Viscous control of shallow elastic fracture

Tim Large[1], John Lister[2], and Dominic Skinner[2]

[1]M.I.T., USA

[2]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK

(Received xx; revised xx; accepted xx)

This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/relatedlink/jfm-keywords.pdf for the full list)

1. Introduction

Here we review the literature as well as describe the problem in more detail. We have the vertical displacement h, the horizontal displacement g, the thickness of the arm l, and the pressure p. We look for a travelling wave solution (propagating left), with speed c.

2. Formulation of problem

From lubrication, we expect Poiselle flow in the crack. This gives us the flux, q, as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3 \tag{2.1}$$

We also have the conservation equation

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.2}$$

Which combined gives us the equation

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2 \tag{2.3}$$

From the linear theory of elasticity, due to others who have studied this problem, we have

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix} = \int_0^\infty \boldsymbol{K}(x-\tilde{x}) \begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix} \mathrm{d}\tilde{x} \tag{2.4}$$

Where the integral kernel is

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2.5}$$

Here we change into a set of dimensionless variables, and will spend the rest of the paper working with them. We have a length scale l, a pressure scale $p^* = E/12(1-\nu^2)$, and a time scale $t^* = 12\mu/p^*$. From these, we can define the following dimensionless parameters,

$$\mathcal{M} = \frac{M}{p^* l^2}, \qquad \mathcal{C} = \frac{c}{l/t^*} = \frac{12\mu c}{p^* l}, \qquad \mathcal{K}_I = \frac{K_I}{p^* l^{1/2}}, \qquad \mathcal{K}_{II} = \frac{K_{II}}{p^* l^{1/2}} \tag{2.6}$$

and variables

$$x = l\xi, \quad K_{ij} = U_{ij}/l, \quad h = \alpha l H(\xi), \quad g = \alpha l G(\xi), \quad p = \beta p^* \mathit{\Pi}(\xi) \tag{2.7}$$

The preferred scalings to be used in this paper are $\alpha = \pi\beta/3 = \mathcal{M}$, $\lambda = \pi\mathcal{C}/3\mathcal{M}^2$, which give

$$\begin{bmatrix} \mathit{\Pi} \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{U}(\xi - \tilde{\xi}) \begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H^2 \frac{\mathrm{d}\mathit{\Pi}}{\mathrm{d}\xi} = \lambda \tag{2.8a, b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2} \tag{2.9a, b}$$

$$\lim_{\xi\to 0} 3\sqrt{2\pi\xi} H' = \frac{K_I}{Ml^{-3/2}} \equiv \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi} G' = \frac{K_{II}}{Ml^{-3/2}} \equiv \kappa_{II}, \tag{2.10a, b}$$

These shall be the governing equations for the rest of this paper, athough we will often rewrite equation 2.8b as an integral, choosing $\mathit{\Pi} \to 0$ as $\xi \to \infty$,

$$\mathit{\Pi}(\xi) = \int_\xi^\infty \lambda / H(\tilde{\xi})^2 \mathrm{d}\tilde{\xi} \tag{2.11}$$

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking n points $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ at which we measure H', G', and $n-1$ intermediate points $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{n-1})$ at which to measure $\mathit{\Pi}$, so that $\xi_1 < \zeta_1 < \ldots < \zeta_{n-1} < \xi_n$. We take $\xi_1 = 0$, where the crack tip is situated. Linear interpolation of G', H' would work poorly near the crack tip, since both functions are singular there. However, both $\sqrt{\xi}G'(\xi)$, and $\sqrt{\xi}H'(\xi)$ are regular functions, so we work with these instead, near the tip. Away from the tip, we treat H', G' as linear. So for $\xi_i < \xi < \xi_{i+1}$ we have that

$$G'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad H'(\xi) = \begin{cases} \xi^{-1/2}(c_i\xi^{1/2} + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for } \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.1}$$

Where $1 < t < n$ is just a number governing where to interpolate as linear, and where not to; typically $t = n/2$ was used. Note that H', G are interpolated slightly differently near the tip, here the interpolating function was simply based on the appearance of $\sqrt{\xi}G'(\xi)$,$\sqrt{\xi}H'(\xi)$, near the tip. We will also define a_n, b_n, c_n, d_n for interpolation beyond ξ_n. We will define $\boldsymbol{a} = [a_1, \ldots, a_n]$, and similarly with $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, we will also define, for convenience, the column vector $\boldsymbol{\gamma} = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]$.

