Viscous Control Of Shallow Elastic Fracture

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Introduction

We consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the "arm" of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions c, the bending moment M, and the toughness of the solid K_I, K_{II}.
Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

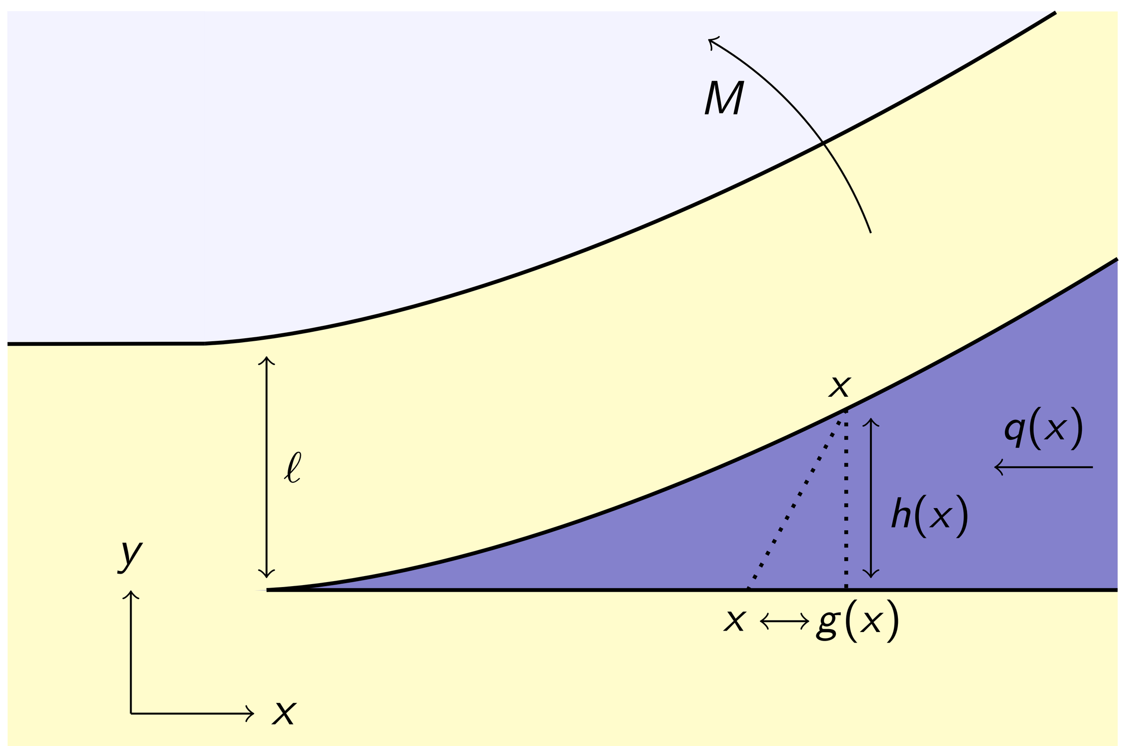

Figure 1 : Diagram to show the geometry of the problem. $q(x)$ is the flux, $g(x)$ the horizontal displacement, $h(x)$ the vertical displacement, and ℓ is the thickness of the arm.

Governing Equations

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h(x)^2 \frac{dp}{dx}.$$

From elasticity, using Muskhelishiveli methods, we can derive the equation

$$[p, 0] = E/(4\pi(1-\nu^2)) \int_0^\infty \mathbf{K}(x - \tilde{x})\ [g'(\tilde{x}), h'(\tilde{x})]\ d\tilde{x},$$

where K_{ij} is the integral kernel specific to this geometry.

