Viscous control of shallow elastic fracture

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This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/ relatedlink/jfm-keywords.pdf for the full list)

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1. Introduction

Here we review the literature as well as describe the problem in more detail. We have the vertical displacement h, the horizontal displacement g, the thickness of the arm l, and the pressure p. We look for a travelling wave solution (propagating left), with speed c.

2. Formulation of problem

From lubrication, we have Poiseulle flow in the crack. We obtain the flux, and conservation of mass as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3\,, \qquad \frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0\,, \tag{2.1}$$

which combined gives

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2\,. \tag{2.2}$$

Setting $p \to 0$ at $x \to \infty$, we can write this in integral form,

$$p(x) = -\int_x^\infty 12\mu c/h(\tilde{x})^2 \mathrm{d}\tilde{x}\,. \tag{2.3}$$

From the linear theory of elasticity, due to others who have studied this problem, we have

$$\begin{bmatrix} -\sigma_y \\ -\tau_{xy} \end{bmatrix} = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} = \frac{1}{l}\int_0^\infty \boldsymbol{K}\left(\frac{\tilde{x}-x}{l}\right)\begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix}\mathrm{d}\tilde{x}\,, \tag{2.4}$$

where the integral kernel is

$$\boldsymbol{K}(\xi) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{(32-24\xi^2)}{(\xi^2+4)^3} & \frac{(48\xi^2-64)}{\xi(\xi^2+4)^3} \\ -\frac{(16\xi^4+16\xi^2+4)}{\xi(\xi^2+4)^3} & -\frac{(32-24\xi^2)}{(\xi^2+4)^3} \end{bmatrix}. \tag{2.5}$$

The boundary conditions near $x=0$ are governed by fracture mechanics

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, h'(x)\,, \qquad K_{II} = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, g'(x)\,. \tag{2.6a, b}$$

As we go to $x \gg l$, we are looking at the problem of peeling off a thin strip from an elastic half space. We can then use beam theory approximations, which give

$$M(x) = \frac{El^3}{12(1-\nu^2)}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = \frac{El^3}{6(1-\nu^2)}\frac{\mathrm{d}g}{\mathrm{d}x}, \qquad p = \frac{El^3}{12(1-\nu^2)}h^{(4)}(x) \tag{2.7a, b}$$

As $x \to \infty$, $M(x) \to M$, the applied bending moment, so this gives us boundary conditions on h'', g'.

2.1. *Rescaling*

We can define the following dimensionless variables

$$x = l\xi, \quad h(x) = \frac{12M(1-\nu^2)}{El}H(\xi), \quad g(x) = \frac{12M(1-\nu^2)}{El}G(\xi)\,, \tag{2.8}$$

$$p = \frac{3M}{\pi l^2}\varPi(\xi), \quad K_I = Ml^{-3/2}\kappa_I, \quad K_{II} = Ml^{-3/2}\kappa_{II}, \quad \lambda = \frac{4\pi\mu p^* l^3}{M^2}\,. \tag{2.9}$$

With these scalings, the equations become

$$\begin{bmatrix} \varPi \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\tilde{\xi}-\xi)\begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi} \tag{2.10}$$

$$H^2\frac{\mathrm{d}\varPi}{\mathrm{d}\xi} = \lambda \quad \text{or} \quad \varPi(\xi) = -\int_\xi^\infty \lambda/H(\tilde{\xi})^2\mathrm{d}\tilde{\xi} \tag{2.11a, b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2}, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' = \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}G' = \kappa_{II}, \tag{2.12}$$

These shall be the governing equations for the rest of this paper.

The equations of the linear perturbation problem:

$$\varPi = \varPi_0 + \mathcal{E}\varPi_1 + O(\mathcal{E}), \quad H = H_0 + \mathcal{E}H_1 + O(\mathcal{E}) \tag{2.13}$$

$$\begin{bmatrix} \varPi_1 \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\xi-\tilde{\xi})\begin{bmatrix} G_1'(\tilde{\xi}) \\ H_1'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi}, \qquad H_0^2\varPi_1' + 2H_0H_1\varPi_0' = \lambda_1 \tag{2.14a, b}$$

$$H_1'' \to 0 \text{ as } \xi\to\infty, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3} + \ldots \text{ as } \xi \to 0 \tag{2.15a, b}$$

But these can be made into a more convenient form, by considering instead $\tilde{\varPi} = \varPi_0 - 3\lambda_0/\lambda_1\varPi_1$, and similar for $\tilde{H}$, $\tilde{G}$. The equations become

$$\begin{bmatrix} \tilde{\varPi} \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\xi-\tilde{\xi})\begin{bmatrix} \tilde{G}'(\tilde{\xi}) \\ \tilde{H}'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi}, \qquad H_0^2\tilde{\varPi}' + 2H_0\tilde{H}\varPi_0' = 0 \tag{2.16a, b}$$

$$\tilde{H}'' \to 1 \text{ as } \xi \to \infty, \qquad \tilde{H} \sim -\frac{3\lambda_0}{\lambda_1}\xi^s + \dots \text{ as } \xi \to 0 \tag{2.17a,b}$$

