = 0, \alpha_1 = 1$  and all other  $\alpha$ 's zero yields

$$s(t) = 1 \cdot \sin(0 \cdot t + \pi) + 1 \cdot \sin(1 \cdot t + 0) + 0 + \dots = 1 + \sin(t)$$

thus our problematic function above **will** be representable by this series. Seems promising.

If you've worked with sinusoids before, however, you know it's annoying. There's trigonometric identities everywhere and integration is a pain. Although this series is promising, we'd really like to work with simpler terms. In the following section I'd like to argue that the series is actually equivalent to a much nicer series of complex exponentials.

### 3.1.3 A third attempt (complex exponentials, optional<sup>1</sup>)

We begin by replacing all  $\phi_k$  with  $\theta_k + \pi$ , thus transforming all sines into cosines.  $\phi_k$  is just an arbitrary constant of our choosing so there's nothing stopping us from doing this.

$$\sum_{n=0}^{\infty} \alpha_n \sin(n\omega_0 t + \pi + \theta_n) = \sum_{n=-\infty}^{\infty} \alpha_n \cos(n\omega_0 t + \theta_n)$$

Now we invoke eulers formula

$$\cos(n\omega_0 t + \theta_n) = \frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2}$$

which, when replaced into the series, becomes

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2} (e^{jn\omega_0 + j\theta_n} + e^{-jn\omega_0 - j\theta_n}) = \sum_{n=0}^{\infty} \left( \frac{\alpha_n}{2} (e^{j\theta_n} e^{jn\omega_0}) + \frac{\alpha_n}{2} (e^{-j\theta_n} e^{-jn\omega_0}) \right)$$

We now split up the sum and replace  $\frac{\alpha_k}{2} e^{j\theta_n}$  with  $\beta_k$ . (Notice that the complex conjugate  $\overline{\beta_k} = \frac{\alpha_k}{2} e^{-j\theta_n}$ .) So we get

$$\sum_{n=0}^{\infty} \beta_n e^{jn\omega_0} + \sum_{n=0}^{\infty} \overline{\beta_n} e^{-jn\omega_n} = \sum_{n=0}^{\infty} \beta_n e^{jn\omega_0} + \sum_{n=-\infty}^0 \overline{\beta_{-n}} e^{jn\omega_n}$$

Now we finally combine the sums again to get

$$\sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0}$$

And we're done. (Note that for  $n < 0$ ,  $\beta_n = \overline{\beta_{-n}}$ , and that  $\beta_0 = \alpha_0$ .)

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<sup>1</sup>Here we just show that we can represent the infinite sum of sinusoids as an infinite sum of complex exponentials. It can be skimmed if you don't care for long derivations.

### 3.1.4 Orthonormal

So where are we? We're trying to find an orthonormal basis for the set of periodic functions. We began with the hunch that sinusoids might be appropriate. Sinusoids are annoying but luckily we found that these were equivalent (due to eulers formula) to a set of complex exponentials. We'd now like to provide some rationale as to why these are an orthonormal basis for the set of periodic functions, meaning we should be able to represent *any* periodic signal as a linear combination (a sum of scaled versions) of these complex exponentials. Not only that, but since they're an *orthonormal* basis, we'll have a really simple formula for calculating what each scalar should be!

**Unfortunately** we won't be able to show that they're a basis<sup>2</sup>, but we will show that they're orthonormal!

Before starting we'll need to assume that the period is 1, thus  $\omega_0 = 2\pi$  (this is just to simplify things, it's also possible to show for arbitrary periods<sup>3</sup>).

We will denote the  $n$ :th complex exponential by  $e_n(t)$

$$e_n(t) = e^{j2\pi nt}$$

We begin by showing that they're orthogonal. (Recall that  $\omega_0 = 2\pi$ .)

**Orthogonal** We want to show that if  $n \neq m$  then  $\langle e_n, e_m \rangle = 0$ . Suppose  $n \neq m$

$$\langle e_n, e_m \rangle = \int_0^1 e^{j2\pi nt} \overline{e^{j2\pi mt}} dt = \int_0^1 e^{j2\pi(n-m)t} dt = \frac{1}{2\pi(n-m)} (e^{j2\pi(n-m)} - e^0) = 0$$

**Normalized** We begin with the fact that

$$\langle e_n, e_n \rangle = \int_0^1 e^{j2\pi nt} \overline{e^{j2\pi nt}} dt = \int_0^1 e^{j2\pi(n-n)t} dt = \int_0^1 1 dt = 1$$

Recall that (by definition)  $\|e_n\| = \sqrt{\langle e_n, e_n \rangle}$ , thus  $\|e_n\| = 1$ , which is what we wanted.

