CS 237 – Final Exam – Fall 2022

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Please read the following instructions very carefully.

- You have **120 minutes** to earn **120 points** on this exam. You will find next to each question, the amount of points that that specific question is worth. You can pace yourself by spending at most one minute for each point.
- You can leave your answer in closed form (e.g. $1 - \binom{120}{7}(0.1)^{200}$ or $\frac{0.5+0.3\cdot 1.2}{3\cdot 4}$).
- Summations ($\sum$) and integrals ($\int$) are not, in general, considered closed form. Unless explicitly stated otherwise, any answer left as a summation or integral will be awarded only partial credit.
- *Show your work.* Answers without justification will be given little credit. Justify each step in your solutions e.g. by stating that the step follows from an axiom of probability, a definition, algebra, etc.; for example, your answer could include a line like this:

$$\Pr(X \cap Y \cap Z)\Pr(A \cup B) = \Pr(X \cap Y \cap Z) \cdot (\Pr(A) + \Pr(B)) \qquad (A \text{ and } B \text{ are disjoint})$$

- Please clearly indicate which parts of your solution you want us to grade.
- Do not detach any page from the exam, including the scratch pages.
- Electronic devices (phones, calculators, laptops, etc.) are **not allowed**.
- One handwritten two-sided page **is allowed**. All other resources are **not allowed**.

Good luck!

Name:

Student ID:

PROBLEM 1 (30 pts). Suppose X is a continuous random variable on $[0,1]$ whose probability distribution is unknown.

(a) (3 pts) What is the minimum possible value of $\mathrm{Var}(X)$?

(b) (3 pts) What is the maximum possible value of $\mathrm{Var}(X)$?

(c) (3 pts) Y is a continuous random variable that is uniform on $[0,2]$. Compute $\mathrm{Ex}(Y^3)$.

(d) (6 pts) Suppose we have a discrete random variable N and a non-negative discrete random variable Z for which we know $\mathrm{Ex}(N) = -5$, $\mathrm{Ex}(Z) = 20$, and $\mathrm{Var}(N) = \mathrm{Var}(Z) = 30$. Fill in the following table with a Yes/No answer: answer Yes if the inequality can be used to upper bound the probability.

	Markov Inequality	Chebyshev's Inequality	Chernoff Bound
$\Pr[N \geq 20 \times \mathrm{Ex}(N)]$			
$\Pr[Z \geq 500]$			

In each of the following scenarios, identify the distribution of the random variable described. Write down the name of the distribution and the values of all of its parameters (e.g. Normal($\mu = 0, \sigma^2 = 1$)).

(e) (3 pts) The number of days until we see a shooting star, assuming that there is a probability of 0.01 of seeing a shooting star on any given day, independently of other days.

(f) (3 pts) The time until a lightbulb fails, if bulb failure is memoryless with expected failure time of 50 hours.

(g) (3 pts) The number of times we spin the spinner until we obtain a number that is between 0.6 and 0.95. (Recall that each spin samples a number uniformly at random from the interval $[0,1]$.)

(h) (3 pts) The number of wins for a football team F that plays a 17 game season and has 0.4 probability of winning each game, independently of all other games.

(i) (3 pts) The value of the 0/1 indicator variable which is 1 if and only if team F has exactly 2 wins after 3 games.

Solution:

(a) Variance could be zero.

(b) Variance is at most 1/4, as we saw twice: biased coin and Bernoulli.

(c-FRI) We want them to use LOTUS and see:

$$\mathrm{Ex}[Y^2] = \int_0^2 y^2 \frac{1}{2}\, dy = \left. , y^3/6 \right]_0^2 = 4/3$$

(c-MON) We want them to use LOTUS and see:

$$\mathrm{Ex}[Y^3] = \int_0^2 y^3 \frac{1}{2}\, dy = \left. , y^4/8 \right]_0^2 = 16/8 = 2$$

(d-1) Random variable is negative so no for Markov, yes for Chebyshev, no for Chernoff. [We realized later that Chebyshev would not work the way the Monday exam was posed, and awarded credit for a No answer as well.]

(d-2) Random variable is non-negative so yes for Markov, yes for Chebyshev, no for Chernoff.