Now given this interpolation, suppose that we know $\boldsymbol{\gamma}$. Then the elasticity integral (2.8a), is exact, that is to say there is an known analytic expression for $\mathit{\Pi}(\xi)$, given our choice of interpolation. We also have a different expression for Π, due to the lubrication integral (2.11). As before, given the interpolation, this integral becomes an analytic expression.

We see that in equation 2.8a, the integral depends linearly on $\boldsymbol{\gamma}$. Therefore, we can write

$$[\Pi(\zeta_1), \ldots, \Pi(\zeta_{n-1}), \underbrace{0, \ldots, 0}_{n-1}] = \boldsymbol{J}\boldsymbol{\gamma} \tag{3.2}$$

where $\boldsymbol{J}$ is a matrix in lieu of the integral kernel. We also know that $[\Pi(\zeta_1), \ldots, \Pi(\zeta_{n-1})] = f_1(\boldsymbol{c}, \boldsymbol{d})$, from the lubrication integral. This time, Π does not depend linearly on $\boldsymbol{\gamma}$.

We define $\boldsymbol{\theta}_G = [a_1\xi_1 + b_1, \ldots, a_n\xi_n + b_n]$, $\boldsymbol{\theta}_H = [c_1\xi_1^{1/2} + d_1, \ldots, c_n\xi_n^{1/2} + d_n]$, as well as $\boldsymbol{\theta} = [\boldsymbol{\theta}_G, \boldsymbol{\theta}_H]$. We would prefer to work with $\boldsymbol{\theta}$ over $\boldsymbol{\gamma}$, since it contains half as many elements. Continuity of G', H' impose $2(n-1)$ equations, which are typically of the form

$$\begin{aligned} a_i\xi_{i+1} + b_i &= a_{i+1}\xi_{i+1} + b_{i+1}, \text{ for } i < n,\ i \neq t-1 \\ \xi^{-1/2}(a_i\xi_{i+1} + b_i) &= a_{i+1}\xi_{i+1} + b_{i+1}, \text{ for } i = t-1 \end{aligned} \tag{3.3}$$

with a slightly modified equation to account for the different interpolation in H'. We can also introduce some boundary conditions here to get another two equations. We know $G'(\xi_n)$ from our asymptotic expansion and beam theory, and we also know $G'(\xi_{n-1})$ (provided ξ_{n-1}, ξ_n are suitably large). Therefore we know $G'(\xi_n)/G'(\xi_{n-1})$, which provides a linear constraint on $\boldsymbol{a}$, $\boldsymbol{b}$ (not similar for H'). We have enough equations here to calculate a matrix $\boldsymbol{T}$, so that

$$\boldsymbol{\gamma} = \boldsymbol{T}\boldsymbol{\theta} \tag{3.4}$$

Although, since this matrix is fairly sparse, with a known structure, it is not ever actually calculated in the program. To recap, we now have that

$$\boldsymbol{J}\boldsymbol{T}\boldsymbol{\theta} = f_2(\boldsymbol{\theta}) \tag{3.5}$$

With $\boldsymbol{J}\boldsymbol{T}$ a $2(n-1) \times 2n$ matrix, and f_2 a function of $\boldsymbol{\theta}$ (really just of $\boldsymbol{\theta}_H$), with the first $n-1$ components calculating Π, and the last $n-1$ components set to 0. We add another two rows to $\boldsymbol{J}\boldsymbol{T}$, to make it a square matrix. Since we know $G'(\xi_n)$, and $H''(\xi_n) \approx (H'(\xi_n) - H'(\xi_{n-1}))/(\xi_n - \xi_{n-1}) = (\boldsymbol{\theta}_{2n} - \boldsymbol{\theta}_{2n-1})/(\xi_n - \xi_{n-1})$, we can replace f_2 with $f = [f_2, G'(\xi_n), H''(\xi_n)]$. The two rows added have the property that when multiplied by $\boldsymbol{\theta}$ equal $\boldsymbol{\theta}_n$, and $(\boldsymbol{\theta}_{2n} - \boldsymbol{\theta}_{2n-1})/(\xi_n - \xi_{n-1})$ respectively. Let the enlarged matrix be called $\boldsymbol{A}$. Then

$$\boldsymbol{A}\boldsymbol{\theta} = f(\boldsymbol{\theta}) \tag{3.6}$$

This can be solved by Newton's method; if $\boldsymbol{A}\boldsymbol{\theta} \approx f(\boldsymbol{\theta})$ and $\boldsymbol{A}(\boldsymbol{\theta} + \boldsymbol{\delta}) = f(\boldsymbol{\theta} + \boldsymbol{\delta})$, so to leading order in $\boldsymbol{\delta}$, we have that

$$\boldsymbol{\delta} = (\boldsymbol{A} - D_{\boldsymbol{\theta}} f)^{-1} (f(\boldsymbol{\theta}) - A\boldsymbol{\theta}) \tag{3.7}$$

3.2. *Double Tip*

Here we mention details specific to the problem with both a fluid and a dry tip. Perhaps details of the $\sin^2$ spacing or the interpolation to $K_I = 0$ should be mentioned here.