## 3.2 The final result

Now that we know the complex exponentials are an orthonormal basis for the set of periodic signals, what does that mean? It means every periodic signal  $x(t)$  can be represented as a linear combination of complex exponentials

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j\omega_0 nt}$$

and since the basis is orthonormal, we know that we can write each scalar  $\alpha_k$  as

$$\alpha_k = \langle x(t), e^{j\omega_0 kt} \rangle = \int_{T_0} x(t) \overline{e^{j\omega_0 kt}} dt = \int_{T_0} x(t) e^{-j\omega_0 kt} dt$$

And now we're done! We've successfully written a periodic signal in terms of a linear combination of complex exponentials. In the next section we'll extend this idea to signals which *aren't* periodic.

<sup>2</sup>There exists a proof, I've just never seen it...

<sup>3</sup>I'm lying here... Kind of. The problem is that there are two choices of basis functions. The first choice is orthonormal and the other is just orthogonal. TSDT84 uses the orthogonal basis, and I'm trying to stay true to the course, but the formulas and motivation become much cleaner/easier with the orthonormal case. The result is me trying to combine the two with lies.



## 4 Fourier transform

The idea behind the fourier transform is really the same as fourier series, but instead of limiting ourselves to periodic signals, we'll also be able to deal with non-periodic signals (with one caveat, that their norm has to be finite, but we'll get to that shortly).

Let's get right into it!

We originally learned that since the complex exponentials is an orthonormal basis for the set of periodic functions (signals), that for every periodic signals  $x(t)$  there exists a set of coefficients  $\alpha_n$ , where  $n$  ranges from  $-\infty$  to  $\infty$ , such that

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j\omega_0 n t}$$

In particular (since it's an *orthonormal* basis), that each term  $\alpha_n$  could be written as<sup>4</sup>

$$\alpha_n = \langle x(t), e^{j\omega_0 n t} \rangle = \int_{T_0} x(t) e^{-j\omega_0 n t} dt.$$

We now want to extend this notion to aperiodic functions. To do this, we simply<sup>5</sup> let the period  $T_0 \rightarrow \infty$ . Remember that  $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0} \implies \omega_0 = \frac{2\pi}{T_0}$ , so as  $T_0 \rightarrow \infty$ ,  $\omega_0 \rightarrow 0$ . We're kind of getting more and more terms in the series representation of  $x(t)$ , until we're essentially dealing with a continuum of  $\omega$ 's, and the sum turns into an integral, called the *inverse fourier transform*,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_\omega e^{j\omega t} d\omega.$$

(The  $\frac{1}{2\pi}$  factor is just something we engineers like because it makes some identities easier. The "mathematicians" fourier transform does not have the factor<sup>6</sup>.) where  $\alpha_\omega$  is the *fourier transform*, denoted  $\alpha_\omega = \mathcal{F}\{x(t)\}(\omega)$ , and is defined (as you'd expect)

$$\mathcal{F}\{x(t)\} = \langle x(t), e^{j\omega t} \rangle = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

(Recall that the negative exponent is due to taking the conjugate of the second factor, which we motivated earlier.)

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<sup>4</sup>I feel I must remind the reader that using the definition of the dot-product as defined here is not really valid as it will not make the complex exponentials we've chosen an *orthonormal* basis. The proper definition should include a  $\frac{1}{T_0}$  factor in front, but motivating the reason why complicates the concept of the dot-product and would take away from the main point this (informal) section is trying to make.

<sup>5</sup>Formally, the integral doesn't really make sense when we let  $T_0 \rightarrow \infty$ , but remember that we're talking informally here.

<sup>6</sup>This is a consequence of the different choices of basis functions mentioned above.

## 4.1 Convergence and finite length

Unfortunately there's one caveat with the fourier transform, and that is that only functions with finite length can be transformed, because only these will make the integral converge. Recall that the length of a vector/function  $f(t)$  is defined as

$$\|f(t)\| = \sqrt{\langle f(t), f(t) \rangle} = \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt}$$

Note that in the book the requirement is that the squared length  $\|f\|^2$  be finite. This is done simply because then we don't have to deal with the square root. But of course these requirements are equivalent. (If the length is infinite then the squared length is infinite, and vice versa.)

Why make this restriction? So what if the fourier transform diverges! Well suppose we had a two functions  $g(t)$  and  $h(t)$  with infinite energy and  $g(t) \neq h(t)$ , thus

$$\mathcal{F}\{g(t)\} = \mathcal{F}\{h(t)\} = \infty$$

But how do we go about inverting this? Should  $\mathcal{F}^{-1}\{\infty\} = g(t)$  or should  $\mathcal{F}^{-1}\{\infty\} = h(t)$ ? Clearly we cannot have it both ways. Instead we avoid the problem by just not letting the transform be defined for these types of functions.

In the next section we'll investigate an way of dealing with these types of functions.

## 5 Laplace transform

There are two types of laplace transforms. The unilateral and the bilateral. We'll begin with the bilateral.

### 5.1 The bilateral laplace transform

The bilateral laplace transform  $\mathcal{L}_{II}$  of a signal  $x(t)$  is

$$\mathcal{L}_{II}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

where  $s$  is the complex, dependent variable of the laplace. It can be seen as a generalization of the fourier transform. To see the motivation behind this statement, we have to dig into what the definition really means.

