Viscous control of shallow elastic fracture

Tim Large[1], John Lister[2], and Dominic Skinner[2]

[1]M.I.T., USA

[2]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK

(Received xx; revised xx; accepted xx)

This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/relatedlink/jfm-keywords.pdf for the full list)

1. Introduction

Here we review the literature as well as describe the problem in more detail. We have the vertical displacement h, the horizontal displacement g, the thickness of the arm l, and the pressure p. We look for a travelling wave solution (propagating left), with speed c.

2. Formulation of problem

From lubrication, we have Poiselle flow in the crack. We have the flux, and conservation of mass as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3 \qquad \frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.1}$$

Which combined gives us

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2 \tag{2.2}$$

Setting $p \to 0$ at $x \to \infty$, we can write this in integral form.

$$p(x) = -\int_x^\infty 12\mu c/h(\tilde{x})^2 \mathrm{d}\tilde{x} \tag{2.3}$$

From the linear theory of elasticity, due to others who have studied this problem, we have

$$\begin{bmatrix} -\sigma_y \\ -\tau_{xy} \end{bmatrix} = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\tilde{x} - x) \begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix} \mathrm{d}\tilde{x} \tag{2.4}$$

Where the integral kernel is

$$\boldsymbol{K}(x) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{l^3(32l^2-24x^2)}{(x^2+4l^2)^3} & \frac{l^4(48x^2-64l^2)}{x(x^2+4l^2)^3} \\ -\frac{l^2(16x^4+16x^2l^2+4l^4)}{x(x^2+4l^2)^3} & -\frac{l^3(32l^2-24x^2)}{(x^2+4l^2)^3} \end{bmatrix} \tag{2.5}$$

We have the boundary conditions near $x = 0$ governed by fracture mechanics

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\,h'(x)\,, \qquad K_{II} = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\,g'(x) \tag{2.6a, b}$$

As we go to $x \gg l$, we are looking at the problem of peeling off a thin strip from an elastic half space. We can then use beam theory approximations, which give

$$M(x) = \frac{El^3}{12(1-\nu^2)}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = \frac{El^3}{6(1-\nu^2)}\frac{\mathrm{d}g}{\mathrm{d}x}, \qquad p = \frac{El^3}{12(1-\nu^2)}h^{(4)}(x) \tag{2.7a, b}$$

As $x \to \infty$, $M(x) \to M$, the applied bending moment, so this gives us boundary conditions on h'', g'.

2.1. *Rescaling*

We change into a set of dimensionless variables. We have a length scale l, a pressure scale $p^* = E/12(1-\nu^2)$, and a time scale $t^* = 12\mu/p^*$. From these, we can define the following dimensionless variables

$$x = l\xi, \quad K_{ij}(x) = U_{ij}(\xi)/l, \quad h(x) = \frac{M}{p^* l}H(\xi), \quad g(x) = \frac{M}{p^* l}G(\xi) \tag{2.8}$$

$$p = \frac{3M}{\pi l^2}\Pi(\xi), \quad K_I = Ml^{-3/2}\kappa_I, \quad K_{II} = Ml^{-3/2}\kappa_{II}, \quad \lambda = \frac{4\pi\mu p^* l^3}{M^2} \tag{2.9}$$

With these scalings, the equations become

$$\begin{bmatrix} \Pi \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{U}(\tilde{\xi} - \xi)\begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi} \tag{2.10}$$

$$H^2\frac{\mathrm{d}\Pi}{\mathrm{d}\xi} = \lambda \quad \text{or} \quad \Pi(\xi) = -\int_\xi^\infty \lambda/H(\tilde{\xi})^2\mathrm{d}\tilde{\xi} \tag{2.11a, b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2}, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' = \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}G' = \kappa_{II}, \tag{2.12}$$

These shall be the governing equations for the rest of this paper.

The equations of the linear perturbation problem:

$$\Pi = \Pi_0 + \mathcal{E}\Pi_1 + O(\mathcal{E}), \quad H = H_0 + \mathcal{E}H_1 + O(\mathcal{E}) \tag{2.13}$$

$$\begin{bmatrix} \Pi_1 \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{U}(\xi - \tilde{\xi})\begin{bmatrix} G_1'(\tilde{\xi}) \\ H_1'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi}, \qquad H_0^2\Pi_1' + 2H_0H_1\Pi_0' = \lambda_1 \tag{2.14a, b}$$

$$H_1'' \to 0 \text{ as } \xi \to \infty, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3} + \dots \text{ as } \xi \to 0 \tag{2.15a, b}$$

But these can be made into a more convenient form, by considering instead $\tilde{\Pi} = \Pi_0 -$

$3\lambda_0/\lambda_1\Pi_1$, and similar for $\tilde{H}$, $\tilde{G}$. The equations become

$$\begin{bmatrix}\tilde{\Pi}\\0\end{bmatrix} = \int_0^\infty \boldsymbol{U}(\xi-\tilde{\xi})\begin{bmatrix}\tilde{G}'(\tilde{\xi})\\\tilde{H}'(\tilde{\xi})\end{bmatrix}\mathrm{d}\tilde{\xi}, \qquad H_0^2\tilde{\Pi}'+2H_0\tilde{H}\Pi_0'=0 \tag{2.16a,b}$$

$$\tilde{H}''\to 1 \text{ as } \xi\to\infty, \qquad \tilde{H}\sim -\frac{3\lambda_0}{\lambda_1}\xi^s+\ldots \text{ as } \xi\to 0 \tag{2.17a,b}$$

