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Affine Models of the Term Structure of Interest Rates, Inflation and Corporate Credit Spread: A Kalman Filter Estimation

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Abstract

In this work we estimate affine term structure models of interest rates, BEI rates and default rates in a period of Crisis. We estimate one factor, two factor and three factor model on both interest rates and BEI rates. We use information criteria to decide the number of factors to include in the models. We also perform the PCA analysis on interest rates, BEI rates and default rates bootstrapped from European OIS rates, European ZC inflation swap and CDS of Italian firms and Italian Government to analyze the determinants of interest rates, BEI rates and default rates changes. We compare the results found in interest rates and BEI rates models with the results of interest rates and BEI rates found by the PCA. We use the results on both interest rates models and BEI rates models to estimate a multifactor model which jointly considers the effect of interest rates and the effect of BEI rates on default rates of Italian corporations and Italian Government.

Contents

Introduction

One factor is not enough to explain interest rates variability over time. This evidence is well known in the related literature starting from the 80s (see Vasicek and Fong 1982). Indeed, the literature on the term structure of interest rates has moved towards a multifactor approach, to overcome the shortcomings provided by first one-factor modeling approaches (e.g. Vasicek 1977, and Brennan and Schwarz 1977). With Litterman and Scheinkman (1991), it became clear among market practitioners and academics that at least three factors are needed to explain most of the variability of interest rates in the market.

Starting from the end of 2008, European market practitioners have been working in a completely new macroeconomic framework, originated by the subprime crisis, that busted a long period of recession in the global economy. In the mid 2011, the sovereign debt crisis brought the European economy in a further period of economic recession. Indeed, the ECB put forward unconventional monetary policies, such as the Asset Purchase Program, with the aim of taking the European economy out of recession. This policy has led, among other things, to negative nominal interest rates.

The questions we address in this work are:

- ▷ Do we still need a multifactor approach to explain the variability of interest rates in this new macroeconomic framework?
- ▷ How many factor are needed to explain the variability of negative interest rates?
- ▷ The new macroeconomic framework led to low European inflation. How do these low inflation rates move in the ongoing macroeconomic situation?
- ▷ The credit default swap (CDS) spread is often considered to be an indicator of the optimism of the overall economy (see Geneakoplos 2009, 2010). How do default rates of financial and non financial firms move in time of crisis?
- ▷ Furthermore, how do interest rates and inflation rates affect default rates of firms and Government?

- ▷ It is well known among academics and market practitioners, that interest rates affect negatively default rates of corporations and Governments (see Duffee 1998, 1999). How does this relationship change during this crisis period?
- ▷ How do interest rates affect credit default risk premia? How do they affect inflation rates?

The thesis is organized as follows:

- In chapter 1 we present the data used for our empirical analysis and describe the standard manipulation techniques (bootstrap) we use.
- Chapter 2 is devoted to a principal component analysis (PCA) on our data, useful to decide the dimension of our multifactor models for interest rates, inflation rates and default rates.
- In chapter 3 we build univariate, bivariate and threevariate affine term structure models and estimate them on the interest rates time series bootstrapped from European overnight index swap rates (OIS).
- Analogous models are constructed in chapter 4 for break even inflation (BEI) rates, and estimated on European ZC inflation swaps.
- In chapter 5 we set up an affine model for default rates, using the results of interest rates and BEI rates found in the previous chapters. The model is estimated on the data bootstrapped from European CDS on some financial and industrial Italian names and on the Italian Government.
- Chapter 6 concludes.

To estimate the models presented here, a Bayesian approach is used. As stated by De Jong (2000), a Bayesian approach in the estimation of Term Structure models allows to exploit the information of the Cross Section of the data matrix and not only the information provided by the times series. In particular, we use the Kalman Filter (Kalman 1960) for the overall estimation procedure.

The evidence of negative interest rates and negative inflation rates in financial markets after the sovereign debt crisis allows us to model the two rates using a Gaussian Dynamics. Indeed, the Kalman Filter is the most appropriate method to estimate interest rates and BEI rates Gaussian term structure models.

We use a Gaussian dynamics and the Kalman Filter to estimate also models for default rates of Italian firms and Government, even if, in this case, the state variables are default intensities, and hence intrinsically non-negative. This is somewhat a compromise, to maintain analytical tractability and sound estimation methods. This compromise is often used in literature (e.g. D'Amato et al. 2006, Corlois et al. 2013 and Russo et al. 2017).

Chapter 1

Bootstrapping interest rates, BEI rates and default rates

1.1 Interest rates

To perform the analysis on interest rates, the OIS rates are considered. OIS contracts are interbank bilateral contracts, where two parties exchange a fixed leg, a stream of annual fixed rate coupons, versus a floating leg, a stream of annual floating rate coupons, each of them is computed as the interest generated on the reference year by a rollover of the notional at the daily overnight interest (OI) rate. The OIS rate for time to maturity m years at a certain day is the rate of the fixed leg of the m-year OIS contract, that can be entered in with no upfront at that day.

The OIS rate is the rate at which banks lend overnight unsecured money (i.e. without guarantees) one with another. In Europe the reference OIS rate is the *Eonia*, in the US the *FED fund rate*. OI rates are perceived less risky than other interest rates, like e.g. *Libor* or *Euribor* rates, as the counterparty risk is limited.

Starting from the bursting of the subprime crisis, as the counterparty credit risk perception on the market increased, Libor and Euribor rates started to be higher than OIS rates and the latter rates started to be considered as a reference proxy of risk free rates (Hull and White 2015). In our analysis we refer therefore to OIS rates as the risk free rates.

1.1.1 Data

We consider a panel of European OIS rates from DataStream over the period November 3, 2008 - November 3, 2016. We refer to the first subperiod November 3, 2008 - July 4, 2011 as the *subprime crisis period*; July 5, 2011 - November 3, 2016 is the *European sovereign debt crisis* subperiod.

For each day in our sample period, we consider OIS rates for time to maturities 1–10 years, and 15, 20, 25 and 30 years. We fill in the missing integral maturities (e.g. 11, 12, ... years) by linear interpolation, as shown e.g. in West and Hagan (2006).

1.1.2 Bootstrapping interest rates

The procedure of *bootstrapping* is a standard method to extract market discount factors (and hence market rates) from swap rates of zero-upfront swap contracts. It is based on the following standard results (see e.g. Bernhart 2013):

1. At time t, the floating leg of an m-year swap contract, with notional N and with risk free underlying rate, has market value given by
$$N[1 - P(t, t + m)] \ ,$$
where $P(t, t+m)$ is the market risk free discount factor at time t for maturity $t + m$.

2. At the same time, the market value of the fixed leg of the N notional, m-year swap contract, with swap rate s_m, tenor τ years (and therefore $p = n/\tau$ fixed coupons) is given by
$$\tau s_m N \sum_{i=1}^{p} P(t, t + \tau i) \ .$$

Since our OIS rates are zero-upfront rates, the market values of the two legs have to be the same at time t. Equating hence the two formulae (with $\tau = 1$) ad solving with respect to s_m, we get
$$s_m = \frac{1 - P(t, t + m)}{\sum_{i=1}^{m} P(t, t + i)} \ . \tag{1.1}$$

The bootstrap procedure is based on inverting (1.1) and solving it with respect to the discount factors. To this end, we consider a set of M OIS rates, for all maturities $m = 1, 2, \ldots, M$, and proceed iteratively:

– For $m = 1$, we solve (1.1) with respect to $P(t, t + 1)$, obtaining
$$P(t, t + 1) = \frac{1}{1 + s_1} \ .$$

– After obtaining $P(t, t + 1)$, $P(t, t + 2)$, ..., $P(t, t + m - 1)$, we solve (1.1) with respect to $P(t, t + m)$:
$$P(t, t + m) = \frac{1 - s_m \sum_{i=1}^{m-1} P(t, t + i)}{1 + s_m} \ .$$

Once the discount factors are computed, (spot instantaneous) interest rates for each m are computed by the standard formula:

$$y(t, t+m) = -\frac{1}{n} \ln P(t, t+m) \ .$$

Figure 1.1 plots the bootstrapped interest rate time series for some maturities.

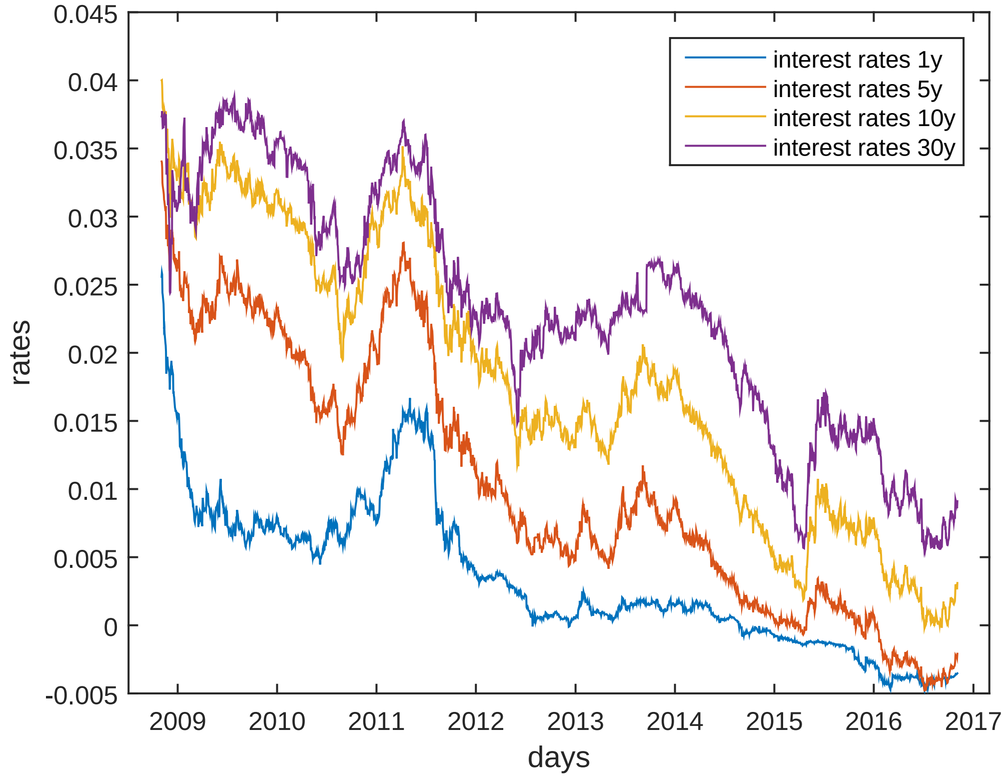

Figure 1.1: Bootstrappend interest rates

1.2 Inflation rates

An zero coupon inflation swap (ZCIS) with maturity T, entered at time t, is a contract where two parties exchange a fixed leg versus a floating leg, linked to realized inflation. The fixed is composed by a unique fixed payoff, given by the interest on the notional for the whole contract time span, computed at the annual compounded interest rate $b(t,T)$. On the other hand, the floating leg pays at time T the realized inflation between t and T on the notional.

The ZCIS rate that makes the m-year contract a zero-upfront contract at time t is called the *break even inflation* (BEI) rate at time t for m years, or simply the m-year ZCIS rate at time t.

Standard European ZICS are no-upfront and are linked to the European Harmonised Index of Consumer Prices (HICP). In the US, a standard ZICS is linked to the US Consumer Price Index, whereas standard British ZICS are linked to Great Britain's Retail Price Index (RPI).

1.2.1 Data

We consider a panel of European ZCIS rates from DataStream over the period November 3, 2008 - November 3, 2016. We consider also the same two subperiods considered for interest rates. For each day in our sample period, we consider ZCIS rates for time to maturities 1–10 years, and 15, 20, 25 and 30 years. We fill in the missing integral maturities by using linear interpolation.

From what seen in the previous subsection, ZCIS rates are readily zero-coupon rates and no bootstrap is needed. Figure 1.2 plots the time series of BEI rates for some maturities.

1.2.2 BEI, real and nominal rates

Let $I(t)$ be the reference price index at time t and consider a no-upfront, m-year ZCIS at rate $b(t,t+m)$ with notional N. Now:

1. The fixed leg pays the fixed amount $N\left([1+b(t,t+m)]^m-1\right)$ at time $t+m$ and has therefore market value at time t

$$N\left([1+b(t,t+m)]^m-1\right)P(t,t+m)\ .$$

2. The floating leg pays the amount

$$N\frac{I(t+m)-I(t)}{I(t)}$$

at time T. By standard arguments (see e.g. Kaminska et al. 2018), its market price at time t is given by

$$N[P^r(t,t+m)-P(t,t+m)]\ ,$$

where $P^r(t,t+m)$ is the real market discount factor at time t for time to maturity m.

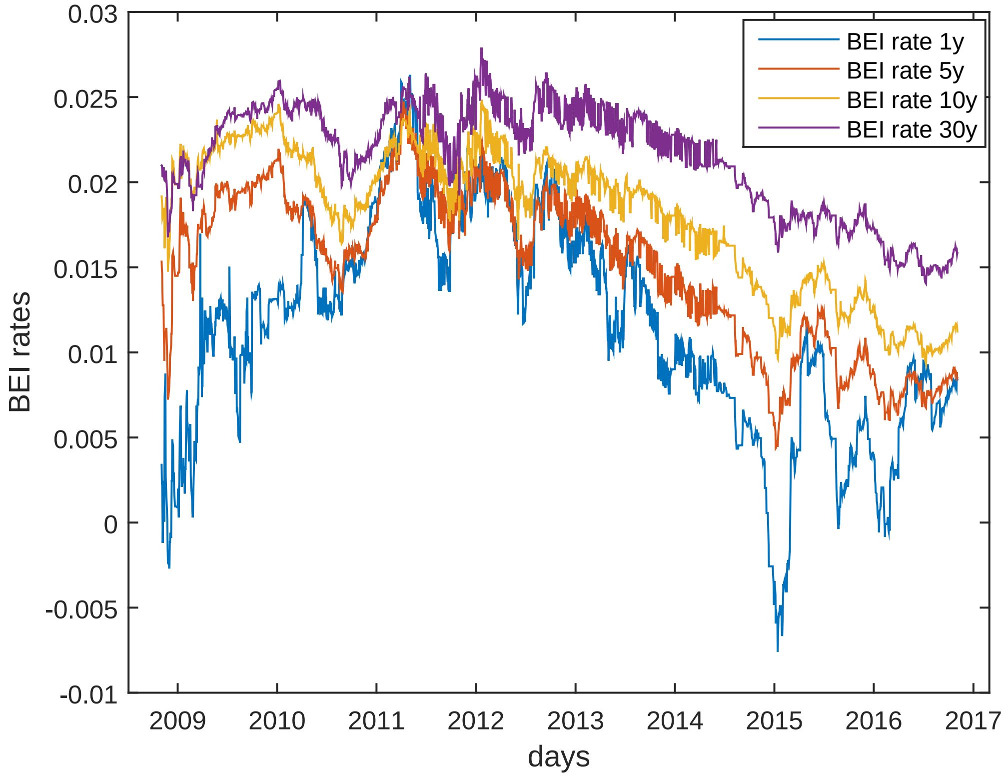

Figure 1.2: BEI rates

Equating the market price of the two legs and solving with respect to the ZICS rate we get

$$b(t, t+m) = \left[\frac{P^r(t, t+m)}{P(t, t+m)}\right]^{\frac{1}{m}} - 1 \ .$$

At the level of spot instantaneous rates, i.e. applying logarithms, we obtain

$$\begin{aligned} y^b(t, t+m) = \ln[1 + b(t, t+m)] &= \frac{1}{m} \ln P^r(t, t+m) - \frac{1}{m} \ln P(t, t+m) \\ &= -y^r(t, t+m) + y(t, t+m) \ , \end{aligned}$$

where $y^r(t, t+m)$ is the real (spot instantaneous) rate.

1.3 Default rates

The default rates considered in our analysis are computed from CDS spreads. An m-year CDS contract with notional N is a contract where a party called the *protection buyer* pays a yearly fixed amount, an interest at the CDS spread s_m on N, to the *protection seller*, to insure against a credit event that could happen to a reference bond. The protection seller, on the other hand, pays the loss on the reference instrument, if and when the credit event happens. In case of a credit event, the protection buyer receives the loss and the contract terminates without further payments. What is a credit event and reference bond are contractually specified; standard CDS contracts are written using ISDA contractual conventions and have no upfront.

1.3.1 Data

We consider a panel of standard CDS spreads from DataStream over the period November 3, 2008 - November 3, 2016, written on the default of the following names: Unicredit, Ubi Bank, MedioBanca, and Generali (financial), ENI, Telecom, and Enel (industrial), and the Republic of Italy. The names have been chosen to be representative on the Italian financial and industrial sector, also considering problems of data availability for the whole sample time span. The subperiods considered are the same as for interest and BEI rates.

We consider this subset of firms because:

- Their CDS spread available on DataStream do cover the whole sample period. This not happens e.g. for FIAT or Alleanza, having incurred in M&A or restructuring of their corporate structure during the time span under analysis.
- The firms analyzed did not face situations of financial distress over the period taken into account (differently e.g. from Monte dei Paschi).

For each day in our sample period, we consider CDS rates for time to maturities 1–5 years, 7 and 10 years (standard CDS durations). We fill in the missing integral maturities by using linear interpolation.

1.3.2 Bootstrapping default rates and zero-recovery defaultable zero-coupon prices and rates

To obtain default rates from CDS spreads we used the standard JP Morgan method (see e.g. Castellacci 2008).

Consider at time t an m-year standard CDS contract, with notional N and spread m, written on a certain name. Let R be the *recovery rate*, that is, the fraction of the notional recovered by the reference bond holder in case of default. Market practice assumes $R = 40\%$. Let $Q(t, T)$ be such that the price of a contingent claim paying 1 unit of cash at time T in case of non-default of the reference entity at time T is $Q(t,T)p(t,T)$. Therefore $Q(t,T)$ is the *forward risk-neutral* survivor probability at time T. Assuming survivor at time t, it holds $Q(t,t) = 1$.

Under these assumptions and notations, the market price at time t of the protection payments of our CDS is

$$N s_m \sum_{i=1}^{m} \frac{Q(t,t+i-1) + Q(t+i)}{2} P(t,t+i) \ .$$

On the other hand, the market price of the protection at time t is

$$(1-R)N \sum_{i=1}^{m} [Q(t,t+i-1) - Q(t+i)] P(t,t+i) \ .$$

A standard CDS has zero upfront, hence the two values have to be the same. By equating the two formulae and solving with respect to s_m we get

$$s_m = (1-R) \frac{\sum_{i=1}^{m} [Q(t,t+i-1) - Q(t,t+i)] P(t,t+i)}{\frac{1}{2} \sum_{i=1}^{m} [Q(t,t+i-1) + Q(t,t+i)] P(t,t+i)} \ . \tag{1.2}$$

As in the case of OIS rates, the bootstrap procedure is based on inverting (1.2) and solving it with respect to the risk neutral probabilities. To this end, we consider a set of M CDS rates, for all maturities $m = 1, 2, \ldots, M$, and proceed iteratively:

- For $m = 1$, we solve (1.2) with respect to $Q(t,t+1)$, obtaining

$$Q(t,t+1) = \frac{1 - R - \frac{1}{2}s_1}{1 - R + \frac{1}{2}s_1} \ .$$

- After obtaining $Q(t,t+1)$, $Q(t,t+2)$, $\ldots$, $Q(t,t+m-1)$, we solve (1.2) with respect to $Q(t,t+m)$:

$$\begin{aligned} Q(t,t+m) = & \frac{1 - R - \frac{1}{2}s_m}{1 - R + \frac{1}{2}s_m} Q(t,t+m-1) \\ & + \frac{1-R}{1 - R + \frac{1}{2}s_m} \sum_{i=1}^{m-1} [Q(t,t+i-1) - Q(t,t+i)] \frac{P(t,t+i)}{P(t,t+m)} \\ & - \frac{\frac{1}{2}s_m}{1 - R + \frac{1}{2}s_m} \sum_{i=1}^{m-1} [Q(t,t+i-1) + Q(t,t+i)] \frac{P(t,t+i)}{P(t,t+m)} \ . \end{aligned}$$

After obtaining the forward risk neutral default probability, we consider the (forward risk neutral instantaneous spot) default rate:

$$y^d(t,t+m) = -\frac{1}{m} \ln Q(t,t+m) \ ,$$

and the (instantaneous spot) zero-recovery risky rate

$$y^*(t, t+m) = -\frac{1}{m} \ln[Q(t, t+m)P(t, t+m)] = y^d(t, t+m) + y(t, t+m) \ .$$

Default rate $y^d(t, t+m)$ is therefore the spread we have to add to the risk-free rate $y(t, t+m)$ to obtain the zero-recovery risky rate.

Figures 1.3 and 1.4 report the boostrapped default rates for some of the considered names and for some maturities.

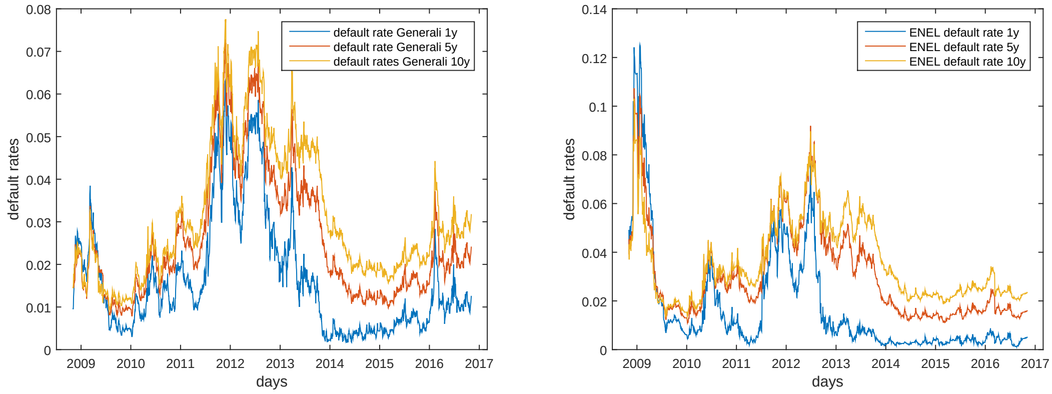

Figure 1.3: Bootstrapped default rates for Generali (left) and Enel (right)

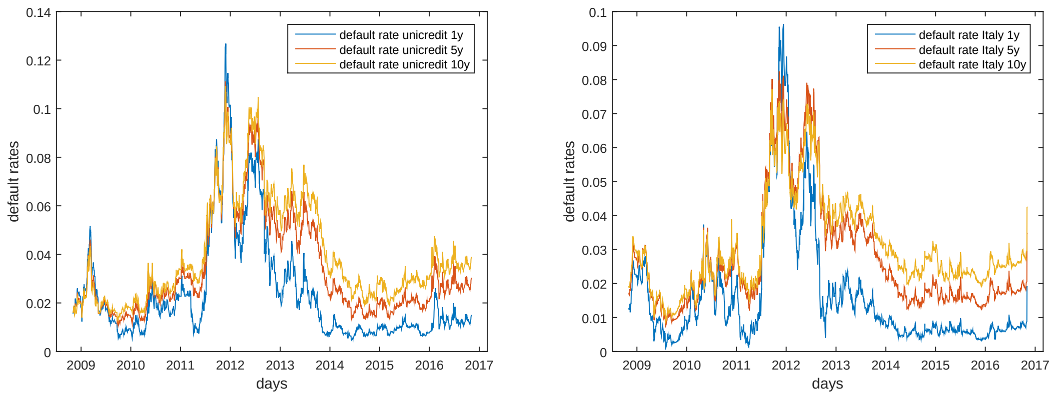

Figure 1.4: Bootstrapped default rates for Unicredit (left) and Republic of Italy (right)

Chapter 2

How many factors for interest rates BEI rates and default rates?

2.1 Introduction

The presence of non parallel shifts in the term structure of interest rates is well know and has important implications e.g. in hedging strategies: the immunization procedure proposed by Fisher and Weil (1971) no longer works; the duration hedging yields losses as it does not consider the possibility the non parallel shift of the term structure.

But also to simple interest rate models, even allowing for non parallel shift, may have similar problems. As an example, Vasicek and Fong (1982) show that actual yield curves show different shapes than those permitted by univariate models such as Vasicek (1977) and Brennan and Schwarz (1977).

The questions to be answered in this chapter are:

- ▷ How do interest rates evolve in times of crisis?
- ▷ How do inflation rates and default rates of financial and non financial firms move over time?

The approach followed here is based on the *Principal Component Analysis* (PCA). It is a widely used statistical method for factor extraction, very popular among practitioners and academics. It is more handy and easier to implement than alternative methods such as the *Factor Analysis* using a computer. It is often easy to find dedicated functions in many software packages such as SAS or Matlab. Furthermore, it does not require any distributional assumption nor other restrictions such as the no-arbitrage assumption.

Despite of its popularity, the PCA has not yet been employed on some financial data. Although it is widely used for US interest rates, (see Barber and Copper

1996, Reisman and Zohar 2004) and for risk management of fixed income securities (see Golup and Tilman 2000), the PCA on rates bootstrapped from European interest rates swap is not much considered, especially in times of crisis.

A PCA on BEI rates bootstrapped from European inflation swaps has never been performed in literature.

Also PCA on corporate default rates bootstrapped from CDS is an original contribution of this work. Longstaff et al. (2012) computes directly the PCA on sovereign CDS spreads of different countries. Bertocchi et al. (2005) analyses the factors which affects the default rates changes of US industrial firms, considering defaultable bond spreads. Truck et al. (2018) applies the PCA on sovereign bond spreads of different countries.

Furthermore, an analysis of factors of default rates of Italian corporations is completely neglected in literature.

The analysis of the drivers of default rates of Italian corporations, creates a link between the term structure evolution of default rates of Italian corporations and the ongoing macroeconomic environment.

The joint analysis of components driving the dynamics of interest rates, BEI rates and default rates could suggest to both market practitioners and academics, new ways of hedging inflation risk, interest rate risk and credit risk of Italian corporations.

2.2 Principal component analysis for term structure models

The principal component analysis is a parsimonious tool to describe the variation of the shape of the term structure over time. The yield curve can be represented as a linear combination of uncorrelated latent factors.

Following Jolliffe (2002), let us consider a data matrix Y, composed by n observations along time of $p < n$ rates:

$$Y = \underbrace{\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \vdots & y_{np} \end{bmatrix}}_{p \times n}$$

Let μ_i be the sample mean of the rates of the ith maturity (ith column of Y), for each $i = 1, \ldots, p$. That is

$$\mu_i = \frac{\sum_{j=1}^{n} y_{ji}}{n} \quad .$$

By subtracting from the ith column its mean μ_i, we obtain the new matrix $\bar{Y}$:

$$\bar{Y} = \underbrace{\begin{bmatrix} \bar{y_{11}} & \bar{y_{12}} & \dots & \bar{y_{1p}} \\ \bar{y_{21}} & \bar{y_{22}} & \dots & \bar{y_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{y_{n1}} & \bar{y_{n2}} & \vdots & \bar{y_{np}} \end{bmatrix}}_{p \times n},$$

whose elements are $\bar{y_{ji}} = y_{ji} - \mu_i$. Furthermore, we have that. $\forall i = 1, \dots, p$ and $\forall j = 1, \dots, n$,

$$(y_{ji} - \mu_i) w_{ji} = f_{ji} \tag{2.1}$$

where latent the factors f_{ji} are elements of matrix $\mathcal{P}$ and w_{ji} are elements of the non singular matrix W. The two are assumed to be of the form:

$$\mathcal{P} = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1p} \\ f_{21} & f_{22} & \dots & f_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \vdots & f_{np} \end{bmatrix}, \qquad W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1p} \\ w_{21} & w_{22} & \dots & w_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p1} & w_{p2} & \vdots & w_{pp} \end{bmatrix}.$$

Matrix W can be found through the Singular Value Decomposition SVD of data.

The data matrix can be decomposed as:

$$\underbrace{Y}_{p \times n} = \underbrace{U}_{n \times n} \underbrace{\Sigma}_{n \times p} \underbrace{W^T}_{p \times p}$$

where,

- U is the matrix YY^T;
- Σ contains the volatility λ of the factor scores which are the square roots of the eigenvalues of the found by the eigendecomposition of the data matrix;
- W the matrix of the eigenvectors of data;

By rearranging the eigenvalues of matrix Σ in decreasing order, the variance explained by each factor is:

$$\sigma_i = \frac{\lambda_i^2}{\sum_{i=1}^{n} \lambda_i^2} \tag{2.2}$$

Following Litterman and Scheinkman (1991), three factors are needed to explain most of the variability of data. Although factors of the PCA cannot be identified, Litterman and Scheikman find, for yields obtained from time series of bond prices, that:

- The first factor is the *level*. The level determines the long run trend made by interest rates. It is positively correlated with interest rates at longer maturities.
- The second factor is the *slope*. The second factor is correlated with the slope of the term structure (i.e. $20y - 2y$);
- The third factor is the *curvature*. The different movements of rates at extreme maturities and central maturities in the term structure is described by the curvature. The curvature is positively correlated with the butterfly strategy (i.e. $10y - 0,5(1y + 30y)$).

The computation of PCA is based on the sample covariance matrix and therefore there are problems if our time series are not stationary, as in such a case the sample covariance is not an estimator of the covariance. A PCA performed on non stationary time series yields biased results of the variance explained by factors (Lansangan et al. 2009). Nevertheless, the related literature on PCA applied to term structure dynamics, is often silent on this stationarity problem. Litterman and Scheinkman (1991) give a brief and not complete explanation on the stationarity of data employed. The same holds in Bertocchi et al. (2005), Longstaff et al. (2012) and Barber and Copper (1996).

2.2.1 Stationarity test on the series

To test the time series stationarity of the data considered, we performed the augmented Dickey-Fuller (ADF) test with 10 lags.

The results of the stationary test (tables 2.1–2.9) show that:

- In the first subsample, for all series of interest rates, BEI rates and default rates, we cannot reject the null hypothesis of unit root.
- In the second subsample, the time series of interest rates at shorter maturity do reject the null hypothesis and can be therefore considered as stationary. This not happens for longer maturities, nor for BEI rates.
- Default rates of financial firms are non stationary in the second subsample, while default rates of industrial firms can be considered stationary at shorter maturities.

The evidence found in subsample analysis, holds also in the analysis over the whole sample.

To solve the problem, the PCA on the series in first differences, i.e. a $\mathcal{I}(1)$ has been done.

2.2.2 PCA results

The PCA on the time series in first differences (tables 2.10–2.18) shows that:

- On average, around 98% of variability of interest rates data is explained by seven factors in every sample.

- On average, in every sample around 90% of variability of BEI rates data is explained by nine factors.

- In both the subsamples and in the whole sample, three factors do explain more than 99% of variability of default rates of the corporations analyzed; two factor do explain more than 98% of the variance.

- For all data analyzed, PCA yields a higher concentration of variance in the first factor over the second subsample. The variance unexplained by the first factor, is spread among a higher number of factors. This effect is stronger for interest rates and BEI rates.

To conclude, PCA on interest rates, BEI rates and default rates, shows that:

- Both after the failure of Lehman Brothers and after the bursting of the sovereign debt crisis, more than one factor is needed to explain the variability of interest rates, BEI rates and default rates.

- More factors are needed for interest rates and BEI rates after the bursting of the sovereign debt crisis.

- On average, the number of factors needed to explain the variability of default rates is stable over the two subsamples.

