

Introduction

We consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the “arm” of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions c , the bending moment M , and the toughness of the solid K_I , K_{II} . Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

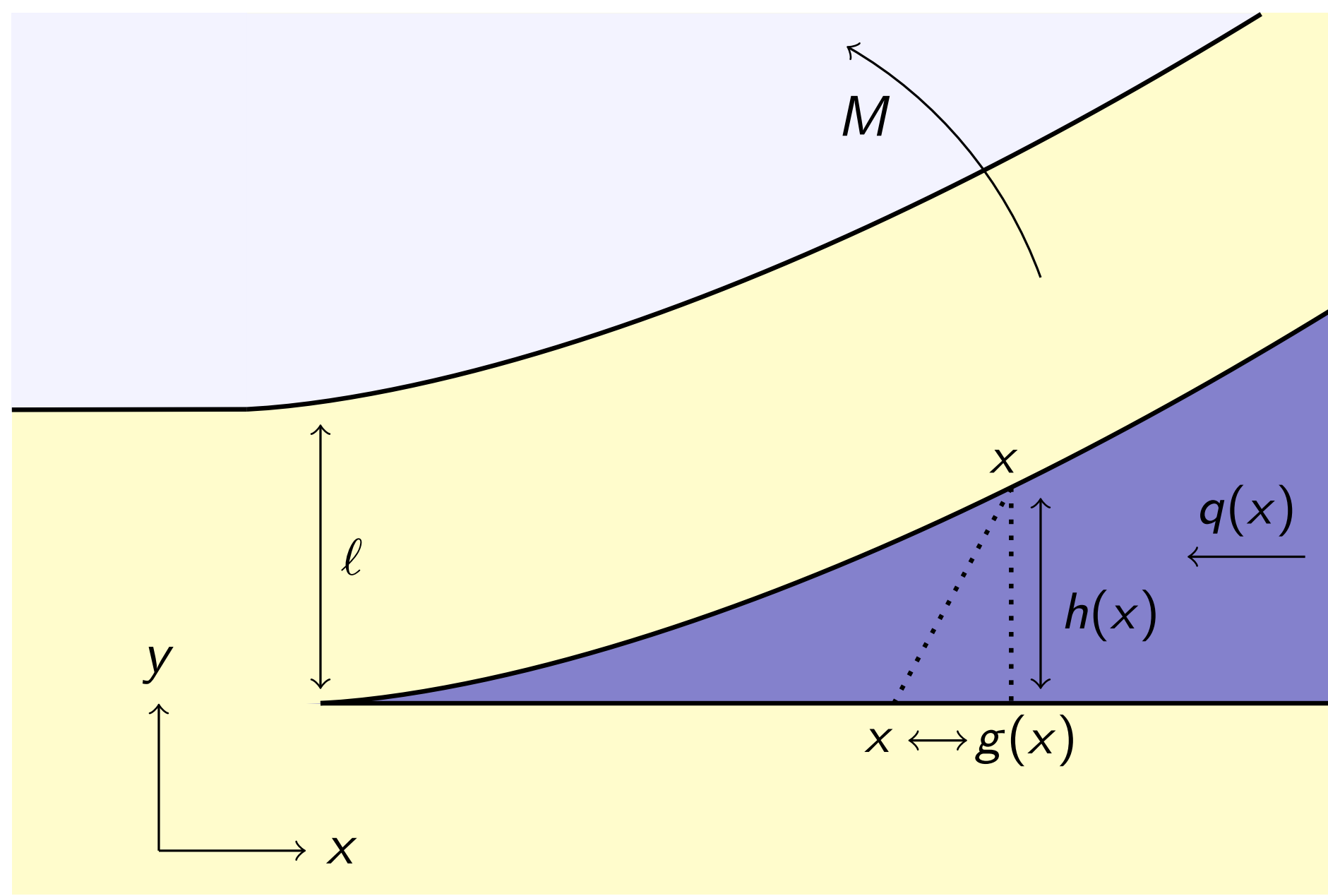


Figure 1 : Diagram to show the geometry of the problem. $q(x)$ is the flux, $g(x)$ the horizontal displacement, $h(x)$ the vertical displacement, and ℓ is the thickness of the arm.

