A MULTI INTEREST RATE CURVE MODEL FOR EXPOSURE MODELLING

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ABSTRACT. The tenor basis phenomenon became significant with the 2007 financial crisis and has altered the traditional way of one-curve pricing and risk management to a multi-curve phenomenon. The stochastic nature of basis spreads between curves particularly poses a challenge for forward looking applications like XVA or real world measure exposure analytics. This paper presents a Two-factor Gaussian approach for modelling multiple fixing curves and basis spreads in the risk neutral and spot measure, shows the impact on basis swap exposure, investigates the correlation structure, and discusses the pros and cons of interpreting as a spread or multi curve model respectively.

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1. Introduction

Since the 2007 financial crisis, the inception value of tenor basis swaps (i.e. float- vs. float swaps exchanging two different index tenors) is not negligible any more. Credit and liquidity issues driven by the length of the index tenor have become more significant. A fair zero-valued basis swap can be constructed by adding a fixed payment, the *tenor basis spread*, onto the variable shorter tenor index payment in order to compensate for these uncertainties.
Picture 1 depicts the 1Y maturity 1M vs. 3M USD tenor basis swap spread quotes from 2005 to 2012 in order to illustrate the 2007 financial crisis impact. Before 2007, basis spreads were negligible. In the crisis spreads rose up to almost 50bps, which is significant. From this diagram we

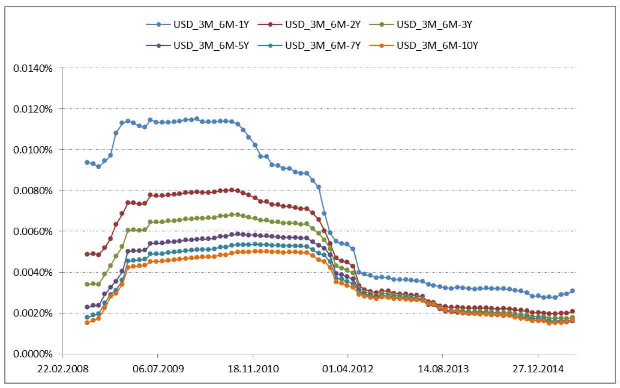
FIGURE 1. Courtesy Bloomberg. 1M vs. 3M USD tenor basis spread time series for 1 year maturity from 2005-2012.

can further deduce that tenor basis spread is far from being deterministic. The stochasticity of tenor basis spread has an impact on future exposure calculations for basis spread sensitive instruments, i.e. those instruments traded on off major tenor (e.g. 3M for USD and 6M for EUR) Libor fixing indices.
In the remainder of this text a model for stochastic tenor basis is suggested and discussed. The structure is as follows: The next chapter investigates from historical data how many factors are necessary to drive the stochastic tenor basis spread term structure. Based on these insights Gaussian spread dynamics are suggested and discussed in Chapter 3 with emphasis on curve and spread modelling duality. Model predictions are then compared with empirical results in Chapter 4. Finally, the phenomenon of negative basis spreads, being a consequence of Gaussian modelling, is addressed in the last chapter.

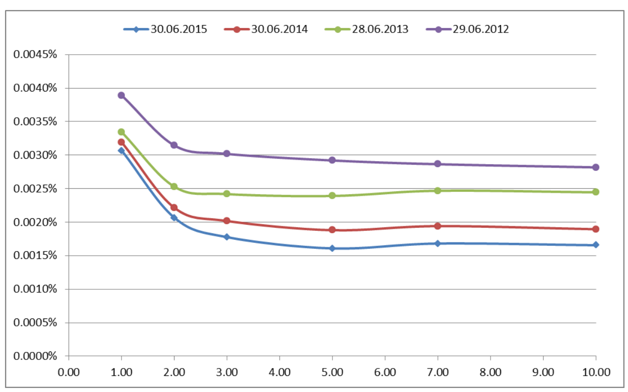
2. Historical Tenor Basis Spread Dynamics

2.1. **Historical Basis Spread Sample Volatility.** Historical time series [1] analysis was performed to get a view on variance behavior of tenor basis spreads. Investigation was done on the major currencies USD, EUR, GBP, CHF and JPY but results are displayed for USD only since they are qualitatively similar.

In figure 2, the historical tenor basis spread sample volatility of a 3M vs. 6M USD basis swap is displayed. The historical window is backward looking and has a length of 3 years. On the left, the results for a moving window starting mid of 2008 are shown until mid of 2015. The swap maturities considered are 1Y, 2Y, 3Y, 5Y, 7Y and 10Y. We observe that just after the crisis volatility is high and comes down to levels of 3 bps in the past 3 years. On the right, the basis spread term structures for 4 dates in the past are depicted. We observe a monotone decreasing behaviour on the swap maturity for all dates.



(A) Sample volatility on 3Y moving window over time



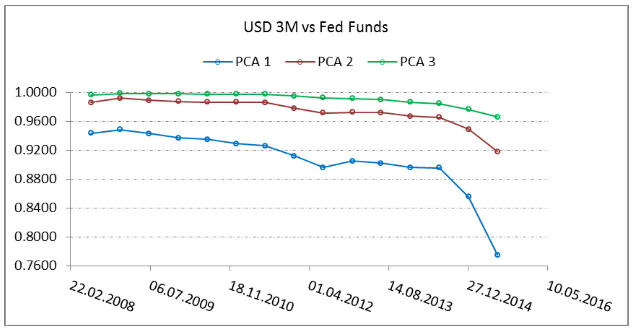
(B) Sample volatility over swap maturities

Figure 2. Historical 3M vs. 6M tenor basis spread sample volatility on 3Y window over time (left) and over swap maturities

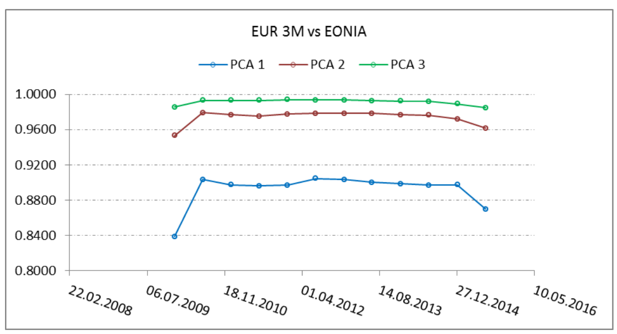
2.2. **PCA of Basis Spread Time Series.** In order to obtain an intuition on how many driving factors are necessary to describe the dynamics of the entire basis swap spread tenor curve, a principal component analysis (PCA) was performed on the historical time series. Under investigation were the 3M against overnight tenor basis swap spreads in the major currencies. We illustrate the PCA

[1] Historical basis spread data from Bloomberg

results for USD and EUR, they are representative for the other currencies as well. For the two diagrams below, the USD 3M vs. Fed Funds and the EUR 3M vs. EONIA tenor basis swap spreads are PC analysed from beginning of 2008 until end of June 2014 with a rolling window of 3 years. The time series are given by the 1Y, 2Y, 3Y, 5Y, 7Y and 10Y maturities, hence the PCA generates 6 variance components represented by ordered eigenvalues of the covariance matrix. Figure 3 shows the PCA time evolution for the USD (left) and EUR tenor basis swap term structure, both displaying the amount of total variance being captured by the first three eigenvalues (components) of the covariance matrix. In both cases, we observe that a large part of variance can be captured with the first component, but there is still a significant increase of captured variance by including the second component. Stepping from two to three components changes the situation marginally.



(A) USD 3M vs. Fed Funds



(B) EUR 3M vs. EONIA

FIGURE 3. PCA of tenor basis swap spreads on swap maturity against 3Y moving window

These outcomes suggest the appropriateness of a 2-factor basis swap spread term structure model. We will formulate the dynamics within the HJM framework in the next chapter. The 2-factor model can easily be deflated to a 1-factor model in the special case by choosing zero volatility for the second factor.

3. The Model

We consider a single currency interest rate market (extending to the multi currency case is straight forward through quanto adjusting the dynamics in the domestic risk neutral measure) where instruments on various tenors (1M, 3M, 6M, 12M etc.) are traded. The classical instruments being sensitive on the tenor in terms of risk managment are forward rate agreements, vanilla swaps and tenor basis swaps. We assume the existence of a discounting curve, which describes the yield on a risk neutral cash investment. Moreover we introduce the notion of a *fixing curve*, which describes the yield of investing into some tenor and a *base curve* corresponding to a specific (major) tenor. To make use of similarities between different tenors, we eventually spread into all potential tenors from a representative base curve, which might not possess the characteristics of the unique discount curve.

We base our formulation on the multi curve Heath-Jarrow-Morton (HJM) framework of [1], [4] and use subscripts d, b, f for discount, base and general tenor fixing curve respectively. Before we step into spreading from base curves, we start with a short recap of the HJM multi curve outlined in [1] and apply it to a two factor Gaussian model.

3.1. **Assumptions in the Multi Curve Framework.** We follow [2] in terms of idea and notations and assume there exists a submarket M_τ for every tenor $\tau \in \{d, 1\text{D}, 1\text{M}, 3\text{M}, 6\text{M}, \ldots\}$, where d stands for discounting and represents a special choice of the discounting curve.[2]. For every market M_τ we assume an existence of a banking account $M_\tau(t)$ and M_τ-zero coupon bond prices $P_\tau(0, \cdot)$. The associated M_τ-short rate $r_\tau(t)$ and M_τ-forward rate $F_\tau(t, T_1, T_2)$ are defined such that

$$M_\tau(t) = \exp\left(\int_0^t r_\tau(u)du\right)$$

and

$$\frac{1}{P_\tau(t, T_1)}(1 + F_\tau(t, T_1, T_2)\tau(T_1, T_2)) = \frac{1}{P_\tau(t, T_2)}$$

In each market we assume the standard no arbitrage relation between the (forward-) bond prices

$$P_\tau(t, T_2) = P_\tau(t, T_1)P_\tau(t, T_1, T_2),$$

where $P(t, T_1, T_2)$ is the forward bond price. Furthermore, we require the M_τ-forward rates $F_\tau(t, T_1, T_2)$ to hold the martingale property in the M_τ-T_2-forward measure induced by the numeraire $P_\tau(t, T_2)$, i.e

$$F_\tau(t, T_1, T_2) := \mathbb{E}^{T_2}_{\mathbb{Q}_\tau}\left(F(T_1, T_1, T_2)|\mathcal{F}(t)\right)$$

The same holds for M_τ-instantaneous forward rate defined as

$$f_\tau(t, T) := \lim_{\hat{T}\searrow T} F_\tau(t, T, \hat{T}).$$

Now we choose a particular fixing curve f and consider a T-claim with a payoff $\pi(T, F_f)$ at time T, which depends on the forward rate F_f. It's uniqe arbitrage free price at time t is then defined as

$$\pi(t) := P_d(t, T)\mathbb{E}^T_{\mathbb{Q}_d}\left[\pi(T, F_f)\right] \tag{1}$$

Hence the pricing in the context of a multi-curve framework involves the dynamic properties of the two interest rate markets M_d and M_f. In order to compute the expectation in (1) we need to know the relation between those two markets which would lead to a well known quanto adjustment resulting from the change of measure. An idea presented in [2] is to follow the natural analogy with the cross-currency derivatives pricing and introduce an exchange rate between the markets M_d and M_τ. The forward exchange rate $X_{fd}(t, T)$ must evolve according to a martingale process under the associated discounting T-forward measure $\mathbb{Q}^T_d$.

$$\frac{dX_{fd}(t, T)}{X_{fd}(t, T)} = \sigma_X(t)dW^T_X$$

[2]The discounting curve d is not necessarily one of the tenor curves but is constructed by every market participant based on the funding or collateral setting using the preferred selection of interest rate instruments and bootstrapping algorithms

Assuming a lognormal (or normal) martingale dynamics for $F_\tau(t, T_1, T)$ in the respective T-forward measure with volatility $\sigma_f(t)$ and correlation $\rho_{fX}(t)$ between forward exchange rate and fixing rate Brownian motions, the expression for the quanto adjustment in (1) will collapse to a product of $\sigma_X(t)$, $\rho_{fX}(t)$ and $\sigma_f(t)$.

