Algorithm Design and Analysis (ECS 122A)
Study Guide

Davis Computer Science Club
Tutoring Committee

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Contents

Chapter 1

Asymptotic Notation

1.1 O-Notation (Big O)

Notation

$$f(n) \in O(g(n))$$

Formal Definition

For a given function $g(n)$, $O(g(n))$ is the set of functions for which there exists positive constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$O(g(n)) = \{f(n) : \exists\, c, n_0 \text{ s.t. } 0 \leq f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0\}$$

Informal Definition

The function $g(n)$ is an asymptotic upper bound for the function $f(n)$ if there exists constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for $n \geq n_0$.

Another way to perceive Big O notation is that for $f(n) \in O(g(n))$, the function f's asymptotic[1] growth is no faster than that of function g's.

Limit Definition

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$$

1.1.1 Example

Prove that asymptotic upper bound of $f(n) = 2n + 10$ is $g(n) = n^2$.

$$\begin{aligned} 0 \leq f(n) &\leq c \cdot g(n) \text{ for } n \geq n_0 \\ 0 \leq 2n + 10 &\leq c \cdot n^2 \text{ for } n \geq n_0 \end{aligned}$$

Arbitrarily choose c and n_0 values. Simplest is to turn one of the variables into the value 1 and solve. For this example, we will assign the value 1 to n_0.

$$\begin{aligned} 0 \leq 2n + 10 &\leq c \cdot n^2 \text{ for } n \geq 1 \\ 2(1) + 10 &\leq c \cdot (1)^2 \\ 12 &\leq c \end{aligned}$$

By picking $n_0 = 1$ and $c = 12$, the inequality of $2n + 10 \leq 12n^2$ will hold true for all $n \geq 1$. Since there exists a constant c and n_0 that fulfill this inequality, we have proven that $f(n) = 2n + 10 = O(n^2)$.

[1] Asymptotic: As given variable approaches infinity.

1.2 o-Notation (Little O)

Notation

$$f(n) \in o(g(n))$$

Formal Definition

For a given function $g(n)$, $o(g(n))$ is the set of functions for which every positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$o(g(n)) = \{f(n) : \exists\ n_0 \text{ s.t. } 0 \leq f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0, c \geq 0\}$$

Informal Definition

The function g(n) is an upper bound that is not asymptotically tight. For all positive constant values of c, there must exists a constant n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. The value of n_0 may not depend on n, but may depend on c.

Another way to perceive Little O notation is that for $f(n) \in o(g(n))$, the function f's asymptotic growth is strictly less than that of the function g's. In this sense, Little O can be seen as a "stronger" bound in comparison to Big O. By proving that a function is an element of Little O, it also proves that the function is an element of Big O.

Limit Definition

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

1.2.1 Example

Prove that $f(n) = 2n$ has an upper bound $o(n^2)$.

$$\begin{aligned} 0 \leq c \cdot g(n) &\leq f(n) \text{ for } n \geq n_0 \\ 0 \leq c \cdot 2n &\leq n^2 \text{ for } n \geq n_0 \\ 2c &\leq n \text{ for } n \geq n_0 \\ 2c &\leq n_0 \end{aligned}$$

For Little O to hold true, the inequality needs to hold true for all $c > 0$ and for all $n > n_0$. From simplifying the inequality, we assert that the inequality will hold true as long as the value of n_0 is twice the value of c. Given that they are both constants, then there exists a constant value of n_0 for all positive constant c that fulfill this inequality.

Another method to solve this problem is to use the limit definition.

$$\begin{aligned} &\lim_{n\to\infty} \frac{2n}{n^2} \\ &\lim_{n\to\infty} \frac{2}{n} = 0 \end{aligned}$$

1.2.2 Example

Prove that $f(n) = 2n^2$ does not have the upper bound $o(n^2)$.

$$\begin{aligned}
0 \leq c \cdot g(n) &\leq f(n) \text{ for } n \geq n_0 \\
0 \leq c \cdot 2n^2 &\leq n^2 \text{ for } n \geq n_0 \\
2c &\leq 1 \text{ for } n \geq n_0
\end{aligned}$$

For a function to have the Little O bound, the inequality must hold true for all positive c. However, simplification of the inequality asserts that the inequality will only hold true for all $c < \frac{1}{2}$. Therefore, $f(n) = 2n^2$ does not have the upper bound $o(n^2)$.

1.3 Ω-Notation (Big Omega)

Notation

$$f(n) \in \Omega(g(n))$$

Formal Definition

For a given function $g(n)$, $\Omega(g(n))$ is the set of functions for which there exists positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

$$\Omega(g(n)) = \{f(n) : \exists\ c, n_0 \text{ s.t. } 0 \leq c \cdot g(n) \leq f(n)\ \forall\ n \geq n_0\}$$

Informal Definition

The function $g(n)$ is an asymptotic lower bound for the function $f(n)$ if there exists constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for $n \geq n_0$.

Limit Definition

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

1.4 ω-Notation (Little Omega)

Notation

$$f(n) \in \omega(g(n))$$

Formal Definition

For a given function $g(n)$, $\omega(g(n))$ is the set of functions for which every positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

$$\omega(g(n)) = \{f(n) : \exists\ n_0 \text{ s.t. } 0 \leq c \cdot g(n) \leq f(n)\ \forall\ n \geq n_0, c \geq 0\}$$

Informal Definition

The function g(n) is a lower bound that is not asymptotically tight. For all positive constant values of c, there must exists a constant n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$. The value of n_0 may not depend on n, but may depend on c.

Another way to perceive Little ω notation is that for $f(n) \in \omega(g(n))$, the function f's asymptotic growth is strictly greater than that of the function g's. In this sense, Little ω can be seen as a "stronger" bound in comparison to Big Ω. By proving that a function is an element of Little ω, it also proves that the function is an element of Big Ω.

Limit Definition

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

1.5 Θ-notation (Big Theta)

Notation

$$f(n) \in \Theta(g(n))$$

Formal Definition

For a given function $g(n)$, $\Theta(g(n))$ is the set of functions for which there exists positive constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

$$\Theta(g(n)) = \{f(n) : \exists\ c_1, c_2, n_0 \text{ s.t. } 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\ \forall\ n \geq n_0\}$$

Informal Definition

The function g(n) is an asymptotic tight bound for the function f(n) if there exists constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for $n \geq n_0$.

Big theta implies that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Limit Definition

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} \in \mathbb{R}_{>0}$$

Chapter 2

Recurrence Relations

2.1 Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence of values. After the initial terms are given, each subsequent term is defined as a function of the previous terms.

Fibonacci

Fibonacci is an example of a recurrence relation.

$$F_n = \begin{cases} F_{n-1} + F_{n-2}, & n \geq 2 \\ 1, & n = 1 \\ 0, & n = 0 \end{cases}$$

The first two terms are defined while the subsequent terms are a function of the two previous.

2.2 Solving Recurrence Relations

- Substitution Method
- Recursion-Tree Method
- Master Theorem

2.3 Substitution Method

1. Guess the bounds.
2. Apply mathematical induction to prove the bounds.

2.3.1 Example

Find the asymptotic upper bound for the following function:

$$T(n) \begin{cases} 2T(n-1)+1, & n \geq 1 \\ 1, & n = 0 \end{cases}$$

Guess

$$T(n) \in O(2^n)$$

Inductive Basis

$$\begin{aligned} T(0) &= 2^0 \\ &= 1 \end{aligned}$$

Inductive Hypothesis
Assume that $T(n) = 2^n$ holds true for all $n = k$.

