Maxwell Equations in PTC's Sector Bend

E. Forest

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1 Useful formulae

$$\nabla \times u = \left(\partial_y u_\phi - \frac{1}{r}\partial_\phi u_y\right)\widehat{r} + \frac{1}{r}\left(\partial_\phi u_r - \partial_r r u_\phi\right)\widehat{y} + \left(\partial_r u_y - \partial_y u_r\right)\widehat{\phi} \tag{1}$$

$$\nabla f = \partial_r f\ \widehat{r}\ +\ \partial_y f\ \widehat{y}\ +\ \frac{1}{r}\partial_\phi f \tag{2}$$

$$\nabla \cdot u = \frac{1}{r}\partial_r r u_r\ + \partial_y u_y + \frac{1}{r}\partial_\phi u_\phi \tag{3}$$

If we use the more convenient (x, y, s) variable defined as

$$\begin{aligned} h &= \frac{1}{r_0} \\ 1 + hx &= hr \\ hs &= \phi \end{aligned} \tag{4}$$

Then the curl, gradient and divergence can be rewritten:

$$\begin{aligned} \nabla \times u = &\left(\partial_y u_s - \frac{1}{1+hx}\partial_s u_y\right)\widehat{x} \\ &+\frac{1}{1+hx}\left(\partial_s u_x - \partial_x (1+hx) u_s\right)\widehat{y} \\ &+\left(\partial_x u_y - \partial_y u_x\right)\widehat{s} \end{aligned} \tag{5}$$

$$\nabla f = \partial_x f\ \widehat{x}\ +\ \partial_y f\ \widehat{y}\ +\ \frac{1}{1+hx}\partial_s f \tag{6}$$

$$\nabla \cdot u = \frac{1}{1+hx}\partial_x \left(1+hx\right) u_r\ + \partial_y u_y + \frac{1}{1+hx}\partial_s u_s \tag{7}$$

2 Magnetic field with vector potential: old way of PTC

We will say that a bend has sector geometry if magnetic field and/or the electric field is invariant as we travel on a circle of radius r down the center of the element. The symmetry would be perfect if the entire element was a cyclotron; so here we neglect fringe fields at both ends.

In the case of a magnetic field, the vector potential a_s must obey the equation:

$$\left\{ \nabla_\perp^2 - \frac{h}{1+hx}\frac{\partial}{\partial x} \right\} F \quad = 0 \quad \text{where} \quad F = (r+x)\, a_s \text{ and } h = \tfrac{1}{r} \tag{8}$$

$\nabla_\perp^2$ is the transverse Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

In a straight element, where $r = \infty$, the solutions if Eq. (8) used in PTC, are just:

$$F_n^{h=0} \quad = -\tfrac{b_n}{n} Re\left\{ (x+iy)^n \right\} + \tfrac{a_n}{n} Im\left\{ (x+iy)^n \right\} \tag{9}$$

These solutions have the property that, in the midplane $y = 0$, the magnetic field is given by:

$$b_y = \sum_n b_n x^{n-1} \quad \text{and} \quad b_x = \sum_n a_n x^{n-1} \tag{10}$$

Of course the full solution is given by an infinite sum:

$$F_n^h \quad = \sum_{k=0,\infty} \Delta_n^k \tag{11}$$

The idea is to set up a recursive relation to solve for the F_n^k while continuously respecting Eqs. (8) and (10). Let us see how that works at order k.

$$\Delta_n^{k+1} \quad = \quad \nabla_\perp^{-2} \sum_{\alpha=0,k} \frac{h}{1+hx}\frac{\partial}{\partial x} \Delta_n^\alpha + C_{k+1} F_{n+k}^0 \tag{12}$$

To solve this equation, one first initializes it at $k = 0$ with $F_n^{h=0}$ of Eq. (9). The operator $\nabla_\perp^{-2}$ of Eq. (12) find a solution for the inverse Laplacian. In phasors variable $\vec{u} = (x+iy, x-iy)$ the inverse Laplacian is given by

$$\nabla_\perp^{-2} f(\vec{u}) \quad = \tfrac{1}{4} \int^u \int^{\overline{u}} f du d\overline{u} + C f^0 \quad \text{where} \quad \nabla_\perp^2 f^0 = 0 \tag{13}$$

in Eq. (13) the first term is an antiderivative with with respect to u and $\overline{u}$. The additive term is an harmonic solution of the type of Eq. (9).

To obtain the solution to the order of truncation one iterates Eq. (12) starting with the harmonic solution $\Delta_n^{k=0} = F_n^0$.

The only issue is how does one set the constant C_{k+1}? One notices that the harmonic solution F_n^0 already obeys Eq. (10). Therefore it suffices to choose C_{k+1} so as to remove leading order terms of the form Cx^{n+k+1} in Δ_n^{k+1} by adding a term proportional to F_n^{k+1}. Higher order terms which violate Eq. (10) while be handled at the next iterations.

