Gravitational memory of Neutrinos

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We calculate the gravitational memory effect when a spherically symmetric shell of energy passes through a spacetime region. In particular, this effect includes a longitudinal component, such that two geodesics can pick up a relative velocity proportional to their separation. Such a measurement will allow us to obtain the total energy released by a supernova explosion in the form of neutrinos. We study the possibility to measure such an effect by space-based interferometers such as LISA and BBO, and also by astrophysical interferometers such as pulsar scintillometry.

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I. INTRODUCTION AND SUMMARY

The recent detection of gravitational waves [1] has proved that gravitational waves leave an oscillating pattern in the amplitude of waveforms measured at detectors such as LIGO. It is also known that this is not the only effect that is potentially detectable. Strong gravitational waves imply a large flow of energy. Just like any other flow of energy, it leads to a gravitational memory effect [2, 3].

The memory effect discussed in [2, 3] causes permanent relative displacements between geodesics. It contains only transverse-traceless components and can leave an imprint in an interferometer. In this paper, we will introduce another memory effect that is different in two ways:

- It has a longitudinal component. The transverse-traceless limitation only applies to freely-propagating changes of the metric (i.e gravitational waves). While coupled to matter, which often have longitudinal (density) waves, it is natural to have an accompanying longitudinal change in the metric.

- In addition to displacements of geodesics, it also causes permanent changes in relative velocities between geodesics, with magnitudes proportional to their separations.

In terms of the dynamics, a change in velocity is higher order than a change in distance measured in gravitational waves. This however does not mean that our effect is harder to measure. The conventional gravitational memory effect needs a physical event that significantly breaks spherical symmetry to be detectable whereas our effect does not and therefore it can occur more generally. In addition, a change in velocity implies a distance change that grows in time even after the initial effect. That is an advantage for some detection methods.

We will present a simple and natural occurrence of this effect. During a supernova explosion (SNe), most of the energy is released in a highly relativistic shell of neutrinos. As illustrated in Fig.1, when a neutrino shell passes through, the three free-falling points A, B and C, will pick up different velocities due to the change of geometry. If AB and BC are two arms of an interferometer, we will see a time-dependent change after the shell passes through. Note that this is purely a geometric change which happens even without the three actual objects. We will demonstrate this by showing that it is possibility to detect the same effect using pulsar interferometry, in which two parallel light rays get different time-delays after being hit by a neutrino shell.

The SKA telescope is expected to detect thousands of pulsars, [4], and one of them might be close enough to a SN explosion to provide us with a good estimation of the total energy in the neutrino shell of the explosion. Currently there is no other way to obtain such information. Thus in addition to direct neutrino detections such as in Super-Kamiokande [5], this memory effect can provide a new handle on constraining the explosion mechanism.

The rest of the paper is organized as follows. In II we discuss how an astrophysical interferometer formed by pulsar scintillometry can measure the memory effect caused by a SNe. In section III we derive the change in velocities caused by shell crossing with the Israel Junction Conditions (IJC) [6], treating the neutrino shell as a co-dimension-one delta function. In the final section V we discuss potential observation of such an effect by experiments that are currently being planned such as LISA and BBO.

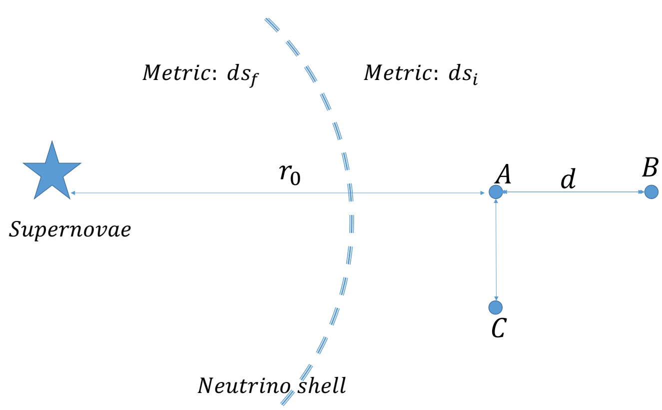

FIG. 1. Schematic of the effect being considered by a neutrino shell passing through the interferometer. The points A, B, C represent ends of the interferometer of arm length d. The three points A, B and C will pick up velocities v_A, v_B and v_C respectively after the shell crosses them. They are all different since they cross the shell at different locations.