(e FRI/MON) Geometric (0.02) / Geometric (0.01)

(f FRI/MON) Exponential (0.01) / Exponential (0.02)

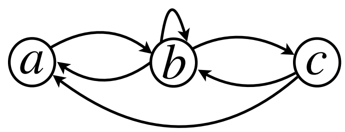
(g) Geometric (0.35), both exams.

(h) FRI/MON) $\mathit{Binomial}(17, 0.6)$ / $\mathit{Binomial}(17, 0.4)$

(i) $\mathit{Bernoulli}(3 \cdot (0.7)^2 \cdot 0.3)$

Correct name of the distribution - 1 pts
Correct parameter(s) - 2 pts
No justification needed, only the name of the distribution and the parameter values

PROBLEM 2 (8 pts). Compute the stationary probabilities on the nodes for the PageRank walk on the following graph. (Recall from lecture that the transition probabilities are proportional to the out-degrees.)



Solution: Let r be the PageRank stationary probabilities. We have

$$\begin{aligned} r(a) &= \frac{1}{3}r(b) + \frac{1}{2}r(c) \\ r(b) &= r(a) + \frac{1}{3}r(b) + \frac{1}{2}r(c) \\ r(c) &= \frac{1}{3}r(b) \\ r(a) + r(b) + r(c) &= 1 \end{aligned}$$

We obtain $r(a) = \frac{3}{11}$, $r(b) = \frac{6}{11}$, $r(c) = \frac{2}{11}$.

PROBLEM 3 (15 pts). A student is going to apply for a credit card. The approval of her application depends on her credit score. She has 0.8 probability to be approved if her score is classified as high, 0.5 probability to be approved if her score is classified as medium, and 0.1 probability if her scored is classified as low. She estimates that her score is classified as high, medium, and low with probability 0.2, 0.6, and 0.2 respectively.

(a) (4 pts) Find the probability that her application for the credit card is approved.

(b) (5 pts) Given that her application for the credit card is approved, find the probability that her score is classified as high.

(c) (6 pts) Given that she did not get approved, find the probability that her score is classified as medium or low.

Solution:

(a) Let H, M, and L be the event that the student's score is classified as high, medium and low respectively. Let J be the event that her application gets approved. Then the problem specifies

$$P(J|H) = 0.8$$

$$P(J|M) = 0.5$$

$$P(J|L) = 0.1$$

, with priors on the type of score given by

$$P(H) = 0.2$$

$$P(M) = 0.6$$

$$P(L) = 0.2$$

We are asked to compute P(J), using law of total probability,

$$P(J) = P(J|H)P(H) + P(J|M)P(M) + P(J|L)P(L) = 0.8(0.2) + (0.5)(0.6) + (0.1)(0.2) = 0.48$$

(b) Given the event J is held true then we are asked to compute $P(H|J)$, which is

$$P(H|J) = \frac{P(J|H)P(H)}{P(J)} = \frac{0.8 \cdot 0.2}{0.48} = \frac{1}{3}$$

(c) For this we are asked to compute $Pr((M \cup L)|\bar{J})$, which is equivalent to $Pr(M|\bar{J}) + Pr(L|\bar{J})$. From what we've had so far, we can calculate

$$Pr(\bar{J}|M) = 1 - Pr(J|M) = 1 - 0.5 = 0.5$$

,

$$Pr(\bar{J}|L) = 1 - Pr(J|L) = 1 - 0.1 = 0.9$$

and

$$Pr(\bar{J}) = 1 - Pr(J) = 1 - 0.48 = 0.52$$

, so

$$Pr(M|\bar{J}) = \frac{Pr(\bar{J}|M)Pr(M)}{Pr(\bar{J})} = \frac{(0.5)(0.6)}{0.52} = \frac{15}{26}$$

$$Pr(L|\bar{J}) = \frac{Pr(\bar{J}|L)Pr(L)}{Pr(\bar{J})} = \frac{(0.9)(0.2)}{0.52} = \frac{9}{26}$$

