$$P(X_i = k) = p(1-p)^k$$

$$\hat{p} = \frac{1}{1+\bar{X}}, \quad E\left(\frac{1}{1+\bar{X}}\right) \neq \frac{1}{1+E(\bar{X})} = p$$

$$\underbrace{f(x) = \frac{1}{1+x}}, \quad E(\bar{X}) = E(X)$$

$$f'(x) = -\frac{1}{(1+x)^2} < 0, \quad x \in [0, +\infty)$$

$$f'' = \frac{2}{(1+x)^3} > 0$$

$$\underbrace{E\left(\frac{1}{1+\bar{X}}\right)}_{\neq p} > \frac{1}{1+E(\bar{X})} = \frac{1}{1+\frac{1-p}{p}} = p$$

N4

$$E(h(x_1, \dots x_m)) = \mu^2$$

$$E(x_1 \cdot x_2) = E(x_1) E(x_2) = \mu^2$$

$$E(h(x_1, \dots x_m)) = \sigma^2$$

$$h(x_1, x_2) = \underline{x_1}^2 - x_1 x_2$$

$$E(h(x_1, x_2)) = E(x_1^2) - E(x_1 x_2) = \sigma^2$$

$$\frac{2}{n(n-1)} \sum_{i,j} (x_i^2 - x_i x_j)$$

N6

$$\hat{\theta} = \bar{X} \sim N(\theta, \sigma^2)$$

N 7 $\quad H_0: \mu \le \mu_0, \; H_1: \mu > \mu_0$

$$t = \frac{\sup_{\theta \in \Theta_0} L(x, \theta)}{\sup_{\theta \in \Theta_1} L(x, \theta)}$$

$$x_1, \dots x_n \sim N(\mu, \sigma^2)$$

$$L(x_1, \dots x_n, \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} =$$

$$= (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\sum \frac{(x_i - \mu)^2}{2\sigma^2}}$$

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

1) $\bar{x} \le \mu_0, \; \Theta_0$ $\qquad \sum (x_i - \mu)^2 = \sum (x_i - \bar{x})^2 + n(\mu - \bar{x})^2$

$$\ln L(x_1, \dots x_n, \mu, \sigma^2) = -\frac{n}{2} \ln 2\pi - n \ln \sigma - \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}$$

$$= C - n \ln \sigma - \frac{1}{2\sigma^2} \left( \sum (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right) =$$

$$= C - n \ln \sigma - \frac{1}{2\sigma^2} \sum (x_i - \bar{x})^2 - \frac{n}{2\sigma^2} (\mu - \bar{x})^2$$

$$= g(\sigma)$$

$$g'(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum (x_i - \bar{x})^2 =$$

$$= \frac{n}{\sigma^3} \Big( \underbrace{\frac{1}{n} \sum (x_i - \bar{x})^2}_{\hat{\sigma}} - \sigma^2 \Big)$$

$\hat{\sigma}$

$\hat{\sigma} > \sigma \to g'(\sigma) > 0$

$\hat{\sigma} < \sigma \to g'(\sigma) < 0$

$$t = \frac{\sup_{\Theta_0} L(\dots)}{\sup_{\Theta} L(\dots)} = \underline{1}$$

$\hat{\mu}\ \mu_0$ $\quad \mu \in (-\infty, \mu_0)$

$(\mu - \bar{x})^2$

2) $\bar{X} > \mu_0 \quad \hat{\mu} = \mu_0$

$$\ln L(x_1, \ldots x_n, \mu, \sigma^2) =$$

$$= C - n \ln \sigma - \frac{1}{2\sigma^2} \sum_i (x_i - \bar{x})^2 - \frac{n}{2\sigma^2} (\mu - \bar{x})^2$$

$$\leq C - n \ln \sigma - \frac{1}{2\sigma^2} \sum_i (x_i - \bar{x})^2 - \frac{n}{2\sigma^2} (\mu_0 - \bar{x})^2$$

$$= C - n \ln \sigma - \frac{1}{2\sigma^2} \sum (x_i - \mu_0)^2 = h(\sigma)$$

$$h'(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum (x_i - \mu_0)^2 =$$

$$= \frac{n}{\sigma^3} \left( \frac{1}{n} \sum (x_i - \mu_0)^2 - \sigma^2 \right)$$

$$\hat{\sigma}_0 = \frac{1}{n} \sum (x_i - \mu_0)^2$$

$$\ln L(x_1, \ldots, \mu, \sigma^2) \leq C - n \ln \hat{\sigma} - \frac{1}{2\hat{\sigma}^2} \sum (x_i - \mu_0)^2$$

$$= \ln L(x_1, \ldots, \mu_0, \hat{\sigma}_0^2)$$

$$t = \frac{L(x_1, \ldots, \mu_0, \hat{\sigma}_0^2)}{L(x_1, \ldots \hat{\mu}, \hat{\sigma}^2)} =$$

$$= \frac{(2\pi \hat{\sigma}_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}_0^2} \sum_i (x_i - \mu_0)^2\right)}{(2\pi \hat{\sigma}^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}^2} \sum (x_i - \hat{\mu})^2\right)}$$

$$= \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n \exp\left(-\frac{1}{2\hat{\sigma}_0^2}\overbrace{\sum (x_i-\mu_0)^2}^{n\hat{\sigma}_0^2} + \right.$$

$$\left. + \frac{1}{2\hat{\sigma}^2}\overbrace{\sum (x_i-\hat{x})^2}^{n\hat{\sigma}^2}\right) =$$

$$= \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n \cdot \exp\left(-\frac{n}{2}+\frac{n}{2}\right) = \left(\frac{\hat{\sigma}}{\hat{\sigma}_0}\right)^n$$