Since  $s$  is complex, let's rewrite it into  $s = \sigma + j\omega$ . What do we get?

$$\int_{-\infty}^{\infty} x(t)e^{-st} dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t} dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t} dt$$

But this looks suspiciously like the fourier transform of  $x(t)e^{-\sigma t}$ . Why would we ever want to do that? What's the point of all this?

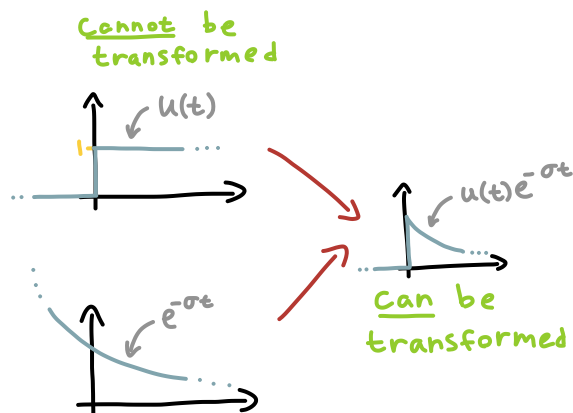
Recall what the central problem with the fourier transform was. We cannot analyze signals using the fourier transform if they have infinite energy. A simple example of this is the signal (called the "unit step" function)

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

which has energy

$$\langle u(t), u(t) \rangle = \int_{-n}^n u(t) dt = \int_0^n 1 dt \rightarrow \infty \text{ as } n \rightarrow \infty$$

thus  $u(t)$  is not analyzable with the fourier transform. But can we analyze it with the laplace transform? Yes!



The power of the laplace is that uses an exponentially decreasing function  $e^{-\sigma t}$  to sort of "push down" signals which have infinite energy into finite energy.

## 5.2 The unilateral laplace transform

The unilateral laplace transform, denoted  $\mathcal{L}_I$  or just  $\mathcal{L}$ , of a signal  $x(t)$  is

$$\mathcal{L}\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st} dt$$

where  $0^-$  means we also include the value of  $x(t)$  at  $t = 0$ , which becomes important when you laplace transform the (or functions containing the) dirac.

Following analogy of the bilateral laplace, the unilateral laplace can be seen as taking the signal  $x(t)$ , then just ignoring the parts where  $t < 0$ .

## 5.3 Region of convergence

Some care should be taken when evaluating the laplace. The main problem is that it (usually) doesn't converge for all possible choices of  $s$ . Instead we get a "region" where  $s$  will make the transform converge to a useful value (i.e. not diverge into  $\pm\infty$ ). This is called the "*region of convergence*", abbreviated ROC.

### 5.3.1 Example

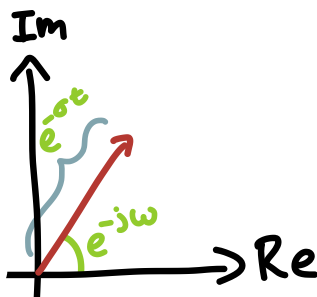
Let's take an example to illustrate this. Consider what happens when we try to evaluate the bilateral<sup>7</sup> laplace transform of  $u(t)$ . We'll assume  $s \neq 0$  and then consider  $s = 0$  in a separate case below

$$\mathcal{L}_{II}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_{t=0}^{t=\infty} = [s = \sigma + j\omega] = -\frac{1}{s} \left( e^{-\sigma t} e^{-j\omega t} \Big|_{t=0}^{t=\infty} \right) \quad (1)$$

Now let's consider the upper and lower limits separately. We begin with the simple case;  $t = 0$ .

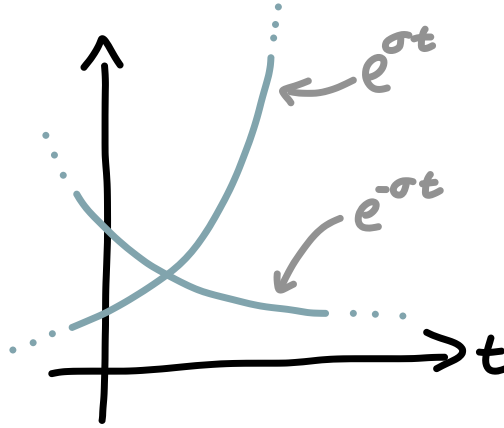
$$e^{-\sigma t} e^{-j\omega t} \Big|_{t=0} = e^0 e^0 = 1$$

What about  $t = \infty$ ? The intuitive way<sup>8</sup> of thinking about this is;  $e^{-\sigma t} e^{-j\omega t}$  is a complex number ( $re^{j\phi}$ ). As  $t \rightarrow \infty$ ,  $e^{-j\omega t}$  just spins round and round, what really determines whether the limit diverges or converges is how the length,  $e^{-\sigma t}$ , behaves as  $t \rightarrow \infty$ , thus we can ignore the second term and instead just think about  $e^{-\sigma t}$  as  $t \rightarrow \infty$ .