II. MEMORY EFFECT IN PULSAR SCINTILLATION

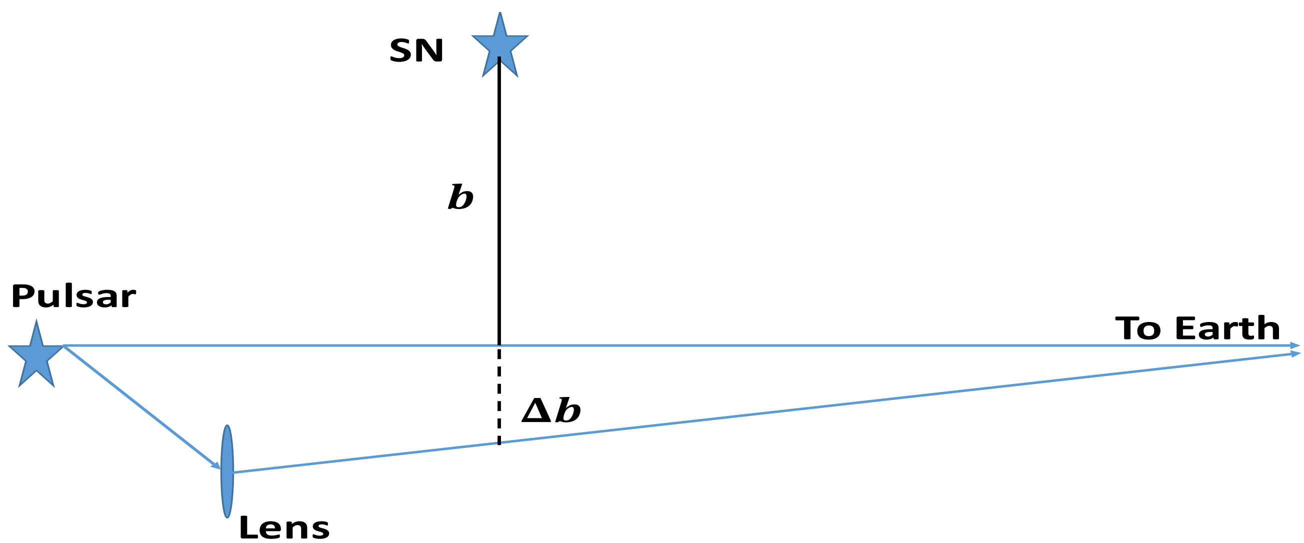

FIG. 2. Geometry of the astrophysical interferometer formed by pulsar scintillometry. Due to scattering or lensing, the image we see is an interference pattern of two light rays represented by the blue lines. If the separation of the two light rays has a component along the longitudinal (radial) direction from the SN, the spacetime distortion of the neutrino shell will change the interference pattern we see. We draw the lens to be behind the SN, but it could have been in front of it and the effect is the same.

It is known that the images of many astronomical bodies scintillate [10]. A general reason for scintillation is that due to scattering or lensing, we receive multiple light rays from the same objects. These light rays are very close to each other, so they cannot be individually resolved and have to interfere. The scintillation pattern we see is the time dependence of their interference. If we consider two light rays from a faraway pulsar which happen to pass by a SN progenitor, as illustrated in Fig.2, they can probe the spacetime distortion when it explodes.

The scintillation/interference pattern is directly related to the path lengths of these light rays. The change in such path length during a SN explosion has been worked out in [11]

$$\Delta t = 2\delta M\left[\ln\left(1+\frac{t^2}{b^2}\right) - \frac{t^2}{b^2+t^2}\right] . \tag{1}$$

Here b is the impact parameter as shown in Fig.2, the shortest distance between the light ray and the SN. t is the proper time on earth, with $t = 0$ the time we directly observe the SN explosion. δM is the total energy of the neutrino shell, and Δt is the resulting time shift. A photon which should have reached the earth at time t, will arrive earlier at $(t - \Delta t)$ instead.

When the separation between two light rays has a component in the radial direction from the SN, Δb, there will be a nonzero relative change between their path lengths.

$$(\Delta t|_b - \Delta t|_{b+\Delta b}) \approx \frac{\partial \Delta t}{\partial b}\Delta b = -\frac{4\delta M t^4}{b(b^2+t^2)^2}\Delta b \ . \tag{2}$$

We can see that this effect grows from zero and approaches an asymptotic value,

$$(\Delta t|_b - \Delta t|_{b+\Delta b}) \longrightarrow \frac{4\delta M \Delta b}{b} \ , \tag{3}$$

at a characteristic time scale given by b. This result can be obtained by calculating the change in time

We estimate b by assuming that the next SN is somewhere near the galactic centre. A sample of ~ 9000 pulsars from the SKA catalog in [4] shows that among those pulsars, the shortest b is about $10\ ly \sim 10^{14}\ km$. Δb is related to the scattering-broadening of images. We use the data from [12] that was observed on a scattering screen near the galactic centre. Scaling the frequency to $1\ GHz$ which is usually a good window to observe pulsar signals. We found that such a scattering screen can produce images separated by $\Delta b \sim 1000 A.U. \sim 10^{10}\ km$. We again use $\delta M \sim 1\ km$, and combining all these numbers we get $(\delta M \Delta b/b) \sim 1\ m$. This is comparable to the wavelength at $1\ GHz$, thus making the change in interference pattern easy to detect. Therefore, if we can monitor the pulsar scintillation pattern over ten years after a SN explosion we should see an order one change in the scintillation pattern predicted by Eq. (2).

