Second definition of the finite and infinite

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First published in the second edition (1893) of the text "Was sind und was sollen die Zahlen?" page XVII, in the form:

> A system S is called finite if it can be mapped into itself in such a way that no proper part of S is mapped into itself; in the opposite case, S is called an infinite system.

Pursuing this definition of a finite system S without using the natural numbers. Let ϕ be a mapping of S into itself, through which no proper part [echter Teil] of S is mapped into itself. Small Latin letters $a, b \dots z$ always mean elements of S, capital Latin letters $A, B \dots Z$ mean *parts* [Teile] of S; the images of $\dots a, A$ generated by ϕ are respectively denoted by a', A'. That A is part of B is denoted by $A \ 3 \ B$.[1] The system consisting of the elements $a, b, c \dots$ is denoted by $[a, b, c \dots]$. So it [*the definition of finiteness*] is

(1) $$S' \ 3 \ S$$

and

(2) $$\text{from } A' \ 3 \ A \text{ it follows that } A = S.$$

Theorem 1. $S' = S$. Every element of S is an image of (at least) one element r of S. Because from (1) it follows $(S')' \ 3 \ S'$, hence by (2), our theorem holds.

Every system $[s]$ consisting of a single element is finite because it has no proper part and is mapped into itself by the identity mapping. This case is excluded below, S means a finite system that does not consist of a single element.

Theorem 2. Every element s is different from its image s', in symbols: $s \neq s'$. Because if $s = s'$, then $[s]' = [s'] = [s] \ 3 \ [s]$, so according to (2) also $[s] = S$ in contradiction to our assumption about S.

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*First pass machine translation, tweaked by David Michael Roberts, March 2024. The start of page number n in the original is denoted by **[n]** in the current text. Thanks to . . . for improvement comments

[1] [*In modern notation, $\subseteq$. Note also that Dedekind doesn't permit a 'part' to be empty, so that strictly speaking $A \ 3 \ B$ means $\emptyset \neq A \subseteq B$. —DMR*]

Definition 3. If s is a certain element of S, then H_s shall be used to denote any part of S that satisfies the following two conditions:

I. s is element of H_s, so $[s]$ 3 H_s, also

$$[s] + H_s = H_s.$$

II. If h is an element of H_s different from s, then h' is also an element of H_s; So if H 3 H_s, but s is not contained in H, then H' 3 H_s.

Theorem 4. S and $[s]$ are special systems H_s, and $[s]$ is the intersection [Durchschnitt] (the commonality Gemeinheit][2]) of all systems H_s corresponding to the element s. Obvious.

Theorem 5. $H_s = S$ or a proper part of S, depending on whether s' lies in H_s or not.— For if s' lies in H_s, then it follows from II in 3. that H'_s 3 H_s, therefore by (2) that $H_s = S$; and vice versa, if $H = S$, then s' also lies in H_s.

Theorem 6. If H_s is a proper part of S, then s' is the only element of H'_s that lies outside H_s. — Because every element k of H'_s is image h' of at least one element h in H; If $k = h'$ is different from s', then h is also different from s, and consequently (according to II in 3.) $k = h'$ lies in H_s, while the element s' of H'_s (according to 5.) lies outside H_s.

Theorem 7. Every system H'_s is a system $H_{s'}$, that is (Definition 3.):

I'. s' is element of H'_s

II'. If k is an element of H'_s that is different from s', then k' also lies in H'_s.

The first follows from the fact that s lies in H_s, the second from the fact that k lies in H_s (Theorem (6).

Theorem 8. If A, B, C ... are special systems H_s corresponding to the same s, then their intersection H is also a system H_s.

Because according to 3.I. s is a common element of $A, B, C, \ldots$, and therefore also an element of H. Furthermore, if h is an element of H that is different from s, then (according to 3.II.) the image h' is an element of A, of B, of C, ..., and therefore also of H. H therefore fulfills the two conditions I, II in 3. that are characteristic of every H_s.

Definition 9. If a, b are certain elements of S, then the symbol ab (*segment* [Strecke] ab) should mean the intersection [Durchschnitt] of all those systems H_b which (such as S) contain the element a.