A non-trivial quanto adjustment implies that the today's yield curves $T \to P_f(t_0, T)$ will not be "repriced" within the multicurve framework, which implies that the current pricing approach is in-principle not arbitrage free. However, there are not enough instruments traded in the market in order to exploit this arbitrage opportunity. At the same time the model becomes underdetermined and one has to resort to historical estimates for the "synthetic" forward exchange rate parameters or use certain functional form assumtions (one suggestion in [2] being $\rho_{fX} \simeq 1$).

Another practical consideration is the extention of the model to several fixing curves in the context of calculating XVA, where even vanilla instruments will become volatility dependent. In order to capture the basis spread dynamics between two fixing curves f_1 and f_2 (e.g. 3M6M or OIS-LIBOR spread), special care needs to be taken with regards to the curve covariance matrix ρ_{f_1,f_2}. The historical curve-curve correlations are very close to 1 and the spread volatility is hightly sensitive to the curve correlations in that region. Therefore constructing a PSD correlation matrix without losing control over the spread volatilities can become a challenging task.

A further shortcoming of directly modelling the forward fixing rates $F_{f_1}(t, T_1, T_2)$ and $F_{f_2}(t, T_1, T_2)$ for simulating future exposures are possible "unrealistic" realisations of the fixing curves in relation to each other, i.e. "unrealistic" basis spread term structure. For example, the OIS-LIBOR basis could well become negative in the simuation.

In order to address the above considerations we choose to model the fixings curves by spreading them from a specified base curve. The approach is a special case of the above outlined multi-curve framework, with a special choice of the synthetic quanto term $\rho_{fX}(t)\sigma_X(t) = 0$. In addition, we choose to formulate the model dynamics in the HJM framework.

Remark 1. *Given a vanishing synthetic quanto term, the instantaneous fixing forward rate is a martingale in the discounting T-forward measure.*

3.2. **Construction of Drift in Multi Curve HJM.** Let $\{W(t)\}_{t\geq 0}$ be an n-dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, Q)$, $\{\mathcal{F}(t)\}_{t\geq 0} \subset \mathcal{F}$ equipped with a risk neutral measure Q. In HJM theory, the zero bond price is the solution to the following SDE

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt - \sigma_{P_d}(t,T)^T dW(t)$$

where $\sigma_{P_d}(t,T)$ is some n-dimensional adapted discount bond volatility process in "orthogonal" coordinates. The instantaneous forward rate with maturity T defined by

$$f_d(t,T) = -\frac{\partial}{\partial T} \ln\left(P_d(t,T)\right)$$

obeys the dynamics

$$df_d(t,T) = \sigma_d(t,T)^T \sigma_{P_d}(t,T)dt + \sigma_d(t,T)^T dW(t) \tag{2}$$

with $\sigma_d(t,T) = \frac{\partial}{\partial T}\sigma_{P_d}(t,T)$ or equivalently $\sigma_{P_d}(t,T) = \int_t^T \sigma_d(t,u)du$. The short rate is defined by $r_d(t) := f_d(t,t)$ and the banking account, which induces the risk neutral measure, has the following price process

$$M_d(t) = \exp\left(\int_0^t r_d(u)du\right)$$

In Q any tradable asset price is a martingale relative to the discount banking account price process, hence the discount bond price can be retrieved via

$$P_d(t,T) = \mathbb{E}\left(\exp\left(-\int_t^T r_d(u)du\right)\middle|\mathcal{F}(t)\right)$$

The T-forward measure is induced by the zero bond $P_d(t,T)$ and we have the familiar drift adjustment

$$dW^T(t) = \sigma_{P_d}(t,T)dt + dW(t)$$

such that W^T is an n-dimensional Brownian motion in the T-forward measure. Note that the instantaneous discount forward rate is a martingale in the T-forward measure.

We go ahead with spreading the discount curve and extend the idea to spreading the base curve. Before going there we introduce the following

Definition 1. *Let $T > 0$, $\sigma(\cdot,T)$ denote an n-dimensional adapted volatility process on $[0,T]$ and $\sigma_Z(t,T) := \int_t^T \sigma(t,u)du$. We say that a stochastic process $Z(\cdot,T)$ satisfies HJM (risk neutral) forward rate dynamics with n-dimensional volatility processes (or simply "volatility" σ), if*

$$dZ(t,T) = \sigma(t,T)^T\sigma_Z(t,T)dt + \sigma(t,T)^T dW(t)$$

Remark 2. *From (2) it follows that the instantaneous discount forward rate follows HJM dynamics with volatility σ_d.*

3.2.1. *Spreading Discount Curves.* We describe a general fixing bond by spreading the discount bond, i.e.

$$P_f(t,T) = P_d(t,T)\exp\left(\int_t^T s(t,u)du\right)$$

for some Ito spread process s. The above can be rewritten in terms of (instantaneous) forward rates as

$$f_f(t,T) = f_d(t,T) + s(t,T)$$

We can either model the fixing forward rates directly or via s. In a market scenario with relatively low spreads (being the case at the time of writing the paper), it turns out that modelling spreads directly is of advantage for reasons that we elaborate on in the sequel. Therefore we define spread dynamics in the risk neutral measure according to

$$ds(t,T) = \alpha(t,T)dt + \sigma_s(t,T)^T dW(t) \tag{3}$$

for some n-dimensional adapted spread volatility process σ_s in orthogonal coordinates. The drift process α is chosen such that f_f is a martingale in the $T-forward$ measure.

Proposition 1. *Under the assumptions in the multi curve framework and given that the spread s_f obeys (3), it follows that*

$$\alpha(t,T) = \sigma_s(t,T)^T\sigma_{P_d}(t,T) \tag{4}$$

Proof. Follows from Remark 2. For details, cf. [1], noting that the spread has reversed sign. □

Theorem 1. *Let*

$$\sigma_{P_s}(t,T) := \int_t^T \sigma_s(t,s)du$$
$$\tilde{\alpha}(t,T) := \sigma_s(t,T)^T\left(\sigma_{P_d}(t,T) - \sigma_{P_s}(t,T)\right)$$

Then the spread dynamics can be written as

$$ds(t,T) = \tilde{\alpha}(t,T)dt + d\tilde{s}(t,T)$$

where $\tilde{s}$ satisfies the classical HJM forward rate dynamics in the risk neutral measure with volatility σ_{s_f}. Setting $\tilde{s}(0,T) = s(0,T)$, we obtain the fixing forward and bond price solutions as drift adjusted composition of HJM solutions, i.e.

$$f_f(t,T) = \int_0^t \tilde{\alpha}(u,T)du + \tilde{s}(t,T) + f_d(t,T)$$
$$P_f(t,T) = \exp\left(-\int_t^T \int_0^t \tilde{\alpha}(u,v)dudv\right)\tilde{P}_s(t,T)P_d(t,T)$$

where $\tilde{P}_s(t,T) := \exp\left(-\int_t^T \tilde{s}(t,u)du\right)$.

Proof. Use the result from Proposition 1 to obtain the asserted spread dynamics. The forward rate is obtained by integration and the bond price by integrating one more time and exponentiating. □

Corollary 1. *Let*

$$\sigma_f(t,T) := \sigma_d(t,T) + \sigma_s(t,T)$$
$$\sigma_{P_f}(t,T) := \int_t^T \sigma_f(t,u)du$$
$$\tilde{\beta}(t,T) := \sigma_f(t,T)^T\left(\sigma_{P_d}(t,T) - \sigma_{P_f}(t,T)\right)$$

Then the fixing forward rate dynamics can be written as

$$df_f(t,T) = \tilde{\beta}(t,T)dt + d\tilde{f}_f(t,T)$$

where $\tilde{f}_f$ satisfies the HJM forward rate dynamics in the risk neutral measure with volatility σ_f. Choosing $\tilde{f}_f(0,T) = f_f(0,T)$, we obtain after integration the fixing forward and bond price solutions as drift adjusted HJM solutions

$$f_f(t,T) = \int_0^t \tilde{\beta}(u,T)du + \tilde{f}_f(t,T)$$
$$P_f(t,T) = \exp\left(-\int_t^T \int_0^t \tilde{\beta}(u,v)dudv\right)\tilde{P}_f(t,T)$$

where $\tilde{P}_f(t,T) := \exp\left(-\int_t^T \tilde{f}_f(t,u)du\right)$

Proof. The SDE for the instantaneous forward fixing rate can be retrieved by using the results from theorem 1 and inserting the definition for $\sigma_f(t,T)$. The remainder follows in analogy to Theorem 1. □

3.2.2. *Spreading base curves.* Historical observations of the covariance structure show that the various tenor curves are similar. We wish to make use of these similarities in that we model one representative tenor curve as our base curve and spread the remaining tenor curves from that base curve.

Let the subscript b denote the base curve as a special fixing curve, f a general fixing curve (the discount curve d being a special case). We model the base curve and fixing curve dependent spread s_f, s_d to obtain a general fixing curve

$$f_f(t,T) = f_b(t,T) + s_f(t,T) \tag{5}$$

We note that the discount curve is still unique as it represents the yield on a risk neutral cash investment and therefore the drift considerations from section 3.2.1 are still valid and necessary. For $f = d$, we particularly obtain

$$f_d(t,T) = f_b(t,T) + s_d(t,T) \tag{6}$$

Proposition 2. *Suppose the general fixing curve is defined through (5) and that*

$$\begin{aligned} df_b(t,T) &= \tilde{\alpha}_b(t,T)dt + d\tilde{f}_b(t,T) \\ ds_f(t,T) &= \tilde{\alpha}_{s_f}(t,T)dt + d\tilde{s}_f(t,T) \end{aligned}$$

where $\tilde{f}_b, \tilde{s}$ *are HJM forward rate processes with n-dimesnional volatility processes* $\sigma_b(t,T), \sigma_s(t,T)$ *respectively. Then*

$$\begin{aligned} \tilde{\alpha}_b(t,T) &= \sigma_b(t,T)^T \sigma_{P_{s_d}}(t,T) \\ \tilde{\alpha}_{s_f}(t,T) &= \sigma_{s_f}(t,T)^T \left(\sigma_{P_b}(t,T) + \sigma_{P_{s_d}}(t,T) - \sigma_{P_{s_f}}(t,T)\right) \end{aligned}$$

Proof. We note from (6) that

$$f_b(t,T) = f_d(t,T) - s_d(t,T)$$

This matches the representation in [1] for the base curve being a special fixing curve. It implies the dynamics of s_d to be

$$\begin{aligned} ds_d(t,T) &= \sigma_{s_d}(t,T)^T \sigma_{P_d}(t,T)dt + \sigma_{s_d}(t,T)^T dW(t) \\ &= \sigma_{s_d}(t,T)^T \left(\sigma_{P_d}(t,T) - \sigma_{P_{s_d}}(t,T))\right) dt + d\tilde{s}_d(t,T) \end{aligned}$$

where $\tilde{s}_d$ obey HJM forward rate dynamics. Noting that by Corollary 1, $\sigma_{P_d}(t,T) = \sigma_{P_b}(t,T) + \sigma_{P_{s_d}}(t,T)$, the claim for s_d follows.
The base curve is a special fixing curve, hence its dynamics follow also from Corollary 1 with

$$df_b(t,T) = \sigma_b(t,T)^T \left(\sigma_{P_d}(t,T) - \sigma_{P_b}(t,T)\right) + d\tilde{f}_b(t,T)$$

where $\tilde{f}_b$ obeys HJM forward rate dynamics. Replacing $\sigma_{P_d}(t,T)$ implies the assertion for f_b.
The claim for the general s follows analogously to the proof of Proposition 1 using the dynamics of the base curve and the fact that $f_f(\cdot,T)$ is a martingale in T-forward measure. □

