Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game

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Problems that are today considered to be part of modern graph theory originally appeared in a variety of different connections and contexts. Some of these original questions appear little more than games or puzzles. In the instance of the 'Icosian Game', this seems quite literally true. Yet for the game's inventor, the Icosian Game encapsulated deep mathematical ideas which we will explore in this project.

Sir William Rowan Hamilton (1805–1865) was a child prodigy with a gift for both languages and mathematics. His academic talents were fostered by his uncle James Hamilton, an Anglican clergyman with whom he lived from the age of 3. Under his uncle's tutelage, Hamilton mastered several languages — including Latin, Greek, Hebrew, Persian, Arabic and Sanskrit — by the age of 10. His early interest in languages was soon eclipsed by his interests in mathematics and physics, spurred in part by his contact with an American calculating prodigy. Hamilton entered Trinity College in Dublin in 1823, and quickly distinguished himself. He was appointed Astronomer Royal of Ireland at the age of 22 based on his early work in optics and dynamics. Highly regarded not only by his nineteenth century colleagues, Hamilton is today recognized as a leading mathematician and physicist of the nineteenth century.

In mathematics, Hamilton is best remembered for his creation of a new algebraic system known as the 'quarternions' in 1843. The system of quarternions consists of 'numbers' of the form $Q = a + bi + cj + dk$ subject to certain basic 'arithmetic' rules. The project that led Hamilton to the discovery of quarternions was the search for an algebraic system that could be reasonably interpreted in the three-dimensional space of physics, in a manner analogous to the interpretation of the algebra of complex numbers $a + bi$ in a two-dimensional plane. Although this geometrical interpretation of the complex numbers is now standard, it was discovered by mathematicians only in the early 1800's and thus was relatively new in Hamilton's time. Hamilton was one of several nineteenth century British mathematicians interested in developing a purely *algebraic* foundation for complex numbers that would capture the essence of this geometrical interpretation. His algebraic development of the complex numbers as ordered pairs of real numbers (a, b) subject to certain operations appeared in a landmark 1837 essay entitled "Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time."

Hamilton concluded his 1837 essay with a statement that he hoped to soon publish a similar work on the algebra of triplets. After years of unsuccessful work on this problem, Hamilton was able to solve it in 1843 only by abandoning the property of commutativity. For example, two of the basic multiplication rules of the quarternion system are $ij = k$ and $ji = -k$, so that $ij \neq ji$. Hamilton also replaced 'triplets' by the 'four-dimensional' quarternion $a+bi+cj+dk$. Soon after Hamilton's discovery, physicists realized that only the 'vector part' $bi + cj + dk$ of a quarternion was needed to represent three-dimensional space. Although vectors replaced the use of quarternions in physics by the end of the nineteenth century, the algebraic system of vectors retains the non-commutativity of quarternions.

*This project has been adapted by the author from the published version [1] that appears (together with two companion projects) in MAA Notes #74, *Resources for Teaching Discrete Mathematics: Classroom Project, History Modules, and Articles* [5].

