Hamiltonian Circuits and The Icosian Game

A TERM PROJECT REPORT

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Bachelor of Technology in

Theory of Computation CSD-225

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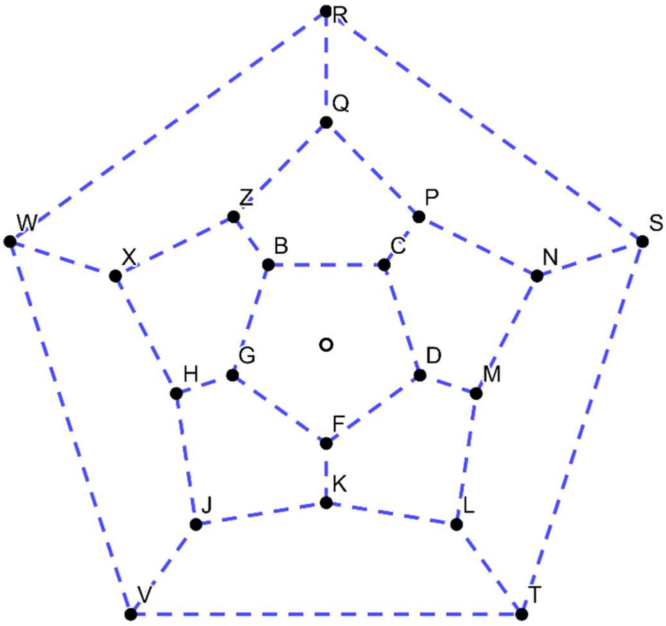
EXAMPLES OF ICOSIAN PROBLEMS (FIRST PROBLEM)

Five initial points are given; cover the board, and finish cyclically. A succession is said to be cyclical when the last piece is adjacent to the first.

Example 3. Given B C P N M as initial: two solutions exist.
one is the succession, D F K L T S R Q Z X W V J H G.
the other is D F G H X W V J K L T S R Q Z.

Example 5. Five initials, J V T S R. Two solutions exist.

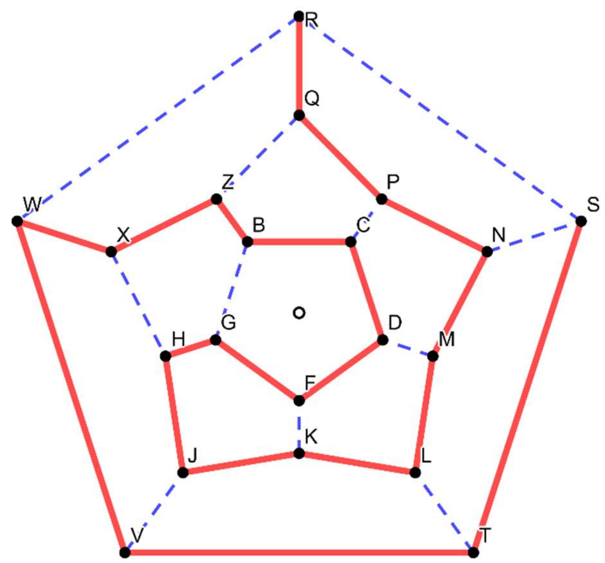


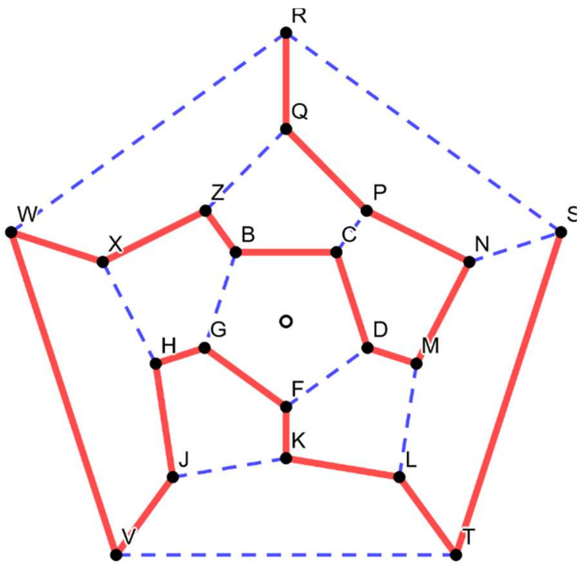
An Empty Icosian Game Board.