These are the equations for the two tip problem

$$\begin{bmatrix} \Pi \\ 0 \end{bmatrix} = \int_{-L}^{\infty} \boldsymbol{K}(\tilde{\xi}-\xi) \begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad \Pi = \int_{\xi}^{\infty} \lambda/H(\tilde{\xi})^2 \mathrm{d}\tilde{\xi} \tag{2.18a,b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2} \tag{2.19a,b}$$

$$\lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' = 0, \quad \lim_{\xi\to -L} 3\sqrt{2\pi\xi}G' = \kappa_{II} \tag{2.20a,b}$$

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking $n+1$ points $\boldsymbol{\xi} = (\xi_0 = 0, \xi_1 \dots, \xi_n)$ at which we measure H', G', and n intermediate points $\boldsymbol{\zeta} = (\zeta_0, \dots, \zeta_{n-1})$ at which to measure Π, so that $\xi_0 < \zeta_0 < \dots < \zeta_{n-1} < \xi_n$. We work with $\sqrt{\xi}G'(\xi)$, $\sqrt{\xi}H'(\xi)$ near the tip to avoid singularities. We define $\boldsymbol{\theta}_G = [\sqrt{\xi_0}G'(\xi_0), \dots, \sqrt{\xi_{t-1}}G'(\xi_{t-1}), G'(\xi_t), \dots G'(\xi_n)]$, $\boldsymbol{\theta}_H = [\sqrt{\xi_0}H'(\xi_0), \dots, \sqrt{\xi_{t-1}}H'(\xi_{t-1}), H'(\xi_t), \dots H'(\xi_n)]$, as well as $\boldsymbol{\theta} = [\boldsymbol{\theta}_G, \boldsymbol{\theta}_H]$, (where $\sqrt{\xi_0}G'(\xi_0) = \lim_{\xi\to\xi_0} \sqrt{\xi}G'(\xi)$). Typically $t \approx n/2$ was used. The elasticity integral is linear in G', H', and so the discretized integration is linear in $\boldsymbol{\theta}$. Such a linear relation may be written as

$$[\Pi(\zeta_1), \dots, \Pi(\zeta_{n-1}), \underbrace{0, \dots, 0}_{n-1}] = \boldsymbol{J\theta}\,. \tag{3.1}$$

By imposing $H(0) = 0$, and choosing a sensible interpolation, one can recover $H(\xi_i)$ from $\boldsymbol{\theta}_H$. Therefore, from the lubrication integral, there exists another expression for $[\Pi(\zeta_0), \dots, \Pi(\zeta_{n-1})]$ as some function of $\boldsymbol{\theta}_H$. So we can write

$$[\Pi(\zeta_1), \dots, \Pi(\zeta_{n-1}), \underbrace{0, \dots, 0}_{n-1}] = \boldsymbol{J\theta} = \boldsymbol{f}(\boldsymbol{\theta}_H)\,, \tag{3.2}$$

for some function $\boldsymbol{f}$.

Both $G'(\xi_n)$, and $H''(\xi_n)$ are known from our beam theory asymptotic expansion. But these are linear in $\boldsymbol{\theta}$, as $G'(\xi_n) = \theta_n$, and $H''(\xi_n) \approx (\theta_{2n} - \theta_{2n-1})/(\xi_n - \xi_{n-1})$, Therefore we can add another two rows to $\boldsymbol{J}$, so that

$$\boldsymbol{A\theta} = [\boldsymbol{f}(\boldsymbol{\theta}), G'(\xi_n), H''(\xi_n)]\ . \tag{3.3}$$

Where the $\boldsymbol{A}$ is the enlarged matrix. This can be solved by Newton's method from quite arbitrary initial guesses.

For $\xi_i < \xi < \xi_{i+1}$, we interpolate as

$$G'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad H'(\xi) = \begin{cases} \xi^{-1/2}(c_i\xi^{1/2} + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for} \quad \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.4}$$

The choice of interpolating function was based on the appearance of the relevant functions. We will also define a_n, b_n, c_n, d_n for interpolation beyond ξ_n. With this choice of interpolation, there exist exact closed form expressions for both the lubrication integral, and the elasticity integral, in terms of the $a_i - d_i$ coefficients.

We therefore want to determine $a_i - d_i$ in terms of $\boldsymbol{\theta}$. Continuity of G', H' imposes $2(n-1)$ equations. For G' they are

$$\begin{aligned} a_i\xi_{i+1} + b_i &= a_{i+1}\xi_{i+1} + b_{i+1} = \theta_{i+1}, \text{ for } i < n,\, i \neq t-1 \\ \xi^{-1/2}(a_{t-1}\xi_t + b_{t-1}) &= a_t\xi_t + b_t = \theta_t, \end{aligned} \tag{3.5}$$

with similar equations for H' (accounting for the slightly different interpolation). We also have the $2n$ equations following from the definition of $\boldsymbol{\theta}$, such as $a_i\xi_i + b_i = \theta_i$ for $t \leqslant i \leqslant n$.