Inductive Step

$$\begin{aligned} T(n) &= 2T(n-1)+1 && \text{Base equation} \\ &= 2T((k+1)-1)+1 && \text{Substitute n with } k+1 \\ &= 2T(k)+1 && \text{Simplify parameters to T(n)} \\ &= 2(2^k)+1 && \text{Substitute T(n) with inductive hypothesis} \\ &= 2^{k+1}+1 && \text{Property of exponents} \\ & && \text{Q.E.D} \end{aligned}$$

2.4 Master Theorem

Used for divide and conquer recurrences that follow the generic form:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n) \text{ where } a \geq 1, b > 1$$

2.4.1 Case 1

Condition

$$f(n) \in O(n^c)$$

$$c < log_b(a)$$

Solution

$$T(n) \in \Theta(n^{log_b(a)})$$

2.4.2 Case 2

Condition

$$f(n) \in \Theta(n^c)$$

$$c = log_b(a)$$

Solution

$$T(n) \in \Theta(n^{log_b(a)} \cdot log_2(n))$$

2.4.3 Case 3

Condition

$$f(n) \in \Omega(n^c)$$

$$c > log_b(a)$$

Regularity Condition

This case must also fulfill the regularity condition.

$$a \cdot f(\frac{n}{b}) \leq k \cdot f(n) \text{ where } k < 1$$

Solution

$$T(n) \in \Theta(f(n))$$

Remark

The idea behind this case is that given the generic form, the function $f(n)$ will grow far quicker than $a \cdot T(\frac{n}{b})$ and will be the primary influence of $T(n)$'s asymptotic behavior.

2.4.4 Example

$$T(n) = 64T(\frac{n}{4}) + 1000n^2$$

Given

$$f(n) = 1000n^2 \in \Theta(n^2)$$

$$a = 64$$

$$b = 4$$

$$c = 2$$

Condition

$$\begin{array}{ccc} c & ? & log_b(a) \\ 2 & ? & log_4(64) \\ 2 & < & 3 \end{array}$$

Condition satisfied for case 1

Solution

$$\therefore T(n) \in \Theta(n^{log_4(64)}) = \Theta(n^3)$$

2.4.5 Example

$$T(n) = 32T(\frac{n}{2}) + 20n^5$$

Given

$$f(n) = 20n^5 \in \Theta(n^5)$$

$$a = 32$$

$$b = 2$$

$$c = 5$$

Condition

$$\begin{array}{ccc} c & ? & log_b(a) \\ 5 & ? & log_2(32) \\ 5 & = & 5 \end{array}$$

Condition satisfied for case 2

Solution

$$\therefore T(n) \in \Theta(n^{log_2(32)} \cdot log_2(n)) = \Theta(n^5 \cdot lg(n))$$

2.4.6 Example

$$T(n) = 7T(\frac{n}{7}) + 19n^{11}$$

Given

$$f(n) = 19n^{11} \in \Theta(n^{11})$$

$$a = 7$$

$$b = 7$$

$$c = 11$$

Condition

$$\begin{array}{rcl} c & ? & log_b(a) \\ 11 & ? & log_7(7) \\ 5 & > & 1 \end{array}$$

Condition partially fulfilled for case 3. Must also check regularity condition.

$$\begin{aligned} a \cdot f(\frac{n}{b}) &\leq k \cdot f(n) \\ 7 \cdot \left[19(\frac{n}{7})^{11}\right] &\leq k \cdot 19n^{11} \\ 7 \cdot \frac{n^{11}}{7^{11}} &\leq k \cdot n^{11} \\ \frac{1}{7^{10}} \cdot n^{11} &\leq k \cdot n^{11} \end{aligned}$$

Choosing $k = \frac{1}{7^{10}} < 1$ fulfills the regularity condition.

Solution

$$\therefore T(n) \in \Theta(19n^{11})$$

Chapter 3

Divide and Conquer Paradigm

3.1 Steps

1. **Divide** the problem into a number of independent subproblems.
2. **Conquer** the subproblems by solving them recursively.
3. **Combine** the solutions of the subproblems into the solution of the original problem.

3.2 Case Study: Merge Sort

Steps

1. **Divide** the list of n elements into two sublists with $\frac{n}{2}$ elements each.
2. **Conquer** the sublists by sorting the two sublists recursively using merge sort. When the sublists are of size 1, it becomes sorted.
3. **Combine** the elements of the two sublists by mering them in a sorted sequence.

Recurrence Relation

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + cn, & n \geq 2 \\ c, & n = 1 \end{cases}$$

Complexity

$$T(n) = \Theta(n \cdot lg(n))$$

3.3 Case Study: Fibonacci Sequence

Theorem

Fibonacci Sequence Starting with 0

Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$$

Fibonacci Sequence Starting with 1

Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Derivation

$$\begin{aligned}
\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-3} \\ F_{n-4} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} F_{n-4} \\ F_{n-5} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}
\end{aligned}$$

To verify, let's choose $n = 5$

$$\begin{aligned}
\begin{bmatrix} F_5 \\ F_4 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\end{aligned}$$

The fifth Fibonacci number (assuming that the sequence starts at 0) is 3.

Recurrence Relation

$$T(n) = T(\frac{n}{2}) + O(1)$$

Complexity

$$T(N) \in \Theta(lg(n))$$

3.4 Case Study: Maximum Subarray

Steps

1. Divide the array in half into two subarrays (left subarray and right subarray).

2. Recursively repeat this process until each subarray consists of only one element. At this point, the maximum sum of each subarray is the single element.

3. Calculate the maximum sum for the cross section.

   (a) Start from the mid-point of the subarray.

   (b) Sum up all numbers from the mid-point to the first element. Whenever the sum exceeds its previous value, that value becomes the left sum.

   (c) Sum up all numbers from the mid-point+1 to the last element. Whenever the sum exceeds its previous value, that values becomes the right sum.

   (d) The summation of the left sum and the right sum becomes the maximum sum for the cross section. Note: If all the elements in the subarrays are negative, then the left and right sum will return 0 by default.

4. Compare the maximum sum from the left array, right array, and cross section. The largest of the three get returned.

Recurrence Relation

$$T(n) = 2T(\frac{n}{2}) + \Theta(n)$$

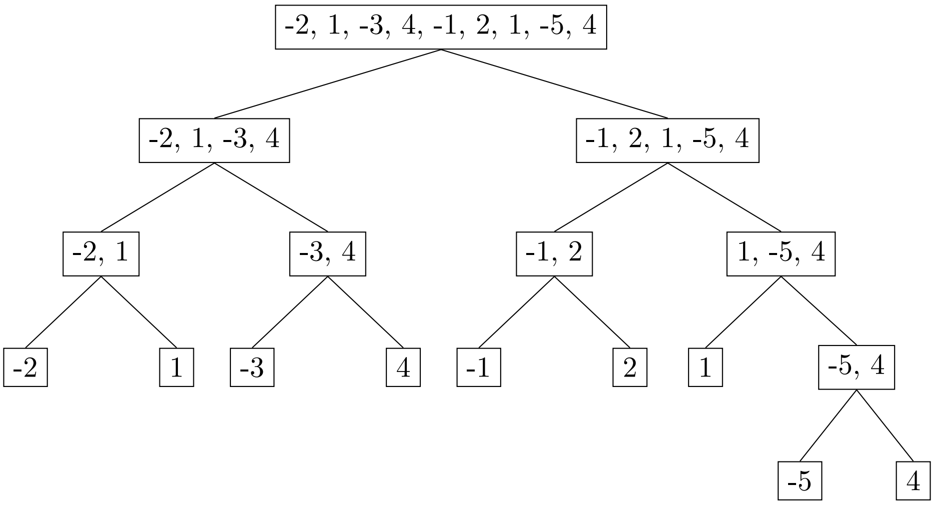
Complexity

$$T(n) \in \Theta(n \cdot lg(n))$$

3.4.1 Example

Find the maximum subarray of the following array: $\{-2, 1, -3, 4, -1, 2, 1, -5, 4\}$

Divide



Combine

Depth	Left Subarray	Right Subarray	Max(Left)	Max(Right)	Max(Cross)	Return
4	$\{-5\}$	$\{4\}$	-5	4	4	4
3	$\{-2\}$	$\{1\}$	-2	1	1	1
	$\{-3\}$	$\{4\}$	-3	4	4	4
	$\{-1\}$	$\{2\}$	-1	2	2	2
	$\{1\}$	$\{-5, 4\}$	1	4	4	4
2	$\{-2, 1\}$	$\{-3, 4\}$	1	4	1	4
	$\{-1, 2\}$	$\{1, -5, 4\}$	2	4	3	4
1	$\{-2, 1, -3, 4\}$	$\{-1, 2, 1, -5, 4\}$	4	4	6	6

The maximum sum is 6 from indices 3 to 6.