Five Initial points can be chosen anywhere but sequentially. For example, we may choose R Q P N M but not R Q P Z B, P is not sequential to Z.

If chosen R Q P N M, then the last point must be W or S.

Two solutions for the given R Q P N M is as given below.
R Q P N M L K J H G F D C B Z X W V T S &
R Q P N M D C B Z XW V J H G F K L T S

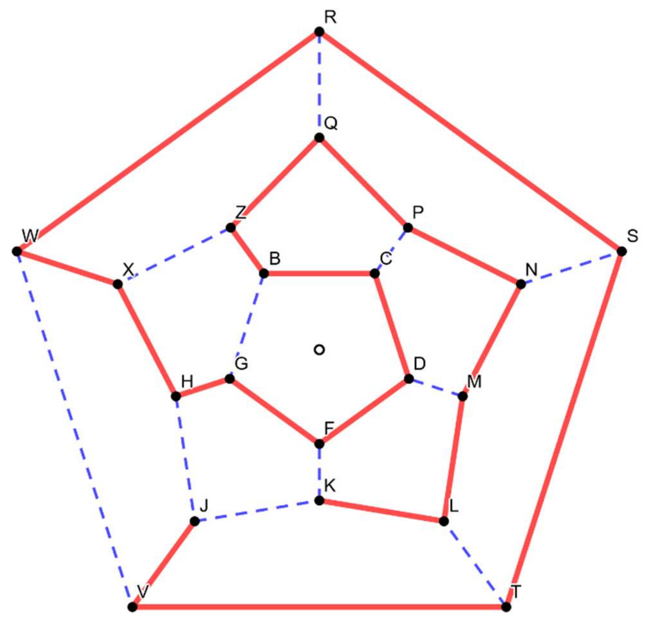




In this case there are no solutions ending to W because if it had to then point S will be left out and there will be no way to cover S. So, both solutions end to S.

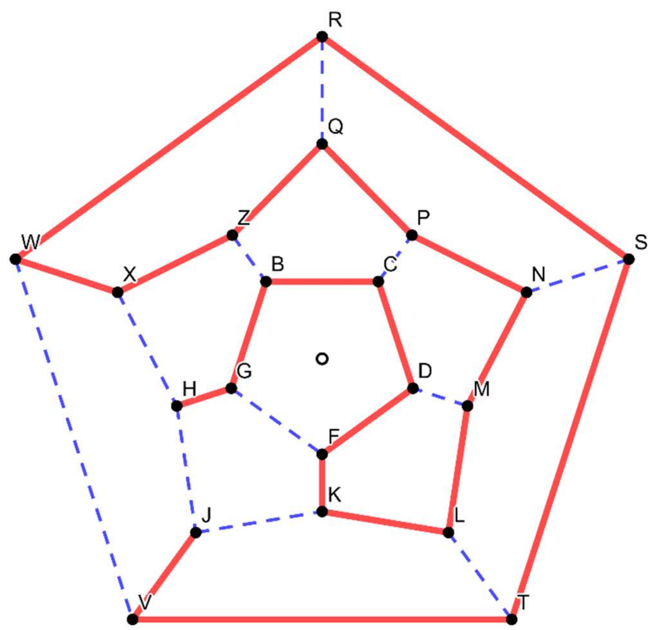
Task 4: In Example 4, Hamilton claimed there are two Hamiltonian circuits that begin with the vertices J V T S R. Find them. (You do not need to prove these are the only two.)

Given 5 initial points J V T S R, the two solutions may end at K or H. Given Below are the two solutions.