4. Results

4.1. *Single tip*

Start off with some of the basic graphs showing H',G', and Π against ξ.

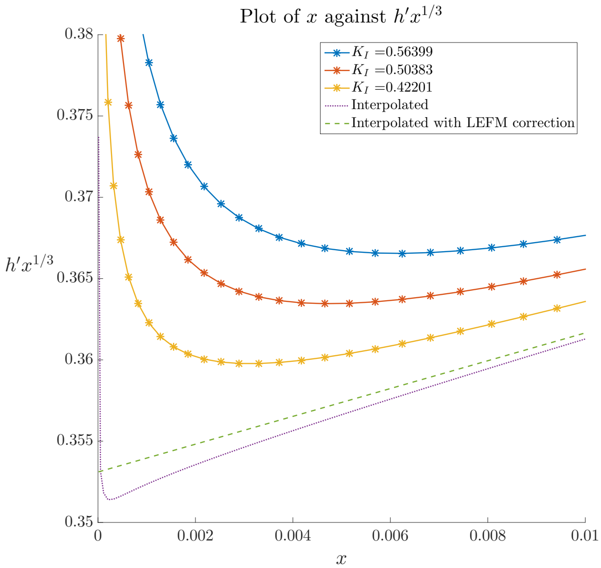

FIGURE 1. This shows some evidence of the LEFM boundary layer

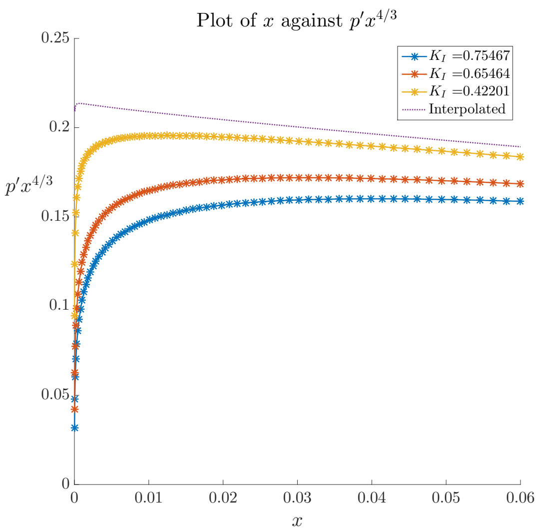

FIGURE 2. This also shows some evidence of the LEFM boundary layer

Obvious questions to ask at this point are: How are you sure this is the right answer, what is the effect of n, t, ξ_{end}? Well t really doesn't have too much of an effect. This is not unexpected, in a region away from zero, interpolating with base functions ξ, 1 is similar to interpolating with $\xi^{-1/2}$, $\xi^{1/2}$, as both are smooth, regular functions. So as long as $t > O(10)$ the difference is not too noticable. In the next graph, we determine the effect of extending ξ_{end}, by adding on extra points (so maintaining the same resolution near the tip). There is a satisfactory demonstration of convergence.

By adding points on in a geometric progression, it becomes quite cheap to extend out to $\xi_{\text{end}} \approx 800$ or so. Once one has done this, it is apparent that the effect of the tip resolution dominates the effect of finite truncation, as the following figure shows.