Governing Equations

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h(x)^2 \frac{dp}{dx}.$$

From elasticity, using Muskhelishvili methods, we can derive the equation

$$[p, 0] = E/(4\pi(1-\nu^2)) \int_0^\infty \mathbf{K}(x-\tilde{x}) [g'(\tilde{x}), h'(\tilde{x})] d\tilde{x},$$

where K_{ij} is the integral kernel specific to this geometry.

► Boundary conditions as $x \rightarrow \infty$ are governed by the bending moment. We have (via beam theory),

$$M(x) = \frac{E\ell^3}{12(1-\nu^2)} \frac{d^2 h}{dx^2} \rightarrow M_0, \text{ as } x \rightarrow \infty.$$

► The boundary conditions as $x \rightarrow 0$ are governed by “Linear Elastic Fracture Mechanics”, (LEFM). This gives the condition

$$K_I = \lim_{x \rightarrow 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x), \quad K_{II} = \lim_{x \rightarrow 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} g'(x).$$

We move into dimensionless variables,

$$(x, h, g, p, K_I, K_{II}, K_{ij}, c) \rightarrow (\xi, H, G, \Pi, \kappa_I, \kappa_{II}, \Lambda_{ij}, \lambda).$$

The new equations and boundary conditions are

$$[\Pi, 0] = \int \Lambda \cdot [G', H'] d\xi, \quad H^2 \Pi' = \lambda$$

$$\lim_{\xi \rightarrow \infty} H'' = 1, \quad \lim_{\xi \rightarrow 0} 3\sqrt{2\pi\xi} H' = \kappa_I, \quad \lim_{\xi \rightarrow 0} 3\sqrt{2\pi\xi} G' = \kappa_{II}.$$