Visual Method of Finding the Max of Cross Section

Taking depth = 1 with left subarray = $\{-2, 1, -3, 4\}$ and right subarray = $\{-1, 2, 1, -5, 4\}$.

Cross Section Left Sum

$$\{-2, 1, -3, 4, \underbrace{-1}_{\text{Mid}}, 2, 1, -5, 4\}$$

$$\{-2, 1, -3, 4, \underbrace{-1}_{\text{-1}}, 2, 1, -5, 4\}$$

$$\{-2, 1, -3, \underbrace{4, -1}_{\text{3}}, 2, 1, -5, 4\}$$

$$\{-2, 1, \underbrace{-3, 4, -1}_{\text{0}}, 2, 1, -5, 4\}$$

$$\{-2, \underbrace{1, -3, 4, -1}_{\text{1}}, 2, 1, -5, 4\}$$

$$\{\underbrace{-2, 1, -3, 4, -1}_{\text{-1}}, 2, 1, -5, 4\}$$

Max Left Sum = 3

Cross Section Right Sum

$$\{-2, 1, -3, 4, -1, \underbrace{2}_{\text{Mid + 1}}, 1, -5, 5\}$$

$$\{-2, 1, -3, 4, -1, \underbrace{2}_{\text{2}}, 1, -5, 5\}$$

$$\{-2, 1, -3, 4, -1, \underbrace{2, 1}_{\text{3}}, -5, 5\}$$

$$\{-2, 1, -3, 4, -1, \underbrace{2, 1, -5}_{\text{-2}}, 4\}$$

$$\{-2, 1, -3, 4, -1, \underbrace{2, 1, -5, 4}_{\text{2}}\}$$

Max Right Sum = 3

Max Sum = $3 + 3 = 6$

Chapter 4

Greedy Algorithm

4.1 Properties

Greedy Choice

A globally optimal solution can be arrived at by making a locally optimal (greedy) choice.

Optimal Substructure Property

An optimal solution to the problem contains within it optimal solution to the subproblems.

4.2 Case Study: Activity-Selection

Formal Problem Statement

Assume there exists n activities, each with a start time s_i and finish time f_i. Two activities i and j are said to be non-conflicting if $s_i \geq f_j$ or $s_j \geq f_i$. The objective is to find the maximum solution set of non-conflicting activities.

Informal Problem Statement

Given n activities and their respective start (s_i) and finish (f_i) times, find the maximum number of activities that can be performed.

Greedy Choice

Choose the next activity with a start time greater than or equal to the previous activity's finish time and has the next smallest finish time.

Steps

1. Sort the activities according to their finish times.
2. Select the first activity from the sorted list.
3. Repeat this process for the remaining activities with the condition that the start time of subsequent activities are greater than or equal to the preceding activity's finish time.

Pseudocode

```
procedure ActivitySelection(A)
    Sort(A)                                          ▷ Sort by finish times

    Let F be the set of finish times corresponding to the sorted list A
    Let B be the set of start times correspdonding to the sorted list A

    S = { A[1] }
    f = F_0

    for i=2 to n do
        if F_i ≥ f then
            S ∪ { A[i] }
            f = F_i
        end if
    end for
end procedure
```

Complexity

$$O(n \cdot lg(n))^{[1]}$$

[1] Total Time = O($n \cdot lg(n)$) + $\Theta(n)$. Sort Time + Greedy Activity Selection. Sort time will dominate.

4.3 Case Study: Huffman Coding

Formal Problem Statement

Let A be defined as the set of alphabets. ($A = \{a_0, a_1, a_2, ..., a_n\}$)
Let W be defined as the set of weights for which $w_i = \text{Weight}(a_i)$. ($W = \{w_0, w_1, w_2, ..., w_n\}$)
Let C be defined as the set of (binary) codewords for which $c_i = \text{CodeWord}(a_i)$.

Assume there exists n alphabets, each with a weight w_i. Find and define the codewords c_i for each respective alphabet a_i such that $\sum\limits_{i=0}^{n} w_i \cdot length(c_i)$ is the smallest possible.

Informal Problem Statement

Given a set of symbols and their weights (probabilities), find a prefix-free binary code with minimum expected codeword length.

Greedy Choice

Choose the two alphabets with the lowest weight.

Steps

1. Pick two letters x and y from the alphabet A with the lowest frequencies or weight w_i.
2. Create a subtree with x and y as leaves. We will define the root as z.
3. The frequency or weight of node z will be define as $w_z = w_x + w_y$.
4. Remove x and y from alphabet.
   $A' = A - \{x, y\}$
5. Insert z into the alphabet.
   $A' = A + \{z\}$
6. Repeat this process until the set of alphabets A consists of only one alphabet.

Complexity

$$O(n \cdot lg(n))$$

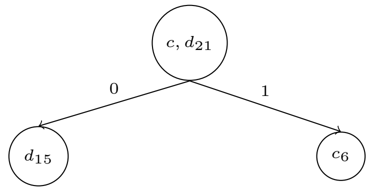
Total = $O(n \cdot lg(n)) + \Theta(n)$. Cost to sort alphabet by weight and cost to iterate through all alphabets.

4.3.1 Example

Let $A = \{a, b, c, d, e\}$ and $W = \{30, 16, 6, 15, 35\}$. Find their corresponding Huffman codes.