- Boundary conditions as $x \to \infty$ are governed by the bending moment. We have (via beam theory),

$$M(x) = \frac{E\ell^3}{12(1-\nu^2)} \frac{d^2h}{dx^2} \to M_0, \text{ as } x \to \infty.$$

- The boundary conditions as $x \to 0$ are governed by "*Linear Elastic Fracture Mechanics*", (LEFM). This gives the condition

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}h'(x), \quad K_{II} = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}g'(x).$$

We move into dimensionless variables,

$$(x, h, g, p, K_I, K_{II}, K_{ij}, c) \to (\xi, H, G, \Pi, \kappa_I, \kappa_{II}, \Lambda_{ij}, \lambda).$$

The new equations and boundary conditions are

$$(\Pi, 0) = \int \Lambda \cdot (G', H') d\xi, \quad H^2\Pi' = \lambda$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' = \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}G' = \kappa_{II}.$$

Results

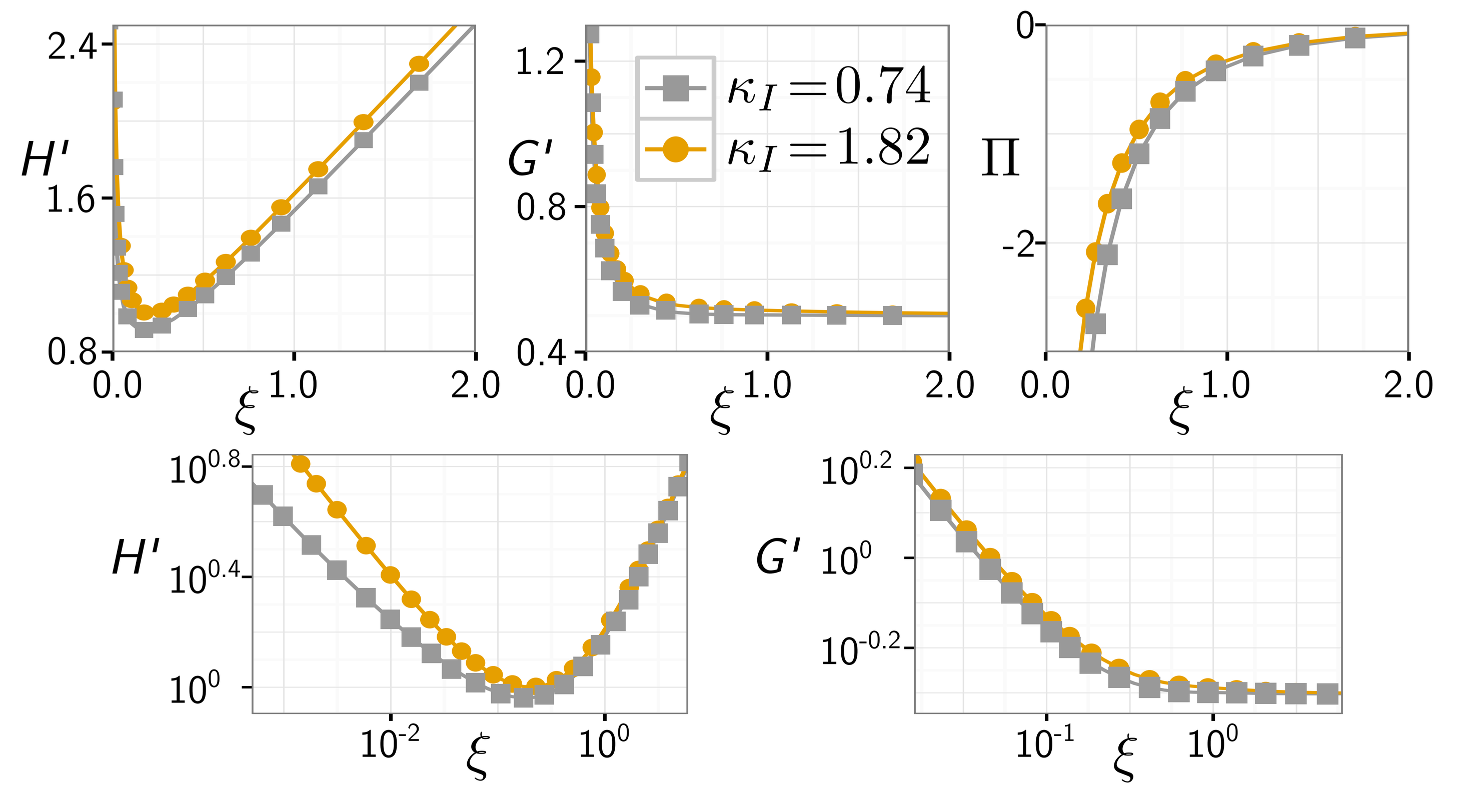

Figure 2 : Numerical results for two typical κ_I values, including logarithmic scales for H' and G' (lower two graphs).