From our asymptotic expansion (via beam theory) we know $\theta_n = G'(\xi_n)$ and $a_n = G''(\xi_n)$. Therefore we can write

$$a_n = \frac{G''(\xi_n)}{G'(\xi_n)}\theta_n, \qquad b_n = \theta_n - a_n\xi_n = \left(1 - \frac{G''(\xi_n)}{G'(\xi_n)}\right)\theta_n \tag{3.6}$$

With H, we know that $c_n = H''(\xi_n)$, $c_{n-1} = H''(\xi_{n-1})$, and so we have that

$$c_n = \frac{H''(\xi_n)}{H''(\xi_{n-1})}c_{n-1}, \qquad d_n = -c_n\xi_n + c_{n-1}\xi_n + d_{n-1} \tag{3.7}$$

Therefore, we have enough equations to calculate a matrix $\boldsymbol{T}$, so that

$$[a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n] = \boldsymbol{T\theta} \tag{3.8}$$

Note that we choose a value of λ, fix the boundary conditions at $\xi \to \infty$, then solve the problem and subsequently recover the boundary conditions at $\xi = 0$ (κ_I, κ_{II}). This can then be inverted, so that we think of $\lambda = \lambda(\kappa_I)$. Physically, we know κ_I, and want to find λ, but in numerically solving the problem, it makes more sense to choose λ and recover κ_I.

The spacing of the points should reflect that the important part of the problem is happening near the tip, and this is where the points should be concentrated. The spacing that was typically used in numerical calculations was

$$\xi_i = \tan^2(\chi\, i/m), \quad i = 1, \ldots, m < n \tag{3.9}$$

where χ is chosen so that $\tan^2(\chi) = O(10)$, and the remaining points are added in a geometric progression, so that

$$\xi_{i+1} = (\xi_m/\xi_{m-1})\xi_i, \quad i = m, \ldots, n-1 \tag{3.10}$$

3.2. *Linear Perturbation Problem*

From equation 2.17b, we anticipate a singularity of the form ξ^{s-1} in $\tilde{H}'$, (we still expect a $\xi^{-1/2}$ singularity in $\tilde{G}'$). Therefore, the interpolation used is

$$\tilde{G}'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad \tilde{H}'(\xi) = \begin{cases} \xi^{s-1}(c_i\xi + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for } \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.11}$$

Some of the integrals no longer have exact expressions. In this case, they are calculated by a numerical integration routine. We define $\tilde{\boldsymbol{\theta}}$ to be like $\boldsymbol{\theta}$, but with $\tilde{H}'$, $\tilde{G}'$ instead of H, G.

The lubrication equation for the linear perturbation problem (2.16b) may be written as

$$\tilde{\Pi}(\zeta) = \int_\zeta^\infty \frac{2\lambda_0\tilde{H}}{H_0^3}\mathrm{d}\xi\,. \tag{3.12}$$

This is linear in $\tilde{H}$, and so in $\tilde{\boldsymbol{\theta}}$. The integrals that appear were determined numerically. We end up with a matrix $\boldsymbol{R}$, such that

$$[\tilde{\Pi}(\zeta_1), \dots \tilde{\Pi}(\zeta_{n-1})] = \boldsymbol{R}\tilde{\boldsymbol{\theta}} \tag{3.13}$$

Padding out $\boldsymbol{R}$ with zeros, until it is of size $2n \times 2n$, we see that

$$\left(\boldsymbol{A} - \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix}\right)\boldsymbol{\theta} = [0, \dots, 0, \tilde{G}'(\xi_n), \tilde{H}''(\xi_n)] \tag{3.14}$$

Since we haven't changed the integral kernel, the beam theory asymptotics remain the same, and together with 3.12, we can calculate an asymptotic expression for $\tilde{H}''(\xi_n)$, $\tilde{G}'(\xi_n)$. Here, we do not need to deploy Newton's method, as we can simply solve the linear set of equations, 3.14.

3.3. *Double Tip*

In solving the problem of two tips situated at $-L$ and 0, it is given that $H = 0$ for $\xi < 0$, and thus $H' = 0$ for $\xi < 0$. We take n points to cover $0 \leqslant \xi < \infty$, and r points to cover $-L \leqslant \xi < 0$. we label these points so that

$$\boldsymbol{\xi} = [\xi_{-r} = -L, \xi_{1-r}, \dots, \xi_0 = 0, \xi_1, \dots, \xi_n]. \tag{3.15}$$

We interpolate G' expecting a $\xi^{-1/2}$ singularity at $\xi = -L$, and H' expecting a $\xi^{-1/2}$ singularity at $\xi = 0$. We do not calculate Π for $\xi < 0$ (although it is easily done), but just require that $\sigma_{xy} = 0$ for $\xi < 0$. This provides enough equations for the problem to be solved as before, with Newton's method.

Note that we input $-L$ and λ and recover κ_I, κ_{II}, where κ_I is measured at 0. Physically, for $L > 0$, we must have $\kappa_I = 0$. Numerically we solve for some λ, L, find $\kappa_I > 0$ and extrapolate to $\kappa_I = 0$.

The spacing of points for $\xi < 0$ was chosen so that there was a concentration of points near $-L$ and near 0.