By increasing n (for large ξ_{end}), we have been able to determine λ_0 and D

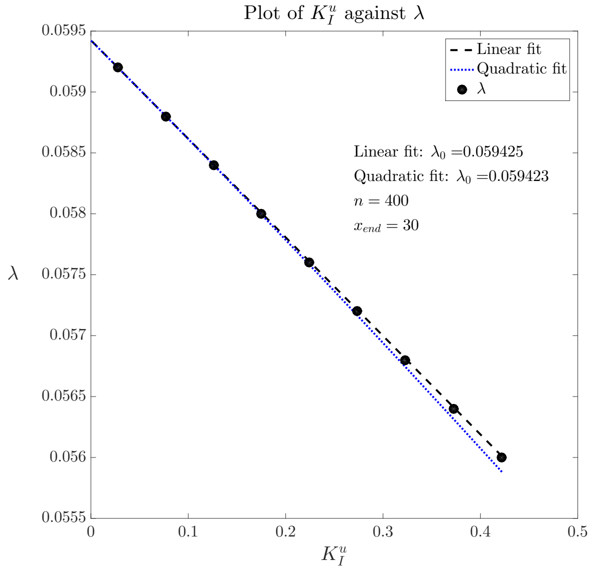

FIGURE 3. Good agreement between theory and numerics here

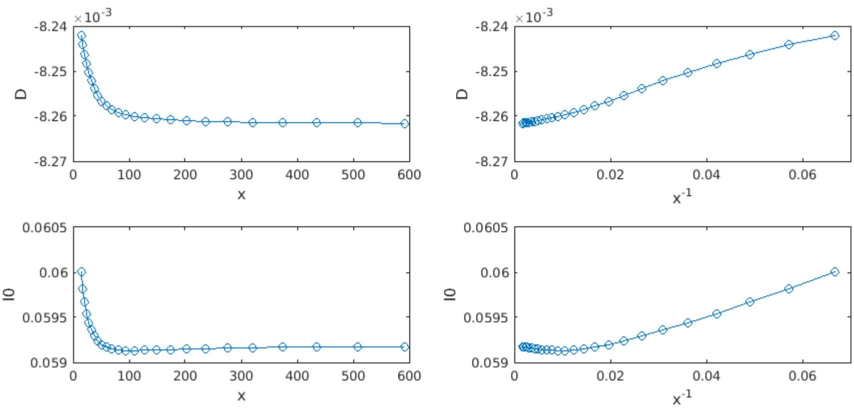

FIGURE 4. Satisfactory convergence as $\xi_{\text{end}} \to \infty$

4.2. *Linear perturbation problem*

We solve the linear perturbation problem. All that we really want to know is that we see the ξ^{s-1} behaviour that we expect, and we ask what the intercept of $\tilde{H}_1$ is. It is perhaps worth mentioning the difficulties in measuring the intercept and perhaps a notion of the sensitivity of the result on the estimate provided for H_0. Illustrating that is the next figure

Then we include the figure that shows convergence with different n values to something approaching the right answer.

4.3. *Two tips*

After the success of the linear perturbation problem, we move on to the two tip problem. Perhaps some graphs that show an outline of the full numerical problem with non-zero κ_I and κ_{II}, although these are not physical.

What would be nice, although it doesn't exist yet, is some sort of record of how we

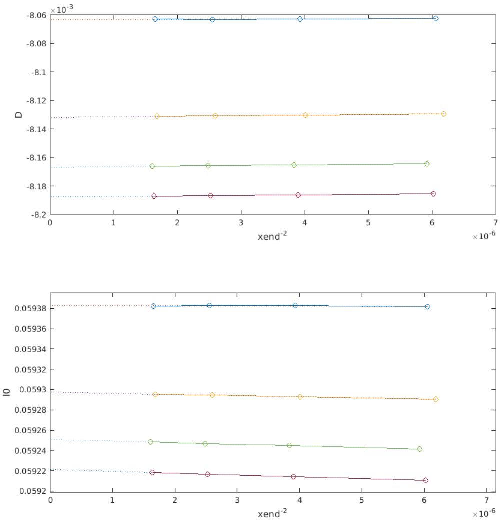

FIGURE 5. Shows the relatively small effect that ξ_{end} has compared to the tip resolution, once ξ_{end} has become large enough

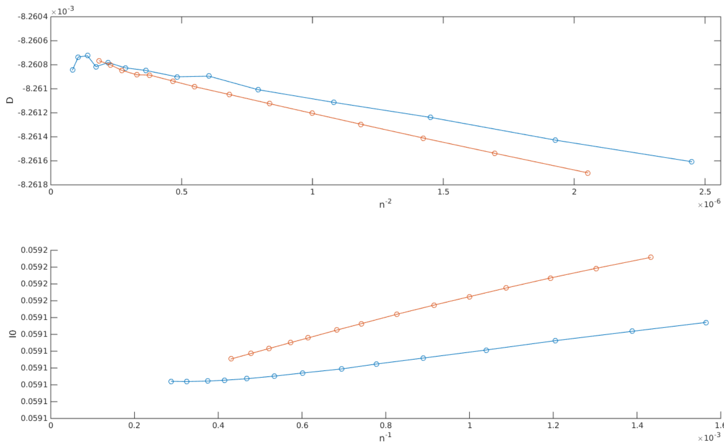

FIGURE 6. Our best guess at λ_0 and D, and the approximate error one can expect in them

now extrapolate to $\kappa_I = 0$. This is certainly a plot that needs to be made. We now move on to the $\kappa_I = 0$ set of relations.

