Viscous Control Of Shallow Elastic Fracture

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Introduction

Consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the "arm" of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions c, the magnitude of the bending moment M, and the toughness of the solid K_I.

Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

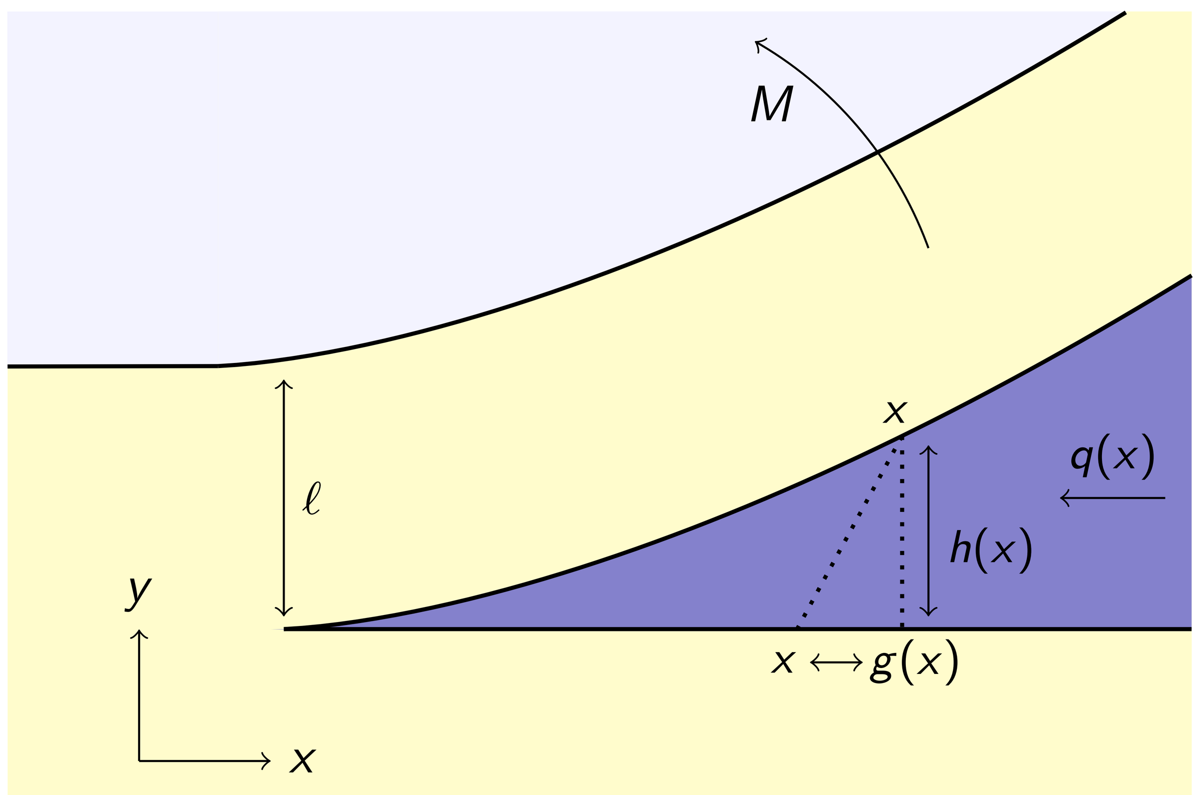


Figure 1 : Diagram to show the geometry of the problem. $q(x)$ is the flux, $g(x)$ the horizontal displacement, $h(x)$ the vertical displacement, and ℓ is the thickness of the arm.

Governing Equations

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h(x)^2 \frac{dp}{dx}$$

Where $p(x)$ is the pressure, and μ the viscosity.

From elasticity, using Muskhelishiveli methods, we can derive the equation

$$\begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \begin{pmatrix} K_{11}(x-\tilde{x}) & K_{12}(x-\tilde{x}) \\ K_{21}(x-\tilde{x}) & K_{22}(x-\tilde{x}) \end{pmatrix} \begin{pmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{pmatrix} d\tilde{x}$$

Where K_{ij} is the integral kernel specific to this geometry, E is the Young's modulus, ν is Poisson's ratio.