, thus combined,

$$Pr((M \cup L)|\bar{J}) = \frac{15}{26} + \frac{9}{26} = \frac{24}{26}$$

Alternative solution for c): the question could be reframed as given she doesn't receive the offer, what's the likelihood she didn't receive a strong recommendation, so

$$Pr((M \cup L)|\bar{J}) = 1 - \frac{Pr(H)}{Pr(\bar{J})} = 1 - \frac{Pr(\bar{J}|H)Pr(H)}{Pr(\bar{J})} = 1 - \frac{(1-0.8)(0.2)}{0.52} = 1 - \frac{4}{52} = \frac{24}{26}$$

PROBLEM 4 (16 pts). Kristoff, Haoyu, and Alice are deciding what to watch for movie night. They have two choices, a Christmas movie, and a Vampire movie. They agree to decide on a purchase in the following manner: Each will choose exactly one of the two movies and the group will watch the movie that the majority of them chose. (Example: if Kristoff chooses the Christmas movie, Haoyu chooses the Vampire movie, and Alice chooses the Christmas movie, they watch the Christmas movie.) Each person chooses one of the two movies uniformly at random, and their choices are independent of each other.

(a) (4 pts) Find the sample space and the probability of each outcome.

(b) (4 pts) Let C be the event that the group watches the Christmas movie. Find the outcomes in C and the probability of C.

(c) (4 pts) Let K be the event that Kristoff chooses the Christmas movie. Determine whether C and K are independent. Justify your answer.

(d) (4 pts) Let B be the event that Haoyu chooses the Vampire movie and Alice chooses the Christmas movie. Determine whether B and C are independent. Justify your answer.

Solution:

(a) The outcome records the choice of each person. We represent each outcome by, e.g., ccv to denote that Kristoff chooses Christmas, Haoyu chooses Christmas, Alice chooses Vampire. The sample space is the set of all such sequences, i.e., sequences of length 3 where each entry is either c (Christmas) or v (Vampire):

$$\Omega = \{ccc, ccv, cvc, vcc, cvv, vcv, vvc, vvv\}$$

Since each person chooses uniformly and independently at random, the probability of each outcome is $\frac{1}{8}$.

(b) We have $\Pr(C) = \Pr(\{ccc, ccv, cvc, vcc\}) = \frac{1}{2}$.

(c) We need to check the definition. We have $\Pr(C)\Pr(K) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, which is not the same as $\Pr(C \cap K) = \Pr(\{ccc, ccv, cvc\}) = \frac{3}{8}$. Thus the events are not independent.

(d) We need to check the definition. We have $\Pr(B) = \Pr(\{ccv, cvv\}) = \frac{1}{2}$ and $\Pr(C \cap B) = \Pr(\{ccv\}) = \frac{1}{8}$. Thus $\Pr(C)\Pr(B) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = \Pr(C \cap B)$, and thus the events are independent.

PROBLEM 5 (15 pts). Suppose you are at the mall with your family. The time that you spend there can be modeled by an exponential distribution with an expectation of 80 minutes.

(a) (3 pts) What is the probability that you spend more than 60 minutes there?

(b) (4 pts) Suppose the probability that you spend at least m minutes at the mall is 0.4. What is the value of m?

(c) (4 pts) So far, you've spent 45 minutes at the mall. How much more time do you expect to spend there?

Suppose you can also model the amount that your family spends in total at the mall using the Normal distribution, with an expectation of 120 dollars, and a variance of 100 dollars^2.

(d) (4 pts) What is the probability your family will spend between 125 and 140 dollars at the mall? You may leave your answer in terms of the CDF Φ of the standard Normal distribution, evaluated at non-negative values $x \geq 0$.

Solution:

(a) Let X be the amount of time that you spend at the mall. We are given $\text{Ex}(X) = 80$, thus $\lambda = \frac{1}{80}$. Then:

$$\Pr(X > 60) = 1 - \Pr(X \leq 60) = 1 - (1 - e^{-\frac{60}{80}}) = e^{-0.75}$$

(b) We are given the statement $\Pr(X \geq m) = 0.4$. By using the CDF of X, we can write:

$$1 - (1 - e^{-\frac{m}{80}}) = 0.4 \Rightarrow e^{-\frac{m}{80}} = 0.4$$

By taking the natural log and rearranging the terms, we obtain:

$$-\frac{m}{80} = \ln 0.4 \Rightarrow m = -80 \ln 0.4$$

(c) The exponential distribution is memoryless, so we can conclude that you can still expect to wait $\text{Ex}(X) = 80$ minutes.