<sup>7</sup>Note that the choice of bilateral vs unilateral laplace transform in this case is arbitrary, the bilateral laplace quickly turns into the unilateral one. This is true in general when the signal is causal (i.e. when  $x(t) = 0$  for  $t < 0$ )

<sup>8</sup>The rigorous way is not part of the course, and requires complex analysis I suppose.



Now think about<sup>9</sup> what happens if  $\sigma > 0$  and  $\sigma < 0$ . For the case of  $\sigma > 0$ ,  $e^{-\sigma t}$  becomes exponentially decreasing, which will converge to 0 as  $t \rightarrow \infty$ . What about  $\sigma < 0$ ? Here it becomes an exponentially increasing, so very divergent!

Here's where the region of convergence comes in. We're interested in finding for what values of  $s$  the transformation converges. We just found that it will only converge for  $\text{Re}\{s\} = \sigma > 0$ , thus this will be our ROC!

Now let's continue with (1). We'd found that (still assuming  $s \neq 0$ )

$$\mathcal{L}_{II}\{u(t)\} = -\frac{1}{s} \left( e^{-\sigma t} e^{-j\omega t} \Big|_{t=0}^{t=\infty} \right) = -\frac{1}{s} (0 - 1) = \frac{1}{s}$$

Provided  $\sigma > 0$ .

What if  $s = 0$ ?

$$\mathcal{L}_{II}\{u(t)\}(0) = \int_{-\infty}^{\infty} u(t) e^{-0 \cdot t} dt = \int_0^{\infty} dt \rightarrow \infty$$

Clearly  $s = 0$  is not in the region of convergence of  $u(t)$ . This was already included in the  $\text{Re}\{s\} > 0$  finding, so not terribly exciting.

### 5.3.2 Final result

The final result we got was that

$$\mathcal{L}_{II}\{u(t)\} = \frac{1}{s}$$

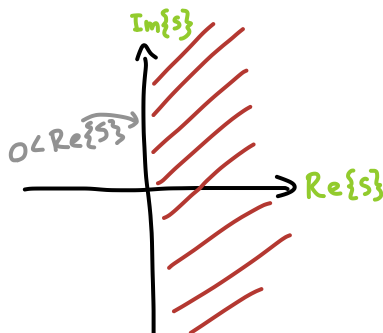
with a region of convergence of  $\text{Re}\{s\} > 0$ .

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<sup>9</sup>I conveniently ignored the case of  $\sigma = 0$ . In this case we'll need to return to the  $e^{-j\omega t}$  term, since  $e^{-\sigma t} \Big|_{\sigma t} = e^0 = 1$ . The motivation behind ignoring this is that  $e^{-j\omega t}$  will just spin around and round and, just like  $\sin(t)$ , will not converge as  $t \rightarrow \infty$ , thus  $\sigma = 0$  is not included in the ROC.

## 5.4 Why is it called a region?

So we got that the region of convergence for  $\mathcal{L}_{II}\{u(t)\}$  to be  $\text{Re}\{s\} > 0$ . But why do we call it a *region* of convergence? It seems more appropriate to call it an "interval" of convergence! But remember that  $s$  is a complex number, and that we're only restricting the real part of  $s$ . Think about the whole complex plane. How would we visualize the ROC?



As you see in the picture above, although the ROC can really be described as an interval on the real line, in the complex plane it's really a region.

## 5.5 Relation between laplace and fourier transform

Earlier we said that the laplace can be seen as a generalization of the fourier transform. Now that we know what the ROC is, I'd like to briefly return to this concept. Recall that

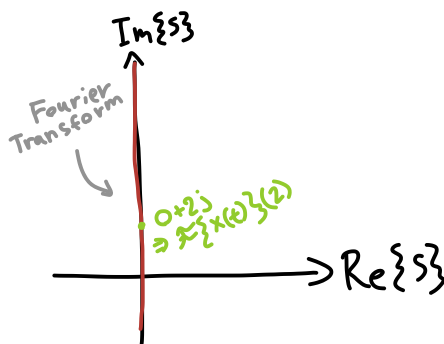
$$\mathcal{L}_{II}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st} dt = [s = \sigma + j\omega] = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t}) e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

If we now let  $\sigma = 0$  we get

$$\mathcal{F}\{x(t)\} = \mathcal{L}_{II}\{x(t)\} \Big|_{\text{Re}\{s\}=0}$$

It's important to realize that this wonderful little relation is **only** true if  $\text{Re}\{s\} = 0$  is in the region of convergence.

Visualizing it on the complex plane, we can see the fourier transform as the imaginary axis! We let  $\text{Re}\{s\} = 0$  then plug in some value  $\omega = \text{Im}\{s\}$  on the horizontal axis (the imaginary axis) to get the value of the fourier transform at that point!