III. VELOCITY CHANGE FROM JUNCTION CONDITIONS

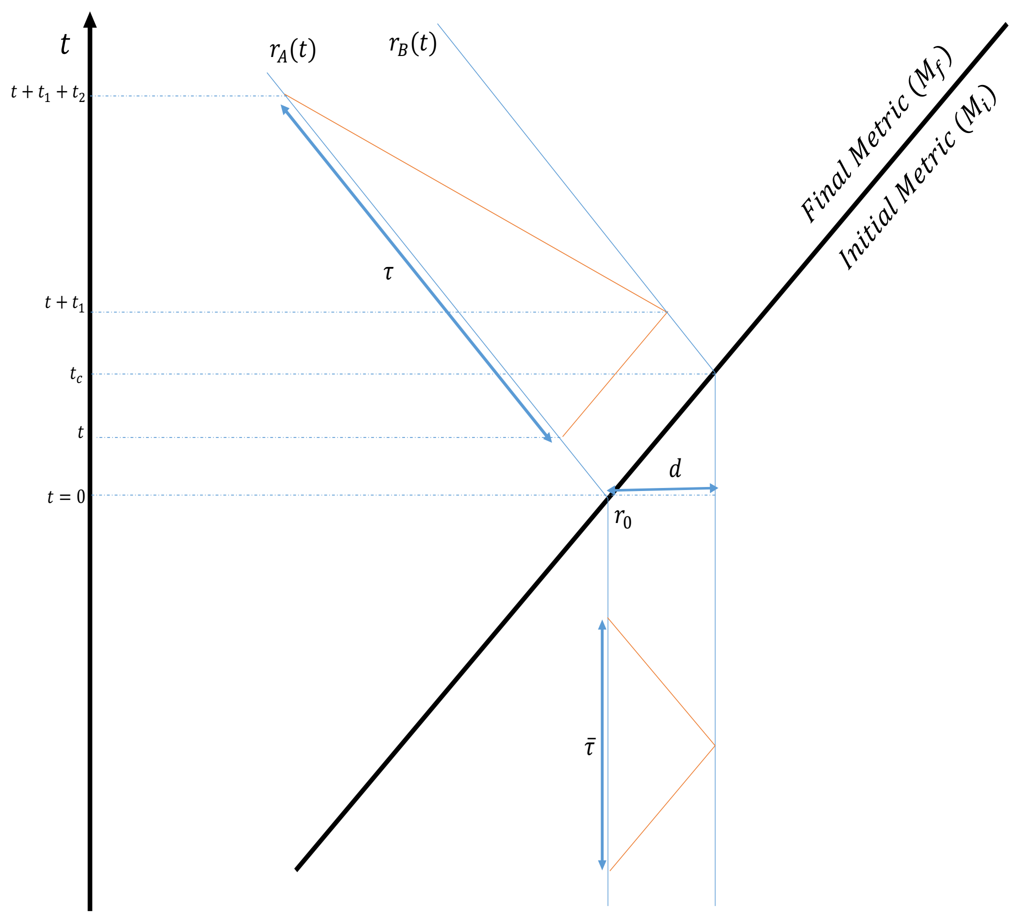

FIG. 3. Schematic of a spacetime diagram showing the paths of photons that are used in the interferometer to measure the change in the length of the interferometer arms. The orange lines represent the photon trajectories. The solid blue lines represent the trajectories of the two points A and B in figure 1. The dark black line is the null trajectory of the neutrino shell.

In this section we discuss how a similar effect can be also seen in traditional interferometers such as LIGO. We assume the geometry of the spacetime is governed by the SN progenitor star. Assuming the star is not rotating very rapidly and thus the surrounding spacetime is parametrized by the Schwarzschild metric,

$$ds_i^2 = -\left(1 - \frac{2M_i}{r}\right) dt_i^2 + \left(1 - \frac{2M_i}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{4}$$

where are working in units with $G = c = 1$. After the star explodes, it emits a shell of neutrinos carrying energy δM. The metric surrounding the original star then changes since the mass has changed. In particular the i index used above to denote the "initial" metric is replaced by f to denote the "final" metric with a smaller mass $M_f = M_i - \delta M$. Notice that the time component of the metric is different in both geometries whereas the radial component is the same as it corresponds to the radius of a two sphere separating the two geometries. Since we are describing the shell as a delta function travelling at roughly the speed of light it will follow a null geodesic. The null vector of the shell can be written in both metrics as follows[1]

$$\Sigma^{\mu}_{(i,f)} = \left(\left(1 - \frac{2M_{(i,f)}}{r}\right)^{-1}, 1, 0, 0\right). \tag{5}$$

Since we expect the interferometer points (denoted by the points A, B, C in figure 1) to be very far for the massive object we assume they have a small velocity[2] v_i and are described by the following geodesic in either metric

$$\zeta^{\mu}_{i} = \left(\left(1 - \frac{2M_i}{r}\right)^{-\frac{1}{2}}, v_i, 0, 0\right). \tag{6}$$

The r will be different for points A, B and C as shown in figure 1. After the shell has crossed the points the velocity of the interferometer points will change from v_i to v_f. Given v_i, we can calculate v_f using the IJC,

$$g_{\mu\nu}\Sigma^{\mu}_{i}\zeta^{\nu}_{i} = \hat{g}_{\mu\nu}\Sigma^{\mu}_{f}\zeta^{\nu}_{f}, \tag{7}$$

where $\hat{g}_{\mu\nu}$ represents the metric of the "final" spacetime and $g_{\mu\nu}$ is the metric of the initial spacetime. This expression comes from demanding the continuity of geodesics from one geometry to another and was used in [7] to calculate a similar effect. Throughout this calculation we assume that the interferometer is very far from the SNe. Thus we assume there are three small quantities, $\frac{M}{r_0}, \frac{\delta M}{r_0}, \frac{d}{r_0} \ll 1$. To simplify notation we will use the symbol $\mathcal{O}(r^{-1})$ to represent suppression by any one of the three small quantities. Substituting Eqs (5, 6) into Eq (7) gives the final vector for the interferometer, to order $\mathcal{O}(r^{-2}_{crossing})$,

$$\zeta^{\mu}_{f} = \left(\left(1 - \frac{2M_f}{r_{crossing}}\right)^{-\frac{1}{2}}, v_i - \frac{\delta M}{r_{crossing}}\left(1 + \frac{M + \frac{1}{2}\delta M}{r_{crossing}}\right), 0, 0\right). \tag{8}$$

