Maxwell Equations in PTC's Sector Bend

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1 Magnetic field

We will say that a bend has sector geometry if magnetic field and/or the electric field is invariant as we travel on a circle of radius r down the center of the element. The symmetry would be perfect if the entire element was a cyclotron; so here we neglect fringe fields at both ends.

In the case of a magnetic field, the vector potential a_s must obey the equation:

$$\left\{\nabla_\perp^2 - \frac{h}{1+hx}\frac{\partial}{\partial x}\right\}F \quad = 0 \quad \text{where} \quad F = (r+x)\,a_s \text{ and } h = \tfrac{1}{r} \tag{1}$$

$\nabla_\perp^2$ is the transverse Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

In a straight element, where $r = \infty$, the solutions if Eq. (1) used in PTC, are just:

$$F_n^{h=0} \quad = -\tfrac{b_n}{n} Re\left\{(x+iy)^n\right\} + \tfrac{a_n}{n} Im\left\{(x+iy)^n\right\} \tag{2}$$

These solutions have the property that, in the midplane $y = 0$, the magnetic field is given by:

$$b_y = \sum_n b_n x^{n-1} \quad \text{and} \quad b_x = \sum_n a_n x^{n-1} \tag{3}$$

Of course the full solution is given by an infinite sum:

$$F_n^h \quad = \sum_{k=0,\infty} \Delta_n^k \tag{4}$$

The idea is to set up a recursive relation to solve for the F_n^k while continuously respecting Eqs. (1) and (3). Let us see how that works at order k.

$$\Delta_n^{k+1} \quad = \quad \nabla_\perp^{-2} \sum_{\alpha=0,k} \frac{h}{1+hx}\frac{\partial}{\partial x}\Delta_n^\alpha + C_{k+1}F_{n+k}^0 \tag{5}$$

To solve this equation, one first initializes it at $k = 0$ with $F_n^{h=0}$ of Eq. (2). The operator $\nabla_\perp^{-2}$ of Eq. (5) find a solution for the inverse Laplacian. In phasors variable $\vec{u} = (x+iy, x-iy)$ the inverse Laplacian is given by

$$\nabla_\perp^{-2} f(\vec{u}) \quad = \textstyle\int^u \int^{\overline{u}} f du d\overline{u} + C f^0 \quad \text{where} \quad \nabla_\perp^2 f^0 = 0 \tag{6}$$

in Eq. (6) the first term is an antiderivative with with respect to u and $\overline{u}$. The additive term is an harmonic solution of the type of Eq. (2).

To obtain the solution to the order of truncation on iterates Eq. (5) starting with the harmonic solution $\Delta_n^{k=0} = F_n^0$.

The only issue is how does one set the constant C_{k+1}? One notices that the harmonic solution F_n^0 already obeys Eq. (3). Therefore it suffices to choose C_{k+1} so as to remove leading order terms of the form Cx^{n+k+1} in Δ_n^{k+1} by adding a term proportional to F_n^{k+1}. Higher order terms which violate Eq. (3) while be handled at the next iterations.

2 Electric Field

The electric field should also obey Maxwell's equation. If we represent it by a potential F, its Laplacian must vanish. In cylindrical coordinates, it is given by the equation:

$$\begin{aligned}\nabla_\perp^2 F &= \left\{\frac{1}{1+hx}\frac{\partial}{\partial x}(1+hx)\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right\} F \\ &= \left\{\nabla_\perp^2 + \frac{h}{1+hx}\frac{\partial}{\partial x}\right\} F \end{aligned} \tag{7}$$