Algorithm Design and Analysis (ECS 122A) Study Guide

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1 Asymptotic Notation

1.1 O-Notation (Big O)

Notation:

$$f(n) \in O(g(n))$$

Formal Definition:
For a given function $g(n)$, $O(g(n))$ is the set of functions for which there exists positive constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$O(g(n)) = \{f(n) : \exists\ c, n_0 \text{ s.t. } 0 \leq f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0\}$$

Informal Definition:
The function $g(n)$ is an asymptotic upper bound for the function $f(n)$ if there exists constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for $n \geq n_0$.

Another way to perceive Big O notation is that for $f(n) \in O(g(n))$, the function f's asymptotic[1] growth is no faster than that of function g's.

Limit Definition:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$$

1.1.1 Example

Prove that asymptotic upper bound of $f(n) = 2n + 10$ is $g(n) = n^2$.

$$\begin{aligned} 0 \leq f(n) &\leq c \cdot g(n) \text{ for } n \geq n_0 \\ 0 \leq 2n + 10 &\leq c \cdot n^2 \text{ for } n \geq n_0 \end{aligned}$$

Arbitrarily choose c and n_0 values. Simplest is to turn one of the variables into the value 1 and solve. For this example, we will assign the value 1 to n_0.

$$\begin{aligned} 0 \leq 2n + 10 &\leq c \cdot n^2 \text{ for } n \geq 1 \\ 2(1) + 10 &\leq c \cdot (1)^2 \\ 12 &\leq c \end{aligned}$$

By picking $n_0 = 1$ and $c = 12$, the inequality of $2n + 10 \leq 12n^2$ will hold true for all $n \geq 1$. Since there exists a constant c and n_0 that fulfill this inequality, we have proven that $f(n) = 2n + 10 = O(n^2)$.

[1] Asymptotic: As given variable approaches infinity.

1.2 o-Notation (Little O)

Notation:

$$f(n) \in o(g(n))$$

Formal Definition:
For a given function $g(n)$, $o(g(n))$ is the set of functions for which every positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$o(g(n)) = \{f(n) : \exists\ n_0 \text{ s.t. } 0 \leq f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0, c \geq 0\}$$

Informal Definition:
The function g(n) is an upper bound that is not asymptotically tight. For all positive constant values of c, there must exists a constant n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. The value of n_0 may not depend on n, but may depend on c.

Another way to perceive Little O notation is that for $f(n) \in o(g(n))$, the function f's asymptotic growth is strictly less than that of the function g's. In this sense, Little O can be seen as a "stronger" bound in comparison to Big O. By proving that a function is an element of Little O, it also proves that the function is an element of Big O.

Limit Definition:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

1.2.1 Example

Prove that $f(n) = 2n$ has an upper bound $o(n^2)$.

$$\begin{aligned} 0 \leq c \cdot g(n) &\leq f(n) \text{ for } n \geq n_0 \\ 0 \leq c \cdot 2n &\leq n^2 \text{ for } n \geq n_0 \\ 2c &\leq n \text{ for } n \geq n_0 \\ 2c &\leq n_0 \end{aligned}$$

For Little O to hold true, the inequality needs to hold true for all $c > 0$ and for all $n > n_0$. From simplifying the inequality, we assert that the inequality will hold true as long as the value of n_0 is twice the value of c. Given that they are both constants, then there exists a constant value of n_0 for all positive constant c that fulfill this inequality.

Another method to solve this problem is to use the limit definition.

$$\begin{aligned} &\lim_{n\to\infty} \frac{2n}{n^2} \\ &\lim_{n\to\infty} \frac{2}{n} = 0 \end{aligned}$$

1.2.2 Example

Prove that $f(n) = 2n^2$ does not have the upper bound $o(n^2)$.

$$\begin{aligned} 0 \leq c \cdot g(n) &\leq f(n) \text{ for } n \geq n_0 \\ 0 \leq c \cdot 2n^2 &\leq n^2 \text{ for } n \geq n_0 \\ 2c &\leq 1 \text{ for } n \geq n_0 \end{aligned}$$

For a function to have the Little O bound, the inequality must hold true for all positive c. However, simplification of the inequality asserts that the inequality will only hold true for all $c < \frac{1}{2}$. Therefore, $f(n) = 2n^2$ does not have the upper bound $o(n^2)$.

1.3 Ω-Notation (Big Omega)

Notation:

$$f(n) \in \Omega(g(n))$$

Formal Definition:
For a given function $g(n)$, $\Omega(g(n))$ is the set of functions for which there exists positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

$$\Omega(g(n)) = \{f(n) : \exists\ c, n_0 \text{ s.t. } 0 \leq c \cdot g(n) \leq f(n)\ \forall\ n \geq n_0\}$$

Informal Definition:
The function $g(n)$ is an asymptotic lower bound for the function $f(n)$ if there exists constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for $n \geq n_0$.