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Theorem 10. a is an element of ab, i.e. $[a]$ 3 ab. Because ab is the intersection of all systems H_b in which a lies. - (a is the *start* [...] of ab.)

[2][*Beman's 1901 translation is 'community' —DMR*]

Theorem 11. ab is a system H_b, i.e. $[b]$ 3 ab, and if s is an element of ab different from b, then $[s']$ 3 ab. — This follows from 8. — So b is an element (the *end* [...]) of ab. If H 3 ab but b is not contained in H, then H' 3 ab.

Theorem 12. From $[a]$ 3 H_b, follows from ab 3 H_b. Immediate consequence of 9.

Theorem 13. $aa = [a]$. This follows from 4., because aa is the intersection of all H_a that contain the element a (according to 3. I.).

Theorem 14. If b' is an element of ab, then $ab = S$. — This follows from 11 and 5.

Theorem 15. $b'b = S$. — This follows from 14 and 10.

Theorem 16. If c is an element of ab, then cb 3 ab. — This follows from 12, because ab is an H (according to 11) which contains the element c.

Theorem 17. If $A + B$ means the system composed of A, B, then

$$a'b + b'aS.$$

For if s is an element of ab, then s' is contained in $b'a$ or $a'b$, depending on $s = b$ or different from b (according to 10 or 11 and 3. II), and likewise if s is an element of $b'a$, then s' is contained in $a'b$ or $b'a$; therefore $(ab + b'a)'$ 3 $a'b + b'a$; This leads to the theorem according to (2).

Theorem 18. If a is different from b, then $ab = [a] + a'b$. For since a is an element of ab different from b, then a' is an element of ab (by 10, 11), and consequently (by 16) ab 3 ab; since furthermore (by 10) we also have $[a]$ 3 ab, therefore

$$[a] + a'b \ 3 \ ab.$$

Furthermore: every element s of $[a] + a'b$ that is different from b is either $= a$ or an element of $a'b$ that is different from b, in both cases s' is (by 10, 11) an element of $a'b$, therefore also from $[a] + a'b$, and since (by 11) also $[b]$ 3 $[a] + a'b$, then $[a] + a'b$ is a system H_b; Finally, since $[a]$ 3 $[a] + a'b$, so (by 12)

$$ab \ 3 \ [a] + a'b.$$

The theorem follows from the comparison of both results.

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Theorem 19. If a, b are different elements of S, then a lies outside $a'b$, and b lies outside $b'a$.

Proof. If one assumes the opposite, that there is an element a that is different from b and lies in $a'b$, and that A denotes the system of all such elements a, the following holds. If one puts $a' = s$, then a lies in sb, and since a is different from b, and therefore (according to 13) is not in bb, then s is different from b, and from this it follows (according to 18) that $sb = [s] + s'b$. Furthermore, since a (according to 2) is different from s and lies in sb, then a must lie in $s'b$, and from this it follows again (according to 1) that s (as the image a') also lies in $s'b$. Therefore, the image a' of every element a of A is also contained in A, i.e. A' 3 A. But since $A = S$ would follow from this, while A does not contain the element b, our assumption is inadmissible, so the theorem is true, Qed. The second part follows by exchanging a with b. □

Theorem 20. If a, b are different, then the segments $a'b$, $b'a$ have no common element.

Proof. If one assumes the opposite, that there is a common element m of $a'b$, $b'a$, then it follows from the preceding Theorem 19 that m is different from b and from a; therefore (according to 11) the image m' must also be a common element of $a'b$ and $b'a$; Therefore, if M denotes the system of all such elements m, then M' 3 M, i.e. $M = S$. But this is impossible because a, b are elements of S but not elements of M. So our theorem is true. □

Theorem 21. If a, b are different, then the images a', b' are also different.

Proof. Otherwise the segments $a'b$, $b'a$ would have a common element $a' = b'$, because (according to 10) a' is an element of $a'b$ and b' is an element of $b'a$. □

Theorem 22. From $cb = S$ follows $c = b$.