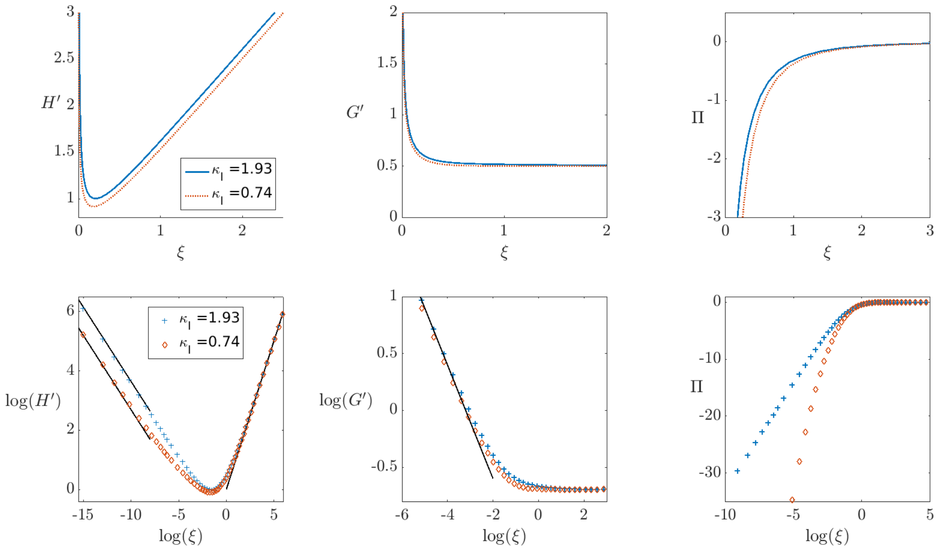

FIGURE 1. Numerical solutions for two typical κ_I values. log-log plots are shown for H',G', with solid lines indicating the predicted asymptotics; $\log(H') \approx -\frac{1}{2}\kappa_I \log(\xi)$, $\log(G') \approx -\frac{1}{2}\kappa_{II} \log(\xi)$ near $\xi = 0$, and $\log(H') \approx \log(\xi)$, $\log(G') \approx -\log(2)$, as $\xi \to \infty$. Figure produced with $n = 465$, $x_n = 819$.

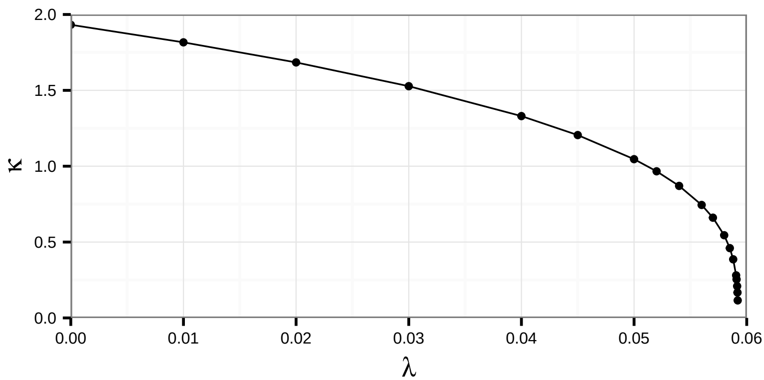

FIGURE 2. Here we vary the parameter λ and plot the change in κ_I. Figure produced with $n = 465$, $x_n = 819$.

4. Results

4.1. *Single tip*

Start off with some of the basic graphs showing H',G', and Π against ξ.

Obvious questions to ask at this point are: How are you sure this is the right answer, what is the effect of n, ξ_{end}? In the next graph, we determine the effect of extending ξ_{end}, by adding on extra points (so maintaining the same resolution near the tip). There is a satisfactory demonstration of convergence.

By adding points on in a geometric progression, it becomes quite cheap to extend out to $\xi_{\text{end}} \approx 800$ or so. Once one has done this, it is apparent that the effect of the tip resolution dominates the effect of finite truncation, as the following figure shows.

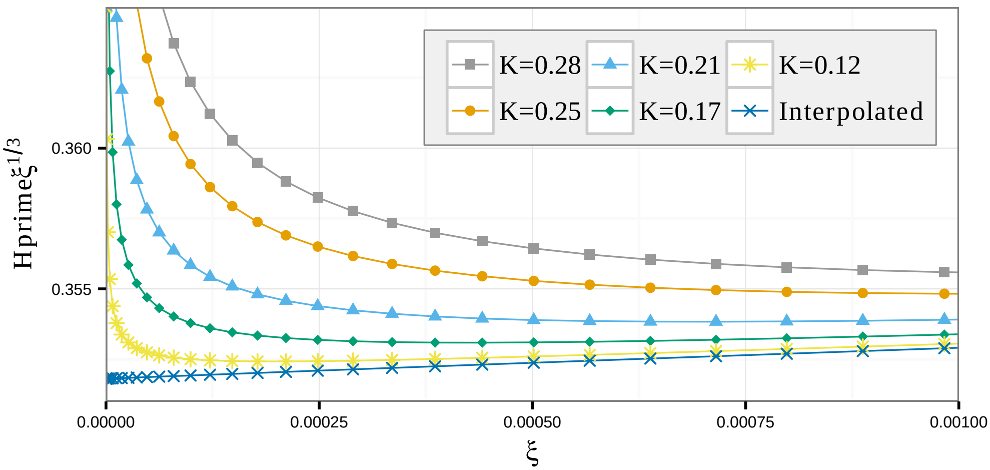

FIGURE 3. As $\kappa_I \to 0$, H' moves from a $\xi^{-1/2}$ singularity to a $\xi^{-1/3}$ singularity. We can not calculate $\kappa_I = 0$, but the extrapolation to it is shown. Figure produced with $n = 465$, $x_n = 819$.

