Méthode des volumes finis :

On doit résoudre l'équation suivante :

$$\mathrm{div}\left(\lambda\, \vec{\mathrm{grad}}(\theta)\right) - \mathrm{div}\left(\rho C_p \theta . \vec{v}\right) = \rho C_p \frac{\partial \theta}{\partial t}$$

(1) Mise en place de l'équation intégrale

On introduit $\bar{\theta}_P = \dfrac{1}{\Omega_P} \displaystyle\iint_{\Omega_P} \theta(x,y)\, d\Omega_P$ avec $\Omega_P = \delta x . \delta y$

On effectue une prise de "moyenne" sur l'EDP sur un volume élémentaire Ω_P :

$$\frac{1}{\Omega_P}\iint_{\Omega_P} \mathrm{div}\left(\lambda\, \vec{\mathrm{grad}}(\theta)\right) d\Omega_P - \frac{1}{\Omega_P}\iint_{\Omega_P} \mathrm{div}\left(\rho C_p \theta . \vec{v}\right) d\Omega_P$$

$$= \frac{1}{\Omega_P}\iint_{\Omega_P} \rho C_p \frac{\partial \theta}{\partial t}\, d\Omega_P$$

On applique sur les 2 intégrales le Théorème Flux-Divergence (Stockes)

$$\boxed{\iint_{\Omega_P} \mathrm{div}(\vec{F})\, d\Omega_P = \oint_{S\Omega_P} \vec{F} . \vec{n}\, dS_P}$$

normale "sortante"

$$\rho C_p \frac{\partial}{\partial t}\Bigg[\underbrace{\frac{1}{\Omega_P}\iint_{\Omega_P} \theta\, d\Omega_P}_{= \bar{\theta}_P \text{ par définition}}\Bigg] = \rho C_p \frac{\partial \bar{\theta}_P}{\partial t}$$

d'où :

$$\frac{1}{\Omega_P}\oint_{S\Omega_P} \underbrace{\lambda\, \vec{\mathrm{grad}}(\theta) . \vec{n}\, dS_P}_{\text{Flux conductif sortant}} - \frac{1}{\Omega_P}\oint_{S\Omega_P} \underbrace{\rho C_p\, \vec{v} . \vec{n}\, dS_P}_{\text{Flux convectif sortant}} = \rho C_p \frac{\partial \bar{\theta}_P}{\partial t}$$

2 Interpolation linéaire :

On fait un développement limité de type Taylor :

ⓐ $$\underbrace{\theta(x_{i+1})}_{\theta_E} = \underbrace{\theta(x_{i+1/2})}_{\theta_e} + \underbrace{(x_{i+1} - x_{i+1/2})}_{\delta x/2} \left.\frac{\partial \theta}{\partial x}\right|_e + \frac{(x_{i+1} - x_{i+1/2})^2}{2} \left.\frac{\partial^2 \theta}{\partial x^2}\right|_e + \frac{(x_{i+1} - x_{i+1/2})^3}{3!} \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e$$

$$\rightarrow \boxed{\theta_E = \theta_e + \frac{\delta x}{2} \left.\frac{\partial \theta}{\partial x}\right|_e + \frac{1}{2}\left(\frac{\delta x}{2}\right)^2 \left.\frac{\partial^2 \theta}{\partial x^2}\right|_e + \frac{1}{3!}\left(\frac{\delta x}{2}\right)^3 \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e}$$

ⓑ $$\underbrace{\theta(x_i)}_{\theta_P} = \underbrace{\theta(x_{i+1/2})}_{\theta_e} + \underbrace{(x_i - x_{i+1/2})}_{-\delta x/2} \left.\frac{\partial \theta}{\partial x}\right|_e + \frac{(x_i - x_{i+1/2})^2}{2} \left.\frac{\partial^2 \theta}{\partial x^2}\right|_e + \frac{(x_i - x_{i+1/2})^3}{3!} \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e$$

$$\rightarrow \boxed{\theta_P = \theta_e - \frac{\delta x}{2} \left.\frac{\partial \theta}{\partial x}\right|_e + \frac{1}{2}\left(\frac{\delta x}{2}\right)^2 \left.\frac{\partial^2 \theta}{\partial x^2}\right|_e - \frac{1}{3!}\left(\frac{\delta x}{2}\right)^3 \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e}$$

- On fait ⓐ + ⓑ :

