Small toughness solution

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July 11, 2016

1 Setup

Recall we have a system goverened by the equations

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \int_0^\infty \begin{pmatrix} K_{11}(x-z) & K_{12}(x-z) \\ K_{21}(x-z) & K_{22}(x-z) \end{pmatrix} \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$p(z) = -\int_z^\infty \frac{\lambda}{h(x)^2} dx \tag{2}$$

where the kernel terms are given by

$$K_{11}(z) = \frac{32 - 24z^2}{(z^2+4)^3} \qquad K_{12}(z) = \frac{48z^2 - 64}{z(z^2+4)^3}$$

$$K_{21}(z) = -\frac{(16z^3 + 16z^2 + 4)}{z(z^2+4)^3} \qquad K_{22}(z) = -\frac{(32 - 24z^2)}{(z^2+4)^3}$$

with boundary conditions at infinity

$$h''(x) \to 1, \quad g'(x) \to \frac{1}{2}$$

For a given speed parameter λ, we wish to find the material toughness K, which is given by

$$K(\lambda) = \lim_{x \to 0} 3\sqrt{2\pi}\sqrt{x}\, h'(x)$$

We are interested in the value $\lambda = \lambda_0$ for which $K(\lambda_0) = 0$, since this is then the propagation speed of a zero-toughness system. We are also interested in $\lambda \approx \lambda_0$.

2 Zero toughness solution

Consider setting $K = 0$. We investigate only the nature of the solution near $x = 0$. We suspect, (and will verify later) that p is singular near the crack tip. Thus in equation 1, we can neglect terms that are non singular.

$$p(z) = \int_0^\infty \frac{h'(x)}{x - z} dx$$

Also have that $p' = \lambda/h^2$. We try the ansatz $h \sim x^\alpha$. From our two equations linking h and p, this gives that

$$p \sim x^{\alpha - 1}$$

$$p' \sim x^{-2\alpha}$$

and so $\alpha = 2/3$. We have made use of the integral

$$\int_0^\infty \frac{z^{s-1}}{z-x} dz = -\pi \cot(\pi s) z^{s-1}$$

So starting with a solution $h_0 = A_0 x^{2/3} + o(x^{2/3})$ near the crack tip, we get $p_0(x) = -\frac{3\lambda_0}{A_0^2} x^{-1/3} + O(1)$. Putting this into the lubrication equation $p' h^2 = \lambda_0$, we find that

$$A_0 = \left(\frac{243\lambda_0}{4\pi^2}\right)^{1/6}$$

We also can take $g_0 = Bx^{1/2} + \dots$ for $x \to 0$. B can only be found numerically. (N.B. in red since this isn't an issue in [1], and I'm not sure where this came from or if it's even needed.) To recap the zero tougness solution takes the form

$$\begin{aligned} h_0(x) &= A_0 x^{2/3} &&+ \dots \\ p_0(x) &= -\frac{3\lambda_0}{A_0^2} x^{-1/3} &&+ \dots \\ g_0(x) &= Bx^{1/2} &&+ \dots \end{aligned}$$

This holds only when $K = 0$ exactly, and the above is a good approximation for small x, all the way to $x = 0$.

3 Small toughness solution

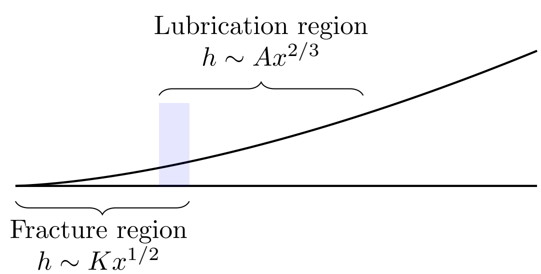
Now let us consider $K > 0$ but take K arbitrarily small. One expects for K small, that the new solution will look much like the $K = 0$ solution. However for any small, but non-zero, value of K, we must have that $h(x) \sim \frac{2K}{3\sqrt{2\pi}} x^{1/2}$ as $x \to 0$. Thus the zero toughness solution cannot be a good approximation for the entire domain. This is resolved by a LEFM boundary layer, following [1]. Outside this boundary layer, we expect behaviour close to the lubrication solution. So outside the boundary layer, look for a solution

$$\begin{aligned} g(x) &= g_0(x) + \mathcal{E}(K) g_1(x) + o(\mathcal{E}) \\ h(x) &= h_0(x) + \mathcal{E}(K) h_1(x) + o(\mathcal{E}) \\ p(x) &= p_0(x) + \mathcal{E}(K) p_1(x) + o(\mathcal{E}) \\ \lambda &= \lambda_0 + \mathcal{E}(K) \lambda_1 + o(\mathcal{E}) \end{aligned}$$

Where $\mathcal{E}(K)$ is an unknown function of K which will be determined. We work with K small enough such that $\mathcal{E}(K) \ll 1$, (will be easily verified later).

It is important to remember that the above only holds outside the LEFM boundary layer. To find $\mathcal{E}$, we will take the outer asymptotics of the LEFM boundary layer and attempt to match that with the inner asypmtotics of the above solution.

Figure 1: Matching outer asymptotics of the LEFM boundary later with the inner asymptotics of the Lubrication region



References

[1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution,* Journal of Applied Mechanics, 2005.