Linear perturbation problem

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Here we consider the linear perturbation problem, and how it can be solved numerically. First we rescale into dimensionless parameters. Recall the full equations are:

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$12\mu c = h^2 p' \tag{2}$$

$$\begin{cases} \lim\limits_{x\to\infty} h''(x) &= \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim\limits_{x\to\infty} g'(x) &= \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \tag{3}$$

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \tag{4}$$

1 Rescaling

Let us use a length scale ℓ, pressure scale $p^* = \frac{E}{12(1-\nu^2)}$, and a time scale $t^* = 12\mu/p*$. We define the following dimensionless parameters.

$$\mathcal{M} = \frac{M}{p^*\ell^2}, \qquad \mathcal{C} = \frac{c}{\ell/t^*} = \frac{12\mu c}{p^*\ell}, \qquad \mathcal{K} = \frac{K_I}{p^*\ell^{1/2}}$$

We also define the variables (with α and β dimensionless rescalings to be determined)

$$x = \ell\xi, \qquad K_{ij} = 3p^*\Lambda_{ij}/\pi\ell, \qquad h = \alpha\ell H(\xi), \qquad p = \beta p^*\Pi(\xi)$$

So that

$$\begin{pmatrix} \Pi \\ 0 \end{pmatrix} = \frac{3\alpha}{\pi\beta} \int \Lambda \begin{pmatrix} G' \\ H' \end{pmatrix} d\xi, \qquad H^2\Pi' = \frac{\mathcal{C}}{\alpha^2\beta},$$

$$H'' \to \mathcal{M}/\alpha, \qquad 3\sqrt{2\pi\xi}H' \sim \frac{K_I}{(4\pi\mu x p^{*2}\ell^{1/2})^{1/3}}$$

Choosing $\alpha = \pi\beta/3 = \mathcal{M}$, $\lambda = \dfrac{\pi\mathcal{C}}{3\mathcal{M}^3} = \dfrac{4\pi\mu c p^{*2}\ell^5}{M^3}$ gets Tim's scalings.

$$\begin{pmatrix} \Pi \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G' \\ H' \end{pmatrix} d\xi, \qquad H^2\Pi' = \lambda,$$

$$H'' \to 1, \qquad 3\sqrt{2\pi\xi}H' \sim \frac{K_I}{M\ell^{-3/2}} \equiv \kappa$$

Now suppose that $(G_0, H_0, \Pi_0, \lambda_0)$ gives the solution for $\kappa = 0$. The outer limit of the LEFM solution is

$$H \sim H_0 + \mathcal{E}(\kappa)\left(\frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3} + \xi^s + \dots\right)$$

where $\mathcal{E} = C\kappa^{4-6s}\lambda_0^{2s-1}$, $s = 0.138673$, and $C = \beta_1(2/9\pi)^{2-3s}(1/4\pi)^{2-3s} = 8.99 \times 10^{-5}$. Working to first order in $\mathcal{E}$ we have the linear outer problem

$$\begin{pmatrix} \Pi_1 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_1' \\ H_1' \end{pmatrix} d\xi, \qquad H_0^2\Pi_1' + 2H_0H_1\Pi_0' = \lambda_1,$$

$$H_1'' \to 0, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3}$$

But we also have to zeroth order

$$\begin{pmatrix} \Pi_0 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_0' \\ H_0' \end{pmatrix} d\xi, \qquad H_0^2\Pi_0' = \lambda_0,$$

$$H_0'' \to 1, \qquad H_0 \sim \tilde{A}\lambda_0^{1/3}\xi^{2/3}$$

Subtracting a scaled version of the first order solution from the zeroth order solution, we get that

$$\begin{pmatrix} \Pi_0 - a\Pi_1 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_0' - aG_1' \\ H_0' - aH_1' \end{pmatrix} d\xi, \quad H_0^2(\Pi_0 - a\Pi_1)' + 2H_0(H_0 - aH_1)\Pi_0' = 3\lambda_0 - a\lambda_1,$$

$$(H_0 - aH_1)'' \to 1, \qquad H_0 - aH_1 \sim \frac{\tilde{A}}{3\lambda_0^{2/3}}\xi^{2/3}(3\lambda_0 - a\lambda_1) - a\xi^s$$

Setting $a = 3\lambda_0/\lambda_1$ and defining $\tilde{H} = H_0 - aH_1$ etc. gets the equations

$$\begin{pmatrix} \tilde{\Pi} \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} \tilde{G}' \\ \tilde{H}' \end{pmatrix} d\xi, \quad H_0^2\tilde{\Pi}' + 2H_0\tilde{H}\Pi_0' = 0,$$

$$\tilde{H}'' \to 1, \qquad \tilde{H} \sim -\frac{3\lambda_0}{\lambda_1}\xi^s$$

Note that this is a slightly different scaling than proposed before, but the idea is, that this has imposable boundary conditions at ∞, and the elasticity integral equation is exactly the same as before. This means the old code can hopefully be reused.