(d) Let Y be the amount of money that your family spends at the mall. We are looking for $\Pr(125 \leq Y \leq 140)$. We can define $Z \sim Normal(0, 1)$ where $Z = \frac{Y-\mu}{\sigma} = \frac{Y-120}{10}$ (as $\text{Var}(Y) = \sigma^2$). Then, we know that $Y = 10Z + 120$. We can then break down the probability:

$$\begin{aligned}
\Pr(125 \leq Y \leq 140) &= \Pr(Y \leq 140) - \Pr(Y \leq 125) \\
&= \Pr(10Z + 120 \leq 140) - \Pr(10Z + 120 \leq 125) \\
&= \Pr(Z \leq 2) - \Pr(Z \leq 0.5) \\
&= \Phi(2) - \Phi(0.5)
\end{aligned}$$

PROBLEM 6 (16 pts). Mike goes to picnic with his friends every weekend. However, he sometimes forgets to bring a few items when he leaves his apartment and heads to the picnic spot. He has 20 items that he would like to bring, and he forgets each item independently at random with probability 0.4.

(a) (4 pts) Let X be the number of items that Mike forgets to bring. Find the distribution of X (state the name of the distribution and its parameter(s), e.g., $X \sim \text{Normal}(\mu = 0, \sigma^2 = 1)$). Find the expectation and variance of X.

(b) (4 pts) Use the Markov inequality to upper bound the probability that Mike forgets at least 12 items.

(c) (4 pts) Use the Chebyshev inequality to upper bound the probability that Mike forgets at least 11 items.

(d) (4 pts) Use the Chernoff inequality to upper bound the probability that Mike forgets at least 14 items.

Solution:

(a) $X \sim \text{Binomial}(n = 20, p = 0.4)$. Using the formulas from class, we have $\text{Ex}(X) = np = 8$ and $\text{Var}(X) = np(1-p) = 4.8$.

(b) We have

$$\Pr(X \geq 12) \leq \frac{\text{Ex}(X)}{12} = \frac{8}{12} = \frac{2}{3}$$

(c) We have

$$\Pr(X \geq 11) = \Pr(X - \text{Ex}(X) \geq 3) \leq \Pr(|X - \text{Ex}(X)| \geq 3) \leq \frac{\text{Var}(X)}{3^2} = \frac{4.8}{3^2}$$

(d) We have

$$\Pr(X \geq 14) = \Pr(X \geq (14/8) \cdot \text{Ex}(X)) \leq \exp(-\beta(14/8)\text{Ex}(X))$$

where $\beta(c) = c \ln c - c + 1$.

PROBLEM 7 (20 pts). Consider the reservoir sampling algorithm from lecture. Recall that the algorithm processes a sequence of items as follows. The algorithm stores a single item in memory, which is initially the first item to arrive. When a new item arrives, the algorithm replaces the item in memory with the current item independently with probability $1/n$, where n is the number of items that have arrived so far. As in lecture, we refer to the step where the algorithm processes the i-th item as the i-th iteration.

(a) (2 pts) Find the probability that, at the end of the 12-th iteration, the item in memory is the 1st item.

(b) (4 pts) Let s be the item in memory at the end of the 12-th iteration. Find the probability that the algorithm replaces s in the next three iterations, i.e., iterations 13 through 15.

(c) (4 pts) Suppose that the stream is comprised of the natural numbers $1, 2, 3, \ldots$ (i.e., in iteration i, the item that arrives is the natural number i). Let X be the random variable that is equal to the item in memory at the end of the 12-th iteration. Find the distribution of X (write the name of the distribution and its parameter(s), e.g., $X \sim \text{Normal}(\mu = 0, \sigma^2 = 1)$). Find the expectation and variance of X.

Let Y be the random variable that is equal to the number of times the algorithm replaced the item in memory in iterations 9 through 12.

(d) (2 pts) Find the range of Y.

(e) (4 pts) Find $\Pr(Y = 3)$.

(f) (4 pts) Find the expectation and variance of Y.

Solution:

(a) As shown in class, the item in memory is equally likely to be any one of the items that have arrived so far. Thus the probability is $1/12$.

