

Summary of summer project

Dominic Skinner

Introduction

Consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the “arm” of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions c , the magnitude of the bending moment M , and the toughness of the solid K_I .

Relevant physical problems include the formation of hydrofractures in an oil reservoir, or of igneous intrusions beneath a volcano, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

This is a joint work with Tim Large, who started this project in summer 2014.

Set up

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h^2 p'$$

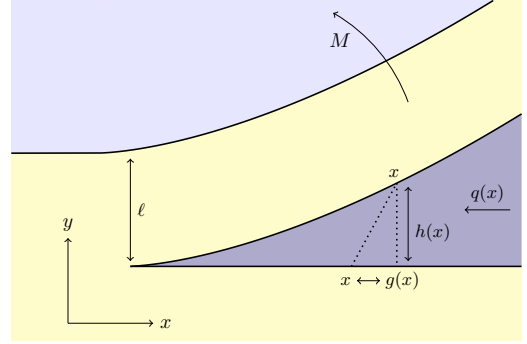
From elasticity, using Muskhelishvili methods, we can derive the equation

$$\begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix}$$

Where K_{ij} is the integral kernel specific to this geometry.

E is the Young's modulus of the solid, ν is Poisson's ratio, μ is the viscosity of the fluid. The boundary conditions on the solid at ∞ are governed by the bending moment, using the beam approximation. The boundary conditions near the crack tip are governed by “Linear Elastic Fracture Mechanics”.

A problem of particular interest is the “Small toughness” problem, when $K_I \ll M\ell^{-3/2}$. The near tip asymptotics when $K_I = 0$, namely $h \sim x^{2/3}$ need to be reconciled with the asymptotics for $K_I > 0$, where $h \sim x^{1/2}$. The result is a near tip boundary layer.



Results

The two equations above (together with boundary conditions) have been solved numerically. Analytic methods have also been applied; it was found that by approaching asymptotically close to the crack tip, the geometry reduces to a known problem. This allows us to obtain the “Small toughness” solution

$$c = \frac{36(1-\nu^2)^2 M^3}{\pi \mu E^2 \ell^5} \left(\lambda_0 + C \lambda_0^{2s-1} \lambda_1 (\ell^{3/2} K_I / M)^u \right)$$

The constants λ_0, λ_1 , C , u, s , have been determined numerically. This result is in close agreement with numerical solutions.