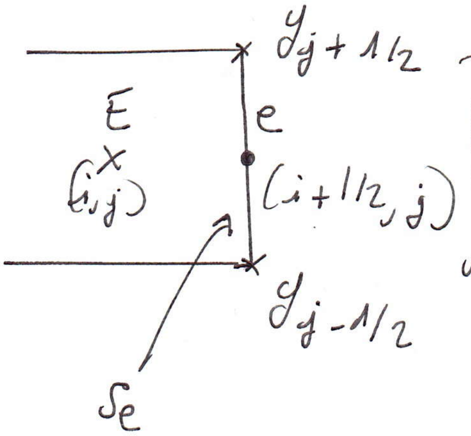
$$\theta_E + \theta_P = 2\theta_e + \left(\frac{\delta x}{2}\right)^2 \frac{\partial^2 \theta}{\partial x^2} \rightarrow \boxed{\theta_e = \frac{\theta_P + \theta_E}{2} + \underbrace{\mathcal{O}(\delta x^2)}_{\text{ordre } 2}}$$

- On fait ⓐ − ⓑ :

$$\theta_E - \theta_P = \delta x \left.\frac{\partial \theta}{\partial x}\right|_e + \frac{2}{3!}\left(\frac{\delta x}{3}\right)^3 \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e$$

d'où : $$\boxed{\left.\frac{\partial \theta}{\partial x}\right|_e = \frac{\theta_E - \theta_P}{\delta x}} \underbrace{- \frac{2}{3!} \frac{(\delta x^2)}{3^3} \left.\frac{\partial^3 \theta}{\partial x^3}\right|_e}_{\mathcal{O}(\delta x^2) \rightarrow \text{ordre } 2}$$

3 Approximation d'une intégrale de surface :



$$\int_{S_e} f(x,y)\, dS_e \simeq f_e \times \delta y + ?$$

Rq : $f(x_{i+1/2}, y_j) = f_e$

Même approche, on fait un DL de la fonction $f(x,y)$ au point $(x_{i+1/2}, y_j)$

$$\int_{S_e} f(x,y)\, dS_e = \int_{y_{j-1/2}}^{y_{j+1/2}} f(x_{i+1/2}, y)\, dy$$

$$= \int_{y_{j-1/2}}^{y_{j+1/2}} \left[f(x_{i+1/2}, y_j) + (y-y_j) \frac{\partial f}{\partial y}\Big|_e + \frac{1}{2}(y-y_j)^2 \frac{\partial^2 f}{\partial y^2}\Big|_e \right] dy$$

$f_e'' = C^{ste}$

$$= f_e \cdot \delta y + \frac{\partial f}{\partial y}\Big|_e \left[\frac{1}{2}(y_{j+1/2}-y_j)^2 - \frac{1}{2}(y_{j-1/2}-y_j)^2 \right]$$

$$+ \frac{\partial^2 f}{\partial y^2}\Big|_e \left[\frac{1}{3}(y_{j+1/2}-y_j)^3 - \frac{1}{3}(y_{j+1/2}-y_j)^3 \right] \times \frac{1}{2}$$

$$= f_e\, \delta y + \frac{1}{2}\frac{\partial f}{\partial y}\Big|_e \left[(\cancel{\delta y}/2)^2 - (-\cancel{\delta y}/2)^2 \right] + \frac{1}{6}\frac{\partial^2 f}{\partial y^2}\left[\left(\frac{\delta y}{2}\right)^3 - \left(-\frac{\delta y}{2}\right)^3 \right]$$

d'où :

$$= f_e \cdot \delta y + \frac{1}{24}\frac{\partial^2 f}{\partial y^2}\, \delta y^3$$

$$= \delta y \left(f_e + \underbrace{\frac{1}{24}\frac{\partial^2 f}{\partial y^2}\, \delta y^2}_{} \right)$$

→ Ordre 2 (Très bonne approximation)

④ Mise en place du schéma linéaire _ linéaire :

$$\underbrace{\frac{1}{\Omega_p}\int_{\partial\Omega_p} \lambda\, \vec{grad}(\theta).\vec{n}\, dS_p}_{\text{"1"}} - \underbrace{\frac{1}{\Omega_p}\int_{\partial\Omega_p} \rho C_p \theta. \vec{v}.\vec{n}\, dS_p}_{\text{"2"}} = \underbrace{\rho C_p \frac{\partial \theta_p}{\partial t}}_{\text{"3"}}$$

→ Terme "3" : $\rho C_p \dfrac{\partial \theta_p}{\partial t} \rightarrow \rho C_p \dfrac{\left(\bar\theta_p^{\,n+1} - \bar\theta_p^{\,n}\right)}{\delta t} + \underbrace{\mathcal{O}(\delta t)}_{\text{Ordre 1}}$

Schéma décentré amont

→ Pour les termes "1" et "2" $\displaystyle\int_{S_p} = \int_e + \int_n + \int_w + \int_s$

On regarde $\displaystyle\int_e$ en premier :