Results

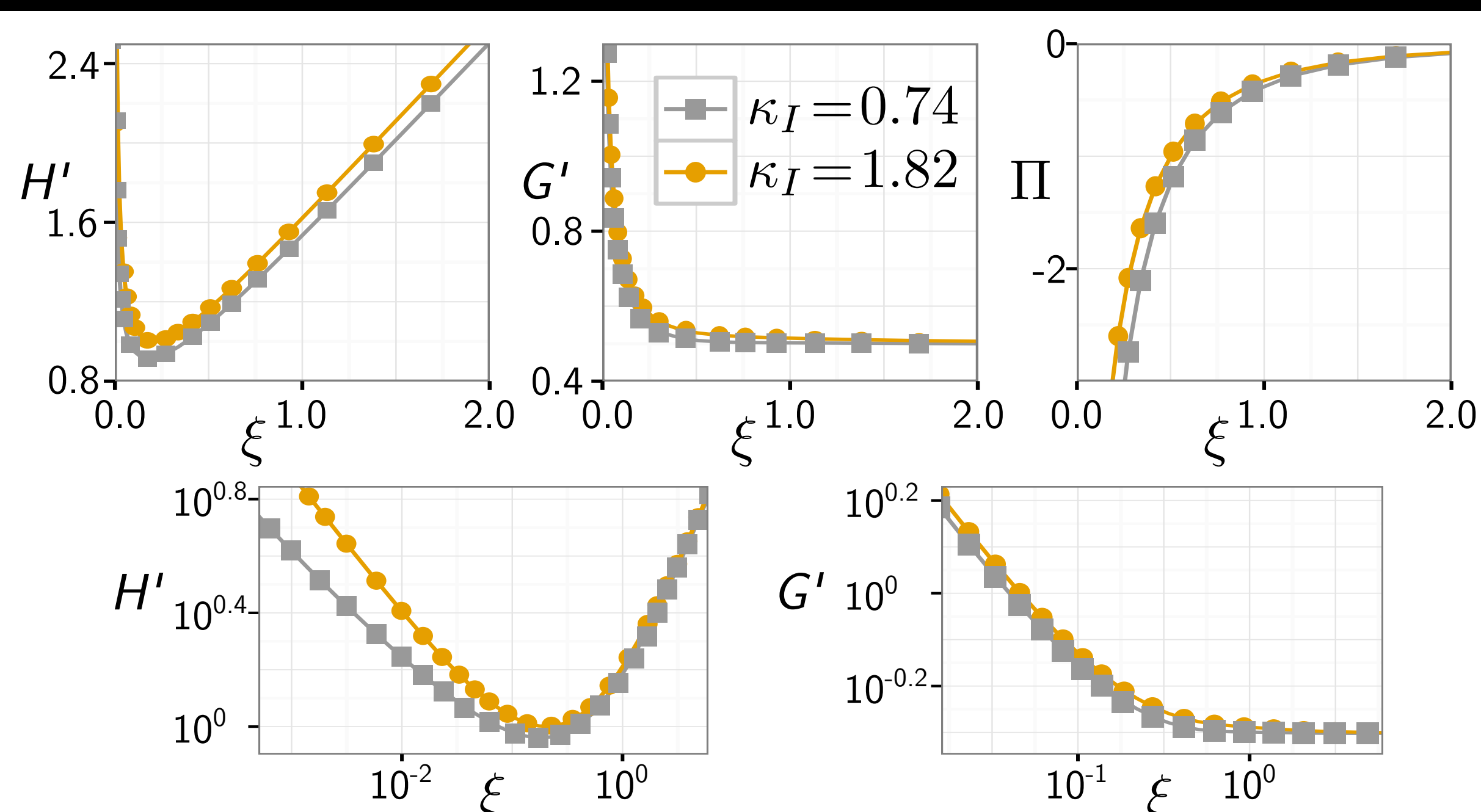


Figure 2 : Numerical results for two typical κ_I values, including logarithmic scales for H' and G' (lower two graphs).

Small Toughness Solution

We can plot how the speed λ varies with the toughness κ_I . For the *small toughness solution*, $\kappa_I \ll 1$, we use the theory of Garagash and Detournay [1] who consider fluid driven fracture in a different geometry. The theory states

► Near the tip there is the “LEFM boundary layer”.

► Away from the tip, the solution behaves as

$$H(\xi) = H_0(\xi) + \mathcal{E}(\kappa_I) H_1(\xi) + o(\mathcal{E}),$$

where $H_0(\xi) = H(\xi; \kappa_I = 0)$ is the zero toughness solution, (similar for G, Π, λ), and $\mathcal{E} = C \kappa_I^u \lambda_0^{2-u/2}$, $u \approx 3.17$.

This is in good agreement with the numerical results.

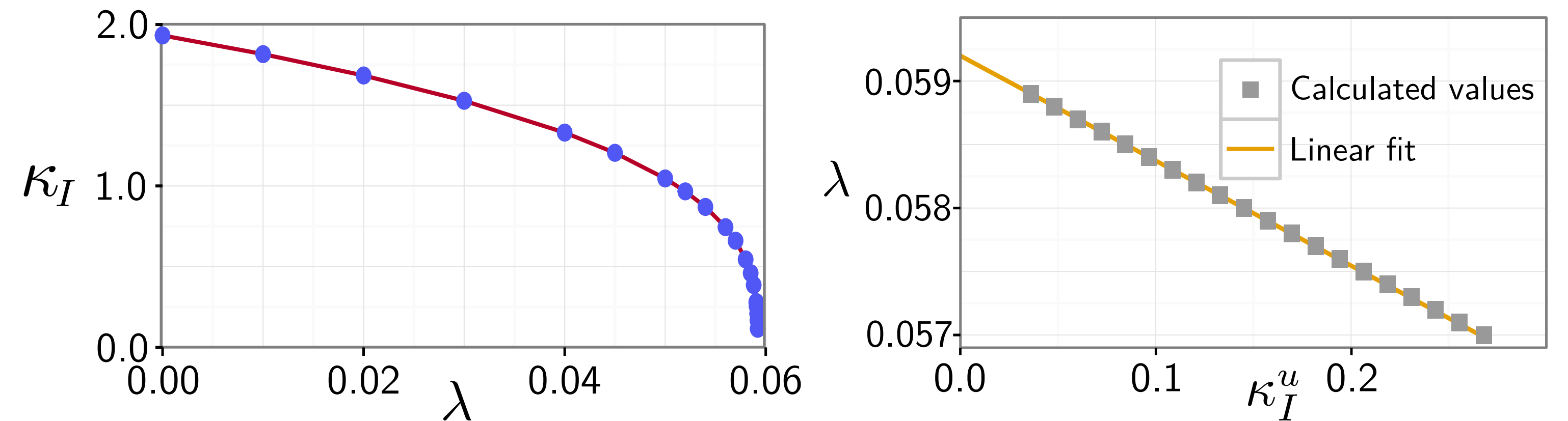


Figure 3 : The relationship between κ_I and λ is plotted on the left. Strong evidence that $\lambda = \lambda_0 + \mathcal{E}\lambda_1$, is plotted on the right.