Table 2.1: Dickey-Fuller test with 10 lags augmentation for interest rates and BEI rates (whole sample)

	BEI rates		Interest rates	
Maturity	t-stat	p-value	t-stat	p-value
1	-1,468	0,133	-5,346***	0,001
2	-1,188	0,216	-3,705***	0,001
3	-1,082	0,255	-3,416***	0,001
4	-1,095	0,250	-2,725***	0,006
5	-1,059	0,263	-2,902***	0,004
6	-1,001	0,285	-2,760***	0,005
7	-1,002	0,284	-2,413**	0,015
8	-0,975	0,294	-2,548**	0,011
9	-0,961	0,299	-2,452**	0,013
10	-0,905	0,320	-2,370**	0,017
15	-0,866	0,334	-2,035**	0,040
20	-0,849	0,340	-1,810*	0,067
25	-0,784	0,364	-1,701*	0,084
30	-0,790	0,362	-1,573	0,109

* week significant rejection of non-stationarity
**significant rejection of non-stationarity
***strong significant rejection of non-stationarity

Table 2.2: Dickey-Fuller test with 10 lags augmentation for interest rates and BEI rates (first subsample)

	BEI rates		Interest rates	
Maturity	t-stat	p-value	t-stat	p-value
1	0,134	0,701	-1,138	0,234
2	0,523	0,828	-1,061	0,262
3	0,422	0,804	-1,000	0,285
4	0,572	0,839	-1,038	0,271
5	0,481	0,818	-0,922	0,313
6	0,584	0,842	-0,880	0,329
7	0,309	0,765	-0,857	0,337
8	0,503	0,823	-0,841	0,343
9	0,379	0,790	-0,889	0,325
10	0,494	0,821	-0,829	0,347
15	0,253	0,744	-0,655	0,411
20	0,334	0,774	-0,491	0,471
25	0,432	0,806	-0,413	0,500
30	0,563	0,837	-0,348	0,524

* week significant rejection of non-stationarity
**significant rejection of non-stationarity
***strong significant rejection of non-stationarity

Table 2.3: Dickey-Fuller test with 10 lags augmentation for interest rates and BEI rates (second subsample)

	BEI rates		Interest rates	
Maturity	t-stat	p-value	t-stat	p-value
1	-1,297	0,179	-4,749***	0,001
2	-1,363	0,161	-4,638***	0,001
3	-1,513	0,122	-4,370***	0,001
4	-1,562	0,111	-3,964***	0,001
5	-1,487	0,128	-3,725***	0,001
6	-1,461	0,135	-3,472***	0,001
7	-1,485	0,128	-3,409***	0,001
8	-1,434	0,141	-3,067***	0,003
9	-1,363	0,160	-2,935***	0,004
10	-1,494	0,126	-2,770***	0,006
15	-1,287	0,183	-2,416**	0,015
20	-1,211	0,208	-2,254**	0,026
25	-1,256	0,192	-2,165**	0,029
30	-1,285	0,183	-2,020*	0,041

* week significant rejection of non-stationarity
** significant rejection of non-stationarity
*** strong significant rejection of non-stationarity

Table 2.4: Dickey-Fuller test with 10 lags augmentation for default rates of Unicredit, Ubi, Generali, MedioBanca (whole sample)

	Unicredit		Ubi		Generali		MedioBanca	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-1,236	0,257	-1,044	0,269	-1,364	0,160	-1,10	0,246
2	-1,076	0,298	-0,905	0,320	-1,216	0,206	-1,08	0,255
3	-0,965	0,341	-0,776	0,367	-0,998	0,286	-0,90	0,319
4	-0,847	0,363	-0,675	0,405	-0,921	0,314	-0,83	0,345
5	-0,787	0,383	-0,616	0,426	-0,791	0,362	-0,70	0,393
6	-0,734	0,399	-0,587	0,436	-0,697	0,396	-0,65	0,411
7	-0,688	0,408	-0,530	0,457	-0,643	0,416	-0,60	0,430
8	-0,663	0,415	-0,537	0,454	-0,618	0,425	-0,57	0,442
9	-0,646	0,420	-0,561	0,446	-0,597	0,433	-0,55	0,448
10	-0,632	0,443	-0,633	0,419	-0,581	0,439	-0,51	0,461

* week significant rejection of non-stationarity
** significant rejection of non-stationarity
***strong significant rejection of non-stationarity

Table 2.5: Dickey-Fuller test with 10 lags augmentation for default rates of Unicredit, Ubi, Generali, MedioBanca (first subsample)

	Unicredit		Ubi		Generali		MedioBanca	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-0,82	0,348	0,037	0,665	-1,22	0,201	-0,86	0,333
2	-0,53	0,454	0,354	0,781	-0,79	0,361	-0,75	0,376
3	-0,37	0,515	0,678	0,861	-0,73	0,382	-0,39	0,505
4	-0,16	0,592	0,943	0,908	-0,37	0,515	-0,24	0,562
5	-0,05	0,632	1,085	0,927	-0,30	0,538	0,088	0,684
6	0,058	0,673	1,152	0,936	-0,23	0,565	0,166	0,712
7	0,169	0,713	1,294	0,950	-0,15	0,593	0,239	0,739
8	0,215	0,730	1,287	0,950	-0,10	0,612	0,288	0,757
9	0,260	0,747	1,277	0,949	-0,05	0,631	0,310	0,765
10	0,306	0,764	1,273	0,948	0,000	0,651	0,363	0,784

* week significant rejection of non-stationarity

** significant rejection of non-stationarity

***strong significant rejection of non-stationarity

Table 2.6: Dickey-Fuller test with 10 lags augmentation for default rates of Unicredit, Ubi, Generali, MedioBanca (second subsample)

	Unicredit		Ubi		Generali		MedioBanca	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-1,56	0,111	-1,72*	0,079	-1,28	0,183	-1,00	0,285
2	-1,39	0,151	-1,59	0,105	-1,19	0,214	-0,97	0,296
3	-1,24	0,195	-1,44	0,139	-1,10	0,245	-0,90	0,321
4	-1,13	0,235	-1,32	0,172	-1,03	0,271	-0,82	0,349
5	-1,07	0,256	-1,25	0,192	-0,98	0,291	-0,78	0,363
6	-1,02	0,277	-1,22	0,202	-0,94	0,306	-0,74	0,378
7	-0,96	0,297	-1,15	0,228	-0,90	0,320	-0,70	0,393
8	-0,94	0,304	-1,13	0,237	-0,88	0,326	-0,68	0,402
9	-0,93	0,309	-1,11	0,242	-0,87	0,329	-0,66	0,407
10	-0,92	0,314	-1,08	0,253	-0,86	0,335	-0,64	0,416

* week significant rejection of non-stationarity

** significant rejection of non-stationarity

***strong significant rejection of non-stationarity

Table 2.7: Dickey-Fuller with test 10 lags augmentation for default rates of Eni, Telecom, Enel (whole sample)

	Eni		Telecom		Enel	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-2,152**	0,030	-2,44**	0,0142	-1,954**	0,048
2	-1,621*	0,099	-2,030**	0,0407	-1,771*	0,073
3	-1,240	0,197	-1,665*	0,091	-1,607	0,102
4	-0,991	0,288	-1,406	0,149	-1,486	0,128
5	-0,748	0,377	-1,161	0,226	-1,359	0,162
6	-0,630	0,420	-1,039	0,271	-1,301	0,178
7	-0,522	0,460	-0,914	0,317	-1,239	0,198
8	-0,423	0,485	-0,864	0,335	-1,21	0,208
9	-0,363	0,496	-0,842	0,343	-1,197	0,213
10	-0,363	0,518	-0,789	0,362	-1,166	0,225

* week significant rejection of non-stationarity
** significant rejection of non-stationarity
*** strong significant rejection of non-stationarity

Table 2.8: Dickey-Fuller with test 10 lags augmentation for default rates of Eni, Telecom, Enel (first subsample)

	Eni		Telecom		Enel	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-1,95**	0,048	-2,08**	0,035	-1,36	0,159
2	-1,48	0,127	-1,81*	0,066	-1,33	0,169
3	-1,04	0,268	-1,49	0,125	-1,27	0,186
4	-0,65	0,412	-1,30	0,177	-1,22	0,202
5	-0,22	0,567	-1,09	0,251	-1,15	0,227
6	-0,00	0,648	-0,99	0,288	-1,12	0,238
7	0,216	0,731	-0,88	0,328	-1,09	0,252
8	0,363	0,784	-0,82	0,347	-1,06	0,261
9	0,428	0,805	-0,80	0,355	-1,05	0,266
10	0,573	0,839	-0,74	0,378	-1,01	0,278

* week significant rejection of non-stationarity
** significant rejection of non-stationarity
*** strong significant rejection of non-stationarity

Table 2.9: Dickey-Fuller with test 10 lags augmentation for default rates of Eni, Telecom, Enel (second subsample)

	Eni		Telecom		Enel	
Maturity	t-stat	p-value	t-stat	p-value	t-stat	p-value
1	-1,70*	0,083	-1,74*	0,077	-1,62*	0,098
2	-1,41	0,145	-1,52	0,118	-1,43	0,142
3	-1,24	0,194	-1,34	0,166	-1,31	0,174
4	-1,10	0,245	-1,15	0,230	-1,18	0,217
5	-0,97	0,295	-0,97	0,294	-1,05	0,263
6	-0,88	0,326	-0,86	0,335	-0,99	0,288
7	-0,81	0,354	-0,75	0,375	-0,92	0,313
8	-0,77	0,368	-0,70	0,393	-0,88	0,327
9	-0,75	0,375	-0,68	0,401	-0,86	0,334
10	-0,71	0,388	-0,63	0,419	-0,82	0,348

* week significant rejection of non-stationarity
** significant rejection of non-stationarity
*** strong significant rejection of non-stationarity

Table 2.10: PCA interest rates and BEI rates (whole sample)

Factor	Interest rate	BEI rates
1	0,747	0,523
2	0,114	0,137
3	0,048	0,058
4	0,044	0,043
Tot. Var.	0,95	0,762

Table 2.11: PCA interest rates and BEI rates (first subsample)

Factor	Interest rate	BEI rates
1	0,667	0,520
2	0,157	0,223
3	0,070	0,109
4	0,059	0,039
Tot. Var.	0,956	0,763

Table 2.12: PCA interest rates and BEI rates (second subsample)

Factor	Interest rate	BEI rates
1	0,837	0,538
2	0,077	0,106
3	0,032	0,049
4	0,013	0,043
Tot. Var.	0,961	0,739

Table 2.13: PCA default rates financial firms (whole sample)

Factor	Unicredit	Generali	Ubi	MedioBanca
1	0,967	0,918	0,942	0,850
2	0,025	0,04	0,045	0,087
3	0,004	0,033	0,011	0,057
4	0,001	0,002	0,001	0,002
Tot. Var.	0,998	0,998	0,999	0,997

Table 2.14: PCA default rates financial firms (first subsample)

Factor	Unicredit	Generali	Ubi	MedioBanca
1	0,959	0,864	0,792	0,882
2	0,027	0,104	0,193	0,060
3	0,009	0,023	0,013	0,043
4	0,002	0,004	0,000	0,006
Tot. Var.	0,998	0,998	0,999	0,993

Table 2.15: PCA default rates financial firms (second subsample)

Factor	Unicredit	Generali	Ubi	MedioBanca
1	0,969	0,924	0,980	0,846
2	0,025	0,036	0,017	0,090
3	0,003	0,027	0,001	0,058
4	0,001	0,009	0,000	0,002
Tot. Var.	0,998	0,999	0,998	0,998

Table 2.16: PCA default rates industrial firms (whole sample)

Factor	ENI	Telecom	Enel
1	0,963	0,966	0,973
2	0,028	0,026	0,020
3	0,004	0,004	0,004
4	0,002	0,001	0,001
Tot. Var	0,998	0,999	0,999

Table 2.17: PCA default rates industrial firms (first subsample)

Factor	ENI	Telecom	Enel
1	0,925	0,970	0,977
2	0,067	0,022	0,016
3	0,004	0,005	0,004
4	0,001	0,001	0,001
Tot. Var.	0,998	0,998	0,998

Table 2.18: PCA default rates industrial firms (second subsample)

Factor	ENI	Telecom	Enel
1	0,971	0,965	0,971
2	0,020	0,028	0,020
3	0,003	0,004	0,003
4	0,002	0,001	0,002
Tot. Var.	0,999	0,999	0,999

Chapter 3

Modeling interest rates in state space form

3.1 Introduction

In recent times, interest rates started to be negative, especially for shorter maturities. There is a wide array of macroeconomic explanations of this phenomenon. The bankruptcy of Lehman Brothers in 2008 and the sovereign debt crisis in 2011 led the ECB to change the monetary policy to increase the stock of liquidity in Financial Markets. In particular, ECB started to buy Treasury Bonds for this purpose, performing a Quantitative Easing, following what it was previously done by the FED in the aftermath of the Lehman Brothers bankruptcy. This led to a drop in interest rates, which became negative at the end of 2015. To this extent, market practitioners trading interest rate instruments in the markets, started to change the way in which they forecast interest rates. This macroeconomic scenario let practitioners to model the term structure of interest rates according to a multifactor Vasicek model whose dynamics is Gaussian, ruling out the CIR (Cox, Ingersoll and Ross 1985) model, where rates follow a non-central chi-square dynamics.

Although the CIR model allows to model positive rates, it has mostly been used by academics until interest rates became negative. On the other hand, market practitioners have always used the Gaussian dynamics to model interest rates, as it is more analytical tractable than the non-central chi-square dynamics.

The introduction of more factors yields a more flexible model which could be more suitable for explaining negative rates. The univariate model proposed by Vasicek (1977) it is not enough for explaining some phenomena which comes from the market, such as the non perfect correlation of interest rates among different maturities as described in Brigo and Mercurio (2007).

Starting from the late 70s, multifactor models of interest rates has become very popular in the academic framework. This choice has been pursued to overcome the shortcomings of modeling the term structure of interest rates according to one factor approach.

As stressed separately by Brennan and Schwarz (1979), Schwarz and Schaerfer (1984, 1987) and Longstaff and Schwarz (1992), interest rates can be modeled with a two factors process. In particular, in the model developed by Brennan and Schwarz (1979), interest rates are modeled according to two factors CIR processes, one for the short rate dynamics and the other for the long term dynamics. In the Longstaff and Schwarz (1992) model, the first factor describes the short rate dynamics and the second factor the volatility of interest rate dynamics.

From the beginning of 90s, it became clear that a multifactor model was needed to explain most of the variability of the term structure of interest rates over time.

The affine relation between rates and factors shown by Duffie and Kan (1996) in the realm of affine models for the term structure, made state space models very popular. Geyer and Pichler (1999) and Chen and Scott (2003) tested multifactor interest rates models in state space form using a CIR dynamics with QML. Following the work of Lagentieg (1980), Babbs and Nowmann (1999) tested Gaussian multifactor models in state space form.

We build and estimate interest rate models in state space form on (spot istantaneous) interest rates bootstrapped from OIS rates. In particular, what is stressed here is the advantage arising from shifting from a one factor Gaussian dynamics to a multifactor Gaussian dynamics in a time span where two crises occurs.

Following the standard literature on affine term structure models (see Duffee 2002), we consider *essentially affine* models to have a better fitting of term structure models. The analysis considers univariate, bivariate and three variate affine term structure models. We estimate these models both in the whole sample and in the two subsamples.

This subsample analysis makes economic sense as it allows to identify how the dynamics of interest rates has been modified within the new macroeconomic conditions brought by the failure of Lehman in 2008 and the bursting of sovereign debt crisis, respectively.

3.2 The model

We consider the class of Gaussian affine models named $A_0(n)$ by Dai and Singleton (2000). Here n is the number of factors and 0 means that the diffusion coefficient of each factor is constant i.e. each factor is Gaussian. Another terminology often used in literature for these models is Gn. The choice of a Gaussian dynamics of the factors of the interest rate model allow us to model some phenomena such as negative rates, which are difficult to model if non-Gaussian dynamics are chosen.

We choose the model parameterization following Joslin et al. (2011).

In our model the short rate $r(t)$ is given by the sum of all the elements in the vector $X(t) = (x_1(t), x_2(t), x_3(t), \ldots, , x_n(t))'$ whose dynamics under the $\mathbb{Q}$ measure is the following:

$$dX(t) = K^{\mathbb{Q}}(\Gamma^{\mathbb{Q}} - X(t))dt + \Sigma dW^{\mathbb{Q}}(t), \tag{3.1}$$

The mean reversion speed matrix $K^{\mathbb{Q}}$ is assumed to be of this form:

$$K^{\mathbb{Q}} = \begin{bmatrix} \kappa_1 & 0 & \ldots & 0 \\ 0 & \kappa_2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & \kappa_n \end{bmatrix}$$

and the non zero element assumed to be positive.

The long term mean vector $\Gamma^{\mathbb{Q}}$ is assumed to be:

$$\Gamma^{\mathbb{Q}} = \begin{bmatrix} \gamma_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The diffusion matrix Σ is assumed to be:

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 & \ldots & 0 \\ \sigma_{21} & \sigma_{22} & \ldots & 0 \\ \sigma_{31} & \sigma_{32} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nn} \end{bmatrix}$$

We also assume $\sigma_{ii} > 0 \ \forall i = 1, \ldots, N$, where no assumption is made on the off diagonal elements. This assumption guarantees matrix Σ to be positive defined, as trivial consequence of *Silverster's* criterion.

Finally,

$$W^{\mathbb{Q}}(t) = \begin{bmatrix} W_1^{\mathbb{Q}}(t) \\ W_2^{\mathbb{Q}}(t) \\ \vdots \\ W_n^{\mathbb{Q}}(t) \end{bmatrix}$$

is a vector of independent Wiener processes.

An alternative but equivalent formulation of the model under the $\mathbb{Q}$ measure can be written by performing a Wiener process rotation and pass independent Wieners to dependent Wieners, but with simpler diffusion matrix:

$$dx_i(t) = \kappa_i(\gamma_i - x_i(t))dt + \sigma_i dZ_i^{\mathbb{Q}}(t) \tag{3.2}$$

$\forall i = 1, \dots, N$ where

$\sigma_i = \sqrt{(\sum_{j=1}^{i} \sigma_{ij}^2)}$

and $Z_i^{\mathbb{Q}}(t)$ is a Wiener process, correlated with $Z_j^{\mathbb{Q}}(t)$ by $corr(dZ_i^{\mathbb{Q}}, dZ_j^{\mathbb{Q}}(t)) = \rho_{ij} \in [-1, 1]$, for all $i \neq j$ where:

$$\begin{cases} \rho_{ij} = \frac{\sum_{l=1}^{i} \sigma_{jl}\sigma_{il}}{\sigma_i \sigma_j} & i \leq j \\ \rho_{ij} = \rho_{ji} & i > j \end{cases}$$

Let ρ be the correlation matrix of this form:

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & 1 & \dots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \dots & 1 \end{bmatrix}$$

Under this model, the short rate follows a multidimensional correlated Gaussian process, with mean reversion in each factor; a process also called *Elastic random walk*

Furthermore, we assume for each factor an affine market price of risk of the form:

$$\Lambda_i(t, x_i(t)) = \lambda_{0i} + \lambda_{1i} x_i(t) \tag{3.3}$$

Therefore, the natural drift is of the form:

$$\begin{aligned} \bar{\kappa_i}(\bar{\gamma_i} - x_i(t)) &= \kappa_i(\gamma_i - x_i(t)) + \Lambda_i(t, x_i(t))\sigma_i \\ &= \kappa_i(\gamma_i - x_i(t)) + (\lambda_{0i} + \lambda_{1i} x_i(t))\sigma_i \\ &= \underbrace{(-\kappa_i + \lambda_{1i}\sigma_i)}_{-\bar{\kappa_i}} x_i(t) + \underbrace{\kappa_i\gamma_i + \lambda_{0i}\sigma_i}_{\bar{\kappa_i}\bar{\gamma_i}} \end{aligned}$$

The dynamics of the short rate according to the natural probability measure is the following:

$$dX(t) = K^{\mathbb{P}}(\Gamma^{\mathbb{P}} - X(t))dt + \Sigma dW^{\mathbb{P}}(t) \tag{3.4}$$

$$K^{\mathbb{P}} = \begin{bmatrix} \bar{\kappa_1} & 0 & \dots & 0 \\ 0 & \bar{\kappa_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \bar{\kappa_n} \end{bmatrix}, \quad \Gamma^{\mathbb{P}} = \begin{bmatrix} \bar{\gamma_1} \\ \bar{\gamma_2} \\ \vdots \\ \bar{\gamma_n} \end{bmatrix}, \quad W^{\mathbb{P}}(t) = \begin{bmatrix} W_1^{\mathbb{P}}(t) \\ W_2^{\mathbb{P}}(t) \\ \vdots \\ W_n^{\mathbb{P}}(t) \end{bmatrix},$$

is a vector of independent Wiener processes.

Given a time t and a time step Δt, the $\mathcal{F}_t$-conditional mean and variance of the short rates under the $\mathbb{P}$ dynamics are:

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{n} x_{i(t+\Delta t)}|\mathcal{F}_t\right] = \sum_{i=1}^{n} \bar{\gamma}_i(1-\exp\left(-\bar{\kappa}_i\Delta t\right)) + \exp(-\bar{\kappa}_i\Delta t)x_{i(t)} \tag{3.5}$$

$$\begin{aligned} Var^{\mathbb{P}}\left[\sum_{i=1}^{n} x_{i(t+\Delta t)}|\mathcal{F}_t\right] &= \sum_{i=1}^{n} \frac{(\sigma_i)^2}{2\bar{\kappa}_i}(1-\exp(-2\bar{\kappa}_i\Delta t)) \\ &+2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\sigma_i\sigma_j\rho_{ij}}{\bar{\kappa}_i+\bar{\kappa}_j}(1-\exp(-(\bar{\kappa}_i+\bar{\kappa}_j)\Delta t)) \end{aligned} \tag{3.6}$$

A standard no-arbitrage argument shows that the price $F = F(t, x_1, x_2, x_3, \ldots, x_n)$ of ZCB with maturity T satisfies the following boundary value problem:

$$\begin{cases} \dfrac{\partial F}{\partial t}\displaystyle\sum_{i=1}^{n}\kappa_i(\gamma_i - x_i(t))\dfrac{\partial F}{\partial x_i} + \dfrac{1}{2}\sigma_i^2\dfrac{\partial^2 F}{\partial x_i^2} + \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\dfrac{\partial^2 F}{\partial x_i\partial x_j} = F\left(\displaystyle\sum_{i=1}^{n} x_i\right) \\ F(T, x_1, x_2, x_3, \ldots, x_n) = 1 \end{cases}$$

The solution of the boundary value problem is an affine function of the factors as in Duffie and Kan (1996).

$$F(t, x_1, x_2, x_3, \ldots, x_n) = \exp\left(A(\tau) - \sum_{i=1}^{n} B_i(\tau)x_i\right)$$

where $\tau = T - t$ and

$$B(\tau) = \begin{bmatrix} B_1(\tau) \\ B_2(\tau) \\ \vdots \\ B_n(\tau) \end{bmatrix} = \begin{bmatrix} \frac{1-\exp(-\kappa_1\tau)}{\kappa_1} \\ \frac{1-\exp(-\kappa_2\tau)}{\kappa_2} \\ \vdots \\ \frac{1-\exp(-\kappa_n\tau)}{\kappa_n} \end{bmatrix}$$

$$\begin{aligned} A = \sum_{i=1}^{n}(\gamma_i - \frac{\sigma_i^2}{2\kappa_i})(B_i(\tau)-\tau) + \sum_{i,j:i\neq j}\frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_i\kappa_j}(\tau - B_i(\tau) - B_j(\tau) \\ + \frac{1}{\kappa_i+\kappa_j}(1-\exp(-(\kappa_i+\kappa_j)\tau)) \end{aligned}$$

The yield at time t of the ZCB maturing in T is:

$$y(t, T) = -\frac{A(\tau)}{\tau} + \sum_{i=1}^{n}\frac{B(\tau)}{\tau}x_i(t) \tag{3.7}$$

3.3 Estimation procedure through state space models

Being the ZCB yields affine with respect to vector $X(t)$ shown by (3.7), it is not complex to compute the interest rate model parameters through a *state space* approach, further simplified by the Gaussian property of factors. Similar analysis has been also performed by De Jong (2000) who stressed the importance of the state space models for estimation of term structure models within the affine framework. Our choice is the *Kalman Filter* (Kalman, 1960).

3.3.1 Term structure models in state space form

Considering a set of calendar times $t_1, t_2, \ldots, t_m$ and with constant time step $\Delta t = t_{k+1} - t_k$ for every $k = 1, 2, \ldots, m-1$ and $(n \times 1)$ vector of latent variables $X(t)$ and $\forall k$ a vector $\zeta(t_k) = (\zeta_1(t_k), \zeta_2(t_k), \ldots, \zeta_p(t_k))'$ zero coupon yields at fixed maturities $\tau_1, \tau_2, \ldots, \tau_p$, our model can be set in a State Space form.

Considering (3.7) as a reference for the ZCB yields maturing at fixed maturities $\tau_1, \tau_2, \ldots, \tau_p$, the *measurement equation* is:

(measurement equation) $\qquad \underbrace{z(t_k)}_{p\times 1} = \underbrace{A}_{p\times 1} + \underbrace{B}_{p\times n} \underbrace{X(t_k)}_{n\times 1} + \underbrace{\eta(t_k)}_{p\times 1} \qquad \eta(t_k) \sim \mathcal{IID}(0,\ R)$

$$\underbrace{\begin{bmatrix} z_1(t_k) \\ z_2(t_k) \\ \vdots \\ z_p(t_k) \end{bmatrix}}_{z(t_k)} = \underbrace{\begin{bmatrix} \frac{-a_1}{\tau_1} \\ \frac{-a_2}{\tau 2} \\ \vdots \\ \frac{-a_p}{\tau p} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} \frac{b_1(\tau_1)}{\tau_1} & \frac{b_2(\tau_1)}{\tau_1} & \cdots & \frac{b_n(\tau_1)}{\tau_1} \\ \frac{b_1(\tau_2)}{\tau_2} & \frac{b_2(\tau_2)}{\tau_2} & \cdots & \frac{b_n(\tau_2)}{\tau_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_1(\tau_p)}{\tau_p} & \frac{b_2(\tau_p)}{\tau_p} & \cdots & \frac{b_n(\tau_p)}{\tau_p} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ \vdots \\ x_n(t_k) \end{bmatrix}}_{X(t_k)} + \underbrace{\begin{bmatrix} \eta_1(t_k) \\ \eta_2(t_k) \\ \vdots \\ \eta_p(t_k) \end{bmatrix}}_{\eta(t_k)},$$

where,

$\eta(t_k) = \zeta(t_k) - z(t_k)$

is the vector of measurement errors observed $\forall$ fixed maturity $\tau_1, \tau_2, \ldots, \tau_p$ at time t_k.

The vector of measurement errors $\eta(t_k)$ is also called the vector of *innovations*, i.e. information which cannot derived from the data at time t_{k-1}.

Furthermore we have that:

$$\hat{z}(t_k) = A + B\hat{X}(t_k)$$

$$cov[z(t_k) - \hat{z}(t_k)|\mathcal{F}t_k] = S(t_k) = \underbrace{BVar[X(t_k)]B'}_{Variance\ explained\ by\ latent} + \underbrace{R}_{Variance\ explained\ by\ errors}$$

where,

$$R = \begin{bmatrix} R_1 & 0 & \dots & 0 \\ 0 & R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & R_p \end{bmatrix},$$

is the covariance matrix of the measurement errors.

Considering (3.5) as the mean of the latent variables and (3.6) as the variance, the *transition equation* which describes the evolution of the vector of unknown latent variables $X(t_k)$ is:

(Trasition equation) $$\underbrace{X(t_k)}_{n\times 1} = \underbrace{C}_{n\times 1} + \underbrace{F}_{n\times n}\underbrace{X(t_{k-1})}_{n\times 1} + \underbrace{\varepsilon(t_k)}_{n\times 1} \quad \varepsilon(t_k) \sim \mathcal{IID}(0,\, Q)$$

where:

$$X(t_k) = \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ \vdots \\ x_n(t_k) \end{bmatrix}, \qquad C = \begin{bmatrix} \bar{\gamma_1}(1 - \exp{(-\bar{\kappa_1}\Delta t)}) \\ \bar{\gamma_2}(1 - \exp{(-\bar{\kappa_2}\Delta t)}), \\ \vdots \\ \bar{\gamma_n}(1 - \exp{(-\bar{\kappa_n}\Delta t)}) \end{bmatrix},$$

$$F = \begin{bmatrix} \exp{(-\bar{\kappa_1}\Delta t)} & 0 & \dots & 0 \\ 0 & \exp{(-\bar{\kappa_2}\Delta t)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp{(-\bar{\kappa_n}\Delta t)} \end{bmatrix},$$

$$\varepsilon(t_k) = \begin{bmatrix} \varepsilon_1(t_k) \\ \varepsilon_2(t_k) \\ \vdots \\ \varepsilon_n(t_k) \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_1 & \varrho_{12} & \dots & \varrho_{1n} \\ \varrho_{12} & \sigma_2 & \dots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{1n} & \varrho_{2n} & \dots & \sigma_n \end{bmatrix},$$

$$\begin{cases} \sigma_i = \dfrac{(\sigma_i)^2}{2\bar{\kappa_i}}(1 - \exp{(-2\bar{\kappa_i}\Delta t)}) \\ \varrho_{ij} = \dfrac{\sigma_i\sigma_j\rho_{ij}}{\bar{\kappa_i} + \bar{\kappa_j}}(1 - \exp{(-\bar{\kappa_i} + \bar{\kappa_j}\Delta t)}) \end{cases} \tag{3.8}$$

Furthermore, we assume that $\forall k = 1, 2, \dots, m-1$

$$\mathbb{E}\begin{bmatrix} \varepsilon(t_k) \\ \eta(t_k) \end{bmatrix}\begin{bmatrix} \varepsilon(t_k) \\ \eta(t_k) \end{bmatrix}' = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix},$$

The variance of the latent variables is:

$$\mathbb{E}[(X(t_k) - \hat{X}(t_k))|\mathcal{F}t_k]^2 = P(t_k) = FVar[X(t_k)|\mathcal{F}t_{k-1}]F' + Q \tag{3.9}$$

In the Kalman Filter algorithm estimates of the latent variables are calculated recursively over time through incoming measurement $z(t_k)$.

In this recursive method, the true value of the latent variables are assumed to be *hidden Markov processes* and the measurements are the observed states of such variables.

Indeed we have that:

$$\Pr(X(t_k)|X(t_{k-1}), X(t_{k-2}), \ldots, X(t_{k-n})) = \Pr(X(t_k)|X(t_{k-1})) \tag{3.10}$$

The true value of the latent variables is only conditionally dependent from the value of the previous state. Similarly, the measurements of the time (t_k) are conditionally dependent from the current value of the latent variables.

$$\Pr(z(t_k)|X(t_k), X(t_{k-1}), \ldots, X(t_{k-n}) = \Pr(z(t_k)|X(t_k)) \tag{3.11}$$

To predict the value of the latent variables at time t_k, we use the information from the measurements of the previous times.

Given $Z(t_k) = (z(t_1), z(t_2), \ldots, z(t_k))$ the measurements set of the latent variables up to time t_k, by applying the Chapman- Kolgomorov equation and combine it with the Markovian property of the probability distribution, the predicted step is:

$$\Pr(X(t_k)|Z(t_{k-1})) \propto \int \Pr(X(t_k)|X(t_{k-1})) \Pr(x(t_{k-1})|Z(t_{k-1}))dX(t_{k-1}) \tag{3.12}$$

When new information at time t_k is incoming, the updated step can be computed by applying the Bayes theorem:

$$\underbrace{\Pr(X(t_k)|Z(t_k))}_{\textit{Posterior density}} \propto \frac{\underbrace{\Pr(z(t_k)|X(t_k))}_{\textit{Likelihood}}\underbrace{\Pr(X(t_k)|Z(t_{k-1}))}_{\textit{Prior density}}}{\Pr(z(t_k)|Z(t_{k-1}))} \tag{3.13}$$

where the probability of the measurement set at time t_k is:

$$\Pr(z(t_k)|Z(t_{k-1})) \propto \int \Pr(z(t_k)|X(t_k)) \Pr(x(t_k)|Z(t_{k-1}))dX(t_k) \tag{3.14}$$

The equations (3.12) and (3.14) are the marginalization of the prediction of the future value of the latent variables and the observed measurements respectively.

3.3.2 Kalman Filter implementation

Assuming that the initial state of the vector of the latent variables is $\sim \mathcal{N}(X(t_0), P(t_0))$ where $X(t_0)$ and $P(t_0)$ known in advance and independent from $\eta(t_k)$ and $\varepsilon(t_k)$.
$\forall k$ let:

- $X^-(t_k)$ be the prior forecast of $X(t_k)$ given the information set at time t_{k-1},
- $X^+(t_k)$ be the posterior estimate of $X(t_k)$, given the information set at time t_k,
- $P^-(t_k)$ be the prior estimate of the error covariance matrix.
- $P^+(t_k)$ be the posterior estimated error covariance matrix.

Starting from t_0, the initialization about the prior probability distribution of the latent variables is:

$$X^+(t_0) = \begin{bmatrix} \bar{\gamma}_1 \\ \bar{\gamma}_2 \\ \vdots \\ \bar{\gamma}_n \end{bmatrix}, \qquad P^+(t_0) = \begin{bmatrix} \sigma_1 & \varrho_{12} & \dots & \varrho_{1n} \\ \varrho_{12} & \sigma_2 & \dots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{1n} & \varrho_{2n} & \dots & \sigma_n \end{bmatrix},$$

$$\begin{cases} \sigma_i = \dfrac{(\sigma_i)^2}{2\bar{\kappa}_i} \\ \varrho_{ij} = \dfrac{\sigma_i \sigma_j \rho_{ij}}{\bar{\kappa}_i + \bar{\kappa}_j} \end{cases} \tag{3.15}$$

A priori prediction and covariance:

$$X^-(t_k) = C + FX^+(t_{k-1}) \tag{3.16}$$
$$P^-(t_k) = FP^+(t_{k-1})F' + Q \tag{3.17}$$

Measurement forecast, prediction error and covariance:

$$z(t_k) = A + BX^-(t_k) \tag{3.18}$$
$$\eta(t_k) = \zeta(t_k) - z(t_k) \tag{3.19}$$
$$S(t_k) = BP^-(t_k)B' + R \tag{3.20}$$

Posterior prediction and covariance (i.e.: updating):

$$X^+(t_k) = X^-(t_k) + K(t_k)\eta(t_k) \tag{3.21}$$

Since,

$$
\begin{aligned}
P^+(t_k) &= cov[(X(t_k) - X^+(t_k))(X(t_k) - X^+(t_k))] \\
&= cov[(X(t_k) - (X^-(t_k) + K(t_k)\eta(t_k)))] \\
&= cov[(X(t_k) - (X^-(t_k) + K(t_k)(A + BX^-(t_k) + \eta(t_k) - (A + BX^-(t_k))] \\
&= cov[(I - K(t_k)B(X(t_k) - x^-(t_k))] + cov[(K(t_k)\eta(t_k))] \\
&= (I - K(t_k)B)cov[(X(t_k) - X^-(t_k))](I - K(t_k)B)' + K(t_k)cov[(\eta(t_k))K(t_k)'] \\
&= (I - K(t_k)B)P^-(t_{k-1})(I - K(t_k)B)' + K(t_k)RK'(t_k) \qquad (3.22)
\end{aligned}
$$

The trace of $P^+(t_k)$, is the sum of the *Mean Squared Error*.
By differentiating the trace of $P^+(t_k)$ with respect to K we get that:

$$
K(t_k) = P^-(t_k)B'S^{-1}(t_k) \qquad (3.23)
$$

i.e. the *Kalman Gain*. The Kalman Gain gives the $X^+(t_k)$ that minimizes the *Mean Squared Error* (MSE). The Kalman Gain is positively related to the prior covariance of the latent variables $P^-(t_k)$, clearly:
$\lim_{P^-(t_k)\to 0} \frac{P^-(t_k)B'}{BP^-(t_k)B'+R} = 0$

When $P^-(t_k) \approx 0$, the Kalman Gain ≈ 0. Indeed there is no updating step. This means that $X(t_k) - X^+(t_k) \approx 0$ and the Mean Squared Error is ≈ 0.
The $X^+(t_k)$ that minimizes *MSE* is also the value of the latent variables which maximize the Likelihood with respect to the observed ZCB yields:
The *Likelihood function* to be maximized is:

$$
\Pr(\zeta(t_k)|X(t_1), X(t_2), \ldots, X(t_{m-1})) = \prod_{k=1}^{m} (2\pi)^{-\frac{m}{2}} \sum_{k=1}^{m} \det S^{-\frac{1}{2}} \exp(-\frac{1}{2} \sum_{k=1}^{m} \eta(t_k) S^{-1} \eta(t_k)') \qquad (3.24)
$$

The log-likelihood is then:

$$
\log \mathcal{L} = \sum_{k=1}^{m} \left[-\frac{p}{2}\log(2\pi) - \frac{1}{2}\log\det S(t_k) - \frac{1}{2}\eta S^{-1}(t_k)\eta'(t_k) \right] \qquad (3.25)
$$

$$
= -\frac{mp}{2}\log(2\pi) - \frac{1}{2}\sum_{k=1}^{m}\log\det S(t_k) - \frac{1}{2}\sum_{k=1}^{m}\eta(t_k)S^{-1}(t_k)\eta'(t_k). \qquad (3.26)
$$

Clearly, $\max[\log \mathcal{L}] = -\min[\log \mathcal{L}]$.

3.3.3 Optimization procedure

To minimize the log likelihood function we use the Matlab function *fmincon*. Given $\theta = (\kappa_i, \gamma_i, \lambda_{i0}, \lambda_{i1}\sigma_{ii}, \sigma_{ij}, R)$ the vector of model parameters, $\forall$ factors $x_i \in X$ with $i = 1, \dots, n$ and $\forall$ maturities $\tau = 1, \dots, p$, the optimization problem is of this form:

$$-\min_{\theta} \log \mathcal{L} \tag{3.27}$$

$$\text{u.c.} \begin{cases} 0 \leq \kappa_i & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \gamma & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \lambda_{i0} & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \lambda_{i1} & \leq +\infty & \forall i = 1, \dots, n \\ \kappa_i - \lambda_{1i}\sigma_i & > 0 & \forall i = 1, \dots, n \\ 0 \leq \sigma_{ii} & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \sigma_{ij} & \leq +\infty & \forall i = 1, \dots, n, \forall i = j, \dots, n \\ 0 \leq r_h & \leq +\infty & \forall h = 1, \dots, p \end{cases} \tag{3.28}$$

We allow the price of risk parameters λ_{i0}, λ_{i1} to be free from constraints to have the maximum degree of flexibility. Furthermore, as price of risk parameters effect the risk natural mean reversion speed $\bar{\kappa_i}$, we impose a further non linear constraint to avoid the non negativity of the mean reversion speed under $\mathbb{P}$.

A negative mean reversion speed makes the model to explode as $\exp(-\bar{\kappa_i}\Delta t)$ is always greater than one.

We also allow the risk neutral long term mean γ_i to be negative. Interest rates are negative after the bursting of the sovereign debt crisis. It is reasonable for the economic point of view to allow the long the mean of interest rates to be negative.

3.4 Comments

3.4.1 Whole sample

Table 3.1 shows that all estimated parameters of the three models (G1, G2 and G3) are statistically significant. Price of risk parameters λ_{i1} are negative in most of the cases except for the second factor of the G2. The effect of the price of risk on the mean reversion speed is higher for the G1 model than in G2 and G3. This fact may suggest that the expectation about the future dynamics of interest rates in $\mathbb{P}$ only affects the long run dynamics of the rates. This evidence is stronger for the G1.

The long run mean under $\mathbb{P}$ measure is strongly negative only for the G2 models shown in table 3.2

The significance of the volatility parameters shows us that:

- The first and the third factor of the G3 model are strongly negatively correlated as shown in table 3.2. The second factor is positively correlated with the first factor and negatively correlated with the third factor. Like in the second sub sample the first and the third factors are negatively correlated;

- The G2 shows a strong and negative correlation between factors.