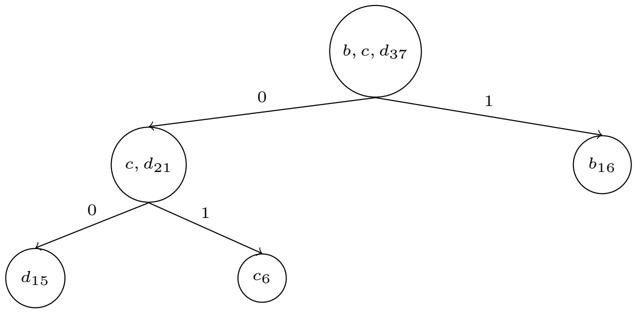
Merge c **and** d

Alphabet	Weight
e	35
a	30
b	16
d	15
c	6



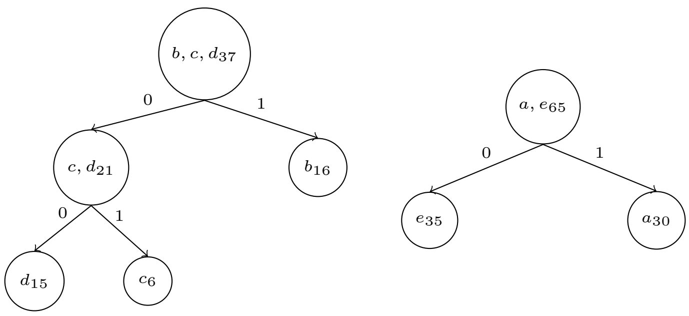
Merge c, d **and** b

Alphabet	Weight
e	35
a	30
c,d	21
b	16



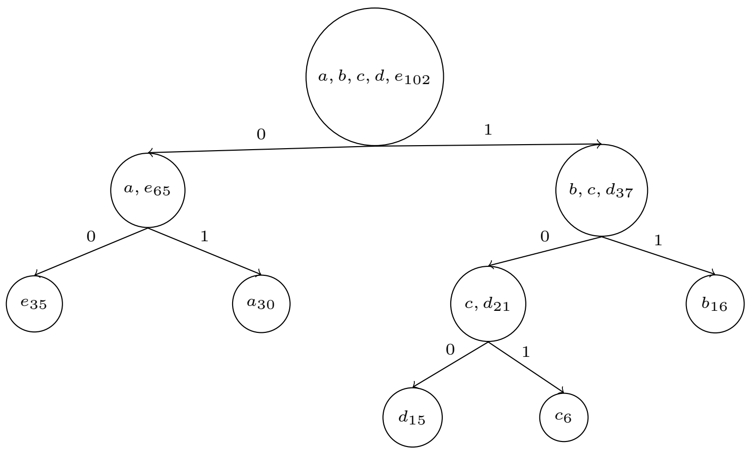
Merge e **and** a

Alphabet	Weight
b,c,d	37
e	35
a	30



Merge a, b, c, d **and** e

Alphabet	Weight
a,e	65
b,c,d	37



Solution

Alphabet	Weight	Codeword
e	35	00
a	30	01
b	16	11
d	15	100
e	6	101

Chapter 5

Dynamic Programming

5.1 Sequence

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up fashion
4. Construct an optimal solution from computed information

5.2 Case Study: Rod Cutting

Problem Statement

Given a rod of length n and a set of prices $P = \{p_1, p_2, ...p_n\}$ such that p_i denotes the price of a piece of rod with length i, find the optimal (maximum) revenue r_i for cutting the rod into pieces whose length sum to n.

Steps

1. Start from rod length = 1.
2. With each subrod length, there will always be one "cut" – splitting the rod into a left half and right half. If the cut is equivalent to the length of the subrod, then it means that the entire length of the subrod was used (Left half will have the full length and the right half will have zero length).
3. Iterate from rod length = 1 to rod length = n.
   On each iteration of length i:
   (a) Assume that the left half has the full length and the right half has length 0.
   (b) Decrement the left half's left by 1 and increase the right half's length by 1.
   (c) Sum the revenue of the left half with the price of the right half.
   (d) Repeat this process until the left half is of length 0.
   (e) The maximum of all these sums become the maximum revenue of length i.

Complexity

$$O(n^2)$$

5.2.1 Example

Given a rod of length = 8 and P defined as $\{1, 5, 8, 9, 10, 17, 17, 20\}$, find the maximum revenue.

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20

Subrod Length = 1

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
0	1	$0 + 1 = 1$

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1							

Subrod Length = 2

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
1	1	$1 + 1 = 2$
0	2	$0 + 5 = 5$

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5						

Subrod Length = 3

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
2	1	$5 + 1 = 6$
1	2	$1 + 5 = 6$
0	3	$0 + 8 = 8$

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8					

Subrod Length = 4

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
3	1	8 + 1 = 9
2	2	5 + 5 = 10
1	3	1 + 8 = 9
0	4	0 + 9 = 9

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8	10				

Subrod Length = 5

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
4	1	10 + 1 = 11
3	2	8 + 5 = 13
2	3	5 + 8 = 13
1	4	1 + 9 = 10
0	5	0 + 10 = 10

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8	10	13			

Subrod Length = 6

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
5	1	13 + 1 = 14
4	2	10 + 5 = 15
3	3	8 + 8 = 16
2	4	5 + 9 = 14
1	5	1 + 10 = 11
0	6	0 + 17 = 17

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8	10	13	17		

Subrod Length = 7

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
6	1	17 + 1 = 18
5	2	13 + 5 = 18
4	3	10 + 8 = 18
3	4	8 + 9 = 17
2	5	5 + 10 = 15
1	6	1 + 17 = 18
0	7	0 + 17 = 17

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8	10	13	17	18	

Subrod Length = 8

Length(Left)	Length(Right)	Revenue(Left) + Price(Right)
7	1	18 + 1 = 19
6	2	17 + 5 = 22
5	3	13 + 8 = 21
4	4	10 + 9 = 19
3	5	8 + 10 = 18
2	6	5 + 17 = 22
1	7	1 + 17 = 18
0	8	0 + 20 = 20

Length	1	2	3	4	5	6	7	8
Price	1	5	8	9	10	17	17	20
Revenue	1	5	8	10	13	17	18	22

The maximum revenue for a rod of length 8 is 22.

5.3 Case Study: Matrix Chain Multiplication

Problem Statement

Given a sequence of matrices $A_1, A_2, ...A_n$ with order $p_{i-1} \times p_i$, find the ordering for the product of $A_1 \times A_2 \times A_3... \times A_n$ such that it minimizes the number of scalar multiplications.

Important

The matrix is 1-indexed while the sequence for the order of matrices is 0-indexed.

Steps

1. Given the orders of the matrices, create a matrix m and a matrix s of order $n \times n$.
2. Zero out the main diagonal ($m[i, i] = 0$ and $s[i, i] = 0$).
3. Each iteration creates a new diagonal that builds the upper right triangle of the m and s matrix.
4. Start from the $i = 1$ and build down the diagonal.
5. For each diagonal:
   (a) For each cell on the diagonal, find the minimum such that: $m[i, j] = \min\limits_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$.
   (b) $s[i, j]$ will be the k that attains the minimum value of $m[i, j]$.
6. A more visual method and informal method for each diagonal (Same as Step 4):
   (a) For a given cell we are trying to calculate, we will denote it as $m[i, j]$.
   (b) Imagine that there exists a sliding window of size $i + j$ number of cells that curves down at $m[i, j]$.
   (c) The end points of the sliding window are the left cell ($m[i, k]$) and the bottom cell ($m[k + 1, j]$).
   (d) Set the left cell as a cell on the main diagonal and on the same row as the cell we are trying to calculate.
   (e) Based on the constraint, this will also set the bottom cell is immediately below the current cell we are trying to calculate.
   (f) Two of the orders are constant – p_{i-1} and p_j. The only order that changes is p_k and k can be easily determined by the column index of the left cell. From that you can calculate the product of orders $p_{i-1}p_kp_j$.
   (g) Sum the value of the left cell, right cell, and the product of orders.
   (h) Slide the window by increasing the column index in the left cell. Based on the sliding window constraint, the row index of the bottom cell must increase. Repeat steps 6a–6h until the entire sliding window is on row j (This also means that the left cell is $m[i, j]$).
   (i) The minimum of all these calculated values will be the value of $m[i, j]$. The left cell that achieved the minimum value will have its column index be the value of $s[i, j]$.

Parenthesizing Based on s Matrix

1. Start from the upper-right corner ($i = 1, j = n$).
2. Split into a binary tree and wrap the root with a pair of parentheses.
   - The left will repeat this process, but from $i = i$ and $j = s[i, j]$.
   - The right will repeat this process, but from $i = s[i, j] + 1$ and $j = j$.
3. Whenever $i = j$, then the print A_i.