Thus the change in velocity due a shell crossing is $v_f - v_i = -\frac{\delta M}{r_{crossing}}\left(1 + \frac{3}{2}\frac{\delta M}{r_{crossing}}\right)$. $r_{crossing}$ is a fixed distance at which the shell crosses a point. For A it is r_0, for B it is $r_0 + d$ and for C it is $r_0 + \mathcal{O}(d/r_0)$. Note that this is the proper velocity $\frac{dr}{d\tau}$. We will need the coordinate velocity for the our calculations which is given by $\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{dr}{d\tau}\left(1 - \frac{2(M-\delta M)}{r_{crossing}}\right)^{\frac{1}{2}} \approx -\frac{\delta M}{r_{crosssing}}\left(1 + \frac{3}{2}\frac{\delta M}{r_{crossing}}\right)$

IV. MEMORY EFFECT IN INTERFEROMETER

A. Distance between A and B

An interferometer can calculate the displacement between geodesics. There are two ways to calculate this displacement. The first is to directly calculate the photon geodesics from the metric and find the difference between them. The second is to integrate the geodesics deviation equation. In this paper we will use the first approach and calculate the trajectories of the interferometer points. We consider points A, B and C in figure 1 as the ends of the arms of the interferometer. We parametrize the trajectories of the points in a piecewise form. For point A we have

$$r_A(t) = \begin{cases} r_0 - \frac{1}{2}\frac{M}{r_0^2}t^2 & (t \leq 0), \\ r_0 - \frac{1}{2}\frac{M-\delta M}{r_0^2}t^2 - \frac{\delta M}{r_0}\left(1 + \frac{3}{2}\frac{\delta M}{r_0}\right)t & (t > 0). \end{cases} \tag{9}$$

$t = 0$ is defined to be the time when the shell crosses point A and the distance $r_A(0) = r_0$. For $t < 0$ we assume there is no velocity and the only term that contributes to the trajectory is the acceleration of point A towards mass

[1] One could write this vector in different ways and still have it satisfy the null normalization condition, however, the fact the shell has some surface area with a *fixed* radius, the two vectors must have the same radial component and this enforces the given form of the time component.

[2] We assume $v_i^2 = 0$.

M. For $t > 0$ there is still the acceleration term but the mass is different since the shell has carried δM away. Along with that there is a term that corresponds to the velocity change (as we calculated from the IJC) coming from shell crossing. In principle the r_0 on the RHS of Eq (9) should be $r_A(t)$ however we know $r_A(t) = r_0 + \mathcal{O}(M/r_0)$. Thus the effect of replacing r_0 with $r_A(t)$ will not affect the leading order answer. The trajectory of point B is given by

$$r_B(t) = \begin{cases} r_0 + d - \frac{1}{2}\frac{M}{(r_0+d)^2}t^2 & (t \leq t_c), \\ r_0 + d - \frac{1}{2}\frac{M}{(r_0+d)^2}t_c^2 - \frac{1}{2}\frac{M-\delta M}{(r_0+d)^2}(t-t_c)^2 - \frac{\delta M}{r_0+d}\left(1+\frac{3}{2}\frac{\delta M}{r_0}\right)(t-t_c) - \frac{Md}{(r_0+d)^2}(t-t_c) & (t > t_c). \end{cases} \tag{10}$$

For $t < t_c$ there is just the acceleration of the point towards mass M. After $t > t_c$ the shell has crossed and now there is a change in distance in the time t_c that the point is accelerating towards M. This is the second term in the expression. The next term corresponds to the acceleration of the point towards mass $M - \delta M$. The next two terms contain the velocity terms for this point. The first term is the comes from the IJC. The second term is the velocity picked up by this point in the time the shell reaches it. Using these trajectories we can calculate the proper time τ shown in figure 3. We know $t_c = d$ to leading order. Since we are interested in terms up to $\mathcal{O}(r_0^{-2})$, we can substitute t_c with d in terms that are already $\mathcal{O}(r_0^{-2})$. Using the trajectories we can calculate the proper time it takes for the photon to complete a round trip to from point A to B. This can then be compared to the proper time it would take without shell crossing. The difference between the proper times will represent the physical distance change between the geodesics. The full calculation is presented in the appendix and the result is a constant change in distance

$$\Delta\tau_{AB} = \tau - \bar{\tau} = \frac{d\delta M}{r_0^2}(d - 3\delta M) \tag{11}$$

There is a simple way to appreciate the lack of time dependence. We have assumed point A has velocity zero at the time of shell crossing. After the shell crosses point A, it will pick up a velocity $\frac{\delta M}{r_0}$. In the time the shell takes to go from A to B, A will also gain a velocity from its acceleration towards the mass $M - \delta M$. The magnitude of the velocity gain from acceleration is $\frac{M-\delta M}{r_0^2}d$. The final velocity of A at time d is, therefore, given by

$$v_A(t = t_c) = \frac{\delta M}{r_0}\left(1 + \frac{3}{2}\frac{\delta M}{r_0}\right) + \frac{M - \delta M}{r_0^2}t_c. \tag{12}$$

Similarly, point B will gain a velocity from the acceleration towards mass M with magnitude $\frac{M}{r_0^2}d$. The velocity change from shell crossing is $\frac{\delta M}{r_0+d}$. The final velocity of point B at time d is

$$v_B(t = t_c) = \frac{\delta M}{r_0 + d}\left(1 + \frac{3}{2}\frac{\delta M}{r_0}\right) + \frac{M}{(r_0 + d)^2}t_c. \tag{13}$$

We see that v_A is the same as v_B to $\mathcal{O}(r_0^{-3})$ and therefore we don't expect the distance to change with time at leading order. Next we show an effect which is much more interesting since it grows with time.