Theorem 2. *Given that* $\tilde{f}_b(0,T) = f_b(0,T)$ *and* $\tilde{s}_f(0,T) = s_f(0,T)$*, the general fixing curve can be expressed through a drift adjusted composition of HJM solutions.*

$$f_f(t,T) = \int_0^t \tilde{\alpha}_f(u,T)du + \tilde{f}_b(t,T) + \tilde{s}_f(t,T)$$

and

$$P_f(t,T) = \exp\left(-\int_t^T \int_0^t \tilde{\alpha}_f(u,v)dudv\right) \tilde{P}_b(t,T)\tilde{P}_{s_f}(t,T) \tag{7}$$

where

$$\tilde{\alpha}_f(t,T) = \left(\sigma_b(t,T) + \sigma_{s_f}(t,T)\right)^T \sigma_{P_{s_d}}(t,T) + \sigma_{s_f}(t,T)^T \left(\sigma_{P_b}(t,T) - \sigma_{P_{s_f}}(t,T)\right)$$

Proof. Insert the results from Proposition 2 and apply arguments along the lines of Theorem 1. □

Remark 3. *Given that* $\tilde{f}_b(0,T) = f_b(0,T)$ *and* $\tilde{s}(0,T) = s(0,T)$*, the general fixing curve dynamics reduce to the result of corollary 1 given that*

$$\sigma_f(t,T) = \sigma_b(t,T) + \sigma_{s_f}(t,T)$$

3.3. **The 2-factor Hull-White Model.**

Definition 2. *A vector of Ito processes is given in "marginal" representation, if the Brownian motions are instantaneously correlated.*

We proceed with a useful Remark, which translates dynamics from marginal (generally used to formulate the model) to the standard orthogonal representation, where Brownian motions are independent.

Remark 4. *Consider HJM dynamics with a volatility process $\widehat{\sigma}(t,T)$ in marginal coordinates. Then the orthogonal volatility process is given by*

$$\sigma(t,T) = C^T \widehat{\sigma}(t,T)$$

where $P = CC^T$ is a square root decomposition of the instantaneous correlation matrix.

Proof. The result follows from the observation that correlated Brownian motions can be generated via square root decomposition, i.e.

$$\sigma(t,T)^T dW =: \widehat{\sigma}(t,T)^T d\widehat{W} = \widehat{\sigma}(t,T)^T C dW$$

which implies

$$\sigma(t,T) = (\widehat{\sigma}(t,T)^T C)^T = C^T \widehat{\sigma}(t,T)$$

where $\tilde{W}$ denotes the 2-dimensional vector of correlated Brownian motions. □

Next we assume that the n-dimensional HJM volatility is bounded (deterministic) and separable, i.e. there exists a matrix function $g \in \mathbb{R}^{n \times n}$ and a vector function $h \in \mathbb{R}^n$, such that

$$\sigma(t,T) = g(t)h(T)$$

Standard results show that in this case, dynamics can be generated by an n-dimensional Hull-White model. The number of factors is not unique. We decide on 2 factors for curves to allow for parallel shifts and rotations. Principal component analysis shows that with two factors, a sufficiently large part of curve dynamics are captured. By interpreting the spread as the difference of curves, the spread is a two factor model, once the two mean reversion parameters (defining the two time scales) are aligned. Therefore we model the spread with two factors as well and comment on the 1-factor special case in the sequel.
We formulate the 2-factor Hull White framework in "marginal" rather then orthogonal coordinates, i.e. the Brownian drivers are correlated. The orthogonal vector forward rate volatility processs will then be determined.
In an additive 2-factor Hull-White model (cf. [1]), the short rate is defined as

$$r(t) = f(0,t) + z_1(t) + z_2(t)$$

where the dynamics of the two factors z_1 and z_2 in risk neutral measure are given by

$$dz_1(t) = (-a_1(t)z_1(t) + y_1(t) + y_3(t))\,dt + \sigma_1(t)dW_1(t) \tag{8}$$

$$dz_2(t) = (-a_2(t)z_2(t) + y_2(t) + y_3(t))\,dt + \sigma_2(t)dW_2(t), \tag{9}$$

a_i are the mean reversion speeds, σ_i are time dependent, bounded volatility functions, W_i are instantaneously correlated Brownian motions with

$$dW_1(t)dW_2(t) = \rho(t)dt$$

and we define

$$
\begin{aligned}
g_i(t) &= \exp\left(-\int_0^t a_i(u)du\right)\\
y_i(t) &= \int_0^t \frac{g_i^2(t)}{g_i^2(u)}\sigma_i^2(u)du\\
y_3(t) &= \int_0^t \frac{g_1(t)g_2(t)}{g_1(u)g_2(u)}\sigma_1(u)\sigma_2(u)\rho(u)du
\end{aligned}
$$

for $i = 1, 2$. Given a maturity $T > 0$, the zero-bond and forward rate price in the model is defined by

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\left(\exp\left(-\int_t^T r(u)du\right)\Big|\mathcal{F}(t)\right)\\
f(t,T) &= -\frac{\partial}{\partial T}\ln\left(P(t,T)\right)
\end{aligned}
$$

and the short rate can be retrieved via the known relation $r(t) = f(t,t)$.

Theorem 3. *Let the dynamics of the two factors z_1, z_2 be given by equations (8) and (9) repectively. Then for i=1,2, the solutions are given by*

$$
z_i(t) = g_i(t)z_i(0) + \int_0^t \frac{g_i(t)}{g_i(u)}(y_i(u) + y_3(u))du + \int_0^t \frac{g_i(t)}{g_i(u)}\sigma_i(u)dW_i(u)
$$

Proof. Follows from the 2-factor Hull-White analysis in [1]. □

Proposition 3. *Let the short rate r be governed by an additive 2-factor Hull-White model with mean reversion parameters a_1, a_2 and volatility functions $\sigma_1(\cdot), \sigma_2(\cdot)$. Then the vector of HJM marginal volatilities is given by*

$$
\widehat{\sigma}(t,T) = \begin{pmatrix} \sigma_1(t)\frac{g_1(T)}{g_1(t)} \\ \sigma_2(t)\frac{g_2(T)}{g_2(t)} \end{pmatrix}
$$

where the corresponding vector of correlated Brownian motions is given by

$$
\widehat{W}(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}
$$

Proof. Follows from the 2-factor Hull-White analysis in [1]. □

Theorem 4. *Let the short rate r be governed by an additive 2-factor Hull-White model with mean reversion speeds $a_1(\cdot), a_2(\cdot)$ and volatility functions $\sigma_1(\cdot), \sigma_2(\cdot)$. Define the initial termstructures $P(0) := P(0,\cdot), f(0) := f(0,\cdot)$ and*

$$
\begin{aligned}
G_i(t) &:= \int_0^t g_i(u)du\\
G_i(t,T) &:= \frac{G_i(T) - G_i(t)}{g_i(t)}
\end{aligned}
$$

Then zero bond and forward rate prices are given by

$$\begin{aligned}
P(t,T) =& P^{2FHW}(t,T,z_1(t),z_2(t),P(0)) \\
:=& \frac{P(0,T)}{P(0,t)} \exp\Big(- G_1(t,T)z_1(t) - G_2(t,T)z_2(t) - \frac{1}{2}G_1^2(t,T)y_1(t) - \frac{1}{2}G_2^2(t,T)y_2(t) \\
& -G_1(t,T)G_2(t,T)y_3(t)\Big) \\
f(t,T) =& f^{2FHW}(t,T,z_1(t),z_2(t),f(0)) \\
:=& f(0,T) + \frac{g_1(T)}{g_1(t)}z_1(t) + \frac{g_2(T)}{g_2(t)}z_2(t) + G_1(t,T)\frac{g_1(T)}{g_1(t)}y_1(t) + G_2(t,T)\frac{g_2(T)}{g_2(t)}y_2(t) \\
& + \left(G_1(t,T)\frac{g_2(T)}{g_2(t)} + G_2(t,T)\frac{g_1(T)}{g_1(t)}\right)y_3(t)
\end{aligned}$$

Proof. The results are standard and can be taken e.g. from [1]. □

Remark 5. *We have the identity*

$$f^{2FHW}(t,T,z_1(t),z_2(t),f(0)) = f(0,T) + \int_0^t \widehat{\sigma}(u,T)^T P(u)\widehat{\sigma}_P(u,T)du + \int_0^t \widehat{\sigma}(u,T)^T d\widehat{W}(u)$$

where

$$P(t) = \begin{pmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{pmatrix}$$

is the system's instantaneous correlation matrix.

Proof. Follows from integrating the HJM forward rate dynamics. □

3.4. **Application to the HJM Multi Curve Setting.** Having the results from section 3.2.2 in mind, it suffices to concentrate on the HJM variables denoted by the tilde accent. We focus on modelling the fixing curve through the base curve and a general fixing curve spread (the discount curve being a special case), i.e.

$$f_f(t,T) = f_b(t,T) + s(t,T) = \int_0^t \tilde{\alpha}_f(u,T)du + \tilde{f}_b(t,T) + \tilde{s}(t,T) \tag{10}$$

where the spread subscript f is dropped for readability. We now assume that the HJM variables $\tilde{f}_b$ and $\tilde{s}$ are driven by additive 2-factor Hull-White short rate models:

$$\begin{aligned}
\tilde{r}_b(t) &= f_b(0,t) + z_{1b}(t) + z_{2b}(t) \\
\tilde{r}_s(t) &= s(0,t) + z_{1s}(t) + z_{2s}(t)
\end{aligned}$$

with

$$\begin{aligned}
dz_{1x}(t) &= \left(-a_{1x}(t)z_{1x}(t) + y_1(t) + y_{3x}(t)\right)dt + \sigma_{1x}(t)dW_{1x}(t) \\
dz_{2x}(t) &= \left(-a_{2x}(t)z_{2x}(t) + y_2(t) + y_{3x}(t)\right)dt + \sigma_{2x}(t)dW_{2x}(t) \\
g_{ix}(t) &= \exp\left(-\int_0^t a_{ix}(u)du\right) \\
y_{ix}(t) &= \int_0^t \frac{g_{ix}^2(t)}{g_{ix}^2(u)}\sigma_{ix}^2(u)du \\
y_{3x}(t) &= \int_0^t \frac{g_{1x}(t)g_{2x}(t)}{g_{1x}(u)g_{2x}(u)}\sigma_{1x}(u)\sigma_{2x}(u)du
\end{aligned}$$

and the instantaneous correlation structure

$$dW_{ix}dW_{jy} = \rho_{ix,jy}(t)dt$$

where $i, j = 1, 2$ and $x, y \in \{b, s\}$. We then obtain the HJM base curve and spread forward rates through

$$\begin{aligned}
\tilde{P}_x(t,T) &= \mathbb{E}\left(\exp\left(-\int_t^T \tilde{r}_x(u)du\right)\middle|\mathcal{F}(t)\right)\\
\tilde{f}_b(t,T) &= -\frac{\partial}{\partial T}\ln\left(\tilde{P}_b(t,T)\right)\\
\tilde{s}(t,T) &= -\frac{\partial}{\partial T}\ln\left(\tilde{P}_s(t,T)\right)
\end{aligned}$$

In order to make the model time homogeneous, we choose correlation, mean reversion speeds and volatility ratio time independent, meaning

$$\begin{aligned}
a_{ix}(t) &= a_{ix}\\
\rho_{ix,jy}(t) &= \rho_{ix,jy}\\
\sigma_{2x}(t) &= \alpha_x \sigma_{1x}(t)
\end{aligned}$$

for some constants $a_{ix} > 0, \rho_{ix,jy} \in [-1,1], \alpha_x > 0$, $i = 1, 2, x \in \{b, s\}$.