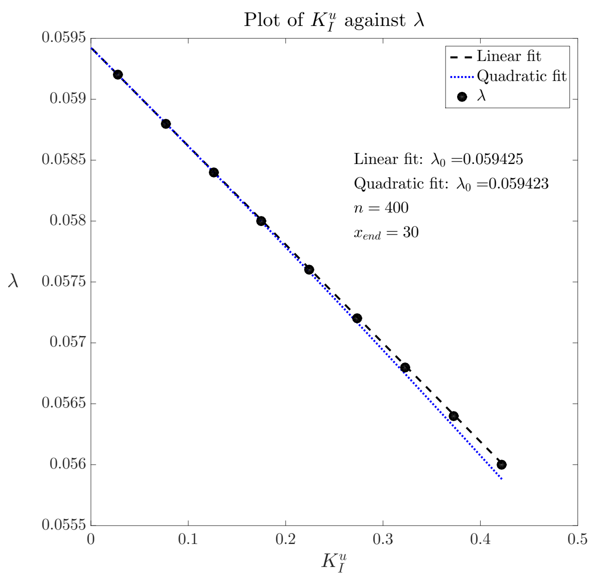

FIGURE 4. Good agreement between theory and numerics here

By increasing n (for large ξ_{end}), we have been able to determine λ_0 and D

4.2. *Linear perturbation problem*

We solve the linear perturbation problem. All that we really want to know is that we see the ξ^{s-1} behaviour that we expect, and we ask what the intercept of $\tilde{H}_1$ is. It is perhaps worth mentioning the difficulties in measuring the intercept and perhaps a notion of the sensitivity of the result on the estimate provided for H_0. Illustrating that is the next figure

Then we include the figure that shows convergence with different n values to something approaching the right answer.

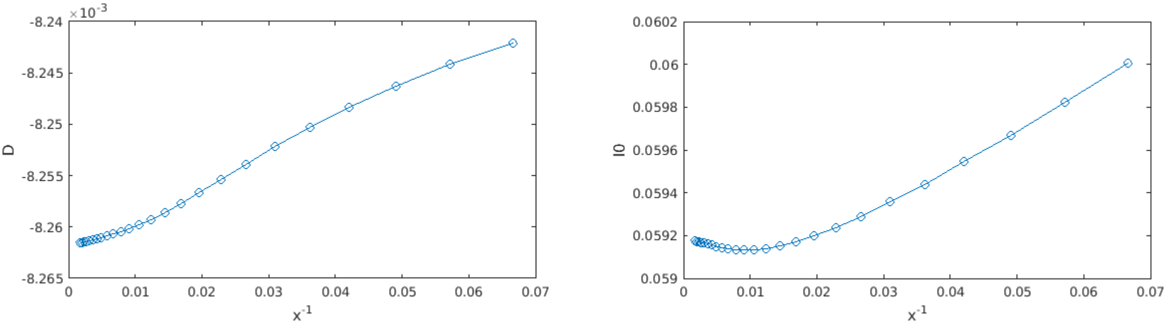

FIGURE 5. As we increase ξ_n, we can estimate the effect of finite truncation. The figure starts with $n = 400$, and increases ξ_n by adding more points until $n = 544$.

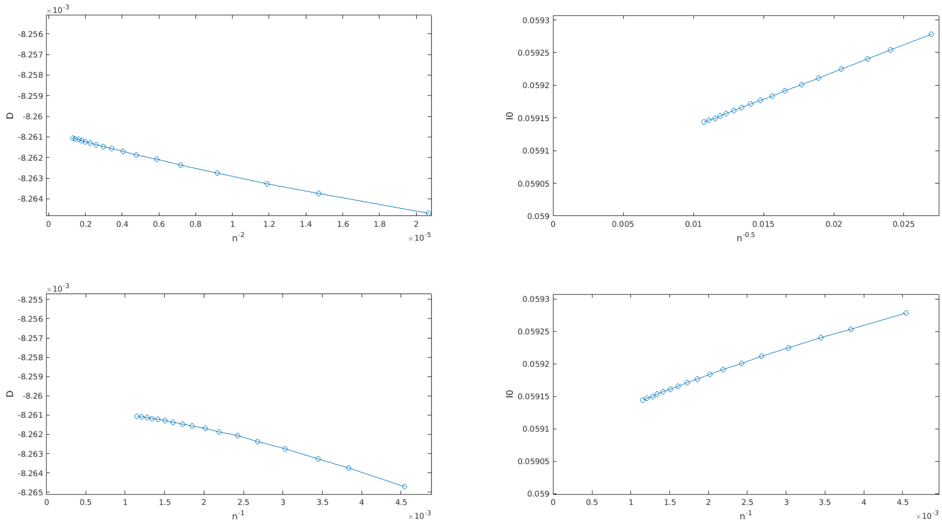

FIGURE 6. Our best guess at λ_0 and D, and the approximate error one can expect in them

4.3. *Two tips*

After the linear perturbation problem, we move on to the two tip problem. Perhaps some graphs that show an outline of the full numerical problem with non-zero κ_I and κ_{II}, although these are not physical.