Complexity

$$\text{Time: } O(n^3)$$

$$\text{Space: } O(n^2)$$

5.3.1 Example

Let A be defined as the sequence $\{A_1, A_2, A_3, A_4\}$ and their orders P = { 10, 100, 5, 50, 1 }. Find the optimal parenthesization.

Initialization

$$m\text{-table} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Cell $i = 1, j = 2$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[1, 1] = 0$	$m[2, 2] = 0$	$p_0 \cdot p_1 \cdot p_2 = 10 \cdot 100 \cdot 5 = 5000$	5000

$$m\text{-table} = \begin{bmatrix} 0 & 5000 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Cell $i = 2, j = 3$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[2, 2] = 0$	$m[3, 3] = 0$	$p_1 \cdot p_2 \cdot p_3 = 100 \cdot 5 \cdot 50 = 25000$	25000

$$m\text{-table} = \begin{bmatrix} 0 & 5000 & & \\ & 0 & 25000 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Cell $i = 3, j = 4$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[3,3] = 0$	$m[4,4] = 0$	$p_2 \cdot p_3 \cdot p_4 = 5 \cdot 50 \cdot 1 = 250$	250

$$m\text{-table} = \begin{bmatrix} 0 & 5000 & & \\ & 0 & 25000 & \\ & & 0 & 1000 \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & 0 \end{bmatrix}$$

Cell $i = 1, j = 3$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[1,1] = 0$	$m[2,3] = 2500$	$p_0 \cdot p_1 \cdot p_3 = 10 \cdot 100 \cdot 50 = 50000$	52500
$m[1,2] = 5000$	$m[3,3] = 0$	$p_0 \cdot p_2 \cdot p_3 = 10 \cdot 5 \cdot 50 = 2500$	7500

$$m\text{-table} = \begin{bmatrix} 0 & 5000 & 7500 & \\ & 0 & 25000 & \\ & & 0 & 1000 \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & 2 & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & 0 \end{bmatrix}$$

Cell $i = 2, j = 4$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[2,2] = 0$	$m[3,4] = 1000$	$p_1 \cdot p_2 \cdot p_4 = 100 \cdot 5 \cdot 1 = 500$	1500
$m[2,3] = 25000$	$m[4,4] = 0$	$p_1 \cdot p_3 \cdot p_4 = 100 \cdot 50 \cdot 1 = 5000$	30000

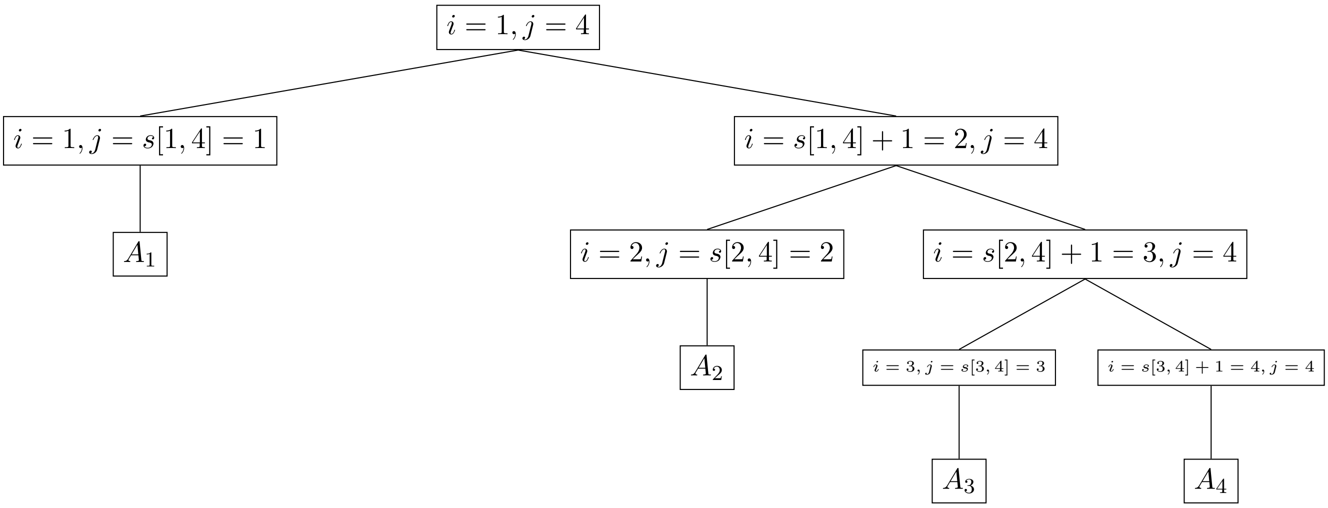
$$m\text{-table} = \begin{bmatrix} 0 & 5000 & 7500 & \\ & 0 & 25000 & 1500 \\ & & 0 & 1000 \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & 2 & \\ & 0 & 2 & 2 \\ & & 0 & 3 \\ & & & 0 \end{bmatrix}$$

Cell $i = 1, j = 4$

Left Cell	Bottom Cell	Product of Orders	Sum
$m[1,1] = 0$	$m[2,4] = 1500$	$p_0 \cdot p_1 \cdot p_4 = 10 \cdot 100 \cdot 1 = 1000$	2500
$m[1,2] = 25000$	$m[3,4] = 0$	$p_0 \cdot p_2 \cdot p_4 = 10 \cdot 5 \cdot 1 = 50$	25050
$m[1,3] = 25000$	$m[4,4] = 0$	$p_0 \cdot p_3 \cdot p_4 = 10 \cdot 50 \cdot 1 = 500$	25500

$$m\text{-table} = \begin{bmatrix} 0 & 5000 & 7500 & 2500 \\ & 0 & 25000 & 1500 \\ & & 0 & 1000 \\ & & & 0 \end{bmatrix} \qquad s\text{-table} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ & 0 & 2 & 2 \\ & & 0 & 3 \\ & & & 0 \end{bmatrix}$$

Building the Parenthesization



Solution

$$(A_1(A_2(A_3 \cdot A_4)))$$

5.4 Case Study: Longest Common Subsequence

Problem Statement

Given that a sequence is defined as $s = \{a_1, a_2, a_3, ...a_n\}$ and given a set of sequences $S = \{s_1, s_2, ...s_n\}$, find the longest subsequence common to all sequences in the set.

Note that subsequence is different from substring in that subsequences do not need to occupy consecutive positions.

Steps

*** These steps assume that the of sequences consists of only two sequences – s_1 and s_2. Nonetheless this algorithm can be made to be applied over an entire set.

1. Define m as the length of the subsequence s_1.

2. Define n as the length of the subsequence s_2.

3. Create a c table and a b table each of whose dimensions are $(m+1) \times (n+1)$. The c table will represent the continuous sum for a given subsequence. The b table will represent the direction to build the subsequence.

4. For both tables, set all cells with column index 0 to 0. For future references, this column will remain 0 and unalterable.

5. For both tables, set all cells with row index 0 to 0. For future references, this row will remain 0 and unalterable.

6. For row indices 1 to m, each will represent an element in the subsequence s_1. Example: Row index 1 represents the element a_1 from the sequence s_1.

7. For column indices 1 to n, each will represent an element in the subsequence s_2. Example: Column index n represents the element a_n from the sequence s_2.

8. Iterate through all cells starting from $i = 1, j = 1$. Whenever $j = m$, increase the value of i and set $j = 1$. This is to avoid entering the 0-column.