Theorem 5. *Suppose the general instantaneous fixing forward rate is governed by (10) with the above stated dynamics for base and spread forward rates. Then the corresponding solutions for fixing forward rate and bond price are given by*

$$\begin{aligned}
f_f(t,T) =& \int_0^t \tilde{\alpha}_f(u,T)du + f^{2FHW}(t,T,x_{1b}(t),x_{2b}(t),f_b(0)) + f^{2FHW}(t,T,x_{1s}(t),x_{2s}(t),s(0))\\
P_f(t,T) =& \exp\left(-\int_t^T\int_0^t \tilde{\alpha}_f(u,v)dudv\right)\cdot\\
& P^{2FHW}(t,T,x_{1b}(t),x_{2b}(t),P_b(0))P^{2FHW}(t,T,x_{1s}(t),x_{2s}(t),P_s(0))
\end{aligned}$$

and in marginal representation

$$\tilde{\alpha}_f(t,T) = \left(\widehat{\sigma}_b(t,T) + \widehat{\sigma}_s(t,T)\right)^T P\widehat{\sigma}_{P_{s_d}}(t,T) + \widehat{\sigma}_s(t,T)^T P\left(\widehat{\sigma}_{P_b}(t,T) - \widehat{\sigma}_{P_s}(t,T)\right)$$

$$\begin{aligned}
\widehat{\sigma}_b(t,T)^T &= \left(\sigma_{1b}(t)\frac{g_{1b}(T)}{g_{1b}(t)}, \sigma_{2b}(t)\frac{g_{2b}(T)}{g_{2b}(t)}, 0, 0, 0, 0\right)\\
\widehat{\sigma}_{s_d}(t,T)^T &= \left(0, 0, \sigma_{1s_d}(t)\frac{g_{1s_d}(T)}{g_{1s_d}(t)}, \sigma_{2s_d}(t)\frac{g_{2s_d}(T)}{g_{2s_d}(t)}, 0, 0\right)\\
\widehat{\sigma}_s(t,T)^T &= \left(0, 0, 0, 0, \sigma_{1s}(t)\frac{g_{1s}(T)}{g_{1s}(t)}, \sigma_{2s}(t)\frac{g_{2s}(T)}{g_{2s}(t)}\right)
\end{aligned}$$

where $P = (\rho_{ix,jy})_{i,x}, i, j = 1, 2$ and $x, y \in \{b, s_d, s\}$ denotes the system's instantaneous correlation matrix.

Proof. The results follow from combining the results of Theorems 2 and 4. □

Proposition 4. *The doubly integrated Hull-White drift correction $\tilde{\alpha}_f(t,T)$ can be interpeted as the quanto adjustment resulting from modeling the integrated base curve and spread short rate. This means*

$$Cov\left(\int_t^T \tilde{r}_b(u)du, \int_t^T \tilde{r}_s(u)du\middle|\mathcal{F}(t)\right) = \int_t^T\int_t^v \tilde{\alpha}_f(u,v)dudv$$

Proof. Note that by Equation (7), we can write

$$P_f(t,T) = \exp\left(-\int_t^T \int_0^t \tilde{\alpha}_f(u,v)dudv\right)\tilde{P}_b(t,T)\tilde{P}_s(t,T) \tag{11}$$

On the other hand by Theorem 5, we can make use of the bond price being a conditional expectation with respect to the banking account, i. e.

$$\begin{aligned} r_f(t) = f_f(t,t) &= \int_0^t \tilde{\alpha}_f(u,t)du + \tilde{f}_b(t,t) + \tilde{s}(t,t) \\ &= \int_0^t \tilde{\alpha}_f(u,t)du + \tilde{r}_b(t) + \tilde{r}_s(t) \end{aligned}$$

and hence

$$\begin{aligned} P_f(t,T) =& \mathbb{E}\left(\exp\left(-\int_t^T \tilde{r}_b(u) + \tilde{r}_s(u) + \int_0^u \tilde{\alpha}_f(v,u)dvdu\right)\Big|\mathcal{F}(t)\right) \\ =& \exp\left(-\int_t^T \int_0^u \tilde{\alpha}_f(v,u)dvdu\right)\mathbb{E}\left(\exp\left(-\int_t^T \tilde{r}_b(u)+\tilde{r}_s(u)du\right)\Big|\mathcal{F}(t)\right) \\ =& \exp\left(-\int_t^T \int_0^u \tilde{\alpha}_f(v,u)dvdu\right)\exp\left(-\mathbb{E}\left(\int_t^T \tilde{r}_b(u)du\Big|\mathcal{F}(t)\right) - \mathbb{E}\left(\int_t^T \tilde{r}_s(u)du\Big|\mathcal{F}(t)\right)\right) \\ & \exp\left(\frac{1}{2}Var\left(\int_t^T \tilde{r}_b(u)du\Big|\mathcal{F}(t)\right) + \frac{1}{2}Var\left(\int_t^T \tilde{r}_s(u)du\Big|\mathcal{F}(t)\right)\right) \\ & \exp\left(Cov\left(\int_t^T \tilde{r}_b(u)du, \int_t^T \tilde{r}_s(u)du\Big|\mathcal{F}(t)\right)\right) \\ =& \exp\left(-\int_t^T \int_0^u \tilde{\alpha}_f(v,u)dvdu + Cov\left(\int_t^T \tilde{r}_b(u)du, \int_t^T \tilde{r}_s(u)du\Big|\mathcal{F}(t)\right)\right)\tilde{P}_b(t,T)\tilde{P}_s(t,T) \end{aligned}$$

where in the third equation we made use of the fact that the short rate integral is normally distributed. Comparing the last equation above with Equation (11), we obtain the assertion by switching the integration variables u, v and merging the integrals. ☐

Note that by matching the mean reversion parameters for each component and therefore aligning to two time scales in the multi curve model, the fixing curve dynamics collapses to a two factor model as well. The next theorem defines the proposed model.

Theorem 6. *Let the conditions of Theorem 5 be satisfied. Furthermore, let $a_{ix} =: a_i$ and define*

$$\sigma_{ix}(t) = \sqrt{\sigma_{ib}^2(t) + 2\sigma_{ib}(t)\sigma_{is_x}(t)\rho_{ib,is_x} + \sigma_{is_x}^2(t)}$$

and

$$\rho_{ix,jy}(t) = \frac{\sigma_{is_x}(t)\sigma_{js_y}(t)\rho_{is_x,js_y} + \sigma_{is_x}(t)\sigma_{jb}(t)\rho_{jb,is_x} + \sigma_{ib}(t)\sigma_{js_y}(t)\rho_{ib,js_y} + \sigma_{ib}(t)\sigma_{jb}(t)\rho_{ib,jb}}{\sqrt{\sigma_{is_x}^2(t) + 2\sigma_{ib}(t)\sigma_{is_x}(t)\rho_{ib,is_x} + \sigma_{ib}^2(t)}\sqrt{\sigma_{js_y}^2(t) + 2\sigma_{jb}(t)\sigma_{js_y}(t)\rho_{jb,js_y} + \sigma_{jb}^2(t)}}$$

for $i = 1,2$, $x, y \in \{d, f\}$ (i.e. the multi curve/ spread system possesess two time scales defined by a_1, a_2).
Then the instantaneous forward fixing rate can be represented by a drift adjusted 2-factor additive

Hull-White model and

$$f_f(t,T) = \int_0^t \tilde{\beta}(u,T)du + f^{2FHW}(t,T,x_{1f}(t),x_{2f}(t),f_f(0))$$
$$P_f(t,T) = \exp\left(-\int_t^T \int_0^t \tilde{\beta}(u,v)dudv\right) P^{2FHW}(t,T,x_{1f}(t),x_{2f}(t),P_f(0))$$

where in marginal representation

$$\tilde{\beta}(t,T) = \widehat{\sigma}_f(t,T)^T P \left(\widehat{\sigma}_{P_d}(t,T) - \widehat{\sigma}_{P_f}(t,T)\right)$$
$$\widehat{\sigma}_d(t,T) = \left(\sigma_{1d}(t)\frac{g_1(T)}{g_1(t)}, \sigma_{2d}(t)\frac{g_2(T)}{g_2(t)}, 0, 0\right)$$
$$\widehat{\sigma}_f(t,T) = \left(0, 0, \sigma_{1f}(t)\frac{g_1(T)}{g_1(t)}, \sigma_{2f}(t)\frac{g_2(T)}{g_2(t)}\right)$$

and $P = (\rho_{ix,jy}(t))_{i,j,x,y}$.

Proof. Start with the forward rate solution given in Theorem 5. Make use of the identity in Remark 5. Observe that since the mean reversion parameters are aligned according to this theorem's premise, we obtain

$$\begin{aligned}
&\frac{g_{ib}(T)}{g_{ib}(t)}\sigma_{ib}(t)dW_{ib}(t) + \frac{g_{is}(T)}{g_{is}(t)}\sigma_{is}(t)dW_{is}(t) \\
=&\frac{g_i(T)}{g_i(t)}\left(\sigma_{ib}(t)dW_{ib}(t) + \sigma_{is}(t)dW_{is}(t)\right) \\
=&\frac{g_i(T)}{g_i(t)}\sigma_{if}(t)dW_{if}(t)
\end{aligned}$$

given that

$$\sigma_{if}(t) = \sqrt{\sigma_{ib}^2(t) + 2\sigma_{ib}(t)\sigma_{is}(t)\rho_{ib,is} + \sigma_{is}^2(t)}$$

and that W_{1f}, W_{2f} are correlated Brownian motions with instantaneous correlation

$$\rho_{1f,2f} = \frac{\sigma_{1s}(t)\sigma_{2s}(t)\rho_{1s,2s} + \sigma_{1s}(t)\sigma_{2b}(t)\rho_{2b,1s} + \sigma_{1b}(t)\sigma_{2s}(t)\rho_{1b,2s} + \sigma_{1b}(t)\sigma_{2b}(t)\rho_{1b,2b}}{\sqrt{\sigma_{1s}^2(t) + 2\sigma_{1b}(t)\sigma_{is_x}(t)\rho_{1b,1s} + \sigma_{1b}^2(t)}\sqrt{\sigma_{2s}^2(t) + 2\sigma_{2b}(t)\sigma_{2s}(t)\rho_{2b,2s} + \sigma_{2b}^2(t)}}$$

Note that the analogous argument holds for determining discount curve volatilities and correlations, since the discount curve is a special fixing curve. Finally group all drift terms and obtain $\tilde{\beta}(t,T)$ after some lengthy calculation. □

3.5. **Calibration.** With Theorem 6 from Section 3.4 we are equipped with a 2-factor Hull-White model for the base curve and a 2-factor Gaussian model for the spread curve. Moreover, by matching the two time scales a_1 and a_2, we ensure that the fixing curve is a Gaussian two factor model as well, which is consistent with historical observations. In this section we want to focus on calibrating the model parameters. Since focus is on stochastic tenor basis spread, we assume that the two factor interest rate model for the base curve is already calibrated. In particular, the mean reversion parameters a_1 and a_2, the instantaneous volitility termstructure $\sigma_b(t)$ and the intra base curve correlation $\rho_{1b,2b}$ as well as the intra base curve volatility scaling α_b are known.
The parameters left to calibrate are given by the intra spread correlation $\rho_{1s,2s}$, the spread volatility termstructure $\sigma_s(t)$ with its volatility scaling α_s and the cross correlations $\rho_{1b,1s}, \rho_{2b,2s}, \rho_{1b,2s}, \rho_{2b,1s}$. Since there are no liquid non-linear instruments on off-major tenor indices, we choose to calibrate

the parameters via historical observations. Because yield curve construction is an already arrived procedure in general, we choose the zero base rate

$$z_x(t,T) := -\frac{1}{T-t}\ln\left(P_x(t,T)\right)$$

for $x \in \{b, f\}$ and zero spread $z_s(t,T) := z_f(t,T) - z_b(t,T)$ as our historical observables.
To avoid under-determining, we make use of the base instantaneous volatility term structure to imply the spread instantaneous volatility term structure by imposing

$$\sigma_s(t) = \gamma_s \sigma_b(t)$$

for some constant parameter γ_s. This choice is also motivated by the fact that historically spread-base zero rate correlation is very low and the intuition that fixing rate volatility increases proportionally with base curve volatility. The following proposition helps us to quantify the parameters with historical samples.