Two tip problem

We can also consider a different mode of fracture. So far, we have been imposing both a κ_I and κ_{II} fracture condition at the origin but only looking at fracture controlled by the κ_I condition. If the κ_{II} value is small, the solid will fracture by slipping, and a second dry crack will extend a length L beyond the wet tip. The fracture is then controlled by the κ_{II} value, with the various relationships plotted in figure 4.

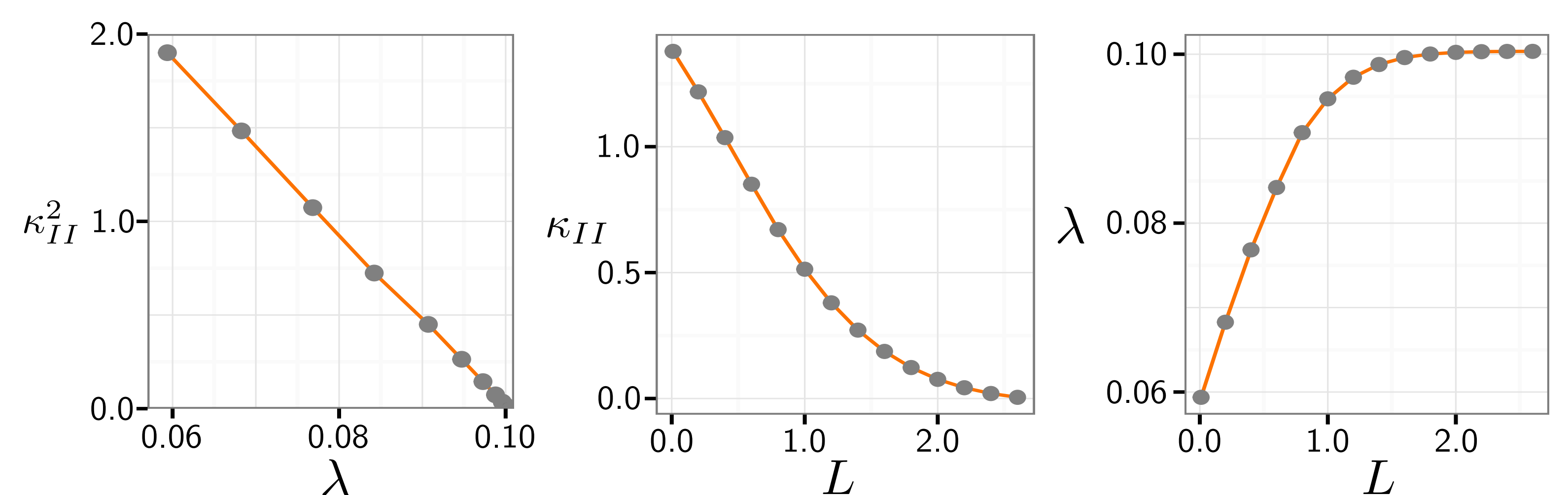


Figure 4 : These graphs show the interdependence of L , λ and κ_{II} , for the two tip problem. Physically κ_{II} is the independent variable (but not numerically).

Note the (approximately) linear relationship between κ_{II}^2 and λ . From conservation of energy, and fracture mechanics, one expects $\alpha\lambda + \beta\kappa_{II}^2 = \text{const.}$, where in theory α, β depend on H , and so κ_{II} . In practice α and β are almost constant, as seen from the graph.

Overall results

Solving the one tip problem for various κ_I, κ_{II} values gives a graph in the κ_I, κ_{II} -plane. From this we can determine where the fracture is controlled by κ_I and where it is controlled by κ_{II} . For $\kappa_I > 1.9$ and $\kappa_{II} > 1.5$, the fracture cannot propagate, the solid is too tough.

