# Viscous Control Of Shallow Elastic Fracture

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## Introduction

Consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the "arm" of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions $c$, the magnitude of the bending moment $M$, and the toughness of the solid $K_I$.
Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

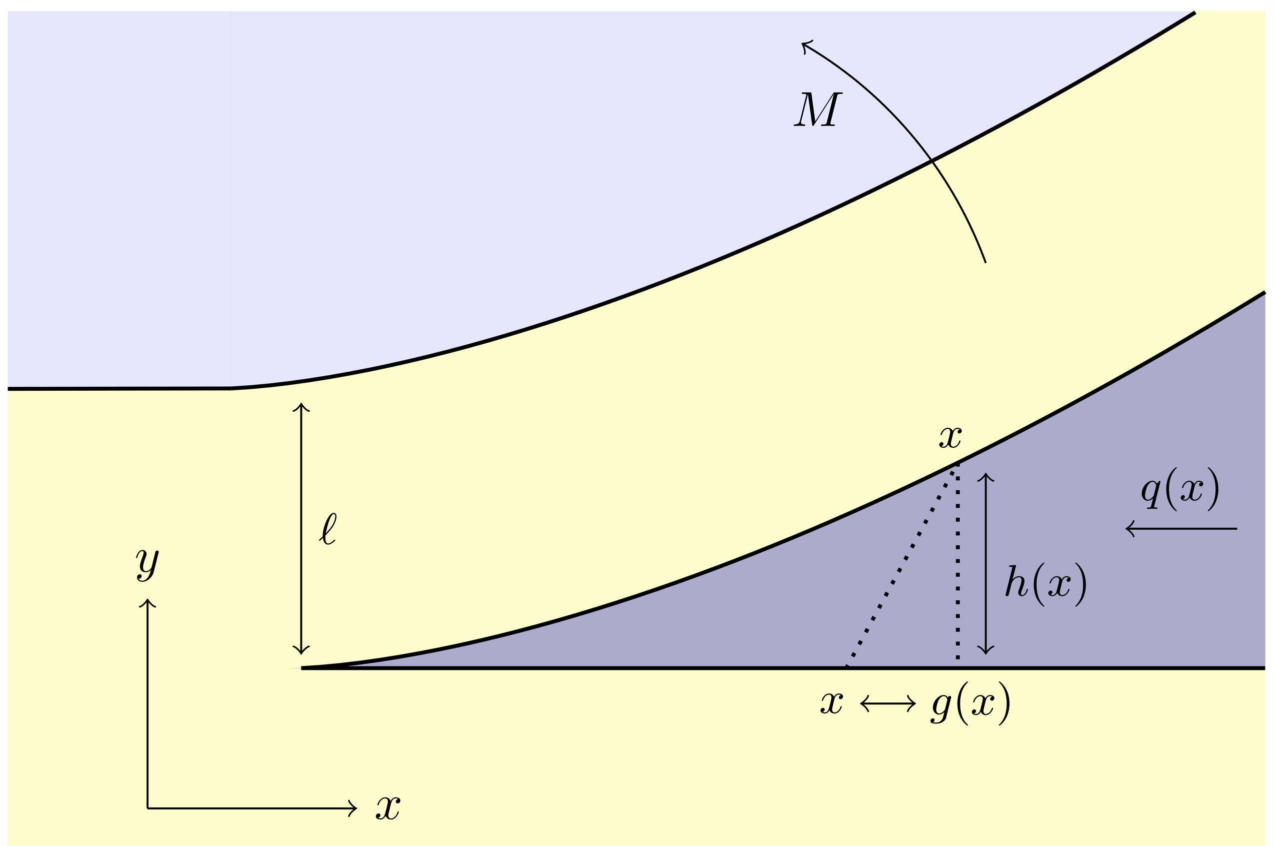


Figure 1 : Diagram to show the geometry of the problem. $q(x)$ is the flux, $g(x)$ the horizontal displacement, $h(x)$ the vertical displacement, and $\ell$ is the thickness of the arm.

## Governing Equations

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h(x)^2 \frac{dp}{dx}$$

Where $p(x)$ is the pressure, and $\mu$ the viscosity.
From elasticity, using Muskhelishiveli methods, we can derive the equation

$$\begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \begin{pmatrix} K_{11}(x-\tilde{x}) & K_{12}(x-\tilde{x}) \\ K_{21}(x-\tilde{x}) & K_{22}(x-\tilde{x}) \end{pmatrix} \begin{pmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{pmatrix} d\tilde{x}$$

Where $K_{ij}$ is the integral kernel specific to this geometry, $E$ is the Young's modulus, $\nu$ is Poisson's ratio. The kernel terms are

$$K_{11}(z) = \frac{32-24z^2}{(z^2+4)^3} \qquad K_{12}(z) = \frac{48z^2-64}{z(z^2+4)^3}$$

$$K_{21}(z) = -\frac{(16z^3+16z^2+4)}{z(z^2+4)^3} \qquad K_{22}(z) = -\frac{(32-24z^2)}{(z^2+4)^3}$$

Note that some terms are singular, so the integral is really a Cauchy Principal value integral.