What would be nice, although it doesn't exist yet, is some sort of record of how we now extrapolate to $\kappa_I = 0$. This is certainly a plot that needs to be made. We now move on to the $\kappa_I = 0$ set of relations.

5. Discussion

This is where we discuss the figures, possibly include more figures, and draw the results and conclusions of this paper.

Perhaps the first thing worth mentioning is the somewhat contrived, but pretty accurate formulae for λ in terms of κ_I. This holds for any toughness in the single tip case.

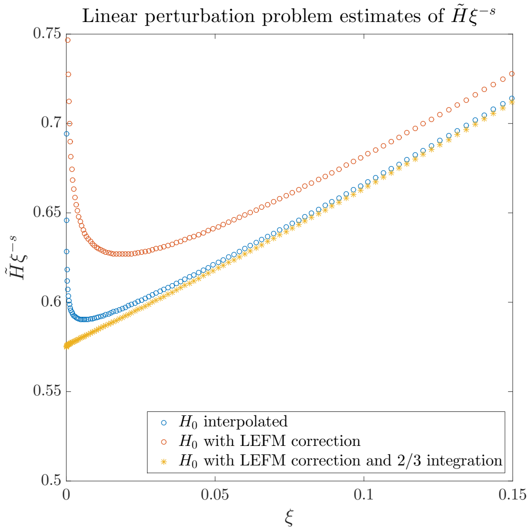

FIGURE 7. Sensitivity of linear perturbation problem on H_0

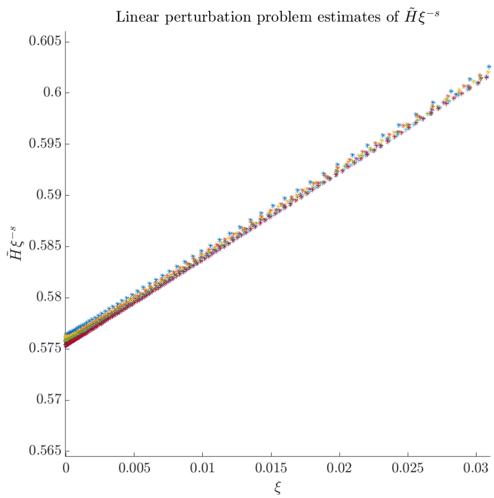

FIGURE 8. Sensitivity of linear perturbation problem on H_0

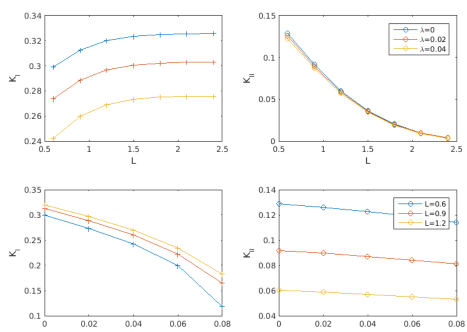

FIGURE 9. Visualisation of what is really a surface in 4D

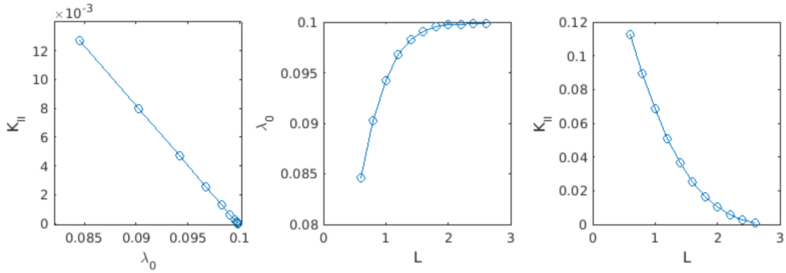

FIGURE 10. The results of extrapolating to $\kappa_I = 0$

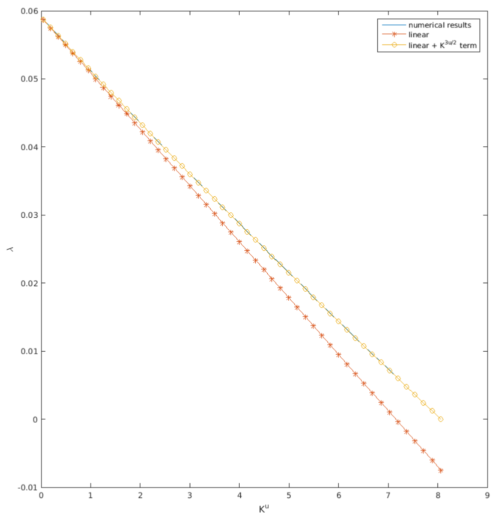

FIGURE 11. The formula valid for all κ_I

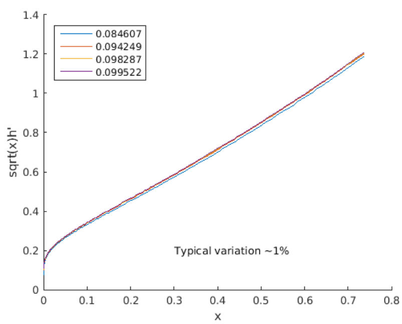

FIGURE 12. Demonstrating the decoupling of fluid and solid fracture

Then we could move on to talk about the decoupling between the fluid problem and the dry fracture problem. Relavent graphs to include would show that H really doesn't vary much with λ_0, and that given a reference H', one can construct G' with relative ease.

At this point, I would like to construct another contrived formulae for the two tip

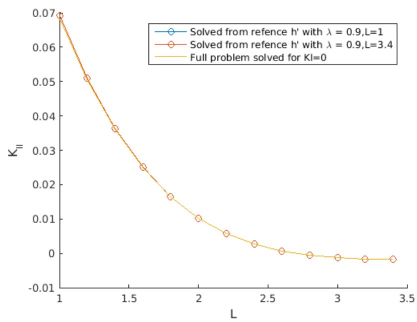

FIGURE 13. Reconstructing the full solution given a reference H'

problem. Then I would like to plot a graph of κ_I against κ_{II} in the full fluid problem. This provides a guide of when it is appropriate to take the single tip, and when it is appropriate to take the double tip.

a/d	$M=4$	$M=8$	Callan *et al.*
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

TABLE 1. Values of kd at which trapped modes occur when $\rho(\theta) = a$

6. Citations and references

All papers included in the References section must be cited in the article, and vice versa. Citations should be included as, for example "It has been shown (Rogallo 1981) that..." (using the `\citep` command, part of the natbib package) "recent work by Dennis (1985)..." (using `\citet`). The natbib package can be used to generate citation variations, as shown below.
`\citet[pp. 2-4]{Hwang70}`:
Hwang & Tuck (1970, pp. 2-4)