- Boundary conditions as $x \to \infty$ are governed by the bending moment. For large x the geometry is well approximated by beam theory. This gives the equation

$$M(x) = \frac{E\ell^3}{12(1-\nu^2)} \frac{d^2h}{dx^2}$$

  Where $M(x)$ tends to a constant bending moment as $x \to \infty$.
- The boundary conditions as $x \to 0$ are governed by "*Linear Elastic Fracture Mechanics*", (LEFM). This gives the condition

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x)$$

We move into dimensionless variables now

$$(x, h, g, p, K_I, K_{ij}) \to (\xi, H, G, \Pi, \kappa, \Lambda_{ij})$$

Where the new equations and boundary conditions become

$$(\Pi, 0) = \int \Lambda \cdot (G', H') d\xi, \quad H^2 \Pi' = \lambda$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to 0} 3\sqrt{2\pi} H' = \kappa$$

Zero Toughness Solution

Instead of tackling the general problem, (which we expect to not have an analytic solution) we investigate the case where $\kappa \ll 1$, the "*small toughness solution*." Perhaps an even simpler problem to consider is the "*zero toughness solution*" for $\kappa = 0$. However, we have the following dichotomy,

- For $\kappa = 0$, one can show that the leading order behaviour as $\xi \to 0$ is $H(\xi) \sim \xi^{2/3}$
- For any $\kappa > 0$, no matter how small, near $\xi = 0$, $H(\xi) \sim \xi^{1/2}$

Small Toughness Solution

Here we take after Garagash and Detournay [1]. Their paper examines a similar problem of fluid driven fracture in a different geometry, with the propagation being driven by fluid injection. They construct a small toughness solution in the following way:

- Near the tip there is the "LEFM boundary layer" which accounts for the $h \sim x^{1/2}$ behaviour, and does not resemble the zero toughness solution.
- Away from the tip, the solution behaves as

$$h(x) = h_0(x) + \mathcal{E}(K_I) h_1(x) + o(\mathcal{E})$$

  where h_0 is the zero toughness solution, and $\mathcal{E}(K_I)$ is an as yet unknown function of K_I. (Similar for p,g).

We can do a similar construction, after moving into dimensionless variables:

$$(x, h, g, p, K_I, K_{ij}) \to (\xi, H, G, \Pi, \kappa, \Lambda_{ij})$$

Where the new equations and boundary conditions become

$$(\Pi, 0) = \int \Lambda \cdot (G', H') d\xi, \quad H^2 \Pi' = \lambda, \quad \lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to 0} 3\sqrt{2\pi} H' = \kappa$$

We look for a solution like $H(\xi) = H_0(\xi) + \mathcal{E}(\kappa) H_1(\xi) + o(\mathcal{E})$ (and again similar for Π, G).

By matching the outer asymptotics of the LEFM boundary layer solution, and the inner asymptotics of the expansion in $\mathcal{E}$, in a region that they overlap, one can show that $\mathcal{E} = C\kappa^{4-6s}\lambda_0^{2s-1}$. $s \approx 0.1386$ comes from solving a transcendental equation, C can be determined numerically, and λ_0 is the value of λ when $\kappa = 0$, also determined numerically.

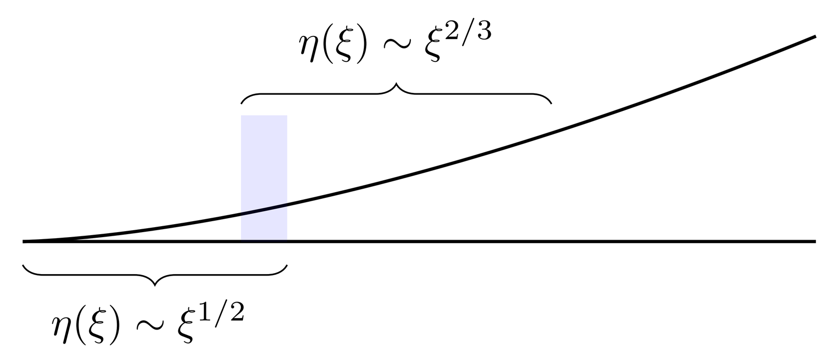


Figure 2 : Matching region of outer and inner asymptotics.

An additional problem not present in [1] is the asymptotic region as $\xi \to \infty$, but it can be shown that with our rescaling, this does not affect the near tip behaviour.

Numerical Solution of Equations

The set of scaled equations can be discritized, and then solved numerically as follows. We choose a set of points ξ to measure G, H, and an intermediate set of points z to measure Π, so $\xi_1 < z_1 < \xi_2 < \cdots < z_{n-1} < \xi_n$. The simplest thing to do, would be to approximate H', G' as piecewise linear functions. However, since both H', G' are singular near the origin, they are badly approximated by linear functions. The solution is to approximate $G'(\xi) = \frac{1}{\sqrt{\xi}}(a_i\xi + b_i)$ near the tip and to approximate $G'(\xi) = a_i\xi + b_i$ away from the tip for $\xi_i < \xi < \xi_{i+1}$ (similar with H'). We store the values $\theta = (a_1\xi_1 + b_1, \ldots, a_n\xi_n + b_n)$.