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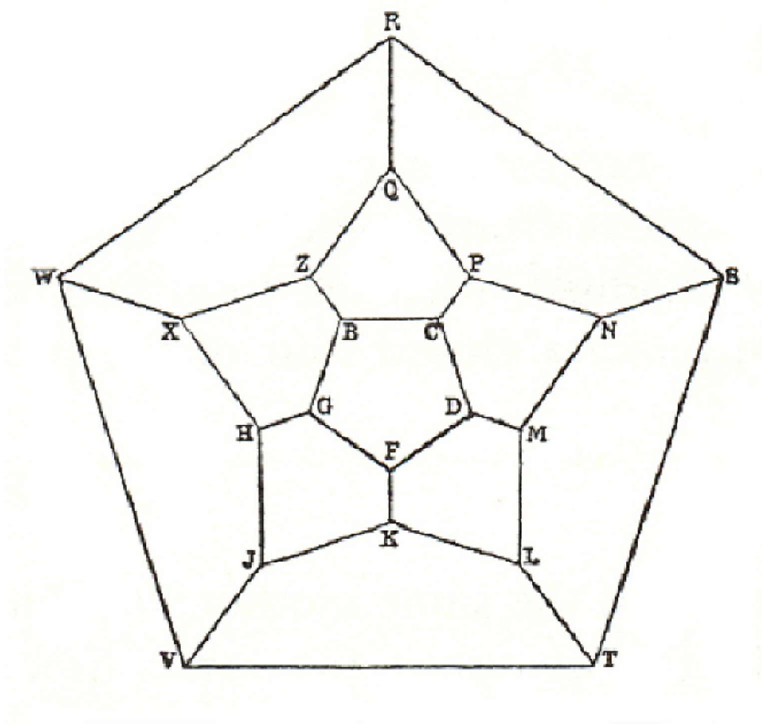
Today's students of mathematics are familiar with a variety of non-commutative algebraic operations, including vector cross-product and matrix multiplication. In their day, however, Hamilton's quarternions constituted a major breakthrough comparable to the discovery of non-Euclidean geometry. Immediately following Hamilton's 1843 announcement of his discovery, at least seven other new numbers systems were discovered by several other British algebraists. In the 'Icosian Game', Hamilton himself developed yet another example of a non-commutative algebraic system. In this project, we explore both the algebra of that system and the graph theoretical notion of 'Hamiltonian circuit' on which Hamilton's interpretation of this algebra is based. We begin with the preface to the instructions pamphlet [2, pp. 32 - 35] which Hamilton prepared for marketing the game after selling it for 25 pounds to 'John Jacques and Son,' a wholesale dealer in games in 1859.

THE ICOSIAN GAME



In this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., &c., of Dublin, and by him named *Icosian* from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed *along the lines* of the figure, and also to fulfil certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

In graph theoretic terminology, the holes of the game board are referred to as *vertices* (singular: *vertex*) and the lines that join two holes (vertices) are called *edges*. The collection of vertices and edges in a given relationship (as represented by a diagram such as the game board) is called a *graph*. Two vertices that are joined by an edge in the graph are said to be *adjacent*. Thus, the instruction 'always to proceed *along the lines* of the figure' requires the player to find a sequence of adjacent vertices; such a sequence is known as a *path*. In the case where no vertex is repeated in the sequence, the path is said to be a *simple path*. In the case where every vertex of the graph is used exactly once in the sequence, the path is said to be a *Hamiltonian path*.

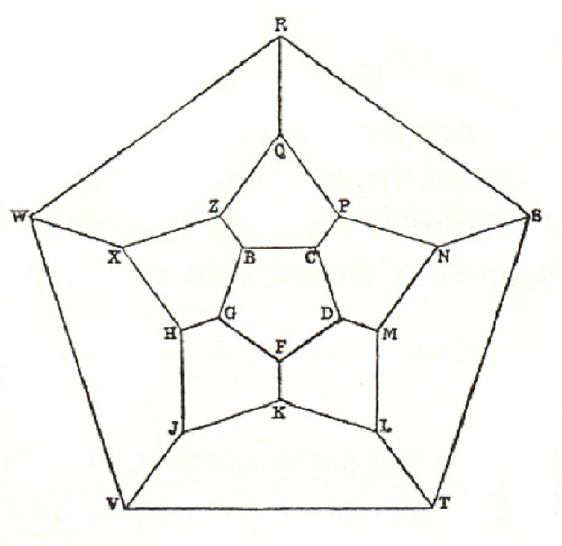
Task 1 According to the instruction booklet, the following two sequences of vertices give Hamiltonian paths that start from the five initial points B C D F G, and place the pegs 'in such a manner that the succession is cyclical'.