- Boundary conditions as $x \to \infty$ are governed by the bending moment. For large $x$ the geometry is well approximated by beam theory. This gives the equation

$$M(x) = \frac{E\ell^3}{12(1-\nu^2)} \frac{d^2h}{dx^2}$$

  Where $M(x)$ tends to a constant bending moment as $x \to \infty$.
- The boundary conditions as $x \to 0$ are governed by "*Linear Elastic Fracture Mechanics*", (LEFM). This gives the condition

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x)$$

## Zero Toughness Solution

Instead of tackling the general problem, (which we expect to not have an analytic solution) we investigate the case where $K_I \ll M\ell^{-3/2}$, the "small toughness solution." Perhaps an even simpler problem to consider is the "zero toughness solution" for $K_I = 0$. However, we have the following dichotomy,

- For $K_I = 0$, one can show that the leading order behaviour as $x \to 0$ is $h(x) \sim x^{2/3}$
- For any $K_I > 0$, no matter how small, near $x = 0$, $h(x) \sim x^{1/2}$

## Small Toughness Solution

Here we take after Garagash and Detournay [1]. Their paper examines a similar problem of fluid driven fracture in a different geometry, with the propagation being driven by fluid injection. They construct a small toughness solution in the following way:

- Near the tip there is the "LEFM boundary layer" which accounts for the $h \sim x^{1/2}$ behaviour, and does not resemble the zero toughness solution.
- Away from the tip, the solution behaves as

$$h(x) = h_0(x) + \mathcal{E}(K_I)h_1(x) + o(\mathcal{E})$$

  where $h_0$ is the zero toughness solution, and $\mathcal{E}(K_I)$ is an as yet unknown function of $K_I$. (Similar for $p$,$g$).

We can do a similar construction, after moving into dimensionless variables:

$$(x, h, g, p, K_I, K_{ij}) \to (\xi, H, G, \Pi, \kappa, \Lambda_{ij})$$

Where the new equations and boundary conditions become

$$(\Pi, 0) = \int \Lambda \cdot (G', H')d\xi, \quad H^2\Pi' = \lambda, \quad \lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to 0} 3\sqrt{2\pi}H' = \kappa$$

We look for a solution like $H(\xi) = H_0(\xi) + \mathcal{E}(\kappa)H_1(\xi) + o(\mathcal{E})$ (and again similar for $\Pi, G$).
By matching the outer asymptotics of the LEFM boundary layer solution, and the inner asymptotics of the expansion in $\mathcal{E}$, in a region that they overlap, one can show that $\mathcal{E} = C\kappa^{4-6s}\lambda_0^{2s-1}$. $s \approx 0.1386$ comes from solving a transcendental equation, $C$ can be determined numerically, and $\lambda_0$ is the value of $\lambda$ when $\kappa = 0$, also determined numerically.

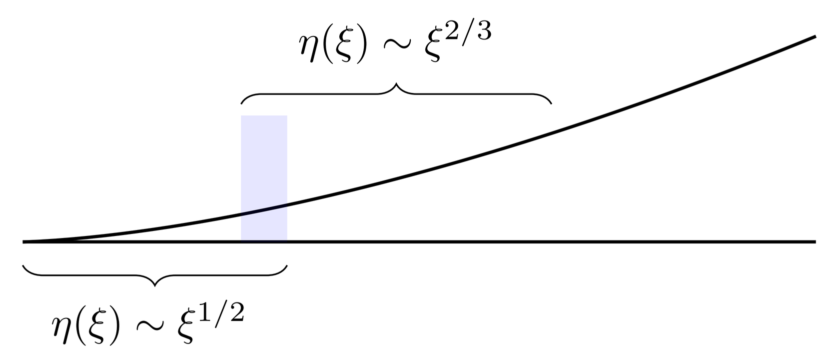


Figure 2 : Matching region of outer and inner asymptotics.

An additional problem not present in [1] is the asymptotic region as $\xi \to \infty$, but it can be shown that with our rescaling, this does not affect the near tip behaviour.

## Numerical Solution of Equations

The set of scaled equations can be discritized, and then solved numerically as follows. We choose a set of points $\xi$ to measure $G, H$, and an intermediate set of points $z$ to measure $\Pi$, so $\xi_1 < z_1 < \xi_2 < \cdots < z_{n-1} < \xi_n$. The simplest thing to do, would be to approximate $H', G'$ as piecewise linear functions. However, since both $H', G'$ are singular near the origin, they are badly approximated by linear functions.
We can rewrite the lubrication integral as $\Pi(z) = \int_z^\infty \lambda / H^2 d\xi$

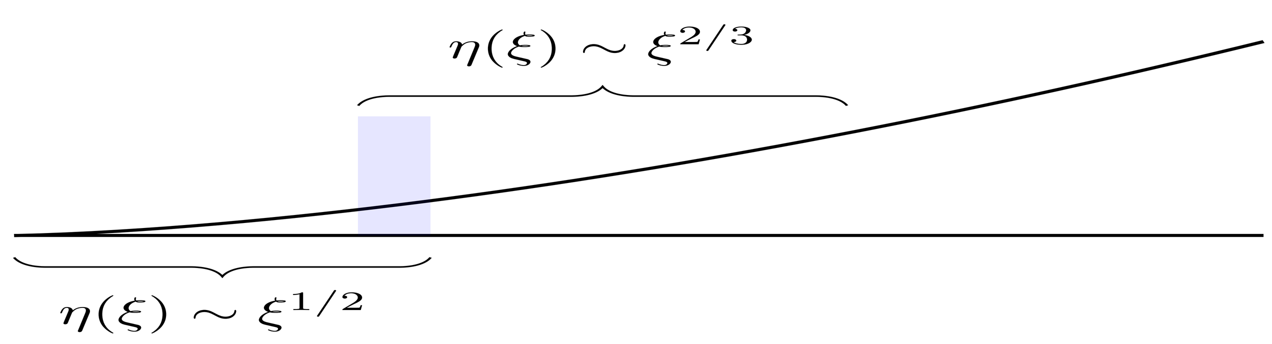


Figure 3 : Contrast for all stimulation train parameter estimates for the pre-drug-administration session (red) and the acute fluoxetine administration session (green). Note the considerably different scales.

## References

[1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.

[2] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.


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