1 Integration av triginomiska uttryck

1.1 Exempel

$$\int \frac{dx}{\cos^3 x} = \int \frac{\cos x \ dx}{\cos^4 x} = \int \frac{\cos x \ dx}{(1-\sin^2 x)^2} = \begin{bmatrix} s = \sin x \\ ds = cosx \ dx \end{bmatrix} = \int \frac{ds}{(1-s^2)^2} = \int \frac{ds}{(1-s)^2(1+s)^2} =$$

$$= \begin{bmatrix} \frac{1}{(1-s)^2(1+s)^2} = \frac{A}{1-s} + \frac{B}{(1-s)^2} + \frac{C}{1+s} + \frac{D}{(1+s)^2} \ (*) \\ B = D = \frac{1}{4} \end{bmatrix}$$

Multiplicera $(*)$ med s och låt $s \to \infty \implies 0 = -A + 0 + C + 0$
Låt $s = 0$ i $(*) \implies 1 = A + B + C + D \implies A + C = \frac{1}{2}$
Alltså $A = B = C = D = \frac{1}{4}$

$$= \frac{1}{4} \int \left(\frac{1}{1-s} + \frac{1}{(1-s)^2} + \frac{1}{1+s} + \frac{1}{(1+s)^2} \right) ds =$$

$$= \frac{1}{4} \left(-1 \ln|1-s| + \frac{1}{1-s} \ln|1+s| - \frac{1}{1+s} + C \right)$$

$$\frac{1}{4} \left(\ln|\frac{1+s}{1-s}| + \frac{2s}{1-s^2} + C \right)$$

$$= \frac{1}{4} \ln|\frac{1+\sin x}{1-\sin x}| + \frac{\sin x}{2\cos^2 x} + C$$

Kan även skrivas:

$$\frac{1}{4} \ln|\frac{1+\sin x}{1-\sin x}| = \frac{1}{4} \ln|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}| = \frac{1}{4} \ln|\frac{1+\sin x}{\cos} x|^2 = \frac{1}{2} \ln \frac{1+\sin x}{\cos} x$$

Alltihop skrivs ibland:

$$\int \sec^3 x \ dx = \frac{1}{2} \ln|\sec x + \tan x| + \frac{1}{2} \sec x \tan x + C$$

1.2 Exempel (Produkt-till-summa-omskrivning)

$$\sin^2 x \cos 3x = \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 \left(\frac{e^{i3x} + e^{-i3x}}{2} \right) = \left[t = e^{ix} \right] =$$

$$= \left(\frac{t^1 - t^{-1}}{2i} \right)^2 \left(\frac{t^3 + t^{-3}}{2} \right) = -\frac{1}{8} \left(t^2 - 2 + t^{-2} \right) \left(t^3 + t^{-3} \right) =$$

$$= -\frac{1}{8} \left(\left(t^5 + t^{-5} \right) - 2 \left(t^3 + t^{-3} \right) + \left(t + t^{-1} \right) \right) =$$

$$= -\frac{1}{4} \left(\cos 5x - 2\cos 3x + \cos \right)$$

ger

$$\int \sin^2 x \cos 3x \ dx = -\frac{1}{4} \left(\frac{\sin 5x}{5} - 2\frac{\sin 3x}{3} + \sin x \right) + C =$$

$$= -\frac{1}{20}\sin 5x + \frac{1}{6}\sin 3x - \frac{1}{4}\sin x + C$$

Notera särskilt:

$$\int \cos^2 x \ dx = \frac{1}{2}\int (1 + \cos 2x \ dx) = \dots$$

$$\int \sin^2 x \ dx = \frac{1}{2}\int (1 - \cos 2x \ dx) = \dots$$

1.3 Exempel (När inget annat funkar)

Variabelbytet $v = 2\arctan t$ (alltså $t = \tan\frac{v}{2}$ med $-\pi < v < \pi$)

Omvandlar trig integral till rationell integral:

$$\cos v + i\sin v = e^{iv} = e^{i2\arctan t} = \left(e^{i\arctan t}\right)^2$$

$$= \left(\frac{1+it}{\sqrt{1+t^2}}\right)^2 = \frac{(1-t^2)+i(2t)}{1+t^2} = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}$$

$$\frac{dv}{dt} = \frac{d}{dt}(2\arctan t) = \frac{2}{1+t^2} \implies dv = \frac{2dt}{1+t^2}$$

1.4 Exempel

$$\int \frac{dv}{5 - 4\cos v} = \begin{bmatrix} t = \tan\frac{v}{2} \\ \vdots \end{bmatrix} = \int \frac{\frac{2}{1+t^2}dt}{5 - 3\frac{1-t^2}{1+t^2}} =$$

$$= \int \frac{2 \ dt}{5(1+t^2) - 3(1-t^2)} = \int \frac{dt}{1+4t^2} = \int \frac{dt}{1+(2t)^2} = \frac{1}{2}\arctan 2t + C$$

$$= \frac{1}{2}\arctan\left(2\tan\frac{v}{2}\right) + C$$

2 Integration av rotuttryck

2.1 Exempel

$$\int \frac{x}{\sqrt{x+1} \ dx} = \begin{bmatrix} t = \sqrt{x+1} \\ x = t^2 - 1 \\ dx = 2t \ dt \end{bmatrix} = \int \frac{t^2-1}{t} 2t \ dt =$$

$$= \frac{2}{3}t^3 - 2t + C = \frac{2}{3}t\left(t^2 - 3\right) + C = \frac{2}{3}\sqrt{x+1}\,(x-2) + C$$

2.2 Exempel

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \begin{bmatrix} t = x+2 \\ dt = dx \end{bmatrix} = \int \frac{t-2}{\sqrt{t^2+1}}dt =$$

$$= \int 2t\frac{1}{2}\left(t^2+1\right)^{-1/2}dt - 2\int \frac{dt}{\sqrt{t^2+1}} =$$

$$= (t^2+1)^{1/2} - 2\ln\left(t+\sqrt{t^2+1}\right) + C =$$

$$= \sqrt{x^2+4x+5} - 2\ln\left(x+2+\sqrt{x^2+4x+5}\right) + C$$

2.3 Exempel

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \begin{bmatrix} x = \tan v \\ \frac{dx}{dv} = \frac{1}{\cos^2 v} \\ \sqrt{1+x^2} = \sqrt{1+\tan^2 v} = \sqrt{1+\frac{\sin^2 v}{\cos^2 v}} = \sqrt{\frac{1}{\cos^2 v}} = \frac{1}{|\cos v|} \end{bmatrix} =$$

$$\int \frac{\frac{dv}{cos^2 v}}{\tan^2 v \frac{1}{\cos v}} = \int \frac{\cos v}{\sin^2 v}dv = -\frac{1}{\sin v} + C$$

$$\frac{-1}{\frac{x}{\sqrt{1+x^2}}} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

2.3.1 Annat sätt

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \begin{bmatrix} t = \frac{1}{x}\ (anta\ x>0,\ och\ t>0) \\ dt = \frac{-dx}{x^2} \end{bmatrix} =$$

$$\int \frac{-dt}{\sqrt{1+\left(\frac{1}{t}\right)^2}} = \int \frac{-dt}{\sqrt{\frac{t^2+1}{t^2}}} = \int \frac{-t\ dt}{\sqrt{t^2+1}} =$$

$$= -\sqrt{t^2+1} + C = -\sqrt{\left(\frac{1}{x}\right)^2 + 1} + C = -\frac{\sqrt{1+x^2}x}{+}C$$

(Stämmer även för $x < 0$, ty räkningarna när man kontrollderiverar påverkas inte av vilket tecken x har.)