First Solution:

J V T S R W X H G F D C B Z Q P N M L K



Second Solution:

J V T S R W X Z Q P N M L K F D C B G H

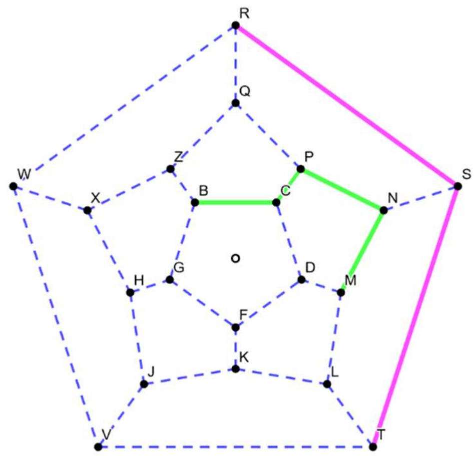
In this case we have one solution ending at K and another at H. Both solutions contain three similar sequences J V T S R W X, Z Q P N M L K and F D C B. We can infer by above observation that all the solutions may contain some same sequences in them.

Task 5: In Example 3, Hamilton specified B C P N M as the first five vertices in the desired circuit. He then claimed that there are only two solutions of this problem:
B C P N M D F K L T S R Q Z X W V J H G and
B C P N M D F G H X W V J K L T S R Q Z
Prove that these are in fact the only two solutions that satisfy the initial conditions of the problem by completing the details of the following argument.
Use current graph terminology (e.g., vertex, edge) to justify your reasoning. Include copies of the diagram illustrating each step of the argument in a different colour as part of your proof.

It is assumed initially that these are not the only solution we know there may exist more solutions. Since we must find a Hamiltonian circuit starting with BCPNM. We might do that by covering each vertex thereafter.

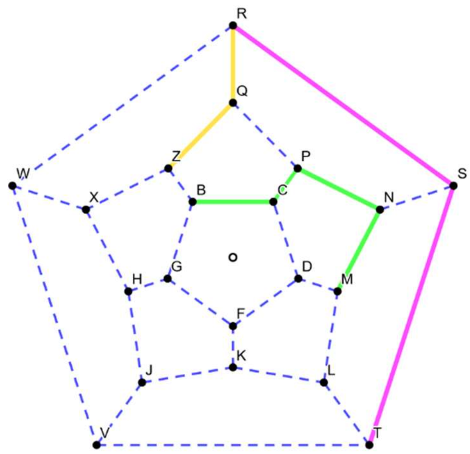
a) **Explain why a solution must include either the sequence RST or the sequence TSR.**

For vertex S we can reach from R or T we cannot use the edge NS because N is already covered by the initial condition. And once at S from either vertex we have only one vertex left to jump. If we choose RS edge initially then we can jump to T or if we choose TS then we may jump to R. Either way we must include the sequence TSR or RST otherwise the vertex S will be left and it will contradict out goal to find a Hamiltonian circuit. This verify the argument (a) that is TSR or RST must be present in the solution.

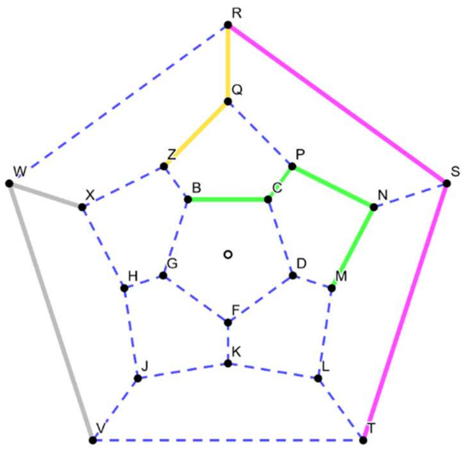


b) **Explain why a solution must include either the sequence RQZ or the sequence ZQR.**

From vertex R we can either jump to Q or W. If we choose W, then Q must be an end vertex because it can then only be reached through vertex Z and that would contradict out expected end vertices Z or G. Hence Q can only be approached from R and then from Q we can only jump to Z. This way we must include the edges RQ and QZ in our solution. This verifies the argument (b) that is RQZ or ZQR must be present in the solution.