Proposition 5. *Let the multi curve model be defined by Theorem 6. Let the constant tenor Δ zero rate be defined as $z_\Delta(t) := z(t, t+\Delta)$ and $G_i(\Delta) \equiv G_i(t, t+\Delta)$. Then for two tenors Δ_1, Δ_2*

$$\begin{aligned}
(a) \quad \Delta_1\Delta_2 \frac{\int_0^T dz_{b,\Delta_1}(t)dz_{b,\Delta_2}(t)}{\int_0^T \sigma_{1b}^2(t)dt} &= G_1(\Delta_1)G_1(\Delta_2) + \alpha^2 G_2(\Delta_1)G_2(\Delta_2) \\
&\quad + \alpha\left(G_1(\Delta_1)G_2(\Delta_2) + G_1(\Delta_2)G_2(\Delta_1)\right)\rho_{1b,2b} \\
(b) \quad \frac{\Delta_1\Delta_2}{\gamma_s^2} \frac{\int_0^T dz_{s,\Delta_1}(t)dz_{s,\Delta_2}(t)}{\int_0^T \sigma_{1b}^2(t)dt} &= G_1(\Delta_1)G_1(\Delta_2) + \alpha^2 G_2(\Delta_1)G_2(\Delta_2) \\
&\quad + \alpha\left(G_1(\Delta_1)G_2(\Delta_2) + G_1(\Delta_2)G_2(\Delta_1)\right)\rho_{1s,2s} \\
(c) \quad \frac{\Delta^2}{\gamma_{\tilde{s}}^2} \frac{\int_0^T dz_{s,\Delta}(t)dz_{\tilde{s},\Delta}(t)}{\int_0^T \sigma_{1b}^2(t)dt} &= \rho_{1s,1\tilde{s}} G_1(\Delta)^2 + \alpha^2 \rho_{2s,2\tilde{s}} G_2(\Delta)^2 \\
&\quad + \alpha\left(\rho_{1s,2\tilde{s}} + \rho_{2s,1\tilde{s}}\right) G_1(\Delta)G_2(\Delta)
\end{aligned}$$

Proof. Use the bond price solutions for base and fixing bonds from Theorem 6 to obtain zero rate expressions. Then do straight forward Ito calculus and note that the modulator functions $G_i(\Delta), i = 1, 2$ are merely tenor dependent. $\square$

The two expressions for total quadratic covariation can be used to imply γ_s from historical calculations with known results stated in the appendix. In fact, if $\sigma_h(\Delta)$ is the historical sample volatility of zero rates with tenor Δ on a time window of length T with high frequent observations, then

$$\sigma_h(\Delta) = \sqrt{\frac{1}{T}\int_0^T dz_{s,\Delta}^2(t)}$$

which with Proposition 5 implies

$$\gamma_s = \sqrt{\frac{\sigma_{s,h}(T)}{\sigma_{b,h}(T)}}$$

Moreover, if $\rho_h(\Delta_1, \Delta_2)$ is the historical sample correlation of zero rates with tenor Δ_1 and Δ_2 respectively on a time window of length T with high frequent observations, then

$$\rho_h(\Delta_1, \Delta_2) = \frac{\int_0^T dz_{s,\Delta_1}(t)dz_{s,\Delta_2}(t)}{\sqrt{\int_0^T dz_{s,\Delta_1}^2(t)\int_0^T dz_{s,\Delta_2}^2(t)}}$$

which implies the instantaneous $\rho_{1s,2s}$. We make use of

Proposition 6. *In the 2F HW model the instantaneous zero rate correlation between two tenors $z(t,T_1)$ and $z(t,T_2)$ is given by*

$$\frac{dz_{s,\Delta_1}(t)dz_{s,\Delta_2}(t)}{dt} = \frac{1+(\frac{G_2(\Delta_1)}{G_1(\Delta_1)}+\frac{G_2(\Delta_2)}{G_1(\Delta_2)})\alpha\rho_{1s,2s}+\frac{G_2(\Delta_1)}{G_1(\Delta_1)}\frac{G_2(\Delta_2)}{G_1(\Delta_2)}\alpha^2}{\sqrt{1+2\frac{G_2(\Delta_1)}{G_1(\Delta_1)}\alpha\rho+(\frac{G_2(\Delta_1)}{G_1(\Delta_1)}\alpha)^2}\sqrt{1+2\frac{G_2(\Delta_2)}{G_1(\Delta_2)}\alpha\rho+(\frac{G_2(\Delta_2)}{G_1(\Delta_2)}\alpha)^2}} \tag{12}$$

Proof. Follows from the zero rate dynamic under 2F HW model together with Proposition 5. □

Finally, historical observations show verly low correlation between spread zero rates and base zero rates, hence we consider zero cross correlations as an adequate estimator for the mid-2015 market regime, i.e.

$$\rho_{1b,1s} = \rho_{2b,2s} = \rho_{1b,2s} = \rho_{2b,1s} = 0$$

In order to relax this choice and allow for non-trivial cross correlations, more combinations of spread and base historical sample correlations will have to be used in order to at least have an overdetermined system of equations. We make use of Proposition 12 and resort to a historical estimation of the correlation parameter on the left hand side in equation (12). We choose the 1Y spread zero rate and the 10Y spread zero rate and observe as of June 2015 historical correlations in the area between 0.2 and 0.4. This holds for all USD spreads with exception of the long tenor 12M, where the observed correlation is 0. It is possible that the "rotation" point in the 12M/3M spread curve is shifted more to the right due to the long tenor 12M, the correlation between 2Y and 20Y spread zero rates is 0.33 which lies again within the above range. Using the 2F HW parameters from 30th June 2015, equation (12) implies that the correlation $\rho_{1s,2s}$ lies between -0.8 and -0.7. This is in line with the calibrated base correlation $\rho_{1b,2b}$. Therefore, in order to reduce the number of parameters, a potential choice for the 2F spread correlation could be $\rho_{1s,2s} = \rho_{1b,2b}$.

There exists a relationship for the inter-curve correlations of the base curve, the spread and the fixing curve, which can be helpful in the calibration. Since we identify the factors of the base, the spread and the fixing curve with each other, i.e. $x_{j,f}(t) = x_{j,b}(t) + x_{j,s}(t)$ for $j \in 1,2$ and all $t \in [0,T]$, we get the following.

Proposition 7. *With the notations as above*

$$(1+\gamma_s^2)\rho_{1f,2f} = \rho_{1b,2b} + \gamma_s(\rho_{1b,2s}+\rho_{2b,1s}) + \gamma_s^2\rho_{1s,2s}.$$

Proof. With the identification of the risk factors

$$\begin{aligned} Cov(x_{1,f}(t),x_{2,f}(t)) =& Cov(x_{1,b}(t)+x_{1,s}(t),x_{2,b}(t)+x_{2,s}(t)) \\ =& Cov(x_{1,b}(t),x_{2,b}(t)) + Cov(x_{1,b}(t),x_{2,s}(t)) \\ &+ Cov(x_{1,s}(t),x_{2,b}(t)) + Cov(x_{1,s}(t),x_{2,s}(t)). \end{aligned}$$

Dividing by the standard deviations $\sqrt{Var(x_{1,f}(t))Var(x_{2,f}(t))}$ and keeping in mind the relations between the standard deviations, i.e. $Var(x_{j,f}(t)) = (1+\gamma_s^2)Var(x_{j,b}(t))$ and $Var(x_{j,s}(t)) = \gamma_s^2 Var(x_{j,b}(t))$, for $j=1,2$, we arrive at

$$(1+\gamma_s^2)\rho_{1f,2f} = \rho_{1b,2b} + \gamma_s(\rho_{1b,2s}+\rho_{2b,1s}) + \gamma_s^2\rho_{1s,2s},$$

which finishes the proof. □

The model described in this section produces negative spreads with positive probability. This feature has rarely been observed in the market and does not make sense from a point of view of economic theory. Therefore, a treatment for negative spreads might be in order - see Appendix B for more details.

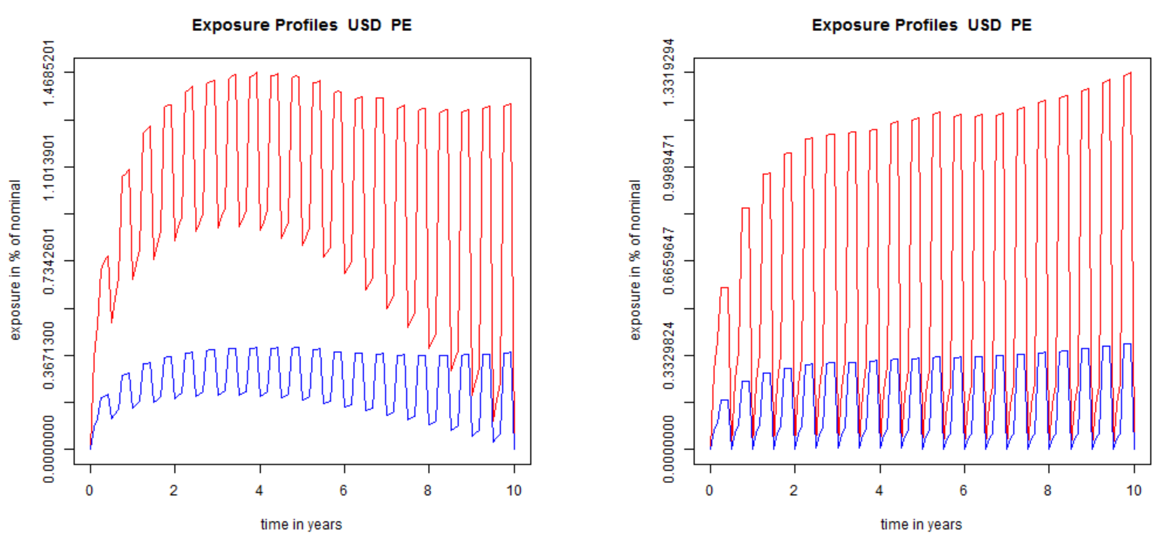
4. Empirical Exposure Profiles

In this section we present some graphs that come from a simulation of the model. We will show in particular exposure profiles of basis swaps and explain the main drivers of their shape. The exposure numbers we will use are expected exposure (EE) and potential exposure at a 95%-quantile (PE). These quantities are defined by

$$\mathrm{EE}(t) := \mathbb{E}\left(\max(V(t), 0)\right) \quad \text{and} \quad \mathrm{PE}_\alpha(t) := \inf\left\{X(t);\, \mathbb{P}(V(t) \geq X(t)) \leq 1-\alpha\right\},$$

where $V = (V(t))_{t\geq 0}$ denotes the value of a portfolio of financial instruments.