- $$\frac{1}{\Omega_p}\int_{S_e} \lambda\, \vec{grad}(\theta).\vec{n}\, dS_e \simeq \frac{1}{\Omega_p}\lambda\, \underbrace{\vec{grad}(\theta)\big|_e . \vec{n}_e}_{\begin{Bmatrix}\frac{\partial\theta}{\partial x}\\ \frac{\partial\theta}{\partial y}\end{Bmatrix}.\begin{Bmatrix}1\\0\end{Bmatrix}} \times \delta y$$

$$= \frac{1}{\delta x \delta y}\lambda \underbrace{\frac{\partial\theta}{\partial x}\Big|_e}_{\text{d'après ②}} \times \delta y = \frac{\lambda}{\delta x}\,\frac{\theta_E - \theta_P}{\delta x}$$

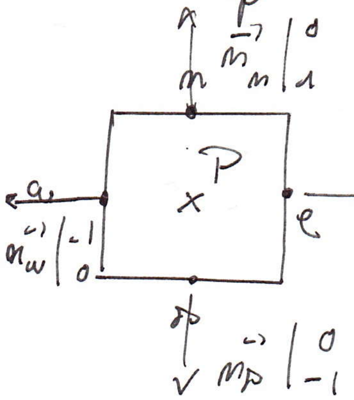
$$\boxed{= \frac{\lambda}{\delta x^2}\left(\theta_E - \theta_P\right)}$$

- $$\frac{1}{\Omega_p}\int_{S_e} \rho C_p \theta . \vec{v}.\vec{n}\, dS_e = \frac{1}{\Omega_p}\rho C_p \theta_e . \vec{v}_e . \vec{n}_e\, \delta y$$

$$\vec{v}_e.\vec{n}_e = \begin{Bmatrix}u_e\\v_e\end{Bmatrix}.\begin{Bmatrix}1\\0\end{Bmatrix} = u_e \Rightarrow \frac{1}{\delta x \delta y}\rho C_p\left(\frac{\theta_P + \theta_E}{2}\right)\delta y = \boxed{\frac{\rho C_p}{\delta x}\left(\theta_P + \theta_E\right)}$$

Idem pour les autres surfaces (⚠ à l'orientation des normales)

$$\frac{1}{\Omega_P}\int_{\partial\Omega_P} \lambda\,\vec{grad}(\theta).\vec{n}\,dS_P = \frac{\lambda}{\Omega_P}\left(\frac{\partial\theta}{\partial x}\Big|_e dy - \frac{\partial\theta}{\partial x}\Big|_w dy\right)$$



$$+ \frac{\lambda}{\Omega_P}\left(\frac{\partial\theta}{\partial y}\Big|_n dx - \frac{\partial\theta}{\partial y}\Big|_s dx\right)$$

$$\simeq \frac{\lambda}{\delta x\,\delta y}\left(\frac{\theta_E-\theta_P}{\delta x}\times\delta y - \frac{\theta_P-\theta_W}{\delta x}\times\delta y\right)$$

$$+ \frac{\lambda}{\delta x\,\delta y}\left(\frac{\theta_N-\theta_P}{\delta y}\times\delta x - \frac{\theta_P-\theta_S}{\delta y}\,\delta x\right)$$

d'où :

- $$\frac{1}{\Omega_P}\int_{\partial\Omega_P} \lambda\,\vec{grad}(\theta).\vec{n}\,dS_P = \frac{\lambda}{\delta x^2}\left(\theta_W - 2\theta_P + \theta_E\right) + \frac{\lambda}{\delta y^2}\left(\theta_S - 2\theta_P + \theta_N\right)$$

- On a ainsi pour le modèle <u>diffusif</u> pur :

$$\underbrace{\frac{\lambda}{\delta x^2}\left(\theta_W^n - 2\theta_P^n + \theta_E^n\right)}_{\text{diffusion selon "x"}} + \underbrace{\frac{\lambda}{\delta y^2}\left(\theta_S^n - 2\theta_P^n + \theta_N^n\right)}_{\text{diffusion selon "y"}} = \rho C_p\,\frac{\theta_P^{n+1}-\theta_P^n}{\delta t}$$

d'où le schéma numérique discret :

$$\boxed{\theta_P^{n+1} = \theta_P^n + \underbrace{\frac{a.\delta t}{\delta x^2}}_{=F}\left(\theta_W^n - 2\theta_P^n + \theta_E^n\right) + \overset{\frac{a\,\delta t}{\delta y^2}}{\left(\theta_S^n - 2\theta_P^n + \theta_N^n\right)}}$$