2 Numerical strategy

The old method of linearising via $\tilde{H}'(\xi) \approx \frac{1}{\sqrt{\xi}}(a\xi + b)$ may no longer be too helpful, since we do not predict such a $\xi^{-1/2}$ singularity. It is possible (and even likely) that $\tilde{G}'$ still has such a singularity, but we predict $\tilde{H}'$ to have a ξ^{s-1} singularity near the origin. The $\tilde{H}'(\xi) \approx \xi^{s-1}(a\xi + b)$ approximation, makes the analytic expressions inside the elasticity integal become worse than they already were.

For now, I propose sticking to the old scheme. Even though it doesn't correctly predict the behaviour near zero, the code should be written to see if this works at all. If it doesn't, then at least we have a rough structure of the code which shouldn't be too time consuming to change for a new representation.

Since we are using the same representation as before, we get that

$$\begin{pmatrix} \tilde{\Pi}(z_1) \\ \vdots \\ \tilde{\Pi}(z_{n-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} B_{1,1} & \cdots & B_{1,2n} \\ \vdots & \ddots & \vdots \\ B_{2(n-1),1} & \cdots & B_{2(n-1),2n} \end{pmatrix} \boldsymbol{\gamma} = BT\boldsymbol{\theta}$$

Where B is in lieu of the Kernel integral, T is the interpolation matrix, and recall that

$$\boldsymbol{\gamma} = (a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots d_n)$$

$$\boldsymbol{\theta} = (a_1\xi_1 + b_1, \ldots, a_n\xi_n + b_n, \ldots, c_1\xi_1 + d_1, \ldots, c_n\xi_n + d_n)$$

Now, the difference from before is that the second equation for Π is linear in $\tilde{H}$.

$$\tilde{\Pi} = \int_z^\infty \frac{2\tilde{H}\Pi_0'}{H_0} d\xi$$

So (after a struggle) one might be able to alternatively represent $\tilde{\Pi}$ via

$$\begin{pmatrix} \tilde{\Pi}(z_1) \\ \vdots \\ \tilde{\Pi}(z_{n-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} R_{1,1} & \cdots & R_{1,2n} \\ \vdots & \ddots & \vdots \\ R_{n-1,1} & \cdots & R_{n-1,2n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \boldsymbol{\theta}$$

To get that $(BT - R)\boldsymbol{\theta} = 0$. We can add another two rows to the matrix by demanding $\theta_n = 1/2$ and $\dfrac{\theta_{2n} - \theta_{2n-1}}{x_{2n} - x_{2n-1}} = 1$. This gives us a matrix equation to solve: $A\boldsymbol{\theta} = \boldsymbol{c}$, where $\boldsymbol{c} = 0$ except $c_n = 1/2$, $c_{2n} = 1$. Inverting A should get the required answer.

3 Numerical Details

We have inherited the matrix BT from Tim's code. It may well be that this needs changing in the future, but for now, we shall take this as given. Therefore, the only real new part to code is the matrix R.

Recall we have also inhereted a matrix H_{Coeff}. It has the property that

$$(w_1, e_1, r_1, \dots, w_n, e_n, r_n) = H_{Coeff}\boldsymbol{\theta}$$

We also recall that

$$\tilde{H}(\xi) = \begin{cases} \sqrt{\xi}(w_i\xi + e_i) + r_i & i < t \\ w_i\xi^2 + e_i\xi + r_i & i \geq t \end{cases}$$

For $\xi \in [\xi_i, \xi_{i+1}]$. We have that

$$(H(\xi_1), \dots, H(\xi_n)) = SH_{coeff}\boldsymbol{\theta}$$

Where

$$S = \begin{pmatrix} x_1^{3/2} & x_1^{1/2} & 1 & & & & & & & & & 0 \\ & & & \ddots & & & & & & & & \\ & & & & x_{t-1}^{3/2} & x_{t-1}^{1/2} & 1 & & & & & \\ & & & & & & & x_t^2 & x_t & 1 & & \\ & & & & & & & & & & \ddots & \\ 0 & & & & & & & & & x_n^2 & x_n & 1 \end{pmatrix}$$

