1 Kontinuerliga stokastiska variabler

Discrete:

$$X : S \to C \subset \mathcal{R}$$

A countable set of real numbers.
Continous:

$$X : S \to I \subseteq \mathcal{R}$$

I is an uncountable set = with the cardinality of the real numbers.

1.1 Definition

- X is called a continous random variable if it can take all values from a real interval or union of F (finite) intervals.
  Example: Lifespan of a lightbulb. Weight of a raindrop. Circumference of a randomly chosen tree.
- A function $f_x : \mathcal{R} \to [0, \infty)$ which satisfies

  $$P(a \leq X \leq b) = \int_a^b f_x(x)dx$$

  is called "density" function.
- $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(y)dy$ is called the "comulative" distribution function.

  $$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a)$$

1.2 Properties

- $f_X(x) \leq 0$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $P(a < X \leq b) = P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b)$

1.3 Remark

Since $f_X(x)$ is non-negative $\int f_X$ is an area

1.4 Example

Assume that X is a continous random variable with density function

$$f(x) = \begin{cases} c(4x - 2x^2), 0 < x < 2 \\ 0, otherwise \end{cases}$$

Find the constant c and $P(X > 1)$

$$\int_{-\infty}^{\infty} f(x) = 1 \to 1 = \int_{-\infty}^{\infty} c(4x - 2x^2)dx - \int_{0}^{2} c(4x - 2x^2)dx =$$

$$= \left[c(2x^2 - \frac{2}{3}x^3)\right]_0^2 = c(8 - \frac{16}{3}) = 1 \implies c = \frac{3}{8}$$

Now our density is

$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2), 0 < x < 2 \\ 0, otherwise \end{cases}$$

$$P(X > 1) = \int_{1}^{2} f(x)dx = \frac{3}{8}\int_{1}^{2} \frac{3}{8}(4x + 2x^2) = \cdots = \frac{1}{2}$$

2 Expectation and variance

2.1 Definitions

Let X be a continous random variable.

- $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
  is called expectation (or expected value) of X.
- $Var[X] = \int_{-\infty}^{\infty}(x - E[X])^2 f_X(x)dx$
  is called variance of X.
- $D[X] = \sqrt{Var(X)}$
  is called standard deviation

2.2 Remark

$$Var(X) = E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_x(x)dx - \left(\int_{-\infty}^{\infty} x f_x(x)dx\right)$$

2.3 Theorem

The first and second properties are the same as in the descrete case.

- $E[a] = a, Var(a) = 0, \forall a \in \mathbb{R}$
- $E[aX + b] = aE[X] + b$
- $Var(aX + b) = a^2 Var(X)$
- $E[g(x)] = \int_{\infty}^{\infty} g(x)f_X(x)dx$

2.4 Example

$$f_X(x) = \begin{cases} 1, 0 \leq x \leq 2 \\ 0, otherwise \end{cases}$$

$$g(x) = e^x.E[e^x] =?, Var(e^x) =?$$

- $E[e^X] = \int_{-\infty}^{\infty} (e^x)(f_X(x))dx = \int_0^1 e^x * 1dx = e - 1$
- $Var(X) = E[X^2] - E[X]^2$
  $E[X^2] = E[(e^x)^2] = E[e^{2x}] = \int_0^1 e^{2x} * 1dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1)$
  $Var = \frac{1}{2}(e^2 - 1) - (e - 1)^2$

3 Common distributions

3.1 Uniform distribution

Uniform distribution on $[a, b]$ (the symbol $U([a, b])$)

3.1.1 Definition

$$f_X(x) = \begin{cases} \frac{1}{b-a}, a \leq x \leq b \\ 0, otherwise \end{cases}$$

3.1.2 Theorem

- $f_X(x) = \begin{cases} 0, -\infty \leq x < a\frac{x-a}{b-a}, a \leq x \leq b \\ 1, x \geq b \end{cases}$
- $E[X] = \frac{a+b}{2}, Var(X) = \frac{(b-a)^2}{12}$

  Proof not included

3.2 Exponential distribution

$$X \sim Exp(\lambda)$$

Given a $0 < \lambda \in \mathbb{R}$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, \ x \geq 0 \\ 0, \ otherwise \end{cases}$$

3.2.1 Theorem

- $F_X(x) = \begin{cases} 0, \ -\infty \leq x \leq 0 \\ 1 - e^{-\lambda x}, \ x \geq 0 \end{cases}$
- $E[X] = \frac{1}{\lambda}; Var(X) = \frac{1}{\lambda^2}$

3.2.2 Theorem (Memoryless property)

Let $X \sim Exp(\lambda)$. We have that $P(X > t + x | X > t) = P(X > x)$

3.2.3 Proof

$$P(X > t+x | X > t) = \frac{P(X > t + x; X > t)}{P(X > t)} = \frac{P(X > t + x)}{P(X > t)} = \frac{1 - F_X(t + x)}{1 - F_X(t)} =$$

$$\frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = 1 - F_X(x) = P(X > x) \quad \blacksquare$$

3.3 Standard normal distribution

$$X \sim \mathcal{N}(0, 1)$$

The random variable has symbol Z, the density funciton is defined as

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in \mathbb{R} = (-\infty, \infty)$$

$F_X(x)$ doesn't have an explicit formula.
The symbol F_X is $\Phi(x) = P(X \leq x)$

3.3.1 Theorem

Let $Z \sim \mathcal{N}(0, 1)$ then

$$E(Z) = 0, Var(Z) = 1$$

Proof comes later in the course.

3.3.2 Example

$$Z \sim \mathcal{N}(0, 1)$$

- $P(Z < 2.11) = \Phi(2.11) = 0.9826, \ from\ table$
- $P(Z \leq -2.11) = \Phi(-2.11) = 1 - \Phi(2.11) = 1 - 0.9826$
- $P(-2.21 \leq Z \leq 2.11) = P(Z \leq 2.11) - P(Z \leq -2.21) = 0.9826 - 0.0163 = 0.969$

4 Standard deveation

$$X \sim f_X(x) = \frac{1}{(2\pi)\sigma} e^{-\frac{x-\mu}{2\sigma^2}}$$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

4.1 Theorem

$$E[X] = \mu, Var(X) = \sigma^2$$

How do we compute $F_X(x)$?
Standardization: we transform X in Z

$$Z = \frac{x - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$