9. For a given cell $c[i, j]$:

   - If $a_i \in s_i$ is the same letter as $a_j \in s_j$, then set $c[i, j] = c[i-1, j-1] + 1$ and set $b[i, j] = \nwarrow$.
   - If $a_i \in s_i$ does not match $a_j \in s_j$, then compare the cell to the immediate left $(c[i, j-1])$ and the cell immediately above $(c[i-1, j])$. Set $c[i, j]$ equivalent to the largest value and set $b[i, j]$ to the direction of the largest value. If both values are equivalent, then default to the cell immediately above.

10. Once the c and b table are completed, start from the cell $b[m, n]$.

11. Follow the direction of the arrows and until $i = 0$ or $j = 0$.

12. Whenever the direction of the arrow is $\nwarrow$, then that alphabet is part of the longest common subsequence.

Complexity

$$\text{Time: } \Theta(mn)$$

$$\text{Space: } \Theta(mn)$$

5.4.1 Example

Given that $s_1 = \{B, D, C, A, B, A\}$ and $s_2 = \{A, B, C, B, D, A, B\}$, find the longest common subsequence.

The c and b table will be combined into a single table for this example. The value in the corner will denote $b[i, j]$.

Initialization

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0							
D	0							
C	0							
A	0							
B	0							
A	0							

Row 1

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0							
C	0							
A	0							
B	0							
A	0							

Row 2

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0	$\uparrow_0$	$\uparrow_1$	$\uparrow_1$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\leftarrow_2$
C	0							
A	0							
B	0							
A	0							

Row 3

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0	$\uparrow_0$	$\uparrow_1$	$\uparrow_1$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\leftarrow_2$
C	0	$\uparrow_0$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$
A	0							
B	0							
A	0							

Row 4

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0	$\uparrow_0$	$\uparrow_1$	$\uparrow_1$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\leftarrow_2$
C	0	$\uparrow_0$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$
A	0	$\nwarrow_1$	$\uparrow_1$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$	$\nwarrow_3$	$\leftarrow_3$
B	0							
A	0							

Row 5

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0	$\uparrow_0$	$\uparrow_1$	$\uparrow_1$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\leftarrow_2$
C	0	$\uparrow_0$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$
A	0	$\nwarrow_1$	$\uparrow_1$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$	$\nwarrow_3$	$\leftarrow_3$
B	0	$\uparrow_1$	$\nwarrow_2$	$\uparrow_2$	$\nwarrow_3$	$\leftarrow_3$	$\uparrow_3$	$\nwarrow_4$
A	0							

Row 6

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	$\uparrow_0$	$\nwarrow_1$	$\leftarrow_1$	$\nwarrow_1$	$\leftarrow_1$	$\leftarrow_1$	$\nwarrow_1$
D	0	$\uparrow_0$	$\uparrow_1$	$\uparrow_1$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\leftarrow_2$
C	0	$\uparrow_0$	$\uparrow_1$	$\nwarrow_2$	$\leftarrow_2$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$
A	0	$\nwarrow_1$	$\uparrow_1$	$\uparrow_2$	$\uparrow_2$	$\uparrow_2$	$\nwarrow_3$	$\leftarrow_3$
B	0	$\uparrow_1$	$\nwarrow_2$	$\uparrow_2$	$\nwarrow_3$	$\leftarrow_3$	$\uparrow_3$	$\nwarrow_4$
A	0	$\nwarrow_1$	$\uparrow_2$	$\uparrow_2$	$\uparrow_3$	$\uparrow_3$	$\nwarrow_4$	$\uparrow_4$

Longest Common Subsequence: $BDAB$

5.5 Case Study: 0-1 Knapsack

Problem Statement

Given a knapsack with maximum capacity W and a set of items $I = \{i_1, i_2, ...i_n\}$, in which each item carries a weight wt_i and value v_i, find the maximum value possible.

General Algorithm Idea

The algorithm slowly builds by first setting the number of items in the system to 0 and the weight of the knapsack to 0. It will then increase the number of items by one and then determine the maximum value achievable with weights $w = 1$ to $w = W$ in the knapsack.

Steps

1. Create a table K that is $(n+1) \times (W+1)$. Each row of table represents how many items are in the system so computations for $n = 1$ represents only item i_1 in the system. Each column in the table represents how much weight the knapsack is capable of holding.

2. Set all cells with column index 0 ($j = 0$) to the value 0.

3. Set all cells with the row index 0 ($i = 0$) to the value 0.

4. We assume that for a no items ($n = 0$) or no weight to our knapsack ($w = 0$), the optimal and maximum value is 0.

5. Iterate through the cells starting with $n = 1$ and $w = 1$. Fill in all weights for a given row before continuing.

6. For each cell $K[i, j]$:

   (a) Retrieve the weight of the item wt_i.

   (b) If $wt_i > j$ then this item cannot be used. The column index represents the current maximum weight of the knapsack. If the weight of this item is greater than that, then it implies that the item cannot fit in the knapsack. In this case, it means that the maximum value is for that weight j, but with $i - 1$ items.
   Therefore, $K[i, j] = K[i-1, j]$

   (c) If the weight of the item alone is less than the current knapsack weight, then this item is a potential candidate. In this condition, there are two potential results:

   - Given that the knapsack weight does not change, the value of the potential item and a subset of items combined is greater than the maximum value for $i - 1$ items.
     Therefore, $K[i, j] = max(v_i + K[i-1][w - w_i], K[i-1][w])$
   - Given that the knapsack weight does not change, the value of the potential item and a subset of items does not exceed the maximum value for $i - 1$ items.
     Therefore, $K[i, j] = K[i-1, j]$.

   (d) Another way to look at step c:

   - Given the column index j, it also represents the current maximum weight of the knapsack. Then $K[i][j - wt_i]$ represents a cell in which it is the maximum value for i items and when the maximum knapsack weight is $j - wt_i$.

- So $K[i-1][j-wt_i]$ represents the maximum value for when there is one item less and when there is enough room for the new item. The sum of this and the value of the potential candidate item becomes the maximum value for when you add the new item.

Complexity

$$O(nW)$$

5.5.1 Example

Given a knapsack with $W = 2$, $v = \{10, 20, 30\}$, and $wt = \{1, 1, 1\}$, find the maximum value possible.

Initialization

	w = 0	w = 1	w = 2
n = 0	0	0	0
n = 1	0		
n = 2	0		
n = 3	0		

Row 1

	w = 0	w = 1	w = 2
n = 0	0	0	0
n = 1	0	10	10
n = 2	0		
n = 3	0		

Row 2

	w = 0	w = 1	w = 2
n = 0	0	0	0
n = 1	0	10	10
n = 2	0	20	30
n = 3	0		

Row 3

	w = 0	w = 1	w = 2
n = 0	0	0	0
n = 1	0	10	10
n = 2	0	20	30
n = 3	0	30	50

Chapter 6

Graph Theory

6.1 Definitions

- Graph G is an ordered pair such that $G = (V, E)$.
- Edge E is a set of vertex pairs such that $e_i = (u, v)$ where $u, v \in V$.

6.2 Minimum Spanning Tree

6.2.1 Definition

Tree: Graph in which any two vertices are connected by exactly one path.
Spanning Tree: Subgraph of a graph that contains all vertices and is a tree.
Minimum Spanning Tree (MST): A spanning tree with weight less than or equal to the weight of every other spanning tree.

6.2.2 Algorithms

- Kruskal's
- Prim's

6.3 Minimum Spanning Tree: Kruskal's Algorithm

Steps

1. Define a forest F such that each vertex is a separate tree (Effectively remove all the edges so none of the vertices are connected).

2. Sort all the edges by their weights.

3. Add the least weight edge. If the edge combines two different trees, then add it to the forest F. If the two vertices are of the same tree, then this edge is not part of the minimum spanning tree.