These are the equations for the two tip problem

$$\begin{bmatrix}\Pi\\0\end{bmatrix} = \int_{-L}^\infty \boldsymbol{U}(\tilde{\xi}-\xi)\begin{bmatrix}G'(\tilde{\xi})\\H'(\tilde{\xi})\end{bmatrix}\mathrm{d}\tilde{\xi}, \qquad \Pi=\int_\xi^\infty \lambda/H(\tilde{\xi})^2\mathrm{d}\tilde{\xi} \tag{2.18a,b}$$

$$\lim_{\xi\to\infty}H''=1, \quad \lim_{\xi\to\infty}G'=\frac{1}{2} \tag{2.19a,b}$$

$$\lim_{\xi\to0}3\sqrt{2\pi\xi}H'=0, \quad \lim_{\xi\to-L}3\sqrt{2\pi\xi}G'=\kappa_{II} \tag{2.20a,b}$$

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking n points $\boldsymbol{\xi}=(\xi_1=0,\xi_2\ldots,\xi_n)$ at which we measure H', G', and $n-1$ intermediate points $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_{n-1})$ at which to measure Π, so that $\xi_1<\zeta_1<\ldots<\zeta_{n-1}<\xi_n$. We work with $\sqrt{\xi}G'(\xi)$, $\sqrt{\xi}H'(\xi)$ near the tip to avoid singularities. We interpolate as

$$G'(\xi)=\begin{cases}\xi^{-1/2}(a_i\xi+b_i)\\a_i\xi+b_i\end{cases}, \quad H'(\xi)=\begin{cases}\xi^{-1/2}(c_i\xi^{1/2}+d_i)\\c_i\xi+d_i\end{cases}, \quad \xi_i<\xi<\xi_{i+1} \quad \begin{cases}i<t\\i\geqslant t\end{cases} \tag{3.1}$$

Where typically $t=n/2$ was used. The choice of interpolating function was based on the appearance of the relevant functions. We will also define a_n,b_n,c_n,d_n for interpolation beyond ξ_n. We define $\boldsymbol{a}=[a_1,\ldots,a_n]$, and similarly with $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. We also define, for convenience, the column vector $\boldsymbol{\gamma}=[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}]$.

Given this interpolation, the elasticity integral (2.11a), is exact; that is to say there is an known analytic expression for $\Pi(\xi)$. We also have a different expression for Π, due to the lubrication integral (**??**). As before; given the interpolation, we can do the integral exactly.

In equation 2.11a, the integral depends linearly on $\boldsymbol{\gamma}$. Therefore, we can write

$$[\Pi(\zeta_1),\ldots,\Pi(\zeta_{n-1}),\underbrace{0,\ldots,0}_{n-1}]=\boldsymbol{J}\boldsymbol{\gamma} \tag{3.2}$$

where $\boldsymbol{J}$ is a matrix representing the integral kernel. We also know that $[\Pi(\zeta_1),\ldots,\Pi(\zeta_{n-1})]=f_1(\boldsymbol{c},\boldsymbol{d})$, from the lubrication integral. This time, Π does not depend linearly on $\boldsymbol{\gamma}$.

We define $\boldsymbol{\theta}_G=[a_1\xi_1+b_1,\ldots,a_n\xi_n+b_n]$, $\boldsymbol{\theta}_H=[c_1\xi_1^{1/2}+d_1,\ldots,c_n\xi_n^{1/2}+d_n]$, as well as $\boldsymbol{\theta}=[\boldsymbol{\theta}_G,\boldsymbol{\theta}_H]$. We would prefer to work with $\boldsymbol{\theta}$ over $\boldsymbol{\gamma}$, since it contains half as many elements. Continuity of G', H' imposes $2(n-1)$ equations. For G' they are

$$\begin{aligned} a_i\xi_{i+1}+b_i &= a_{i+1}\xi_{i+1}+b_{i+1}, \text{ for } i<n,\ i\neq t-1\\ \xi^{-1/2}(a_i\xi_{i+1}+b_i) &= a_{i+1}\xi_{i+1}+b_{i+1}, \text{ for } i=t-1\end{aligned} \tag{3.3}$$

with similar equations for H' (accounting for the slightly different interpolation). From our asymptotic expansion (via beam theory) we know $\theta_n = G'(\xi_n)$ and $a_n = G''(\xi_n)$. Therefore we can write

$$a_n = \frac{G''(\xi_n)}{G'(\xi_n)}\theta_n, \qquad b_n = \theta_n - a_n\xi_n = \left(1 - \frac{G''(\xi_n)}{G'(\xi_n)}\right)\theta_n \tag{3.4}$$

With H, we know that $a_n = H''(\xi_n)$, $a_{n-1} = H''(\xi_{n-1})$, and so we have that

$$a_n = \frac{H''(\xi_n)}{H''(\xi_{n-1})}a_{n-1}, \qquad b_n = -a_n\xi_n + a_{n-1}\xi_n + b_{n-1} \tag{3.5}$$

Therefore, we have enough equations to calculate a matrix $\boldsymbol{T}$, so that

$$\gamma = \boldsymbol{T\theta} \tag{3.6}$$

We now have that

$$\boldsymbol{JT\theta} = f_2(\boldsymbol{\theta}) \tag{3.7}$$

With $\boldsymbol{JT}$ a $2(n-1)\times 2n$ matrix, and f_2 a function of $\boldsymbol{\theta}$ (really just of $\boldsymbol{\theta}_H$), with the first $n-1$ components calculating Π, and the last $n-1$ components set to 0.