B. Distance between A and C

The coordinates for this calculation are set up as follows; we let the y coordinate be 0 at A and increases as we go down. Thus point C is at y coordinate d. The time is set to zero when the shell crosses point A and C (to leading order the shell crosses both of them at the same time). The interferometer arm that goes from A to C will see the distance between these two points change. We can parametrize the trajectory of $r_C(t)_\perp$ in the perpendicular direction to the radial vector as

$$r_C(t)_\perp = \begin{cases} d - \frac{1}{2}\frac{M}{(r_0^2+d^2)}t^2\sin\theta, & (t \leq 0) \\ d - \frac{1}{2}\frac{M-\delta M}{(r_0^2+d^2)}t^2\sin\theta - \frac{\delta M}{\sqrt{r_0^2+d^2}}t\sin\theta, & (t > 0) \end{cases}. \tag{14}$$

where $\sin\theta \approx \theta = \frac{d}{r_0}$. Point A will only move in the radial direction so we don't need to consider its motion in this case. Using this trajectory we can calculate how the proper time it takes a photon to get to C and come back to A changes due to shell crossing.

$$\Delta\tau_{AC} = -2d\left(\frac{\delta M(t+d)}{r_0^2} + \frac{\delta M}{r_0}\right) \tag{15}$$

We emphasize two things here;

- There is a time dependent term $-\frac{2d\delta Mt}{r_0^2}$.
- There is a constant term at leading order $\frac{-2d\delta M}{r_0}$. Note the similarity to the result seen from pulsar scintillometry in Eq (3).

V. OBSERVATION WITH SPACE-BASED INTERFEROMETERS

Taking the generic form of the change in distance as $\Delta L \sim \frac{\delta Md}{r_0^2}t$ we can estimate the distance a SN would have to be from the interferometer to have an observable change in strain, which is a unitless number quantifying the amount of space-time distortion.

$$h \sim \frac{\Delta L}{d} \sim \frac{\delta Mt}{r_0^2} \ . \tag{16}$$

Since our strain grows linearly with time, we do not expect detections from ground based experiments as for those setups the three points A, B, C cannot remain in free fall for a long enough amount of time such that the signal builds up to an observable value.

If we plot our effect on the strain-frequency diagram [8] that is usually used to compare different interferometers, it will be a 45-degree line. Thus the first point at which the sensitivity curve of a device crosses with a 45-degree line will give the best chance for our effect being detected. In all these estimations, we take δM to be a faction of a solar mass, and take the corresponding Schwarzschild radius to be 1 km for simplicity.

For LISA, the best observing frequency is $\sim 0.5 \times 10^{-2} Hz$ with a sensitivity in strain $\sim 10^{-21}$. Using Eq. (16), we can solve for the distance to the SN, r_0, for our effect to be detectable.

$$r_0 = \left(\frac{\delta M}{h}t\right)^{\frac{1}{2}} \tag{17}$$

$$= \left(\frac{1\ km}{10^{-21}} \times 10^7\ km\right)^{\frac{1}{2}} \approx 10^{14}\ km = 10\ ly. \tag{18}$$

This is clearly too close. It has been estimated that only once in 10^8 years will a SN go off within a distance of 30 ly [9].[3] By a naive volume scaling, an explosion within 10 ly only occurs once every billion years.

If we look at the Big Bang Observer (BBO) instead, the best observing frequency is $\sim 0.5Hz$ with a sensitivity in strain $\sim 10^{-24}$. First of all, this frequency range does not have as many background signals from compact binaries, making it a much better device to measure our effect. The improved sensitivity gives a value for r_0 of $\sim 100\ ly$. This is a factor of 10^3 increase in the volume for detectable events, thus improves the expectation of one SN that is within 10 ly to occur in less than a million years . That is unfortunately still a long shot.

In this type of simple estimation, we cannot go lower in the frequency. The exact duration of the neutrino-shell passage is not known, but we do no expect it to be much less than a second. Thus for higher frequencies, the co-dimension-one delta function approximation breaks down, and the effect will be weaker than Eq. (16).

Finally, we expect 2 to 3 SN per century in our galaxy and we can assume that the next SN would be at a distance comparable to the galactic diameter of $\sim 10^5\ ly$. If we are going to measure such effect at 1 Hz, Again using Eq. (16), we find

$$h = \frac{1\ km \times\ (3 \times 10^8\ m)}{(10^5\ ly)^2} \sim 10^{-30}\ . \tag{19}$$

This requires a measurement of the strain that is six orders of magnitude better than BBO and is not yet achievable by interferometers that are currently being planned.

[3] And if that happens, it might kill us.

ACKNOWLEDGMENTS

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Appendix A: Full calculation of distance between A and B

The trajectories are

$$r_A(t) = \begin{cases} r_0 - \frac{1}{2}\frac{M}{r_0^2}t^2 & (t \le 0) \\ r_0 - \frac{1}{2}\frac{M-\delta M}{r_0^2}t^2 - \frac{\delta M}{r_0}\left(1 + \frac{3}{2}\frac{\delta M}{r_0}\right)t & (t > 0) \end{cases} \tag{A1}$$

$$r_B(t) = \begin{cases} r_0 + d - \frac{1}{2}\frac{M}{(r_0+d)^2}t^2 & (t \le t_c) \\ r_0 + d - \frac{1}{2}\frac{M-\delta M}{(r_0+d)^2}d^2 - \frac{1}{2}\frac{M-\delta M}{(r_0+d)^2}(t-d)^2 - \frac{\delta M}{r_0+d}\left(1 + \frac{3}{2}\frac{\delta M}{r_0}\right)(t-d) - \frac{Md}{(r_0+d)^2}(t-d) & (t > t_c) \end{cases} \tag{A2}$$

1. Crossing time

Assuming that $t = 0$ when the shell crosses r_A. We assume $r_A(t=0) = d$ and $r_B(t=0) = d + r_0$. Solving the metric equation for the shell, which is a null shell, is t_c

$$t_c = \left(1 + \frac{2(M-\delta M)}{r_0}\right)d \tag{A3}$$

This is to leading order.