4. Repeat until all vertices are connected into a single tree.

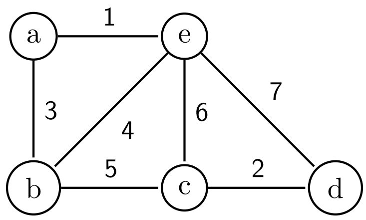
Complexity

$$O(E\log(E)) = O(E\log V)$$

Note that E is at most $|V|^2$ which would make the complexity, $O(E\log(V^2)) = O(2E\log(V)) = O(E\log V)$.

6.3.1 Example

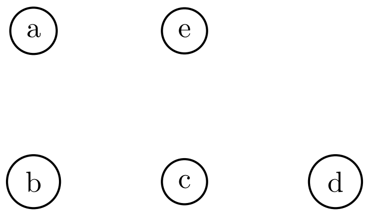
Find the minimum spanning tree of the following graph using Kruskal's algorithm.



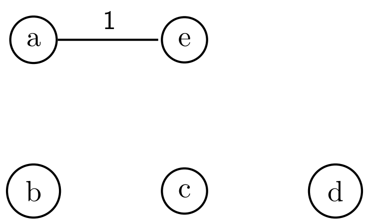
Sort the edges

Edge	(a,e)	(c,d)	(a,b)	(b,e)	(b,c)	(e,c)	(e,d)
Weight	1	2	3	4	5	6	7

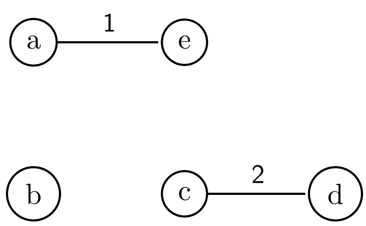
Initial Graph



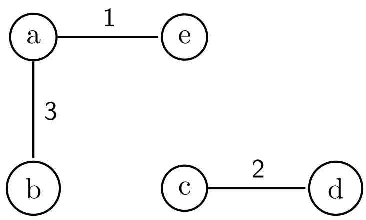
Adding Edge (a,e)



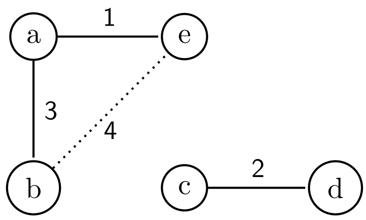
Adding Edge (c,d)



Adding Edge (a,b)

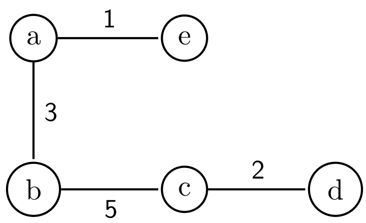


Adding Edge (b,e)



Cannot add this edge since it vertex b and vertex e are already in the same tree.

Adding Edge (b,c)



All vertices in same tree. Therefore, minimum spanning tree has been derived.

6.4 Minimum Spanning Tree: Prim's Algorithm

Steps

1. Define an empty tree.
2. Select an arbitrary vertex to add to the tree.
3. Add the least weight edge that connects to the tree.
4. Repeat until all vertices are connected.

Complexity

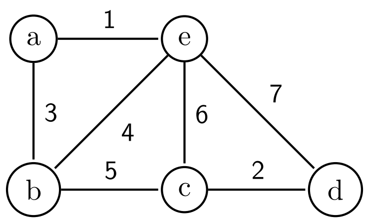
Using adjacency matrix: $O(|V|^2)$

Using binary heap and adjacency list: $O((|V| + |E|)\log(|V|)) = O(|E|\log(|V|))$

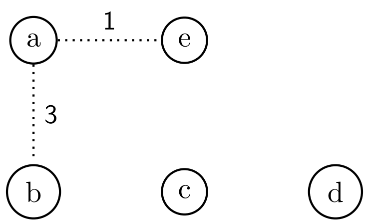
Using Fibonacci heap and adjacency list: $O(|E| + |V|\log(|V|))$

6.4.1 Example

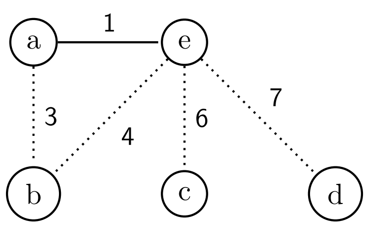
Find the minimum spanning tree of the following graph using Prim's algorithm.



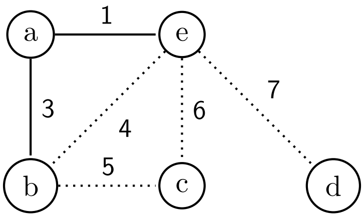
Initial Tree Starting from Vertex a



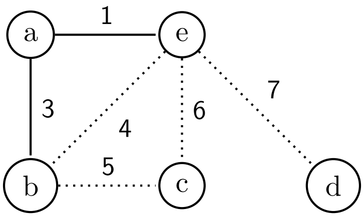
Add edge (a,e)



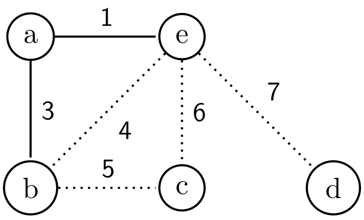
Add edge (a,b)



Add edge (a,b)

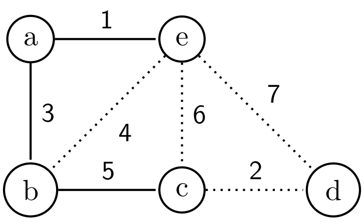


Add edge (b,e)

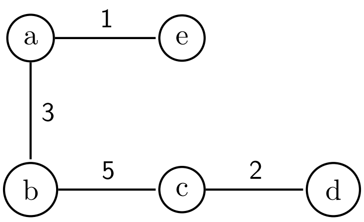


Cannot edge (b,e) since vertex b and vertex e are in the same tree.

Add edge (b,c)



Add edge (c,d)



All vertices in same tree. Therefore, minimum spanning tree has been derived.

6.5 Breadth-First Search (BFS)

Problem Statement

Given a graph G and a vertex $v \in Graph$, find all vertices reachable from v as they are discovered.

Pseudocode

```
procedure BREADTH-FIRST-SEARCH(G,v)
    Let Q be defined as a queue
    Q.enqueue(v)

    v.discovered = true

    while Q is not empty do
        u = Q.dequeue()
        (Arbitrary Processing of Node u)
        for Edge e ∈ E where e = (u, w) do
            if w.discovered ! = true then
                Q.enqueue(w);
                w.discovered = true
            end if
        end for
    end while
end procedure
```

Steps

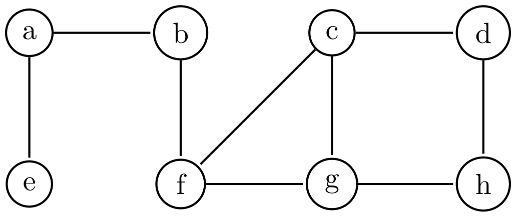
1. Assume there exists some queue Q.
2. Starting from a given vertex v, mark v as discovered and enqueue v into the Q.
3. For every iteration:
   (a) Dequeue the next vertex in Q and define this dequeued vertex as u.
   (b) For every vertex w that is connected to vertex u by some edge, enqueue each of them in lexical order (alphabetical order) and mark them as discovered.
   (c) Repeat until Q is empty.

