Rescaling the equations - Summary

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Here we will recap the scalings, and then explain how we will switch between the "$h'' \to 1$" and the "$\lambda = 1$" scalings.

Full equations:

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$12\mu c = h^2 p' \tag{2}$$

$$\begin{cases} \lim\limits_{x\to\infty} h''(x) &= \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim\limits_{x\to\infty} g'(x) &= \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \tag{3}$$

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \tag{4}$$

1 Scale $h'' \to 1$

The more common used scaling.

$$\begin{aligned} x &= \ell\xi & p(x) &= \beta_1 \mathscr{P}(\xi) \\ h(x) &= \alpha_1 \mathscr{H}(\xi) & g(x) &= \alpha_1 \mathscr{G}(\xi) \\ \underline{\underline{K}}(x) &= \frac{1}{\ell} \underline{\underline{\Lambda}}(\xi) & K_I &= M\ell^{-3/2}\mathscr{K} \end{aligned}$$

Where $\alpha_1 = \dfrac{12(1-\nu^2)M}{E\ell}$ and $\beta_1 = \dfrac{3M}{\pi\ell^2}$. The equations become

$$\begin{pmatrix} \mathscr{P}(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi}-\xi) \begin{pmatrix} \mathscr{G}'(\tilde{\xi}) \\ \mathscr{H}'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{5}$$

$$\lambda = \mathscr{H}^2 \mathscr{P}' \tag{6}$$

$$\begin{cases} \lim_{\xi\to\infty} \mathscr{H}''(x) &= 1 \\ \lim_{\xi\to\infty} \mathscr{G}'(x) &= 1/2 \end{cases} \tag{7}$$

$$\mathscr{K} = \lim_{\xi\to 0} 3\sqrt{2\pi}\sqrt{\xi}\mathscr{H}'(\xi) \tag{8}$$

2 Scale $\lambda = 1$

$$\begin{aligned} x &= \ell\xi & p(x) &= \beta_2 \mathfrak{P}(\xi) \\ h(x) &= \alpha_2 \mathfrak{H}(\xi) & g(x) &= \alpha_2 \mathfrak{G}(\xi) \\ \underline{\underline{K}}(x) &= \frac{1}{\ell}\underline{\underline{\Lambda}}(\xi) & K_I &= M\ell^{-3/2}\mathfrak{K} \end{aligned}$$

Where $\alpha_2 = \left(\frac{48\pi(1-\nu^2)\mu c\ell^2}{E}\right)^{1/3}$, and $\beta_2 = \left(\frac{3\mu c E^2}{4\pi^2(1-\nu^2)^2\ell}\right)^{1/3}$. We then have the relevant equations as

$$\begin{pmatrix} \mathfrak{P}(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi} - \xi) \begin{pmatrix} \mathfrak{G}'(\tilde{\xi}) \\ \mathfrak{H}'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{9}$$

$$1 = \mathfrak{H}^2 \mathfrak{P}' \tag{10}$$

$$\begin{cases} \lim\limits_{\xi\to\infty} \mathfrak{H}''(x) &= \gamma \\ \lim\limits_{\xi\to\infty} \mathfrak{G}'(x) &= \gamma/2 \end{cases} \tag{11}$$

$$\mathfrak{K} = \lim_{\xi\to 0} 3\sqrt{2\pi}\sqrt{\xi}\mathfrak{H}'(\xi) \tag{12}$$

Where $\gamma = M\left(\frac{36(1-\nu^2)^2}{\pi E^2 \mu c \ell^5}\right)^{1/3}$.

3 Relating the two scalings

One can show that

$$\gamma = 1/\lambda^{1/3}$$

$$\mathfrak{G} = \frac{1}{\lambda^{1/3}}\mathscr{G} \qquad \mathfrak{H} = \frac{1}{\lambda^{1/3}}\mathscr{H}$$

$$\mathfrak{P} = \frac{1}{\lambda^{1/3}}\mathscr{P} \qquad \mathfrak{K} = \frac{1}{\lambda^{1/3}}\mathscr{K}$$

The point of this, is that in earlier analysis (done elsewhere), we looked at $\mathscr{H}$ near $\xi = 0$. We found the asymptotic form

$$\lambda = \lambda_0 + \mathcal{E}(\mathscr{K})\lambda_1 + \dots$$

$$\mathscr{H}(\xi) = (A_0\xi^{2/3} + \dots) + \mathcal{E}(\mathscr{K})\left(\frac{A_0\lambda_1}{3\lambda_0}\xi^{2/3} + \xi^s + \dots\right) + \dots$$

$$\mathscr{P}(\xi) = (-\frac{3\lambda_0}{A_0^2}\xi^{-1/3} + \dots) + \mathcal{E}(\mathscr{K})\left(\frac{2\pi A_0\lambda_1}{9\lambda_0\sqrt{3}}\xi^{-1/3} + \frac{4\pi}{9\sqrt{3}(1-s)}\xi^{s-1} + \dots\right) + \dots$$

$$\mathcal{E}(\mathscr{K}) = C\mathscr{K}^u\lambda_0^{2s-1}$$

The reason we've bothered with this other scaling $(\mathfrak{H}, \mathfrak{K}, \dots)$, is that one notices that the equations near zero are almost exactly the same. The only difference is that λ is set to

1. Since all of this is based near $\xi = 0$, the boundary conditions at ∞ are not important. Thus we have that

$$\mathfrak{H}(\xi) = \left(\left(\frac{243}{4\pi^2} \right)^{1/6} \xi^{2/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) \left(\xi^s + \dots \right) + \dots$$

$$\mathfrak{P}(\xi) = \left(- \left(\frac{2\pi}{3} \right)^{2/3} \xi^{-1/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) \left(\frac{4\pi}{9\sqrt{3}(1-s)} \xi^{s-1} + \dots \right) + \dots$$

$$\mathcal{E}(\mathfrak{K}) = C \mathfrak{K}^u$$

from which one can calculate C. The point is that it's the same C in both equations, which must be some numerical constant dependent on the near tip geometry. One hopes that it is easier to calculate C in the $\mathfrak{H}, \mathfrak{K}, \dots$ scaling.