(b) Our intended interpretation was to compute the probability that s be replaced at least once in iterations 13-15. Our answer below is for that case. An alternative interpretation was that a replacement happened in all iterations 13-15, and we also gave full credit for that: $1/13 * 1/14 * 1/15$.

Let A be the event of interest. We find $\Pr(\overline{A})$ and use the complement rule. Let R_i be the event that the algorithm replaces the item in memory in the i-th iteration. We have $\overline{A} = \overline{R_{13}} \cap \overline{R_{14}} \cap \overline{R_{15}}$. Since the events R_i are independent and $\Pr(\overline{R_i}) = 1 - \Pr(R_i) = 1 - \frac{1}{i} = \frac{i-1}{i}$, we have

$$\Pr(\overline{A}) = \overline{R_{13}} \cap \overline{R_{14}} \cap \overline{R_{15}} = \Pr(\overline{R_{13}}) \cdot \Pr(\overline{R_{14}}) \cdot \Pr(\overline{R_{15}}) = \frac{12}{13} \cdot \frac{13}{14} \cdot \frac{14}{15} = \frac{12}{15} = \frac{4}{5}$$

$$\Pr(A) = 1 - \Pr(\overline{A}) = \frac{1}{5}$$

(c) The range of X is $\{1, 2, \ldots, 12\}$. As we have seen in class, each item is equally likely to be the item stored in memory, and thus X is a uniform random variable. Thus we have $X \sim \text{Uniform}(\{1, 2, \ldots, 12\})$. Using the formulas in class, we obtain $\text{Ex}(X) = \frac{13}{2}$ and $\text{Var}(X) = \frac{11 \cdot 13}{12}$.

(d) The range is $\{0, 1, 2, 3, 4\}$.

(e) As in part (b), let R_i be the event that the algorithm replaces the item in memory in the i-th iteration. Since there is exactly one iteration where the algorithm does not replace the item, the event $Y = 3$ is the disjoint union of the events $\overline{R_9} \cap R_{10} \cap R_{11} \cap R_{12}$, $R_9 \cap \overline{R_{10}} \cap R_{11} \cap R_{12}$, $R_9 \cap R_{10} \cap \overline{R_{11}} \cap R_{12}$, $R_9 \cap R_{10} \cap R_{11} \cap \overline{R_{12}}$. Using the additivity axiom and the fact that the events are independent and $\Pr(R_i) = \frac{1}{i}$, we obtain

$$\Pr(Y = 3) = \frac{8}{9} \cdot \frac{1}{10} \cdot \frac{1}{11} \cdot \frac{1}{12} + \frac{1}{9} \cdot \frac{9}{10} \cdot \frac{1}{11} \cdot \frac{1}{12} + \frac{1}{9} \cdot \frac{1}{10} \cdot \frac{10}{11} \cdot \frac{1}{12} + \frac{1}{9} \cdot \frac{1}{10} \cdot \frac{1}{11} \cdot \frac{11}{12}$$

(f) We break it down based on each iteration and use linearity of expectation. Let Y_i be the indicator random variable that is equal to 1 if the algorithm replaces the item in memory in iteration i. We have $Y = Y_9 + Y_{10} + Y_{11} + Y_{12}$. Y_i is a Bernoulli random variable with success probability $p = \Pr(Y_i = 1) = \frac{1}{i}$ and thus it has $\mathrm{Ex}(Y_i) = p = \frac{1}{i}$ and $\mathrm{Var}(X_i) = p(1-p) = \frac{i-1}{i^2}$ via the formulas from class. Since the Y_i's are independent, linearity of variance also applies. Thus we can apply linearity of expectation and variance and obtain

$$\mathrm{Ex}(Y) = \mathrm{Ex}(Y_9) + \mathrm{Ex}(Y_{10}) + \mathrm{Ex}(Y_{11}) + \mathrm{Ex}(Y_{12}) = \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$$

$$\mathrm{Var}(Y) = \mathrm{Var}(Y_9) + \mathrm{Var}(Y_{10}) + \mathrm{Var}(Y_{11}) + \mathrm{Var}(Y_{12}) = \frac{8}{9^2} + \frac{9}{10^2} + \frac{10}{11^2} + \frac{11}{12^2}$$

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