Therefore, the first and the third factor are similar but distinct factors. They work in the opposite direction. The same holds for the G2.

The G1 show a significant and close to 0 risk neutral mean reversion parameter κ and a significant and negative price of risk parameters λ_{10} and λ_{11}. The expectation of future interest rates in $\mathbb{P}$ is more affected by the long term mean $\bar{\gamma}$.

3.4.2 First subsample

The price of risk parameters λ_{i0} and λ_{i1} are negative and not statistically significant as shown in table 3.3. The natural distribution of expected future interest rates is not statistically different from the risk neutral one. This effect is stronger for G3 and G2.

The G3 model shows a better AIC than the G2 and the G1 as shown by table 3.7. On the other hand, the G3 shows an overall lower significance of models parameters than G1 and G2. Only the risk neutral mean reversion speed κ_1 κ_3 and the volatility parameters of the first and the second factor are significant. This may depend on the short time span considered that does not allow to catch the mean reversion phenomena and the role of risk premium in the expectations under $\mathbb{P}$.

The third factor affects bond yields only through the mean reversion speed. On the contrary, the G2 shows significance in all model parameters except for the price of risk parameters. This confirms that, also in the bivariate model, the role of risk premia on the expectation on future interest rates under $\mathbb{P}$ is not catched. Furthermore, by looking at volatility parameters, we are able to see that G2 and G3 have that same ρ_{12} as shown in table 3.4. This is not striking as most of the parameters of the G3 are not statistically significant.

The G1 model in the first sub sample shows significance in all parameters. On the contrary of G3 and G2, the univariate model show significance on price of risk parameters. Indeed, risk premia play a role in the expectations of future rates under $\mathbb{P}$.

3.4.3 Second subsample

Table 3.5 shows a higher degree of significance of estimated models parameters. Price of risk parameters λ_{i1} are significant in all model expect for λ_{21}. Results show in particular a negative value of λ_{i1} for all model, except for the first factor of the G2. Indeed, in the risk natural World $\mathbb{P}$ factors reverts to the mean with a higher speed than in risk neutral World. Expectation about future interest rates are more driven by the long run mean of rates.

Furthermore, the significance of all volatility parameters allows to compare the correlation among factors. Very similar values of both volatilities of the first and the third process and risk neutral mean reversion of the first and the third factors of the G3 explain the strong negative correlation between the two factors as shown in table 3.6.

In this model the second factor plays a smaller role in the explanation of the dynamic of interest rates: it is less correlated both with the third factor and with the first factor. In particular, the lower mean reversion parameter both in $\mathbb{P}$ and in $\mathbb{Q}$ shows that it affects more the expectation of future interest rates through the current level of the factor. We see this evidence both in $\mathbb{Q}$ and in $\mathbb{P}$.

On the contrary, the G2 shows a negative correlation between the first and the second factor. Furthermore, there is a switch of mean reversion speed between $\mathbb{Q}$ and in $\mathbb{P}$. In $\mathbb{Q}$ we observe a higher mean reversion speed of the first factor while in $\mathbb{P}$ we observe the opposite. Expectation in $\mathbb{Q}$ show that the first factor affects more interest rates through the long period while the second factor controls for the short run effects. Expectations in $\mathbb{P}$ are more affected by the second factor through the long run while the first factor effects more the short run.

The G1 model show a low mean reversion parameter both in $\mathbb{Q}$ and in $\mathbb{P}$ and a high value of long run mean γ in $\mathbb{Q}$. This mean that both the distribution of rates$in\mathbb{Q}$ and in $\mathbb{P}$ according to the univariate model is close to a random walk.

Table 3.7 shows that the G3 model is the best model to fit interest rates in the second sub sample.

3.4.4 Errors

By looking the errors we can see that:

- G1 model shows a higher concentration of errors in the first part of the term structure and in the last one as shown in figure 3.1 . This evidence is stronger in the two sub samples (figure 3.3). This evidence is shown also by the G2 model.

- G3 model shows a remarkable reduction of the mean errors in cross section. The errors are concentrated in the last part of the term structure. Not all maturities are considered from 10 years maturity to 30 year maturity. Indeed, it is reasonable to observe higher errors in maturities where data are not considered. This evidence is shown also in the two sub samples.

- RMSE of G1 is higher than G2 and G3 in cross section and in times series. In particular, the G2 model shows a lower RMSE in time series than the G3 model and the G1 model.

Figure 3.4 reports the errors in time series of G1, G2 and G3. In particular, we see that G1 model overestimates data in the first part of the time series (from the failure of Lehman to the bursting of the sovereign debt crisis) and underestimate data in the last part (from the bursting of the sovereign debt crisis to the end of 2016). The G1 underestimates data after the bursting of the sovereign debt crisis until the 5 year maturity. Thereafter the model errors decrease. Results show a further increase of errors from 10 year to 30 years maturity.

In cross section, the G1 overestimates data in the shorter maturities and overestimates data in longer maturities. The opposite holds for shorter maturities in the second sub sample.

Time series errors of G2 indicate smaller overestimation of data from the failure of Lehman to the bursting of the sovereign debt crisis and an underestimation thereafter in the first year maturity. Model errors disappears at $5th$ maturity and increase again after 10 year maturity.

In cross section the G2 model underestimates data at shorter maturities and overestimates data in longer maturities.

Errors of the G3 shows a low underestimation of data in the whole time series at one year maturity. Errors disappear at $5th$ maturity and increase after the 10 year maturity. Unlike the G1 and the G2, the fitting of the data after 10 year maturity is better for the G3 model.

In cross section G3 model, shows a small underestimation at the shorter maturities an overestimation at longer maturities. This effect is stronger in the second sub sample, after the bursting of the sovereign debt crisis.

Results on error's autocorrelation go partially in line with models errors. In particular we see that:

- Errors are strongly autocorrelated at the first maturity for every model estimated . G1 model shows higher autocorrelated errors than G2 and G3 as shown in figures 3.5 3.6 and 3.7;

- Error autocorrelation of G2 and G3 disappear at the $5th$ maturity while it is still present in the G1 model;

- The autocorrelation of errors increases after 10 year maturity. The error autocorrelation of the G3 model increases less than in the G2 and in the G1.

3.5 Conclusions

We estimate univariate (G1), bivariate (G2) and three variate model (G3) on intensity of interest bootstrapped from OIS rates in a time span where two crisis occurs. We find that both in the whole sample and in the two sub sample the Akaike information Criterion suggest that the G3 is the best model to fit the data. This evidence is stronger in the second sample, after the bursting of the sovereign debt crisis.

This result seems to contradicts the results yielded by the PCA. PCA on data in first differences shows that 74% of the variance is explained by the first factor in the whole sample and four factors are needed to explain the 96% of the variance of the data.

Results does not change in the sub sample analysis: in the first sub sample 66% of variance is explained by the first factor and four factors explain 96% of the variance of data. After the bursting of the sovereign debt crisis, the first factor explains 83% of the variance and four factors are needed to explain the 96% of the variance of the data.

Factors found by the PCA, are generic factors where no restriction is imposed to the distribution of rates. Furthermore, the independence assumption among factors postulated by the PCA analysis does not hold here.

Factors modeled here are significantly correlated one with another in almost all samples analyzed.

The no arbitrage restriction between yields and factors imposed by the Duffie and Kan framework, the Gaussian assumption of rates and the relaxing of independence assumption among factors leads to model interest rates with more factors than is shown by the decomposition of the data.

3.5.1 Tables and Graphs

Table 3.1: Interest rates whole sample

$Para$	G1	S.E	G2	S.E	G3	S.E
κ_1	0,0018***	1,9E-06	0,3229***	0,0054	0,4673***	0,0013
γ_1	2,3607***	0,0069	0,1047***	0,0028	0,0973***	0,0007
σ_1	0,0177***	5,5E-06	0,0362***	0,0007	0,4297***	0,0087
λ_{01}	-0,115***	0,0032	-0,149***	0,0307	0,4751***	0,0070
λ_{11}	-5,819***	0,0003	-0,182***	0,0637	-0,011***	0,0001
κ_2			0,0471***	0,0003	0,0261***	0,0001
σ_{21}			-0,028***	0,0001	0,0106***	0,0001
σ_{22}			0,0073***	-0,000	0,0143***	0,0001
λ_{02}			-0,137**	0,0682	-0,109**	0,0558
λ_{12}			0,1685***	0,0046	-0,136***	0,0042
κ_3					0,4405***	0,0012
σ_{31}					-0,435***	0,0087
σ_{32}					-0,016***	0,0001
σ_{33}					0,0080***	0,0002
λ_{03}					-0,506***	0,0064
λ_{13}					-0,118***	0,0013
Loglike	-151982		-178700		-189166	

* significance at 1%

** significance at 5%

*** significance at 10%

Table 3.2: Parameters in P and correlations coefficients whole sample

Para	G1	G2	G3
$\bar{\kappa}_1$	0,1054	0,3279	0,4722
$\bar{\kappa}_2$		0,0422	0,0285
$\bar{\kappa}_3$			0,4924
$\bar{\gamma}_1$	0,0230	0,0865	0,5288
$\bar{\gamma}_2$		-0,1255	-0,0684
$\bar{\gamma}_3$			-0,4491
ρ_{12}		-0,9673	0,5948
ρ_{23}			-0,5794
ρ_{13}			-0,999
σ_2		0,0290	0,0178
σ_3			0,436
$\bar{\kappa}_i = \kappa_i - \sigma_i \lambda_{1i}$		$\bar{\gamma}_i = \frac{\kappa_i \gamma_i + \sigma_i \lambda_{0i}}{\bar{\kappa}_i}$	$\sigma_i = \sqrt{(\sum_{j=1}^{i} \sigma_{ij}^2)}$

Table 3.3: Interest rates first subsample

Para	G1	S.E	G2	S.E	G3	S.E
κ_1	0,0457***	0,0010	0,3693***	0,0532	0,7230***	0,0093
γ_1	0,1751***	0,0038	0,1074***	0,0263	0,2896	0,4279
σ_1	0,0290***	0,0002	0,0261***	0,0032	0,3896	0,6998
λ_{01}	-0,272***	0,0042	-0,364	0,4977	2,1296	5,7693
λ_{11}	1,0624***	0,0371	-0,400	0,5578	-3,349	-7,252
κ_2			0,0276***	0,0043	0,0076	0,0148
σ_{21}			-0,015***	0,0008	0,0053	0,0100
σ_{22}			0,0104***	0,0009	0,0136***	0,0006
λ_{02}			-0,161	0,4608	-0,353	1,6531
λ_{12}			-0,812	1,1375	-0,882***	0,1636
κ_3					0,6573***	0,1484
σ_{31}					-0,387	0,7273
σ_{32}					-0,015	0,0077
σ_{33}					0,0081	0,0118
λ_{03}					-1,307*	0,6832
λ_{13}					-3,678	8,1332
Loglike	-53283,7		-60266,6		-63262,4	

* significance at 10%

** significance at 5%

*** significance at 1%

Table 3.4: Parameters in P and correlations coefficients first subsample

$Para$	G1	G2	G3
$\bar{\kappa_1}$	0,0149	0,3797	2,0283
$\bar{\kappa_2}$		0,0395	0,0205
$\bar{\kappa_3}$			2,0857
$\bar{\gamma_1}$	0,0074	0,0794	0,5123
$\bar{\gamma_2}$		-0,0598	-0,2525
$\bar{\gamma_3}$			-0,2434
ρ_{12}		0,3662	0,3662
ρ_{23}			-0,9990
ρ_{13}			-0,4025
σ_2		0,01470	0,01470
σ_3			0,3882
$\bar{\kappa_i} = \kappa_i - \sigma_i\lambda_{1i}$		$\bar{\gamma_i} = \frac{\kappa_i\gamma_i+\sigma_i\lambda_{0i}}{\bar{\kappa_i}}$	$\sigma_i = \sqrt{(\sum_{j=1}^{i}\sigma_{ij}^2)}$

Table 3.5: Interest rates second subsample

$Para$	G1	S.E	G2	S.E	G3	S.E
κ_1	0,0037***	0,0001	0,1896***	0,0004	0,2860***	0,0013
γ_1	1,0959***	0,0300	0,0960***	0,0008	0,0221***	0,0005
σ_1	0,0165***	0,0000	0,0755***	0,0008	0,2238***	0,0052
λ_{01}	-0,185***	0,1071	-0,122	0,1418	0,8668***	0,1328
λ_{11}	-1,149***	0,0417	1,9989***	0,0234	-0,425***	0,0212
κ_2			0,0862***	0,0007	0,0535***	0,0005
σ_{21}			-0,072***	0,0007	0,0044***	0,0003
σ_{22}			0,0051***	0,0001	0,0162***	0,0001
λ_{02}			-0,815***	0,1379	0,1078***	0,0267
λ_{12}			-3,051***	0,0378	0,0005	0,0004
κ_3					0,2565***	0,0016
σ_{31}					-0,224***	0,0052
σ_{32}					-0,017***	0,0002
σ_{33}					0,0048***	0,0001
λ_{03}					-0,904***	0,1370
λ_{13}					-3,678	8,1332
Loglike	-102064		-120838		-130855	

* significance at 10%
** significance at 5%
*** significance at 1%

Table 3.6: P parameters and correlations coefficients second sub sample

$Para$	G1	G2	G3
$\bar{\kappa}_1$	0,0227	0,0386	0,3813
$\bar{\kappa}_2$		0,4749	0,0535
$\bar{\kappa}_3$			0,3573
$\bar{\gamma}_1$	0,0446	0,0648	0,5255
$\bar{\gamma}_2$		-0,0509	-0,2525
$\bar{\gamma}_3$			-0,5700
ρ_{12}		-0,9974	0,2625
ρ_{23}			-0,0170
ρ_{13}			-0,9968
σ_2		0,0726	0,0167
σ_3			0,2252
$\bar{\kappa_i} = \kappa_i - \sigma_i\lambda_{1i}$		$\bar{\gamma_i} = \frac{\kappa_i\gamma_i+\sigma_i\lambda_{0i}}{\bar{\kappa_i}}$	$\sigma_i = \sqrt{(\sum_{j=1}^{i} \sigma_{ij}^2)}$

Table 3.7: AIC: 2(loglikelihood - number of parameters)

model	first subsample	second subsample	whole sample
$G1 : n = 1$	-10652944	- 20409069	- 30392689
$G2 : n = 2$	- 12048511	- 24162772	- 35575419
$G3 : n = 3$	- 12646489	- 26165017	- 378277192

3.5.2 Errors in cross section (whole sample analysis)

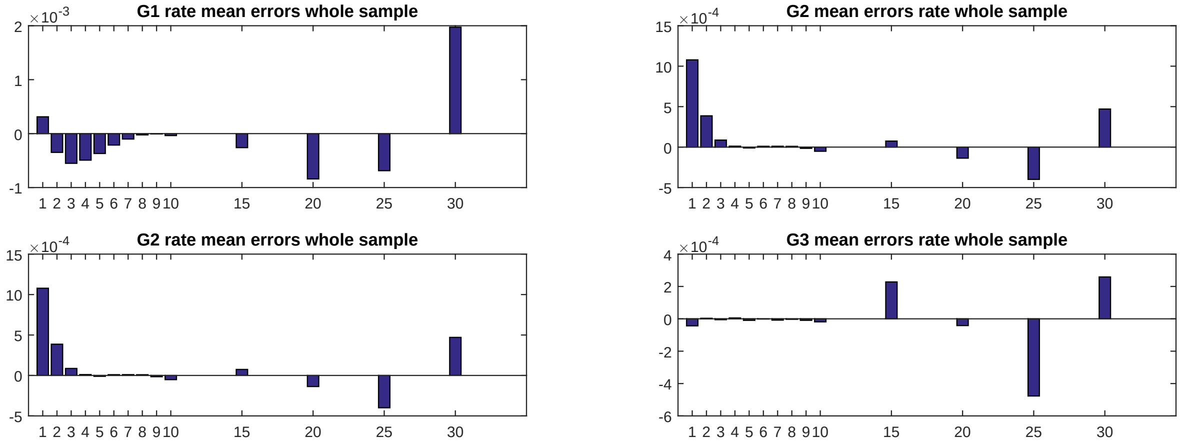

Figure 3.1: G1, G2, G3 mean error (cross section)

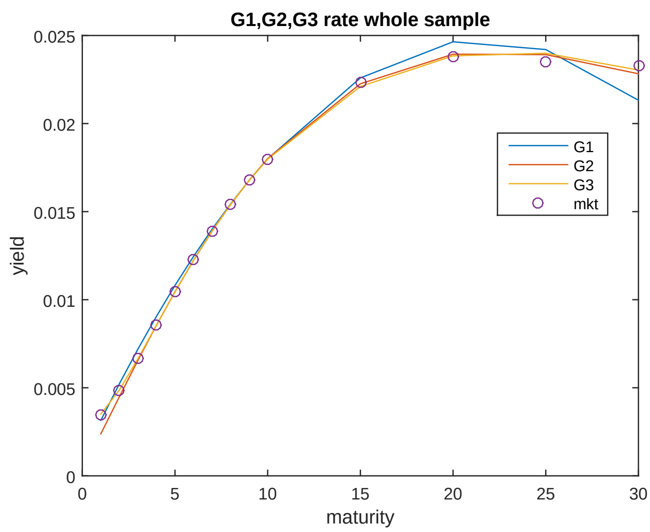

Figure 3.2: G1, G2, G3 term structures (mean data yields in cross section vs mean model yields in cross section)

3.5.3 Errors in cross section (sub sample analysis)

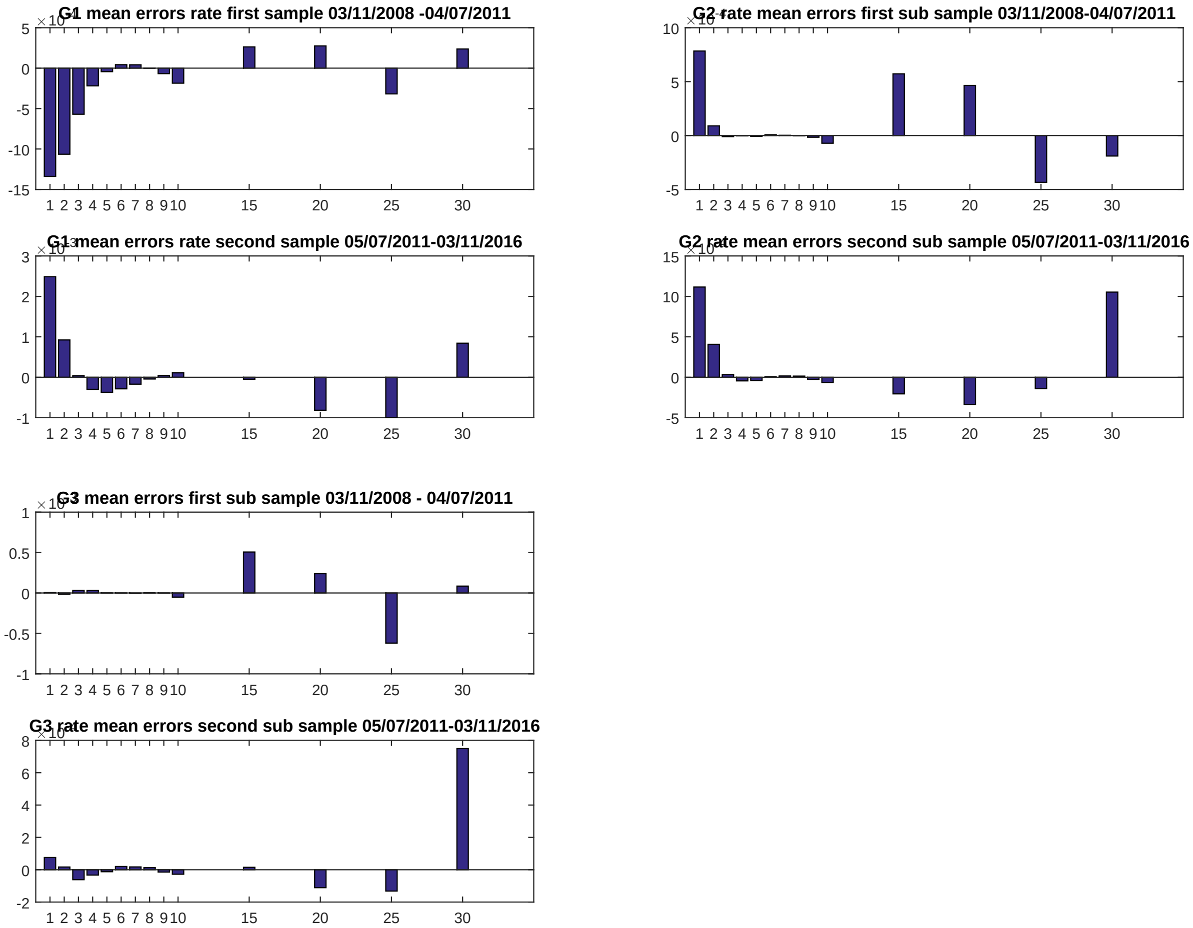

Figure 3.3: G1, G2, G3 mean errors sub sample

3.5.4 Predicted interest rate vs data whole sample analysis

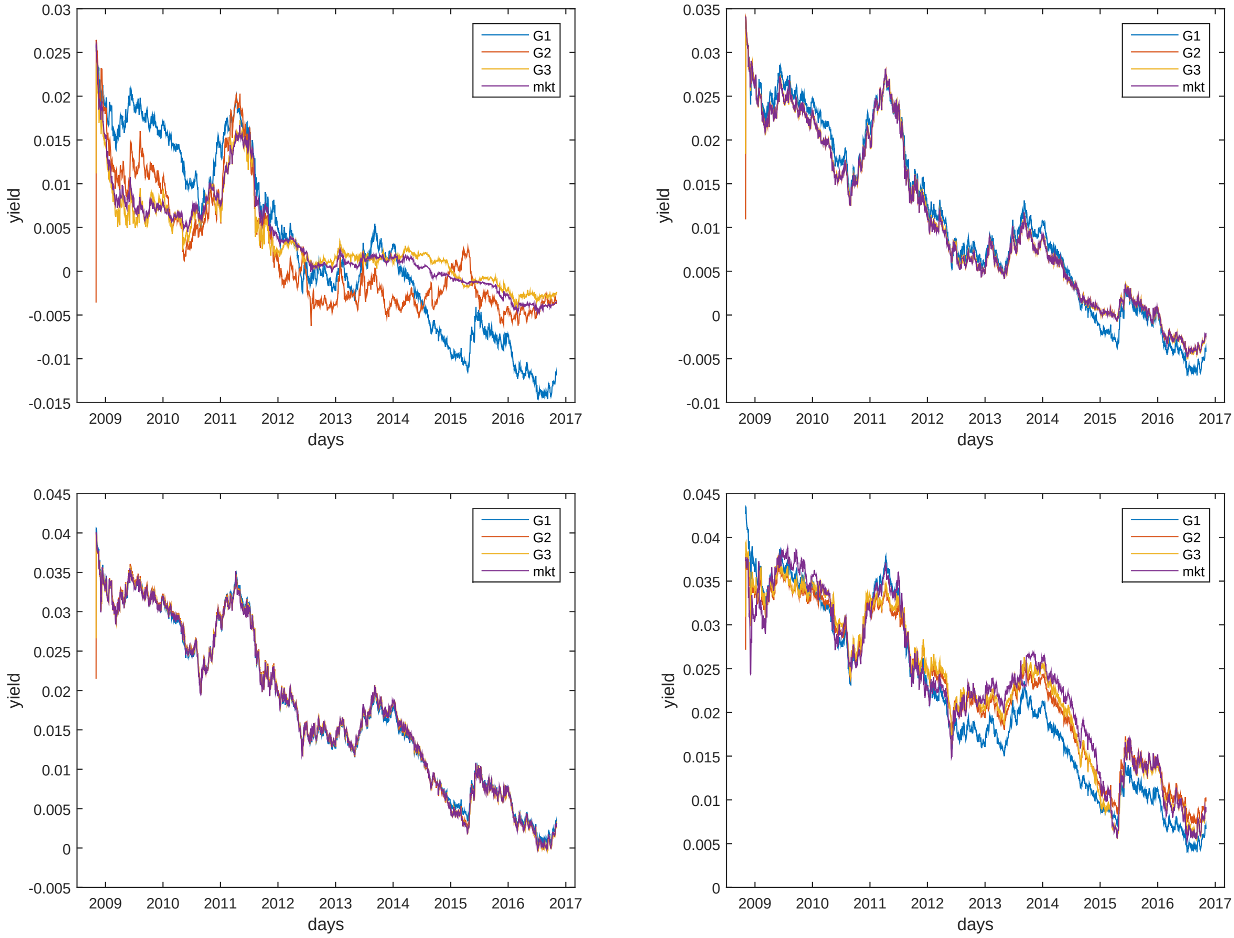

Figure 3.4: G1, G2, G3 time series errors 1y (up left) 5y (up right) 10y (down left) 30y (down right)

3.5.5 Errors autocorrelation (whole sample analysis)

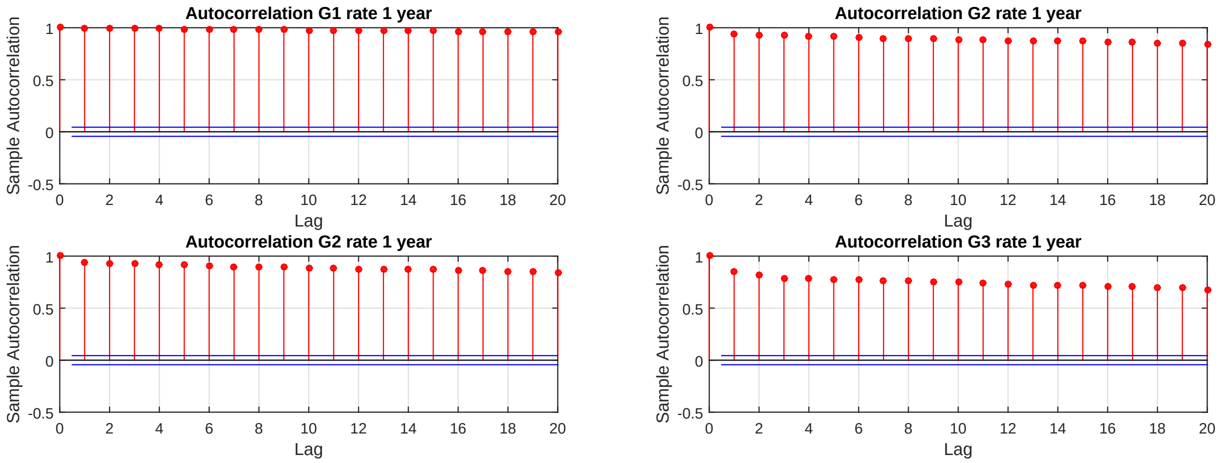

Figure 3.5: G1, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

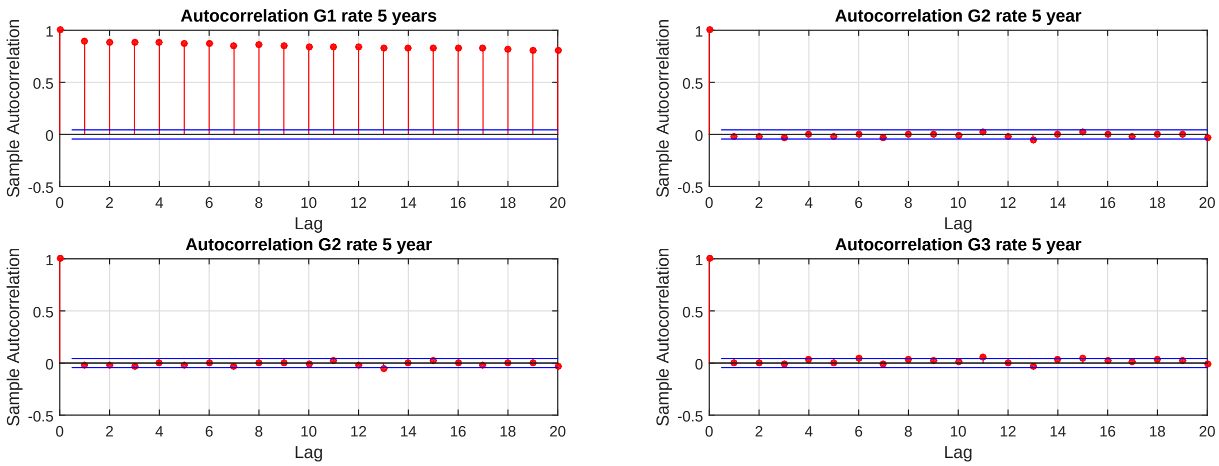

Figure 3.6: G2, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

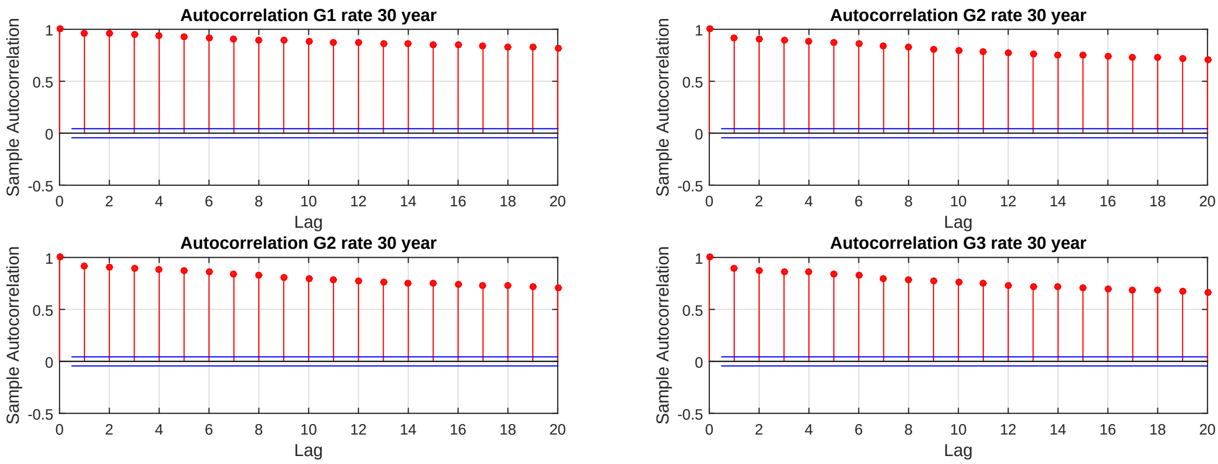

Figure 3.7: G3, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

3.5.6 RMSE (whole sample analysis)

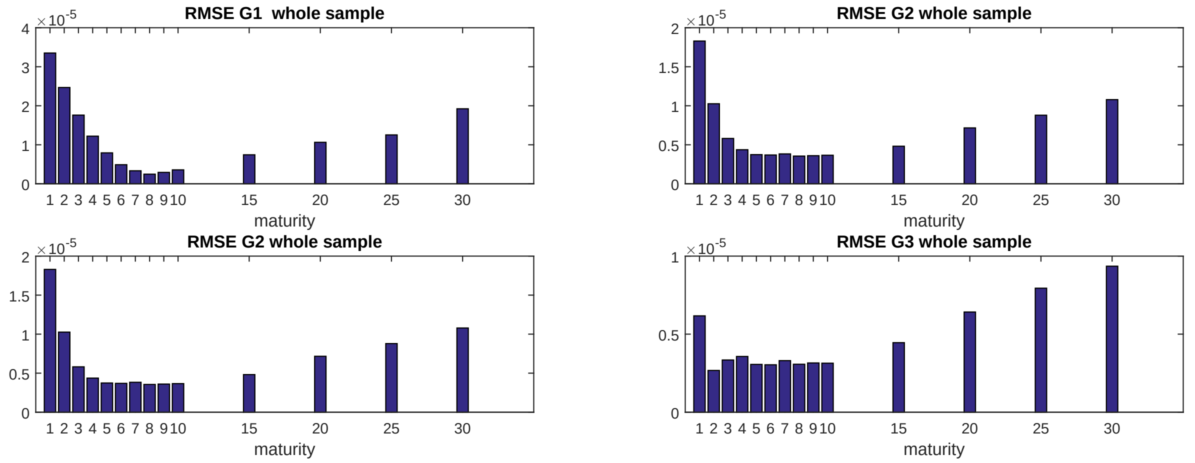

Figure 3.8: RMSE G1 G2 G3 cross section

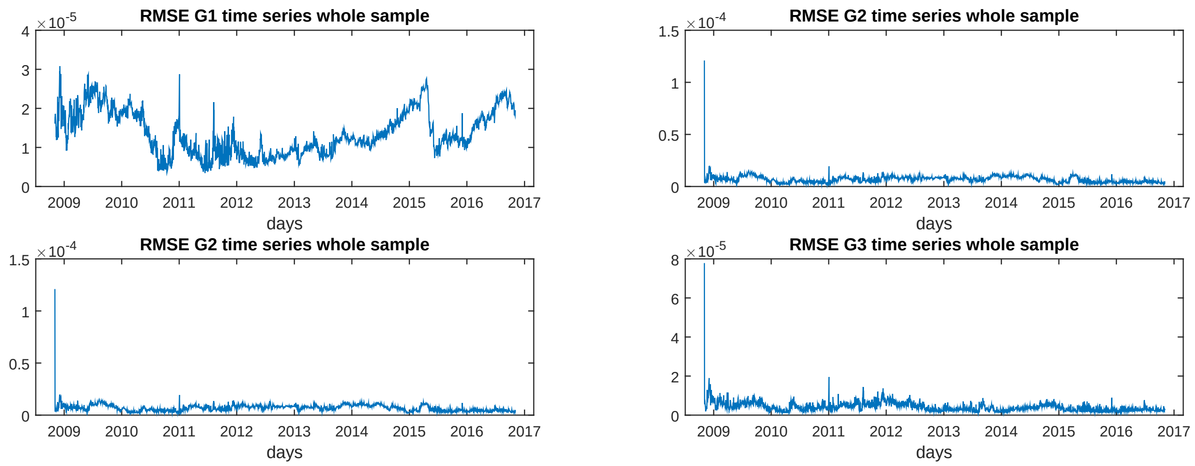

Figure 3.9: RMSE G1 G2 G3 time series

Chapter 4

A multifactor model of inflation rate: a Macro-Finance approach

4.1 Introduction

Modeling expected inflation is an hard and interesting topic. More information is needed, than interest rate modeling.

As the well known relationship with the monetary policy as described in Estralla et al. (1991) and Mishkin (1991), modeling inflation has always been of interest of academics.

The new monetary policy implemented by the FED after the failure of Lehman and the developments of the literature on Term Structure modeling and the macroeconomics see Kim (2009), Ang and Piazzesi (2006) Rudebush et al. (2008) and Joslin et al. (2014) made inflation rates modeling a great challenge among academics and market practitioners.

Jarrow and Yildrim (2003) consider a multifactor model for modeling inflation which consists in the correlated factors : the real dynamics, the nominal rate dynamics and the inflation rate dynamics as the difference between the two. Both real rate dynamics and the nominal rate dynamics are modeled according to HJM model Heath, Jarrow and Morton (1992).

Within this context, modeling inflation is analogous to model the currency exchange rate dynamics, as the exchange rate between nominal and real outcomes is the CPI.

Chernov and Muller (2012) models inflation expectations as an hidden factor in the Term Structure of nominal yields using survey data of US inflation forecasts and Treasury and real yields.

Haubrich et al. (2012) model inflation expectation with a four factor model. Nominal rates, real rates, the volatility of inflation and the long run central tendency. They analyses the inflation dynamics by fitting the model on *TIPS* and inflation swap respectively. They find, that Inflation swap rates yield a lower inflation dynamics because, there is no embedded options against debt-deflation in them.