## 5.6 Poles and Zeros

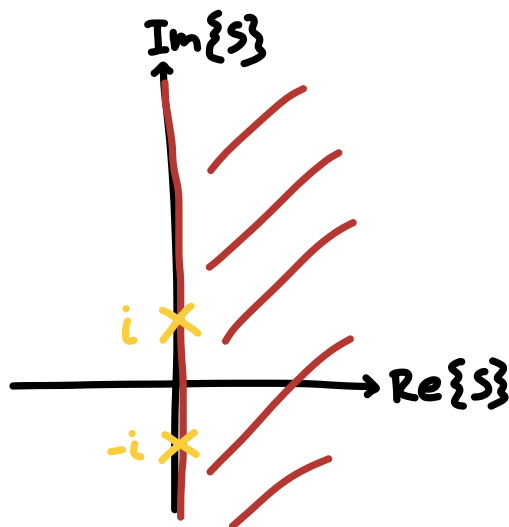
Here are some laplace transforms that we might encounter

$$\frac{1}{s+1}, \frac{s+2}{(s-1)^2(s+3)}, \frac{s-5}{s^2(s+2)}, \dots$$

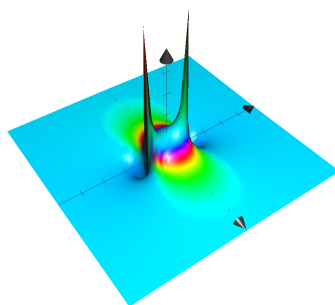
As you can see, it's common that they can be represented as some polynomial divided by another polynomial.

The roots of the numerator are called the zeroes (i.e. where it is identically zero), and the roots of the denominator are called the poles, and are the points where the polynomials explode into  $\pm\infty$ <sup>10</sup>. These points are often drawn together with the ROC on the "s-plane" (which we've drawn above).

For instance  $\sin(t)u(t)$  has laplace transform  $\frac{1}{s^2+1}$ , with ROC  $\text{Re}\{s\} > 0$ . The polynomial has a pole at  $s = \pm i$ , thus the "s-plane" can be visualized as



I won't prove it, but it turns out that the ROC will always have zeroes and/or poles at the edges.



<sup>10</sup>The reason I say  $\pm$  is because it will converge differently depending on which path you take on the complex plane. Though you could say that the poles are where the absolute value of the rational function explode to  $\infty$ .

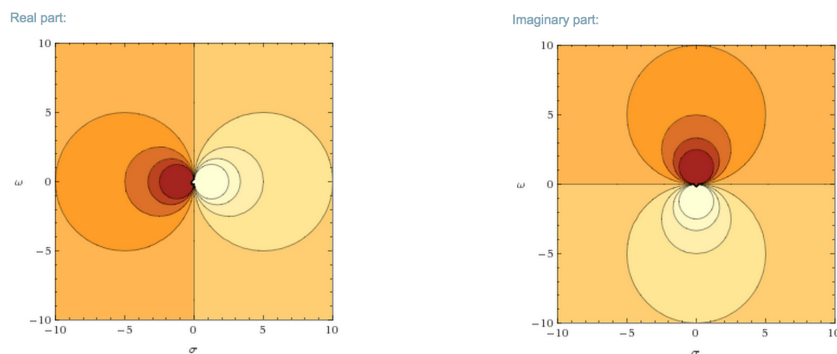
## 5.7 Warnings

Here's a few common misconceptions and things to think about regarding the laplace.

### 5.7.1 Complex heightmap

The visualization of the laplace transform, displaying the ROC can be very useful. But it's important to realize that what we're viewing is not a heightmap, i.e. we can't make the plot 3D to visualize the "actual" transform, because each point is a complex number.

Let's take  $\mathcal{L}\{u(t)\} = \frac{1}{s}$ , with ROC  $Re\{s\} > 0$  as an example. In order to fully see what value the transformation has at a given point, we need two plots, one for the real part and one for the imaginary part.



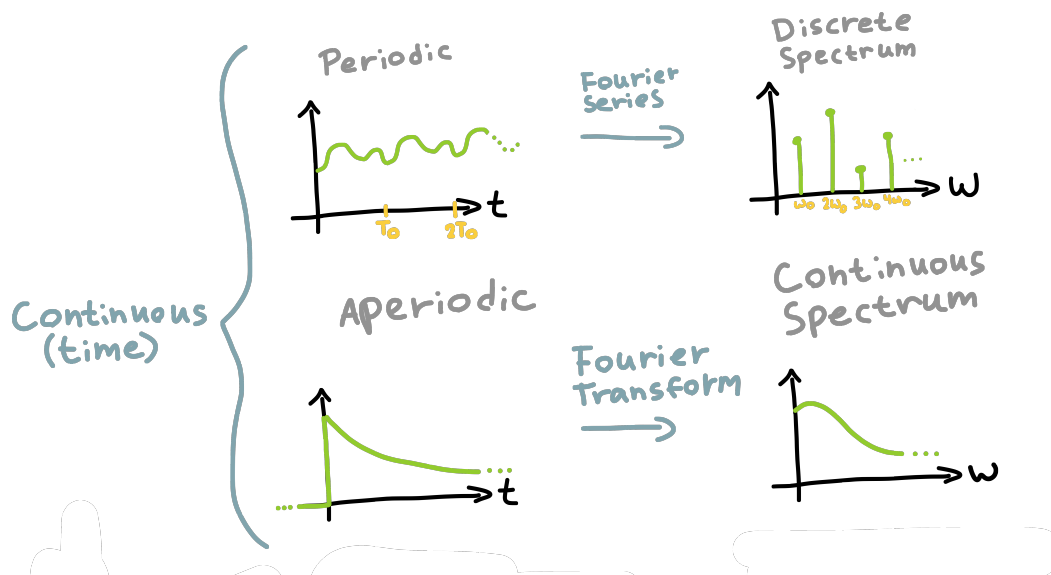
Note that we're displayin the whole plane here, whereas the transformation will only converge for  $Re\{s\} > 0$ . The image may show a finite value for  $Re\{s\} < 0$ , but that does not mean the the transformation would converge there.  $\mathcal{L}\{u(t)\} = \frac{1}{s}$  is only valid for  $\sigma = Re\{s\} > 0$ .