Both $G'(\xi_n) = \theta_n$, and $H''(\xi_n) \approx (\theta_{2n} - \theta_{2n-1})/(\xi_n - \xi_{n-1})$, are known, and are linear in $\boldsymbol{\theta}$. Therefore we can add another two rows to $\boldsymbol{JT}$, and replace f_2 with $f = [f_2, G'(\xi_n), H''(\xi_n)]$. Let the enlarged matrix be called $\boldsymbol{A}$. Then

$$\boldsymbol{A\theta} = f(\boldsymbol{\theta}) \tag{3.8}$$

This can be solved by Newton's method, from quite arbitrary initial guesses.

Note that we choose a value of λ, fix the boundary conditions at $\xi \to \infty$, then solve the problem and subsequently recover the boundary conditions at $\xi = 0$ (κ_I, κ_{II}). This can then be inverted, so that we think of $\lambda = \lambda(\kappa_I)$. Physically, we know κ_I, and want to find λ, but in numerically solving the problem, it makes more sense to choose λ and recover κ_I.

The spacing of the points $\boldsymbol{\xi}$ makes a significant difference to the convergence properties. The spacing should reflect that the important part of the problem is happening near the tip, and this is where the points should be concentrated. The spacing most often used is

$$\xi_i = \tan(\chi\, i/m)^2, \quad i = 1, \dots, m < n \tag{3.9}$$

where χ is chosen so that $\tan(\chi)^2 = O(10)$, and the remaining points are added in a geometric progression, so that

$$\xi_{i+1} = (\xi_m/\xi_{m-1})\xi_i, \quad i = m, \dots, n-1 \tag{3.10}$$

3.2. *Linear Perturbation Problem*

From equation 2.17b, we anticipate a singularity of the form ξ^{s-1} in $\tilde{H}'$, (we still expect a $\xi^{-1/2}$ singularity in $\tilde{G}'$). Therefore, the interpolation used is

$$\tilde{G}'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad \tilde{H}'(\xi) = \begin{cases} \xi^{s-1}(c_i\xi + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for} \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.11}$$

We compute the $\boldsymbol{J}$, $\boldsymbol{T}$ in a similar way as before, but taking into account the new interpolation. Some of the integrals no longer have exact expressions. In this case, they are calculated by a numerical integration routine.

The lubrication equation for the linear perturbation problem (2.16b) is first transformed into an integral, where we replace Π_0' by λ_0/H_0^2 and use the boundary conditions

$\Pi_0, \Pi_1 \to 0$ as $\xi \to \infty$. We get the equation

$$\tilde{\Pi}(\zeta) = \int_{\zeta}^{\infty} \frac{2\lambda_0 \tilde{H}}{H_0^3} \mathrm{d}\xi \tag{3.12}$$

This is linear in $\tilde{H}$, which is linear in $\boldsymbol{c}$, $\boldsymbol{d}$ (after integrating H' over each spline and imposing continuity). The integrals that appear were determined numerically. We end up with a matrix $\boldsymbol{R}$, such that

$$[\tilde{\Pi}(\zeta_1), \dots \tilde{\Pi}(\zeta_{n-1})] = \boldsymbol{R\theta} \tag{3.13}$$

Padding out $\boldsymbol{R}$ with zeros, until it is of size $2n \times 2n$, we see that

$$\left(\boldsymbol{A} - \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix}\right) \boldsymbol{\theta} = [0, \dots, 0, \tilde{G}'(\xi_n), \tilde{H}''(\xi_n)] \tag{3.14}$$

Since we haven't changed the integral kernel, the beam theory asymptotics remain the same, and together with 3.12, we have an asymptotic expression for $\tilde{H}''(\xi_n)$, $\tilde{G}'(\xi_n)$. Here, we do not need to deploy Newton's method, as we can simply solve the linear set of equations, 3.14.

3.3. *Double Tip*

In solving the problem of two tips situated at $-L$ and 0, it is given that $H = 0$ for $\xi < 0$, and thus $H' = 0$ for $\xi < 0$. We take n points to cover $0 \leqslant \xi < \infty$, and r points to cover $-L \leqslant \xi < 0$. we label these points so that

$$\begin{array}{l} \boldsymbol{\xi} = [\ \xi_{1-r} = -L,\ \xi_{2-r},\ \dots,\ \xi_1 \ = 0,\ \xi_2\ ,\ \dots,\ \xi_t\ ,\ \dots,\ \xi_n\ ] \\ \boldsymbol{v} = [\ v_1 \ = \ 0,\ v_2\ ,\ \dots,\ v_{r+1} = L,\ v_{r+2},\ \dots,\ v_{t+r},\ \dots,\ v_{n+r}] \end{array} \tag{3.15}$$

Where $v_i = \xi_i + L$. We similarly define $\boldsymbol{\zeta}$ and $\boldsymbol{w}$, so that $w_i = \zeta_i + L$. It is useful to introduce a new function $F(v) = F(L + \xi) = G(\xi)$. We interpolate H' as before, and F' as,

$$F'(v) = \begin{cases} \xi^{-1/2}(a_i v + b_i) & v_i < v < v_{i+1} \quad i < t + r \\ a_i \xi + b_i & v_i < v < v_{i+1} \quad i \geqslant t + r \end{cases} \tag{3.16}$$

