The Poisson's summation formula

-How to explain the aliasing-

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In a nutshell, the Poisson summation's formula is the link between the Fourier transform of a signal and the Fourier transform of the (interpolated) sampled signal. For the sake of simplicity, and without loss of generality (w.l.o.g.) in the sequel we consider only one dimensional signals (you may assume this signal represents a sound). The whole discussion extends, by separability, to N-dimensional signals, e.g., images. As usual, w.l.o.g. we shall treat the case of functions sampled at a unit rate, i.e., the observed samples are $\dots, f(-1), f(0), f(1), \dots$. The general case, is easily deduced using the Fourier/zoom formula.

We recall that sampling is the art of recording a signal $f : \mathbb{R} \to \mathbb{R}$ on a device that can record discrete entities only, e.g. $\dots, f(-1), f(0), f(1), \dots$.

We compare the (modulus or spectrum) of the Shannon-Whittaker interpolated signal obtained from the two collection of samples $\dots, f(-2), f(-1), f(0), f(1), f(2), \dots$ and $\dots, f(-4), f(-2), f(0), f(2), f(4), \dots$ and obtain figure 1.

The seeming odd effect we see on the right panel of figure 1 deserves an explanation. To this aim, from a collection of samples of f we define the Shannon-Whittaker interpolate given by

$$\mathcal{I}f(x) := \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc(x-n)} \qquad \forall x \in \mathbb{R}, \tag{1}$$

where $\mathrm{sinc(x)} := \frac{\sin(\pi x)}{x}$. (You can assume, e.g., that $(f(n)) \in \ell^2(\mathbb{Z})$). We recall the Shannon-Whittaker sampling theorem: if f is $[-\pi, \pi]$ band limited[1], i.e., if $\hat{f}$ satisfies $\hat{f}(\xi) = 0$ for every $\xi \in \mathbb{R}\backslash[-\pi, \pi]$, then

$$\mathcal{I}f = f. \tag{2}$$

From (1) it is easy to deduce that

$$\widehat{\mathcal{I}f}(\xi) = \sum_{n\in\mathbb{Z}} f(n)e^{-in\xi}\mathbb{1}_{[-\pi,\pi]}(\xi) \qquad \forall \xi \in \mathbb{R}. \tag{3}$$

It is worth noticing that, from (3), we have that any Fourier transform of Shannon-Whittaker interpolated signal is supported on $[-\pi, \pi]$. It should be no surprise: equation (2) is valid for $[-\pi, \pi]$ band limited signals and therefore $\mathcal{I}f$ must be band limited.

We begin by stating, formally, the Poisson's summation formula. In its simplest form the Poisson's summation formula reads as

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(2\pi m). \tag{4}$$

From (4) and the Fourier translation/modulation formula we obtain

$$\sum_{n\in\mathbb{Z}} f(n)e^{-in\xi} = \sum_{m\in\mathbb{Z}} \hat{f}(2\pi m + \xi) \qquad \forall \xi \in [-\pi, \pi]. \tag{5}$$

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[1]Our convention, as usual, is $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$.