Small Toughness Solution

We can plot how the speed λ varies with the toughness κ_I. For the *small toughness solution*, $\kappa_I \ll 1$, we use the theory of Garagash and Detournay [1] who consider fluid driven fracture in a different geometry. The theory states

- Near the tip there is the "LEFM boundary layer".
- Away from the tip, the solution behaves as

$$H(\xi) = H_0(x) + \mathcal{E}(\kappa_I)H_1(\xi) + o(\mathcal{E}),$$

where $H_0(\xi) = H(\xi; \kappa_I = 0)$ is the zero toughness solution, (similar for G,Π,λ), and $\mathcal{E} = C\kappa_I^u\lambda_0^{2-u/2}$, $u \approx 3.17$.

This is in good agreement with the numerical results.

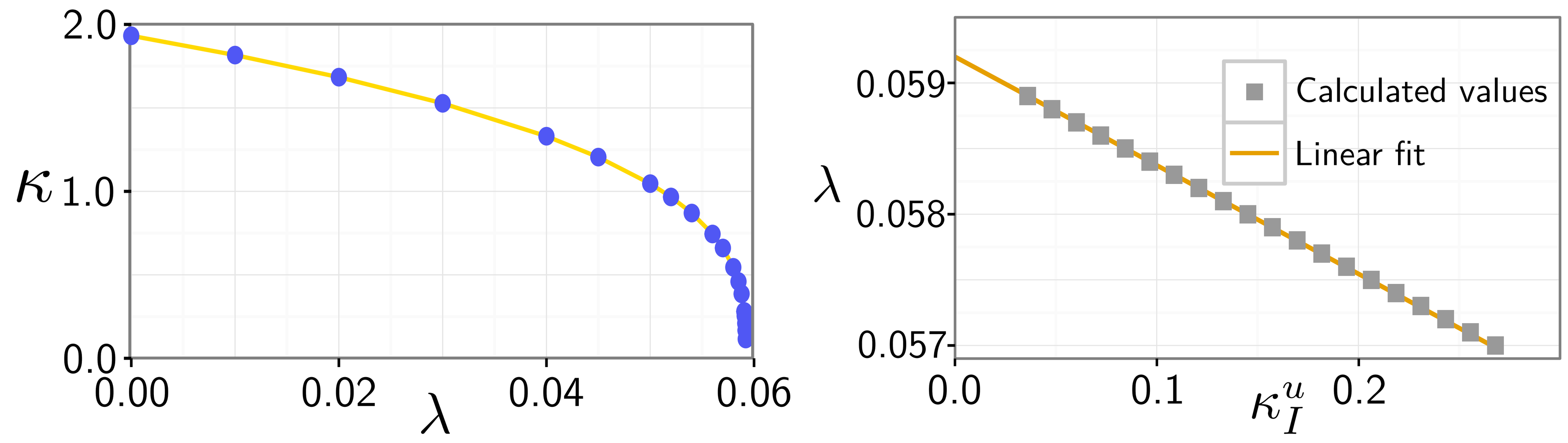

Figure 3 : The relationship between κ_I and λ is plotted on the left. Strong evidence that $\lambda = \lambda_0 + \mathcal{E}\lambda_1$, is plotted on the right.

Two tip problem

We can also consider a different mode of fracture. So far, we have been imposing both a κ_I and κ_{II} fracture condition at the origin but only looking at fracture controlled by the κ_I condition. If the κ_{II} value is small, the solid will fracture by slipping, and a second dry crack will extend a length L beyond the wet tip. The fracture is then controlled by the κ_{II} value, with the various relationships plotted in figure 4.

