Small toughness solution

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1 Setup

Recall we have a system goverened by the equations

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \int_0^\infty \begin{pmatrix} K_{11}(x-z) & K_{12}(x-z) \\ K_{21}(x-z) & K_{22}(x-z) \end{pmatrix} \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$p(z) = -\int_z^\infty \frac{\lambda}{h(x)^2} dx \tag{2}$$

where the kernel terms are given by

$$K_{11}(z) = \frac{32-24z^2}{(z^2+4)^3} \qquad K_{12}(z) = \frac{48z^2-64}{z(z^2+4)^3}$$

$$K_{21}(z) = -\frac{(16z^3+16z^2+4)}{z(z^2+4)^3} \qquad K_{22}(z) = -\frac{(32-24z^2)}{(z^2+4)^3}$$

with boundary conditions at infinity

$$h''(x) \to 1, \quad g'(x) \to \frac{1}{2}$$

For a given speed parameter λ, we wish to find the material toughness K, which is given by

$$K(\lambda) = \lim_{x\to 0} 3\sqrt{2\pi}\sqrt{x}\, h'(x)$$

We are interested in the value $\lambda = \lambda_0$ for which $K(\lambda_0) = 0$, since this is then the propagation speed of a zero-toughness system. We are also interested in $\lambda \approx \lambda_0$.

2 Zero toughness solution

Consider setting $K = 0$. We investigate only the nature of the solution near $x = 0$. We suspect, (and will verify later) that p is singular near the crack tip. Thus in equation 1, we can neglect terms that are non singular.

$$p(z) = \int_0^\infty \frac{h'(x)}{x-z} dx$$

Also have that $p' = \lambda/h^2$. We try the ansatz $h \sim x^\alpha$. From our two equations linking h and p, this gives that

$$p \sim x^{\alpha-1}$$

$$p' \sim x^{-2\alpha}$$

and so $\alpha = 2/3$. We have made use of the integral

$$\int_0^\infty \frac{z^{s-1}}{z-x}dz = -\pi\cot(\pi s)z^{s-1}$$

So starting with a solution $h_0 = A_0 x^{2/3} + o(x^{2/3})$ near the crack tip, we get $p_0(x) = -\frac{3\lambda_0}{A_0^2 x^{1/3}} + O(1)$

References

[1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution,* Journal of Applied Mechanics, 2005.