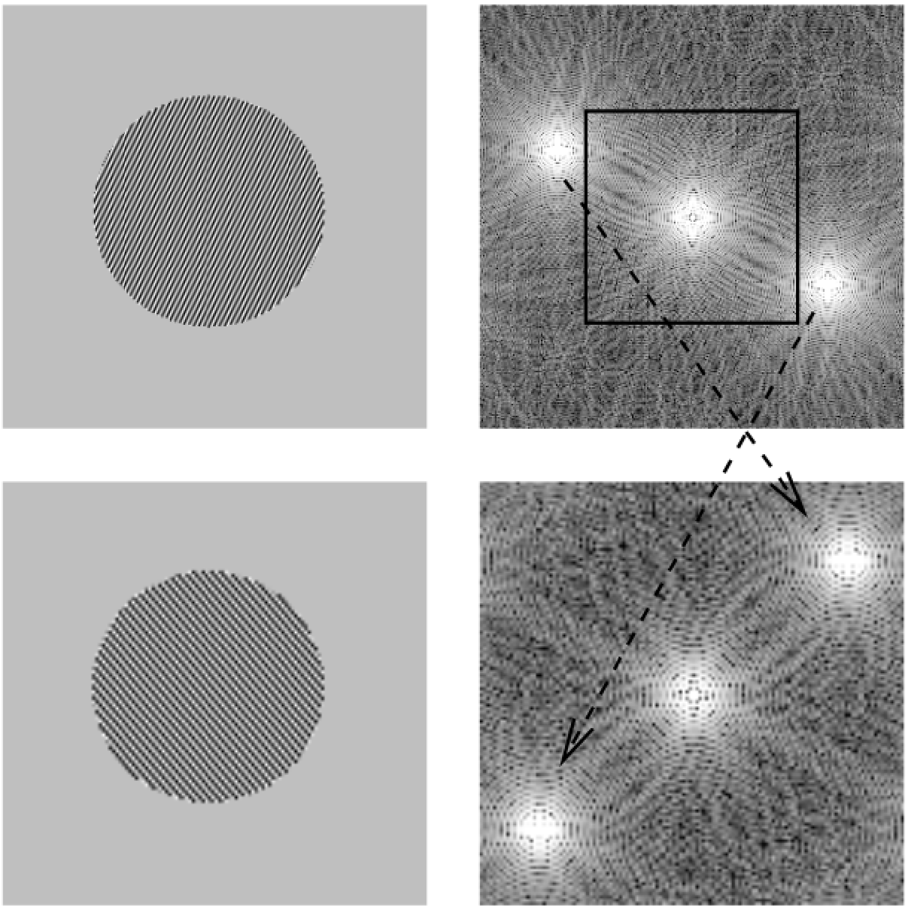

Figure 1: Top left panel: the signal made when observing $\dots, f(-1), f(0), f(1), \dots$. Bottom left panel: the signal made when observing $\dots, f(-4), f(-2), f(0), f(2), f(4), \dots$. Top right panel: the Fourier transform of the interpolated signal of the samples $\dots, f(-1), f(0), f(1), \dots$. Top right panel: the Fourier transform of the interpolated signal of the samples$\dots, f(-4), f(-2), f(0), f(2), f(4), \dots$

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The left-hand side (LHS) of (5), namely the function of the frequency variable ξ given by $g_1 : \mathbb{R} \ni \xi \mapsto \sum_{n \in \mathbb{Z}} f(n) e^{-in\xi}$ (obviously) only depends on the samples $\dots, f(-1), f(0), f(1), \dots$ of the observed signal. In fact, from (3), we have that g_1 coincides with $\hat{f}$ on the interval $[-\pi, \pi]$. In other words, for every $\xi \in [-\pi, \pi]$ the LHS of (5) is the Fourier transform of the Shannon-Whittaker interpolated signal namely $\mathcal{I}f$.

Consider the function of the frequency variable ξ given by $g_2 : \mathbb{R} \ni \xi \mapsto \sum_{m \in \mathbb{Z}} \hat{f}(2\pi m + \xi)$. We first notice that the restriction of g_2 to $[-\pi, \pi]$ is obviously the right-hand side of (5). We also notice that g_2 is, by construction, 2π periodic. In fact, f_2 is the 2π periodized function obtained from $\hat{f}$. Indeed, g_2 is made by summing every $2\pi m$ (for every $m \in \mathbb{Z}$) translated version of $\hat{f}$. We therefore deduce in important fact about sampling: sampling *any* signal makes its Fourier transform periodic.

Now, if $\hat{f}(\xi)$ is $[-\pi, \pi]$ band limited then, by definition, we have $\hat{f}(2\pi m + \xi) = 0$ for any $m \in \mathbb{Z} \backslash \{0\}$. This means that $g_2(\xi) = \hat{f}(\xi)$ for every $\xi \in \mathbb{R}$. In addition, from (5) we obtain that $g_1(\xi) = \hat{f}(\xi)$ for every $\xi \in [-\pi, \pi]$. Here, it is easy to see why the Shannon-Whittaker theorem is valid for $[-\pi, \pi]$ band-limited functions. Indeed, multiplying (5) by in indicator function of

$[-\pi,\pi]$ we obtain

$$\sum_{n\in\mathbb{Z}} f(n)e^{-in\xi}\mathbb{1}_{[-\pi,\pi]}(\xi) = \sum_{m\in\mathbb{Z}} \hat{f}(2\pi m+\xi)\mathbb{1}_{[-\pi,\pi]}(\xi) \qquad \forall\xi\in\mathbb{R}. \tag{6}$$

Now, from (3) we clearly see that the LHS of (6) is $\widehat{\mathcal{I}f}$. Under the assumption that f is $[-\pi,\pi]$ band limited the RHS of (6) is $\hat{f}$. Hence, the Fourier transform of $\mathcal{I}f$ equals $\hat{f}$. Since the Fourier transform is a bijection, we deduce that $\mathcal{I}f$ coincides with f on $\mathbb{R}$ and the equality in (2) is immediate.