D'Amico and Orphadines (2014) analyze how inflation uncertainty affects bond's risk premia by extracting inflation expectations from TIPS rates.

Furthermore, D'Amico at al. (2018) proposes a multivariate inflation rate model where the nominal rate process in combined with the real rate process and the inflation risk premium and estimate it with nominal yields. TIPS rates and US inflation forecasts.

All procedures previously described model inflation dynamics according to multifactor approaches, where at least two sources of data are needed: the real rate data and the nominal rate data.

Furthermore, few space is given to model estimated on Inflation swap data and none of this works model European inflation. A different way of modeling inflation is performed here. The approach proposed here, assumes inflation expectations to be given by*BEI*(Break Even Inflation) rates. The BEI rate is often used as an indicator of inflation expectation (see Ciccarelli and Garcia (2010)). BEI rates coincide with the fixed leg of the widely traded ZC inflation swap contracts see Dean and Pelata (2013).

Following the literature on Gaussian Term Structure models we model BEI rates considering only the information from European inflation swap data.

Here, the analogy with interest rates modeling is straightforward: we simply need a data on inflation swap rates to estimate the Term Structure of BEI rates.

We estimate one factor, two factor and three factor model in State Space form in a time span when two Crisis occur. Like interest rates, also BEI rates have been affected by the macroeconomic conditions brought by the failure of Lehman Brother and the Sovereign Debt Crisis. Therefore, it is of interest of both academics and market practitioners to analyze BEI rates to create a tighter link between financial modeling and the macroeconomics.

4.2 A spanned inflation rate model

Given a fixed time interval $\tau = T - t$, the BEI rate is the amount of money that an inflation linked financial contract pays in time T to compensate the contract holder from the realized inflation.

$\forall t$ the BEI rate is:

$$r(t,T)^N - r(t,T)^R = \underbrace{I(t,T)}_{Expected\ Inflation} + \underbrace{\phi(t,T)}_{Inflation\ risk\ premium} \quad (4.1)$$

Given the nominal and the real discount factor to be of this form;

$$P(t,T)^N = \exp\left(A^N(\tau) - \sum_{i=1}^{3} B^N(\tau) x_i^N(t)\right) \quad (4.2)$$

$$P(t,T)^R = \exp\left(A^R(\tau) - \sum_{i=1}^{3} B^R(\tau) x_i^R(t)\right) \quad (4.3)$$

By exploiting the standard setting of Affine Term Structure models of Duffie and Kan (1996), Dai and Singleton (2000) , Duffee (2002) and merging it with the more recent developments in Term Structure models and the macroeconomics separately developed by Kim (2009), Duffee (2011), Hamilton and Wu (2012) and given the ratio of the nominal and real bond prices of the same maturity is an exponentially affine function of the vector $X^b = (x_{b1}, x_{b2}, \dots, x_{bn})$ our BEI discount factor is of this form:

$$\frac{P(t,T)^N}{P(t,T)^R} = \exp\left(A^b(\tau) + \sum_{i=1}^{3} B^b(\tau) x_{bi}(t)\right) \quad (4.4)$$

where $\forall \tau = T - t$ the BEI rate is:

$$BEI(t,T) = -\frac{A^b(\tau)}{\tau} + \sum_{i=1}^{3} \frac{B^b(\tau)}{\tau} x_{bi}(t) \quad (4.5)$$

As done by Kaminska et all. (2018), we focus directly on the estimation of the Break Even Inflation rate rather than considering the nominal rate and the real rate alone.

This identification strategy considers the information from BEI rates as spanned in the Term Structure.

Indeed , given a generic vector of factors observed without error $X(t_k) = (x_1(t_k), x_2(t_k), \dots, x_n(t_k))'$, a (p $\times$ 1) matrix a and a (p $\times$ n) of factor loading matrix b, this relation must hold:

$$X(t_k) = b^{-1}(a - BEI(t,T)) \quad (4.6)$$

with $b = \frac{B(\tau)}{\tau}$ and $a = \frac{A(\tau)}{\tau}$.

It is always possible to invert the BEI rate formula and right the vector of factor as a function of it. We also have that, $\forall k = 1, \dots, m-1$ times, the vector of factors which describe the expected future BEI rate $X(t_k) = (x_{b1}(t_k), x^{b2}(t_k), \ldots, x_{bn}(t_k))'$, has the dynamics under the neutral probability measure $\mathbb{Q}$ of this form:

$$dX(t) = K_b^{\mathbb{Q}}(\Gamma_b^{\mathbb{Q}} - X^b(t))dt + \Sigma dW_b^{\mathbb{Q}}(t) \tag{4.7}$$

where, like interest rates, the mean reversion speed matrix is:

$$K_{bn}^{\mathbb{Q}} = \begin{bmatrix} \kappa_{b1} & 0 & \ldots & 0 \\ 0 & \kappa_{b2} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & \kappa_{bn} \end{bmatrix}$$

with positive non zero elements.

The risk neutral long term mean matrix $\Gamma_b^{\mathbb{Q}}$ is assumed to be:

$$\Gamma_b^{\mathbb{Q}} = \begin{bmatrix} \gamma_b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The diffusion matrix Σ is assumed to be:

$$\Sigma = \begin{bmatrix} \sigma_{b11} & 0 & \ldots & 0 \\ \sigma_{b21} & \sigma_{b22} & \ldots & 0 \\ \sigma_{b31} & \sigma_{b32}^{b} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{bn1} & \sigma_{bn2} & \ldots & \sigma_{bnn} \end{bmatrix}$$

As for the estimation of interest rates models, We also assume $\sigma_{bii}> 0\ \forall i = 1, \ldots, N$, where no assumption is made on the off diagonal elements.

We assume an affine price of risk as described in Duffee (2002) of this form:

$$\Lambda_i(t, x_i(t)) = \lambda_{0i} + \lambda_{1i} x_{bi}(t) \tag{4.8}$$

Therefore, the natural drift is of the form:

$$\begin{aligned} \bar{\kappa_{bi}}(\bar{\gamma_{bi}} - x_i^b(t)) &= \kappa(\gamma_{bi} - x_{bi}(t)) + \Lambda_i(t, x_{bi}(t))\sigma_{bi} \\ &= \kappa_{bi}(\gamma_{bi} - x_{bi}(t)) + (\lambda_{0i} + \lambda_{1i} x_{bi}(t))\sigma_{bi} \\ &= (-\kappa_{bi} + \lambda_{1i}\sigma_i)x_{bi}(t) + \kappa_{bi}\gamma_{bi} + \lambda_{0i}\sigma_{bi} \end{aligned}$$

$$\bar{\kappa_{bi}} = \kappa_{bi} - \sigma_{bi}\lambda_{1i} \qquad \bar{\gamma_{bi}} = \frac{\kappa_{bi}\gamma_{bi} + \sigma_{bi}\lambda_{0i}}{\bar{\kappa_{bi}}} \tag{4.9}$$

The dynamics of the BEI under the $\mathbb{P}$ measure is:

$$dX(t) = K_b^{\mathbb{P}}(\Gamma_b^{\mathbb{P}} - X^b(t))dt + \Sigma dW_b^{\mathbb{P}}(t) \tag{4.10}$$

$$K_b^{\mathbb{P}} = \begin{bmatrix} \bar{\kappa_{b1}} & 0 & \dots & 0 \\ 0 & \bar{\kappa_{b2}} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \bar{\kappa_{bn}} \end{bmatrix}, \Gamma^{\mathbb{P}} = \begin{bmatrix} \bar{\gamma_{b1}} \\ \bar{\gamma_{b2}} \\ \vdots \\ \bar{\gamma_{bn}} \end{bmatrix}, W^{\mathbb{P}}(t) = \begin{bmatrix} W_{1b}^{\mathbb{P}}(t) \\ W_{2b}^{\mathbb{P}}(t) \\ \vdots \\ W_{bn}^{\mathbb{P}}(t) \end{bmatrix},$$

Given a time t and a time step Δt, the $\mathcal{F}_t$ - conditional mean and variance of the BEI rates under the $\mathbb{P}$ dynamics are :

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{n} x_{i(t+\Delta t)}^b | \mathcal{F}_t\right] = \sum_{i=1}^{n} \bar{\gamma_{bi}}(1 - \exp(-\bar{\kappa_{bi}}\Delta t)) + \exp(-\bar{\kappa_{bi}}\Delta t)x_{i(t)} \tag{4.11}$$

$$\begin{aligned} Var^{\mathbb{P}}\left[\sum_{i=1}^{n} x_{i(t+\Delta t)} | \mathcal{F}_t\right] &= \sum_{i=1}^{n} \frac{(\sigma_{bi})^2}{2\bar{\kappa_{bi}}}(1 - \exp(-2\bar{\kappa_{bi}}\Delta t)) + \\ &2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\sigma_{bi}\sigma_{bj}\rho_{bij}}{\bar{\kappa_{bi}} + \bar{\kappa_{bj}}}(1 - \exp(-(\bar{\kappa_{bi}} + \bar{\kappa_{bj}})\Delta t)) \end{aligned} \tag{4.12}$$

4.3 Estimation procedure

To estimate the multivariate BEI model we use a linear stochastic filtering technique.

As shown by data there are periods of negative BEI rates. Indeed, we assume a Gaussian distribution for BEI rates to fit negative BEI rates.

Merging the Gaussian assumption of the BEI rates with the affine property, we apply the same procedure used for estimating the interest rates. We use the Kalman Filter to estimate the model's parameters.

The measurement and the transition dynamics of the BEI rate are:

$$\text{(measurement equation)} \quad \underbrace{z(t_k)}_{p\times 1} = \underbrace{A}_{p\times 1} + \underbrace{B}_{p\times n}\underbrace{X(t_k)}_{n\times 1} + \underbrace{\eta(t_k)}_{p\times 1} \qquad \eta(t_k) \sim \mathcal{IID}(0,\, R)$$

$$\text{(Trasition equation)} \quad \underbrace{X(t_k)}_{n\times 1} = \underbrace{C}_{n\times 1} + \underbrace{F}_{n\times n}\underbrace{X(t_{k-1})}_{n\times 1} + \underbrace{\varepsilon(t_k)}_{n\times 1} \qquad \varepsilon(t_k) \sim \mathcal{IID}(0,\, Q)$$

$\eta(t_k) = \zeta(t_k) - z(t_k)$

where Q and R are $(n \times n)$ and $(p \times p)$ positive definite matrices and

$$X^b(t_k) = \begin{bmatrix} x_{b1}(t_k) \\ x_{b2}(t_k) \\ \vdots \\ x_{bn}(t_k) \end{bmatrix}, \qquad C^b = \begin{bmatrix} \bar{\gamma_{b1}}(1 - \exp(-\bar{\kappa_{b1}}\Delta t)) \\ \bar{\gamma_{b2}}(1 - \exp(-\bar{\kappa_{b2}}\Delta t)), \\ \vdots \\ \bar{\gamma_{bn}}(1 - \exp(-\bar{\kappa_{bn}}\Delta t)) \end{bmatrix} \tag{4.13}$$

and,

$$F^b = \begin{bmatrix} \exp(-\bar{\kappa_{b1}}\Delta t) & 0 & \dots & 0 \\ 0 & \exp(-\bar{\kappa_{b2}}\Delta t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(-\bar{\kappa_{bn}}\Delta t) \end{bmatrix}, \tag{4.14}$$

$\eta(t_k)$ is the vector of *innovations*, i.e. information which cannot derived from the data at time t_{k-1}.

(Variance of the measurements)

$$S(t_k) = BVar[X(t_k)|\mathcal{F}t_k]B' + R \tag{4.15}$$

(Variance of the latent variables)

$$P(t_k) = FVar[X(t_k)|\mathcal{F}t_{k-1}]F' + Q \tag{4.16}$$

The Log likelihood function is of this form:

$$\log \mathcal{L} = \sum_{k=1}^{m} \left[-\frac{p}{2}\log(2\pi) - \frac{1}{2}\log\det S(t_k) - \frac{1}{2}\eta S^{-1}(t_k)\eta'(t_k) \right]$$
$$= -\frac{mp}{2}\log(2\pi) - \frac{1}{2}\sum_{k=1}^{m}\log\det S(t_k) - \frac{1}{2}\sum_{k=1}^{m}\eta(t_k)S^{-1}(t_k)\eta'(t_k). \tag{4.17}$$

4.3.1 Optimization procedure

To minimize the log likelihood function we use the Matlab function *Fmincon*. Given $\theta = (\kappa_{bi}, \gamma_i, \lambda_{i0}, \lambda_{i1}\sigma_{bii}, \sigma_{bij}, R)$ the vector of model parameters, the optimization problem is of this form:

$$-\min_{\theta} \log \mathcal{L} \tag{4.18}$$

$$\text{u.c.} \begin{cases} 0 \leq \kappa_{bi} & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \gamma & \leq +\infty & i = 1 \\ -\infty \leq \lambda_{i0} & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \lambda_{i1} & \leq +\infty & \forall i = 1, \dots, n \\ \kappa_{bi} - \lambda_{1i}\sigma_{bi} & > 0 & \forall i = 1, \dots, n \\ 0 < \sigma_{bii} & \leq +\infty & \forall i = 1, \dots, n \\ -\infty \leq \sigma_{bij} & \leq +\infty & \forall i = 1, \dots, n, \forall j = 1, \dots, n \\ 0 \leq r_h & \leq +\infty & \forall h = 1, \dots, p \end{cases} \tag{4.19}$$

We set the same constraints of the optimization procedure used for interest rates. We allow the risk neutral long term mean to be negative. Data, after the bursting of the sovereign debt crisis show a negative Break even inflation. Indeed, it makes economic sense not to impose constraints to the risk neutral long term mean parameter of the BEI rate.

4.4 Discussion

4.4.1 whole sample

In the whole sample analysis we see a smaller effect of price of risk parameters than the second sub sample.

In the G3 model we see very similar risk neutral and risk natural mean reversion speed of factor 2 and factor 3. The same holds for the volatilities. The correlation parameter between factor 2 and factor 3 is strongly negative and close to -1.

Factor 2 and factor 3 are distinct factors which work in opposite way. The first factor is weakly positively correlated with the second factor and weakly negatively correlated with the third.

Furthermore, the mean reversion speed of both factor 2 and factor 3 is higher than the mean reversion speed of factor 1 both in $\mathbb{Q}$ and in $\mathbb{P}$. Indeed, factor 2 and factor 3 make a short run effect on the expectation of the future BEI rates and factor 1 effect BEI rates in the long run.

The G2 model, shows a higher effect of price of risk parameters in the expectation of future BEI rates under $\mathbb{P}$. Results on model parameters show a higher difference of mean reversion parameters of factor 1 and factor 2 than in the first sample. This holds also for the volatilities parameters.

Factor 1 and factor 2 can't be considered an unique factor. In particular, the mean reversion of the first factor is higher than the mean reversion speed of the second factor. Factor 1 makes a long run effect on the expectation of future BEI rates and factor 2 the expected BEI rates more in the short run.

The G1 shows a high and positive λ_{11}. The mean reversion speed in $\mathbb{P}$ is smaller than the mean reversion speed in $\mathbb{Q}$.

The expected future BEI rates is influenced more by the short run dynamics of the BEI rates than in the second sample.

The G3 is the best model to fit BEI rates in the whole sample as shown in table 4.7.

4.4.2 First subsample

Table 4.3 shows that most of the parameters are significant except for λ_{01} that are not statistically significant for the G1, G2. In particular, price of risk parameters are lower in the G3 that in the G2 and G1 model.

In the G3 we see similar risk neutral mean reversion parameters κ_{b1} and κ_{b2} and similar volatilities σ_1 and σ_2. This result show that factor 1 and factor 2 of the G3 on BEI rates are strictly negatively correlated.

Indeed, they are distinct factors which work in opposite way. The third factor shows a lower risk neutral and risk natural mean reversion speed than factor 1 and factor 2. This may suggest that the third factor affects more the expectation of the future BEI rate under $\mathbb{P}$ and under $\mathbb{Q}$ through the current level of BEI rate. The opposite holds with factor 1 and factor 2.

The G2 model shows different risk neutral mean reversion parameters κ_{b1} and κ_{b2} and similar volatilities σ_1 and σ_2. The correlation parameter ρ_{12} is strongly negative and close to -1. On the contrary to G3, the parameterλ_{11} of the G2 is not statistically significant. Expectations do not effect the mean reversion speed of the first factor. Furthermore,λ_{21} is negative and significant.

The mean reversion speed in $\mathbb{P}$ is different for the factor 2 and equal for the factor 1. Factor 2 shows a higher mean reversion speed in $\mathbb{P}$ than in $\mathbb{Q}$ still lower than mean reversion of the first factor. In particular, in $\mathbb{P}$ factor 1 reverts almost twice as much the factor 2. The first factor affects more the expectation of future BEI rates through the long term mean $\bar{\gamma_{b1}}$. On the contrary the second factor effects more the expectation of the future BEI rates under $\mathbb{P}$ through the current value of the rate and less by the long term mean.

Factor 1 and factor 2 in the G2 cannot be considered as a unique factor.

The G1 model shows higher value of price of risk parameters than the G3. The two price of risk parameters are negative and significant. This indicates that in $\mathbb{P}$ the dynamic of BEI rates is more explained by a lower long run mean.

Table 4.7 shows us that the G3 is that best model to fit BEI rates.

4.4.3 Second subsample

Table 4.5 shows a higher degree of significant of model parameters than in the first sub sample. In particular, we observe a higher value of price of risk parameters in all model estimated.

The long term mean in $\mathbb{Q}$ is higher in the G1 and the G2 and lower in the G3 with respect to the first sub sample.

The G3 shows different mean reversion parameters of factor 1 and factor 2 both in $\mathbb{Q}$ and $\mathbb{P}$. Also the volatilities of factor 1 σ_1 and factor 2 are more different than in the first sub sample. Factor 1 and factor 2 are less correlated than in the first sub sample. In the second sub sample, factor 1 and factor 2 cannot be considered as a unique factor.

The third factor of the G3 model in the second sub sample is not correlated with both factor 1 and factor 2 as the volatility parameter σ_{31} is not statistically significant.

The mean reversion speed in $\mathbb{Q}$ and in $\mathbb{P}$ are similar in the G3. The higher values of price of risk parameters is compensated by the lower values of the volatility parameters.

The first factor show a higher mean reversion speed than the second factor. Furthermore, the second factor has a higher mean reversion speed than the third factor.

This results may suggest that the first factor affects more the expectations of the future BEI rates both in $\mathbb{P}$ and in $\mathbb{Q}$ through the long term mean than the second and the third factor. In particular, the third factor effects the short run tendency of BEI rates, the second factor the medium tendency and the first factor the long run tendency.

The G2 shows a lower correlation coefficient ρ_{12} between factor 1 and factor 2 than in the first sub sample. The difference between the volatilities σ_1 and σ_2 and the mean reversion speed of factor 1 and factor 2 both in $\mathbb{Q}$ and in $\mathbb{P}$ is higher than in the first sub sample.

Factor 1 and factor 2 cannot be consider a unique factor in the second sub sample. Furthermore, the mean reversion parameter of the first factor is higher than the mean reversion parameter of the second factor. This result holds both in $\mathbb{Q}$ and in $\mathbb{P}$.

This may suggest that the first factor affects more the expectation on the future BEI rate both in $\mathbb{Q}$ and in $\mathbb{P}$ through the long run mean while the second factor makes a short run effect on expected future BEI rates.

The G1 model shows a higher effect of the price of risk parameters in the expectation on future BEI rates under $\mathbb{P}$. The risk natural mean reversion speed under $\mathbb{P}$. is higher than the mean reversion speed under $\mathbb{Q}$. The future expected BEI rates under $\mathbb{P}$ are more affected by the long term mean. The short run effect is smaller than in the other models estimated.

Table4.7 shows that the G3 model is the best model to fit BEI data in the second sub sample.

4.4.4 Errors

Figures 4.1 and 4.3 show that:

- In cross section G1 and G2 show a higher concentration of errors at shorter maturities than the G3 model in the whole sample. This effect is stronger in the two sub samples;
- In time series, except for the first maturity, error are concentrated in the second sub sample (around 2012 2013);

Furthermore:

- The G1 underestimates data at shorter and longer maturities in cross section. This evidence is stronger in the whole sample analysis.
- The G2 and G3 overestimate data at shorter maturities and underestimate data at longer ones in the whole sample analysis. In the first sub sample the G2 and G3 overestimate data at shorter maturities and underestimates data at longer ones. In the second sub sample both the G2 and the G3 overestimates data at shorter maturities and overestimate data at longer ones;

In time series as shown by figure 4.4, the G1 shows a higher errors at the first maturity than the G2 and the G3. The model's error both in the G1, G2 and the G3 decreases from the $5th$ maturity but they increases again after $10y$ maturity. In most of the maturities errors are concentrated in the period 2012, 2013.

Results about the RMSE show that:

- The G1 shows higher RMSE in cross section than the G2 and the G3 in the whole sample as show in figure 4.8;
- The G2 shows a lower RMSE in times series than G1 and G3. as shown by 4.9. The G2 show a higher RMSE than the G2 and the G3 respectively;

- In G1, G2, G3 models RMSE are concentrated in period 2012-2013;

Figures 4.5 4.6 and 4.7 show that errors of G1, G2 and G3 are autocorrelated at the first maturity. In particular the autocorrelation is stronger for the G1.

At the $5th$ maturity error autocorrelation almost disappears in the G2 and the G3, while it is still present in the G1.

In the G2, the autocorrelation of errors increases after $10y$ maturity while it is still very low in the G3.

4.5 Conclusions

We have estimated the univariate, bivariate and three variate model on BEI rates bootstrapped from Inflation swap.

Both in the whole sample and in the sub sample analysis the Akaike information Criteria suggest that the G3 is the best model to fit BEI rates.

This result seems to contradict the result found by the Principal Component analysis of the data. The PCA analysis, suggests that seven factors are needed to explain the 80% of the variance of the data in first differences. We obtain similar results both in the first and in the second subsample.

Factors of the PCA are generic factors free from constraints. Factors estimated here are constrained factors where BEI rates are assumed to be Gaussian distributed an the no arbitrage restriction is imposed.

On the other hand, G3 models shows two similar but distinct factors in which one is the opposite of the other. This holds for all samples analyzed except for the period after the bursting of the sovereign debt crisis.

Although, the no arbitrage restriction imposed by the affine class of models estimated here and the Gaussian restriction on the distribution of BEI rates, both in the whole sample and in the first sub sample, results found by models on the number factor needed for describing the variance of the BEI data converge with those found by PCA. This convergence does not appear after the bursting of the sovereign debt crisis.

4.6 Tables and Graphs

Table 4.1: BEI rates whole sample

Para	G1	S.E	G2	S.E	G3	S.E
κ_{b1}	0,0498***	0,0004	0,7279***	0,0101	0,2822***	0,0119
γ_{b1}	0,0602***	0,0003	0,0505***	0,0005	0,1021***	0,0079
σ_{b1}	0,0152***	7,6E-05	0,0243***	0,0006	0,1317***	0,0204
λ_{01}	-0,190***	0,0208	-0,177***	0,2325	-0,016***	0,0185
λ_{11}	1,5784***	0,0930	4,1428***	0,0535	-0,528***	0,0061
κ_{b2}			0,0589***	0,0007	0,0101***	0,0009
σ_{b21}			-0,013***	0,0002	-0,001***	0,0003
σ_{b22}			0,0100***	0,0001	0,0052***	-0,041
λ_{02}			-0,341***	0,0911	-0,169	0,1011
λ_{12}			-4,315***	0,0478	-0,063	0,1391
κ_{b3}					0,2008***	0,0534
σ_{b31}					-0,119***	0,0205
σ_{b32}					-0,000***	0,0005
σ_{b33}					0,0088***	0,0001
λ_{03}					0,0253***	0,0106
λ_{13}					-1,196***	0,0378
Loglike	-162167		-174658		-182816	

* significance at 10%
** significance at 5%
*** significance at 1%

Table 4.2: P parameters and correlations coefficients

$Para$	G1	G2	G3
$\bar{\kappa_{b1}}$	0,0253	0,6269	0,3519
$\bar{\kappa_{b2}}$		0,1318	0,0105
$\bar{\kappa_{b3}}$			0,3443
$\bar{\gamma_{b1}}$	0,0042	0,0517	0,0758
$\bar{\gamma_{b2}}$		-0,0437	-0,0863
$\bar{\gamma_{b3}}$			0,0088
ρ_{b12}		-0,8011	-0,1870
ρ_{b23}			-0,9972
ρ_{b13}			0,1796
σ_{b2}		0,0168	0,0536
σ_{b3}			0,1199

Table 4.3: BEI rates first subsample

$Para$	G1	S.E	G2	S.E	G3	S.E
κ_{b1}	0,1016***	0,0022	0,4084***	0,0057	0,2154***	0,0016
γ_{b1}	0,0505***	0,0004	0,0318***	0,0003	0,2857***	0,0262
σ_{b1}	0,0246***	0,0004	0,1161***	0,0048	0,5136***	0,0472
λ_{01}	-0,156	0,2261	0,0972	0,0643	-0,031***	0,0088
λ_{11}	-0,565***	0,0958	0,0177*	0,0091	0,0670***	0,0110
κ_{b2}			0,2284***	0,0037	0,1912***	0,0027
σ_{b21}			-0,092***	0,0045	-0,498***	0,0548
σ_{b22}			0,0110***	0,0003	0,0103***	0,0003
λ_{02}			-0,188***	0,0754	0,0071	0,0071
λ_{12}			-1,689***	0,0689	0,0231***	0,0023
κ_{b3}					0,0026***	0,0003
λ_{03}					0,0025***	0,0003
λ_{13}					-0,001***	0,0002
σ_{b31}					0,0046***	0,0001
σ_{b32}					-0,066	0,0626
σ_{b33}					0,2168***	0,0546
Loglike	-53685,1		-59312,8		-63249,8	

* significance at 10%
** significance at 5%
*** significance at 1%

Table 4.4: Parameters in P and correlations coefficients

$Para$	G1	G2	G3
$\bar{\kappa_{b1}}$	0,1155	0,3996	0,3971
$\bar{\kappa_{b2}}$		0,2494	0,1796
$\bar{\kappa_{b3}}$			0,0014
$\bar{\gamma_{b1}}$	0,0111	0,0611	0,2518
$\bar{\gamma_{b2}}$		-0,0839	0,0199
$\bar{\gamma_{b3}}$			-0,2596
ρ_{b12}		-0,9950	-0,9997
ρ_{b23}			0,4683
ρ_{b13}			-0,4732
σ_{b2}		0,1113	0,4986
σ_{b3}			0,0055

Table 4.5: BEI rates second sub sample

$Para$	G1	S.E	G2	S.E	G3	S.E
κ_{b1}	0,0360***	0,0005	0,4996***	0,0102	0,6530***	0,0269
γ_{b1}	0,0700***	0,0008	0,0903***	0,0030	0,0591***	0,0024
σ_{b1}	0,0129	7,1283	0,0121***	0,0005	0,0258***	0,0007
λ_{01}	0,5424***	0,0403	2,0623***	0,0071	0,0453***	0,0046
λ_{11}	-79,30***	2,8757	-27,24***	0,0022	0,5493***	0,0089
κ_{b2}			0,0224***	0,0010	0,2965***	0,0117
σ_{b21}			-0,007***	0,0005	-0,015***	0,0007
σ_{b22}			0,0093***	0,0002	0,0090***	0,0006
λ_{02}			-0,159*	0,0953	-0,164***	0,0254
λ_{12}			-0,186***	0,0185	-0,630***	0,0343
κ_{b3}					0,0241***	0,0017
σ_{b31}					-0,001***	0,0005
σ_{b32}					-0,002***	0,0005
σ_{b33}					0,0083***	0,0002
λ_{03}					-0,105***	0,0069
λ_{13}					0,2503**	0,0221
Loglike	-110487		-119279		-1,2E+11	

* significance at 10%
** significance at 5%
*** significance at 1%

Table 4.6: Parameters in P and correlations coefficients

$Para$	G1	G2	G3
$\bar{\kappa_{b1}}$	1,0642	0,8297	0,6388
$\bar{\kappa_{b2}}$		0,0246	0,3076
$\bar{\kappa_{b3}}$			0,0219
$\bar{\gamma_{b1}}$	0,0089	0,0845	0,0623
$\bar{\gamma_{b2}}$		-0,0764	-0,0094
$\bar{\gamma_{b3}}$			-0,0425
ρ_{b12}		-0,6082	-0,8590
ρ_{b23}			-0,1269
ρ_{b13}			-0,0487
σ_{b2}		0,1179	0,1773
σ_{b3}			0,0088

Table 4.7: AIC : 2(loglikelihood- number of parameters)

model	first subsample	second subsample	whole sample
G1 : n=1	- 1074302469	- 2210349358	- 3243734019
G2 : n=2	- 1186913736	- 2385951466	- 3493641000
G3 : n=3	- 1265596084	- 2434200000	- 3656927524

4.6.1 Errors in cross section (whole sample analysis)

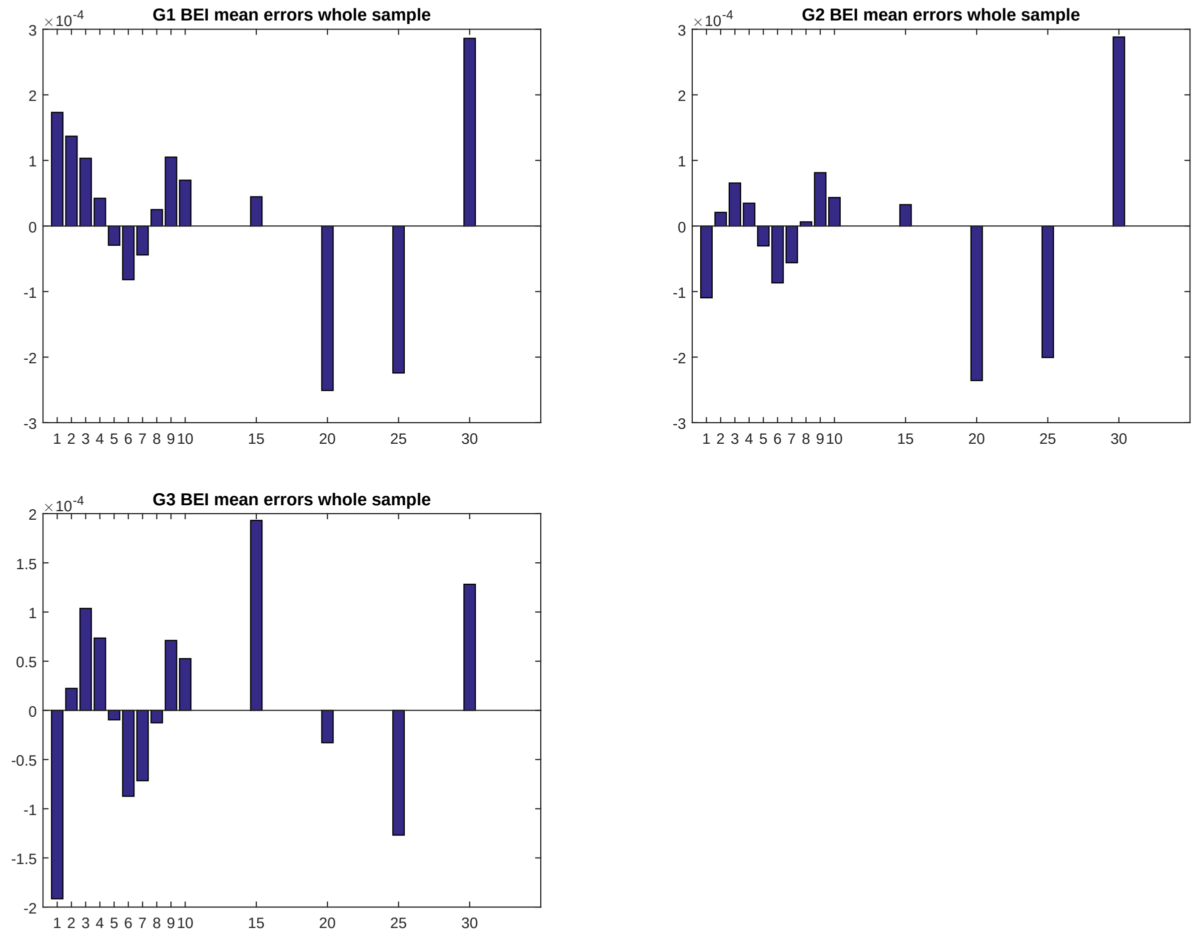

Figure 4.1: G1, G2, G3 mean error (cross section) y axis $10^{-}4$

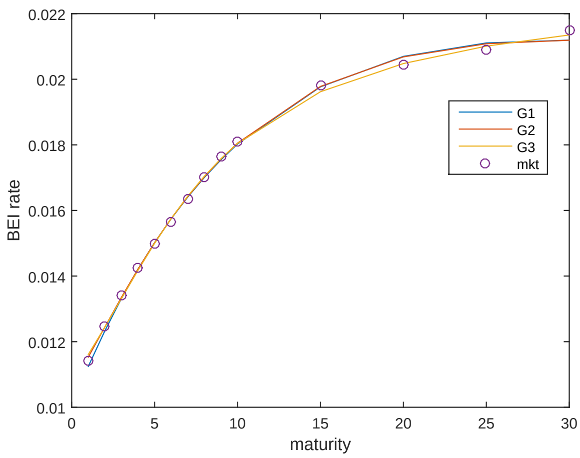

Figure 4.2: G1, G2, G3 model vs mkt (cross section mean of BEI rates vs cross section mean of model rates)