5. Discussion

This is where we discuss the figures, possibly include more figures, and draw the results and conclusions of this paper.

Perhaps the first thing worth mentioning is the somewhat contrived, but pretty accurate formulae for λ in terms of κ_I. This holds for any toughness in the single tip case.

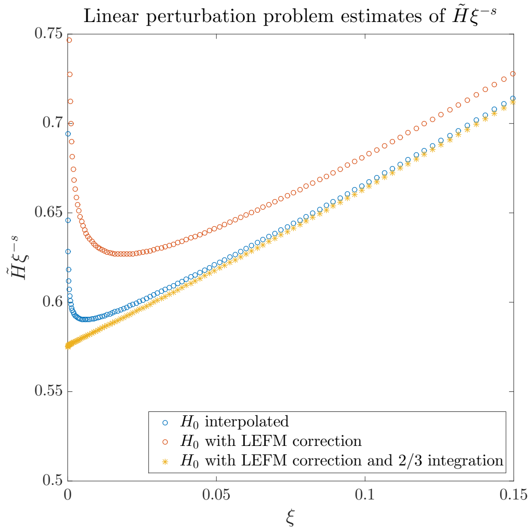

FIGURE 7. Sensitivity of linear perturbation problem on H_0

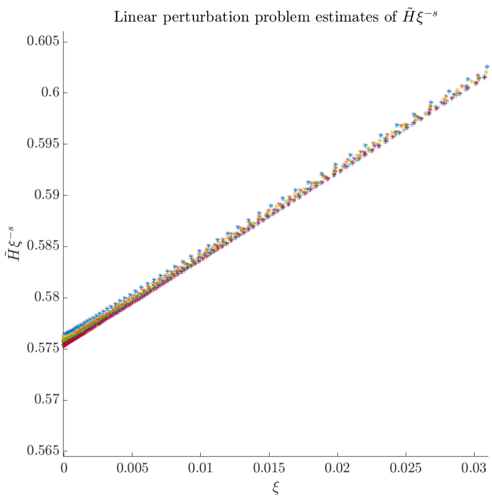

FIGURE 8. Sensitivity of linear perturbation problem on H_0

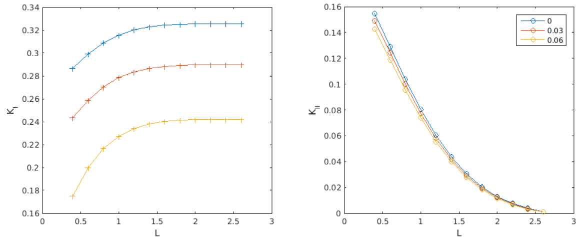

FIGURE 9. Different colours indicate different λ values

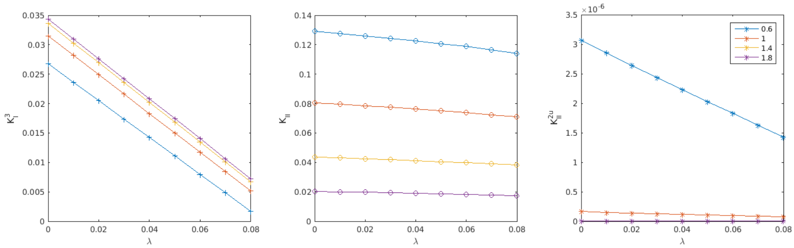

FIGURE 10. Different colours indicate different L values

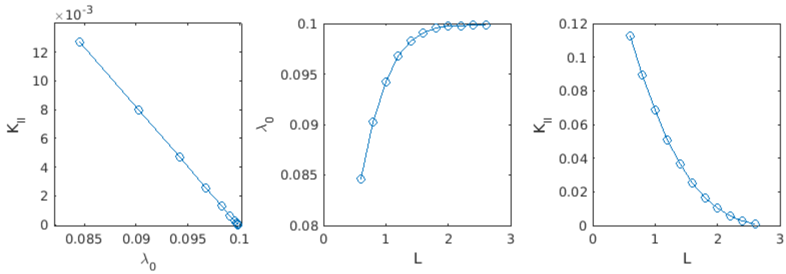

FIGURE 11. The results of extrapolating to $\kappa_I = 0$

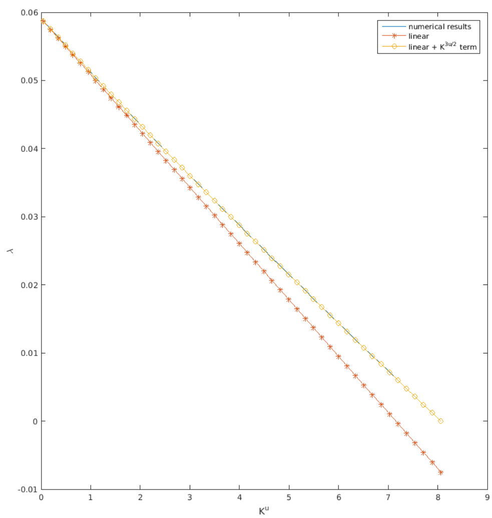

FIGURE 12. The formula valid for all κ_I

Then we could move on to talk about the decoupling between the fluid problem and the dry fracture problem. Relavent graphs to include would show that H really doesn't vary much with λ_0, and that given a reference H', one can construct G' with relative ease.

At this point, I would like to construct another contrived formulae for the two tip problem. Then I would like to plot a graph of κ_I against κ_{II} in the full fluid problem. This provides a guide of when it is appropriate to take the single tip, and when it is appropriate to take the double tip asymptotics.

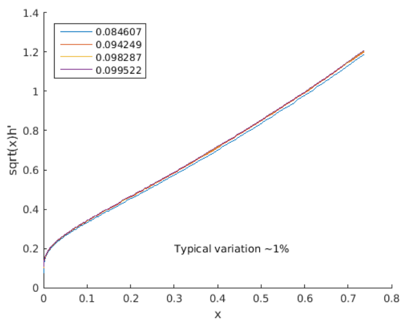

FIGURE 13. Demonstrating the decoupling of fluid and solid fracture

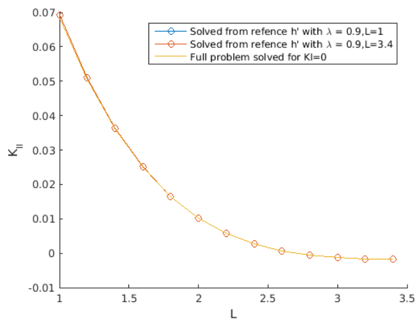

FIGURE 14. Reconstructing the full solution given a reference H'

a/d	$M = 4$	$M = 8$	Callan *et al.*
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

TABLE 1. Values of kd at which trapped modes occur when $\rho(\theta) = a$

6. Citations and references