In the elasticity integral, the range of integration is $[-L, \infty]$. Since $H = 0$ for $\xi < 0$, these limits can be replaced with $[0, \infty]$ for any integral of H. We can rewrite integrals involving G as

$$\int_{-L}^{\infty} U_{11}(\xi - \zeta) G'(\xi) \mathrm{d}\xi = \int_0^{\infty} U_{11}(v - (z + L)) F'(v) \mathrm{d}v = \int_0^{\infty} U_{11}(v - w) F'(v) \mathrm{d}v \tag{3.17}$$

Writing the equation in this way, it is clear that the numerical problem is almost exactly the same as for the single tip problem. In fact, the lubrication equation **??**, is entirely unchanged. We do not calculate Π for $\xi < 0$ (although it is easily done), but require that $\sigma_{xy} = 0$ for $\xi < 0$. This provides enough equations for the problem to be solved as before, with Newton's method. We input $-L$ and λ and recover κ_I, κ_{II}, where κ_I is measured at 0. Physically, for $L > 0$, we must have $\kappa_I = 0$. Numerically we solve for some λ, L, find $\kappa_I > 0$ and extrapolate to $\kappa_I = 0$.

The spacing of points for $\xi < 0$ was chosen so that there was a concentration of points near $-L$ and near 0.

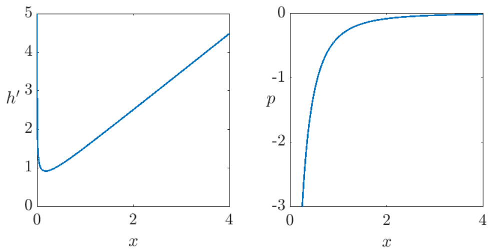

FIGURE 1. This is a typical example of what H' and Π look like

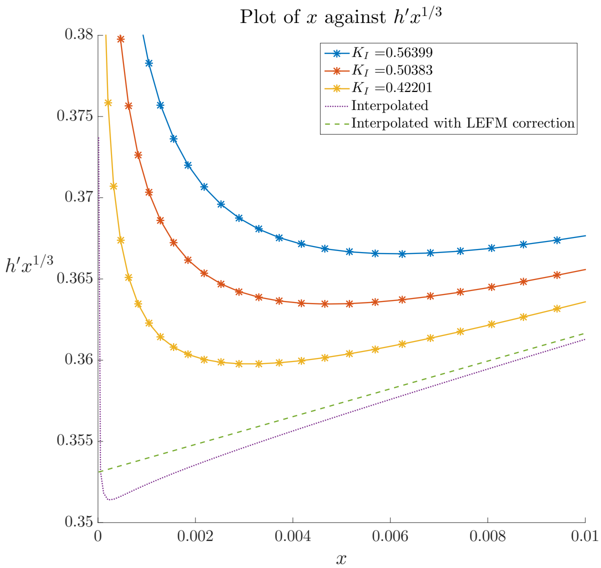

FIGURE 2. This shows some evidence of the LEFM boundary layer

4. Results

4.1. *Single tip*

Start off with some of the basic graphs showing H',G', and Π against ξ.

Obvious questions to ask at this point are: How are you sure this is the right answer, what is the effect of n, ξ_{end}? In the next graph, we determine the effect of extending ξ_{end}, by adding on extra points (so maintaining the same resolution near the tip). There is a satisfactory demonstration of convergence.

By adding points on in a geometric progression, it becomes quite cheap to extend out to $\xi_{\text{end}} \approx 800$ or so. Once one has done this, it is apparent that the effect of the tip resolution dominates the effect of finite truncation, as the following figure shows.

By increasing n (for large ξ_{end}), we have been able to determine λ_0 and D

4.2. *Linear perturbation problem*

We solve the linear perturbation problem. All that we really want to know is that we see the ξ^{s-1} behaviour that we expect, and we ask what the intercept of $\tilde{H}_1$ is. It is perhaps worth mentioning the difficulties in measuring the intercept and perhaps a

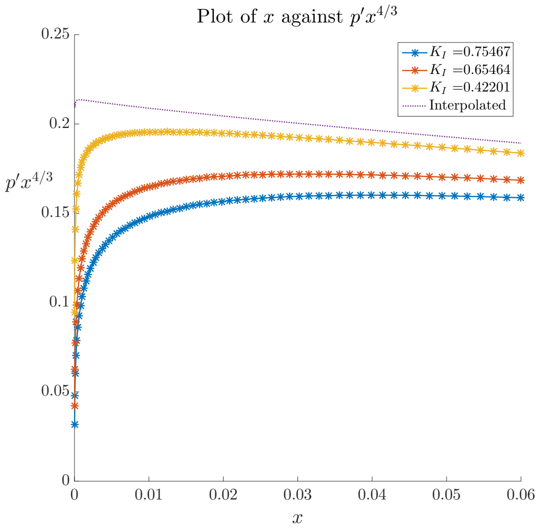

FIGURE 3. This also shows some evidence of the LEFM boundary layer

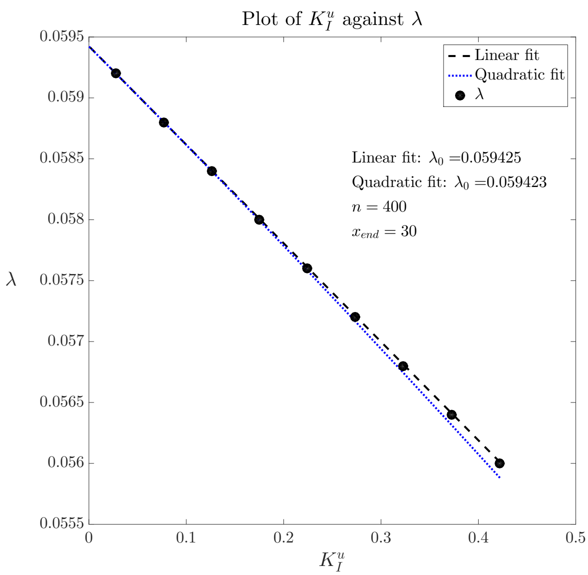

FIGURE 4. Good agreement between theory and numerics here

notion of the sensitivity of the result on the estimate provided for H_0. Illustrating that is the next figure

Then we include the figure that shows convergence with different n values to something approaching the right answer.