4.6.2 Errors in cross section (sub sample analysis)

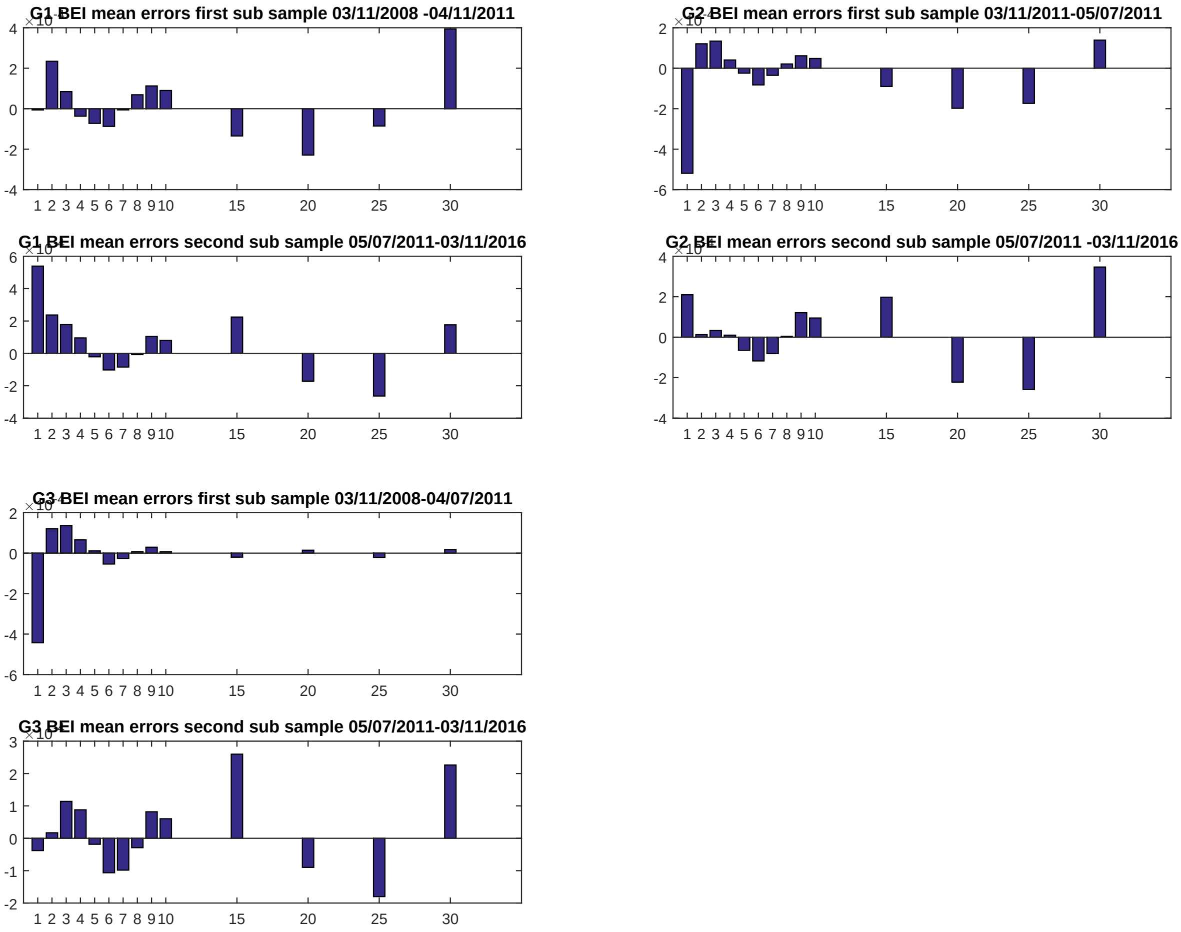

Figure 4.3: G1, G2, G3 mean error (cross section) sub sample y axis $10^{-}4$

4.6.3 Predicted BEI rate vs data whole sample analysis

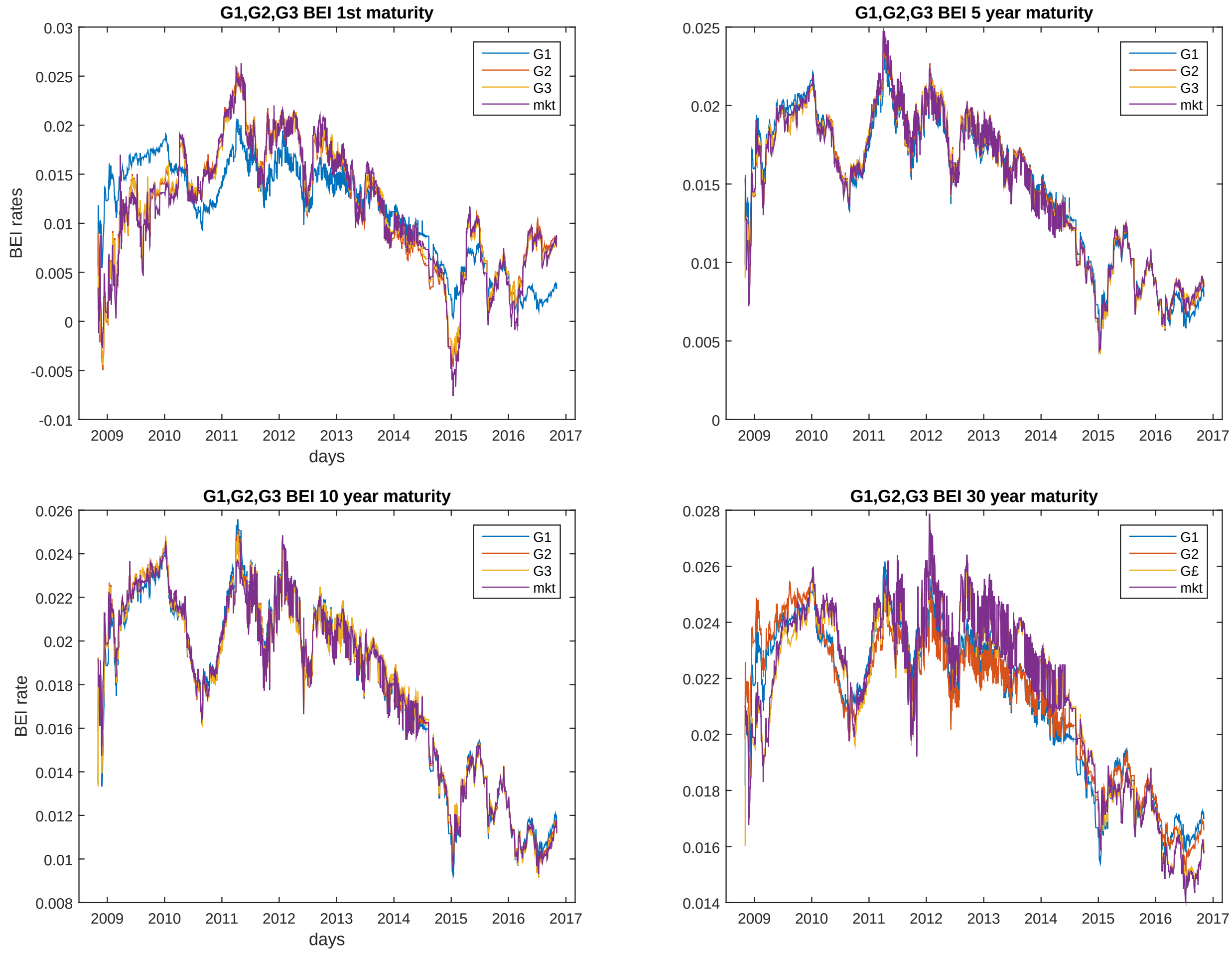

Figure 4.4: G1, G2, G3 predicted vs data (time series)

4.6.4 Errors autocorrelation (whole sample analysis)

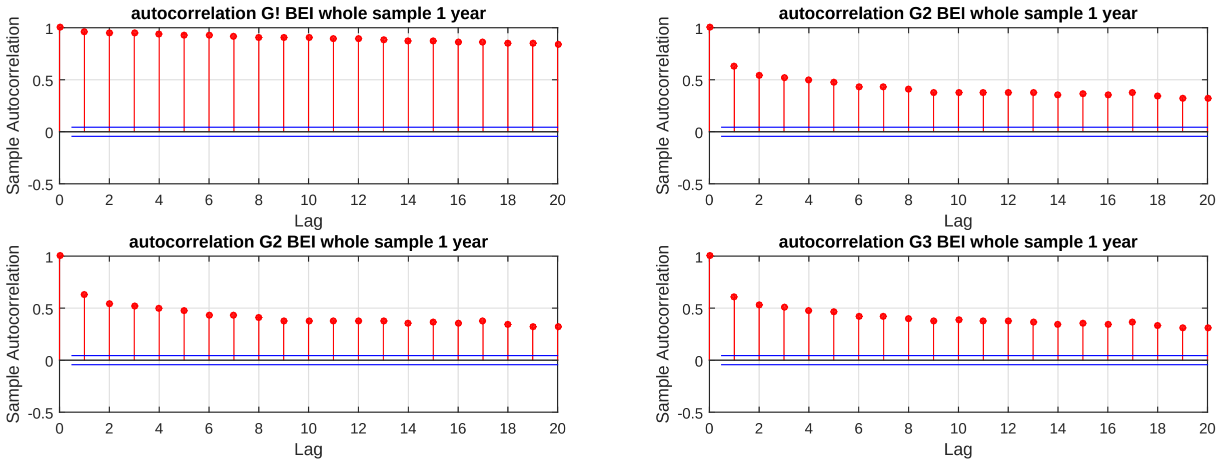

Figure 4.5: G1, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

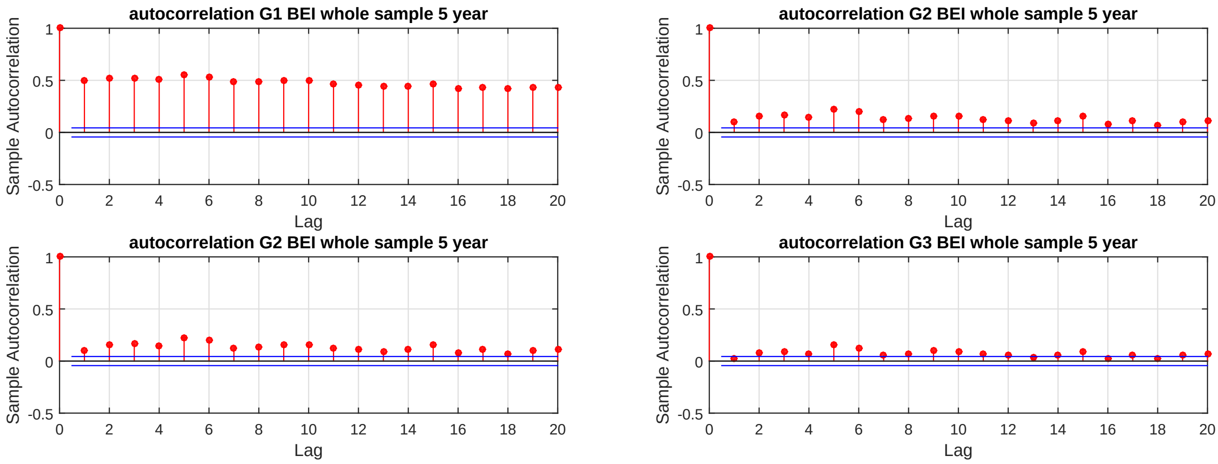

Figure 4.6: G2, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

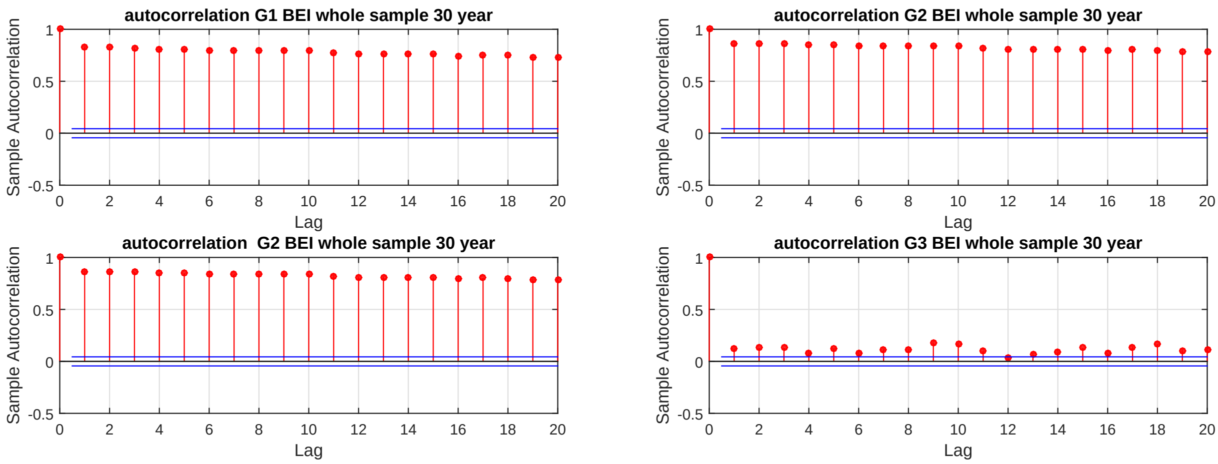

Figure 4.7: G3, autocorrelation 1y vs 5y (left) 5y vs 30y (right)

4.6.5 RMSE (whole sample analysis)

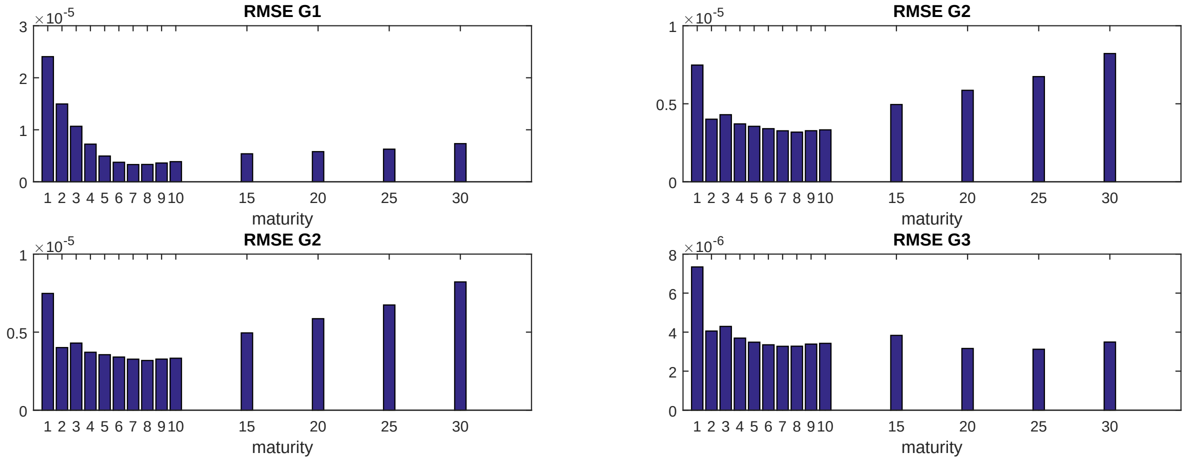

Figure 4.8: RMSE G1 G2 G3 cross section

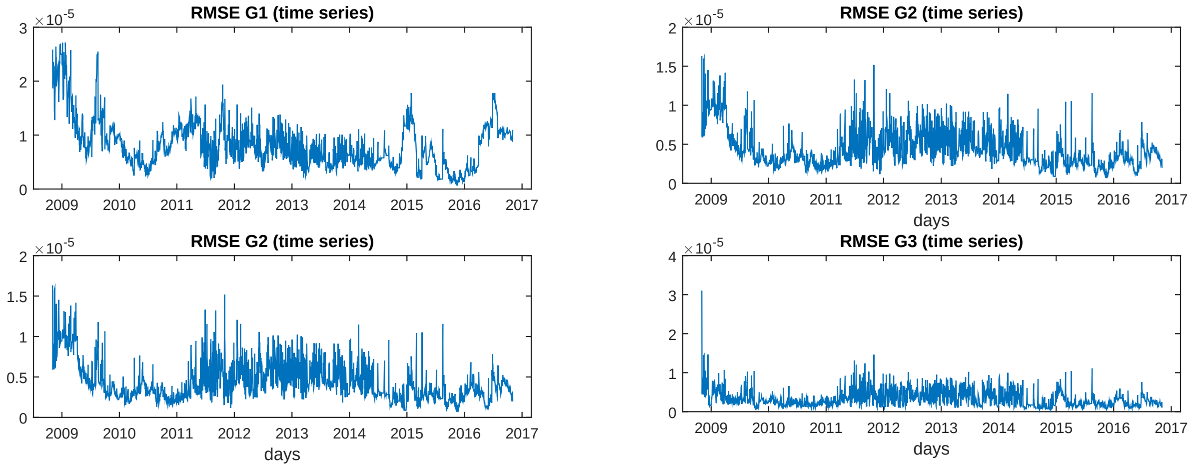

Figure 4.9: RMSE G1 G2 G3 time series

Chapter 5

Credit risk modeling with jointly spanned and unspanned interest rate and unspanned BEI rate: a Macro-Finance approach

5.1 Introduction

The negative effect of interest rate on default rates is well known in the literature as described in Duffee (1998, 1999), Longstaff and Schwarz (1995) and Collin Dufresne, Goldstein, Martin (2001). In all of these works, interest rates are considered spanned in the Term Structure of default rates. Nothing is said on how interest rates affect the future expectations of default rates through the default risk premia.

Neal et al. (2000) test the negative relation between treasury bills and default rates. They find that interest rates and default rates are negatively correlated in the short run and positively correlated in the long run.

Monetary policy affects the expectations that investors have on the future default rates of firms and Governments in Euro area see Gilchrist et al (2018).Indeed, it makes economic sense to analyze also how interest rates affect the expectations of default rates in the long run by relating interest rates with default risk premia.

Furthermore, defaults rates of firms and Governments are also affected by the rate of inflation.

Corporate and Government Bonds are overwhelmingly nominal. Indeed they suffer from the *debt deflation* effect when an unexpected drop of the inflation rate occurs. A low or negative inflation rate worsen the capacity of firms to pay back the debt Fisher (1933).

Knoop (2008) argues that when inflation rates are negative, the probability of default of firms and Governments rise up and spread all over the Economic System. From the failure of Lehman Brothers onwards, the effect of inflation rate on asset prices started to be analyzed.

In particular, a wide stream of literature is on inflation uncertainty and Government yields Viceira, Pflueger, (2011) and D'Amico and Orphanedes (2014) from the previous setting of Fama(1976), forecasting inflation expectations from TIPS, Kajuth and Watska, (2008) Wright(2008) and D'Amico et al. (2018).

The analysis of the effect of inflation rates on default rates is not so much considered and the analysis of the effect made by the rate of inflation of the Creditworthiness of firms and Governments is almost neglected.

David (2008) put the emphasis on the inflation uncertainty using a structural approach to model Credit Risk as described Merton (1974). He argues that the rate of inflation affects positively the Credit spread via the decrease of Sovency ratio.

Bhamra et al (2011) evaluates the effect of inflation on Credit Spread by analyzing the monetary policy put forward by FED.

It argues that the debt deflation proposed by Fisher does not have the same strength as during the Great Depression. Interest rates fell more rapidly than during the Great Recession than during the great Recession. This weakened the prociclical relation between inflation rates and output growth observed during the Great Depression. They conclude that the negative effect on Credit Risk of inflation rate is dampen during the Great Recession.

The paper written by Kang and Pflueger (2015) analyses the effect of the inflation rate on yields of Corporate Bonds, using a structural approach to evaluate Credit Risk as a previous development on the linkage between Credit risk and the macroeconomics made by Gourio (2012, 2013). The latter approach takes a DSGE model into account, analyzing the effect made by the macroeconomics fundamentals through a structural approach.

All of this papers share the same point: neglecting the effect of the rate of inflation on default rates can lead to an incomplete evaluation on the overall Credit Spread charged by firms and Governments.

The main goal of this paper is to analyze the effect of interest rates and inflation rates on the default rates of financial firms, industrial firms and the Italian Government.

The approach followed here is different from the one proposed in the related literature on Credit Risk and the macroeconomics. Instead of taking into account a structural approach to model relationship between default rates and the macroeconomy by following the work of Chen (2010), the default rate is modeled according to a reduced form approach where the default event is modeled as jump of a Poisson process.

This approach is more handy than a structural approach. It takes only the default rate into account but it doesn't allow to look at the effect made by the interest rates and the inflation rates on the capital structure of firms.

Furthermore, a reduced form approach to model Credit Risk which jointly takes into account the interest rate and the inflation rate is neglected in the literature.

The idea of this work is to extend the analysis made by Lucheroni and Pacati (2004) and by Mari and Renò (2005) as the development of the previous study of Duffie and Singleton (1999).

In those papers an affine correlation parameter between interest rates and instantaneous probability of default has been considered. This choice has been made to preserve the exponential affine functional form of the Survival probability with non Gaussian dynamics. Although, that approach gives reliable results about the effect made by interest rates on the Creditworthiness of firms and governments, it lacks to consider how inflation rates affects default rates of those entities. Furthermore, the effect made by interest rates on the default risk premium is completely neglected.

The aim of this chapter is:

- To introduce an *unspanned* (i.e. non perfectly correlated with) relation between BEI rates and default rates .
- To jointly identify how interest rates affect default rates in the short and in the long run.

The idea of the unspanned link between observed factors and rates has firstly discovered by Collon-Dufresne and Goldstein (2002) with the USV (unspanned stochastic volatility) and successively adapted to Macro Finance Term Structure models (MTSM) by Joslin at al. (2014) in dealing with a Term Structure model of interest rates with observable output gap and inflation rates by exploiting the previous work of Duffee (2011b).

As in the bond market, unspanned variables influence only the expectations of the future interest rates through the interest rate risk premium, leaving the price of the Bonds unchanged. Unspanned BEI rates influence only the future default rates through default risk premium. It does not appear in the Survival probability. This choice may have sense, as in the bootstrapping equation of the default rate, the BEI rate plays no role.

Furthermore, the double effect of interest rates on default rates allows us to jointly identify the effect of interest rates on the Survival probability and on the expectations of future default rates expressed by the rating grades given by rating agencies.

This work can be seen as an opportunity to merge the two stream of literature: the one on the Macro Finance Term structure modeling and the other one on the affine Term Structure of Credit Risk modeling. Such a mixing of the two streams of literature creates a tighter link between the Credit Risk and the macroeconomics.

5.2 The model

5.2.1 Risk neutral default intensity

Following Lando (1998), Duffie and Singleton (2003), we model a credit default event as the first jump of a Poisson process with stochastic intensity $\lambda(t)$. Under this assumption, at time t the survival probability at time T is given by

$$E_t\left[e^{-\int_t^T \lambda(u)du}\right]$$

We model the default intensity as the sum of two Gaussian factors: $\lambda(t) = \lambda_1(t)+\lambda_2(t)$. By setting $\Lambda(t) = (\lambda_1(t), \lambda_2(t))'$, we assume the following risk-neutral dynamics:

$$d\Lambda(t) = (\Gamma_\lambda - \Lambda(t) + Cr(t))dt + \Sigma dW^{\mathbb{Q}}(t) \ , \tag{5.1}$$

where $r(t)$ is the (nominal) risk free short rate and

$$A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \qquad \Gamma_\lambda = \begin{bmatrix} \gamma_\lambda \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_{\lambda_1} & \rho_\lambda \\ \rho_\lambda & \sigma_{\lambda_2} \end{bmatrix}$$

and $dW^{\mathbb{Q}}$ is a vector of correlated Wiener processes of this form:

$$dW^{\mathbb{Q}} = \begin{bmatrix} dW_1^{\mathbb{Q}}(t) \\ dW_2^{\mathbb{Q}}(t) \end{bmatrix}, \qquad corr(dW_1^{\mathbb{Q}}(t), dW_2^{\mathbb{Q}}(t)) = \rho_\lambda dt$$

The default rate is stochastic, therefore, the Credit Risk is modeled by a so-called *doubly stochastic intensity model* as described in Lando (1998) and Duffie and Singleton (2003). Following Ang and Piazzesi (2003) and Pericoli and Taboga (2008), the dynamics of default rates is a mixture of unobserved factors and observed factors.

5.2.2 Further assumptions

We make the following further assumptions:

Assumption (inflation rate)**.** *We use the BEI rate (Break Even inflation) as a proxy of inflation expectations. Although the two quantities may not coincide, the* *choice of the BEI rate as a proxy of inflation expectations is widely used among market practitioners (see Ciccarelli and Garcia (2010))*

Assumption (Interest rate and BEI rate)**.** *Both the short rate $r(t)$ and the BEI rate $b(t)$ are described by a three factor Gaussian mean reverting models (G3), as seen and estimated in chapters 3 and 4.*

Assumption (Interest rate and BEI rate 1)**.** *In the drift part of the default rate processes there are also the dynamics of $r(t)$ and $b(t)$ linked with the default rate through two affine dependence parameters as described in Lando (2004);*

Assumption (Interest rate and BEI rate 2)**.** *We consider interest rates and BEI rates as observable variables, as previously estimated alone. This approach, widely used in the Macro Finance literature see Joslin et all (2014), Rudebush et al. (2017), don't allow us to forecast the future interest rate and the future BEI rate through the current default rate. As the default rate is perceived as leading indicator of economic activity see Gilchrist et all (2012, 2018), it is unreasonable that a default rate of a firm can effect the shape of the Term Structure of European interest rates.*

Assumption (Independence between Interest rates and BEI rates)**.** *We assume that the Wieners driving the the interest rate dynamics are independent on the Wieners driving the BEI dynamics.*

Assumption (Indipendence between default rates and BEI rates)**.** *We let BEI rates to effect only the future $\mathbb{P}$-expected default rates and not the survival probability. This choice is consistent with the market boostrapping practice seen in chapter 1, where BEI rates play no role in the bootstrapping procedure from CDS spread.*

5.2.3 Zero-recovery bond prices

By Lando (1998), in this model, the price at time t of zero-recovery unit defaultable zero-coupon bond maturing in T is given by

$$P^*(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T[\lambda(u)+r(y)]du}\right] \quad ,$$

so that default intensity has the role of the short credit spread.

Assuming this price to be function of λ_1, λ_2, and of the factors x_1, x_2 and x_3

of the risk free short rate, it solves the boundary value problem (BVP):

$$\begin{cases} \dfrac{\partial P^*}{\partial t} + \displaystyle\sum_{i=1}^{2} \left(\alpha_i(\gamma_{\lambda_i} - \lambda_i + c_i(x_1 + x_2 + x_3)) \dfrac{\partial P^*}{\partial \lambda_i} + \dfrac{1}{2}\sigma_{\lambda_i}^2 \dfrac{\partial^2 P^*}{\partial \lambda_i^2} \right) \\ \qquad\qquad + \sigma_{\lambda_1}\sigma_{\lambda_2}\rho_\lambda \dfrac{\partial P^*}{\partial \lambda_1 \partial \lambda_2} \\ \qquad + \displaystyle\sum_{i=1}^{3} \left(\kappa_i(\gamma_i - x_i(t)) \dfrac{\partial P^*}{\partial x_i} \right) + \dfrac{1}{2} \displaystyle\sum_{i=1}^{3} \left(\sigma_i^2 \dfrac{\partial^2 P^*}{\partial x_i^2} \right) + \\ \dfrac{1}{2} \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} \left(\sigma_i \sigma_j \rho_{ij} \dfrac{\partial^2 P^*}{\partial x_i \partial x_j} \right) = (\lambda_1 + \lambda_2 + x_1 + x_2 + x_3)\, P^* \\ \qquad\qquad P^*(T, x_1, x_2, x_3, \lambda_1, \lambda_2) = 1 \end{cases} \tag{5.2}$$

(for the risk-free short rate we used the same notations as in chapter 3).

By standard arguments, the solution of this BVP is ($\tau = T - t$):

$$P^*(t,T) = \exp\left(A^d(\tau) - \sum_{i=1}^{3} B_i^d(\tau) x_i(t) - \sum_{i=1}^{2} C_i^d(\tau)\lambda_i(t) \right) \ , \tag{5.3}$$

where $A^d(\tau)$, $B_i^d(\tau)$ and $C_i^d(\tau)$ are deterministic functions, solutions of a system of ODEs obtained in the usual way from our PDE:

$$\begin{aligned} \frac{\partial A^d(\tau)}{\partial \tau} &= -B_1^d\,(\tau)\,\kappa_1\gamma_1 - C_1^d(\tau)\alpha_1\gamma_\lambda + \frac{1}{2}\sum_{i=1}^{2} \sigma_{\lambda_i}^2 C_i^d(\tau)^2 + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 B_i^d(\tau)^2 + \\ &\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sigma_i\sigma_j\rho_{ij} B_i^d(\tau) B_j^d(\tau) + \sigma_{\lambda_1}\sigma_{\lambda_2}\rho_\lambda C_1^d(\tau) C_2^d(\tau) \\ &\frac{\partial B_i^d(\tau)}{\partial \tau} = -B_i^d(\tau)\kappa_i + \alpha_1 c_1 C_1^d(\tau) + \alpha_2 c_2 C_2^d(\tau) + 1 \\ &\frac{\partial C_1^d(\tau)}{\partial \tau} = -C_1^d(\tau)\alpha_1 + 1 \\ &\frac{\partial C_2^d(\tau)}{\partial \tau} = -C_2^d(\tau)\alpha_2 + 1 \end{aligned} \tag{5.4}$$

$$\tag{5.5}$$

The zero-recovery risky rate at time t for maturity T is therefore

$$y^*(y,T) = -\frac{A^d(\tau)}{\tau} + \sum_{i=1}^{3} \frac{B_i^d(\tau)}{\tau} x_i(t) + \sum_{i=1}^{2} \frac{C_i^d(\tau)}{\tau}\lambda_i(t) \ , \sum_{i=1}^{2} \frac{C_i^d(\tau)}{\tau}\lambda_i(t) \ , \tag{5.6}$$

The credit spread of a zcb zero-recovery is

$$y^*(y,T) - y^r(y,T) = \left(\frac{A^r(\tau) - A^d(\tau)}{\tau}\right) + \left(\sum_{i=1}^{3} \frac{B_i^d(\tau)}{\tau} - \sum_{i=1}^{3} \frac{B_i^r(\tau)}{\tau}\right) x_i(t) + \sum_{i=1}^{2} \frac{C_i^d(\tau)}{\tau} \lambda_i(t) \ . \tag{5.7}$$

5.2.4 Market price of risk an $\mathbb{P}$-dynamics

The market price of risk is assumed to have the form:

$$\omega_{0i} + \omega_{1i}\lambda(t) + \omega_{bi}b(t) + \omega_{ri}r(t) \ ,$$

where $r(t)$ is the short rate dynamics and $b(t)$ is the instantaneous Break Even Inflation(BEI) dynamics.

Therefore, the natural drift is of the form:

$$\bar{\alpha_i}(\bar{\gamma_{\lambda_i}} - \lambda(t) + \bar{c_i}r(t) + \phi_i b(t)) = \alpha_i(\gamma_\lambda - \lambda(t) + c_i r(t)) + (\omega_{0i} + \omega_{1i}\lambda(t) + \omega_{bi}b(t) + \omega_{ri}r(t))\sigma_{\lambda_i}$$

where,

$$\begin{aligned} \bar{\alpha_i} &= \alpha_i - \omega_{1i}\sigma_{\lambda_i} \\ \bar{\gamma_{\lambda_i}} &= \frac{\alpha_i\gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}} \\ \phi_i &= \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}} \\ \bar{c_i} &= \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}} \end{aligned} \tag{5.8}$$

The dynamics of the default factors Λ under the natural probability measure $\mathbb{P}$ is of this form:

$$d\Lambda(t) = \bar{A}(\bar{\Gamma_\lambda} - \Lambda(t) + \bar{C}r(t) + \Phi b(t))dt + \Sigma dW^{\mathbb{P}}(t) \tag{5.9}$$

where,

$$\bar{A} = \begin{bmatrix} \bar{\alpha_1} & 0 \\ 0 & \bar{\alpha_2} \end{bmatrix} \qquad \bar{\Gamma_\lambda} = \begin{bmatrix} \bar{\gamma_{\lambda_1}} \\ \bar{\gamma_{\lambda_2}} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} \bar{c_1} \\ \bar{c_2} \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

and $dW^{\mathbb{P}}(t)$ is a vector of correlated Wiener processes of the form:

$$dW^{\mathbb{P}} = \begin{bmatrix} dW_1^{\mathbb{P}}(t) \\ dW_2^{\mathbb{P}}(t) \end{bmatrix}, \qquad corr(dW_1^{\mathbb{P}}(t), dW_2^{\mathbb{P}}(t)) = \rho_\lambda dt$$

For a time step $\Delta t = t + \Delta t - t$, the expected future default rate λ is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\lambda(t+\Delta t)|\mathcal{F}_t\right] = \bar{\gamma_{\lambda_1}}(1-\exp\left(-\bar{\alpha_1}\Delta t\right)) + \bar{\alpha_1}\bar{c_1}\int_t^{t+\Delta t}(\mathbb{E}^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right]\exp\left(-\bar{\alpha_1}\Delta s\right))ds \\
+ \bar{\alpha_1}\phi_1\int_t^{t+\Delta t}(\mathbb{E}^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right]\exp\left(-\bar{\alpha_1}\Delta s\right))ds \\
+ \bar{\gamma_{\lambda_2}}(1-\exp\left(-\bar{\alpha_2}\Delta t\right)) + \bar{\alpha_2}\bar{c_2}\int_t^{t+\Delta t}(\mathbb{E}^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right]\exp\left(-\bar{\alpha_2}\Delta t\right))ds \\
+ \bar{\alpha_2}\phi_2\int_t^{t+\Delta t}(\mathbb{E}^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right]\exp\left(-\bar{\alpha_2}\Delta s\right))ds + \\
\int_t^{t+\Delta t}(\exp\left(-\bar{\alpha_1}\Delta s\right)ds)\lambda_{1_{t-1}} + \int_t^{t+\Delta t}(\exp\left(-\bar{\alpha_2}\Delta s\right)ds)\lambda_{2_{t-1}}
\end{aligned} \tag{5.10}$$

where,

$$\mathbb{E}^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right] = \sum_{i=1}^{3}\mathbb{E}^{\mathbb{P}}\left[x_i(t)|\mathcal{F}_{t-1}\right] \tag{5.11}$$

and, denoting by $x_i^b(t)$ the ith factor of the BEI rate $b(t)$,

$$\mathbb{E}^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right] = \sum_{i=1}^{3}\mathbb{E}^{\mathbb{P}}\left[x_i^b(t)|\mathcal{F}_{t-1}\right] \tag{5.12}$$

are the posterior of the short rate and the BEI rate conditioned to time $t-1$. We consider the two rates as *observable* variables as they are previously filtered from ZCB yields and Inflation Swap respectively. The dynamics are of this form:

$$\mathbb{E}^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right] = \sum_{i=1}^{3}\bar{\gamma_i}(1-\exp\left(-\bar{\kappa_i}\Delta t\right)) + \exp(-\bar{\kappa_i}\Delta t)x_i(t-1) \tag{5.13}$$

and,

$$\mathbb{E}^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right] = \sum_{i=1}^{3}\bar{\gamma_i^b}(1-\exp\left(-\bar{\kappa_i^b}\Delta t\right)) + \exp(-\bar{\kappa_i^b}\Delta t)x_i^b(t-1) \tag{5.14}$$

As the short rate and the BEI rate are observable variables we can put them out from the integral.