(This will never appear as a matrix in the code, point is that it can be thought of one). Recall the equation we are trying to solve:

$$\tilde{\Pi} = \int_z^\infty \frac{2\tilde{H}\Pi_0'}{H_0} d\xi$$

We want to solve for $(\tilde{\Pi}(z_1), \dots \tilde{\Pi}(z_{n-1}))$. In a break with the approaches previously used, we will solve this via simple numerical integration using the trapezium rule. We will also use the lubrication equation $\Pi_0' H_0^2 = \lambda_0$ to remove Π_0'.

$$\tilde{\Pi}(z_k) = \int_{z_k}^{\xi_{k+1}} \frac{2\lambda_0\tilde{H}}{H_0^3} d\xi + \sum_{r=k+1}^{n-1} \int_{\xi_r}^{\xi_{r+1}} \frac{2\lambda_0\tilde{H}}{H_0^3} d\xi + \int_{\xi_n}^{\infty} \frac{2\lambda_0\tilde{H}}{H_0^3} d\xi$$

It is simple to approximate some of these integrals with the trapezium rule:

$$\int_{\xi_r}^{\xi_{r+1}} \frac{2\lambda_0\tilde{H}}{H_0^3} d\xi \approx (\xi_{r+1} - \xi_r)\left(\frac{\lambda_0\tilde{H}(\xi_{r+1})}{H_0^3(\xi_{r+1})} + \frac{\lambda_0\tilde{H}(\xi_r)}{H_0^3(\xi_r)}\right) = (\xi_{r+1} - \xi_r)(f(\xi_{r+1}) + f(\xi_r))$$

Where we adopt the new notation $f(\xi) = \lambda_0\tilde{H}(\xi)/H_0^3(\xi)$ for convenience. It is easy to see that

$$(f(\xi_1), \dots, f(\xi_n)) = FSH_{Coeff}$$

where now

$$F = \begin{pmatrix} \lambda_0/H_0^3(\xi_1) & & \\ & \ddots & \\ & & \lambda_0/H_0^3(\xi_n) \end{pmatrix}$$

That leaves two integrals to deal with. The first is also easily despached with the trapezium rule, just needs a little bit of care.

$$\int_{z_k}^{\xi_{k+1}} 2f(\xi)d\xi \approx (\xi_{k+1} - z_k)(f(\xi_{k+1}) + f(z_k))$$

where

$$f(z_k) \approx f(\xi_k) + \frac{z_k - \xi_k}{\xi_{k+1} - \xi_k}(f(\xi_{k+1}) - f(\xi_k))$$

so

$$\int_{z_k}^{\xi_{k+1}} 2f(\xi)d\xi \approx (\xi_{k+1} - z_k)\left(f(\xi_k)\frac{\xi_{k+1} - z_k}{\xi_{k+1} - \xi_k} + f(\xi_{k+1})\frac{z_k - 2\xi_k + \xi_{k+1}}{\xi_{k+1} - \xi_k}\right)$$

As for $\int_{\xi_n}^{\infty} 2f(\xi)d\xi$, this is where I don't have any particularly good ideas so far. From the boundary conditions, expect that $f(\xi) \approx 4\lambda_0/\xi^4$ to leading order. So

$$\int_{\xi_n}^{\infty} 2f(\xi)d\xi \approx \frac{8\lambda_0}{4\xi_n^3} = \frac{2}{3}\xi_n f(\xi_n)$$

This may prove to be a pretty poor approximation, so if this method fails, correcting this (or putting solid bounds on the error) is an important improvement to make. Finally we get that

$$\begin{aligned}\tilde{\Pi}(z_k) \approx &(\xi_{k+1} - z_k)\left(f(\xi_k)\frac{\xi_{k+1} - z_k}{\xi_{k+1} - \xi_k} + f(\xi_{k+1})\frac{z_k - 2\xi_k + \xi_{k+1}}{\xi_{k+1} - \xi_k}\right) \\ &+ \sum_{r=k+1}^{n-1} (\xi_{r+1} - \xi_r)(f(\xi_{r+1}) + f(\xi_r)) + \frac{2}{3}\xi_n f(\xi_n)\end{aligned}$$

Which, one notes, is linear in the $f(\xi_k)$.