Limit Definition:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

1.3.1 Example

Prove that the asymptotic lower bound of $f(n) = 8n^2$ is $g(n) = n$.

$$\begin{aligned} 0 \leq c \cdot g(n) &\leq f(n) \text{ for } n \geq n_0 \\ 0 \leq c \cdot n &\leq 8n^2 \text{ for } n \geq n_0 \end{aligned}$$

Arbitrarily choose c and n_0 values. Simplest is to turn one of the variables into the value 1 and solve. For this example, we will assign the value 1 to c.

$$\begin{aligned} 0 \leq n &\leq 8n^2 \text{ for } n \geq n_0 \\ (n_0) &\leq 8(n_0)^2 \\ 1 &\leq 8n_0 \\ \frac{1}{8} &\leq n_0 \end{aligned}$$

By picking $n_0 = 1$ and $c = 1$, the inequality of $n \leq 8n^2$ will hold true for all $n \geq 1$. Since there exists a constant c and n_0 that fulfill this inequality, we have proven that $f(n) = n^2 = \Omega(n)$.

1.4 ω-Notation (Little Omega)

Notation:

$$f(n) \in \omega(g(n))$$

Formal Definition:
For a given function $g(n)$, $\omega(g(n))$ is the set of functions for which every positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

$$\omega(g(n)) = \{f(n) : \exists\ n_0 \text{ s.t. } 0 \leq c \cdot g(n) \leq f(n)\ \forall\ n \geq n_0, c \geq 0\}$$

Informal Definition:
The function g(n) is a lower bound that is not asymptotically tight. For all positive constant values of c, there must exists a constant n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$. The value of n_0 may not depend on n, but may depend on c.

Another way to perceive Little ω notation is that for $f(n) \in \omega(g(n))$, the function f's asymptotic growth is strictly greater than that of the function g's. In this sense, Little ω can be seen as a "stronger" bound in comparison to Big Ω. By proving that a function is an element of Little ω, it also proves that the function is an element of Big Ω.

Limit Definition:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

1.5 Θ-notation

Notation:

$$f(n) \in \Theta(g(n))$$

Formal Definition:
For a given function $g(n)$, $\Theta(g(n))$ is the set of functions for which there exists positive constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

$$\Theta(g(n)) = \{f(n) : \exists\ c_1, c_2, n_0 \text{ s.t. } 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\ \forall\ n \geq n_0\}$$

Informal Definition:
The function g(n) is an asymptotic tight bound for the function f(n) if there exists constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for $n \geq n_0$.

Limit Definition:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}_{>0}$$

2 Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence of values. After the initial terms are given, each subsequent term is defined as a function of the previous terms.

2.1 Fibonacci

Fibonacci is an example of a recurrence relation.

$$F_n = \begin{cases} F_{n-1} + F_{n-2}, & n \geq 2 \\ 1, & n = 1 \\ 0, & n = 0 \end{cases}$$

The first two terms are defined while the subsequent terms are a function of the two previous.

2.2 Solving Recurrence Relations with Induction

2.2.1 Example

Prove that the following systems of equations has the solution $T(n) = n \cdot lg(n)$.

$$T(n) \begin{cases} 2T(\frac{n}{2}) + n, & n = 2^k \text{ for } k > 1 \\ 2, & n = 2 \end{cases}$$

Basis

$$\begin{aligned} T(2) &= (2) \cdot lg(2) \\ &= 2 \cdot 1 \\ &= 2 \end{aligned}$$

Inductive Hypothesis
Assume that $T(n) = n \cdot lg(n)$ holds true for all $n = 2^k$.

Inductive Step

$$\begin{aligned} T(n) &= 2T(\frac{n}{2}) + n && \text{Base equation} \\ &= 2T(\frac{2^{k+1}}{2}) + 2^{k+1} && \text{Substitute n with } 2^{k+1} \\ &= 2T(2^k) + 2^{k+1} && \text{Simplify parameters to function T(...)} \\ &= 2(2^k \cdot lg(2^k)) + 2^{k+1} && \text{Inductive hypothesis} \\ &= 2^{k+1}\left[lg(2^k) + 1\right] && \text{Distributive property} \\ &= 2^{k+1}\left[lg(2^k) + lg(2)\right] && \text{Logarithmic identity} \\ &= 2^{k+1} \cdot lg(2^k \cdot 2) && \text{Logarithmic identity} \\ &= 2^{k+1} \cdot lg(2^{k+1}) && \text{Exponent property} \\ & && \text{Q.E.D} \end{aligned}$$

2.2.2 Example

Prove that the following systems of equations has the solution $T(n) = 2F(n) - 1$ where $F(n) = F(n-1) + F(n-2)$.

$$T(n) \begin{cases} T(n-1) + T(n-2) + 1, & \text{if } n \geq 2 \\ 0, & \text{if } n = \{0, 1\} \end{cases}$$

Basis

$$T(0) \quad = \quad 1$$

Inductive Hypothesis
Assume that $T(n) = F(n) - 1$ is true for all $n = k$.
Inductive Step

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) + 1 && \text{Base equation} \\ T(k+1) &= T((k+1)-1) + T((k+1)-2) + 1 && \text{Substitute n with k+1} \\ &= T(k) + T(k-1) + 1 && \text{Simplify parameters to function T(...)} \\ &= (2F(k) - 1) + (2F(k-1) - 1) + 1 && \text{Inductive hypothesis} \\ &= 2F(k) + 2F(k-1) - 1 && \text{Simplify equation} \\ &= 2(F(k) + F(k-1)) - 1 && \text{Distributive property} \\ &= 2(F(k+1)) - 1 && \text{Definition of function: } F(k+1) = F(k) + F(k-1) \\ &= 2F(k+1) - 1 && \text{Simplify} \\ & && \text{Q.E.D} \end{aligned}$$