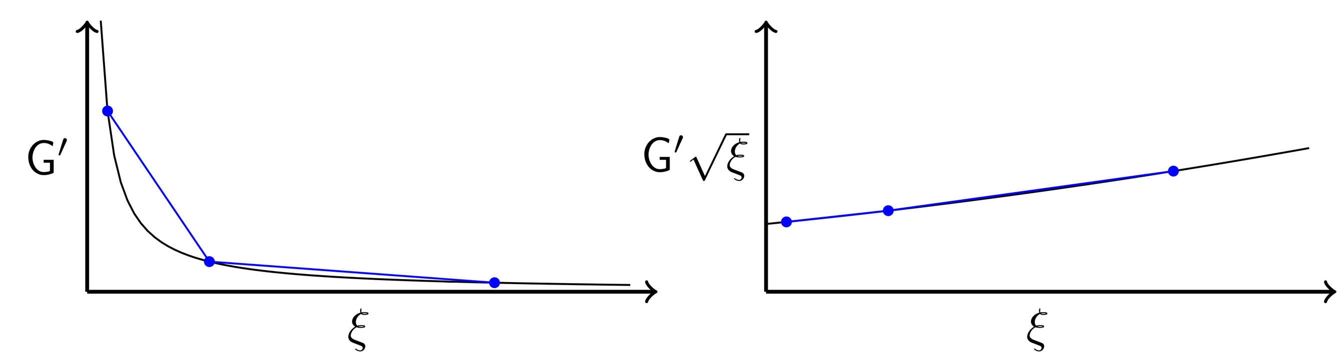


Figure 3 : Relative improvement in interpolation for a given number of points, once known singular behaviour is accounted for.

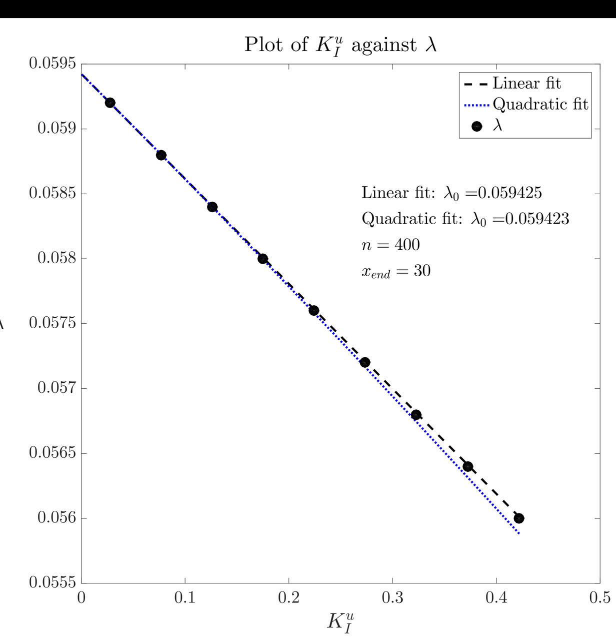
Once we have the values θ, it is possible to linearly recover the coefficients, i.e. $(a_1, \ldots, a_n, b_1, \ldots, b_n) = T\theta$ where T is some matrix. It is clear, given this interpolation, the elasticity integral depends linearly on the θ values. We can rewrite the lubrication integral as $\Pi(z) = \int_z^\infty \lambda / H^2 d\xi$. This depends non-linearly on the θ values.

There are now two different expressions for $(\Pi(z_1), \ldots, \Pi(z_{n-1}))$, so $n-1$ equations. There are an additional $n-1$ equations from the elasticity integral. We can get another two equations from the boundary conditions as $\xi \to \infty$. This gives $2n$ equations for $2n$ unknowns (θ and H' equivalent). This is enough to solve the problem using Newton's method, to give G', H'. We use λ as an input parameter and solve for κ, although in the physical problem we think of κ as the independent variable.

Results

The relationship $\lambda = \lambda_0 + \mathcal{E}(\kappa)\lambda_1$ holds well in practice. It has been calculated that $\lambda_0 \approx 0.0591$, To calculate λ_1, C is harder, since the linear perturbation problem must be solved (linearise and work only to first order in $\mathcal{E}$). We found $C \approx 5.8 \times 10^{-3}$, $\lambda_1 \approx -0.31$. Redimensionalising

$$c = \frac{36(1-\nu^2)^2 M^3}{\pi \mu E^2 \ell^5} \left(\lambda_0 + C\lambda_0^{2s-1} \lambda_1 (\ell^{3/2} K_I / M)^{4-6s} \right)$$



References

[1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.

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