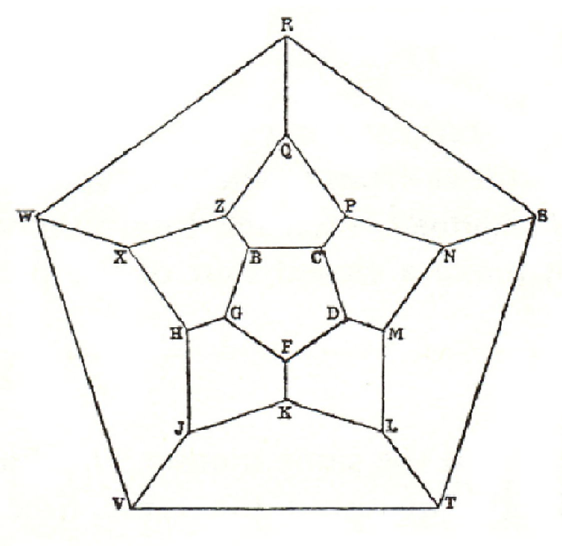
B C D F G H J K L M N P Q R S T V W X Z

B C D F G H X W R S T V J K L M N P Q Z

What did Hamilton mean when he said that 'the succession is cyclical' in these two sequences? (Today, this type of path is called a *cycle*; it can also be called a *circuit*.)

Use the copies of the Icosian graph to verify that the two sequences listed above are Hamiltonian cycles.





Do you think there are any other solutions to this problem? That is, are there other Hamiltonian cycles in the Icosian graph that begin with the five vertices B C D F G? Explain why or why not.

Task 2 Use modern terminology to formally define the terms *cycle* and *Hamiltonian cycle*.

Following the preface, Hamilton included several examples of Icosian Problems in the Instruction Pamphlet. To familiarize ourselves with the concepts of Hamiltonian cycle and Hamiltonian path, we consider only the first two problems.

EXAMPLES OF ICOSIAN PROBLEMS

FIRST PROBLEM

Five initial points are given; cover the board, and finish *cyclically*. (As hinted in the preceding page, a succession is said to be *cyclical* when the *last* piece is adjacent to the *first*.)

[This problem is always possible in at least two, and sometimes in four, different ways.]

. . . .

Example 3. Given B C P N M as initial: two solutions exist; one is the succession, D F K L T S R Q Z X W V J H G; the other is D F G H X W V J K L T S R Q Z.

Example 4. Five initials, L T S R Q. Four solutions.

Example 5. Five initials, J V T S R. Two solutions.

Task 3. In *Example 5*, Hamilton claimed there are only four Hamiltonian circuits that begin with the vertices $L\,T\,S\,R\,Q$. Find them. (You do not need to prove these are the only four.)

Task 4. In *Example 4*, Hamilton claimed there are four Hamiltonian circuits that begin with the vertices $J\,V\,T\,S\,R$. Find them. (You do not need to prove these are the only two.)

Task 5. In *Example 3*, Hamilton specified $B\,C\,P\,N\,M$ as the first five vertices in the desired circuit. He then claimed that there are *only* two solutions of this particular problem:

B C P N M D F K L T S R Q Z X W V J H G and B C P N M D F G H X W V J K L T S R Q Z

Prove that these are in fact the only two solutions that satisfy the initial conditions of the problem by completing the details of the following argument.

Use current graph terminology (e.g., *vertex*, *edge*) to justify your reasoning.

Include copies of the diagram illustrating each step of the argument in a different color as part of your proof.

(a) Explain why a solution must include either the sequence $R\,S\,T$ or the sequence $T\,S\,R$.

(b) Explain why a solution must include either the sequence $R\,Q\,Z$ or the sequence $Z\,Q\,R$.

(c) Explain why a solution must include either the sequence $X\,W\,V$ or the sequence $V\,W\,X$.

(d) Explain why a solution must include either the sequence $F\,D\,M$ or the sequence $M\,D\,F$.