1 Equation discrète = 1 volume "Ω_P"

- Le terme de transport : (schéma linéaire)

$$\frac{1}{\Omega_P}\int_{S_e} \rho C_p \theta \vec{v}\cdot\vec{n}\, dS_e \simeq \frac{\rho C_p}{\Omega_P}\,\theta_e \cdot \underbrace{\vec{v}_e\cdot\vec{n}_e}_{\begin{Bmatrix}u_e\\ v_e\end{Bmatrix}\cdot\begin{Bmatrix}1\\0\end{Bmatrix}}\,\delta y \qquad \vec{v}_e\cdot\vec{n}_e = u_e$$

On rappelle que $\boxed{\theta_e \simeq \frac{\theta_P+\theta_E}{2}}$ d'après la question 2.)

$$\frac{1}{\Omega_P}\int_{S_e} \rho C_p \theta \cdot \vec{v}\cdot\vec{n}\, dS_e \simeq \frac{\rho C_p}{\delta x\cdot\cancel{\delta y}}\left(\frac{\theta_P+\theta_E}{2}\right)\times u_e \times \cancel{\delta y}$$

$$\simeq \frac{\rho C_p u_e}{\delta x}\left(\frac{\theta_P+\theta_E}{2}\right)$$

On généralise sur l'ensemble des surfaces :

$$\frac{1}{\Omega_P}\int_{S\Omega_P} \rho C_p \theta \vec{v}\cdot\vec{n}\, dS_P \simeq \underset{=\delta x\cdot\delta y}{\frac{\rho C_p}{\Omega_P}}\Big[\theta_e \overbrace{\vec{v}_e\cdot\vec{n}_e}^{u_e}\,\delta y - \theta_w \overbrace{\vec{v}_w\cdot\vec{n}_w}^{-u_w}\,\delta y\Big]$$

$$+ \underset{=\delta x\cdot\delta y}{\frac{\rho C_p}{\Omega_P}}\Big[\theta_n \underbrace{\vec{v}_n\cdot\vec{n}_n}_{v_n}\,\delta x - \theta_s \underbrace{\vec{v}_s\cdot\vec{n}_s}_{-v_s}\,\delta x\Big]$$

$$\frac{1}{\Omega_P}\int_{S\Omega_P} \rho C_p \theta \cdot \vec{v}\cdot\vec{n}\, dS_P = \frac{\rho C_p}{\delta x}\Big[\theta_e u_e - \theta_w u_w\Big] + \frac{\rho C_p}{\delta y}\Big[\theta_n v_n - \theta_s v_s\Big]$$

Schéma linéaire :

$$\frac{\rho C_p}{\delta x}\left[\left(\frac{\theta_P+\theta_E}{2}\right)\times u_e - \left(\frac{\theta_P+\theta_W}{2}\right)u_w\right] + \frac{\rho C_p}{\delta y}\left[\left(\frac{\theta_P+\theta_N}{2}\right)v_n - \left(\frac{\theta_P+\theta_S}{2}\right)v_s\right]$$

d'où le schéma final (linéaire - linéaire) :

$$\theta_P^{n+1} = \theta_P^n + \underbrace{\overset{F}{\frac{a\,\delta t}{\delta x^2}}\left(\theta_W^n - 2\theta_P^n + \theta_E^n\right)}_{\text{diffusion \% } x} + \underbrace{\frac{a\,\delta t}{\delta y^2}\left(\theta_S^n - 2\theta_P^n + \theta_N^n\right)}_{\text{diffusion \% } y}$$

$$- \underbrace{\frac{\delta t}{\delta x}\left(\frac{\theta_P^n + \theta_E^n}{2} \times u_e - \frac{\theta_P^n + \theta_W^n}{2} \times u_w\right)}_{\text{convection \% } x}$$

$$- \underbrace{\frac{\delta t}{\delta y}\left(\frac{\theta_P^n + \theta_N^n}{2} \times v_n - \frac{\theta_P^n + \theta_S^n}{2} \times v_s\right)}_{\text{convection \% } y}$$

schéma d'ordre 2 en espace et d'ordre 1 en temps.

- Reste à regarder le schéma "Upwind" et "Quick" (TP).

Schéma "Upwind"

$$\frac{1}{\Delta P}\int_{S_e} \rho C_p \theta\, \vec{v}\cdot\vec{n}\, dS_e \simeq \frac{\rho C_p}{\underset{\Delta P = \delta x \cdot \cancel{\delta y}}{\cancel{\Delta P}}}\, \theta_e \cdot \overset{u_e}{\vec{v}_e \cdot \vec{n}_e}\, \cancel{\delta y}$$

$$\theta_e = \begin{cases} \theta_P \text{ si } \vec{v}_e \cdot \vec{n}_e = u_e > 0 \\ \theta_E \text{ si } \vec{v}_e \cdot \vec{n}_e = u_e < 0 \end{cases}$$

$$= \frac{\rho C_p}{\delta x}\left[\theta_P \max(u_e, 0) + \theta_E \min(u_e, 0)\right]$$