2. Before shell crossing

From the metric, we can set $ds = 0$, for the photon to get

$$\int_t^{t+t_1} dt = \int_{r_A(t)}^{r_B(t+t_1)} \frac{dr}{(1-\frac{2M}{r})} \tag{A4}$$

Integrating this gives

$$t_1 = r_B(t+t_1) - r_A(t) + 2M\ln\left(\frac{r_B(t+t_1)-2M}{r_A(t)-2M}\right) \tag{A5}$$

Now we can expand the log using $\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + ...$ For us $x = \frac{r_B(t+t_1)-2M}{r_A(t)-2M}$ Thus $x - 1 = \frac{r_B(t+t_1)-r_A(t)}{r_A(t)-2M} = \frac{r_B(t+t_1)-r_A(t)}{r_A(t)}\left(1 - \frac{2M}{r_A(t)}\right)^{-1}$ and we get

$$2M\ln\left(\frac{r_B(t+t_1)-2M}{r_A(t)-2M}\right) = 2M\left(\frac{r_B(t+t_1)-r_A(t)}{r_A(t)}\left(1+\frac{2M}{r_A(t)}\right) - \frac{1}{2}\left(\frac{r_B(t+t_1)-r_A(t)}{r_A(t)}\right)^2\left(1-\frac{2M}{r_A(t)}\right)^{-2}\right) \tag{A6}$$

Since we only want to keep terms that are $\mathcal{O}(r_0^{-2})$ we can replace $\frac{M}{r_A(t)}$ with $\frac{M}{r_0}$. Thus Eq (A5) becomes

$$\begin{aligned} t_1 &= (r_B(t+t_1) - r_A(t))\left(1 + \frac{2M}{r_0}\left(1 + \frac{2M}{r_0} - \frac{1}{2}\frac{r_B(t+t_1)-r_A(t)}{r_0}\right)\right) \\ &= (r_B(t+t_1) - r_A(t))\left(1 + \frac{2M}{r_0} + \frac{4M^2 - M(r_B(t+t_1)-r_A(t))}{r_0^2}\right) \end{aligned} \tag{A7}$$

to leading order $r_B(t+t_1) - r_A(t) = d$

$$t_1 = (r_B(t+t_1) - r_A(t)) \left(1 + \frac{2M}{r_0} + \frac{4M^2 - Md}{r_0^2}\right) \tag{A8}$$

$$r_B(t+t_1) - r_A(t) = d - \frac{1}{2}\frac{M}{r_0^2}(t+d)^2 + \frac{1}{2}\frac{Mt^2}{r_0^2} = d\left(1 - \frac{1}{2}\frac{Md}{r_0^2} - \frac{Mt}{r_0^2}\right) \tag{A9}$$

Therefore t_1 becomes

$$t_1 = d\left(1 + \frac{2M}{r_0} + \frac{4M^2}{r_0^2} - \frac{3}{2}\frac{Md}{r_0^2} - \frac{Mt}{r_0^2}\right) \tag{A10}$$

Now we can compute t_2.

$$t_2 = (r_B(t+t_1) - r_A(t+t_1+t_2))\left(1 + \frac{2M}{r_0} + \frac{4M^2 - M(r_B(t+t_1) - r_A(t+t_1+t_2))}{r_0^2}\right) \tag{A11}$$

$$\begin{aligned} r_B(t+t_1) - r_A(t+t_1+t_2) &= d - \frac{1}{2}\frac{M}{r_0^2}(t+d)^2 + \frac{1}{2}\frac{M}{r_0^2}(t+2d)^2 \\ &= d\left(1 + \frac{Mt}{r_0^2} + \frac{3}{2}\frac{Md}{r_0^2}\right) \end{aligned} \tag{A12}$$

Thus t_2 becomes

$$\begin{aligned} t_2 &= d\left(1 + \frac{Mt}{r_0^2} + \frac{3}{2}\frac{Md}{r_0^2}\right)\left(1 + \frac{2M}{r_0} + \frac{4M^2 - Md}{r_0^2}\right) \\ &= d\left(1 + \frac{2M}{r_0} + \frac{4M^2}{r_0^2} + \frac{1}{2}\frac{Md}{r_0^2} + \frac{Mt}{r_0^2}\right) \end{aligned} \tag{A13}$$

Summing t_1 and t_2

$$t_1 + t_2 = 2d\left(1 + \frac{2M}{r_0} + \frac{4M^2}{r_0^2} - \frac{1}{2}\frac{Md}{r_0^2}\right) \tag{A14}$$

The proper time measured is

$$\begin{aligned} \tau &= -\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}(t_1 + t_2) \\ &= -2d\left(1 - \frac{M}{r} - \frac{1}{2}\frac{M^2}{r_0^2}\right)\left(1 + \frac{2M}{r_0} + \frac{4M^2}{r_0^2} - \frac{1}{2}\frac{Md}{r_0^2}\right) \\ &= -2d\left(1 + \frac{M}{r_0} + \frac{3}{2}\frac{M^2}{r_0^2} - \frac{1}{2}\frac{Md}{r_0^2}\right) \end{aligned} \tag{A15}$$