All papers included in the References section must be cited in the article, and vice versa. Citations should be included as, for example "It has been shown (Rogallo 1981) that..." (using the `\citep` command, part of the natbib package) "recent work by Dennis (1985)..." (using `\citet`). The natbib package can be used to generate citation variations, as shown below.
`\citet[pp. 2-4]{Hwang70}`:
Hwang & Tuck (1970, pp. 2-4)
`\citep[p. 6]{Worster92}`:
(Worster 1992, p. 6)
`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
`\citep{Brownell04,Brownell07,Ursell50,Wijngaarden68,Miller91}`:
(Brownell & Su 2004, 2007; Ursell 1950; van Wijngaarden 1968; Miller 1991)
The References section can either be built from individual `\bibitem` commands, or can be built using BibTex. The BibTex files used to generate the references in this document can be found in the zip file at http://journals.cambridge.org/data/relatedlink/jfm-ifc.zip.
Where there are up to ten authors, all authors' names should be given in the reference list. Where there are more than ten authors, only the first name should appear, followed by et al.

Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

Appendix A

This appendix contains sample equations in the JFM style. Please refer to the LaTeX source file for examples of how to display such equations in your manuscript.

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \tag{A 1}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{v} = 0, \quad \nabla^2 P = \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{v} \times \boldsymbol{w}). \tag{A 2}$$

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \quad r \equiv |P - Q| \rightarrow 0, \tag{A 3}$$

$$\left.\begin{aligned} \frac{\partial G_s}{\partial y} = 0 \quad \text{on} \qquad y = 0, \\ G_a = 0 \quad \text{on} \qquad y = 0, \end{aligned}\right\} \tag{A 4}$$

$$-\frac{1}{2\pi}\int_0^\infty \gamma^{-1}[\exp(-k\gamma|y-\eta|)+\exp(-k\gamma(2d-y-\eta))]\cos k(x-\xi)t\,\mathrm{d}t, \qquad 0 < y, \quad \eta < d, \tag{A 5}$$