4.3. *Two tips*

After the linear perturbation problem, we move on to the two tip problem. Perhaps some graphs that show an outline of the full numerical problem with non-zero κ_I and κ_{II}, although these are not physical.

What would be nice, although it doesn't exist yet, is some sort of record of how we

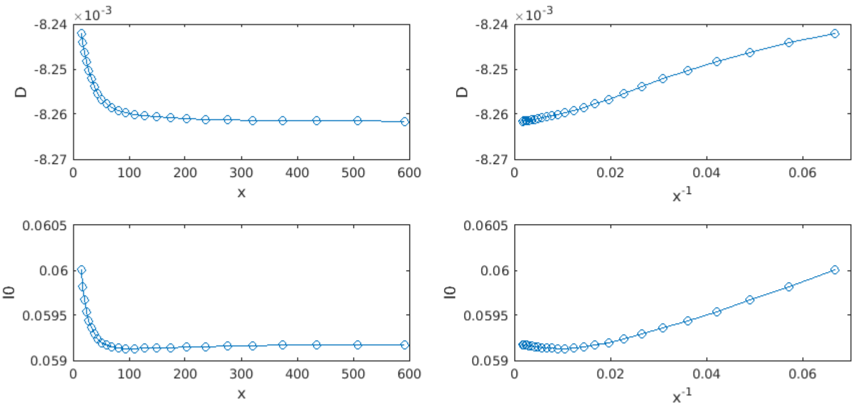

FIGURE 5. Satisfactory convergence as $\xi_{\text{end}} \to \infty$

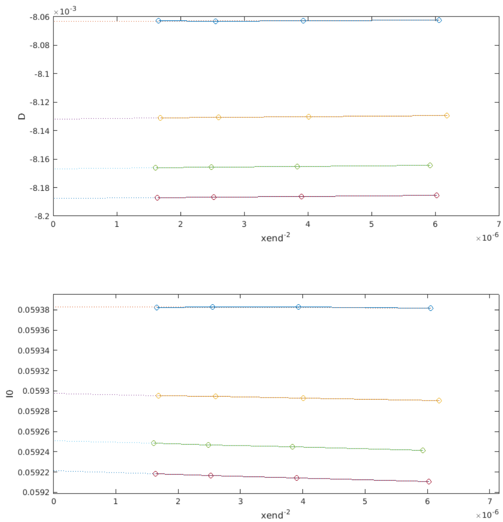

FIGURE 6. Shows the relatively small effect that ξ_{end} has compared to the tip resolution, once ξ_{end} has become large enough

now extrapolate to $\kappa_I = 0$. This is certainly a plot that needs to be made. We now move on to the $\kappa_I = 0$ set of relations.

5. Discussion

This is where we discuss the figures, possibly include more figures, and draw the results and conclusions of this paper.

Perhaps the first thing worth mentioning is the somewhat contrived, but pretty accurate formulae for λ in terms of κ_I. This holds for any toughness in the single tip case.

Then we could move on to talk about the decoupling between the fluid problem and the dry fracture problem. Relavent graphs to include would show that H really doesn't

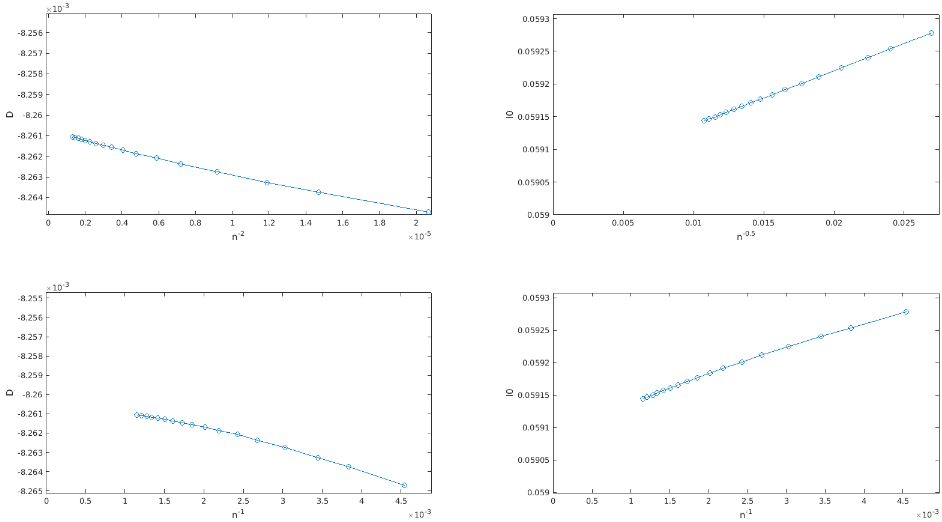

FIGURE 7. Our best guess at λ_0 and D, and the approximate error one can expect in them

