Rescaling the equations and boundary conditions

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In this document, we ignore all of the physics and geometry of the problem, to just consider the governing equations on their own. There are an abundance of dimensional parameters, but we consider most of them to be fixed. What we are really interested in, is the relationship between M, K_I and c.

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$12\mu c = h^2 p' \tag{2}$$

$$\begin{cases} \lim\limits_{x\to\infty} h''(x) & = & \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim\limits_{x\to\infty} g'(x) & = & \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \tag{3}$$

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \tag{4}$$

We switch to working with dimensionless variables,

$$(x, p, h, g, \underline{\underline{K}}) \to (\xi, \Pi, H, G, \underline{\underline{\Lambda}})$$

Let us do this by the following transforms:

$$\begin{aligned} x &= \ell\xi \\ p(x) &= \beta\Pi(\xi) \\ h(x) &= \alpha H(\xi) \\ g(x) &= \alpha G(\xi) \\ \underline{\underline{K}}(x) &= \frac{1}{\ell}\underline{\underline{\Lambda}}(\xi) \end{aligned}$$

At this point we have already made some choices. I would claim the choices made this far are "natural" although I don't have any more justification than that. The equations become

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \frac{E\alpha}{4\pi(1-\nu^2)\beta\ell} \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi}-\xi) \begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{5}$$

$$\frac{12\mu c\ell}{\alpha^2\beta} = H^2\Pi' \tag{6}$$

$$\begin{cases} \lim_{\xi\to\infty} H''(x) &= \frac{12(1-\nu^2)}{E\ell\alpha} M \\ \lim_{\xi\to\infty} G'(x) &= \frac{6(1-\nu^2)}{E\ell\alpha} M \end{cases} \tag{7}$$

$$K_I = \lim_{\xi\to 0} \frac{E\alpha}{1-\nu^2} \sqrt{\frac{\pi}{8\ell}} \sqrt{\xi} H'(\xi) \tag{8}$$

We now have several choices. The first to be explored, is to take equations 5 and 6 and set the dimensionless parameters appearing in them to unity.

1 Scaling out c

Set $\frac{E\alpha}{4\pi(1-\nu^2)\beta\ell} = 1$ and $\frac{12\mu c\ell}{\alpha^2\beta} = 1$ which uniquely determines α, β as

$$\alpha^3 = \frac{48\pi(1-\nu^2)\mu c\ell^2}{E} \qquad \beta^3 = \frac{3\mu c E^2}{4\pi^2(1-\nu^2)^2\ell}$$

We then have the relavent equations as

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{\Lambda}}(\tilde{\xi}-\xi) \begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{9}$$

$$1 = H^2\Pi' \tag{10}$$

$$\begin{cases} \lim_{\xi\to\infty} H''(x) &= \gamma \\ \lim_{\xi\to\infty} G'(x) &= \gamma/2 \end{cases} \tag{11}$$

$$K_I = \lim_{\xi\to 0} \frac{E\alpha}{1-\nu^2} \sqrt{\frac{\pi}{8\ell}} \sqrt{\xi} H'(\xi) \tag{12}$$

Where $\gamma = M\left(\frac{36(1-\nu^2)^2}{\pi E^2\mu c\ell^5}\right)^{1/3}$. Given equations 9,10,11, we can solve for K_I. It is clear from the form of them, that the only way the physical parameters can enter the solution is through γ. Thus, the relationship between K_I and the other physical quantities *must* be of the form

$$K_I = E^{2/3}\mu^{1/3}c^{1/3}\ell^{1/6}(1-\nu^2)^{-2/3}f(\gamma)$$

N.B. if all we are interested in is K_I, c, M, get the relationship

$$K_I = c^{1/3}\tilde{f}(M/c^{1/3})$$

but this does depend on other physical parameters.

2 Scaling out γ

Note that this is the scaling that is used in other documents, and Tim's work. Set $\dfrac{E\alpha}{4\pi(1-\nu^2)\beta\ell} = 1$ and $\dfrac{12(1-\nu^2)}{E\ell\alpha}M = 1$. We get that

$$\alpha = \frac{12(1-\nu^2)M}{E\ell} \qquad \beta = \frac{3M}{\pi\ell^2}$$

We then have the relevant equations as

$$\begin{pmatrix} \Pi(\xi) \\ 0 \end{pmatrix} = \int_0^\infty \underline{\underline{A}}(\tilde{\xi}-\xi)\begin{pmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{pmatrix} d\tilde{\xi} \tag{13}$$

$$\lambda = H^2\Pi' \tag{14}$$

$$\begin{cases} \lim\limits_{\xi\to\infty} H''(x) & = & 1 \\ \lim\limits_{\xi\to\infty} G'(x) & = & 1/2 \end{cases} \tag{15}$$

$$K_I = \lim_{\xi\to 0} 3M\sqrt{\frac{2\pi}{\ell^3}}\sqrt{\xi}H'(\xi) \tag{16}$$

Where $\lambda = \frac{\pi\mu c E^2\ell^5}{36(1-\nu^2)^2M^3}$. Thus K_I must be of the form

$$K_I = M\ell^{-3/2}s(\lambda)$$

for some function s. This is not a new relationship, writing $s = \lambda^{1/3}\tilde{s}(\lambda)$ recovers the exact same relationship that we had earlier. We can also rescale K_I via $K_I = M\ell^{-3/2}\kappa$, so that $\kappa = \lim\limits_{\xi\to 0} 3\sqrt{2\pi}\sqrt{\xi}H'(\xi)$ and so $\kappa = \kappa(\lambda) = s(\lambda)$.

Note that the value of λ_0 s.t. $\kappa(\lambda_0) = 0$ does not depend on any physical parameters. It can be found numerically; $\lambda_0 \approx 0.06$. For the small toughness solution, we have

$$\kappa(\lambda) \propto (\lambda - \lambda_0)^{1/u}$$

where $u \approx 3.17\ldots$ is a number obtained by solving a transcendental equation. Thus, for K_I small we have the relationship

$$\lambda = \lambda_0 + A(\ell^{3/2} K_I / M)^u$$

Where A is a dimensionless constant that can be determined numerically and does not depend on any physical parameters, $A \approx -0.21$. From this we can calculate the dependance of c on the other physical parameters, for the small toughness solution.

$$c = \frac{36(1-\nu^2)^2 M^3}{\pi \mu E^2 \ell^5} \left(\lambda_0 + A(\ell^{3/2} K_I / M)^u \right)$$