Furthermore we have that:

$$\int_t^{t+\Delta t}\exp\left(-\bar{\alpha_i}\Delta s\right))ds = \frac{1-\exp\left(-\bar{\alpha_i}\Delta t\right))}{\bar{\alpha_i}} \tag{5.15}$$

After setting $\mu_i = \bar{\alpha_i}\left(\frac{1-\exp(-\bar{\alpha_i}\Delta t))}{\bar{\alpha_i}}\right)$, the expected default rate is:

$$\mathbb{E}^{\mathbb{P}}\left[\lambda(t+\Delta t)|\mathcal{F}_t\right] = \sum_{i=1}^{2}\left(\bar{\gamma_{\lambda_i}} + \bar{c_i}(\mathbb{E}^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right]) + \phi i(\mathbb{E}^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right])\right)\underbrace{\bar{\alpha_i}(\frac{1-\exp(-\bar{\alpha_i}\Delta t))}{\bar{\alpha_i}})}_{\mu_i}$$
$$+\sum_{i=1}^{2}) \underbrace{\exp(-\bar{\alpha_i}\Delta t)}_{1-\mu_i}\lambda_{t-1} \tag{5.16}$$

The variance of the default rate is:

$$\begin{aligned} Var^{\mathbb{P}}\left[\lambda(t)|\mathcal{F}_t\right] = &\sum_{i=1}^{2}\left(\bar{c_i}^2 Var^{\mathbb{P}}\left[r(t)|\mathcal{F}_{t-1}\right](1-\exp(-2\bar{\alpha_i}\Delta t))\right) \\ &+\sum_{i=1}^{2}\left(\phi_i^2 Var^{\mathbb{P}}\left[b(t)|\mathcal{F}_{t-1}\right](1-\exp(-2\bar{\alpha_i}\Delta t))\right) \\ &+\sum_{i=1}^{2}\left(\frac{\sigma_\lambda(i)^2}{2\bar{\alpha_i}}\right)(1-\exp(-2\bar{\alpha_i}\Delta t)) + \left(\frac{\sigma_{\lambda_1}\sigma_{\lambda_2}\rho_\lambda}{\bar{\alpha_1}+\bar{\alpha_2}}\right)(1-\exp(-(\bar{\alpha_1}+\bar{\alpha_2})\Delta t)) \end{aligned} \tag{5.17}$$

where ,

$$Var^{\mathbb{P}}\left[\sum_{i=1}^{3} x^N_{i(t+\Delta t)}|\mathcal{F}_t\right] = \sum_{i=1}^{3}\frac{(\sigma_i)^2}{2\bar{\kappa_i^N}}(1-\exp(-2\bar{\kappa_i^N}\Delta t))+2\sum_{i=1}^{3}\sum_{j=1}^{n}\frac{\sigma_i^N\sigma_j^N\rho_{ij}}{\bar{\kappa_i^N}+\bar{\kappa_j^N}}(1-\exp(-(\bar{\kappa_i^N}+\bar{\kappa_j^N})\Delta t)) \tag{5.18}$$

$$Var^{\mathbb{P}}\left[\sum_{i=1}^{3} x_{bi(t+\Delta t)}|\mathcal{F}_t\right] = \sum_{i=1}^{3}\frac{(\sigma_{bi})^2}{2\bar{\kappa_{bi}}}(1-\exp(-2\bar{\kappa_{bi}}\Delta t))+2\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\sigma_{bi}\sigma_{bj}\rho_{bij}}{\bar{\kappa_{bi}}+\bar{\kappa_{bj}}}(1-\exp(-(\bar{\kappa_{bi}}+\bar{\kappa_{bj}})\Delta t)) \tag{5.19}$$

are the variance of the short rate and the variance of the BEI rate respectively. and, for $\Delta t \to +\infty$

$$\lim_{\Delta\to+\infty}\mathbb{E}^{\mathbb{P}}\left[\lambda(+\infty)\right] = \sum_{i=1}^{2}\bar{\gamma_{\lambda_i}} + \sum_{i=1}^{3}(\bar{c_i}\bar{\gamma_i}) + \phi\sum_{i=1}^{3}\left(\phi_i\bar{\gamma_i^b}\right) \tag{5.20}$$

$$\lim_{\Delta\to+\infty} Var^{\mathbb{P}}\left[\lambda(+\infty)|\right] = \sum_{i=1}^{2}\left(\bar{c_i}^2 Var^{\mathbb{P}}\left[r(t)\right]\right) + \sum_{i=1}^{2}\left(\phi_i^2 Var^{\mathbb{P}}\left[b(t)\right]\right)$$
$$+ \sum_{i=1}^{2}\left(\frac{\sigma_{\lambda_i}^2}{2\bar{\alpha_i}}\right) + \left(\frac{\sigma_{\lambda_1}\sigma_{\lambda_2}\rho_\lambda}{\bar{\alpha_1}+\bar{\alpha_2}}\right) \tag{5.21}$$

The factors of the expected default rates are composed by three parts: the observed interest rates and BEI rates, and an not observed part. All of the parts describes the dynamics of the expected default rate in a composite way. The same holds for the variance.

5.3 Estimation procedure

In order to estimate the model, a Bayesian approach is used.

The Gaussian assumption of default rates and the affine relation between the zero-recovery risky rates and the factors allow us to use the Kalman Filter to estimate the model parameters.

This choice leads to have:

- Faster convergence towards the optimum than Sequential Montecarlo filters Doucet et al. (2001) and MCMC procedures such as the Gibbs sampling , Geman and Geman (1984) and the Metropolis Hasting algorithm Metropolis, Hasting (1953). All of these estimation procedures needed couple of days to be completed. Thus when lots of data sets are needed to carry out the analysis, the estimation procedure could be dramatically long;
- Less asymptotic biases than the use of Kalman Filter with non-Gaussian dynamics as in the analysis proposed by De Jong (2000) Chen and Scott (2003) Duan and Simonato (1999).

Considering a set of calendar times $t_1, t_2, \ldots, t_m$ and with constant time step $\Delta t = t_{k+1} - t_k$ for every $k = 1, 2, \ldots, m-1$ and (2×1) vector of latent variables $X(t)$ (default rate components), two vectors (3×1) of observed variables $X^N(t)$ (risk free short rate components), $X^b(t)$ (BEI rate components) $\forall k$ a vector $\zeta(t_k) = (\zeta_1(t_k), \zeta_2(t_k), \ldots, \zeta_p(t_k))'$ zero-recovery risky rates at fixed maturities $\tau_1, \tau_2, \ldots, \tau_p$, we have our state space model of the form:

(measurement equation) $\underbrace{z(t_k)}_{p\times 1} = \underbrace{A}_{p\times 1} + \underbrace{C}_{p\times 2}\underbrace{X(t_k)}_{2\times 1} + \underbrace{B}_{p\times 3}\underbrace{X^N(t_k)}_{3\times 1} + \underbrace{\eta(t_k)}_{p\times 1} \quad \eta(t_k) \sim \mathcal{IID}(0,\ R)$

$$\text{(Trasition equation) } \underbrace{X(t_k)}_{2\times 1} = \underbrace{\Theta}_{2\times 1} + \underbrace{F}_{2\times 2}\underbrace{X(t_{k-1})}_{2\times 1} + \underbrace{\varepsilon(t_k)}_{2\times 1} \quad \varepsilon(t_k) \sim \mathcal{IID}(0,\, Q)$$

where,

$$\Theta = \begin{bmatrix} \left(\bar{\gamma_1} + \bar{c_1}\sum_{i=1}^{3}\left(x^N(t_k|\mathcal{F}t_{k-1}\right) + \phi_1\sum_{i=1}^{3}\left(x^b(t_k|\mathcal{F}t_{k-1})\right)(1-\exp\left(-\bar{\alpha}\Delta t\right)\right) \\ \left(\bar{\gamma_2} + \bar{c_2}\sum_{i=1}^{3}\left(x^N(t_k|\mathcal{F}t_{k-1}\right) + \phi_2\sum_{i=1}^{3}\left(x^b(t_k|\mathcal{F}t_{k-1})\right)(1-\exp\left(-\bar{\alpha_1}\Delta t\right)\right) \end{bmatrix} \tag{5.22}$$

and

$$F = \begin{bmatrix} \exp\left(-\bar{\alpha_1}\Delta t\right) & 0 \\ 0 & \exp\left(-\bar{\alpha_2}\Delta t\right) \end{bmatrix} \qquad \varepsilon(t_k) = \begin{bmatrix} \varepsilon_1(t_k) \\ \varepsilon_2(t_k) \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_1 & \varrho_{12} \\ \varrho_{12} & \sigma_2 \end{bmatrix},$$

$$\sigma_i = \frac{(\sigma_{\lambda_i})^2}{2\bar{\alpha_i}}(1-\exp\left(-2\bar{\alpha_i}\Delta t\right) \tag{5.23}$$

$$\varrho_{ij} = \frac{\sigma_{\lambda_i}\sigma_{\lambda_j}\rho_\lambda}{\bar{\alpha_i}+\bar{\alpha_j}}(1-\exp\left(-\bar{\alpha_i}+\bar{\alpha_j})\Delta t\right)) \tag{5.24}$$

$$R = \begin{bmatrix} R_1 & 0 & \dots & 0 \\ 0 & R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & R_p \end{bmatrix}$$

is the covariance of measurement errors, where:

$$\forall r_{ii} \in R, \exists \quad diag(R) : r_{11} \neq r_{22} \neq \dots r_{nn}$$

Although heteroskedastic noises are also considered in the estimation of both the interest rate process and the BEI rate process, the only pricing imperfection considered in this model is the one which comes from the imperfect observation of the bootstrapped default rates through maturities. We assume them to vary across maturities.

This approach is in line with the model proposed by Joslin, Le and Singleton (2013). In particular the model presented here is the TS^f model where only the unobserved part the default rate is filtered and the observable variables are introduced without measurement errors.

The variance of the latent variables is equal to:

$$\mathbb{E}[(X(t_k) - \hat{X}(t_k))|\mathcal{F}t_k]^2 = P(t_k) = \psi V_t \psi + \psi_1 V_t \psi_1 + FVar[X(t_k)|\mathcal{F}t_{k-1}]F' + Q \tag{5.25}$$

where,

$$\psi = \begin{bmatrix} (1 - \exp{(-\bar{\alpha_1}\Delta t)})\bar{c_1} & 0 \\ 0 & (1 - \exp{(-\bar{\alpha_1}\Delta t)}))\phi_1 \end{bmatrix}$$

$$\psi_1 = \begin{bmatrix} (1 - \exp{(-\bar{\alpha_2}\Delta t)})\bar{c_2} & 0 \\ 0 & (1 - \exp{(-\bar{\alpha_2}\Delta t)})\phi_2 \end{bmatrix}$$

and

$$V_t = \begin{bmatrix} Var\left[r(t)|\mathcal{F}_{t-1}\right] & 0 \\ 0 & Var\left[b(t)|\mathcal{F}_{t-1}\right] \end{bmatrix}$$

The variance of the measurement is:

$$S(t_k) = CP(t_k)C' + \sum_{i=1}^{3} \left(B_i Var[x_i^N(t_k)^N|\mathcal{F}t_{k-1}]B_i'\right) + R \tag{5.26}$$

The log likelihood function is:

$$\begin{aligned} \log \mathcal{L} &= \sum_{k=1}^{m} \left[-\frac{p}{2}\log(2\pi) - \frac{1}{2}\log\det S(t_k) - \frac{1}{2}\eta S^{-1}(t_k)\eta'(t_k) \right] \\ &= -\frac{mp}{2}\log(2\pi) - \frac{1}{2}\sum_{k=1}^{m}\log\det S(t_k) - \frac{1}{2}\sum_{k=1}^{m}\eta(t_k)S^{-1}(t_k)\eta'(t_k). \end{aligned} \tag{5.27}$$

5.3.1 Optimization procedure

To minimize the log likelihood function we use the Matlab function *Fmincon*.

Given $\theta = (\alpha_1, \gamma_\lambda, \sigma_{\lambda_1}, \omega_{01}, \omega_{11}, \omega_{b1}, \omega_{r1}, \alpha_2, \sigma_{\lambda_2}, \omega_{02}, \omega_{12}, \omega_{b2}, \omega_{r2}, c_1, c_2, \rho_\lambda, R)$ the vector of model's parameter and

$\forall$ latent variables $n = 1, 2$ and $\forall$ maturities h=1,...,p, the optimization problem is of the form:

$$-\min_{\theta} \log \mathcal{L} \tag{5.28}$$

$$\text{u.c.} \begin{cases} 0 \leq \alpha_i & \leq +\infty & \forall i = 1, \ldots, 2 \\ 0 < \gamma_\lambda & \leq +\infty & \forall i = 1 \\ -1 \leq \omega_{0i} & \leq +1 & \forall i = 1, \ldots, 2 \\ -1 \leq \omega_{1i} & \leq +1 & \forall i = 1, \ldots, 2 \\ -1 \leq \omega_{bi} & \leq +1 & \forall i = 1, \ldots, 2 \\ -1 \leq \omega_{ri} & \leq +1 & \forall i = 1, \ldots, 2 \\ -\infty \leq c_i & \leq +\infty & \forall i = 1, \ldots, 2 \\ \alpha_i - \omega_{1i}\sigma_{\lambda_i} & > 0 & \forall i = 1, \ldots, 2 \\ 0 < \sigma_{\lambda_i} & \leq +\infty & \forall i = 1, \ldots, 2 \\ -1 \leq \rho_\lambda & \leq +1 & \forall i = 1 \\ 0 \leq r_h & \leq +\infty & \forall h = 1, \ldots, p \end{cases} \tag{5.29}$$

We do not set all price of risk parameters, ω_{0i}, ω_{1i}, ω_{bi} ω_{ri}, free from constraints to have the maximum degree of flexibility.

The unconstrained price of risk parameters make the non arbitrage restriction imposed by Duffie and Kan affine models less important (see Duffee (2011a, 2012) and Joslin et al. (2011)). The model is indeed nearly an unconstrained factor VAR as described in Joslin et al. (2013). The shape of instantaneous default rates is mostly driven by the price of risk parameters.

Clearly, by exploiting the property of the affine price of risk we can have a negative mean reversion parameter in the natural probability measure $\mathbb{P}$ when $\omega_{1i} < \frac{\alpha_i}{\sigma_{\lambda_i}}$.

The natural mean reversion parameter $\bar{\alpha}_i$ has to be always positive. The dynamics of the default rate is always determinate as $\exp(-\bar{\alpha}_i \Delta t)$ is always smaller than 1. As default rates are always positive we set the risk natural long term mean γ_λ strictly positive.

We set, the affine correlation parameters between interest rates and default rates in $\mathbb{Q}$, c_i, free from constraints. As for the price of risk parameters, we allow the model to have the maximum degree of flexibility.

5.4 Comments and Conclusions

Equation 5.8 shows that the effect both of interest rates and BEI rates on default rates under $\mathbb{P}$ is significant when ω_{1i}, ω_{bi} and ω_{ri} are statistically significant.

In particular, when ω_{ri} and ω_{1i} are both not statistically significant, the effect made by interest rates on default rates under $\mathbb{Q}$ and under $\mathbb{P}$ is equal. This holds also for BEI rates when ω_{bi} and ω_{1i} are both not statistically significant.

5.4.1 Whole sample

In the whole sample we see higher degree of significance of price of risk parameters with respect the subsample analysis for financial firms, see 5.1 .

ω_{1i} are all negative and statistically significant for almost all firms analyzed. In particular, the 5.7 and 5.8 shows:

- financial firms show a higher difference between the two risk natural mean reversion speed than industrial firms. In particular the mean reversion speed of the second factor is always greater than the mean reversion speed of the first factor. This may suggest that the first factor affects more default risk by long run expectation in the price of risk and the second factor affects more the expected future default rates from the short run prospective;

- For industrial firms we see a smaller difference of risk neutral mean reversion speed between factors (table 5.2).

Financial firms and Republic of Italy, show also a higher degree of significance of ω_{bi} and ω_{ri} than industrial firms.

The two affine correlation parameters between interest rates and default rates c_1 and c_2 are negative and statistically significant. Interest rates make a significant net negative effect on default rates in $\mathbb{Q}$. This effect is stronger for financial firms and Republic of Italy.

Furthermore, financial firms and Republic of Italy are negatively affected by interest rates under $\mathbb{P}$. The contrary holds for industrial firms. BEI rates make a significant and negative effect on default rates of almost all firms analyzed. This effect is stronger for financial firms.

5.4.2 First subsample

In the first sub sample,5.3 and 5.4 we see that ω_{0i} are not significant for some firms in the data set. The risk natural long term mean $\bar{\gamma_{\lambda_i}}$ are not influence by the price of risk. We cannot say anything about the long term mean of default rates in the risk natural World $\mathbb{P}$.

Furthermore we see for all firms analyzed a negative and significant value of ω_{1i}. The mean reversion speed under the risk natural probability measure $\mathbb{P}$ is faster than the mean reversion speed in risk neutral probability measure $\mathbb{Q}$.

The risk neutral long term mean parameter γ_λ is close to 0 and not significant for all firms considered. This may depend on the affine correlation parameter c introduced in the dynamic $\mathbb{Q}$ of the default rate process.

The effect made by interest rates in $\mathbb{Q}$ is significant for almost all the firms considered. The net effect is negative for every firms analyzed. In particular, the effect of interest rates is more negative for financial firms and for Italy.

In almost all firms considered the effect made by interest rates on the expectation of future default rates under $\mathbb{P}$ is negative. For almost all firms analyzed the price of risk parameters ω_{ri} are significant and the net effect is positive. Tables 5.9 and 5.10 show that $\bar{c}_i$ are ,on average, less negative than the affine correlation parameters in $\mathbb{Q}$.

The effect of BEI rates on the default rates is negative for all the firms taken into consideration. This effect is stronger and more statistically significant for financial firms.

Investors asks for a more risk premium for bearing the default rate as the BEI goes down. As the BEI is a proxy of inflation expectation, the latter relation is in line with the debt deflation hypothesis.

5.4.3 Second subsample

As show in tables table 5.5 and 5.6 the parameters ω_{1i} are negative. The risk natural mean reversion speeds are higher than risk neutral speeds.

Financial firms show a higher risk natural mean reversion speed than industrial firms but lower statistically significant ω_{1i}. As the mean reversion parameter gets higher, the effect made by current default rate on the expected future default rate both in $\mathbb{P}$ and in $\mathbb{Q}$ gets lower. The expected default rates depends more on the information provided the long run mean of default rates.

Furthermore, results show a lower ω_{bi} in absolute value, when ω_{1i} are not significant. This result, shown by financial firms, may explain the smaller effect made by BEI rates on expected future default rates under $\mathbb{P}$ of financial firms. This result holds also for Republic of Italy.The contrary happens for industrial firms.

As in the first subsample, the effect made by interest rates on default rates in $\mathbb{P}$ is not equal to the one made in $\mathbb{Q}$ (see tables 5.11 and 5.12) . Interest rates make for most of the firms a negative and significant effect on default rates in $\mathbb{Q}$. This effect in $\mathbb{P}$ is more negative for financial firms and positive for industrial firms.

The effect made by BEI rates on default rates is negative and statistically significant for financial firms. BEI rates do not affect default rates of industrial firms and Republic of Italy.

The value of the volatility parameters, σ_{λ_1} and σ_{λ_2} are higher than in the first subsample. In particolar, industrial firms show higher and significant values of volatility parameters than financial firms and Republic of Italy.

A higher volatility parameters in the model estimated here, lead us to have negative default rates after 2013. This effect is stronger for industrial firms. Although, there is a relevant evidence of negative default rates estimated by the model, the value of estimated default rates hardly go beyond 10 b.p. (basis points) for industrial firms. The evidence of negative rates, is less relevant for financial firms (see figures5.7 and **??**). This value is consistent with the bid-ask spread of CDS that can be found in the market.

Furthermore, as shown by figures 5.15 and 5.16, default rates are higher than in the first subsample at longer maturities. Furthermore, financial firms show default rates higher than in the first subsample for all maturities analyzed. This holds also for Republic of Italy.

On the contrary, the default rate of industrial firms decreases at shorter maturities and increases at longer ones.

5.4.4 Errors

By estimating the model with the data we obtain model's errors given by:

$$\mathcal{E} = y_{data} - \widehat{y} \tag{5.30}$$

where y_{data} are the bootstrapped from CDS default rates and $\widehat{y}$ are the model's default rates.

By looking at figures 5.1 and 5.2 we can say that:

- Both in the whole sample analysis and in the sub sample analysis, errors are concentrated in the first maturity. This effect is stronger for industrial firms.
- In the first sample the mean of errors is smaller than in the second sample. This evidence is stronger for financial firms;
- In the second sub sample, errors are concentrated more in the first maturity as shown by 5.3 5.4 5.5. This evidence is less strong in the first subsample. In particular, industrial firms show a higher error concentration in the first maturity than financial firms and Republic of Italy;
- In times series we see that the model show a good fitting in all maturities except the first one. The model show a better fitting in the first maturity with financial firms. Figures 5.7–5.14 show a worse model fitting in the first maturity after 2013. This evidence is stronger for industrial firms;

- Looking at (tables 5.14–5.19) and table5.13, the volatility of errors are concetrated in the first maturity. This evidence is stronger in the second subsample after the bursting of the sovereign debt crisis;
- Financial firms show a lower RMSE than industrial firms in whole sample.

Furthermore, errors in the whole sample also show that in cross section the model underestimate the data (positive errors) at the first maturity and overestimate the data in the last two maturities. This result is shown by the mean error in cross section of financial firms and industrial firms. On the contrary, Republic of Italy overestimates the data both at the first maturity and in the last two maturities.

This evidence holds also in subsample analysis. It is stronger in the second subsample than in the first one.

5.4.5 Error Autocorrelation

As shown by 5.17 5.18 5.22 and 5.21 models errors are strongly correlated at the first maturity. From the 5th maturity onward the autocorrelation decreases.

Error autocorrelation of financial firms disappears at the $5th$ maturity onwards. In particular, this evidence is stronger for Unicredit and Ubi. On the contrary industrial firms show a decreasing but persistent error autocorrelation also after the $5th$ maturity.

Italy, also shows a decreasing error autocorrelation. In particular, like financial firms it shows, non autocorrelated errors from the $5th$ maturity onwards.

5.4.6 Conclusion

In this work we estimate a bivariate Gaussian default rate model. We assume an unspanned relation between BEI rates and default rates. BEI rates effects only the expectation of future default rate in $\mathbb{P}$ through the default risk premium. Furthermore we also allow interest rates to have a double effect on default rates: one in $\mathbb{Q}$ and another one in the expectation of future default rates in $\mathbb{P}$. We find a significant relation between interest rates and default rates both in $\mathbb{P}$ and in $\mathbb{Q}$. We find the same for BEI rates.

In particular, results on the correlation between interest rates and default rates both in $\mathbb{P}$ and in $\mathbb{Q}$ confirm the state of literature (see Duffee (1998,1999)). The recent period of Crisis did not change the relation between the two variables. After the sovereign debt crisis such a relation is stronger than in the period after the Lehman failure.

The debt deflation do affect future default rates. This is more evident for financial firms. Nevertheless, results obtained here are only in sample results. No out of sample analysis has been performed. Clearly, this weakens the analysis of the default rate performed here. We leave this topic for a further research.

5.4.7 Risk neutral and price of risk parameters of default rate financial firms industrial firms and Republic of Italy

Table 5.1: Default rate financial firms whole sample

Para	Generali	UBI	Unicredit	Mediobanca
α_1	0,0101***	0,0086**	0,0147***	0,0140**
S.E	0,0004	0,0048	0,0001	0,0056
γ_λ	3,1E-06	1E-07	1,9E-07	2,4E-07
S.E	0,0098	0,0328	0,0009	0,0274
σ_{λ_1}	0,0155***	0,0162***	0,0194***	0,0177***
S.E	0,0001	0,0006	0,0001	0,0013
ω_{01}	0,0498***	0,0560***	0,0557***	0,0891***
S.E	0,0024	0,0633	0,0045	0,0199
ω_{11}	-0,997***	-0,999**	-0,997***	-0,997*
S.E	0,0434	0,4619	0,0213	0,5472
ω_{b1}	-0,030***	-0,359***	-0,001	-0,016
S.E	0,0048	0,2427	0,0015	0,0347
ω_{r1}	0,3569***	0,9997***	0,1802***	0,0110
S.E	0,0698	0,3481	0,0267	0,1437
c_1	-3,191***	-3,533*	-4,619***	0,8876***
S.E	0,0683	0,0756	0,0216	0,2577
c_2	-0,315***	-0,345*	-0,014**	-0,995***
S.E	0,0096	0,1917	0,0055	0,1920
α_2	0,3724***	0,6056***	0,5423***	0,3323***
S.E	0,0039	0,0111	0,0027	0,0480
σ_{λ_2}	0,0233***	0,0214***	0,0243***	0,0188
S.E	0,0011	0,0004	0,0007	0,0151
ω_{02}	-0,395***	-0,792***	-0,501***	-0,690
S.E	0,0232	0,0941	0,0144	0,9453
ω_{12}	-0,983***	-0,999***	-0,976***	-0,987
S.E	0,0716	1,6041	0,0314	1,6845
ω_{b2}	-0,978***	-0,999***	-0,044***	-0,193
S.E	0,0127	0,0768	0,0058	0,4139
ω_{r2}	0,0544***	-0,193***	-0,007**	0,0034
S.E	0,0113	0,1676	0,0034	0,0382
ρ_λ	-0,165***	0,0291***	-0,135***	-0,254
S.E	0,0259	0,0571	0,0046	0,2409
Loglike	-121867	-119034	-121231	-119366

* significance at 10%
** significance at 5%
*** significance at 1%

Table 5.2: Default rate industrial firms and Republic of Italy whole sample

Para	Italy	ENI	ENEL	Telecom
α_1	0,1335***	0,1680***	0,1330***	0,1787***
S.E	0,0025	0,0066	0,0022	0,0018
γ_λ	2,4E-06	1,3E-07	1,6E-07	1,3E-07
S.E	0,0018	0,0064	0,0007	0,0118
σ_{λ_1}	0,0547***	0,5958*	0,0949***	3,7989***
S.E	0,0021	0,0275	0,0053	0,0154
ω_{01}	0,1791***	0,9117***	0,4252***	0,9754
S.E	0,0161	0,3848	0,0676	0,7564
ω_{11}	-0,997***	-0,316***	-0,512***	-0,045***
S.E	0,0968	0,0144	0,1355	0,0006
ω_{b1}	-0,273***	-0,604***	-0,996***	-0,257
S.E	0,0262	0,1715	0,1139	0,2274
ω_{r1}	0,9668***	0,9871**	0,9904***	0,7604
S.E	0,0840	0,5026	0,1608	1,0137
c_1	-1,827***	-6,804***	2,6047***	77,945***
S.E	0,0231	0,1622	0,0693	0,3200
c_2	0,6406***	6,2911***	-2,858***	-78,30***
S.E	0,0263	0,1602	0,0691	0,2788
α_2	0,4336***	0,1739***	0,2136***	0,1811***
S.E	0,0126	0,0079	0,0031	0,0018
σ_{λ_2}	0,0461***	0,5927***	0,0911***	3,7961***
S.E	0,0035	0,0259	0,0049	0,0171
ω_{02}	-0,498***	-0,906***	-0,697***	-0,976
S.E	0,0636	0,3794	0,1042	0,7641
ω_{12}	-0,990***	-0,307***	-0,998***	-0,045***
S.E	0,1561	0,0174	0,0572	0,0002
ω_{b2}	-0,971***	-0,985***	-0,996***	-0,721
S.E	0,0794	1,6483	0,0681	0,4323
ω_{r2}	0,9846***	-0,039	0,9919***	0,5375
S.E	0,2187	0,0854	0,1411	0,4403
ρ_λ	-0,892***	-0,999***	-0,950***	-0,999***
S.E	0,0141	2,7E-05	0,0081	3,8E-05
Loglike	-119130	-127095	-114556	-111487

* significance at 10%

** significance at 5%

*** significance at 1%

Table 5.3: Default rate financial firms first subsample

Para	Generali	UBI	Unicredit	Mediobanca
α_1	0,0251***	0,0456***	0,0292***	0,0172
S.E	0,0004	0,0068	0,0009	0,0215
γ_λ	3,5E-07	1E-07	1,9E-07	1E-07
S.E	0,0002	0,0130	0,0036	0,0609
σ_{λ_1}	0,0187***	0,0192***	0,0212***	0,0140
S.E	0,0004	0,0012	0,0005	0,0394
ω_{01}	0,0165	0,0619	0,0015	0,0196
S.E	0,0274	0,1258	0,0529	0,3771
ω_{11}	-0,983***	-0,999	-0,999***	-0,999
S.E	0,1736	2,5185	0,0765	27,017
ω_{b1}	-0,167	0,0450	0,0165*	-0,017
S.E	0,1763	0,1389	0,0094	0,0903
ω_{r1}	0,8342***	0,9996	0,9910***	0,9993
S.E	0,2073	2,7937	0,0562	2,1777
c_1	-2,303***	-1,420***	-2,762***	-3,294
S.E	0,0535	0,1123	0,0603	6,1841
c_2	0,1431***	0,1582***	0,3605***	0,1220
S.E	0,0022	0,0652	0,0076	0,9483
α_2	0,5150***	0,5862***	0,3706***	0,7893
S.E	0,0088	0,0506	0,0041	1,0493
σ_{λ_2}	0,0268***	0,0175***	0,0233***	0,0109
S.E	0,0005	0,0049	0,0002	0,0248
ω_{02}	0,3031***	0,5042	0,4453	0,9828
S.E	0,0634	0,5225	0,4095	8,2851
ω_{12}	-0,917***	-0,403	-0,995	-0,000
S.E	0,2714	1,4423	0,3227	1,2541
ω_{b2}	-0,890***	-0,999***	-0,993**	-0,999
S.E	0,1648	0,3440	0,4679	8,8640
ω_{r2}	0,0044***	0,1928	-0,327***	0,2391
S.E	0,2918	0,1996	0,0951	5,0537
ρ_λ	-0,576***	-0,575***	-0,427***	-0,336
S.E	0,0088	0,0453	0,0112	0,8435
Loglike	-41608,3	-42386	-43470,8	-43060,6

* significance at 10%
** significance at 5%
*** significance at 1%

Table 5.4: Default rate first subsample industrial firms and Republic of Italy

Para	Italy	ENI	ENEL	Telecom
α_1	0,0210***	0,0328***	0,0392	0,0445**
S.E	0,0005	0,0047	0,0669	0,0224
γ_λ	1,2E-07	1,9E-07	1E-07	1,2E-07
S.E	0,0050	0,0318	0,0149	0,1000
σ_{λ_1}	0,0253***	0,0106***	0,0328***	0,0303***
S.E	0,0004	0,0008	0,0130	0,0020
ω_{01}	-0,038	-0,008	0,0862	0,3706***
S.E	0,0469	0,1392	1,1333	0,1539
ω_{11}	-0,997***	-0,998	-0,999	-0,999*
S.E	0,1668	1,7596	4,4682	0,5921
ω_{b1}	-0,000	0,0979	-0,999	-0,087
S.E	0,0094	1,4059	5,4857	0,1888
ω_{r1}	0,9503**	0,9835	0,9989	0,0032
S.E	0,4596	2,2942	10,353	0,0951
c_1	-6,859***	-1,765*	-2,859	2,2601***
S.E	0,1073	0,1356	3,9588	1,1510
c_2	1,0234***	0,2495*	0,1692*	-1,249***
S.E	0,0115	0,1027	0,0800	0,0202
α_2	0,1911***	0,3436***	0,3666***	0,5755***
S.E	0,0017	0,0260	0,1623	0,0463
σ_{λ_2}	0,0162***	0,0111***	0,0306***	0,0626***
S.E	0,0002	0,0061	0,0396	0,0042
ω_{02}	0,4507***	0,7000	-0,130	-0,999*
S.E	0,1415	0,7743	0,5219	0,6149
ω_{12}	-0,989***	-0,007	-0,999	-0,999
S.E	0,0590	1,2303	3,3751	1,5509
ω_{b2}	0,4229***	-0,996	-0,999	-0,998
S.E	0,1758	1,7504	3,6236	2,4236
ω_{r2}	0,9320***	-0,991	0,9994	-0,034
S.E	0,2974	1,5098	7,8592	0,1007
ρ_λ	-0,539***	-0,581***	-0,386***	-0,325***
S.E	0,0070	0,0559	0,1372	0,1098
Loglike	-43108	-45094,8	-38836,3	-39216,1

* significance at 10%
** significance at 5%
*** significance at 1%

Table 5.5: Default rate second subsample financial firms

Para	Generali	UBI	Unicredit	Mediobanca
α_1	2,1E-05	1,7E-05	0,0099*	1E-05
S.E	5,1E-05	0,0023	0,0003	0,0005
γ_λ	6,1E-07	0,0232***	7,8E-07	0,0413
S.E	0,0267	0,0018	0,0093	0,0830
σ_{λ_1}	0,0159***	0,0177***	0,0218***	0,0169***
S.E	0,0001	0,0002	0,0002	0,0005
ω_{01}	0,0334	0,0543***	0,0573	0,0515
S.E	0,0459	0,0284	0,0708	0,6474
ω_{11}	-0,999***	-0,998***	-0,950***	-0,999
S.E	0,1764	0,0348	0,1376	0,9071
ω_{b1}	-0,996***	-0,976***	-0,050**	0,0170
S.E	0,1600	0,1602	0,0258	0,2057
ω_{r1}	0,8135***	-0,017***	0,0264	0,0275
S.E	0,1604	0,0117	0,0448	1,2392
c_1	-0,2709***	0,6961***	-3,687***	-2,904***
S.E	0,0003	0,0749	0,1346	0,8451
c_2	-0,263***	-0,889***	-0,404***	-1,137***
S.E	0,0102	0,0114	0,0162	0,0296
α_2	0,3649***	0,4524***	0,5213***	0,3037***
S.E	0,0041	0,0056	0,0082	0,0083
σ_{λ_2}	0,0160***	0,0210***	0,0265***	0,0196***
S.E	0,0007	0,0006	0,0009	0,0048
ω_{02}	-0,547***	-0,742***	-0,534***	-0,593***
S.E	0,2167	0,0756	0,1668	3,5351
ω_{12}	-0,998*	-0,983***	-0,722***	-0,999
S.E	0,5723	0,0950	0,1642	4,5165
ω_{b2}	-0,639***	-0,562***	-0,455***	0,0349
S.E	0,2098	0,0739	0,1375	0,3896
ω_{r2}	0,9919***	0,9398***	0,0954	0,1045
S.E	0,1397	0,2088	0,1635	1,8748
ρ_λ	-0,030	-0,107***	-0,140***	-0,098
S.E	0,0361	0,0159	0,0364	0,1672
Loglike	-81956	-80622,8	-79575,7	-79031,6

* significance at 10%
** significance at 5%
*** significance at 1%

Table 5.6: Default rate second subsample industrial firms and Republic of Italy

Para	Italy	ENI	ENEL	Telecom
α_1	0,1173	0,1529***	0,1604***	0,1890***
S.E	0,1854	0,0364	0,0009	0,0269
γ_λ	1,6E-06	9,1E-06	1,7E-06	2,9E-06
S.E	0,2322	0,0350	0,0030	0,2635
σ_{λ_1}	0,0491	0,2892***	0,9793***	1,1107***
S.E	0,1455	0,0094	0,0037	0,2642
ω_{01}	0,1528	0,8864	0,9698***	0,9770
S.E	2,1803	0,8667	0,1794	1,7271
ω_{11}	-0,996	-0,963***	-0,316***	-0,258***
S.E	7,3495	0,1152	0,0042	0,0563
ω_{b1}	-0,049	0,1105	-0,558***	-0,839
S.E	1,8708	1,0517	0,0755	3,4539
ω_{r1}	0,9838	0,9463***	0,9990***	0,9674
S.E	12,771	0,4096	0,0162	0,6050
c_1	-1,404***	-16,96***	-12,81***	-39,07***
S.E	0,4791	0,3882	0,0432	1,0106
c_2	0,2345	15,515***	11,643***	37,382***
S.E	1,8583	0,3225	0,0180	1,8255
α_2	0,4630*	0,1660***	0,1670***	0,1984***
S.E	0,2584	0,0411	0,0006	0,0368
σ_{λ_2}	0,0493	0,2858***	0,9779***	1,1085***
S.E	0,0782	0,0101	0,0065	0,2984
ω_{02}	-0,387	-0,824	-0,952***	-0,960
S.E	3,7438	0,6165	0,1760	1,6477
ω_{12}	-0,992	-0,820***	-0,303***	-0,247***
S.E	1,9115	0,2940	0,0043	0,0152
ω_{b2}	-0,984	-0,174	-0,411***	-0,880
S.E	12,742	0,8351	0,1448	4,6059
ω_{r2}	0,9811	0,9476*	0,9990***	0,9654
S.E	2,9929	0,4989	0,0225	2,6911
ρ_λ	-0,865	-0,999*	-0,999	-0,999***
S.E	0,5417	0,0002	1,4283	0,0001
Loglike	-79086,8	-84301,3	-77123,4	-73616,6

* significance at 10%

** significance at 5%

*** significance at 1%

5.4.8 P parameters of default rate financial firms industrial firms and Italian Government

Table 5.7: P parameters whole sample financial firms

$Para$	Generali	UBI	Unicredit	Mediobanca
$\bar{\alpha_1}$	0,0256	0,0248	0,0341	0,0317
$\bar{\alpha_2}$	0,3953	0,6225	0,5661	0,3509
$\bar{\gamma_{\lambda_1}}$	0,0301	0,0365	0,0317	0,0498
$\bar{\gamma_{\lambda_2}}$	-0,023	-0,027	-0,021	-0,037
$\bar{c_1}$	-1,051	-0,579	-1,895	0,3984
$\bar{c_2}$	-0,294	-0,342	-0,014	-0,953
ϕ_1	-0,018	-0,234	-0,001	-0,009
ϕ_2	-0,057	-0,034	-0,001	-0,010

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i \gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.8: P parameters industrial firms and Republic of Italy