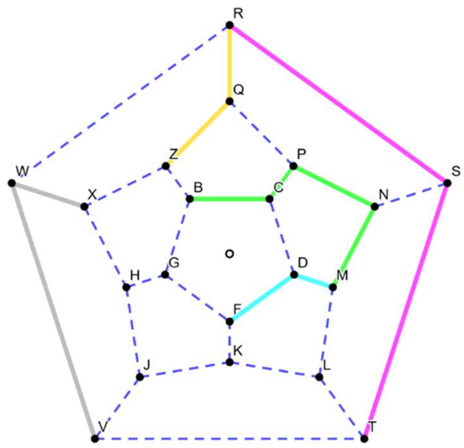
c) Explain why a solution must include either the sequence XWV or the sequence VWX.



After argument (a) and (b) vertex W can only be reached from vertices X or V because vertex R is already covered. Either way we must include edges XW and WV. This verifies the argument (c) that is XWV or VWX must be present in the solution.

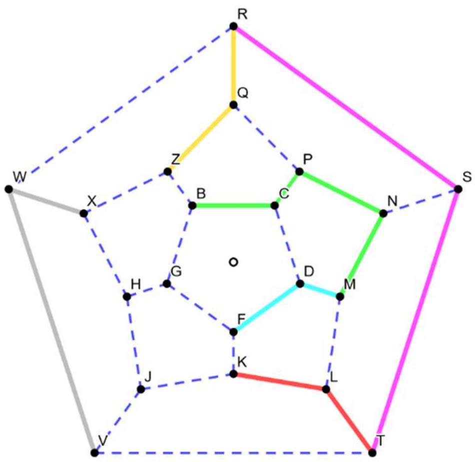
d) Explain why a solution must include either the sequence FDM or the sequence MDF.

When at vertex M we have option to either go to vertex D or vertex L. If we choose L then D must be an end vertex because then we have to reach D via F and we can go any further from D but that contradicts the expected end vertices that is Z or G. Hence, we cannot choose vertex L so we must choose D and from D only vertex reachable is F hence edge MD and edge DF must be present in the solution. This verifies the argument (d) that is MDF or FDM must be present in the solution.



e) Explain why we can now conclude that a solution to this problem must include either the sequence KLT or the sequence TLK.

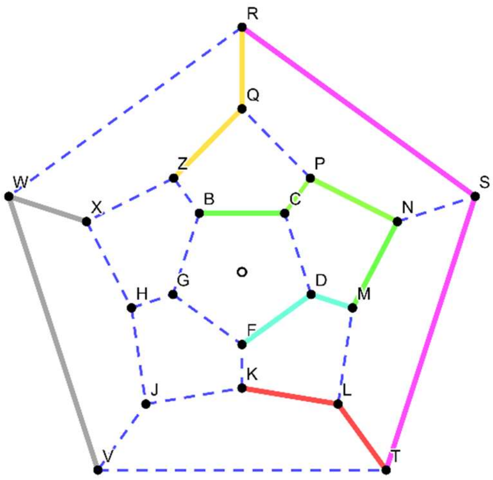
Lastly at vertex T we may jump to L or V. If you choose V, then L must be an end vertex because then it will only be reachable through vertex K this contradicts our expected end vertices G or Z. Hence must include edge TL. At vertex L we then have only choice as K because M is already covered. This way we must include edge LK. This verify argument (e) that is TLK or KLT must be present in the solution.