(e) Explain why we can now conclude that a solution to this problem must include either the sequence $K\,L\,T$ or the sequence $T\,L\,K$.

(f) Use the information from above concerning which edges and vertices we know must be part of a solution to this problem to fully justify the claim that the two circuits listed by Hamilton are the only to solutions to the problem.

Task 6. Although Hamilton claimed that every initial sequence of five vertices will lead to at least two solutions (and possibly four) within the graph of the Icosian Game, he did not offer a proof of this claim. Nor did he claim that this is true of all graphs.

Show that it is not true of every graph that any initial sequence of five vertices will lead to at least one Hamiltonian circuit by finding an example of a graph with at least 5 vertices that has no Hamiltonian circuit.

Then prove that your graph does not contain a Hamiltonian circuit.

The Icosian Game and Non-Commutative Algebra

We now turn to the portion of Hamilton's pamphlet which links the Icosian Game to a non-commutative algebra.

HINTS ON THE ICOSIAN CALCULUS, OF WHICH THE ICOSIAN GAME IS DESIGNED TO BE AN ILLUSTRATION.

I. In a "MEMORANDUM respecting a New System of Roots of Unity," which appeared in the *Philosophical magazine* for December 1856, Sir W. R. Hamilton expressed himself nearly as follows (a few words only being here omitted):

'I have lately been led to the conception of a new system, or rather *family of systems,* of *non-commutative roots of unity*, which are entirely distinct from the *i j k* of quaternions, though having some general analogy thereto; and which admit, even more easily than the quaternion symbols do, of geometrical interpretation. In the system which seems at present to be the most interesting one among those included in this new family, I assume three symbols, ι, κ, λ, such that $\iota^2 = 1, \kappa^3 = 1, \lambda^5 = 1, \lambda = \iota\kappa$; where $\iota\kappa$ must be *distinguished* form $\kappa\iota$, since otherwise we should have $\lambda^6 = 1, \lambda = 1$. As a very simple *specimen* of the symbolical conclusions deduced from these fundamental assumptions I may mention that if we make $\mu = \iota\kappa^2 = \lambda\iota\lambda$, we shall have also $\mu^5 = 1$, $\lambda = \mu\iota\mu$; so that μ is a new fifth root of reciprocity. A long train of such symbolical deductions is found to follow; and every one of the results may be *interpreted* as having reference to the passage from *face to face* (or from corner to corner) of the *icosahedron* (or of the dodecahedron): on which account, I am at present disposed to give the name of 'Icosian Calculus' to this new system of symbols, and of rules for their operations.'

The system of '*non-commutative roots of unity*' described above employs three symbols ι, κ, λ subject to the following (non-commutative) rules:

$$\iota^2 = 1\,,\ \kappa^3 = 1\,,\ \lambda^5 = 1\,,\ \lambda = \iota\kappa\,,\ \iota\kappa \neq \kappa\iota$$

The symbol '1' represents the identity, so that $1\iota = \iota 1 = \iota$, $1\kappa = \kappa 1 = \kappa$, and $1\lambda = \lambda 1 = \lambda$.

Although Hamilton did not explicitly state this in the above excerpt, he also assumed that this new systme of '*non-commutative roots of unity*' satisfies the associative property for multiplication.

In Part II of Hints on the Icosian Calculus, Hamilton described in detail how to interpret his system of 'non-commutative roots of unity' within the Icosian Game. First, let's consider only the symbolic action of ι, κ, λ as defined by the above multiplication rules to complete Tasks 7 and 8 below.

Recall: $\iota^2 = 1\ ,\ \kappa^3 = 1\ ,\ \lambda^5 = 1\ ,\ \lambda = \iota\kappa\ ,\ \iota\kappa \neq \kappa\iota$

Task 7. Prove symbolically that $\kappa = \iota\lambda$.

Task 8. Prove symbolically that $\iota\kappa^2 = \lambda\iota\lambda$.