3 Magnetic field with scalar potential: new way of PTC

See routine get_bend_magnetic_potential in Se_status.f90

We will say that a bend has sector geometry if magnetic field and/or the electric field is invariant as we travel on a circle of radius r down the center of the element. The symmetry would be perfect if the entire element was a cyclotron; so here we neglect fringe fields at both ends.

In the case of a magnetic field, the scalar potential V must obey the equation:

$$\left\{ \nabla_{\perp}^{2} + \frac{h}{1+hx}\frac{\partial}{\partial x} \right\} V \quad = 0 \tag{14}$$

$\nabla_{\perp}^{2}$ is the transverse Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

In a straight element, where $r = \infty$, the solutions if Eq. (14) used in PTC, are just:

$$F_n^{h=0} \quad = \frac{b_n}{n} Re\left\{ i\,(x+iy)^n \right\} - \frac{a_n}{n} Re\left\{ (x+iy)^n \right\} \tag{15}$$

These solutions have the property that, in the midplane $y = 0$, the magnetic field is given by:

$$b_y = \sum_n b_n x^{n-1} \quad \text{and} \quad b_x = \sum_n a_n x^{n-1} \tag{16}$$

Of course the full solution is given by an infinite sum:

$$F_n^h \quad = \quad \sum_{k=0,\infty} \Delta_n^k \tag{17}$$

The idea is to set up a recursive relation to solve for the F_n^k while continuously respecting Eqs. (14) and (16). Let us see how that works at order k.

$$\Delta_n^{k+1} \quad = \quad -\nabla_{\perp}^{-2} \sum_{\alpha=0,k} \frac{h}{1+hx}\frac{\partial}{\partial x}\Delta_n^{\alpha} + C_{k+1}F_{n+k}^{0} \tag{18}$$

To solve this equation, one first initializes it at $k = 0$ with $F_n^{h=0}$ of Eq. (15). The operator $\nabla_{\perp}^{-2}$ of Eq. (18) find a solution for the inverse Laplacian. In phasors variable $\vec{u} = (x+iy, x-iy)$ the inverse Laplacian is given by

$$\nabla_{\perp}^{-2} f(\vec{u}) \quad = \frac{1}{4}\int^{u}\int^{\overline{u}} f du d\overline{u} + C f^0 \quad \text{where} \quad \nabla_{\perp}^{2} f^0 = 0 \tag{19}$$

in Eq. (19) the first term is an antiderivative with with respect to u and $\overline{u}$. The additive term is an harmonic solution of the type of Eq. (15).

To obtain the solution to the order of truncation one iterates Eq. (18) starting with the harmonic solution $\Delta_n^{k=0} = F_n^0$.

The only issue is how does one set the constant C_{k+1}? One notices that the harmonic solution F_n^0 already obeys Eq. (16). Therefore it suffices to choose C_{k+1} so as to remove leading order terms of the form Cx^{n+k+1} in Δ_n^{k+1} by adding a term proportional to F_n^{k+1}. Higher order terms which violate Eq. (16) while be handled at the next iterations.

4 Electric Field

See routine get_bend_electric_coeff in Se_status.f90

The electric field should also obey Maxwell's equation. If we represent it by a potential F, its Laplacian must vanish. In cylindrical coordinates, it is given by the equation:

$$\begin{aligned}
\nabla_\perp^2 F &= \left\{ \frac{1}{1+hx}\frac{\partial}{\partial x}(1+hx)\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} \right\} F \\
&= \left\{ \nabla_\perp^2 + \frac{h}{1+hx}\frac{\partial}{\partial x} \right\} F
\end{aligned} \tag{20}$$

The zeroth order solution is:

$$F_n^{h=0} = -\tfrac{b_n}{n} Re\left\{(x+iy)^n\right\} - \tfrac{a_n}{n} Im\left\{(x+iy)^n\right\} \tag{21}$$

or equivalently

$$F_n^{h=0} = -\tfrac{b_n}{n} Re\left\{(x+iy)^n\right\} + \tfrac{a_n}{n} Re\left\{i\,(x+iy)^n\right\} \tag{22}$$

The electric field is defined as:

$$\vec{E} = -\vec{\nabla} F \tag{23}$$

The electric coefficients a_n and b_n are determined by a relation similar to Eq. (16):

$$E_x = \sum_\alpha b_n x^{n-1} \quad \text{and} \quad E_y = \sum_\alpha a_n x^{n-1} \tag{24}$$

PTC has an electric septum. This is a straight dipole element producing a vertical electric field. Remarkably, the body of that bend can be exactly solved. Of course a zero curvature electric sector bend should produce the same result. This is something we ought to check and document.