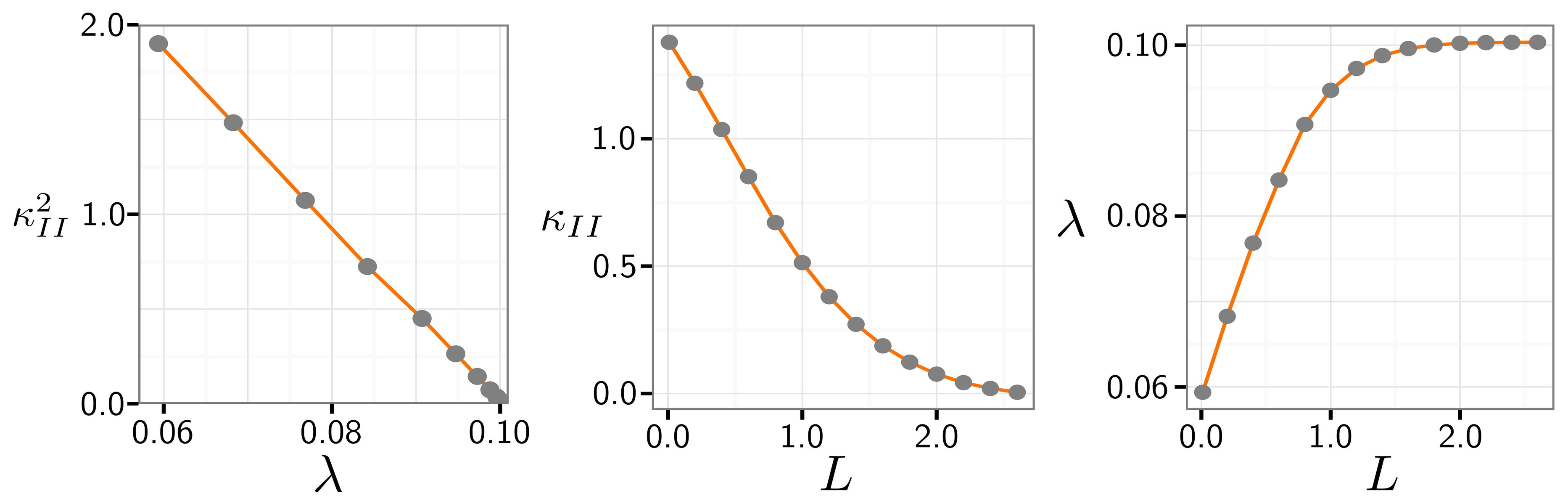

Figure 4 : These graphs show the interdependence of L, λ and κ_{II}, for the two tip problem. Physically κ_{II} is the independent variable (but not numerically).

Note the (approximately) linear relationship between κ_{II}^2 and λ. From conservation of energy, and fracture mechanics, one expects $\alpha\lambda + \beta\kappa_{II}^2 =$ const., where in theory α, β depend on H, and so κ_{II}. In practice α and β are almost constant, as seen from the graph.

Overall results

Solving the one tip problem for various κ_I, κ_{II} values gives a graph in the κ_I, κ_{II}-plane. From this we can determine where the fracture is controlled by κ_I and where it is controlled by κ_{II}. For $\kappa_I > 1.9$ and $\kappa_{II} > 1.5$, the fracture cannot propagate, the solid is too tough.