(This shows that is makes sense to define the new symbol μ by $\mu = \iota\kappa^2 = \lambda\iota\lambda$.)

Bonus Task: Show symbolically that $\mu^5 = 1$.

We now consider Hamilton's interpretation of this algebraic system within the Icosian Game. This interpretation provides a concrete method for deriving new symbolic equations such as those mentioned in the following excerpt. The lithograph referenced by Hamilton in this excerpt appears in [4].

HINTS ON THE ICOSIAN CALCULUS (continued)

II. In a LITHOGRAPH, which was distributed in Section A of the British Association, during its Meeting at Dublin in 1857, Sir W. R. H. pointed out a few other symbolical results of the same kind: especially the equations $\lambda\mu^2\lambda = \mu\lambda\mu$, $\mu\lambda^2\mu = \lambda\mu\lambda$, $\lambda\mu^3\lambda = \mu^2$, $\mu\lambda^3\mu = \lambda^2$; and the formula $(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1$, which serves as a *common mathematical type* for the solution of *all cases* of the First Problem of the Game. He also gave at the same time an oral (and hitherto unprinted) account of his rules of *interpretation* of the principal symbols; which rules, with reference to the present Icosian Diagram (or ICOSIAN), may be briefly stated as follows:

1. The operation ι *reverses* (or reads backwards) a *line* of the figure; changing, for example, B C to C B.
2. The operation κ causes a line to *turn* in a particular direction round it final point; changing, for instance, B C to D C.
3. The operation λ changes a line considered as a *side* of a pentagon to the *following side* thereof, proceeding always *right-handedly* for every pentagon except the large or outer one; thus λ changes B C to C D, but S R to R W.
4. The operation μ is *contrasted* with λ, and changes a line considered as a side of a *different pentagon*, and in the *opposite order* or rotation, to the consecutive side of that *other* pentagon; thus μ changes B C to C P, and S R to R Q; but it changes also R S to S T, whereas λ would change R S to S N.
5. The only operations employed in the *game* are those marked λ and μ; but another operation, $\omega = \lambda\mu\lambda\mu\lambda = \mu\lambda\mu\lambda\mu$, having the property that $\omega^2 = 1$, was also mentioned in the Lithograph above referred to; and to complete the present statement of *interpretations*, it may be added that the effect of this operation ω is to change an *edge* of a pentagonal *dodecahedron* to the *opposite edge* of that *solid*; for example, in the diagram, B C to T V.

Note that proceeding 'right handedly' may be described as moving clockwise around the appropriate pentagon, so that action which is 'in the opposite order' of proceeding 'right handedly' may be described as moving counter-clockwise.

Task 9. Use the interpretation of ι in (1) to explain why $\iota^2 = 1$.

Begin by looking at the effect of applying the operation ι *twice in succession*, beginning with the edge BC. Then explain in general.

Task 10. Use the interpretation of κ in (2) to explain why $\kappa^3 = 1$.

Begin by looking at the effect of applying the operation κ *three times in succession*, beginning with the edge $B\,C$. Then look at the effect of applying the operation κ *three times in succession*, beginning with the edge $P\,N$. Finally, explain in general.

Task 11. Use the interpretation of λ in (3) to explain why $\lambda^5 = 1$.

Begin by looking at the effect of applying the operation λ *five times in succession*, beginning with the edge $B\,C$. Then look at the effect of applying the operation λ *five times in succession*, beginning with the edge $S\,R$. Finally, explain in general.

Task 12. Use the interpretation of μ in (4) to explain why $\mu^5 = 1$.

Begin by looking at the effect of applying the operation μ *five times in succession*, beginning with the edge $B\,C$. Then look at the effect of applying the operation μ *five times in succession*, beginning with the edge $R\,S$. Finally, explain in general. (Note that this provides a geometric solution of the extra credit question stated above, immediately following Task 8.)

Task 13. Beginning with the edge BC, use the interpretations given for the four symbols $\iota, \kappa, \lambda, \mu$ to illustrate that $\mu = \iota\kappa^2 = \lambda\iota\lambda$.