In the following we simulate these quantities for a hypothetical (but realistically parametrized) basis swap, swapping 3M-USD Libor against 6M-USD Libor. We will look at long and short variants of the swap paying-3M and receiving-6M. The swap has a maturity of 10 years and the payments are made according to their tenor, 3M is paid or received every three months and 6M every six months. Furthermore, the swap is uncollateralized so that the impact of the stochastic basis the over the whole lifetime of the swap becomes visible. In the figures below, the value on the ordinate axis is the exposure in % of the nominal of the swap. However, we are more interested in the shape of the exposure profiles and less in the absolute size. The simulation was done by simulating 15000 paths and one point each month.

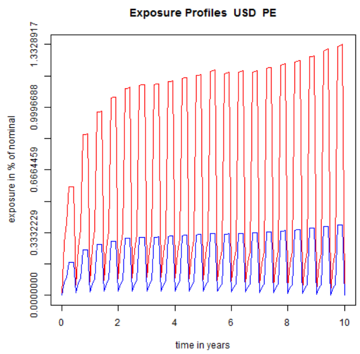


(A) (B)

Figure 4. EE and PE profiles for a 3M-payer-6M-receiver basis swap with and without stochastic basis

In Figure 4 we show the EE and the PE for a basis swap paying 3M-USD Libor and receiving 6M-USD Libor, on the left with stochastic basis and on the right without stochastic basis.

One can see that the inclusion of the stochastic basis has one particular consequence for the profiles of such a basis swap: the appearance of a lower arc for both the EE and the PE profiles. In fact, these profiles contain three particular elements: a lower arc, columns of varying height and an upper arc. The interpretation of the arcs depends on whether the shorter tenor is paid or received. If the shorter tenor is paid, then the lower arc represents the exposure of the remaining swap with no payment mismatch. The upward columns are formed by two effects, firstly the fixed 3M tenor payment and secondly the difference of the subsequent 3M and 6M payment. The upper



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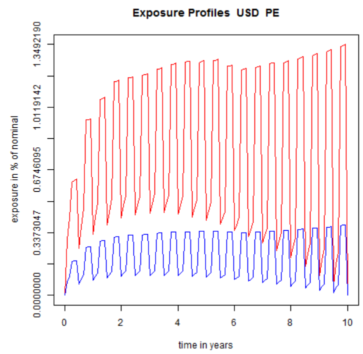

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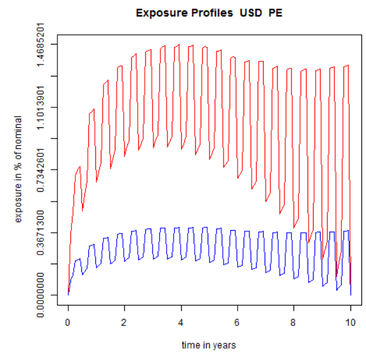

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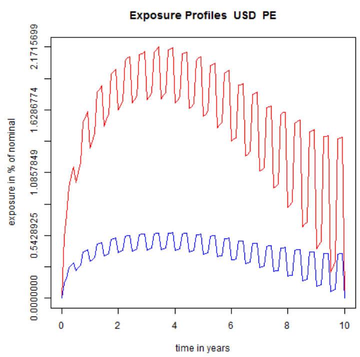

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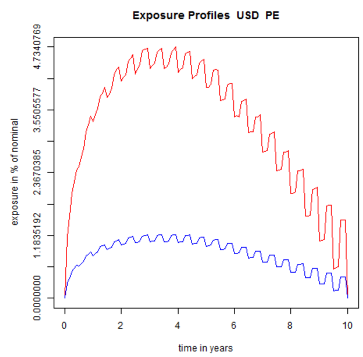

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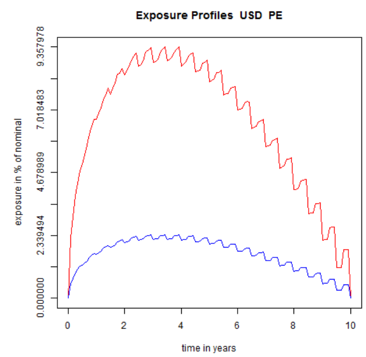

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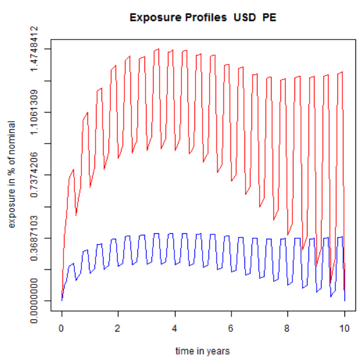

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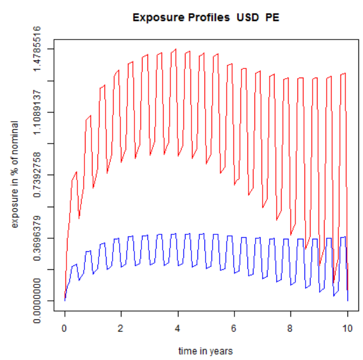

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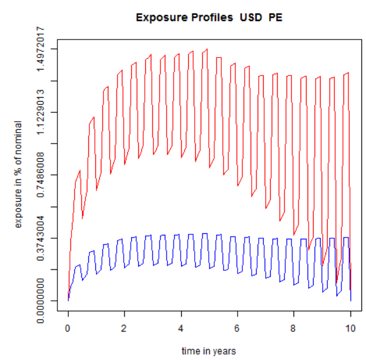

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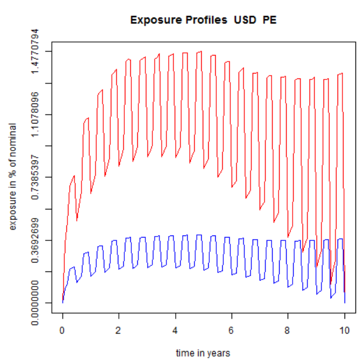

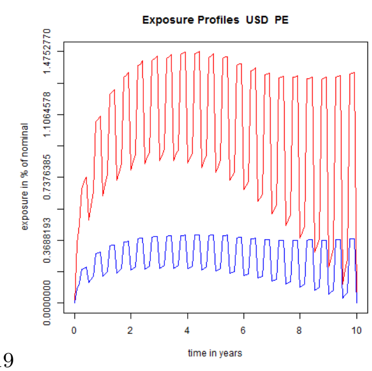

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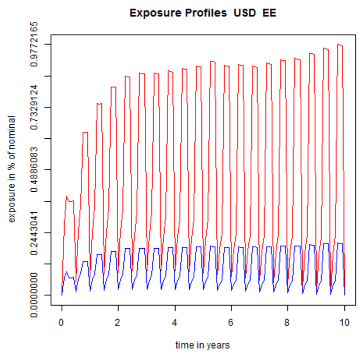
FIGURE 5. EE and PE profiles for a 3M-payer-6M-receiver basis varying the volatility of the basis spread (first and second line) and the correlation of the basis spread with the base curve (3M-USD Libor) (third and fourth line)



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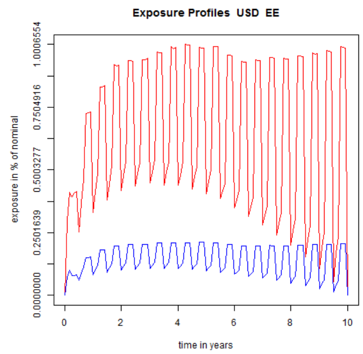

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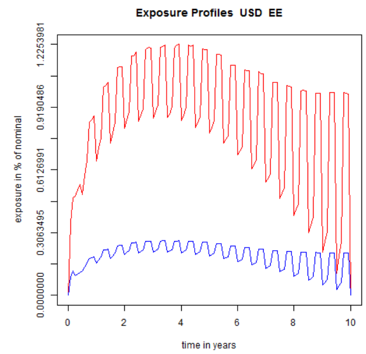

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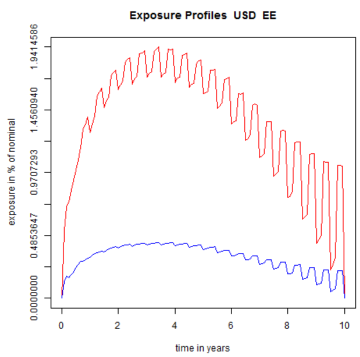

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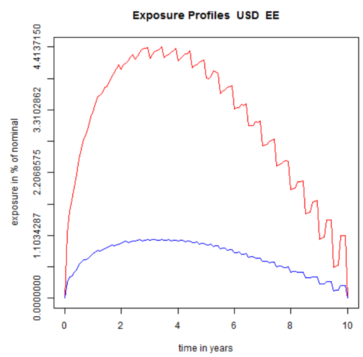

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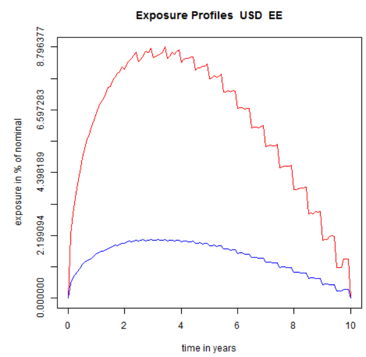

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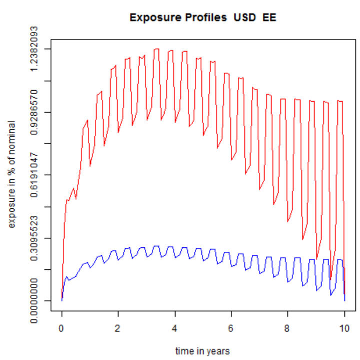

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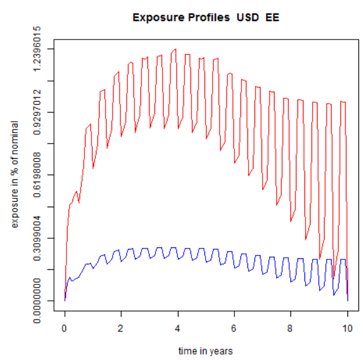

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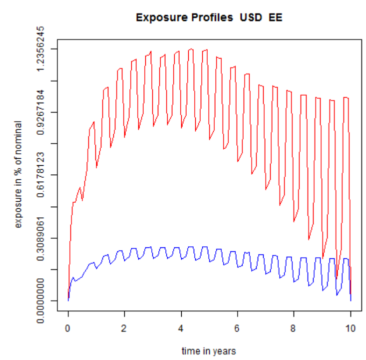

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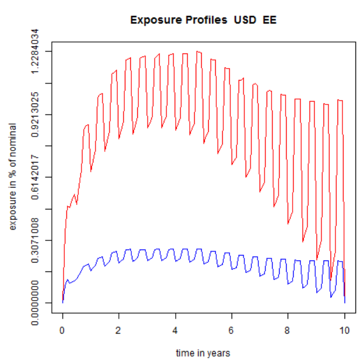

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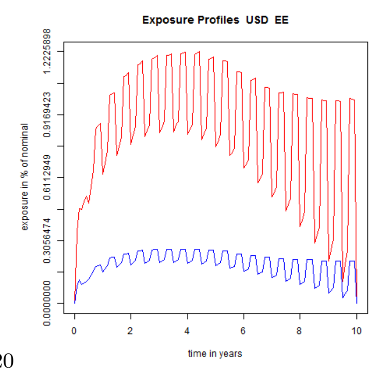

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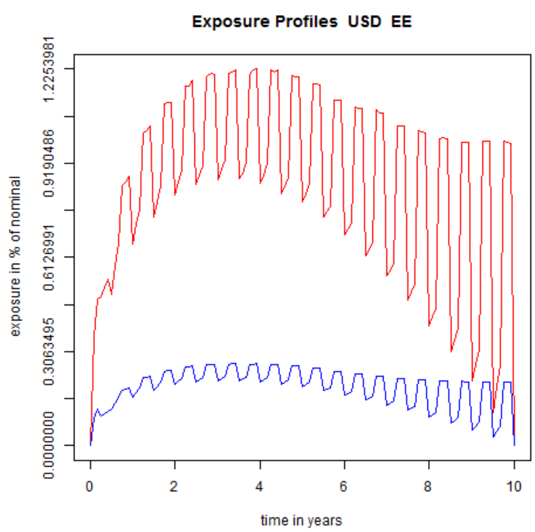
FIGURE 6. EE and PE profiles for a 3M-receiver-6M-payer basis varying the volatility of the basis spread (first and second line) and the correlation of the basis spread with the base curve (3M-USD Libor) (third and fourth line)

arc is the consequence of this payment sequence. Vice versa, if the shorter tenor is received, then the upper arc represents the exposure of the remaining swap with no payment mismatch. The downward columns are formed by receiving the 3M tenor and subsequently receiving the 3M and 6M difference. The lower arc is then a consequencde of this payment sequence.