`\citep[p. 6]{Worster92}`:
(Worster 1992, p. 6)
`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
`\citep{Brownell04,Brownell07,Ursell50,Wijngaarden68,Miller91}`:
(Brownell & Su 2004, 2007; Ursell 1950; van Wijngaarden 1968; Miller 1991)
The References section can either be built from individual `\bibitem` commands, or can be built using BibTex. The BibTex files used to generate the references in this document can be found in the zip file at http://journals.cambridge.org/data/relatedlink/jfm-ifc.zip.
Where there are up to ten authors, all authors' names should be given in the reference list. Where there are more than ten authors, only the first name should appear, followed by et al.

Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

Appendix A

This appendix contains sample equations in the JFM style. Please refer to the LaTeX source file for examples of how to display such equations in your manuscript.

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \tag{A 1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \nabla^2 P = \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{w}). \tag{A 2}$$

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \quad r \equiv |P - Q| \rightarrow 0, \tag{A 3}$$

$$\left.\begin{aligned} \frac{\partial G_s}{\partial y} = 0 \quad \text{on} \quad y=0, \\ G_a = 0 \quad \text{on} \quad y = 0, \end{aligned}\right\} \tag{A 4}$$

$$-\frac{1}{2\pi}\int_0^\infty \gamma^{-1}[exp(-k\gamma|y-\eta|)+\exp(-k\gamma(2d-y-\eta))]\cos k(x-\xi)t\,\mathrm{d}t, \qquad 0<y, \quad \eta<d, \tag{A 5}$$

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{1/2}, & t \leqslant 1 \\ (t^2-1)^{1/2}, & t>1. \end{cases} \tag{A 6}$$

$$-\frac{1}{2\pi}\int_0^\infty B(t)\frac{\cosh k\gamma(d-y)}{\gamma \sinh k\gamma d}\cos k(x-\xi)t\,\mathrm{d}t$$

$$G = -\frac{1}{4}\mathrm{i}(H_0(kr)+H_0(kr_1)) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-k\gamma d}}{\gamma\sinh k\gamma d}\cosh k\gamma(d-y)\cosh k\gamma(d-\eta) \tag{A 7}$$

Note that when equations are included in definitions, it may be suitable to render them in line, rather than in the equation environment: $\boldsymbol{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G_a}$ where $r = \{[x(\theta)-x(\psi)]^2 + [y(\theta)-y(\psi)]^2\}^{1/2}$ and $\widetilde{G_a}$ is regular as $kr \to 0$. However, any fractions displayed like this, other than $\frac{1}{2}$ or $\frac{1}{4}$, must be written on the line, and not stacked (ie 1/3).