Bonus Task Establish the equation $\lambda\mu^2\lambda = \mu\lambda\mu$ symbolically; then illustrate this equation in the Icosian Game, first beginning with the edge BC and then beginning with the edge RS.

Some Closing Remarks

Notice Hamilton's claim that an *algebraic proof* using equations of the type $(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1$ can be used to find all Hamiltonian cycles beginning with a specified initial sequence of five vertices. In general, however, it is not viable to associate an algebraic system with an arbitrary graph as a means to find all Hamiltonian circuits within that graph. In fact, a graph may contain no Hamiltonian circuits at all. Unlike the known situation for other kinds of circuits (e.g., Euler circuits), there is no known simple condition on a graph which allows one to determine in all cases whether a Hamiltonian circuit exists or not. In the case that a graph does contain a Hamiltonian circuit, we say the graph is *Hamiltonian.*

The more general question of determining a condition under which a graph is Hamiltonian was first studied by Thomas Penyngton Kirkman (1840–1892). Unlike Hamilton, who was primarily interested in the algebraic connections of one specific graph, Kirkman was interested in the general study of 'Hamiltonian circuits' in arbitrary graphs. The rector of a small and isolated English parish, Kirkman presented a paper on this subject to the Royal Society on 6 August 1855. Regrettably, his solution of the problem was incorrect. He did, however, present a second paper in 1856 in which he described a general class of graphs which do not contain such a circuit. Kirkman also studied the existence of Hamiltonian circuits on the dodecahedron, a variation of the Icosian Game which Hamilton also studied. In fact, the two men met once in 1861 when Hamilton visited Kirkman at his rectory. That Hamilton's name became associated with the circuits, and not Kirkman's, appears to be one of the accidents of history, or perhaps a credit to the fame of Hamilton's quarternions and work in mathematical physics.

References

[1] Barnett, J., Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game, in Part II: Historical Projects in Discrete Mathematics and Computer Science of *Resources for Teaching Discrete Mathematics: Classroom Project, History Modules, and Articles*, MAA Notes #74, editor B. Hopkins, Washington, DC: Mathematical Association of America, pp. 217–224.

[2] Biggs, N., Lloyd, E., Wilson, R., *Graph Theory: 1736–1936*, Clarendon Press, Oxford, 1976.

[3] Crowe, M. J., *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, Dover Publications, New York, 1994.

[4] Hamilton, W. R., "Account of the Icosian Game," *Proc. Roy. Irish. Acad.* **6** (1853-7), 415–416.

[5] Hopkins, B. (editor), *Resources for Teaching Discrete Mathematics: Classroom Project, History Modules, and Articles*, MAA Notes #74, Washington, DC: Mathematical Association of America.

[6] Katz, V., *A History of Mathematics: An Introduction*, Second Edition, Addison-Wesley, New York, 1998.

Notes to the Instructor

This project contains two sub-sections "The Icosian Game and Hamiltonian Circuits" and "The Icosian Game and Non-Commutative Algebra," both of which were developed specifically for use in an introductory undergraduate course in discrete mathematics. Because no prior background in graph theory is assumed, the connection to symbolic algebra makes the project suitable for use in a junior-level abstract algebra course as well. In a discrete mathematics course, the project could be assigned independently or in conjunction with one or both of the projects "Early Writings on Graph Theory: Euler Circuits and The Königsberg Bridge Problem" and "Early Writings on Graph Theory: Topological Connections," both of which appear in this volume. For students with no prior knowledge of non-commutative algebras, the instructor may wish to provide more explicit directions for Questions 8 and 9, or work these together as a whole class. Otherwise, the project may be completed by students working in small groups over 2–3 in-class days, or assigned as a week-long individual project outside of class. Multiple copies of the Icosian Game diagram will be needed for each student; use of color pencils or markers is also highly recommended.