$Para$	Italy	ENI	ENEL	Telecom
$\bar{\alpha_1}$	0,1881	0,3568	0,1817	0,3504
$\bar{\alpha_2}$	0,4794	0,3563	0,3047	0,3525
$\bar{\gamma_{\lambda_1}}$	0,0521	1,5225	0,2221	10,573
$\bar{\gamma_{\lambda_2}}$	-0,047	-1,506	-0,208	-10,51
$\bar{c_1}$	-1,015	-1,555	2,4243	47,988
$\bar{c_2}$	0,6743	3,0041	-1,707	-34,43
ϕ_1	-0,079	-1,008	-0,520	-2,795
ϕ_2	-0,093	-1,638	-0,298	-7,772

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i \gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.9: P parameters first subsample financial firms

$Para$	Generali	UBI	Unicredit	Mediobanca
$\bar{\alpha_1}$	0,0435	0,0648	0,0504	0,0312
$\bar{\alpha_2}$	0,5397	0,5933	0,3938	0,7894
$\bar{\gamma_{\lambda_1}}$	0,0071	0,0184	0,0006	0,0087
$\bar{\gamma_{\lambda_2}}$	0,0151	0,0149	0,0263	0,0136
$\bar{c_1}$	-0,970	-0,701	-1,181	-1,370
$\bar{c_2}$	0,1368	0,1621	0,3198	0,1254
ϕ_1	-0,072	0,0133	0,0069	-0,007
ϕ_2	-0,044	-0,029	-0,019	-0,013

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i\gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.10: P parameters first subsample industrial firms and Republic of Italy

$Para$	Italy	ENI	ENEL	Telecom
$\bar{\alpha_1}$	0,0463	0,0434	0,0721	0,0748
$\bar{\alpha_2}$	0,2072	0,3437	0,3973	0,6382
$\bar{\gamma_{\lambda_1}}$	-0,021	-0,002	0,0392	0,1502
$\bar{\gamma_{\lambda_2}}$	0,0353	0,0227	-0,010	-0,098
$\bar{c_1}$	-2,595	-1,093	-1,102	1,3452
$\bar{c_2}$	1,0171	0,2172	0,2331	-1,130
ϕ_1	-0,000	0,0239	-0,455	-0,035
ϕ_2	0,0332	-0,032	-0,077	-0,098

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i\gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.11: P parameters second subsample financial firms

$Para$	Generali	UBI	Unicredit	Mediobanca
$\bar{\alpha_1}$	0,0159	0,0176	0,0307	0,0169
$\bar{\alpha_2}$	0,3809	0,4731	0,5405	0,3233
$\bar{\gamma_{\lambda_1}}$	0,0333	0,0544	0,0408	0,0515
$\bar{\gamma_{\lambda_2}}$	-0,023	-0,033	-0,026	-0,036
$\bar{c_1}$	-2,751	-0,017	-1,173	0,0275
$\bar{c_2}$	-0,279	-0,808	-0,385	-1,061
ϕ_1	-0,995	-0,977	-0,035	0,0170
ϕ_2	-0,026	-0,025	-0,022	0,0021

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i \gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.12: P parameters second subsample industrial firms and Republic of Italy

$Para$	Italy	ENI	ENEL	Telecom
$\bar{\alpha_1}$	0,1662	0,4315	0,4702	0,0169
$\bar{\alpha_2}$	0,5120	0,4005	0,4640	0,3233
$\bar{\gamma_{\lambda_1}}$	0,0451	0,5941	2,0198	0,0515
$\bar{\gamma_{\lambda_2}}$	-0,037	-0,588	-2,007	-0,036
$\bar{c_1}$	-0,700	-5,377	-2,291	0,0275
$\bar{c_2}$	0,3067	7,1086	6,2976	-1,061
ϕ_1	-0,014	0,0740	-1,164	0,0170
ϕ_2	-0,094	-0,124	-0,866	0,0021

$\bar{\alpha_i} = \alpha_i - \omega_{1i}\sigma_{\lambda_i}$ $\quad \bar{\gamma_{\lambda_i}} = \frac{\alpha_i \gamma_\lambda + \omega_{0i}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

$\phi_i = \frac{\omega_{bi}\sigma_{\lambda_i}}{\bar{\alpha_i}}$ $\quad \bar{c_i} = \frac{\alpha_i c_i + \omega_{ri}\sigma_{\lambda_i}}{\bar{\alpha_i}}$

Table 5.13: Mean and Standard Deviation of errors

$Firms$	Whole sample		first sample		second sample	
	mean	std	mean	std	mean	std
Unicredit	-0,00017	0,001896	-9,2E-05	0,001618	-0,00013	0,002092
ENI	0,000235	0,001149	1,71E-05	0,001572	0,000118	0,001315
Italy	-0,00013	0,002117	-3,6E-05	0,001692	-0,00026	0,002499
Generali	0,000142	0,001476	4,98E-05	0,001441	8,01E-05	0,001579
Ubi	0,000239	0,001668	2,85E-06	0,001422	0,000196	0,001967
Telecom	-0,00037	0,002287	0,000109	0,002361	0,000343	0,002526
MedioBanca	-1,4E-05	0,001876	-8,5E-05	0,000889	-2,1E-05	0,001974
ENEL	0,000205	0,00219	-1,5E-05	0,002431	0,000206	0,002155

Table 5.14: RMSE financial firms whole sample

Maturity	Firms			
	Unicredit	Ubi	Generali	MedioBanca
1	0,0079	0,0348	0,0152	0,0241
2	0,0027	0,0044	0,0017	0,0081
3	0,0002	0,0028	0,0021	0,0082
4	0,0049	0,0011	0,0016	0,0115
5	0,0029	0,0011	0,0039	0,0031
6	0,0034	0,0043	0,0001	0,0005
7	0,0140	0,0072	0,0136	0,0008
8	0,0003	0,0030	0,0015	0,0008
9	0,0005	0,0039	0,0031	0,0029
10	0,0002	0,0061	0,0006	0,0003

Table 5.15: RMSE industrial firms and Republic of Italy whole sample

Maturity	Firms			
	Italy	ENI	ENEL	Telecom
1	0,0024	0,0295	0,0316	0,0497
2	0,0063	0,0084	0,0088	0,0125
3	0,0005	0,0001	3,8E-05	3,7E-05
4	0,0002	0,0040	0,0057	0,0074
5	0,0013	0,0009	0,0015	0,0040
6	0,0073	0,0003	0,0013	0,0016
7	0,0025	0,0041	0,0034	0,0054
8	0,0004	0,0013	0,0002	0,0003
9	0,0033	0,0045	0,0059	0,0084
10	0,0010	0,0013	0,0029	0,0014

Table 5.16: RMSE financial firms first subsample

Maturity	Firms			
	Unicredit	Ubi	Generali	MedioBanca
1	0,0004	4,1E-06	0,0020	0,0161
2	0,0002	8,6E-06	0,0002	0,0003
3	5,9E-06	0,0005	0,0015	0,0031
4	0,0026	0,0005	0,0006	0,0030
5	0,0008	0,0001	0,0018	0,0020
6	0,0009	0,0011	0,0005	0,0009
7	0,0041	0,0020	0,0038	5,9E-05
8	0,0001	0,0010	0,0007	0,0003
9	0,0003	0,0006	0,0002	0,0001
10	0,0004	0,0014	0,0008	0,0006

Table 5.17: RMSE financial firms second subsample

Maturity	Firms			
	Italy	ENI	ENEL	Telecom
1	0,0038	0,0005	0,0024	0,0131
2	0,0022	0,0004	0,0006	0,0009
3	0,0002	0,0006	2,9E-05	0,0007
4	0,0019	0,0014	0,0025	0,0026
5	9,8E-05	0,0005	0,0013	0,0019
6	0,0019	0,0005	0,0003	0,0007
7	0,0013	0,0001	0,0005	0,0002
8	0,0006	0,0005	0,0001	0,0001
9	3,9E-05	0,0001	0,0001	0,0009
10	0,0006	0,0005	0,0006	0,0003

Table 5.18: RMSE industrial firms and Republic of Italy first subsample

Maturity	Firms			
	Unicredit	Ubi	Generali	MedioBanca
1	0,0005	0,0198	0,0047	0,0190
2	0,0035	0,0004	0,0038	0,0057
3	0,0007	0,0038	0,0002	0,0043
4	0,0044	0,0004	0,0027	0,0098
5	0,0035	0,0001	0,0032	0,0031
6	0,0026	0,0055	0,0001	0,0011
7	0,0135	0,0073	0,0126	0,0002
8	5,6E-05	0,0047	0,0002	0,0001
9	0,0005	0,0004	0,0039	0,0040
10	0,0003	1,5E-05	0,0003	0,0004

Table 5.19: RMSE industrial firms and Republic of Italy second subsample

Maturity	Firms			
	Italy	ENI	ENEL	Telecom
1	0,0173	0,0103	0,0194	0,0254
2	0,0085	0,0029	0,0068	0,0067
3	0,0023	0,0006	0,0007	0,0009
4	0,0007	0,0015	0,0028	0,0027
5	2,3E-05	0,0025	0,0029	0,0077
6	0,0069	0,0001	0,0005	0,0004
7	0,0022	0,0038	0,0044	0,0067
8	7,4E-05	0,0002	8,8E-05	9,7E-05
9	0,0035	0,0056	0,0066	0,0082
10	0,0000	0,0008	0,0001	0,0036

5.5 Graphs

5.5.1 errors in cross section (whole sample analysis)

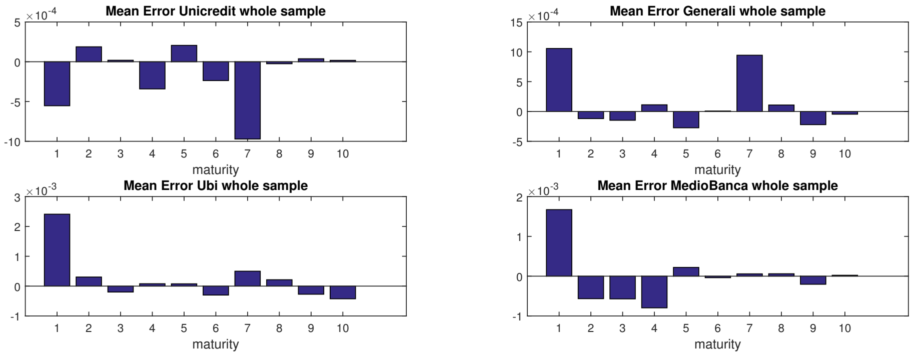

Figure 5.1: Financial firms mean errors (cross section)

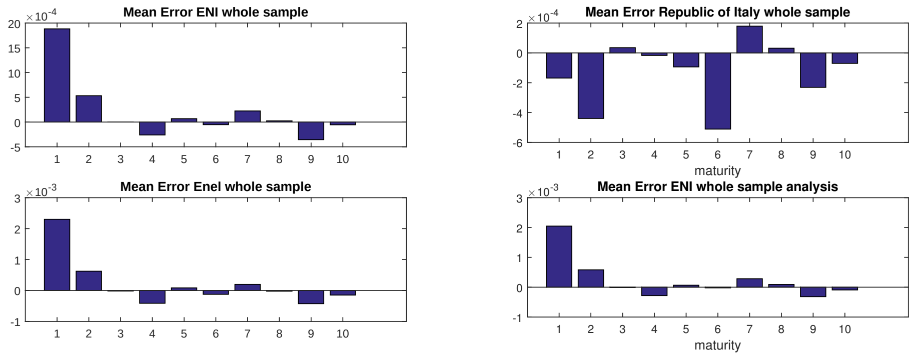

Figure 5.2: industrial firms and Italy mean errors (cross section)

5.5.2 errors in cross section (sub sample analysis)

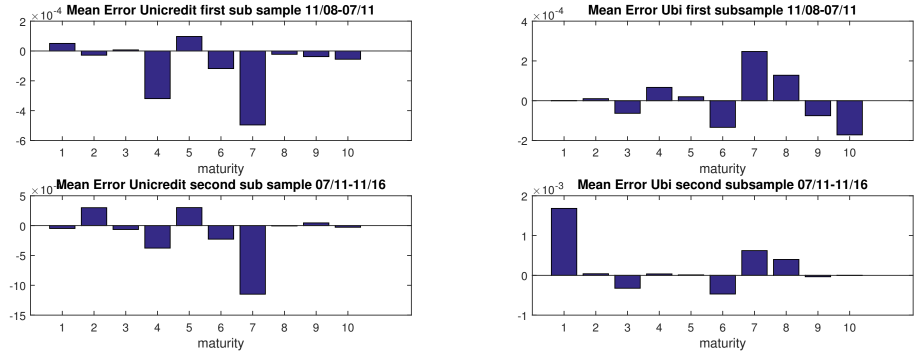

Figure 5.3: Error mean cross section Unicredit (left) UBI (right)

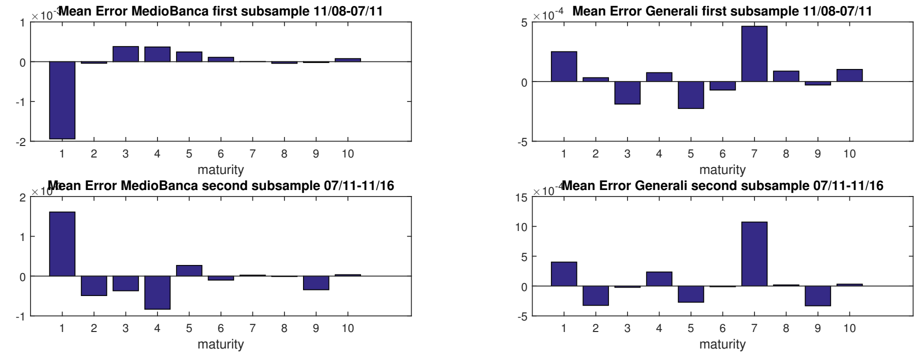

Figure 5.4: Error mean cross section MedioBanca (left) Generali (right)

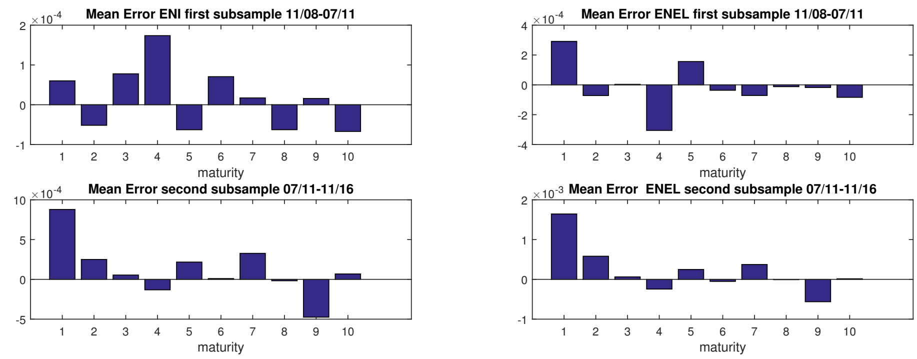

Figure 5.5: Error mean cross section ENI (left) ENEL (right)

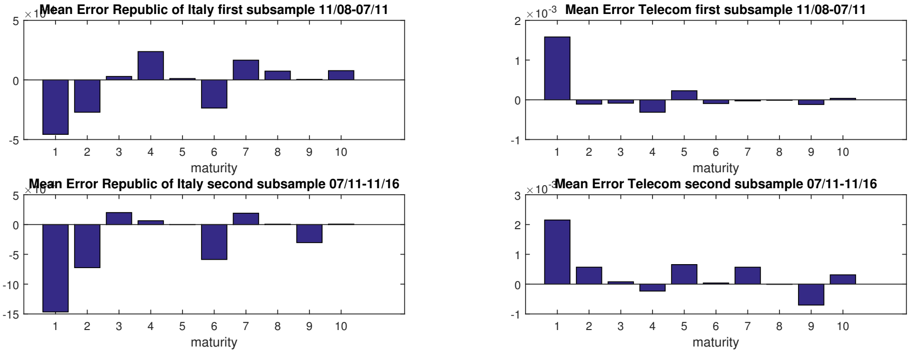

Figure 5.6: Error mean cross section Italy (left) Telecom (right)

5.5.3 Predicted default rate vs data whole sample analysis

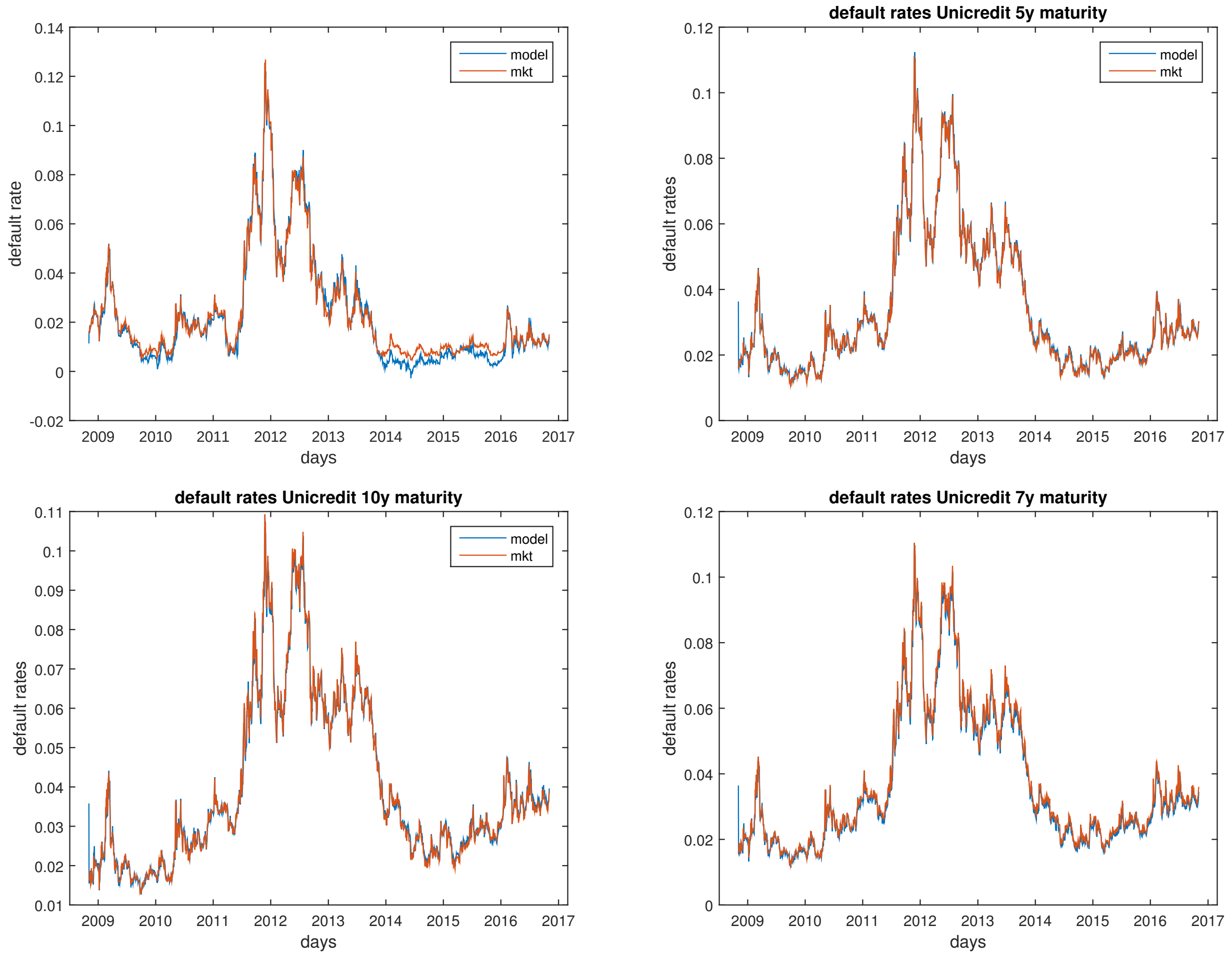

Figure 5.7: Unicredit times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

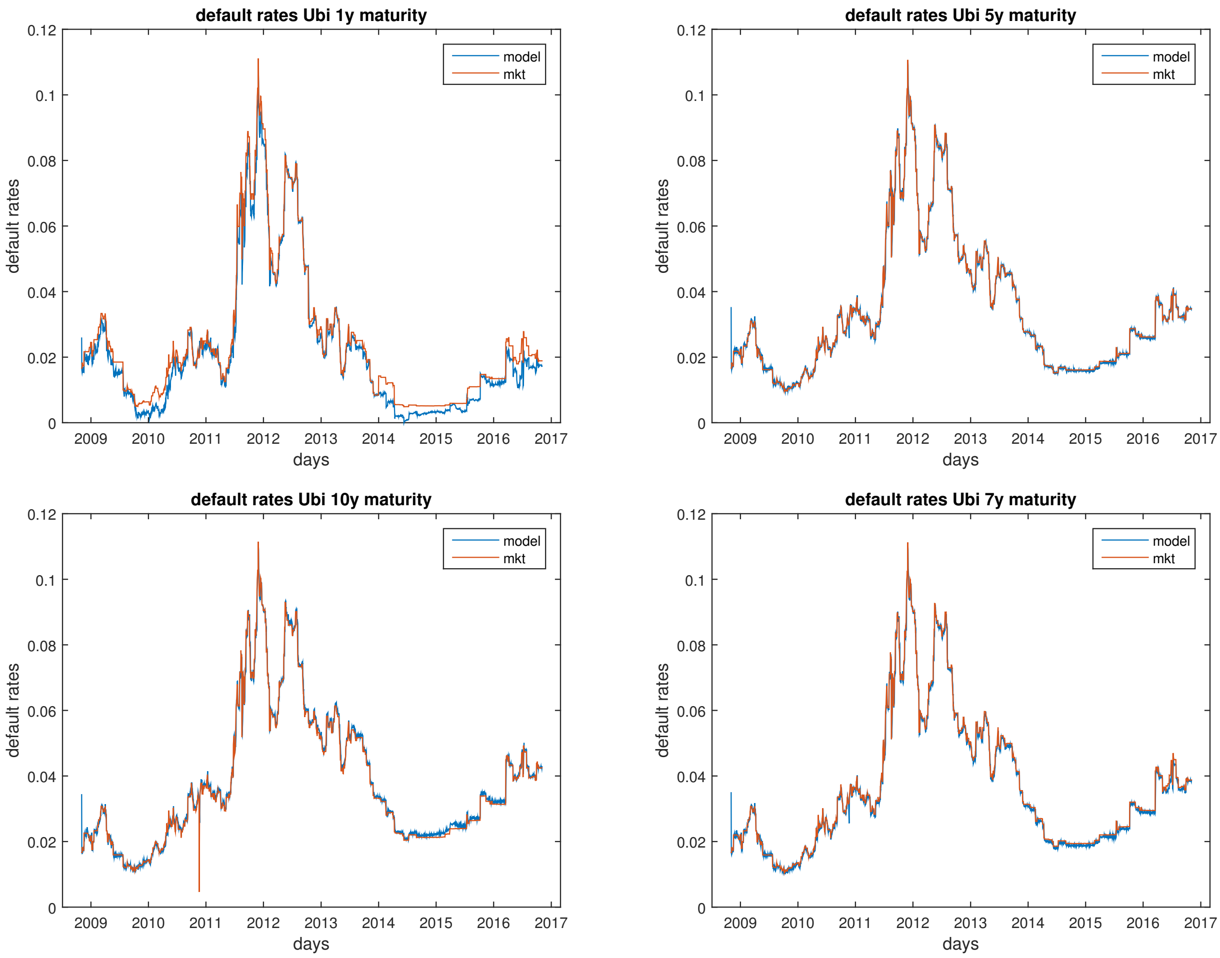

Figure 5.8: Ubi times series errors 1y (up left), 5y (up right) ,7y (down left), 10y (down right)

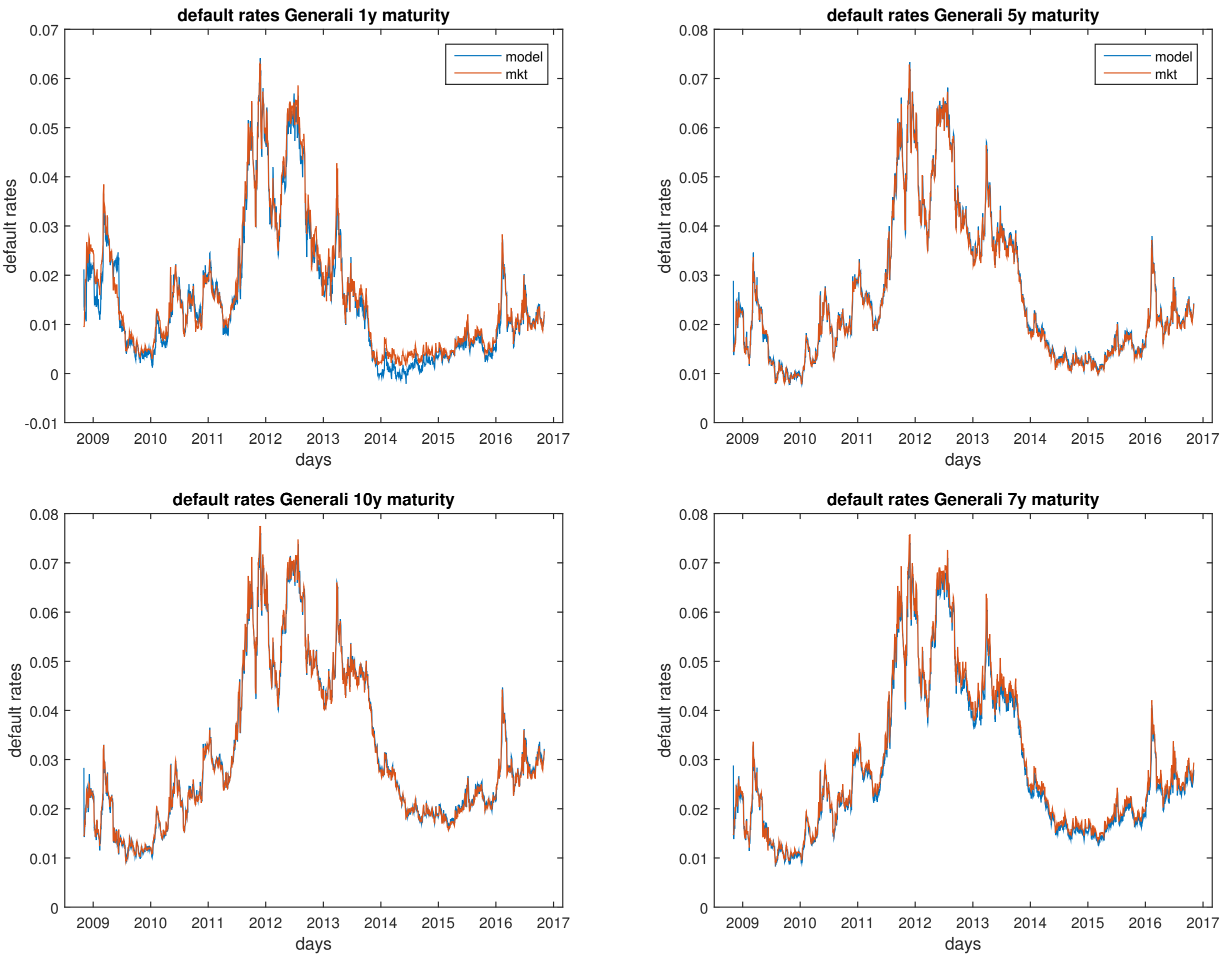

Figure 5.9: Generali times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

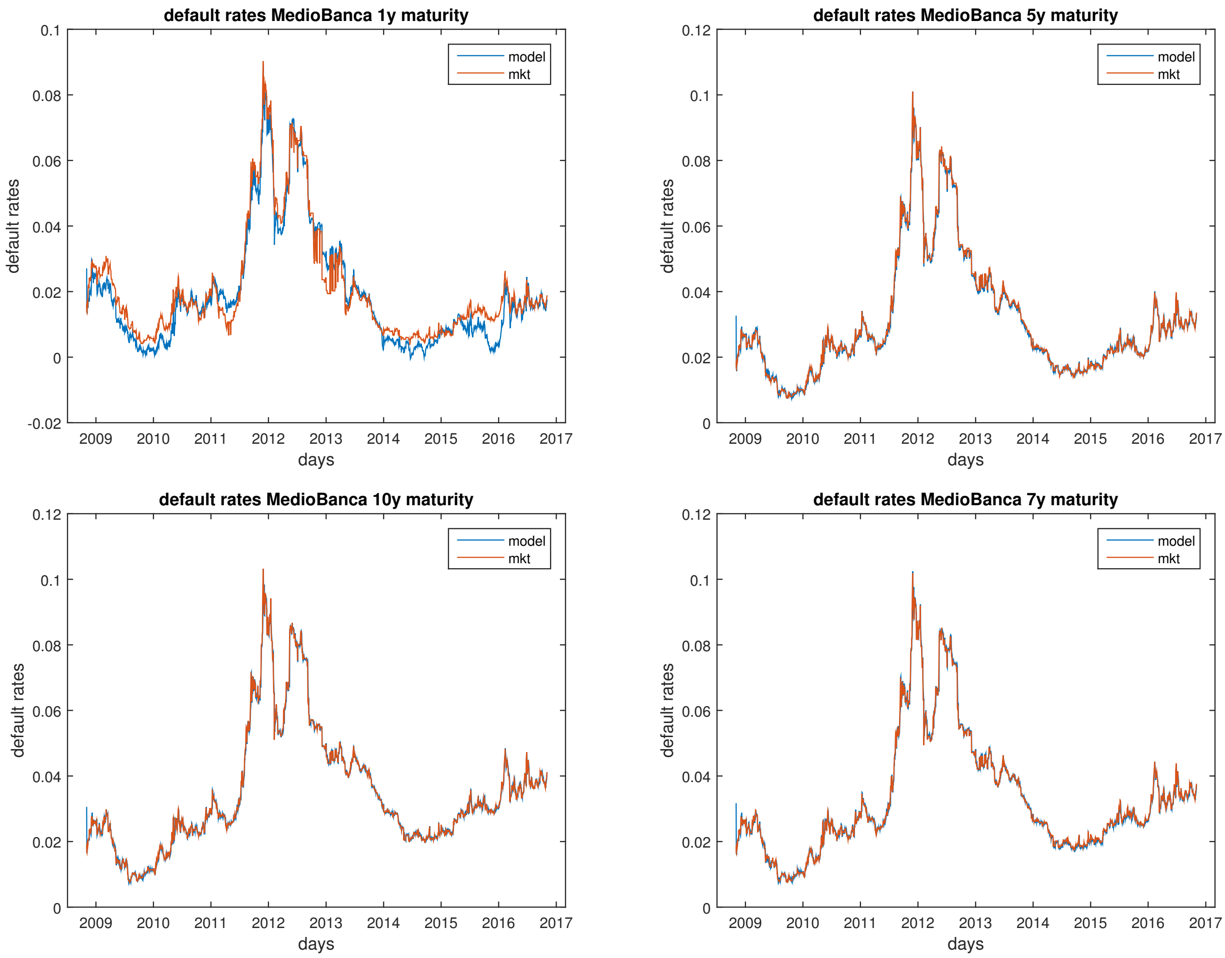

Figure 5.10: MedioBanca times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

