

Numerically calculating dimensionless coefficients

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Recall that

$$\begin{aligned}\gamma &= 1/\lambda^{1/3} \\ \mathfrak{G} &= \frac{1}{\lambda^{1/3}} \mathcal{G} & \mathfrak{H} &= \frac{1}{\lambda^{1/3}} \mathcal{H} \\ \mathfrak{P} &= \frac{1}{\lambda^{1/3}} \mathcal{P} & \mathfrak{K} &= \frac{1}{\lambda^{1/3}} \mathcal{K}\end{aligned}$$

The relevant asymptotic form is

$$\begin{aligned}\lambda &= \lambda_0 + \mathcal{E}(\mathcal{K})\lambda_1 + \dots \\ \mathcal{H}(\xi) &= (A_0\xi^{2/3} + \dots) + \mathcal{E}(\mathcal{K}) \left(\frac{A_0\lambda_1}{3\lambda_0} \xi^{2/3} + \xi^s + \dots \right) + \dots \\ \mathcal{P}(\xi) &= \left(-\frac{3\lambda_0}{A_0^2} \xi^{-1/3} + \dots \right) + \mathcal{E}(\mathcal{K}) \left(\frac{2\pi A_0\lambda_1}{9\lambda_0\sqrt{3}} \xi^{-1/3} + \frac{4\pi}{9\sqrt{3}(1-s)} \xi^{s-1} + \dots \right) + \dots \\ \mathcal{E}(\mathcal{K}) &= C \mathcal{K}^u \lambda_0^{2s-1}\end{aligned}$$

Or, in the alternative scaling

$$\begin{aligned}\mathfrak{H}(\xi) &= \left(\left(\frac{243}{4\pi^2} \right)^{1/6} \xi^{2/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) (\xi^s + \dots) + \dots \\ \mathfrak{P}(\xi) &= \left(-\left(\frac{2\pi}{3} \right)^{2/3} \xi^{-1/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) \left(\frac{4\pi}{9\sqrt{3}(1-s)} \xi^{s-1} + \dots \right) + \dots \\ \mathcal{E}(\mathfrak{K}) &= C \mathfrak{K}^u\end{aligned}$$

We wish to calculate the parameters λ_0, λ_1, C . To do this, we will also introduce the parameter D ,

$$D = C\lambda_1\lambda_0^{2s-1}$$

so that $\lambda = \lambda_0 + D\mathcal{K}^u + \dots$

The good news

Very rough estimates give that $\lambda_0 \approx 0.06$, $D \approx -0.01$, $\lambda_1 \approx -0.2$. For values of $\mathcal{K} < 1$, this gives $|\mathcal{E}(\mathcal{K})| < 0.05$. Further, for $\mathcal{K} < 0.5$ (we can go this low numerically, but not much further) $|\mathcal{E}(\mathcal{K})| < 0.005$. So for moderately low values of \mathcal{K} , ignoring \mathcal{E}^2 terms is an excellent approximation.

Plotting λ against \mathcal{K}^u yields a very good approximation to a straight line. Using simple linear regression we can calculate λ_0 , and D , to around 2.s.f. accuracy.

$$\lambda_0 \approx 0.0059$$

$$D \approx -0.0074$$

Given this value of D , we need only calculate C , to have the full range of constants.

The bad news

Let us move into the alternative scalings.

$$\mathfrak{H}(\xi) = \mathfrak{H}_0(\xi) + \mathcal{E}(\mathfrak{K})\mathfrak{H}_1(\xi)$$

$$\mathfrak{P}(\xi) = \mathfrak{P}_0(\xi) + \mathcal{E}(\mathfrak{K})\mathfrak{P}_1(\xi)$$

$$\mathcal{E}(\mathfrak{K}) = C\mathfrak{K}^u$$

Where this holds away from the LEFM boundary layer. We know that $\mathfrak{H}_1(\xi) \rightarrow \xi^s$ near $\xi = 0$. Something we can do is to fix a point ξ , and consider $\mathfrak{H}(\xi; \mathfrak{K}^u)$. As $\mathfrak{K}^u \rightarrow 0$, plotting $\mathfrak{H}(\xi; \mathfrak{K}^u)$ against \mathfrak{K}^u yields a straight line. From this line, one calculates $\mathfrak{H}_0(\xi)$ and $C\mathfrak{H}_1(\xi)$.

The suspected cause of the problem is that $\mathfrak{H}_1 = \xi^s$ is never that good of an approximation. In particular, plotting $C\mathfrak{H}_1$ against ξ^s should yield a straight line through the origin with gradient C , at least for small ξ . The problem is that it doesn't. I have not yet resolved this issue.

An alternative means of progress, would be to recall that C arises as the coefficient of an asymptotic expansion. Recall the problem:

$$\Pi(\zeta) = - \int_0^\infty \frac{\eta'(\xi)}{\xi - \zeta} d\xi, \quad \eta^2 \Pi' = 1, \quad \eta(\xi) \rightarrow \frac{2}{3\sqrt{2\pi}} \xi^{1/2}$$

The solution to this problem has the far field asymptotics,

$$\eta(\xi) = \left(\frac{243}{4\pi^2} \right)^{1/6} \xi^{2/3} + C\xi^s + \dots$$

This may well turn out to be a much more convenient way to find C , as the problem is certainly simpler.