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{1/2}, & t \leqslant 1 \\ (t^2-1)^{1/2}, & t > 1. \end{cases} \tag{A 6}$$

$$-\frac{1}{2\pi}\int_0^\infty B(t)\frac{\cosh k\gamma(d-y)}{\gamma\sinh k\gamma d}\cos k(x-\xi)t\,\mathrm{d}t$$

$$G = -\frac{1}{4}\mathrm{i}(H_0(kr)+H_0(kr_1)) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-k\gamma d}}{\gamma\sinh k\gamma d}\cosh k\gamma(d-y)\cosh k\gamma(d-\eta) \tag{A 7}$$

Note that when equations are included in definitions, it may be suitable to render them in line, rather than in the equation environment: $\boldsymbol{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G_a}$ where $r = \{[x(\theta)-x(\psi)]^2 + [y(\theta)-y(\psi)]^2\}^{1/2}$ and $\widetilde{G_a}$ is regular as $kr \to 0$. However, any fractions displayed like this, other than $\frac{1}{2}$ or $\frac{1}{4}$, must be written on the line, and not stacked (ie 1/3).

$$\begin{aligned} \frac{\partial}{\partial n_q}\left(\frac{1}{4}Y_0(kr)\right) &\sim \frac{1}{4\pi w^3(\theta)}[x''(\theta)y'(\theta) - y''(\theta)x'(\theta)] \\ &= \frac{1}{4\pi w^3(\theta)}[\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as} \qquad kr \to 0. \end{aligned} \tag{A 8}$$

$$\frac{1}{2}\phi_i = \frac{\pi}{M}\sum_{j=1}^{M}\phi_j K_{ij}^a w_j, \qquad i = 1, \ldots, M, \tag{A 9}$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i,\theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G_a}(\theta_i,\theta_i)/\partial n_q + [\rho_i'\rho_i'' - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \tag{A 10}$$

$$\rho_l = \lim_{\zeta \to Z_l^-(x)} \rho(x,\zeta), \quad \rho_u = \lim_{\zeta \to Z_u^+(x)} \rho(x,\zeta) \tag{A 11a, b}$$

$$(\rho(x,\zeta), \phi_{\zeta\zeta}(x,\zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \tag{A 12}$$

$$\tau_{ij} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + (\overline{\overline{u}_i u_j^{SGS} + u_i^{SGS}\overline{u}_j}) + \overline{u_i^{SGS}u_j^{SGS}}, \tag{A 13a}$$

$$\tau_j^\theta = (\overline{\overline{u}_j\overline{\theta}} - \overline{u}_j\overline{\theta}) + (\overline{\overline{u}_j\theta^{SGS} + u_j^{SGS}\overline{\theta}}) + \overline{u_j^{SGS}\theta^{SGS}}. \tag{A 13b}$$

$$\boldsymbol{Q}_C = \begin{bmatrix} -\omega^{-2}V_w' & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\ \dfrac{\beta}{\alpha\omega^2}V_w' & 0 & 0 & 0 & \mathrm{i}\omega^{-1} \\ \mathrm{i}\omega^{-1} & 0 & 0 & 0 & 0 \\ \mathrm{i}R_\delta^{-1}(\alpha^t+\omega^{-1}V_w'') & 0 & -(\mathrm{i}\alpha^t R_\delta)^{-1} & 0 & 0 \\ \dfrac{\mathrm{i}\beta}{\alpha\omega}R_\delta^{-1}V_w'' & 0 & 0 & 0 & 0 \\ (\mathrm{i}\alpha^t)^{-1}V_w' & (3R_\delta^{-1}+c^t(\mathrm{i}\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0 \end{bmatrix}. \tag{A 14}$$

$$\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[\mathrm{i}(\alpha^t x_1^t - \omega t)], \tag{A 15}$$

where $\hat{\boldsymbol{\eta}}^t = \boldsymbol{b}\exp(\mathrm{i}\gamma x_3^t)$.