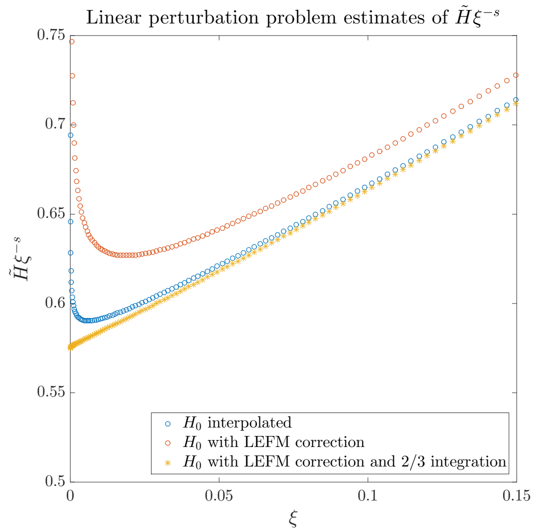

FIGURE 8. Sensitivity of linear perturbation problem on H_0

vary much with λ_0, and that given a reference H', one can construct G' with relative ease.

At this point, I would like to construct another contrived formulae for the two tip problem. Then I would like to plot a graph of κ_I against κ_{II} in the full fluid problem. This provides a guide of when it is appropriate to take the single tip, and when it is appropriate to take the double tip.

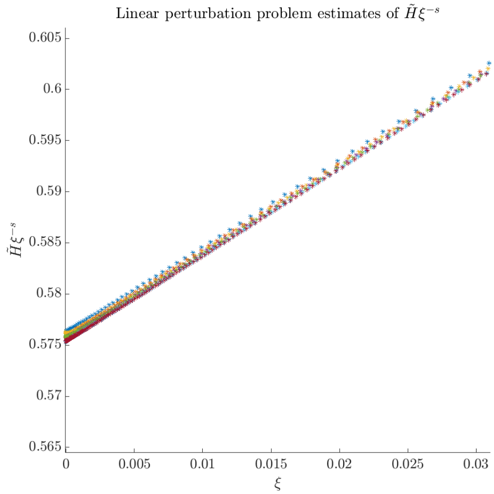

Figure 9. Sensitivity of linear perturbation problem on H_0

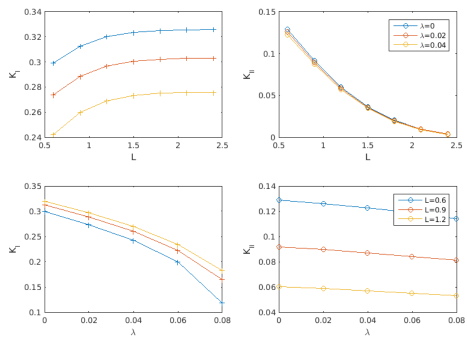

Figure 10. Visualisation of what is really a surface in 4D

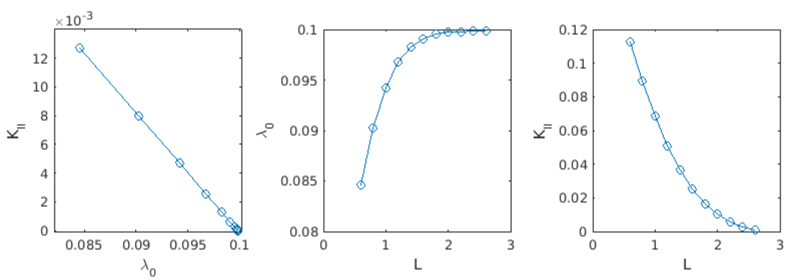

Figure 11. The results of extrapolating to $\kappa_I = 0$

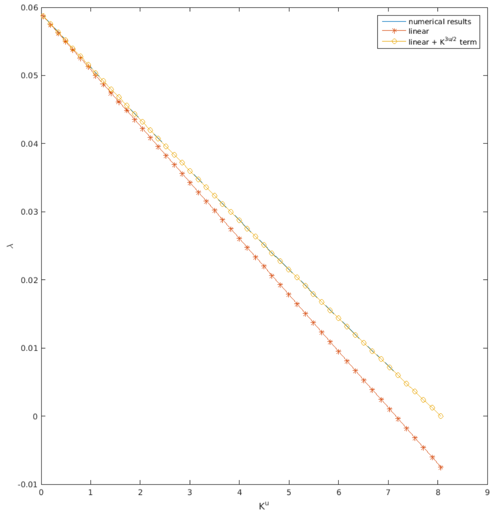

FIGURE 12. The formula valid for all κ_I

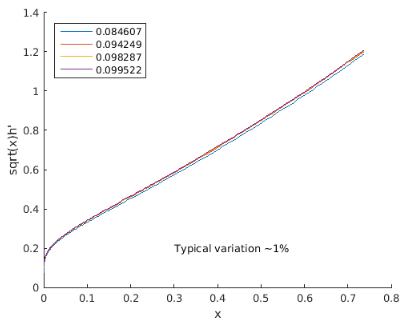

FIGURE 13. Demonstrating the decoupling of fluid and solid fracture

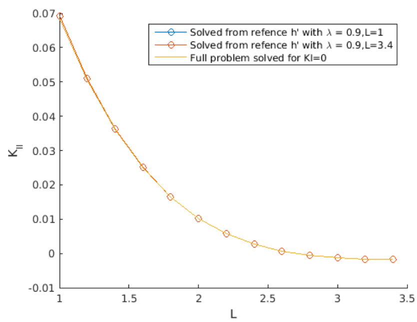

FIGURE 14. Reconstructing the full solution given a reference H'

a/d	$M=4$	$M=8$	Callan *et al.*
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