The following piece of code checks most assertions of this document. Notice that PTC has two new parameters `volt_c=1.e-3` and `volt_i=1`. These are used to enforce megavolts in PTC. In reality the code uses gigavolts. Here by setting `volt_c` to 1, I use gigavolts. `volt_i` is an input parameter. We can enforce megavolts in the code computations (deep in) or at the input level when we create the element. Anyway in this example, everything is in gigavolts `volt_c=1` .

```
call GET_ONE(p0c=p0c)
volt_c=1
voltage=1.d-6*p0c
solve_electric=my_true
B  = SBEND("B", L=2.54948D0,ANGLE=0.d0)
```

```
CALL ADD(B,-1,1,voltage,electric=my_true)

QF =ELSEPARATOR("BB", L=2.54948D0,E=voltage)
!QF%MAG%phas=pi/2d0 ;QF%MAGp%phas=QF%MAG%phas;

X=0.001D0
call init(8,2)
call alloc(y);call alloc(eb);call alloc(phi);call alloc(bf)
y(1)=morph(1.0d0.mono.1);
y(3)=morph(1.0d0.mono.2);
call GETELECTRIC(B%mag%tp10,BFr,phir,EBR,x)

WRITE(6,*) BFR
WRITE(6,*) phir
CALL GETELECTRIC(QF%MAG%SEP15,EBR,phir,X)
WRITE(6,*) EBR
WRITE(6,*) phir
call GETELECTRIC(B%magp%tp10,BF,phi,EB,y)
do i=1,3
call print(bf(i),6)
enddo
call print(phi,6)
b%dir=1
x=0.001d0
call track(b,x,state)
write(6,*)x
x=0.001d0
call track(QF,x,state)
write(6,*)x
stop
```

The result is:

```
 0.000000000000000E+000  1.000000000000000E-006  0.000000000000000E+000
-9.999999999999999E-010
 0.000000000000000E+000  1.000000000000000E-006  0.000000000000000E+000
-1.000000000000000E-009

etall    1, NO =    8, NV =    2, INA =   40
*********************************************

   ALL COMPONENTS ZERO
      NO =     8      NV =     2
    -1   0.000000000000000       0  0
```

```
etall    1, NO =    8, NV =    2, INA =   41
*********************************************

   I  COEFFICIENT          ORDER   EXPONENTS
     NO =     8      NV =     2
  0  0.1000000000000000E-05   0  0
   -1   0.000000000000000       0  0
 0.000000000000000E+000

etall    1, NO =    8, NV =    2, INA =   30
*********************************************

   I  COEFFICIENT          ORDER   EXPONENTS
     NO =     8      NV =     2
  1 -0.1000000000000000E-05   0  1
   -1   0.000000000000000       0  0
 3.546219861200883E-003  1.000000000000000E-003  3.550377891682065E-003
 1.003266041983185E-003  1.000000000000000E-003 -6.323016230930416E-004
 3.546219861200894E-003  1.000000000000000E-003  3.550377891682071E-003
 1.003266041983188E-003  1.000000000000000E-003 -6.323016230926903E-004
```

So all is fine: the electric septum produces a vertical kick,i.e., an E_y. The results agrees with the sector bend integration. Notice that the parameter `solve_electric` is set to true before creating the element. This forces non-symplectic integration of a sector bend with electric and magnetic field.

Now, I will produce an horizontal kick. For the septum, it is achieved using a rotation of $\pi/2$.

```
B  = SBEND("B", L=2.54948D0,ANGLE=0.d0)
CALL ADD(B,1,1,voltage,electric=my_true)

QF =ELSEPARATOR("BB", L=2.54948D0,E=voltage)
QF%MAG%phas=pi/2d0 ;QF%MAGp%phas=QF%MAG%phas;
```

Tracking results agrees.

5 Electric hard edge fringe effects

$$
\begin{aligned}
a_x &= -\frac{yU}{Brho} \\
a_y &= \frac{xU}{Brho}
\end{aligned}
$$

$$
\begin{aligned}
U &= \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n} B_z^{[2n]}}{2^{2n+1} n!(n+1)!} \\
B_z^{[2n]} &= \frac{d^{2n}}{dz^{2n}} B_z(0,0,z).
\end{aligned}
$$

The scaled vector potential for this field is given by

$$
a_z = -x \; b(z). \tag{25}
$$

It is easy to check that this vector potential does not obey Maxwell's equations—$\nabla \times \nabla \times (0, 0, -x \; b(z)) \neq 0$. However we can introduce an x-component to the vector potential of the form

$$
a_x = \sum_{n=1}^{\infty} a_x^n(z) \; y^{2n}, \tag{26}
$$

and require that $(a_x, 0, a_z)$ be curl-free. The solution is found immediately to be

$$
a_x = \sum_{n=1}^{\infty} \frac{(-1)^n b_z^{[2n-1]}}{(2n)!} y^{2n}. \tag{27}
$$

6 Hamiltonian

$$
K = -(1+hx)\sqrt{1 + 2\frac{\Delta}{\beta_0} + \Delta^2 - p_x^2 - p_y^2} - A \quad \text{where } \Delta = \delta - \Phi \tag{28}
$$