Proof. There is (according to 1 and 21) in S one and only one element a which satisfies the condition $a' = c$, and therefore $a'b = S$, therefore $[a]$ 3 $a'b$; Therefore (by 19) $a = b$, thus $c = b'$, Qed. □

Theorem 23. If a, b are different, then every element of S is contained in one and only one of the segments $a'b$, $b'a$. — This follows from 17 and 20.

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Theorem 24. If a, b, c are different, then the segments $b'c$, $c'a$, $a'b$ have no common element, and the same applies to the segments $a'c$, $b'a$, $c'b$.

Proof. Because the opposite assumption, that there is an element m common to the segments $b'c$, $c'a$, $a'b$, leads to a contradiction. Let M be the system of all such elements. Since (according to 19) a is not in $a'b$, b is not in $b'c$, c is not in $c'a$, then m is different from c, a, b, and consequently (by 11) m' is a common element of $b'c$, $c'a$, $a'b$, i.e. an element of M; therefore M' 3 M, hence $M = S$. But this is impossible because M does not contain any of the elements a, b, c. So our theorem is true. — The second part results from the first if one swaps a with b, which does not change the assumption. □

Corollary. If you put (as in the following 25):

$$A = c'b,\ B = a'c,\ C = b'a;\ A_1 = b'c,\ B_1 = c'a,\ C_1 = a'b,$$

then $A - B - C = 0$[3] (empty) and $A_1 - B_1 - C_1 = 0$ (empty) and (according to 17, 20) hence

$$S = A + A = B + B_1 = C + C_1;$$
$$0 = A - A_1 = B - B_1 = C - C_1.$$

This also applies (according to 20) if at least two of the elements a, b, c are different.

Theorem 25. If a, b, c are different, then one and only one of the following two cases occurs: Either

$$(3) \quad b'c = b'a + a'c, \quad c'a = c'b + b'a, \quad a'b = a'c + c'b$$
$$c'b = c'a - a'b, \quad a'c = a'b - b'c, \quad b'a = b'c - c'a$$

and each element of S lies in one, but only one, of the segments c'b, a'c, b'a; or

$$(4) \quad c'b = c'a + a'b, \quad a'c = a'b + b'c, \quad b'a = b'c + c'a$$
$$b'c = b'a - a'c, \quad c'a = c'b - b'a, \quad a'b = a'c - cb$$

and each element of S lies in one, but only one, of the segments $b'c$, $c'a$, $a'b$.

Proof. According to 23, c lies either in $a'b$ or in $b'a$. We only consider the first case because the second arises from it by exchanging a for b. Since c is in $a'b$ and is distinct from b, **[455]** then (according to 11) c' also lies in $a'b$, and consequently (by 16) $c'b \ 3 \ a'b$; from this it follows (by 19) that $c'b$ has no element in common with $b'a$; now (by 17) is $a'b + b'a = b'c + c'b$, therefore $b'a \ 3 \ b'c$, and consequently (by 11) a is in $b'c$. From the assumption that c lies in $a'b$, it follows: $c'b \ 3 \ a'b$, $b'a \ 3 \ b'c$, a lies in $b'c$. In the same way, this last conclusion follows if one assumes c, a, b replaced by a, b, c, respectively, again we have the consequences $a'c \ 3 \ bc$, $cb \ 3 \ c'a$, and that b lies in $c'a$; and from this it follows again $b'a \ 3 \ c'a$, $a'c \ 3 \ a'b$ (and the first assumption: c lies in $a'b$).

[Circular diagrams]

Therefore: $c'b \ 3 \ a'b$, $b'a \ 3 \ b'c$, $a'c \ 3 \ b'c$, $c'b \ 3 \ ca$, $b'a \ 3 \ c'a$, $a'c \ 3 \ a'b$, and also $b'a + a'c \ 3 \ b'c$, $c'b + b'a \ 3 \ c'a$, $a'c + c'b \ 3 \ a'b$. If now an element of, say, $b'c$ is neither in $b'a$ nor in $a'c$, then (according to 23) it would be a common element of $b'c$, $a'b$, $c'a$, which (according to 24) is impossible; therefore $b'c \ 3 \ b'a + a'c$, therefore also $b'c = b'a + a'c$, and likewise follows $c'a = c'b + b'a$, $a'b = a'c + c'b$. If, say, $b'a$, $a'c$ have a common element, then the same would also be a common element of $b'c$, $c'a$, $a'b$, which (according to 24) is not the case. From $S = b'c + c'b$ we finally get $S = b'a + a'c + c'b$, which means our theorem is completely proven. □