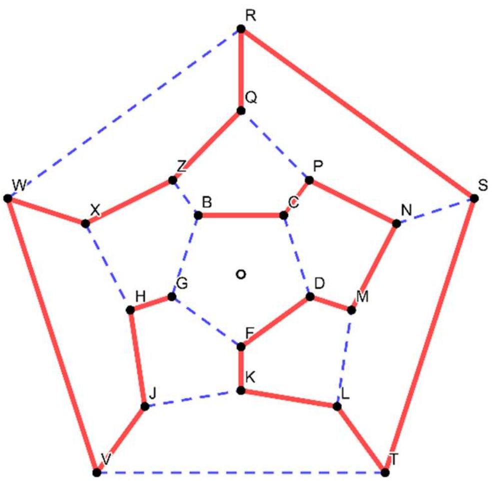


f) **Use the information from above concerning which edges and vertices we know must be part of a solution to this problem to fully justify the claim that the two circuits listed by Hamilton are the only to solutions to the problem.**

Combining all the above arguments we can now plot the final must required sequences of the solutions.

With these sequences in place we have now very few edges left to complete the Hamiltonian Circuit. At this point we must remember that we can have solutions that must end to vertex Z or G. Now we can connect edges and see if we get a Hamiltonian circuit.

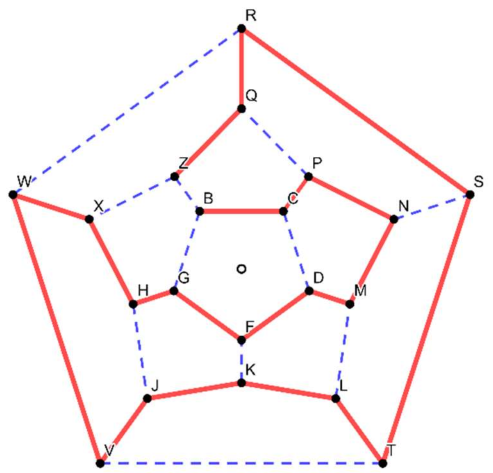




Completing edges ZX we must find a solution that ends to G. So, we complete edges GH, HJ, JV and FK in sequence and avoiding any deadlocks.

We now got our first solution that is:
B C P N M D F K L T S R Q Z X W V J H G

It covers all vertices and ends cyclically so this is a verified solution.



In our next attempt we join the edge XH and HG. Then the solution must contain Z as end vertex. So, we then complete edge GF and JK.

We got our second solution that is:
B C P N M D F G H X W V J K L T S R Q Z

It also covers all the vertices and ends cyclically so this is a verified solution.

Any other set of edges with all arguments included result in a broken circuit. Hence, we can conclude that there are only two solution to this problem.

Task 6: Although Hamilton claimed that every initial sequence of five vertices will lead to at least two solutions (and possibly four) within the graph of the Icosian Game, he did not offer a proof of this claim. Nor did he claim that this is true of all graphs.

Show that it is not true of every graph that any initial sequence of five vertices will lead to at least one Hamiltonian circuit by finding an example of a graph with at least 5 vertices that has no Hamiltonian circuit.

Then prove that your graph does not contain a Hamiltonian circuit.

We did not find any set of initial points that leads to no solution of the Icosian game. However, two theories below can be used to show up to some extent that there may be a no solution condition.

Dirac's Theorem:
A simple graph with *n* vertices in which each vertex has degree at least $\lceil n/2 \rceil$ has a Hamiltonian cycle.

For Icosian number of vertices = 20.
Degree of each vertex = 3.

According to this theorem degree must be at least 10 to compulsorily have a Hamiltonian Cycle in the Icosian. But since we have degree equal to 3, we cannot say if Hamiltonian Cycle exist.

Ore's Theorem:
A simple graph with *n* vertices in which the sum of the degrees of any two non-adjacent vertices is greater than or equal to *n* has a Hamiltonian cycle.

Sum of degrees of any two non-adjacent vertices = 6.
Total number of vertices = 20.

According to this theorem sum of degrees of any two non-adjacent vertices must be greater than 20 that is not true in the case of Icosian. So, we again cannot say if Hamiltonian circuit exist or not.

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