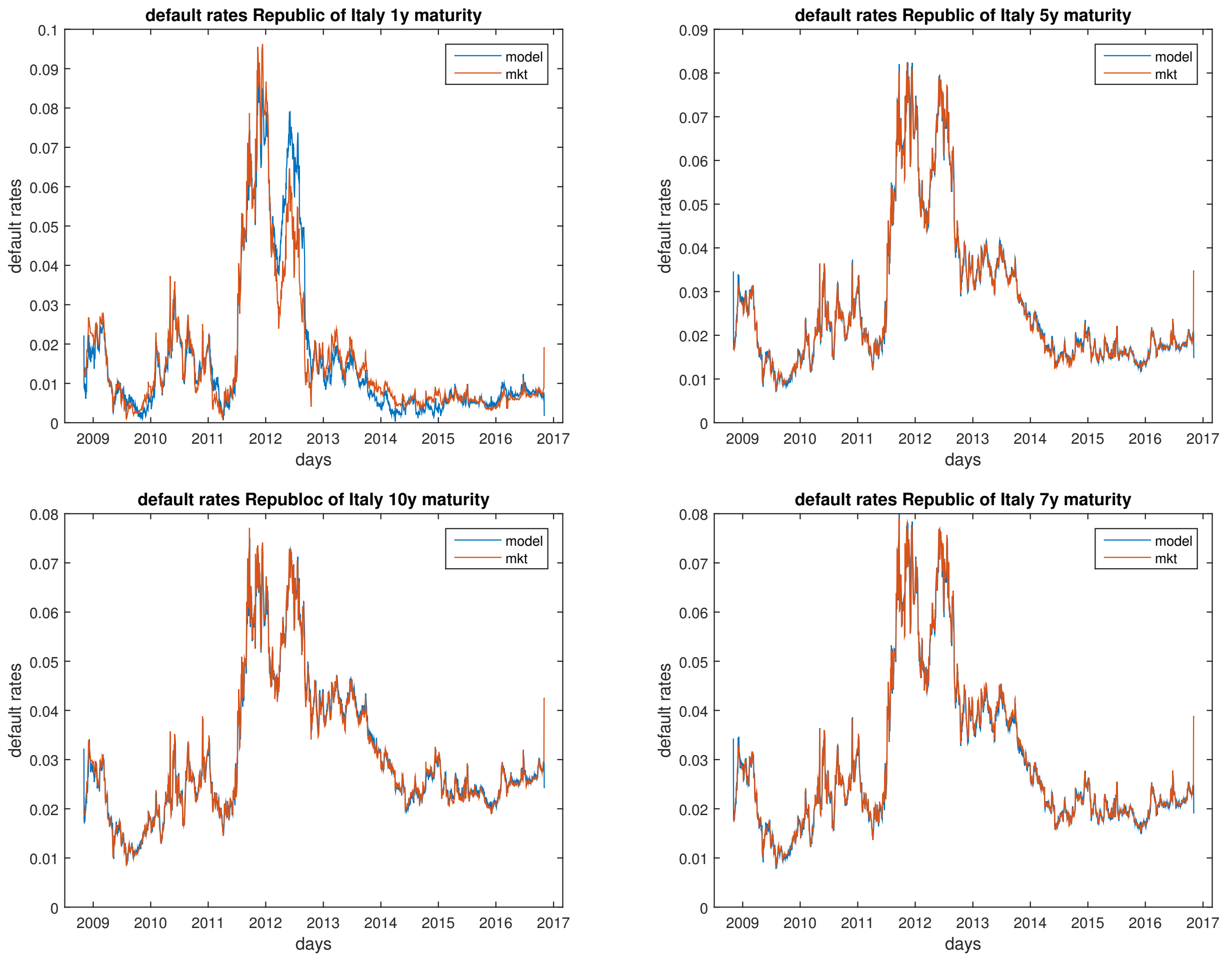

Figure 5.11: Italy times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

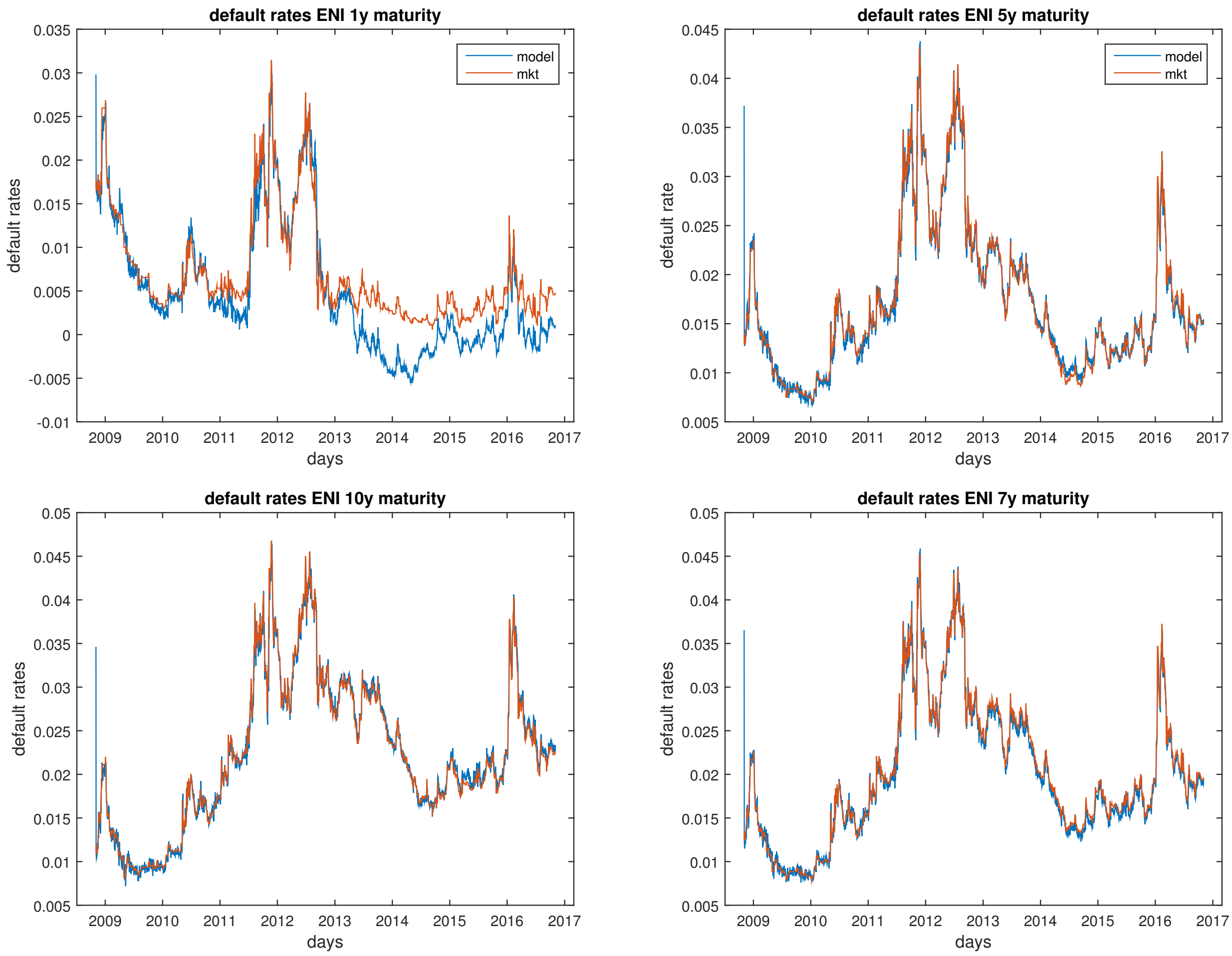

Figure 5.12: ENI times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

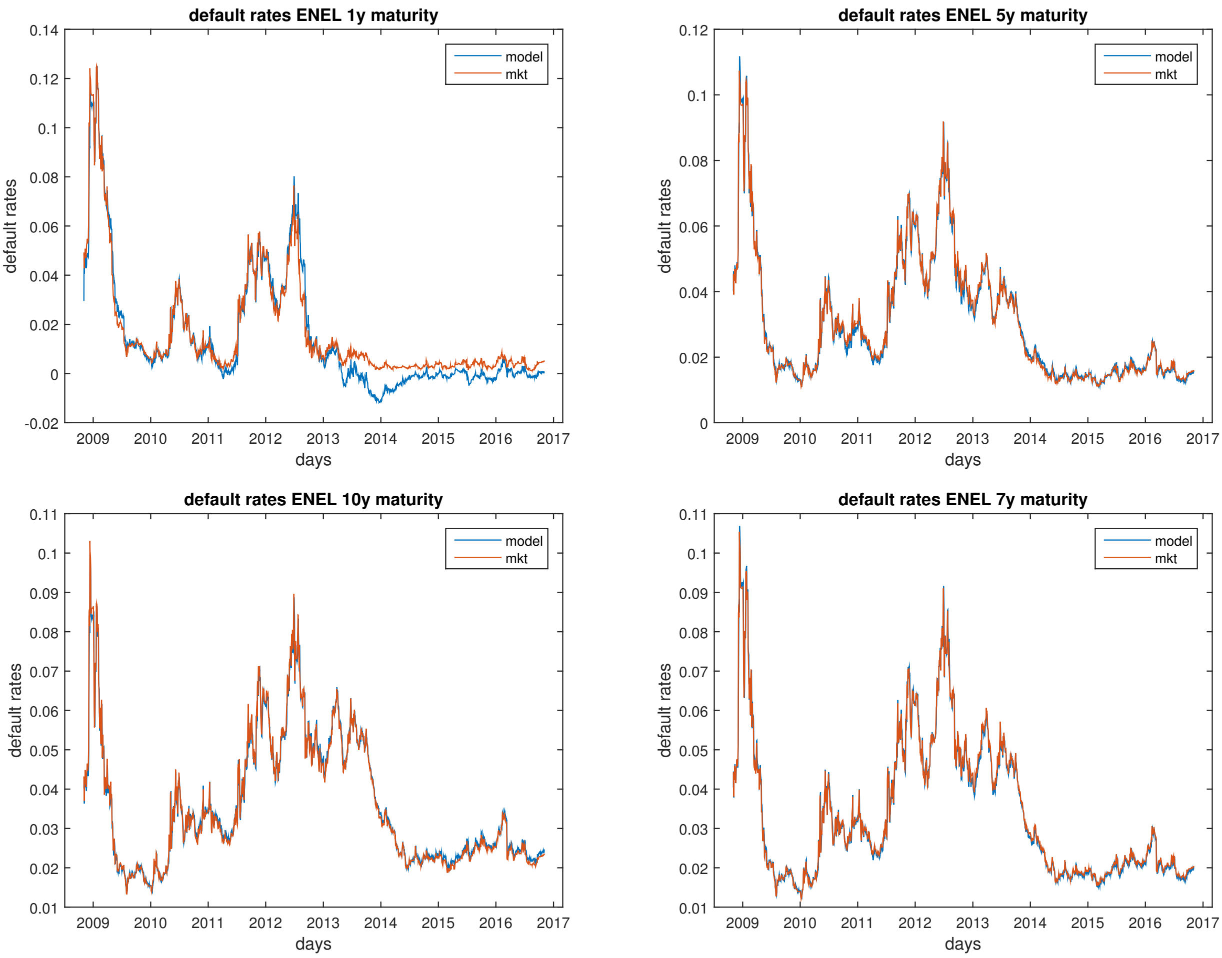

Figure 5.13: ENEL times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

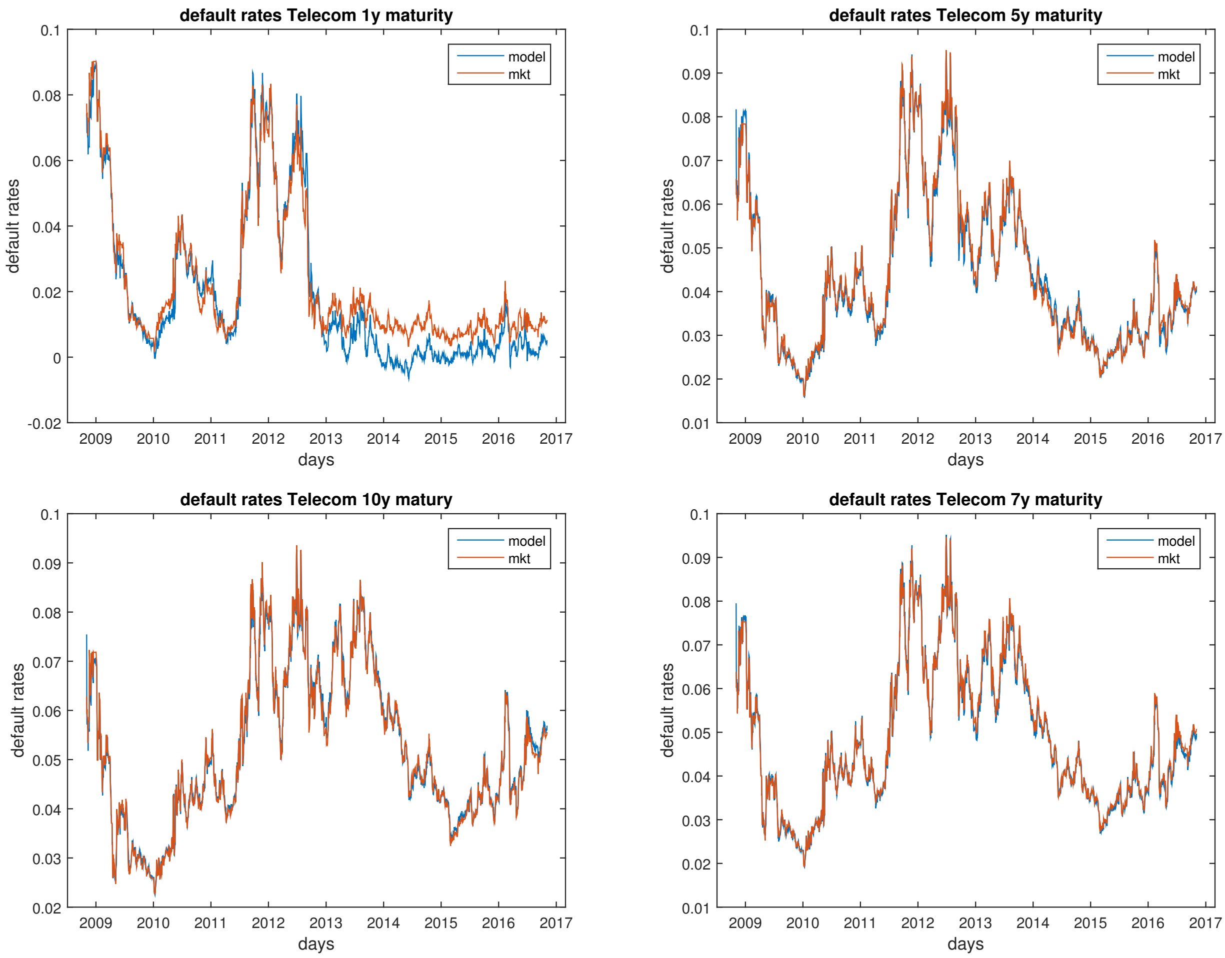

Figure 5.14: Telecom times series errors 1y (up left), 5y (up right) ,7y (down left) ,10y (down right)

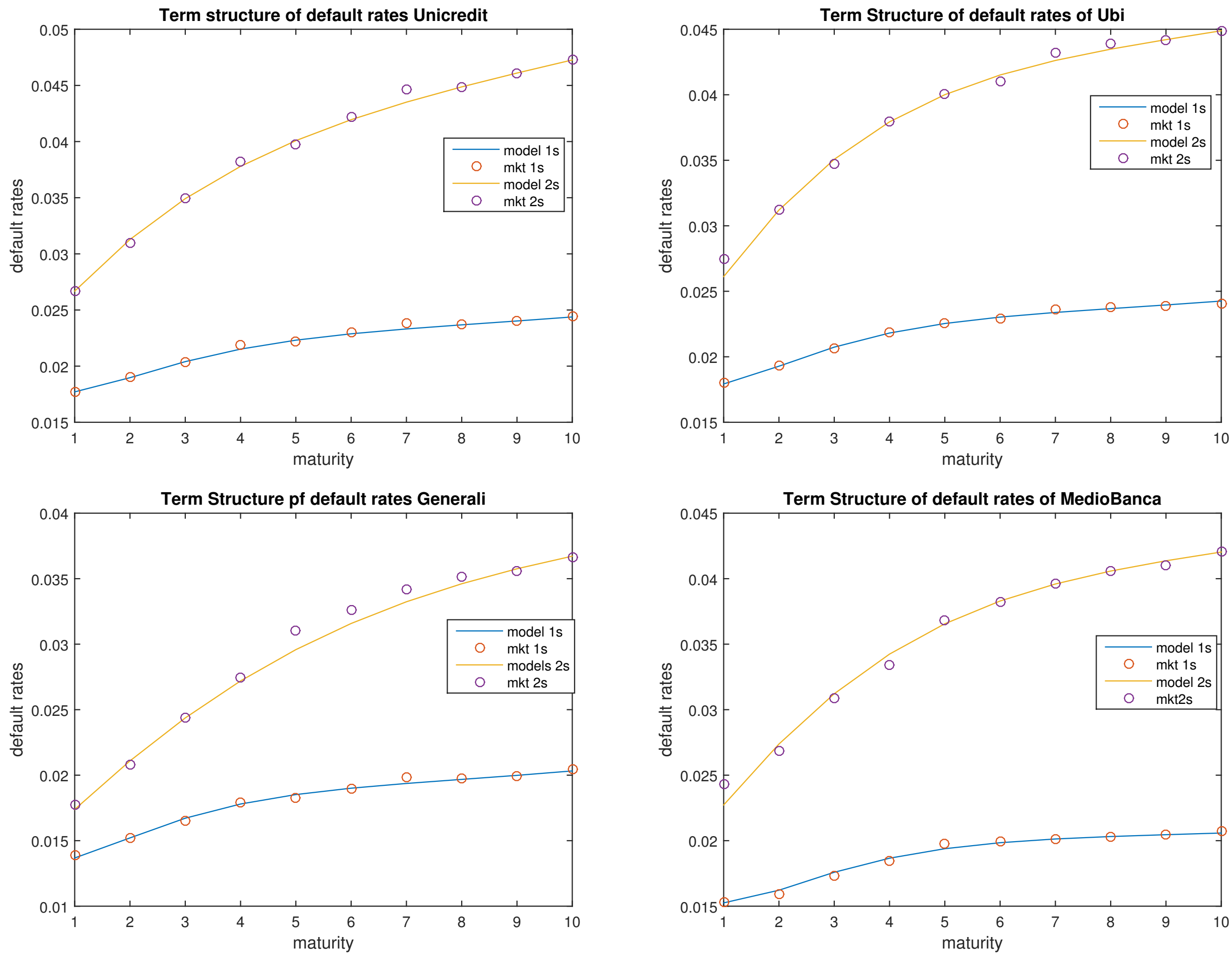

Figure 5.15: Term Structure of default rates of financial firms first subsample (blue) second subsample (yellow)

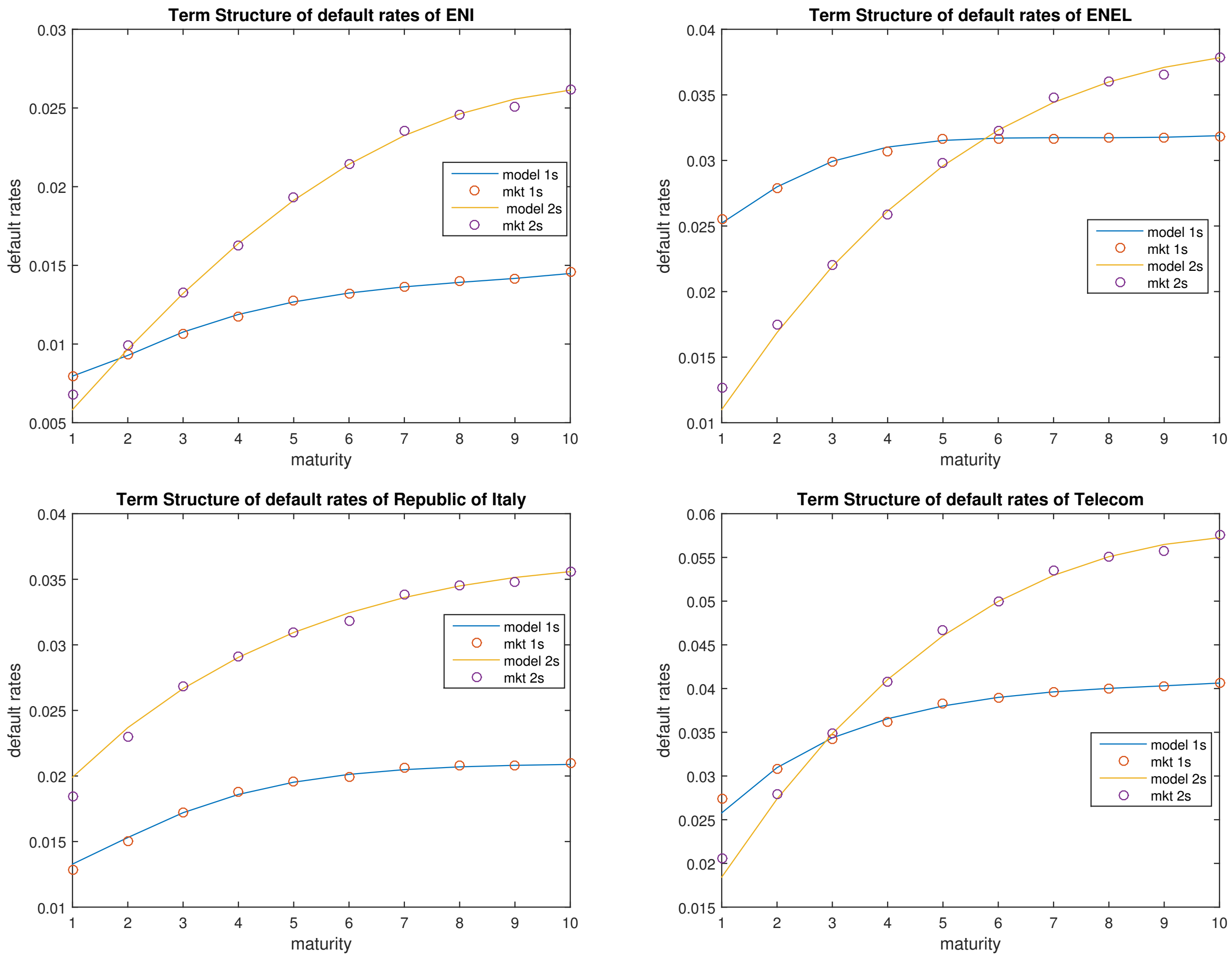

Figure 5.16: Term Structure of default rates of industrial firms and Republic of Italy first subsample (blue) second subsample (yellow)

5.5.4 Errors autocorrelation (whole sample analysis)

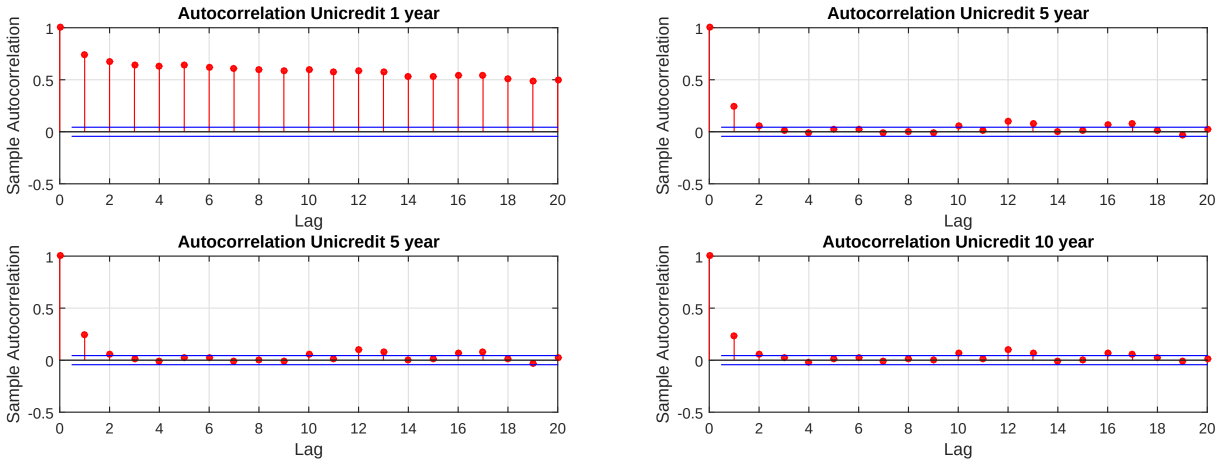

Figure 5.17: Error autocorrelation Unicredit: 1 year vs 5 year left, 5 year vs 10 year right

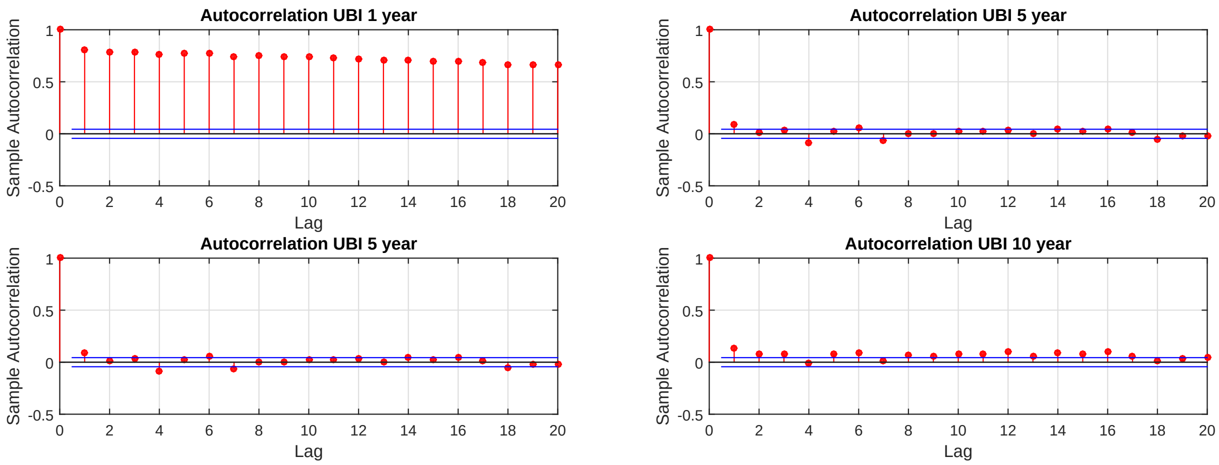

Figure 5.18: Error autocorrelation Ubi: 1 year vs 5 year left, 5 year vs 10 year right

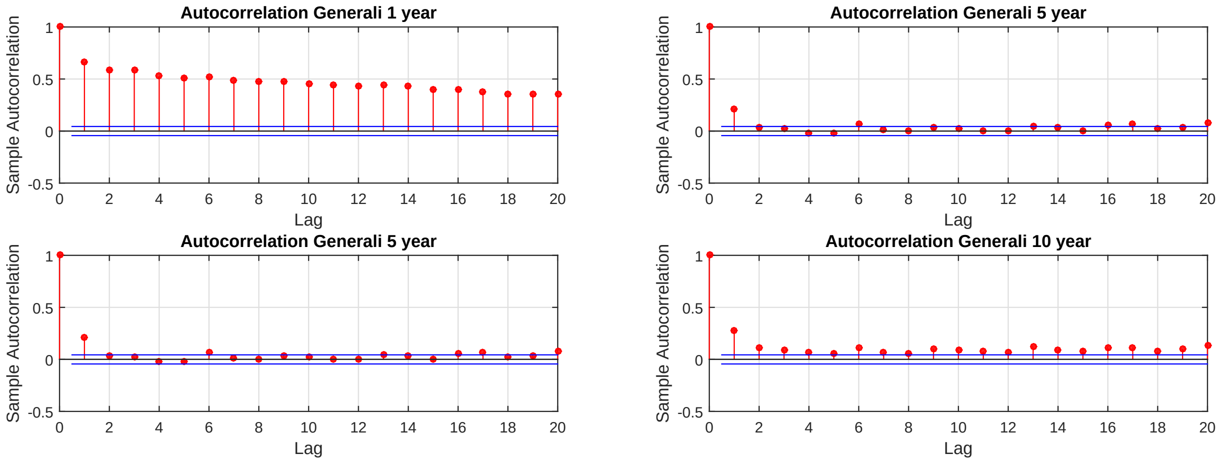

Figure 5.19: Error autocorrelation Generali: 1 year vs 5 year left, 5 year vs 10 year right

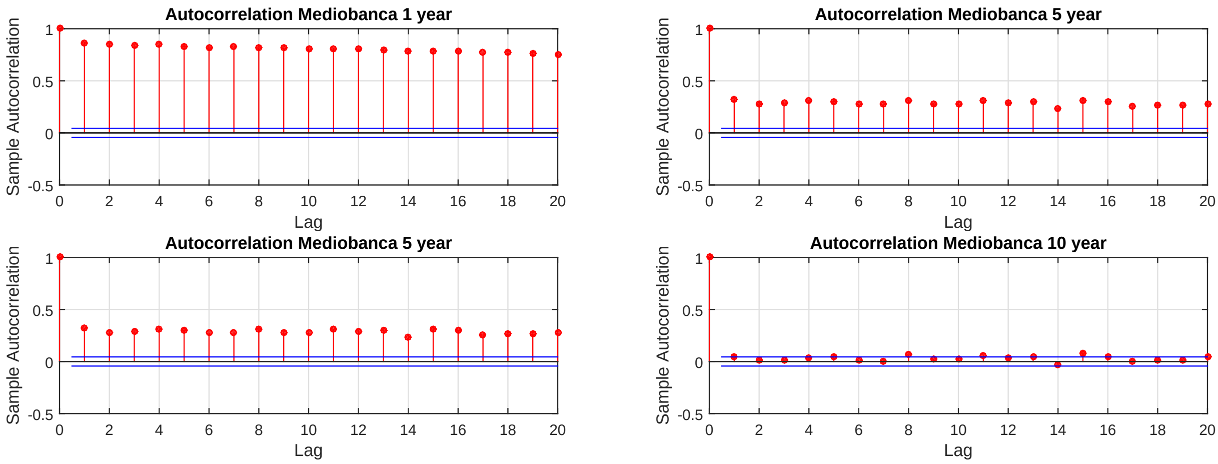

Figure 5.20: Error autocorrelation MedioBanca: 1 year vs 5 year left, 5 year vs 10 year right

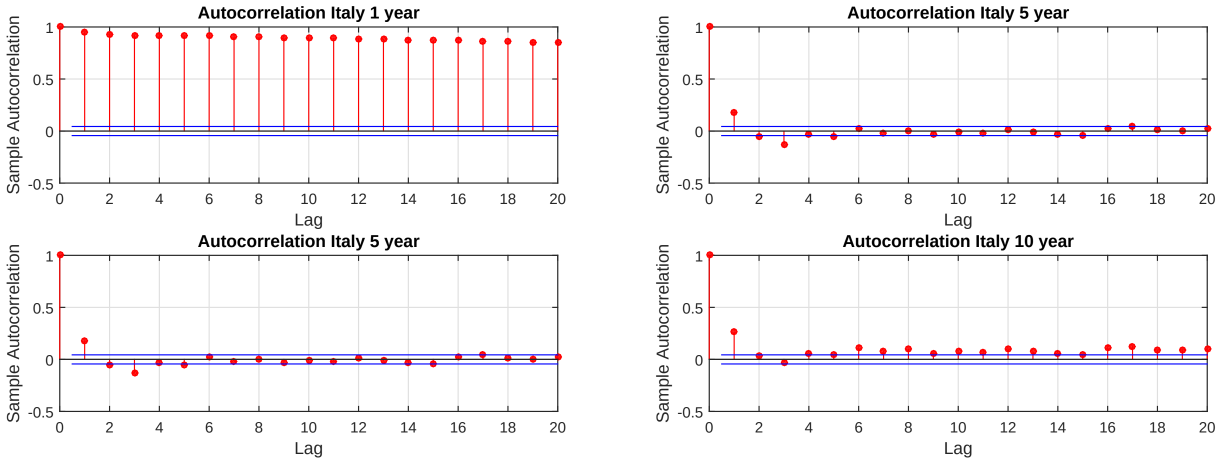

Figure 5.21: Error autocorrelation Italy: 1 year vs 5 year left, 5 year vs 10 year right

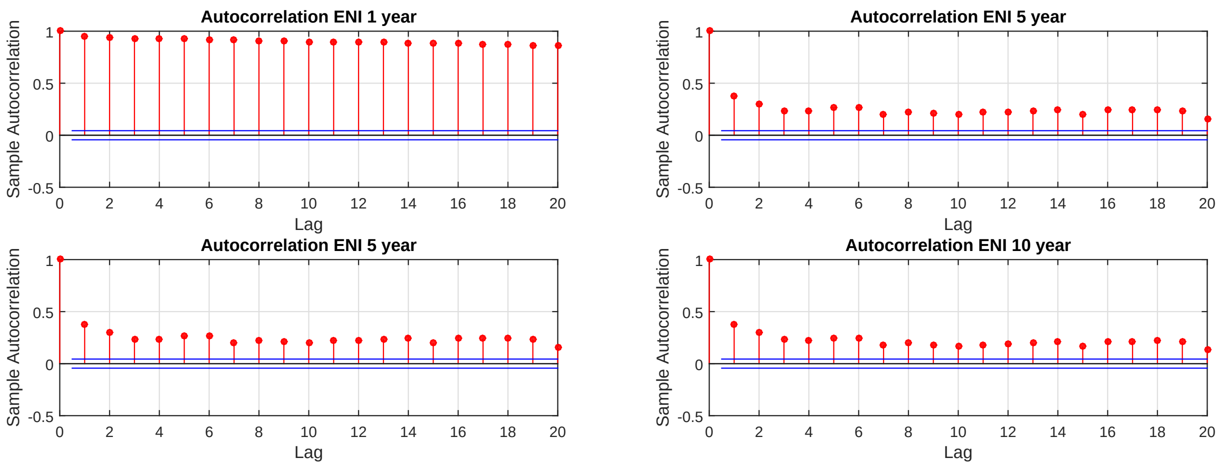

Figure 5.22: Error autocorrelation ENI: 1 year vs 5 year left, 5 year vs 10 year right

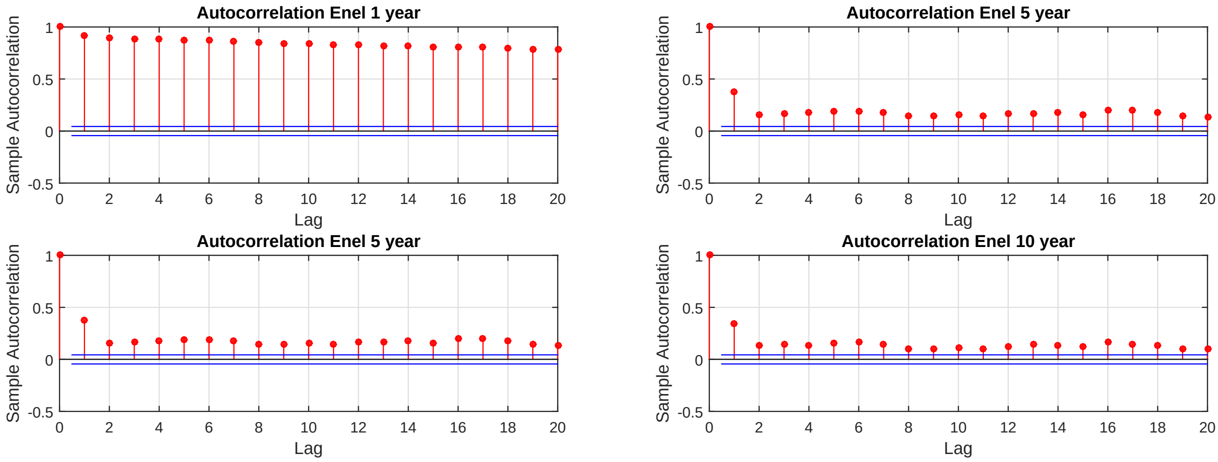

Figure 5.23: Error autocorrelation Enel: 1 year vs 5 year left, 5 year vs 10 year right

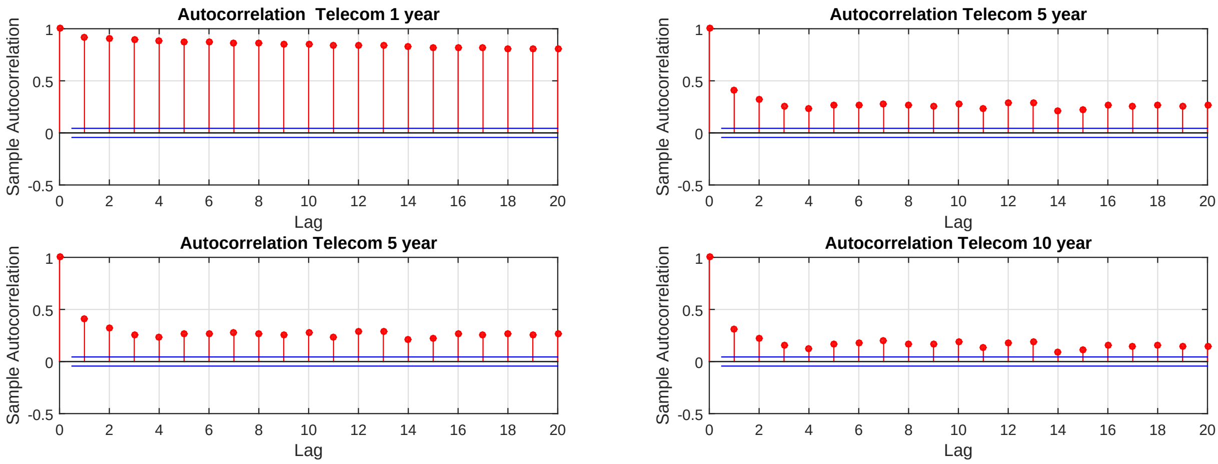

Figure 5.24: Error autocorrelation Telecom: 1 year vs 5 year left, 5 year vs 10 year right

Chapter 6

Conclusions

Multifactor models are needed to model interest rates, BEI rates and default rates. In particular, more factors are needed after the bursting the Sovereign Debt Crisis.

Three factor models on interest rates, are more suitable than two factor models and one factor models for explaining the dynamics of interest rates after the bursting of the Sovereign Debt Crisis than before.

This is evident by looking at the AIC. The difference between the AIC of the three factor model and the AIC of the two factor model, is higher after the bursting of the Sovereign Debt Crisis.

The evidence of more factors needed after the bursting of the Sovereign Debt Crisis, is also confirmed by the Principal Component Analysis.

The PCA in performed in the time series in first differences, show that seven factors are needed to explain most of the variability of the data over that sample.

The results found for interest rates applies also for BEI rates.

Factors found in interest rates and BEI rates models, are different from factors found through the PCA analysis of data. The main difference relies on the distribution assumption of interest rates, BEI rates and by the relaxing assumption of independence of factor present in the PCA analysis.

Default rates of financial firms and the Italy behave differently from the default rates of industrial firms analysed. Default rates of financial firms and Italy raised more than default rates of industrial firms after the bursting of the Sovereign Debt Crisis. On the contrary, default rates of industrial firms were higher than default rates of financial firms and Italy, before the sovereign debt crisis.

Furthermore, default rates of all institutions analysed do depend on both interest rates and inflation rates.

Interest rates affect negatively the short run and the long run tendency of the Term Structure of default rates. This result is in line with the related literature (see Duffee (1998, 1999)) and it is evident for financial firms.

This makes economic sense in a context of low interest rates. Low interest rates affect negatively the net interest margin of banks and make minimum guaranteed insurance products less profitable. This effect is stronger after the bursting of the sovereign debt crisis.

The negative effect made by inflation rates on default rates of financial firms, shows that the debt deflation hypothesis holds.

Industrial firms are negatively influenced by interest rates. This makes economic sense When interest rates are low, industrial firms increase their leverage. This increases the default risk.

On the other hand, industrial firms are less influenced by inflation rates than Republic of Italy and financial firms.

The default rate of industrial firms is less affected by the news coming from the monetary policy environment but depends more on factors like the price of oil, which is not considered here.

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