Figure 5 the exposure profile of a 3M payer swap is displayed. The lower arc is driven by the volatility of the basis spread, because latter is the driver of exposure for the payment unmismatched swap. As we have seen in the simulations in Figure 5 the lower arc always returns to zero at maturity and takes its maximum at about half of the lifetime of the swap. In the first part of the swaps' lifetime, spread variance increases due to a diffusion. After half of the lifetime the lower arc decreases due to amortization effects. As an extreme case, when the volatility of the stochastic basis spread is set to zero, then the lower arc completely disappears and the situation of the second picture in Figure 4 emerges.
The size of the columns is governed by the fixed index cashflows which in turn depend on the standard deviation of the forward rates determined by the underlying interest rate model. Since drift and variance of a forward rate are increasing until its fixing we see an increasing expected exposure for the 3M and 3M vs. 6M cashflows respectively resulting in increasing column size through time.



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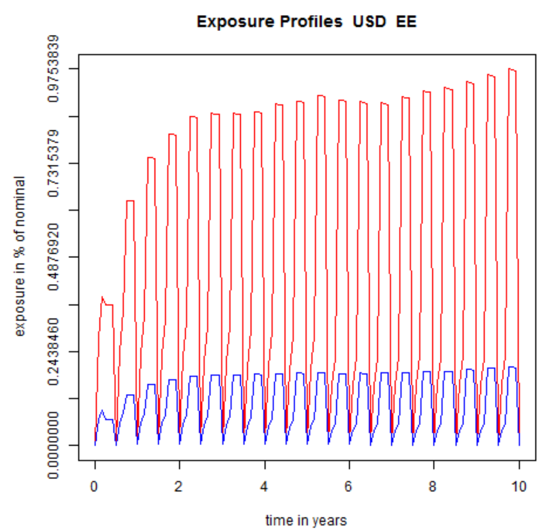


(B)

FIGURE 7. EE and PE profiles for a 3M-receiver-6M-payer basis swap with and without stochastic basis

The exposure profiles for a 3M receiver swap are shown in Figure 7, again comparing the case of with stochastic basis to the case without stochastic basis. The first difference here is that the exposure profiles visibly increase, in particular the PE profile. This is a consequence of negative spreads of significant amount. Thoughts on how to deal with negative spreads based on the given dynamics are illustrated in Appendix B.

In Figure 6 one can see again the influence of the two important quantities for the stochastic spread, its volatility and its correlation with the base curve. Again, the volatility of the basis spread determines the lower arc and the correlation of the stochastic spread with the basis curve controls the lower arc.

A particular feature of the exposure profiles when interest rates are low, is a column flip. This means that the negative interest rate regime determines expected and potential positive exposure negative and hence 6M payer and receiver basis swaps end up with qualitatively the same profile in the tail of their lifetime. Increasing the interest rate level further away from zero, the flip occurs later in the lifetime of the basis swap. This effect is visualized in Figure 8. A similar phenomenon occurs for the expected exposure profiles.

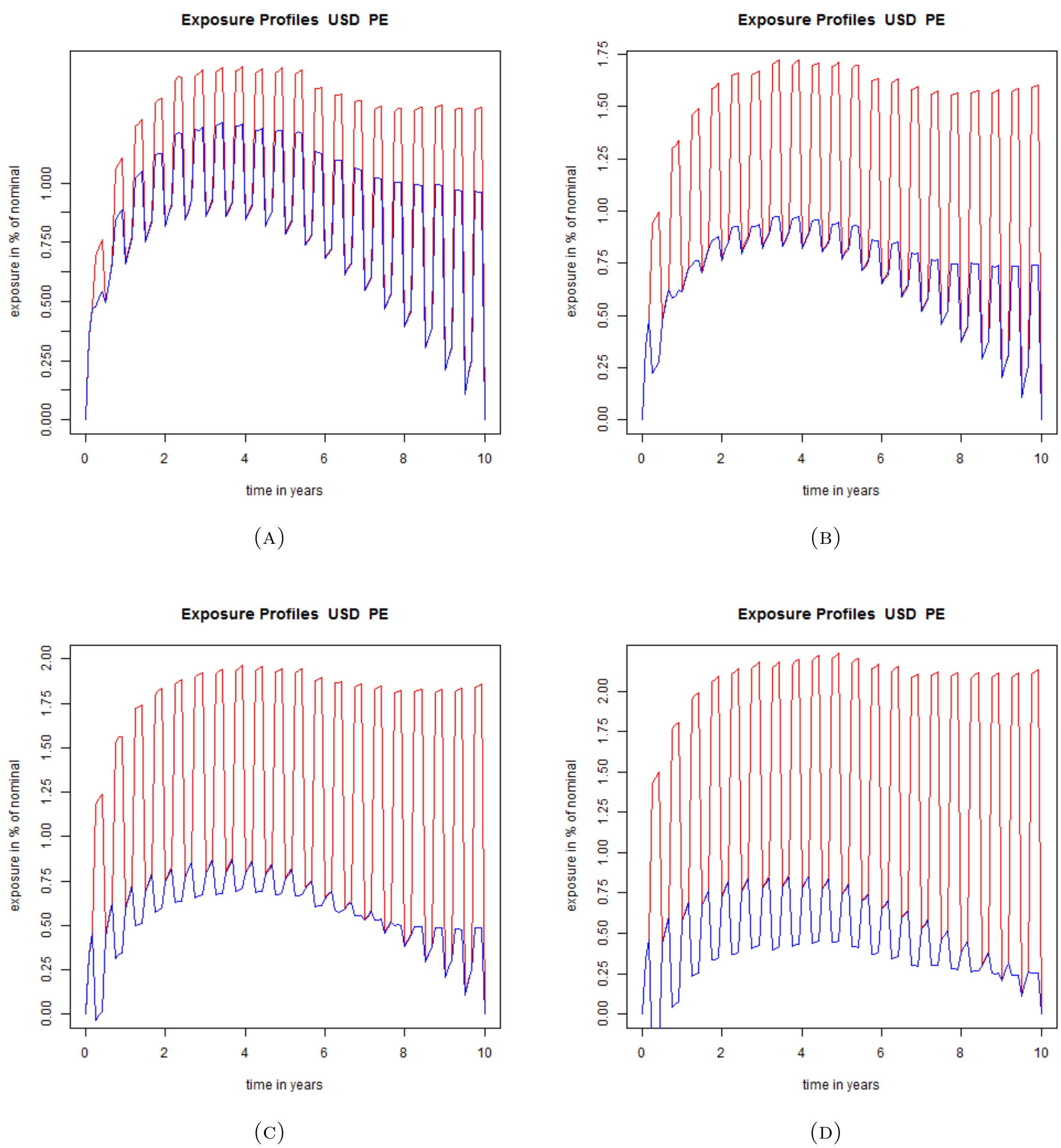


FIGURE 8. PE Profiles for different initial levels of interest rates, increasing from left to right.

5. Conclusion

This paper presents an approach for stochastically modelling tenor basis spreads in the context of credit exposure calculations. The model proposed is of additive two factor Gaussian type. A two factor model is motivated by a principal component analysis of historical market data which shows that above 90% of the systems variance can be captured by the first two components.
In order to keep interest rate dynamics consistent, time scales of the spread and interest rate dynamics are aligned (which is supported by historical observations of the interest rate market - although not being the focus here). Instead of modelling multiple interest curves, spread dynamics are formulated explicitly on top of a base interest rate curve, not being the discount curve but a major tenor depending on the currency. This approach allows us to control spread variances on top of the dominating interest rate dynamics and providing a globally stable multi-curve, multi-currency interest rate instantaneous correlation matrix.
The proposed model is calibrated and each parameter choice is argued carefully. Finally basis swap expected and potential exposure profiles are calculated and explained. Since interest rates are low and the model is Gaussian in nature, the profiles of long and short basis swaps match in the long run. The questionable property of potential negative model tenor basis spreads is discussed in the appendix, where, in contrast to an alternated SDE, a postprocessing of the model results is proposed. This approach has advantages, if interest rate dynamics are already established in terms of implementation.

Appendix A. A results on the convergence to quadratic variation

Above, we have used the result that the empirical covariance of two zero rate paths converges to the quadratic covariation of these paths. To argue this, we follow [3] and consider the independent increments of zero rate processes (i.e. absolute returns)

$$\begin{aligned} X(t_i) &= z(t_i, t_i + \Delta) - z(t_{i-1}, t_{i-1} + \Delta) = z_\Delta(t_i) - z_\Delta(t_{i-1}), \\ Y(t_i) &= \tilde{z}(t_i, t_i + \Delta) - \tilde{z}(t_{i-1}, t_{i-1} + \Delta) = \tilde{z}_\Delta(t_i) - \tilde{z}_\Delta(t_{i-1}), \end{aligned}$$

where we define $z_\Delta(t) = z(t, t+\Delta), \tilde{z}_\Delta(t) = \tilde{z}(t, t+\Delta)$ for some zero rates. We have

$$\begin{aligned} K\bar{X}\bar{Y} &= \frac{1}{K}\sum_{i=1}^{K}\left(z_\Delta(t_i) - z_\Delta(t_{i-1})\right)\sum_{i=1}^{K}\left(\tilde{z}_\Delta(t_i) - \tilde{z}_\Delta(t_{i-1})\right) \\ &= \frac{1}{K}\left(z_\Delta(T) - z_\Delta(t_0)\right)\left(\tilde{z}_\Delta(T) - \tilde{z}_\Delta(t_0)\right) \to 0, \end{aligned}$$

almost surely as $K \to \infty$. Moreover, by definition of quadratic covariation

$$\sum_{i=1}^{K} X(t_i)Y(t_i) = \sum_{i=1}^{K}\left(z_\Delta(t_i) - z_\Delta(t_{i-1})\right)\left(\tilde{z}_\Delta(t_i) - \tilde{z}_\Delta(t_{i-1})\right) \to \langle z_\Delta, \tilde{z}_\Delta \text{âŇł}_T,$$

in probability as $K \to \infty$, with

$$\langle z_\Delta, \tilde{z}_\Delta \rangle_T = \int_0^T dz_\Delta(s) d\tilde{z}_\Delta(s).$$

Now wet

$$\rho(T,K) = \frac{Cov_{X,Y}(T,K)}{\sqrt{Var_X(T,K)}\sqrt{Var_Y(T,K)}} = \frac{\sum_{i=1} KX(t_i)Y(t_i) - K\bar{X}\bar{Y}}{\sqrt{\sum_{i=1} KX(t_i)^2 - K\bar{X}^2}\sqrt{\sum_{i=1} KY(t_i)^2 - K\bar{Y}^2}}.$$

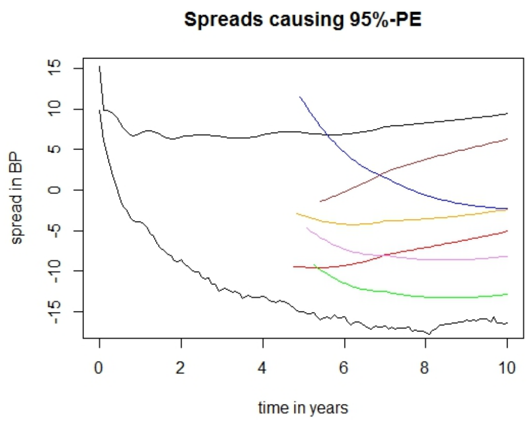
Th above computations implies

$$\rho(T) = \lim_{K\to\infty} \rho(T,K) = \frac{\langle z_\Delta, \tilde{z}_\Delta\rangle_T}{\sqrt{\langle z_\Delta, z_\Delta\rangle_T}\sqrt{\langle \tilde{z}_\Delta, \tilde{z}_\Delta\rangle_T}},$$

where the limit is taken in probability. Note however that the estimator $\rho(T,K)$ is in general biased, and is consistent only asymptotically..