## 6 Brief interlude

Where are we? We started investigating periodic, continuous, signals. We showed that as long as these have finite energy, then we could write these as a sum (linear combination) of complex exponentials. We then moved on to aperiodic signals and showed that by letting the period tend to infinity, these could also be analyzed with the *fourier transform*.

We took a quick sidestep investigating signals with infinite energy, and showed how these could be managed with the *laplace transform*.



Now we're ready to leave the continuous world and enter the discrete, starting with the z-transform and then moving on with the discrete-time fourier transform.

## 7 Sampling

Let's say we have a continuous function  $x(t)$  and we'd like to analyze it with a computer. Computers can only deal with discrete values, so what do we do? We have to find some way of discretizing the signal. This is called sampling.



The most obvious way of sampling the signal is just to periodically, say with period  $T$ , pluck out values and call this our sampled signal, let's call it  $x[n] = x(nT)$ , where  $n$  is an integer. Plug in  $n = 3$  and we get the third sample-value of  $x(t)$ .

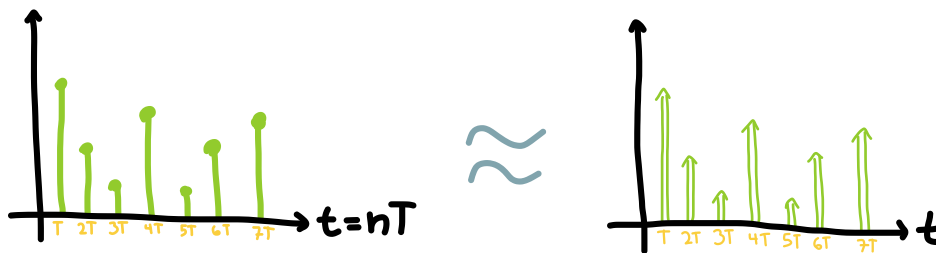
We'll get to how to choose the sampling-period  $T$  later when we get to the Nyquist-Shannon sampling theorem.

## 8 Z-transform

So far we've only been able to deal with (periodic and aperiodic) **continuous** signals. Now we'd like to focus on the more practical, realistic, time-discrete signals. We call them "time-discrete" to emphasize the fact that it's the time-domain that's discrete, not the frequency-domain (like we had in the fourier series case).

### 8.1 Discrete signals in a continuous world

In the previous section we described how to sample a function; just create a new function  $x[n] = x(nT)$ . Unfortunately our tools are useless for dealing with this function, since we can only handle the continuous. Luckily there's a way of constructing this sort of discrete function in the continuous world. Just place a bunch of scaled deltas where the discrete values would be!



Sure, that sounds neat and all, but how would we go about representing this mathematically?

Well perhaps we can make use of the (conveniently named) *sampling property* of the dirac delta!

$$x(t)\delta(t - T) = x(T)\delta(t - T)$$

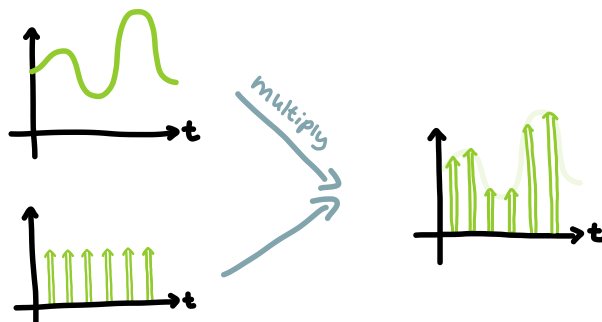
Now I'd like to introduce a new function, the  $\text{III}_T$  function (pronounced "Sha", a.k.a. the dirac comb or the dirac train). This is just an infinite amount of diracs spaced  $T$  apart.

$$\text{III}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Inspired by the sampling property, look what happens if we multiply  $\text{III}_T$  with  $x(t)$

$$\text{III}_T x(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

Let  $\bar{x}(t) = \text{III}_T x(t)$  and we've got our function!