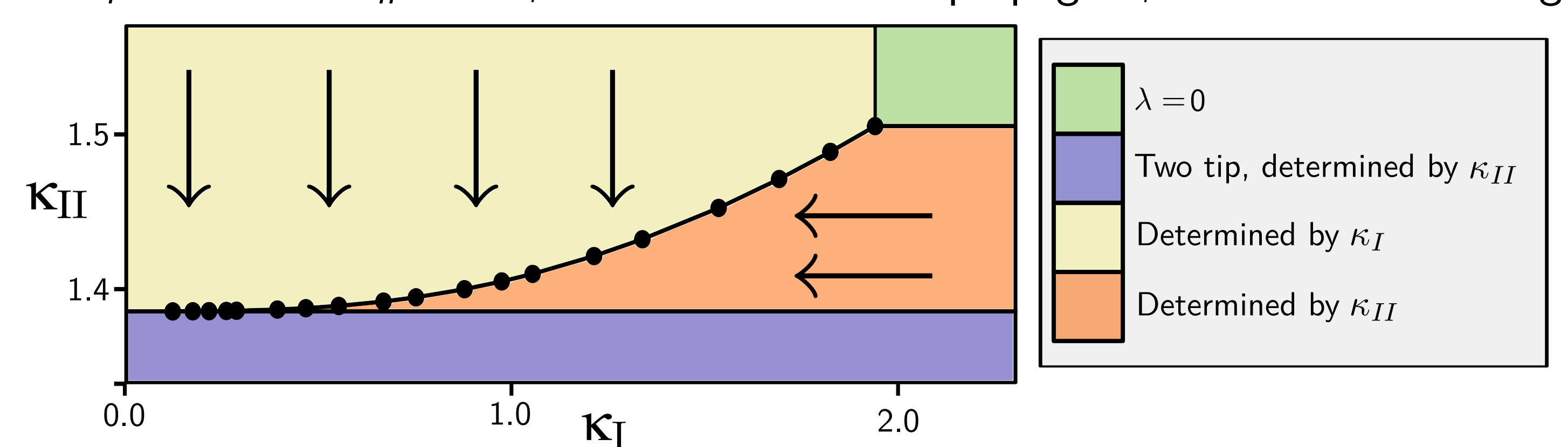


Figure 5 : For a given (κ_I, κ_{II}) value, this diagram shows where the fracture speed is limited by the κ_I or κ_{II} value, and where there is a two tip fracture.

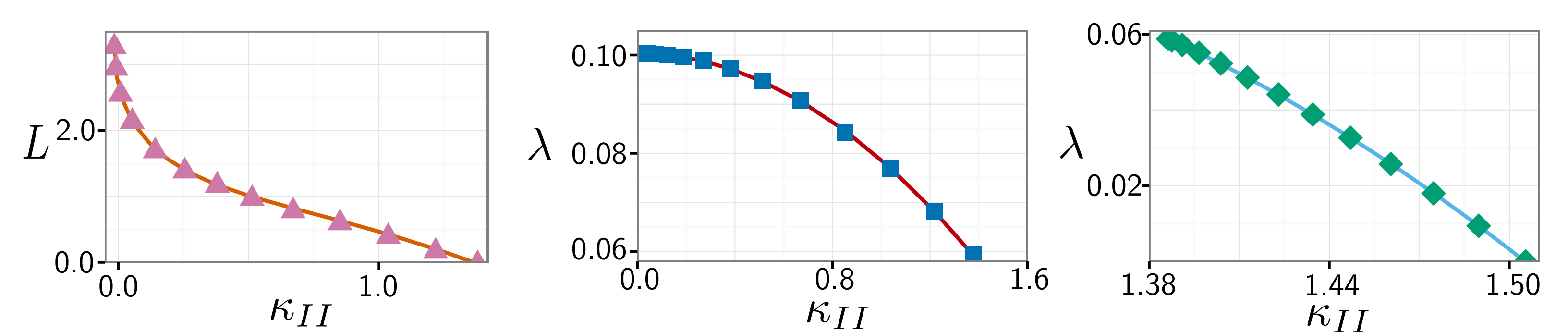


Figure 6 : When the fracture speed is limited by κ_{II} , these graphs provide a way of calculating λ, L in terms of κ_{II} . The two graphs on the left are for $L > 0$ and the graph on the right is for $L = 0$.

References

- [1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.