Corollary. It can never be that $[a] \ 3 \ cb$ and $[b] \ 3 \ ca$ at the same time; because (according to 18) then $[a] \ 3 \ c'b$ and $[b] \ 3 \ c'a$ would have to be at the same time, which is impossible.

[3] [The symbol means the intersection [Durchschnitt].]

Theorem 26. From $ab = cb$ it follows that $a = c$, and if $ab = cd$ is a proper part of S, then $a = c$, $b = d$.

This follows from earlier theorems. Since (by 10) c is in cb, and therefore also in ab, then if $a = b$, then $ab = [a]$, then must also $c = a$. But if a is different from b, then (by 18) $ab = [a] + a'b$, **[456]** and (by 19) $a'b$ is a proper part of ab; If one assumes that c is different from a, then c must lie in $a'b$, so (by 16) cb 3 $a'b$, i.e. cb is a proper part of ab; But since $cb = ab$, this assumption is inadmissible, and therefore always $c = a$, Qed. Furthermore, if $ab = cd$ is a proper part of S, then $b = d$; If b is different from d, then (by 11) b' must also be in cd, and therefore also in ab; But then (by 14) would $ab = S$ violating the assumption, so $b = d$, therefore $ab = cb$, therefore also $a = c$, Qed.

Theorem 27. Every system H (defined in 3.) is a segment [Strecke] $a's$ with the end s and its beginning a' completely determined.

Proof. If $H_s = S$, then $H_s = s's$ (according to 15). But if H_s is a proper part of S, then A is the system of all elements of S that lie outside H_s, so $S = A + H_s$. Since A is a proper part of S, then A' 3 A cannot be, so there is certainly an element a in A, whose image a' lies outside A, hence in H_s; Since (according to 12) $a's$ 3 H_s, then $a's$ and A have no common element. Since a is in A, s is in H_s (even in $a's$), then a and s are different, so (by 20) the segments $a's$, $s'a$ have no common element, and (by 17) $a's + s'a = S = H_s + A$, therefore A 3 $s'a$. If one now assumes that $a's$ is a proper part of H_s, and H denotes the system of all those elements of H_s which are outside $a's$, i.e. in $s'a$, then $H_s = H + a's$, and $s'a = H + A$, so $H = H_s - s'a$ is the intersection of the systems H_s, $s'a$. Since neither s nor a lies in H, it follows from (that?) H 3 H_s, and H 3 $s'a$ (according to 3. and 11.) that H' 3 H_s also, and H' 3 $s'a$, also H' 3 H, hence $H = S$. But this is impossible because s (and also a) lies outside H. Therefore certainly $H_s = a's$, and $A = s'a$, Qed. □

Theorem 28. The intersection of such segments $as, bs \ldots$ which have the same end s is itself such a segment hs, and its beginning h is completely determined.

For every such segment is (according to 11) a system H_s, and (according to 8) the same applies to its intersection, from which the theorem (according to 27) follows.

Corollary (to 28). The intersection of the segments $as, bs, cs \ldots$ is itself one of these segments. — As a proof, let us first state the

Lemma. If hs is a proper part of as, and k is the element whose image is $k' = h$, then hs is also a proper part of ks, and at the same time ks 3 as.