## 8.2 Converting continuous transforms into discrete ones

So now that we know how to construct "discrete" signals in the continuous world, it's easy to just convert our existing transforms into discrete equivalents. We'll begin with the laplace transform and we'll call its discrete equivalent the *z-transform*.

Let's just laplace transform  $\bar{x}(t)$

$$\begin{aligned}
 \mathcal{L}_{II}\{\bar{x}(t)\} &= \int_{-\infty}^{\infty} \bar{x}(n) e^{-st} dt \\
 &= \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT) \right) e^{-st} dt \\
 &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT) e^{-st} dt \\
 &= \sum_{n=-\infty}^{\infty} x[n] e^{-snT} = [z = e^{sT}] \\
 &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
 &\stackrel{\text{def}}{=} \mathcal{Z}\{x[n]\}
 \end{aligned}$$

Thus for discrete signals we'll use the z-transform, but remember that it's really the laplace in disguise.

### 8.3 Region of convergence

We've seen that the z-transform of a discrete signal is equivalent (by definition) to the laplace of a bunch of (equally spaced) diracs, thus in order to gain further insight into the z-transform we can just analyze the laplace.

There's one caveat though. Remember that the laplace transform depended on the variable  $s$ , whereas the z-transform depends on  $z$ . The relationship between them is  $z = e^{sT}$ . Thus when you take your signal, sample it, and wish to laplace transform it as if it was the original (continuous) signal (using the z-transform), you must remember that everything will be in polar coordinates. Lines turn into circles.

Let's take an example to illustrate the point.

#### 8.3.1 Example

Say we want to find the z-transform of  $u[t]$ .

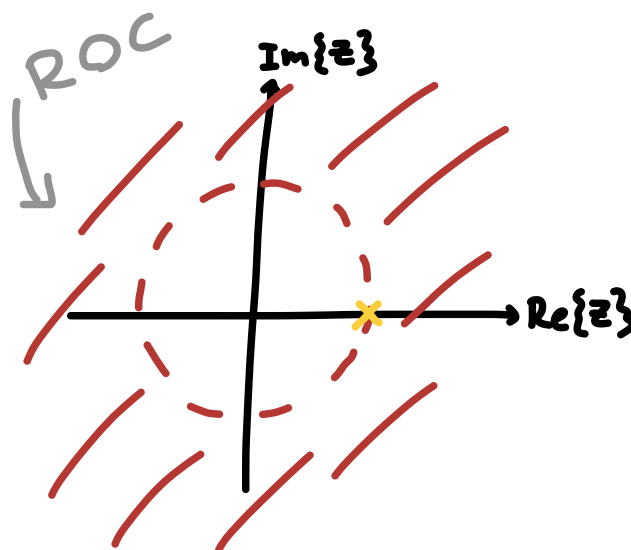
$$\mathcal{Z}\{u[n]\} = \sum_{-\infty}^{\infty} u[n]z^{-n} = \sum_0^{\infty} z^{-n}$$

Here's where the ROC comes in. We already know (from single variable calculus) that this converges to

$$\frac{z}{z-1}$$

if<sup>11</sup>  $|z| > 1$ . Also note that we have a pole at  $z = 1$ .

Put together this transform can be visualized in the "z-plane" as



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<sup>11</sup>and only if?

## 9 Discrete-time fourier transform

There's two ways of deriving the discrete-time fourier transform (DTFT).

- Convert discrete signals into continuous equivalents (by multiplying with the dirac comb, thus sampling) and then apply the normal (continuous) fourier transform which will (hopefully) produce an expression dependent on a discrete signal  $x[n]$ , which we'll take as the definition. This is how we did it when we went from the laplace to the z-transform.
- Alternatively, we could recognize the relation between the z-transform and the laplace, and the relation between the laplace and fourier, and try to find the definition from there instead.

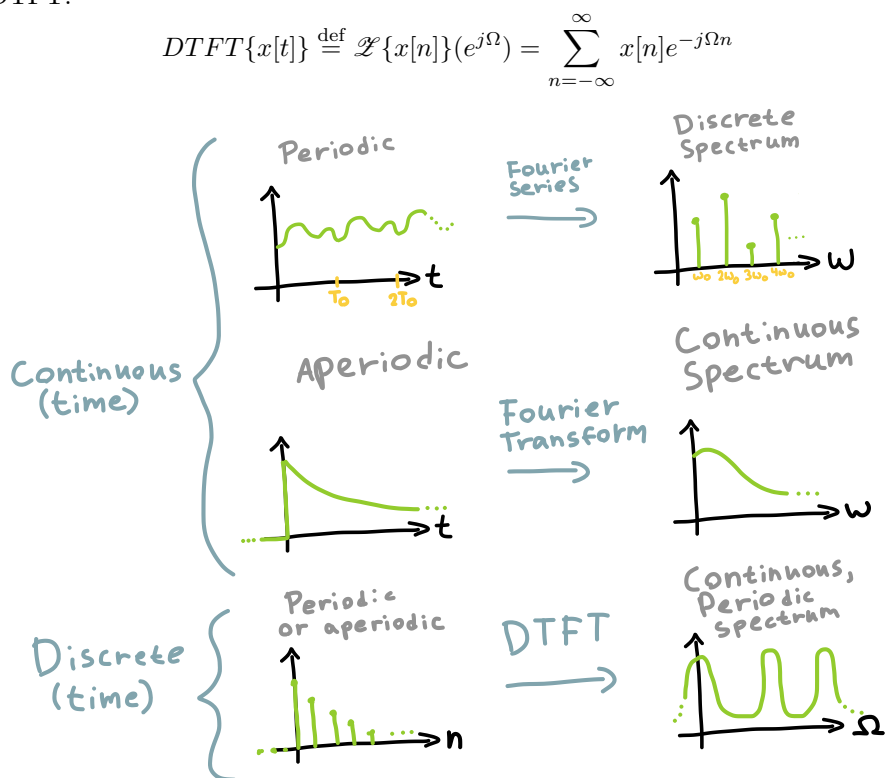
We'll be taking the second route.