Consider now a signal f that is not $[-\pi,\pi]$ band limited. If f that is not $[-\pi,\pi]$ then $g_2 : \mathbb{R}\ni\xi\mapsto\sum_{m\in\mathbb{Z}}\hat{f}(2\pi m+\xi)\neq\hat{f}(\xi)$ because there's at least one $m\neq 0$ such that $\hat{f}(2\pi m+\xi)\neq 0$ for some $\xi\in[-\pi,\pi]$. Hence, $\hat{f}$ and g_2 do not coincide on $[-\pi,\pi]$ because there exists some $\xi\in[-\pi,\pi]$ such that $\hat{f}(\xi)\neq g_2(\xi)$. Since, again, the Fourier transform is a bijection, we deduce that $\mathcal{I}f\neq f$. The $\hat{f}(2\pi m+\xi)\neq 0$ are called alias of $\hat{f}$ and we say that $\mathcal{I}f$ is aliased. This completely explain the effect we see in figure 1.

To enhance the fact that re-sampling a well-sampled signal can produce aliasing we consider the following setup: we store in a computer the unit-rate samples $\dots, f(-1), f(0), f(1),\dots,$ of a $[-\pi,\pi]$ band limited signal f. Shannon-Whittaker theorem applies and from the Poisson's summation formula we can verify that there's no aliasing. We now consider an operation that re-samples this signal (e.g., a sub-sampling of a factor 2). From the Fourier transform/zoom formula and (5) we have that for any $a>0$

$$\sum_{n\in\mathbb{Z}} f(an)e^{-in\xi} = \frac{1}{a}\sum_{m\in\mathbb{Z}}\hat{f}\left(\frac{2\pi m+\xi}{a}\right) \qquad \forall\xi\in[-\pi,\pi]. \tag{7}$$

When zooming out we have $a>1$ (and zooming in $a<1$). For instance if $a=2$ this just means that we re-sample f and keep only $\dots, f(-4), f(-2), f(0), f(2), f(4),\dots$. The RHS of (7) reads as $\frac{1}{2}\sum_{m\in\mathbb{Z}}\hat{f}\left(\pi m+\frac{\xi}{2}\right)$ and if f is $[-\pi,\pi]$ band limited we have

$$\sum_{n\in\mathbb{Z}} f(an)e^{-in\xi} = \frac{1}{2}\left[\underbrace{\hat{f}\left(-\pi+\frac{\xi}{2}\right)}_{\text{Alias}} + \hat{f}\left(\frac{\xi}{2}\right) + \underbrace{\hat{f}\left(\pi+\frac{\xi}{2}\right)}_{\text{Alias}}\right] \qquad \forall\xi\in[-\pi,\pi]. \tag{8}$$

From (8) it is easy to see the aliasing term that corrupt the Fourier transform we can calculate from the $f(2n)$ samples.

We point out the fact that many operations on signals or images can produce aliasing. For instance, when creating a panorama one will likely have to resamples many images and each of these resampling operations can produce aliased images. To avoid aliasing when zooming out a signal: do not forget to apply a low pass filter before modifying the sampling grid. For images, it is better to use a Gaussian filter with standard-deviation $\approx 0.8\sqrt{\text{zoom}-\text{factor}^2-1}$.

It is important to remember that aliasing is non-invertible. In other words, aliasing destroys information. Indeed, at some frequency $\xi\in[-\pi,\pi]$ we observe $g_2(\xi)$ which is the sum of (at least) two complex numbers $f(\xi_1)$ and $f(\xi_2)$ (and $\xi_1=\xi_2+2\pi m$ for some $m\in\mathbb{Z}$). If aliasing was invertible, from the sum $z=z_1+z_2$ of two numbers we would be able uniquely determine z_1 and z_2 which is obviously impossible.

Lastly, we see that aliasing is the name used in signal/image processing communities but the same degradation can also arise in numerical analysis when numerically estimating e.g. an integral. Indeed, $\hat{f}(0)=\int_{\mathbb{R}}f(x)dx$ and if f is $[-\pi,\pi]$ band limited then from (4) $\sum_n f(n) = \int_{\mathbb{R}}f(x)dx$. This means that the Riemann sum $\sum_n f(n)$ is exactly equal to the integral of f. In

Figure 2: Two examples of aliased images. As proved above, aliased images are not representative of the real-worlds observed light-field and calculations or detections using theses images will be inaccurate. For the practitioner, it is useful to be able to detect if an image is obviously aliased and we recall the three clues of aliasing in images: 1) spectral folding of textures, 2) loss of connectivity of fine structures and 3) "stairs-like" effect.

general, the difference between the Riemann sum and $\int f$ comes from aliasing terms and can be evaluated thanks to the Poisson's summation formula.

We conclude this short note by showing two characteristic examples of aliased images, i.e., images that are samples on a too coarse grid as depicted in figure 2. The figure 3 depicts the images obtained by 1) Applying a Gaussian filter and 2) sub-sampling the images.

Figure 3: The images obtained by correctly zooming out images. The effects of aliasing are gone.