

# Rescaling the equations - Summary

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Here we will recap the scalings, and then explain how we will switch between the “ $h'' \rightarrow 1$ ” and the “ $\lambda = 1$ ” scalings.

**Full equations:**

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \quad (1)$$

$$12\mu c = h^2 p' \quad (2)$$

$$\begin{cases} \lim_{x \rightarrow \infty} h''(x) &= \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim_{x \rightarrow \infty} g'(x) &= \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \quad (3)$$

$$K_I = \lim_{x \rightarrow 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \quad (4)$$

## 1 Scale $h'' \rightarrow 1$

The more common used scaling.

$$\begin{aligned} x &= \ell \xi & p(x) &= \beta_1 \mathcal{P}(\xi) \\ h(x) &= \alpha_1 \mathcal{H}(\xi) & g(x) &= \alpha_1 \mathcal{G}(\xi) \\ \underline{\underline{K}}(x) &= \frac{1}{\ell} \underline{\underline{\Lambda}}(\xi) & K_I &= M \ell^{-3/2} \mathcal{K} \end{aligned}$$

Where  $\alpha_1 = \frac{12(1-\nu^2)M}{E\ell}$  and  $\beta_1 = \frac{3M}{\pi\ell^2}$ . The equations become

$$\begin{pmatrix} \mathcal{P}(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi} - \xi) \begin{pmatrix} \mathcal{G}'(\tilde{\xi}) \\ \mathcal{H}'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \quad (5)$$

$$\lambda = \mathcal{H}^2 \mathcal{P}' \quad (6)$$

$$\begin{cases} \lim_{\xi \rightarrow \infty} \mathcal{H}''(x) &= 1 \\ \lim_{\xi \rightarrow \infty} \mathcal{G}'(x) &= 1/2 \end{cases} \quad (7)$$

$$\mathcal{K} = \lim_{\xi \rightarrow 0} 3\sqrt{2\pi} \sqrt{\xi} \mathcal{H}'(\xi) \quad (8)$$

## 2 Scale $\lambda = 1$

$$\begin{aligned} x &= \ell \xi & p(x) &= \beta_2 \mathfrak{P}(\xi) \\ h(x) &= \alpha_2 \mathfrak{H}(\xi) & g(x) &= \alpha_2 \mathfrak{G}(\xi) \\ \underline{K}(x) &= \frac{1}{\ell} \underline{\Lambda}(\xi) & K_I &= M \ell^{-3/2} \mathfrak{K} \end{aligned}$$

Where  $\alpha_2 = \left( \frac{48\pi(1-\nu^2)\mu c \ell^2}{E} \right)^{1/3}$ , and  $\beta_2 = \left( \frac{3\mu c E^2}{4\pi^2(1-\nu^2)^2 \ell} \right)^{1/3}$ . We then have the relevant equations as

$$\begin{pmatrix} \mathfrak{P}(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\Lambda}(\tilde{\xi} - \xi) \begin{pmatrix} \mathfrak{G}'(\tilde{\xi}) \\ \mathfrak{H}'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \quad (9)$$

$$1 = \mathfrak{H}^2 \mathfrak{P}' \quad (10)$$

$$\begin{cases} \lim_{\xi \rightarrow \infty} \mathfrak{H}''(x) &= \gamma \\ \lim_{\xi \rightarrow \infty} \mathfrak{G}'(x) &= \gamma/2 \end{cases} \quad (11)$$

$$\mathfrak{K} = \lim_{\xi \rightarrow 0} 3\sqrt{2\pi} \sqrt{\xi} \mathfrak{H}'(\xi) \quad (12)$$

Where  $\gamma = M \left( \frac{36(1-\nu^2)^2}{\pi E^2 \mu c \ell^5} \right)^{1/3}$ .

## 3 Relating the two scalings

One can show that

$$\begin{aligned} \gamma &= 1/\lambda^{1/3} \\ \mathfrak{G} &= \frac{1}{\lambda^{1/3}} \mathcal{G} & \mathfrak{H} &= \frac{1}{\lambda^{1/3}} \mathcal{H} \\ \mathfrak{P} &= \frac{1}{\lambda^{1/3}} \mathcal{P} & \mathfrak{K} &= \frac{1}{\lambda^{1/3}} \mathcal{K} \end{aligned}$$

The point of this, is that in earlier analysis (done elsewhere), we looked at  $\mathcal{H}$  near  $\xi = 0$ . We found the asymptotic form

$$\lambda = \lambda_0 + \mathcal{E}(\mathcal{K}) \lambda_1 + \dots$$

$$\mathcal{H}(\xi) = (A_0 \xi^{2/3} + \dots) + \mathcal{E}(\mathcal{K}) \left( \frac{A_0 \lambda_1}{3\lambda_0} \xi^{2/3} + \xi^s + \dots \right) + \dots$$

$$\mathcal{P}(\xi) = \left( -\frac{3\lambda_0}{A_0^2} \xi^{-1/3} + \dots \right) + \mathcal{E}(\mathcal{K}) \left( \frac{2\pi A_0 \lambda_1}{9\lambda_0 \sqrt{3}} \xi^{-1/3} + \frac{4\pi}{9\sqrt{3}(1-s)} \xi^{s-1} + \dots \right) + \dots$$

$$\mathcal{E}(\mathcal{K}) = C \mathcal{K}^u \lambda_0^{2s-1}$$

The reason we've bothered with this other scaling ( $\mathfrak{H}, \mathfrak{K}, \dots$ ), is that one notices that the equations near zero are almost exactly the same. The only difference is that  $\lambda$  is set to

1. Since all of this is based near  $\xi = 0$ , the boundary conditions at  $\infty$  are not important. Thus we have that

$$\begin{aligned}\mathfrak{H}(\xi) &= \left( \left( \frac{243}{4\pi^2} \right)^{1/6} \xi^{2/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) (\xi^s + \dots) + \dots \\ \mathfrak{P}(\xi) &= \left( - \left( \frac{2\pi}{3} \right)^{2/3} \xi^{-1/3} + \dots \right) + \mathcal{E}(\mathfrak{K}) \left( \frac{4\pi}{9\sqrt{3}(1-s)} \xi^{s-1} + \dots \right) + \dots \\ \mathcal{E}(\mathfrak{K}) &= C \mathfrak{K}^u\end{aligned}$$

from which one can calculate  $C$ . The point is that it's the same  $C$  in both equations, which must be some numerical constant dependent on the near tip geometry. One hopes that it is easier to calculate  $C$  in the  $\mathfrak{H}, \mathfrak{K}, \dots$  scaling.