$$\mathrm{Det}[\rho\omega^2\delta_{ps} - C_{pqrs}^t k_q^t k_r^t] = 0, \tag{A 16}$$

$$\langle k_1^t, k_2^t, k_3^t\rangle = \langle \alpha^t, 0, \gamma\rangle \tag{A 17}$$

$$\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{A 18}$$

$$f(\psi_1) = \frac{3b}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi} \frac{(\sin\psi_1-\sin\psi)(a+b\cos\psi)^{1/2}}{[1-\cos(\psi_1-\psi)](2+\alpha)^{1/2}}\mathrm{d}x, \tag{A 19}$$

$$\begin{aligned} g(\psi_1) = {} & \frac{3}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\left(\frac{a+b\cos\psi}{2+\alpha}\right)^{1/2}\Bigg\{f(\psi)[(\cos\psi_1 - b\beta_1)S + \beta_1 P] \\ & \times \frac{\sin\psi_1-\sin\psi}{1-\cos(\psi_1-\psi)} + g(\psi)\left[\left(2+\alpha-\frac{(\sin\psi_1-\sin\psi)^2}{1-\cos(\psi-\psi_1)}-b^2\gamma\right)S\right. \\ & \left. + \left(b^2\cos\psi_1\gamma - \frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi,\delta) - (2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi,\delta)\right]\Bigg\}\mathrm{d}\psi, \end{aligned} \tag{A 20}$$

$$\alpha = \alpha(\psi,\psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}, \quad \beta - \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \tag{A 21}$$

$$\left.\begin{aligned} H(0) &= \frac{\epsilon\overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}, \quad H'(0) = -1+\epsilon^{2/3}\overline{C}_u + \epsilon\hat{C}_u'; \\ H''(0) &= \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2}u_P^2}, \quad H'(\infty) = 0. \end{aligned}\right\} \tag{A 22}$$

Lemma 1. *Let $f(z)$ be a trial Batchelor (1971, pp. 231–232) function defined on* $[0,1]$. *Let $\varLambda_1$ denote the ground-state eigenvalue for $-\mathrm{d}^2g/\mathrm{d}z^2 = \varLambda g$, where g must satisfy $\pm\mathrm{d}g/\mathrm{d}z + \alpha g = 0$ at $z = 0,1$ for some non-negative constant α. Then for any f that is not identically zero we have*

$$\frac{\alpha(f^2(0)+f^2(1)) + \displaystyle\int_0^1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\mathrm{d}z}{\displaystyle\int_0^1 f^2\mathrm{d}z} \geqslant \varLambda_1 \geqslant \left(\frac{-\alpha+(\alpha^2+8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \tag{A 23}$$

COROLLARY 1. *Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies*

$$\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{A 24}$$

REFERENCES

BATCHELOR, G. K. 1971 Small-scale variation of convected quantities like temperature in turbulent fluid. part 1. general discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113–133.

BROWNELL, C. J. & SU, L. K. 2004 Planar measurements of differential diffusion in turbulent jets. *AIAA Paper 2004-2335* .

BROWNELL, C. J. & SU, L. K. 2007 Scale relations and spatial spectra in a differentially diffusing jet. *AIAA Paper 2007-1314* .

DENNIS, S. C. R. 1985 Compact explicit finite difference approximations to the Navier–Stokes equation. In *Ninth Intl Conf. on Numerical Methods in Fluid Dynamics* (ed. Soubbaramayer & J. P. Boujot), *Lecture Notes in Physics*, vol. 218, pp. 23–51. Springer.

HWANG, L.-S. & TUCK, E. O. 1970 On the oscillations of harbours of arbitrary shape. *J. Fluid Mech.* **42**, 447–464.

KOCH, W. 1983 Resonant acoustic frequencies of flat plate cascades. *J. Sound Vib.* **88**, 233–242.

LEE, J.-J. 1971 Wave-induced oscillations in harbours of arbitrary geometry. *J. Fluid Mech.* **45**, 375–394.

LINTON, C. M. & EVANS, D. V. 1992 The radiation and scattering of surface waves by a vertical circular cylinder in a channel. *Phil. Trans. R. Soc. Lond.* **338**, 325–357.

MARTIN, P. A. 1980 On the null-field equations for the exterior problems of acoustics. *Q. J. Mech. Appl. Maths* **33**, 385–396.

MILLER, P. L. 1991 Mixing in high schmidt number turbulent jets. PhD thesis, California Institute of Technology.

ROGALLO, R. S. 1981 Numerical experiments in homogeneous turbulence. *Tech. Rep.* 81835. NASA Tech. Mem.

URSELL, F. 1950 Surface waves on deep water in the presence of a submerged cylinder i. *Proc. Camb. Phil. Soc.* **46**, 141–152.

VAN WIJNGAARDEN, L. 1968 On the oscillations near and at resonance in open pipes. *J. Engng Maths* **2**, 225–240.

WORSTER, M. G. 1992 The dynamics of mushy layers. In *In Interactive dynamics of convection and solidification* (ed. S. H. Davis, H. E. Huppert, W. Muller & M. G. Worster), pp. 113–138. Kluwer.