3. After shell crossing

The time t_1 is again given by

$$t_1 = (r_B(t+t_1) - r_A(t))\left(1 + \frac{2(M-\delta M)}{r_0} + \frac{4(M-\delta M)^2}{r_0^2} - \frac{d(M-\delta M)}{r_0^2}\right) \tag{A16}$$

First lets look at

$$\begin{aligned}
r_B(t+t_1)-r_A(t) &= d-\frac{1}{2}\frac{M-\delta M}{r_0^2}d^2-\frac{1}{2}\frac{M-\delta M}{r_0^2}t^2-\frac{\delta M}{r_0}\left(1+\frac{3}{2}\frac{\delta M}{r_0}\right)(t+t_1-d)+\frac{\delta M dt}{r_0^2}-\frac{Mdt}{r_0^2} \\
&+\frac{1}{2}\frac{M-\delta M}{r_0^2}t^2+\frac{\delta M}{r_0}\left(1+\frac{3}{2}\frac{\delta M}{r_0}\right)t+\frac{2\delta M(M-\delta M)d}{r_0^2} \\
&= d-\frac{1}{2}\frac{M-\delta M}{r_0^2}d^2-\frac{\delta M}{r_0}t_1+\frac{\delta M d}{r_0}-\frac{(M-\delta M)dt}{r_0^2}+\frac{2\delta M(M-\delta M)d}{r_0^2}
\end{aligned} \tag{A17}$$

So t_1 becomes

$$\begin{aligned}
t_1 &= \left(d+\frac{2\delta M(M-\delta M)d}{r_0^2}-\frac{1}{2}\frac{M-\delta M}{r_0^2}d^2-\frac{\delta M}{r_0}t_1+\frac{\delta M}{r_0}d-\frac{(M-\delta M)dt}{r_0^2}\right) \\
&\times\left(1+\frac{2(M-\delta M)}{r_0}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{d(M-\delta M)}{r_0^2}\right) \\
&= d-\frac{\delta M}{r_0}t_1+\frac{\delta M}{r_0}d+\frac{2\delta M(M-\delta M)d}{r_0^2}-\frac{1}{2}\frac{M-\delta M}{r_0^2}d^2-\frac{(M-\delta M)dt}{r_0^2}+\frac{2d(M-\delta M)}{r_0}+\frac{4(M-\delta M)^2d}{r_0^2}-\frac{d^2(M-\delta M)}{r_0^2}
\end{aligned} \tag{A18}$$

$$t_1\left(1+\frac{\delta M}{r_0}\right)=d\left(1+\frac{\delta M}{r_0}+\frac{2\delta M(M-\delta M)}{r_0^2}-\frac{1}{2}\frac{(M-\delta M)d}{r_0^2}-\frac{(M-\delta M)t}{r_0^2}+\frac{2(M-\delta M)}{r_0}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{(M-\delta M)d}{r_0^2}\right) \tag{A19}$$

$$t_1=d\left(1+\frac{2(M-\delta M)}{r_0}-\frac{3}{2}\frac{(M-\delta M)d}{r_0^2}-\frac{(M-\delta M)t}{r_0^2}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{2\delta M^2}{r_0^2}\right) \tag{A20}$$

Now we can compute t_2.

$$t_2=(r_B(t+t_1)-r_A(t+t_1+t_2))\left(1+\frac{2(M-\delta M)}{r_0}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{d(M-\delta M)}{r_0^2}\right) \tag{A21}$$

$$\begin{aligned}
r_B(t+t_1)-r_A(t+t_1+r_2) &= d-\frac{1}{2}\frac{M-\delta M}{r_0^2}d^2-\frac{1}{2}\frac{M-\delta M}{r_0^2}t^2-\frac{\delta M}{r_0}\left(1+\frac{3}{2}\frac{\delta M}{r_0}\right)(t+t_1-d)+\frac{\delta M dt}{r_0^2}-\frac{Mdt}{r_0^2} \\
&+\frac{1}{2}\frac{M-\delta M}{r_0^2}(t+2d)^2+\frac{\delta M}{r_0}\left(1+\frac{3}{2}\frac{\delta M}{r_0}\right)(t+t_1+t_2)+\frac{2\delta M(M-\delta M)d}{r_0^2} \\
&= d+\frac{3}{2}\frac{M-\delta M}{r_0^2}d^2+\frac{\delta M d}{r_0}+\frac{\delta M t_2}{r_0}+3\frac{\delta M^2}{r_0^2}d+\frac{(M-\delta M)dt}{r_0^2}+\frac{2\delta M(M-\delta M)d}{r_0^2}
\end{aligned} \tag{A22}$$

Thus t_2 becomes

$$\begin{aligned}
t_2 &= \left(d+\frac{3}{2}\frac{M-\delta M}{r_0^2}d^2+\frac{\delta M}{r_0}d+\frac{\delta M t_2}{r_0}+\frac{3\delta M^2}{r_0^2}d+\frac{(M-\delta M)dt}{r_0^2}+\frac{2\delta M(M-\delta M)d}{r_0^2}\right) \\
&\times\left(1+\frac{2(M-\delta M)}{r_0}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{d(M-\delta M)}{r_0^2}\right) \\
&= d+\frac{3}{2}\frac{(M-\delta M)d^2}{r_0^2}+\frac{\delta M}{r_0}d+\frac{\delta M t_2}{r_0}+\frac{3\delta M^2 d}{r_0^2}+\frac{(M-\delta M)dt}{r_0^2}+\frac{2\delta M(M-\delta M)d}{r_0^2}+\frac{2(M-\delta M)d}{r_0} \\
&+\frac{2\delta M(M-\delta M)d}{r_0^2}+\frac{2\delta M(M-\delta M)t_2}{r_0^2}+\frac{4(M-\delta M)^2d}{r_0^2}-\frac{d^2(M-\delta M)}{r_0^2}
\end{aligned} \tag{A23}$$