Appendix B. Negative spreads and their mitigation

Since the model that we have specified for stochastic spreads is Gaussian in nature, the spreads will - especially after some time - become negative. From the standpoint of economic theory, where spreads are often interpreted as a premium for counterparty default or as a liquidity preference, this is questionable. However, for the spreads between curves of different currencies, this is a realistic feature.



(A)

Figure 9. Spreads which cause the 95%-PE for a 3M-receiver-6M-payer basis swap

Negative spreads have consequences for the EE and (in particular) the PE profile of a 3M-receiver-6M-payer basis swap, see Figure 7, where the PE almost doubles. In fact, these high PE values are caused by very low spreads, see Figure 9, where the following curves are drawn: the mean spread over the lifetime of the swap using todays yield curves, the 5%-quantile of the spreads (for comparison reasons) and the curves of the instantaneous forward spreads starting at different times which cause the PE at 95%. One clearly sees that these spreads are unreasonably negative.

In order to avoid negative spreads, there are a couple of possibilities. One alternative, which is standard, is to directly model the spreads with a stochastic differential equation whose solution has the property of not becoming negative, for instance a CIR-process, a geometric Brownian motion or, more general, a CEV process

$$ds(t) = a(b - s(t))\,dt + \sigma s(t)^\gamma dW(t),$$

with $a, b, \sigma, \gamma > 0$.

The alternative we are following is a mapping approach. Mapping techniques turn out to be quite handy implementation wise, since the SDE core implementation does not need to be altered. One idea is to map the distribution of the spreads to a distribution with support on the positive half-axis, or even with a negative lower bound, if one wishes to allow the spreads to become slightly negative. Another idea is to take the maximum of the spreads and zero (or any other lower bound for the spreads). We will now briefly describe both approaches and discuss their advantages and weaknesses.

The first idea, i.e. mapping the normal distribution of the spreads at any time $t > 0$ to a distribution with positive support, can be done by selecting a distibution which one assumes to describe the spreads reasonably well and then compute the parameters such that key quantities match those of the normal distribution. Here, we work out this approach for two distributions which are closely linked to the normal distribution, the truncated normal distribution and the lognormal distribution. Moreover, we will drop the dependency on time $t > 0$ for ease of presentation.

In the first case of the truncated normal distribution, we truncate the distribution of the spreads from below at 0. The density $\bar{\phi}_{\bar{\mu},\bar{\sigma},0}(x)$ of such a distribution is then given by

$$\bar{\phi}_{\bar{\mu},\bar{\sigma},0}(x) = \frac{\frac{1}{\bar{\sigma}}\phi(\frac{x-\bar{\mu}}{\bar{\sigma}})}{1-\Phi(\frac{-\bar{\mu}}{\bar{\sigma}})} 1_{[0,\infty)}(x),$$

where ϕ, Φ are the density function and the cumulated distribution function of the normal distribution, and $1_{[0,\infty)}$ is the indicator function of the nonnegative real numbers.

Now one can map the normal distribution of the spreads, which we denote by $\mathcal{N}(\mu_s, \sigma_s^2)$ to a truncated normal distribution with lower bound at zero $\mathcal{N}_0(\bar{\mu}, \bar{\sigma})$. The parameters $\bar{\mu}$ and $\bar{\sigma}$ can be determined from their normal counterparts by

$$\bar{\mu} = \mu + \sigma\frac{\phi(-\frac{\mu}{\sigma})}{1-\Phi(-\frac{\mu}{\sigma})} \quad \text{and} \quad \bar{\sigma}^2 = \sigma^2\Big(1 - \frac{\phi(-\frac{\mu}{\sigma})}{1-\Phi(-\frac{\mu}{\sigma})}\Big(\frac{\phi(-\frac{\mu}{\sigma})}{1-\Phi(-\frac{\mu}{\sigma})} + \frac{\mu}{\sigma}\Big)\Big).$$

Solving for the parameters μ and σ such that the right-hand side of the previous equations becomes equal to μ_s and σ_s^2 gives the parameters for the truncated normal distribution.

Another nonnegative distribution that one can map the spreads to is the lognormal distribution $\mathcal{LN}(\bar{\mu}, \bar{\sigma}^2)$ with density function $f_{\bar{\mu},\bar{\sigma}}$ given by

$$f_{\bar{\mu},\bar{\sigma}}(x) = \frac{1}{\sqrt{2\pi}x\bar{\sigma}} \exp\Big(-\frac{(\ln x - \bar{\mu})^2}{\bar{\sigma}^2}\Big) 1_{[0,\infty)}(x),$$

and here

$$\bar{\mu} = \exp(\mu + \sigma^2/2), \quad \text{and} \quad \bar{\sigma}^2 = \big(\exp(\sigma^2) - 1\big)\exp(2\mu + \sigma^2).$$

Again, solving for the parameters μ and σ such that the right-hand side becomes equal to μ_s and σ_s^2 for the lognormal distribution. An alternative to equating the first and second moment of the distribution might be to equate any other desired quantity, such as the first positive central moment $\mathbb{E}\big[(S(t) - \mu)^+\big]$, where S has the normal distribution of the spreads at time $t > 0$.

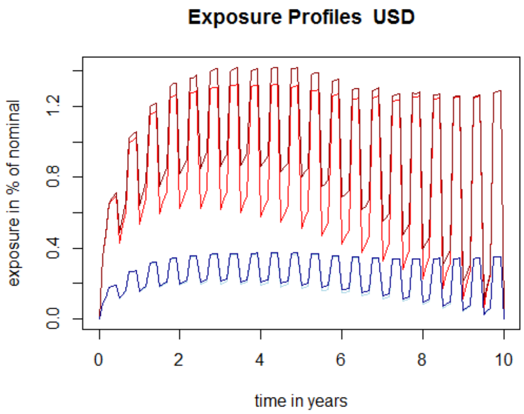
This approach has the advantages to keep some characteristics of the distribution of spreads, but one has to choose a reasonable distribution for the mapped spreads such that it fits well with the empirical one of the spreads and that the distribution of the fixing curve can still be easily treated. In particular the latter point is delicate to achieve. The exposure profiles generated with this approach can be seen in Figure 10.

The other approach is to floor the spreads at a minimum level. As an example, we floor the spreads at 0, i.e. we consider

$$\bar{f}_s(t,T) = \max(f_s(t,T), 0),$$

where $f(t,T)$ denotes the instantaneous forward rate from t to T. Here we see immediately the main shortcoming of this approach, we introduce an atom in the distribution of the spreads at zero. This reduces the volatility incorporated in the model. The advantage here is that we can compute the distribution of the fixing curve explicitly as the maximum

$$\bar{f}_f(t,T) = \max\{f_f(t,T), f_b(t,T)\},$$



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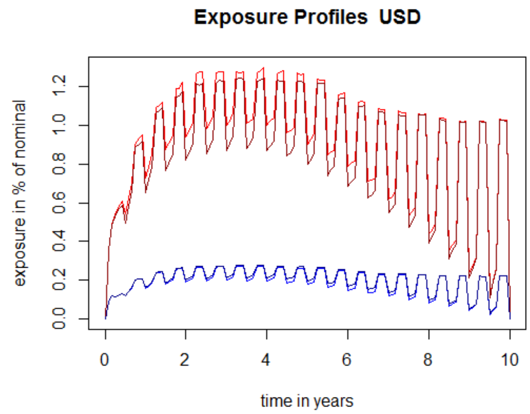

(B)

FIGURE 10. EE and PE profiles of a 6M-payer-3M-receiver swap and a 6M-payer-3M-receiver swap with mapped and unmapped spreads

which is the maximum of the two Gaussian random variables $f_f(t,T)$ and $f_b(t,T)$ with

$$\mu_1 := \mathbb{E}[f_f(t,T)], \quad \mu_2 := \mathbb{E}[f_b(t,T)], \quad \sigma_1^2 := \mathbb{V}[f_f(t,T)] \quad \text{and} \quad \sigma_2^2 := \mathbb{V}[f_b(t,T)].$$

The distribution of $\bar{f}_f(t,T)$ can then be computed to be

$$\begin{aligned} p_{\bar{f}_f(t,T)}(x) =& \frac{1}{\sigma_1}\phi\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Phi\Big(\frac{x(\sigma_1-\rho\sigma_2)-(\sigma_1\mu_2-\sigma_2\mu_1\rho)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\Big) \\ &+\frac{1}{\sigma_2}\phi\Big(\frac{x-\mu_2}{\sigma_2}\Big)\Phi\Big(\frac{x(\sigma_2-\rho\sigma_1)-(\sigma_2\mu_1-\sigma_1\mu_2\rho)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\Big). \end{aligned}$$

This implies that this approach has the advantage of a more explicitly treatable fixing rate distribution. However, as already mentioned above, this advantage decreases the volatility of the spreads. This latter point could however be mitigated considering the flooring

$$\bar{f}_s(t,T) = c\max(f_s(t,T),0),$$

where $c > 0$ is set such that the variance of $\bar{f}_s(t,T)$ equals the variance of $f_s(t,T)$. The exposure profiles generated with this approach can be seen in Figure 11.

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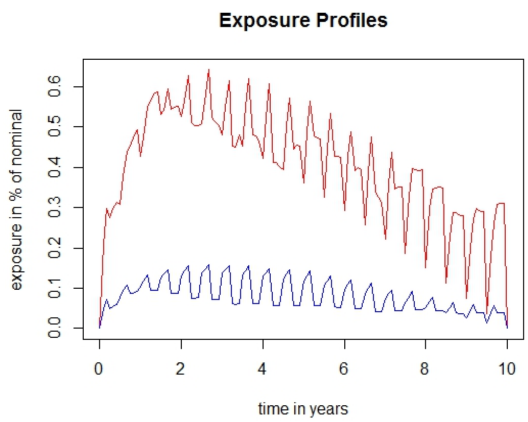

(A)

FIGURE 11. EE and PE profiles of a 6M-payer-3M-receiver swap with a floor of the spreads