$$\begin{aligned} \frac{\partial}{\partial n_q}\left(\frac{1}{4}Y_0(kr)\right) &\sim \frac{1}{4\pi w^3(\theta)}[x''(\theta)y'(\theta) - y''(\theta)x'(\theta)] \\ &= \frac{1}{4\pi w^3(\theta)}[\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as} \quad kr \to 0. \end{aligned} \tag{A 8}$$

$$\frac{1}{2}\phi_i = \frac{\pi}{M}\sum_{j=1}^{M}\phi_j K_{ij}^a w_j, \qquad i = 1, \ldots, M, \tag{A 9}$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i, \theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G_a}(\theta_i, \theta_i)/\partial n_q + [\rho_i'\rho_i'' - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \tag{A 10}$$

$$\rho_l = \lim_{\zeta \to Z_l^-(x)} \rho(x,\zeta), \quad \rho_u = \lim_{\zeta \to Z_u^+(x)} \rho(x,\zeta) \tag{$A\,11a, b$}$$

$$(\rho(x,\zeta), \phi_{\zeta\zeta}(x,\zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \tag{A 12}$$

$$\tau_{ij} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + (\overline{\overline{u}_i u_j^{SGS} + u_i^{SGS}\overline{u}_j}) + \overline{u_i^{SGS}u_j^{SGS}}, \tag{A 13a}$$

$$\tau_j^\theta = (\overline{\overline{u}_j\overline{\theta}} - \overline{u}_j\overline{\theta}) + (\overline{\overline{u}_j\theta^{SGS} + u_j^{SGS}\overline{\theta}}) + \overline{u_j^{SGS}\theta^{SGS}}. \tag{A 13b}$$

$$\boldsymbol{Q}_C = \begin{bmatrix} -\omega^{-2}V_w' & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\ \dfrac{\beta}{\alpha\omega^2}V_w' & 0 & 0 & 0 & \mathrm{i}\omega^{-1} \\ \mathrm{i}\omega^{-1} & 0 & 0 & 0 & 0 \\ \mathrm{i}R_\delta^{-1}(\alpha^t + \omega^{-1}V_w'') & 0 & -(\mathrm{i}\alpha^t R_\delta)^{-1} & 0 & 0 \\ \dfrac{\mathrm{i}\beta}{\alpha\omega}R_\delta^{-1}V_w'' & 0 & 0 & 0 & 0 \\ (\mathrm{i}\alpha^t)^{-1}V_w' & (3R_\delta^{-1} + c^t(\mathrm{i}\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0 \end{bmatrix}. \tag{A 14}$$

$$\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[\mathrm{i}(\alpha^t x_1^t - \omega t)], \tag{A 15}$$

where $\hat{\boldsymbol{\eta}}^t = \boldsymbol{b}\exp(\mathrm{i}\gamma x_3^t)$.

$$\mathrm{Det}[\rho\omega^2\delta_{ps} - C_{pqrs}^t k_q^t k_r^t] = 0, \tag{A 16}$$

$$\langle k_1^t, k_2^t, k_3^t\rangle = \langle \alpha^t, 0, \gamma\rangle \tag{A 17}$$

$$\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{A 18}$$

$$f(\psi_1) = \frac{3b}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\frac{(\sin\psi_1 - \sin\psi)(a+b\cos\psi)^{1/2}}{[1-\cos(\psi_1-\psi)](2+\alpha)^{1/2}}\mathrm{d}x, \tag{A 19}$$

$$\begin{aligned} g(\psi_1) = {}& \frac{3}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\left(\frac{a+b\cos\psi}{2+\alpha}\right)^{1/2}\Bigg\{f(\psi)[(\cos\psi_1 - b\beta_1)S + \beta_1 P] \\ & \times \frac{\sin\psi_1 - \sin\psi}{1-\cos(\psi_1-\psi)} + g(\psi)\left[\left(2+\alpha - \frac{(\sin\psi_1-\sin\psi)^2}{1-\cos(\psi-\psi_1)} - b^2\gamma\right)S\right. \\ & \left. + \left(b^2\cos\psi_1\gamma - \frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi,\delta) - (2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi,\delta)\right]\Bigg\}\mathrm{d}\psi, \end{aligned} \tag{A 20}$$

$$\alpha = \alpha(\psi,\psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}, \quad \beta - \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \tag{A 21}$$

$$\left.\begin{aligned} H(0) &= \frac{\epsilon\overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}, \quad H'(0) = -1 + \epsilon^{2/3}\overline{C}_u + \epsilon\hat{C}_u'; \\ H''(0) &= \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2}u_P^2}, \quad H'(\infty) = 0. \end{aligned}\right\} \tag{A 22}$$

LEMMA 1. *Let $f(z)$ be a trial Batchelor (1971, pp. 231–232) function defined on* $[0,1]$. *Let Λ_1 denote the ground-state eigenvalue for $-\mathrm{d}^2g/\mathrm{d}z^2 = \Lambda g$, where g must satisfy $\pm\mathrm{d}g/\mathrm{d}z + \alpha g = 0$ at $z = 0, 1$ for some non-negative constant α. Then for any f that is not identically zero we have*

$$\frac{\alpha(f^2(0) + f^2(1)) + \displaystyle\int_0^1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\mathrm{d}z}{\displaystyle\int_0^1 f^2\mathrm{d}z} \geqslant \Lambda_1 \geqslant \left(\frac{-\alpha + (\alpha^2 + 8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \tag{A 23}$$

COROLLARY 1. *Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies*

$$\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{A 24}$$

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