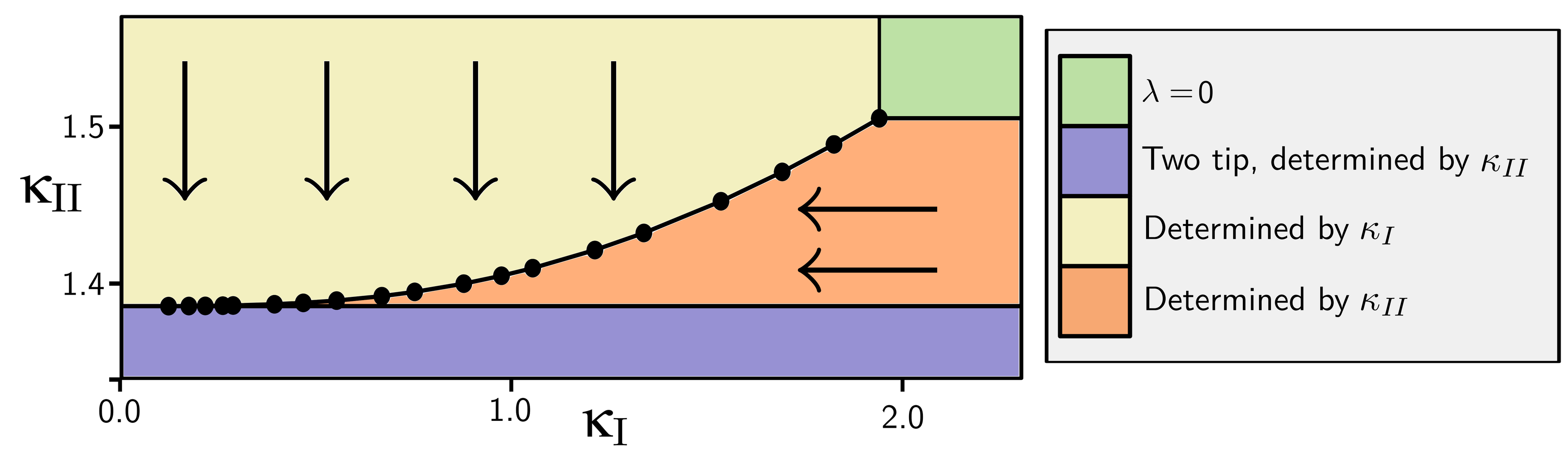

Figure 5 : For a given (κ_I, κ_{II}) value, this diagram shows where the fracture speed is limited by the κ_I or κ_{II} value, and where there is a two tip fracture.

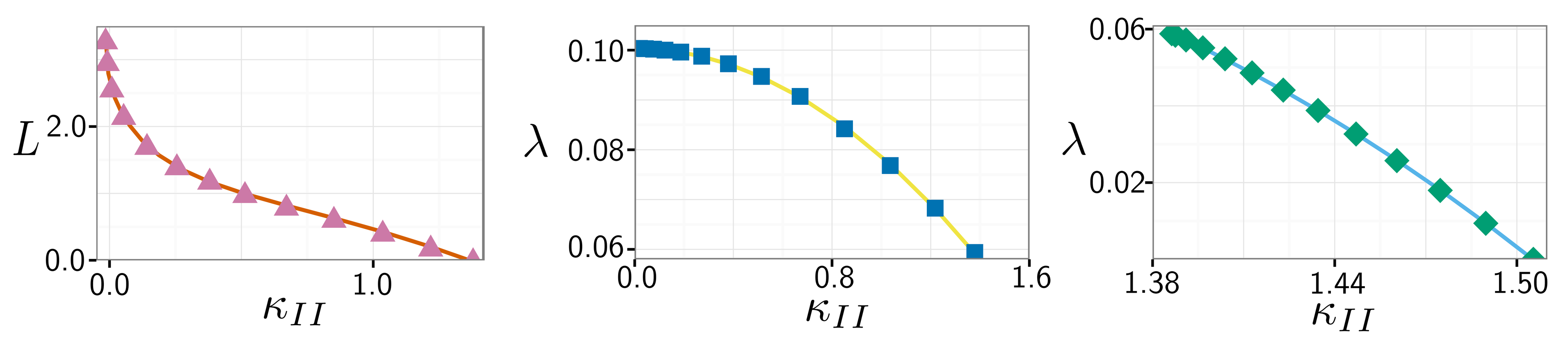

Figure 6 : When the fracture speed is limited by κ_{II}, these graphs provide a way of calculating λ, L in terms of κ_{II}. The two graphs on the left are for $L > 0$ and the graph on the right is for $L = 0$.

References

[1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.

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