## 10 Deriving the DTFT

So suppose we have some signal  $x(t)$ . We begin by sampling it  $\bar{x}(t) = \text{III}_T x(t)$ . Now recall that

$$\mathcal{F}\{\bar{x}(t)\} = \mathcal{L}\{\bar{x}(t)\} \Big|_{\sigma=0} = \mathcal{Z}\{x[n]\}(e^{-j\omega T})$$

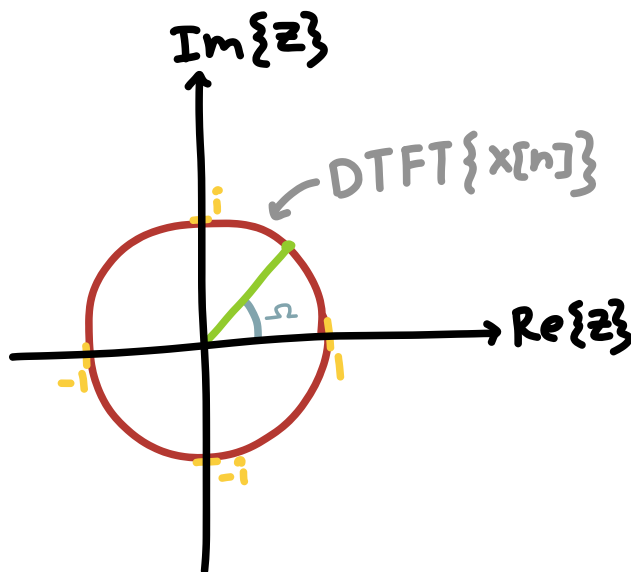
assuming  $\text{Re}\{s\} = 0$  is in the ROC. We now just rename  $\omega T$  to  $\Omega$  (called the "normalized" frequency) and we've got our DTFT!



Notice the duality between fourier series and the DTFT.

## 10.1 Why is the frequency spectrum of the DTFT periodic?

The standard way of explaining why the DTFT is periodic starts with a picture.



As is clear from the derivation above, the values of the DTFT exist on the unit circle in the "z-plane" (assuming it's in the ROC, of course). This is analogous to how the fourier transform can be seen as the imaginary axis in the "s-plane". As you vary the angle  $\Omega$  around the circle you'll eventually come back to where you started, which is why it's periodic.

### 10.1.1 Alternative (optional)

That's the usual way of explaining it, but I'd like to show it from a slightly different direction. Instead of taking the z-transform route we'll go directly from the fourier transform of the sampled signal  $\bar{x}(t) = \text{III}x(t)$ . We assume sampling period<sup>12</sup>  $T = 1$  for simplicity, but the result is analogous for arbitrary periods.

$$DTFT\{x(t)\} = \mathcal{F}\{\bar{x}(t)\} = \mathcal{F}\{\text{III}x(t)\}$$

Now use the convolution/multiplication duality of the fourier transform and the fact<sup>13</sup> that  $\mathcal{F}\{\text{III}\} = \text{III}$

$$\mathcal{F}\{\text{III}x(t)\} = \mathcal{F}\{\text{III}\} * \mathcal{F}\{x(t)\} = \text{III} * \mathcal{F}\{x(t)\}$$

Since  $\text{III}$  convolved with any function will make it periodic, we're done! What we've shown is that the DTFT spectrum is precisely<sup>14</sup> a version of the original signal, but periodized!

Fun fact; this is one of the steps in the Nyquist-Shannon theorem proof! Just "unperiodize" it with a rect function to get

$$\mathcal{F}\{x(t)\} = \Pi(\text{III} * \mathcal{F}\{x(t)\})$$

Taking the inverse fourier transform of both sides yeilds the Nyquist-Shannon formula!

<sup>12</sup>When we write  $\text{III}$  we really mean  $\text{III}_1$ .

<sup>13</sup>Provable using the Poisson summation formula.

<sup>14</sup>For period  $T$  the result would be  $DTFT\{x(t)\} = \frac{1}{T} \text{III}_{\frac{1}{T}} \mathcal{F}\{x(t)\}$ .