Summary of summer project

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Introduction

Consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the "arm" of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions c, the magnitude of the bending moment M, and the toughness of the solid K_I.

Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

This is a joint work with Tim Large, who started this project in summer 2014.

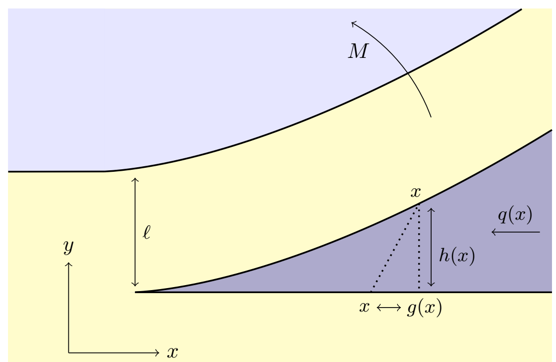
Set up

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h^2 p'$$

From elasticity, using Muskhelishiveli methods, we can derive the equation

$$\begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix}$$



Where K_{ij} is the integral kernel specific to this geometry, E is the Young's modulus of the solid, ν is Poisson's ratio, and μ is the viscosity of the fluid. The boundary conditions on the solid at ∞ are governed by the bending moment, using the beam approximation. The boundary conditions near the crack tip are governed by "Linear Elastic Fracture Mechanics".

A problem of particular interest is the "Small toughness" problem, when $K_I \ll M\ell^{-3/2}$. The near tip asymptotics when $K_I = 0$, namely $h \sim x^{2/3}$ need to be reconciled with the asymptotics for $K_I > 0$, where $h \sim x^{1/2}$. The result is a near tip boundary layer.

The equations are difficult to solve numerically as they are a set of non-linear integro-differential equations, with multiple length scales, and singular solutions. By approximating relevant functions as "known singular behaviour × piecewise linear function", and using a sensible spacing of grid points, the above equations can be discritized and solved numerically.

Results

From using analytic methods, it was found that by approaching asymptotically close to the crack tip, the geomety reduces to a known problem. This allows us to obtain the "Small toughness" solution

$$c = \frac{36(1-\nu^2)^2 M^3}{\pi\mu E^2 \ell^5} \left(\lambda_0 + C\lambda_0^{2s-1}\lambda_1(\ell^{3/2}K_I/M)^u\right)$$

The constants λ_0,λ_1, C, u,s, have been determined numerically. This result is in close agreement with numerical solutions.