Rescaling the equations and boundary conditions

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July 9, 2016

In this document, we ignore all of the physics, geometry of the problem, and the approximations made, to just consider the governing equations on their own. Recall that with equations 1,2 can solve for K_I given the BC's at ∞ or alternatively, given the BC's at ∞, we can solve for K_I.

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$12\mu c = h^2 p' \tag{2}$$

$$\begin{cases} \lim\limits_{x\to\infty} h''(x) & = & \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim\limits_{x\to\infty} g'(x) & = & \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \tag{3}$$

$$K = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \tag{4}$$

We switch to working with dimensionless variables,

$$(x, p, h, g, \underline{\underline{K}}) \to (\xi, \Pi, H, G, \underline{\underline{\Lambda}})$$

Let us do this by the following transforms:

$$\begin{aligned} x &= \ell\xi \\ p(x) &= \beta\Pi(\xi) \\ h(x) &= \alpha H(\xi) \\ g(x) &= \alpha G(\xi) \\ \underline{\underline{K}}(x) &= \frac{1}{\ell}\underline{\underline{\Lambda}}(\xi) \end{aligned}$$

At this point we have already made some choices. I would claim the choices made this far are "natural" although I don't have any more justification than that. The equations become

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \frac{E\alpha}{4\pi(1-\nu^2)\beta\ell} \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi}-\xi) \begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{5}$$

$$\frac{12\mu c\ell}{\alpha^2\beta} = H^2\Pi' \tag{6}$$

$$\begin{cases} \lim_{\xi\to\infty} H''(x) &= \frac{12(1-\nu^2)}{E\ell\alpha}M \\ \lim_{\xi\to\infty} G'(x) &= \frac{6(1-\nu^2)}{E\ell\alpha}M \end{cases} \tag{7}$$

$$K_I = \lim_{\xi\to 0} \frac{E\alpha}{1-\nu^2}\sqrt{\frac{\pi}{8\ell}}\sqrt{\xi}H'(\xi) \tag{8}$$

We now have several choices. The first to be explored, is to take equations 5 and 6 and set the dimensionless parameters appearing in them to unity.

1 Scaling out c

Set $\dfrac{E\alpha}{4\pi(1-\nu^2)\beta\ell} = 1$ and $\dfrac{12\mu c\ell}{\alpha^2\beta} = 1$ which uniquely determines α, β as

$$\alpha^3 = \frac{48\pi(1-\nu^2)\mu c\ell^2}{E} \qquad \beta^3 = \frac{3\mu c E^2}{\pi(1-\nu^2)\ell}$$

We then have the relavent equations as

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi}-\xi) \begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{9}$$

$$1 = H^2\Pi' \tag{10}$$

$$\begin{cases} \lim_{\xi\to\infty} H''(x) &= \gamma \\ \lim_{\xi\to\infty} G'(x) &= \gamma/2 \end{cases} \tag{11}$$

$$K_I = \lim_{\xi\to 0} \frac{E\alpha}{1-\nu^2}\sqrt{\frac{\pi}{8\ell}}\sqrt{\xi}H'(\xi) \tag{12}$$

Where $\gamma = M\left(\frac{36(1-\nu^2)^2}{\pi E^2\mu c\ell^5}\right)^{1/3}$. Given equations 9,10,11, we can solve for K_I. It is clear from the form of them, that the only way the physical parameters can enter the solution is through γ. Thus, the relationship between K_I and the other physical quantities *must* be of the form

$$K_I = E^{2/3}\mu^{1/3}c^{1/3}\ell^{1/6}(1-\nu^2)^{-2/3}f(\gamma)$$

N.B. if all we are interested in is K_I, c, M, get the relationship

$$K_I = c^{1/3}\tilde{f}(M/c^{1/3})$$

but this does depend on other physical parameters.

2 Scaling out γ

Set $\dfrac{E\alpha}{4\pi(1-\nu^2)\beta\ell} = 1$ and $\dfrac{12(1-\nu^2)}{E\ell\alpha}M = 1$. We get that

$$\alpha = \frac{12(1-\nu^2)M}{E\ell} \qquad \beta = \frac{3M}{\pi\ell^2}$$

We then have the relevant equations as

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{A}}(\tilde{\xi}-\xi)\begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{13}$$

$$\lambda = H^2\Pi' \tag{14}$$

$$\begin{cases} \lim_{\xi\to\infty} H''(x) & = & 1 \\ \lim_{\xi\to\infty} G'(x) & = & 1/2 \end{cases} \tag{15}$$

$$K_I = \lim_{\xi\to 0} 3M\sqrt{\frac{\pi}{2\ell^3}}\sqrt{\xi}H'(\xi) \tag{16}$$

Where $\lambda = \frac{\pi\mu c E^2\ell^5}{36(1-\nu^2)^2M^3}$. Thus K_I must be of the form

$$K_I = M\ell^{-3/2}s(\lambda)$$

for some function s. This is not a new relationship, writing $s = \lambda^{1/3}\tilde{s}(\lambda)$ recovers the exact same relationship that we had earlier.