TABLE 1. Values of kd at which trapped modes occur when $\rho(\theta) = a$

6. Citations and references

All papers included in the References section must be cited in the article, and vice versa. Citations should be included as, for example "It has been shown (Rogallo 1981) that..." (using the `\citep` command, part of the natbib package) "recent work by Dennis (1985)..." (using `\citet`). The natbib package can be used to generate citation variations, as shown below.
`\citet[pp. 2-4]{Hwang70}`:
Hwang & Tuck (1970, pp. 2-4)
`\citep[p. 6]{Worster92}`:
(Worster 1992, p. 6)
`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
`\citep{Brownell04,Brownell07,Ursell50,Wijngaarden68,Miller91}`:
(Brownell & Su 2004, 2007; Ursell 1950; van Wijngaarden 1968; Miller 1991)
The References section can either be built from individual `\bibitem` commands, or can be built using BibTex. The BibTex files used to generate the references in this document can be found in the zip file at http://journals.cambridge.org/data/relatedlink/jfm-ifc.zip.
Where there are up to ten authors, all authors' names should be given in the reference list. Where there are more than ten authors, only the first name should appear, followed by et al.

Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

Appendix A

This appendix contains sample equations in the JFM style. Please refer to the LaTeX source file for examples of how to display such equations in your manuscript.

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \tag{A 1}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{v} = 0, \quad \nabla^2 P = \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{v} \times \boldsymbol{w}). \tag{A 2}$$

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \quad r \equiv |P - Q| \rightarrow 0, \tag{A 3}$$

$$\left.\begin{aligned} \frac{\partial G_s}{\partial y} &= 0 \quad \text{on} \quad y = 0, \\ G_a &= 0 \quad \text{on} \quad y = 0, \end{aligned}\right\} \tag{A 4}$$

$$-\frac{1}{2\pi}\int_0^\infty \gamma^{-1}[exp(-k\gamma|y-\eta|)+\exp(-k\gamma(2d-y-\eta))]\cos k(x-\xi)t\,\mathrm{d}t, \qquad 0 < y, \quad \eta < d, \tag{A 5}$$

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{1/2}, & t \leqslant 1 \\ (t^2-1)^{1/2}, & t > 1. \end{cases} \tag{A 6}$$

$$-\frac{1}{2\pi}\int_0^\infty B(t)\frac{\cosh k\gamma(d-y)}{\gamma \sinh k\gamma d}\cos k(x-\xi)t\,\mathrm{d}t$$

$$G = -\frac{1}{4}\mathrm{i}(H_0(kr)+H_0(kr_1)) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-k\gamma d}}{\gamma \sinh k\gamma d}\cosh k\gamma(d-y)\cosh k\gamma(d-\eta) \tag{A 7}$$

Note that when equations are included in definitions, it may be suitable to render them in line, rather than in the equation environment: $\boldsymbol{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G_a}$ where $r = \{[x(\theta)-x(\psi)]^2 + [y(\theta)-y(\psi)]^2\}^{1/2}$ and $\widetilde{G_a}$ is regular as $kr \to 0$. However, any fractions displayed like this, other than $\frac{1}{2}$ or $\frac{1}{4}$, must be written on the line, and not stacked (ie 1/3).

$$\begin{aligned} \frac{\partial}{\partial n_q}\left(\frac{1}{4}Y_0(kr)\right) &\sim \frac{1}{4\pi w^3(\theta)}[x''(\theta)y'(\theta) - y''(\theta)x'(\theta)] \\ &= \frac{1}{4\pi w^3(\theta)}[\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as} \quad kr \to 0. \end{aligned} \tag{A 8}$$

$$\frac{1}{2}\phi_i = \frac{\pi}{M}\sum_{j=1}^{M}\phi_j K_{ij}^a w_j, \qquad i = 1,\, \ldots,\, M, \tag{A 9}$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i,\theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G_a}(\theta_i,\theta_i)/\partial n_q + [\rho_i'\rho_i'' - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \tag{A 10}$$

$$\rho_l = \lim_{\zeta \to Z_l^-(x)} \rho(x,\zeta), \quad \rho_u = \lim_{\zeta \to Z_u^+(x)} \rho(x,\zeta) \tag{$A\,11a, b$}$$

$$(\rho(x,\zeta), \phi_{\zeta\zeta}(x,\zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \tag{A 12}$$

$$\tau_{ij} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + (\overline{\overline{u}_i u_j^{SGS} + u_i^{SGS}\overline{u}_j}) + \overline{u_i^{SGS}u_j^{SGS}}, \tag{A 13a}$$

$$\tau_j^\theta = (\overline{\overline{u}_j\overline{\theta}} - \overline{u}_j\overline{\theta}) + (\overline{\overline{u}_j\theta^{SGS} + u_j^{SGS}\overline{\theta}}) + \overline{u_j^{SGS}\theta^{SGS}}. \tag{A 13b}$$

$$\boldsymbol{Q}_C = \begin{bmatrix} -\omega^{-2}V'_w & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\ \dfrac{\beta}{\alpha\omega^2}V'_w & 0 & 0 & 0 & \mathrm{i}\omega^{-1} \\ \mathrm{i}\omega^{-1} & 0 & 0 & 0 & 0 \\ \mathrm{i}R_\delta^{-1}(\alpha^t+\omega^{-1}V''_w) & 0 & -(\mathrm{i}\alpha^t R_\delta)^{-1} & 0 & 0 \\ \dfrac{\mathrm{i}\beta}{\alpha\omega}R_\delta^{-1}V''_w & 0 & 0 & 0 & 0 \\ (\mathrm{i}\alpha^t)^{-1}V'_w & (3R_\delta^{-1}+c^t(\mathrm{i}\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0 \end{bmatrix}. \tag{A 14}$$

$$\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[\mathrm{i}(\alpha^t x_1^t - \omega t)], \tag{A 15}$$

where $\hat{\boldsymbol{\eta}}^t = \boldsymbol{b}\exp(\mathrm{i}\gamma x_3^t)$.

