1 Tillampningar av Maclaurinutvecklingar

Antag att $f, f', f'', \dots, f^{(n+1)}$ är kontinuerliga.

1.1 Standard Maclaurinutvecklingar

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } |x| < 1$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{for all } |x| < 1 \text{ and all complex } \alpha$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| \leq 1, x \neq \pm i$$

1.2 Sats

$$r(x) = f(x) - p_n(x) = \mathcal{O}(x^{n+1})$$

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^n(0)}{n!} x^n + \mathcal{O}(x^{n+1})$$

1.3 Sats

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \mathcal{O}(x^{n+1}) \implies$$

$$c_0 = f(0), c_1 = f'(0), \dots, c_n = \frac{f^n(0)}{n!}$$

2 Ex. 1

Härled Maclaurinpolynomet av ordning 2 av $f(x) = \sqrt{4-x}$

$$f(x) = \sqrt{4(1 - \frac{x}{4})} = 2\sqrt{1 - \frac{x}{4}}$$

$$g(t) = \sqrt{1+t} \implies g(0) = 1$$

$$g'(t) = \frac{1}{2}(1+t)^{\frac{1}{2}-1} \implies g'(0) = \frac{1}{2}$$

$$g''(t) = \frac{1}{2}(\frac{1}{2} - 1)(1+t)^{\frac{1}{2}-2} \implies g''(0) = \frac{1}{2}(\frac{1}{2} - 1)$$

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2}t^2 + \mathcal{O}(t^3) \Leftrightarrow$$

$$g(t) = 1 + \frac{1}{2}t + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}t^2 + \mathcal{O}(t^3)$$

Sätt $t = -\frac{x}{4}$

$$\sqrt{1 - \frac{x}{4}} = 1 + \frac{1}{2}(-\frac{x}{4}) - \frac{1}{8}(-\frac{x}{4})^2 + \mathcal{O}\left(\left(-\frac{x}{4}\right)^3\right)$$

$$f(x) = 2(1 - \frac{1}{2}\frac{x}{4} - \frac{1}{2}\frac{1}{4}\frac{x^2}{16} + \mathcal{O}(x^3)) \implies$$

$$f(x) = 2 - \frac{x}{4} - \frac{x^2}{64} + \mathcal{O}(x^3)$$

3 Ex. 2

Räkna ut

$$\lim_{x\to 0}\frac{(\sin x)^2}{xe^x - x} = \lim_{x\to 0}\frac{x^2 + \mathcal{O}(x^4)}{x^2 + \mathcal{O}(x^3)}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \mathcal{O}(x^4)$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \mathcal{O}(x^5)$$

$$xe^x - x = x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \mathcal{O}(x^5)$$

$$\sin x = x + \mathcal{O}(x^3)$$
$$(\sin x)^2 = (x + \mathcal{O}(x^3))^2 = x^2 + 2x\mathcal{O}(x^3) + \mathcal{O}(x^3)\mathcal{O}(x^2) =$$
$$x^2 + \mathcal{O}(x^4) + \emptyset(x^6) = x^2 + \mathcal{O}(x^4)$$

$$\lim_{x\to 0}\frac{x^2 + \mathcal{O}(x^4)}{x^2 + \mathcal{O}(x^3)} =$$

$$\lim_{x\to 0}\frac{x^2(1 + \mathcal{O}(x^2))}{x^2(1 + \mathcal{O}(x))} =$$

$$\lim_{x\to 0}\frac{1 + \mathcal{O}(x^2)}{1 + \mathcal{O}(x)} = 1$$

4 Ex. 3

Undersök om $f(x) = 2 + x^2 - \sqrt{1 + 2x^2}$ har lokalt maximum eller minimum i $x = 0$.

$$\sqrt{1+2x^2}; \sqrt{1+t} = 1 + \frac{1}{2}t + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}t^2 + \mathcal{O}(x^3)$$

Sätt $t = 2x^2$

$$\sqrt{1+2x^2} = 1 + \frac{1}{2}(2x^2) - \frac{1}{8}(2x^2)^2 + \mathcal{O}((2x^2)^3) =$$

$$1 + x^2 - \frac{1}{2}x^4 + \mathcal{O}(x^6)$$

$$f(x) = 2 + x^2 - (1 + x^2 - \frac{1}{2}x^4 + \mathcal{O}(x^6)) =$$

$$= 1 + \frac{1}{2}x^4 + \mathcal{O}(x^6); f(0) = 1$$

$$f(x) - f(0) = \frac{1}{2}x^4 + \mathcal{O}(x^6) > 0 \Leftrightarrow$$

för alla x nära 0

$$\Leftrightarrow f(x) > f(0)$$

för alla x nära 0

5 Ex. 4

Undersök om $f(x) = 2 + x - \sin x$ har lokalt max eller min i $x = 0$

$$\sin x = x - \frac{x^3}{3!} + \mathcal{O}(x^5)$$

$$f(x) = 2 + x - \left(x - \frac{x^3}{3!} + \mathcal{O}(x^5)\right) = 2 + \frac{x^3}{3!} + \mathcal{O}(x^5) \Leftrightarrow$$

$$f(x) - f(0) = \frac{x^3}{3!} + \mathcal{O}(x^5)$$

$$\begin{cases} x > 0 \implies f(x) - f(0) > 0 \implies f(x) > f(0) \\ x < 0 \implies f(x) - f(0) < 0 \implies f(x) < f(0) \end{cases}$$

6 Ex. 5

Beräkna $f^{(5)}(0)$ då $f(x) = \sin x e^{x^2}$.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \mathcal{O}(x^6) \implies$$

$$c_0 = f(0), c_1 = f'(0), c_2 = \frac{f''(0)}{2}, \cdots, c_3 = \frac{f^{(5)}(0)}{5!} \implies$$

$$f^{(5)}(0) = c_5 5!$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \mathcal{O}(x^6)$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$

$$+x^3 - \frac{x^5}{3!} + \mathcal{O}(x^7)$$

$$\frac{x^5}{2} + \mathcal{O}(x^7) = x + x^3\left(1 - \frac{1}{3!}\right) + x^5\left(\frac{1}{5!} - \frac{1}{3!} + \frac{1}{2}\right) + \mathcal{O}x^7$$

$$f^{(5)}(0) = c_5 5! \Leftrightarrow f^{(5)}(0) = 5!\left(\frac{1}{5!} - \frac{1}{3!} + \frac{1}{2}\right) = 1 - \frac{5!}{3!} + \frac{5!}{2} =$$

$$1 - \frac{5*4*3*2}{3*2} + \frac{5*4*3*2}{2} = 41$$

7 Ex. 6

Bestäm en eventuell asymptot till $y = \sqrt{x^2 - 2x}$ då $x \to +\infty$

$$y = \sqrt{1 - \frac{2}{x}}$$

$$\sqrt{1 - \frac{2}{x}}; g(t) = \sqrt{1+t}\left[t = -\tfrac{2}{x} \to 0, x \to \infty\right] = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \mathcal{O}(t^3)$$

$$\sqrt{1 - \frac{2}{x}} = x\left(1 + \frac{1}{2}(-\frac{2}{x}) - \frac{1}{8}(-\frac{2}{x})^2 + \mathcal{O}((\frac{1}{x})^3)\right) =$$

$$x - 1 - \frac{1}{8}2^2\frac{1}{x} + \mathcal{O}((\frac{1}{x})^3)$$

$$y = x - 1 + \mathcal{O}(\frac{1}{x}), x \to \infty$$