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Proof. If $k = s$, then $hs = s's = S$, while hs is a proper part of as, and therefore also of S. Since k is different from s, then (by 18) $ks = [k] + hs$, and (according to 19) k is not contained in hs, so hs is a proper part of ks. Since hs is a proper part of as, let $as = M + hs$, where M is the system of all elements m of as that lie outside hs and are therefore also different from s; from this follows $M' \ni as$, and since M' obviously cannot be part of M (because M is not $= S$), there must be in M an element m, the image m' of which lies outside M, hence in hs, from which $m's \ni hs$ follows[4]. □

Theorem 29. If T is a part of S, and s is an element of S, then in S there is always one and only one associated element s_1, which has the following two properties:

1. If a satisfies the condition $T \ni as$, then $s_1s \ni as$

2. $T \ni s_1s$

and from this follow the two properties

3. s_1 is in T

4. The segment ss_1 contains no element of T that is different from s and s_1.

Proof. Since $s's = S$, therefore $T \ni s's$ (by 15), there is at least one element a that satisfies the condition $T \ni as$. If A is the system of all such elements a, then (according to 28) the intersection of all the segments corresponding to them is a segment s_1s, where s_1 is a completely determined element of S. According to the concept of an intersection, s_1 has the property 1., but also property 2., because T is a common part of all as, and therefore also part of their intersection s_1s. If $s_1 = s$, then $s_1s = ss = [s]$, then it follows from 2. that T consists of the single element s; and vice versa, if s lies in T and is the only element of T, then $T = [s] = ss$, so then according to 1. $s_1s \ni ss$, therefore $s_1 = s$; In this case, s_1 therefore has the property 3. and obviously also the property 4. But if s_1 is different from s, then (according to 18) $s_1s = [s_1] + (s_1)'s$. If one now assumes that s_1 lies outside T, if every element of T is different from s_1, it follows from 2. also that $T \ni (s_1)'s$, and from this according to 1. we also have $s_1s \ni (s_1)'s$, but this is impossible because (according to 10) in s_1s is the element s_1 (according to 19) which lies outside $(s_1)'s$; therefore **[458]** our assumption is inadmissible, i.e. s_1 has property 3. We now consider the segment ss_1; if it has an element u that is different from s and s_1, then s is also different from s_1 (because otherwise $ss_1 = [s]$, which would also give $u = s$), and (according to 18) $ss_1 = [s] + ss_1$; therefore u lies in $s's_1$, therefore (according to 19) outside $(s_1)'s$, and since (as above) $s_1s = [s_1] + (s_1)'s$, and u is also different from s_1, then u also lies outside s_1s, therefore according to 2. also outside T', i.e. s_1 also has property 4. □

30. *Mapping of S into T.* Through 29 a mapping ψ of S into T is created, which is defined by the fact that each element s of S is sent by ψ into the element s_1, which is defined there and (according to 3.) lies in T. If A is then

[4] [The proof is apparently incomplete. According to J. Cavaillès, a proof of the Lemma results directly from 25 by replacing the a, b, c there by a, k, s. The Corollary follows from 28 and the Lemma. E.N.]

any part of S, then A_1 should mean the associated image of A (i.e. the system of images a_1 of all elements a of A). So S_1 3 T, also T_1 3 T, i.e. T is mapped by ψ into itself.

Theorem 30. This mapping of T into itself is a similar one, i.e.: if a, b are different elements of T, then their images a_1, b_1 are also different.

Proof. By 29, T 3 $a_1 a$ and T 3 $b_1 b$. Since a, b are elements of T, then $[a]$ 3 $b_1 b$, $[b]$ 3 $a_1 a$. If, although a, b are different, $a_1 = b_1 \doteq c$, so then $[a]$ 3 cb, $[b]$ 3 c; but since c is different from a and b (because otherwise $a = b$), this is impossible (after the Corollary to 25). Therefore a_1, b_1 are different, Qed. □