Complexity

$$O(|V| + |E|)$$

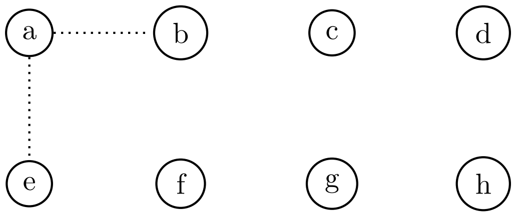
6.5.1 Example

Given the following graph G, perform a breadth-first search from vertex a and print the order that vertices are discovered.



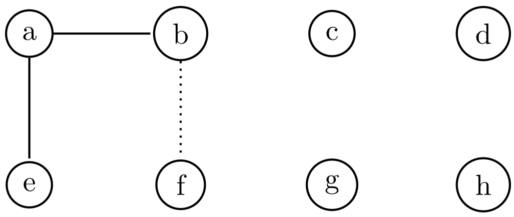
Initialization

$$Q = \{a\}$$



Dequeue a and Enqueue b,e

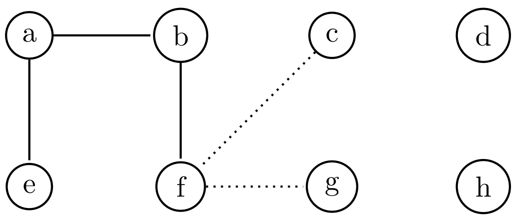
$$Q = \{b, e\}$$



Order of Discovery: a

Dequeue b and Enqueue f

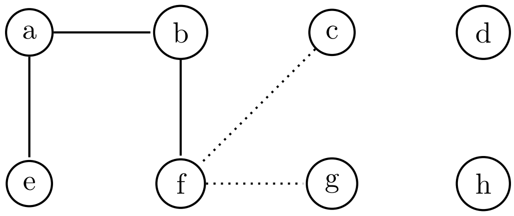
$$Q = \{e, f\}$$



Order of Discovery: a, b

Dequeue e and Enqueue Nothing

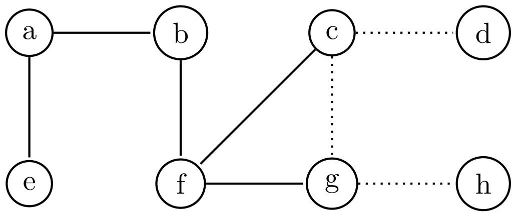
$$Q = \{f\}$$



Order of Discovery: a, b, e

Dequeue f and Enqueue c,g

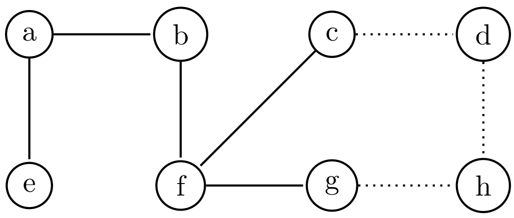
$$Q = \{c, g\}$$



Order of Discovery: a, b, e, f

Dequeue c and Enqueue d

$$Q = \{g, d\}$$

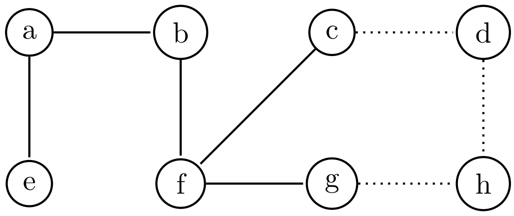


Note that when c is processed, it notices that g is already discovered so it does not attempt that edge. Hence why the dotted edge (c,g) disappears.

Order of Discovery: a, b, e, f, c

Dequeue g and Enqueue h

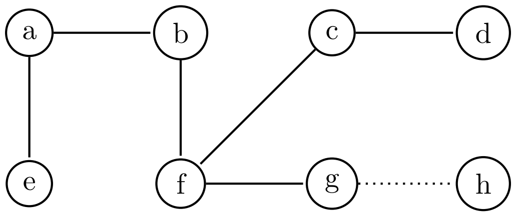
$$Q = \{d, h\}$$



Order of Discovery: a, b, e, f, c, g

Dequeue d and Enqueue Nothing

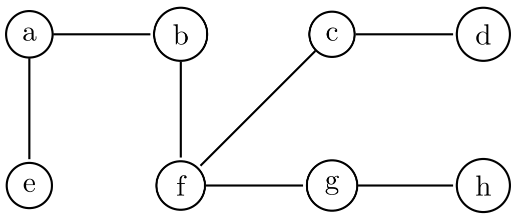
$$Q = \{d, h\}$$



Order of Discovery: a, b, e, f, c, g, d

Dequeue h and Enqueue Nothing

$$Q = \{h\}$$



Order of Discovery: a, b, e, f, c, g, d, h

6.6 Case Study: Depth-First Search (DFS)

Chapter 7

Side Topics

7.1 Proof by Mathematical Induction

Steps

1. Basis (Base Case)
2. Inductive Hypothesis
3. Inductive Step

7.1.1 Example

Prove that the following systems of equations has the solution $T(n) = n \cdot lg(n)$.

$$T(n) = \begin{cases} 2T(\frac{n}{2}) + n, & n = 2^k \text{ for } k > 1 \\ 2, & n = 2 \end{cases}$$

Basis

$$\begin{aligned} T(2) &= (2) \cdot lg(2) \\ &= 2 \cdot 1 \\ &= 2 \end{aligned}$$

Inductive Hypothesis

Assume that $T(n) = n \cdot lg(n)$ holds true for all $n = 2^k$.

Inductive Step

$$\begin{aligned}
T(n) &= 2T(\frac{n}{2}) + n && \text{Base equation} \\
&= 2T(\frac{2^{k+1}}{2}) + 2^{k+1} && \text{Substitute n with } 2^{k+1} \\
&= 2T(2^k) + 2^{k+1} && \text{Simplify parameters to function T(...)} \\
&= 2(2^k \cdot lg(2^k)) + 2^{k+1} && \text{Inductive hypothesis} \\
&= 2^{k+1}\left[lg(2^k) + 1\right] && \text{Distributive property} \\
&= 2^{k+1}\left[lg(2^k) + lg(2)\right] && \text{Logarithmic identity} \\
&= 2^{k+1} \cdot lg(2^k \cdot 2) && \text{Logarithmic identity} \\
&= 2^{k+1} \cdot lg(2^{k+1}) && \text{Exponent property} \\
& && \text{Q.E.D}
\end{aligned}$$

7.1.2 Example

Prove that the following systems of equations has the solution $T(n) = 2F(n) - 1$ where $F(n) = F(n-1) + F(n-2)$.

$$T(n) \begin{cases} T(n-1) + T(n-2) + 1, & \text{if } n \geq 2 \\ 0, & \text{if } n = \{0, 1\} \end{cases}$$

Basis

$T(0) = 1$

Inductive Hypothesis

Assume that $T(n) = F(n) - 1$ is true for all $n = k$.

Inductive Step

$$\begin{aligned}
T(n) &= T(n-1) + T(n-2) + 1 && \text{Base equation} \\
T(k+1) &= T((k+1)-1) + T((k+1)-2) + 1 && \text{Substitute n with k+1} \\
&= T(k) + T(k-1) + 1 && \text{Simplify parameters to function T(...)} \\
&= (2F(k) - 1) + (2F(k-1) - 1) + 1 && \text{Inductive hypothesis} \\
&= 2F(k) + 2F(k-1) - 1 && \text{Simplify equation} \\
&= 2(F(k) + F(k-1)) - 1 && \text{Distributive property} \\
&= 2(F(k+1)) - 1 && \text{Definition of function: } F(k+1) = F(k) + F(k-1) \\
&= 2F(k+1) - 1 && \text{Simplify} \\
& && \text{Q.E.D}
\end{aligned}$$