$$\mathrm{Det}[\rho\omega^2\delta_{ps} - C^t_{pqrs}k^t_q k^t_r] = 0, \tag{A 16}$$

$$\langle k_1^t, k_2^t, k_3^t\rangle = \langle \alpha^t, 0, \gamma\rangle \tag{A 17}$$

$$\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{A 18}$$

$$f(\psi_1) = \frac{3b}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\frac{(\sin\psi_1-\sin\psi)(a+b\cos\psi)^{1/2}}{[1-\cos(\psi_1-\psi)](2+\alpha)^{1/2}}\mathrm{d}x, \tag{A 19}$$

$$\begin{aligned} g(\psi_1) = {} & \frac{3}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\left(\frac{a+b\cos\psi}{2+\alpha}\right)^{1/2}\Bigg\{f(\psi)[(\cos\psi_1-b\beta_1)S+\beta_1 P] \\ & \times\frac{\sin\psi_1-\sin\psi}{1-\cos(\psi_1-\psi)} + g(\psi)\left[\left(2+\alpha-\frac{(\sin\psi_1-\sin\psi)^2}{1-\cos(\psi-\psi_1)}-b^2\gamma\right)S\right. \\ & \left.+\left(b^2\cos\psi_1\gamma-\frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi,\delta)-(2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi,\delta)\right]\Bigg\}\mathrm{d}\psi, \end{aligned} \tag{A 20}$$

$$\alpha = \alpha(\psi,\psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}, \quad \beta - \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \tag{A 21}$$

$$\left.\begin{aligned} & H(0) = \frac{\epsilon\overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}, \quad H'(0) = -1+\epsilon^{2/3}\overline{C}_u+\epsilon\hat{C}'_u; \\ & H''(0) = \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2}u_P^2}, \quad H'(\infty) = 0. \end{aligned}\right\} \tag{A 22}$$

LEMMA 1. *Let $f(z)$ be a trial Batchelor (1971, pp. 231–232) function defined on* $[0,1]$. *Let Λ_1 denote the ground-state eigenvalue for $-\mathrm{d}^2g/\mathrm{d}z^2 = \Lambda g$, where g must satisfy $\pm\mathrm{d}g/\mathrm{d}z+\alpha g = 0$ at $z=0,1$ for some non-negative constant α. Then for any f that is not identically zero we have*

$$\frac{\alpha(f^2(0)+f^2(1))+\displaystyle\int_0^1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\mathrm{d}z}{\displaystyle\int_0^1 f^2\mathrm{d}z} \geqslant \Lambda_1 \geqslant \left(\frac{-\alpha+(\alpha^2+8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \tag{A 23}$$

COROLLARY 1. *Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies*

$$\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{A 24}$$

REFERENCES

BATCHELOR, G. K. 1971 Small-scale variation of convected quantities like temperature in turbulent fluid. part 1. general discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113–133.

BROWNELL, C. J. & SU, L. K. 2004 Planar measurements of differential diffusion in turbulent jets. *AIAA Paper 2004-2335* .

BROWNELL, C. J. & SU, L. K. 2007 Scale relations and spatial spectra in a differentially diffusing jet. *AIAA Paper 2007-1314* .

DENNIS, S. C. R. 1985 Compact explicit finite difference approximations to the Navier–Stokes equation. In *Ninth Intl Conf. on Numerical Methods in Fluid Dynamics* (ed. Soubbaramayer & J. P. Boujot), *Lecture Notes in Physics*, vol. 218, pp. 23–51. Springer.

HWANG, L.-S. & TUCK, E. O. 1970 On the oscillations of harbours of arbitrary shape. *J. Fluid Mech.* **42**, 447–464.

KOCH, W. 1983 Resonant acoustic frequencies of flat plate cascades. *J. Sound Vib.* **88**, 233–242.

LEE, J.-J. 1971 Wave-induced oscillations in harbours of arbitrary geometry. *J. Fluid Mech.* **45**, 375–394.

LINTON, C. M. & EVANS, D. V. 1992 The radiation and scattering of surface waves by a vertical circular cylinder in a channel. *Phil. Trans. R. Soc. Lond.* **338**, 325–357.

MARTIN, P. A. 1980 On the null-field equations for the exterior problems of acoustics. *Q. J. Mech. Appl. Maths* **33**, 385–396.

MILLER, P. L. 1991 Mixing in high schmidt number turbulent jets. PhD thesis, California Institute of Technology.

ROGALLO, R. S. 1981 Numerical experiments in homogeneous turbulence. *Tech. Rep.* 81835. NASA Tech. Mem.

URSELL, F. 1950 Surface waves on deep water in the presence of a submerged cylinder i. *Proc. Camb. Phil. Soc.* **46**, 141–152.

VAN WIJNGAARDEN, L. 1968 On the oscillations near and at resonance in open pipes. *J. Engng Maths* **2**, 225–240.

WORSTER, M. G. 1992 The dynamics of mushy layers. In *In Interactive dynamics of convection and solidification* (ed. S. H. Davis, H. E. Huppert, W. Muller & M. G. Worster), pp. 113–138. Kluwer.