Explanations to the above treatise

The definition of the finite given here is chronologically the first that enables the derivation of all properties without using the axiom of choice — a fact that Dedekind was probably not yet aware of. He himself only draws the first conclusions; In this way, one can still conclude from his last theorem that every subset of a finite set is finite, and the principle of complete induction can be proven and thus move on to Dedekind's original definition (cf. a forthcoming note[5] by J. Cavaillès, Fund. Math. **19**). A comparative overview of the various definitions of finiteness is given by A. Tarski (*Sur les ensembles finis*, Fund. Math. **6**[6]), whose own definition goes like this: A set is said to be finite if in every system of subsets at least one im System minimal is included. The corresponding maximum condition is equivalent to this minimum condition through the transition to the complementary set; all properties of finite [**459**] sets follow from both without using the selection postulate. Tarski concludes that the above definition by Dedekind is equivalent to the minimal condition from the relation that also appears in a different version in Dedekind: ab' 3 $ab + [b']$. In particular, Tarski gets from the above to the original Dedekind definition, while the reverse transition requires the axiom of choice. Dedekind believed — in the preface to the 2nd edition of "Was sind und was sollen die Zahlen?" — that the proof of the agreement of the definitions required the full theory developed there. How he thought about the transition in detail is shown in the following passage Letter to H. Weber:

> "The shortest characterization of the finite and infinite is, as I believe, the one which I found on March 9, 1889 and in the preface (p. XI) to the second edition (1893) of the work, "Was sind und was sollen die Zahlen?". I say it like this: 'A system S is called finite if

[5] [*This paper appeared as "Sur la deuxième des ensembles finis donnée par Dedekind", Fundamenta Mathematicae* ***19*** *(1932) pp 143–148. Available from* `https://www.impan.pl/en/publishing-house/journals-and-series/fundamenta-mathematicae/all/19/0/92980/sur-la-deuxieme-des-ensembles-finis-donnee-par-dedekind` *—DMR*]

[6] [*Fundamenta Mathematicae* ***6*** *(1924) pp 45–95. Available from* `https://www.impan.pl/en/publishing-house/journals-and-series/fundamenta-mathematicae/all/6/0/92568/sur-les-ensembles-finis` *—DMR*]

there is a mapping of S into itself through which no proper part of S is mapped into itself; in the opposite case, S is called an infinite system.'

But if one assumes that one already knows the natural number series and its laws completely, and one replaces the word "called" with the word "is" in the above, then this definition turns into a theorem that can be proven like this:

Let ϕ be a mapping of a system S into itself, through which no proper part of S is mapped into itself. I denote the image of an element a or a part A of S with $a\phi$ or $A\phi$ (much more natural than $\phi(a)$ or $\phi(A)$). If a is any element of S, then all the images

$$a\phi, a\phi^2 = (a\phi)\phi \ldots, a\phi^{n+1} = (a\phi^n)\phi \ldots$$

are elements of S, so the system A of all these images is also a part of S, and since $A\phi$ is the system of all images

$$(a\phi)\phi = a\phi^2, (a\phi^2)\phi = a\phi^3,$$

is also a part of A, then A is represented in itself by ϕ; and consequently $A = S$. Therefore a is also an element of A, so there is a smallest natural number n, that of the condition

$$a\phi^n = a$$

suffices. Then S is the system of n elements

$$a\phi, a\phi^2, \ldots a\phi^n$$

and these are different from each other. For according to the definition of n, the last element is different from all previous ones; would also be $1 \leq r < s < n$ and

$$a\phi^r = a\phi^s$$

that would be the case

$$(a\phi^r)\phi^{n-s} = (a\phi^s)\phi^{n-s}$$

hence

$$a\phi^{r+n-s} = a\phi^n = a$$

although $1 < r + n - s < n$. Finally, the fact that S contains no elements other than these follows from $a\phi^{m+n} = a\phi^m$. So S is really a finite system (in the usual sense), and at the same time it follows that ϕ is a cyclic permutation of the n elements of S, so it is also a similar (i.e. clearly reversible) mapping.

Conversely, a finite (in the usual sense) system S consists of n different elements

$$a_1, a_2 \ldots a_{n-1}, a_n$$

and one defines a mapping ϕ of S by

$$a_n\phi = a_1, a_r\phi = a_{r+1}$$

for $1 \leq r < n$, then $S' = S$, so ϕ is a mapping from S into itself, and it is easy to show that no proper part of S is mapped into itself. Because if a part A of S is represented by ϕ in itself and contains an element a, then A must also contain all elements $a\phi, a\phi^2, a\phi^3, \ldots$, i.e. all elements of S, and therefore $= S$. Qed."

Noether.