Now we use $t_2 = d$ for the term which is $\mathcal{O}(r_0^{-2})t_2$.

$$t_2\left(1-\frac{\delta M}{r_0}\right) = d\left(1+\frac{\delta M}{r_0}+\frac{2(M-\delta M)}{r_0}+\frac{3}{2}\frac{(M-\delta M)d}{r_0^2}+3\frac{\delta M^2}{r_0^2}+\frac{(M-\delta M)t}{r_0^2}+\frac{2\delta M(M-\delta M)}{r_0^2}\right.$$
$$\left.+\frac{2\delta M(M-\delta M)}{r_0^2}+\frac{2\delta M(M-\delta M)}{r_0^2}+\frac{4(M-\delta M)^2}{r_0^2}-\frac{d(M-\delta M)}{r_0^2}\right) \tag{A24}$$

$$t_2 = d\left(1+\frac{2\delta M}{r_0}+\frac{2(M-\delta M)}{r_0}+\frac{1}{2}\frac{(M-\delta M)d}{r_0^2}+\frac{5\delta M^2}{r_0^2}+\frac{(M-\delta M)t}{r_0^2}+\frac{8\delta M(M-\delta M)}{r_0^2}+\frac{4(M-\delta M)^2}{r_0^2}\right) \tag{A25}$$

Combining this with t_1 gives

$$t_1+t_2 = 2d\left(1+\frac{\delta M}{r_0}+\frac{2(M-\delta M)}{r_0}-\frac{1}{2}\frac{(M-\delta M)d}{r_0^2}+\frac{3}{2}\frac{\delta M^2}{r_0^2}+\frac{4\delta M(M-\delta M)}{r_0^2}+\frac{4(M-\delta M)^2}{r_0^2}\right) \tag{A26}$$

Now we need to turn this into proper time, which is surprisingly tricky here.

$$d\tau^2 = -dt^2\left(1-\frac{2(M-\delta M)}{r_0}\right)^{-1}\left(\left(\frac{dr}{dt}\right)^2-\left(1-\frac{2(M-\delta M)}{r_0}\right)^2\right) \tag{A27}$$

Using the expression for velocity we get

$$d\tau = dt\left(1-\frac{2(M-\delta M)}{r_0}\right)^{-\frac{1}{2}}\left(-\frac{\delta M^2}{r_0^2}+\left(1-\frac{2(M-\delta M)}{r_0}\right)^2\right)^{\frac{1}{2}} \tag{A28}$$

Which we can expand to get

$$d\tau = dt\left(1-\frac{M-\delta M}{r_0}-\frac{1}{2}\frac{(M-\delta M)^2}{r_0^2}-\frac{1}{2}\frac{\delta M^2}{r_0^2}\right) \tag{A29}$$

Thus we get

$$\begin{aligned}\tau &= -2d\left(1+\frac{\delta M}{r_0}+\frac{2(M-\delta M)}{r_0}-\frac{1}{2}\frac{(M-\delta M)d}{r_0^2}+\frac{3}{2}\frac{\delta M^2}{r_0^2}+\frac{4\delta M(M-\delta M)}{r_0^2}+\frac{4(M-\delta M)^2}{r_0^2}\right)\\ &\times\left(1-\frac{(M-\delta M)}{r_0}-\frac{1}{2}\frac{(M-\delta M)^2}{r_0^2}-\frac{1}{2}\frac{\delta M^2}{r_0^2}\right)\\ &= -2d\left(1+\frac{M}{r_0}-\frac{1}{2}\frac{(M-\delta M)d}{r_0^2}+\frac{3}{2}\frac{(M-\delta M)^2}{r_0^2}+\frac{\delta M^2}{r_0^2}+\frac{3\delta M(M-\delta M)}{r_0^2}\right)\end{aligned} \tag{A30}$$

Comparing this to the proper time before shell crossing we get

$$\Delta\tau = \tau-\bar{\tau} = \frac{2d\delta M}{r_0^2}(d-3\delta M) \tag{A31}$$

Appendix B: Distance between A and C

We follow the same procedure as we did for the distance between A and B. First we calculate the time, t_1, it takes for a photon starting from A to go to B

$$t_1 = r_C(t+t_1)_\perp - r_A(t)_\perp = d - \frac{\delta M d(t+d)}{r_0^2}. \tag{B1}$$

The time it takes for the photon to go from C to A, t_2, is also equal to t_1. Therefore the proper time is

$$\tau_{AC} = 2t_1\left(1 - \frac{2(M-\delta M)}{r_0}\right)^{\frac{1}{2}} = 2d\left(1 - \frac{\delta M(t+d)}{r_0^2} - \frac{M-\delta M}{r_0}\right) \tag{B2}$$

The proper time without any shell crossing would be

$$\bar{\tau}_{AC} = -2d\left(1 - \frac{M}{r_0}\right). \tag{B3}$$

The change in proper time is

$$\Delta\tau_{AC} \equiv \tau_{AC} - \bar{\tau}_{AC} = -2d\left(\frac{\delta M(t+d)}{r_0^2} + \frac{\delta M}{r_0}\right) \tag{B4}$$

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