Viscous control of shallow elastic fracture

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This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/ relatedlink/jfm-keywords.pdf for the full list)

1. Introduction

Consider a semi infinite elastic solid, with a thin strip peeled off, and the resulting crack filled with an incompressible fluid with viscosity μ, as shown in figure 1. The motion is driven by a constant bending moment M. We look for travelling wave solutions, propagating with speed c. We define the origin to be instantaneously at the crack tip, and the positive x axis to be aligned in the direction of the crack. We define the vertical displacement to be $h(x)$, the horizontal displacement to be $g(x)$, and the thickness of the strip as l.

The flow is assumed to be in lubrication everywhere. The fracture is assumed to obey linear elastic fracture mechanics, which describes well the fracture of brittle solids. Since we have posed a two dimensional problem, only mode I and mode II fractures are of relevance. These are governed by two fracture toughness constants K_I, K_{II}. This paper will calculate the speed of travelling waves c for any combination of K_I, K_{II}. To do that, we also consider a geometry where the mode II fracture preceeds the fluid tip.

The problem considered here is relevant to the physical problem of the expansion of a magma bubble just under the surface, with the motion being driven by a flux of magma into the bubble. Consider just the outer edges of such an expanding bubble. Looking at just the crack tip, the problem becomes the one studied in this paper, where the motion is driven by some far off bending moment.

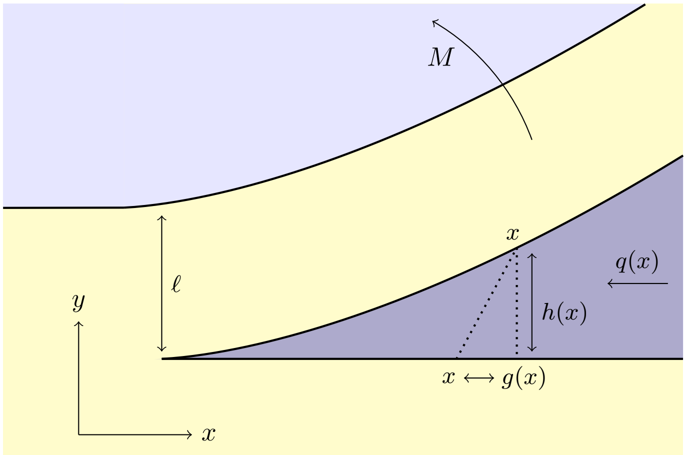

FIGURE 1. Diagram of the geometry. **Redo this diagram**

2. Formulation of problem

2.1. *Single tip*

From lubrication, we have Poiseulle flow in the crack. We obtain the flux, and conservation of mass as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3\,, \qquad \frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0\,, \tag{2.1}$$

which combined gives

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2\,. \tag{2.2}$$

Setting $p \to 0$ at $x \to \infty$, we can write this in integral form,

$$p(x) = -\int_x^\infty 12\mu c/h(\tilde{x})^2 \mathrm{d}\tilde{x}\,. \tag{2.3}$$

From (citations to relevant papers) who have studied an elastic solid with the same geometry, we have

$$\begin{bmatrix} -\sigma_y \\ -\tau_{xy} \end{bmatrix} = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} = \frac{E}{4\pi l(1-\nu^2)}\int_0^\infty \boldsymbol{K}\left(\frac{\tilde{x}-x}{l}\right)\begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix}\mathrm{d}\tilde{x}\,, \tag{2.4}$$

where the integral kernel is

$$\boldsymbol{K}(\xi) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{(32-24\xi^2)}{(\xi^2+4)^3} & \frac{(48\xi^2-64)}{\xi(\xi^2+4)^3} \\ -\frac{(16\xi^4+16\xi^2+4)}{\xi(\xi^2+4)^3} & -\frac{(32-24\xi^2)}{(\xi^2+4)^3} \end{bmatrix}. \tag{2.5}$$

The boundary conditions near $x = 0$ are governed by fracture mechanics,

$$K_I \geqslant \lim_{x\to 0}\frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\,h'(x)\,, \qquad K_{II} \geqslant \lim_{x\to 0}\frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\,g'(x)\,. \tag{2.6a, b}$$

Where equality holds in at least one of the two equations.

As we go to $x \gg l$, we are looking at the problem of peeling off a thin strip from an elastic half space. We can then use beam theory approximations, which give

$$M(x) = \frac{El^3}{12(1-\nu^2)}\frac{\mathrm{d}^2h}{\mathrm{d}x^2} = \frac{El^2}{6(1-\nu^2)}\frac{\mathrm{d}g}{\mathrm{d}x}, \qquad p = \frac{El^3}{12(1-\nu^2)}h^{(4)}(x) \tag{2.7a, b}$$

As $x \to \infty$, $M(x) \to M$, the applied bending moment, so this gives us boundary conditions on h'', g'.

2.2. *Double tip*

Consider the mode II fracture preceeding the fluid tip at $x = 0$ by a distance lL, so $h(x), h'(x) = 0$ for $-lL < x < 0$ (but $g \neq 0$). Since the solid has already fractured, $h'(x)$ does not have an $x^{-1/2}$ singularity at $x = 0$. The boundary conditions at the crack tip become

$$\lim_{x\to 0} \sqrt{x}\, h'(x) = 0\,, \qquad \lim_{x\to -Ll} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, g'(x) = K_{II}. \tag{2.8a, b}$$

2.3. *Rescaling*

We can define the following dimensionless variables

$$x = l\xi, \quad h(x) = \frac{12M(1-\nu^2)}{El} H(\xi), \quad g(x) = \frac{12M(1-\nu^2)}{El} G(\xi)\,, \tag{2.9}$$

$$p = \frac{3M}{\pi l^2}\Pi(\xi), \quad K_I = Ml^{-3/2}\kappa_I, \quad K_{II} = Ml^{-3/2}\kappa_{II}, \quad \lambda = \frac{4\pi\mu p^* l^3}{M^2}\,. \tag{2.10}$$

With these scalings, the equations become

$$\begin{bmatrix} \Pi \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\tilde{\xi} - \xi) \begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi} \tag{2.11}$$

$$H^2 \frac{\mathrm{d}\Pi}{\mathrm{d}\xi} = \lambda \quad \text{or} \quad \Pi(\xi) = -\int_\xi^\infty \lambda / H(\tilde{\xi})^2 \mathrm{d}\tilde{\xi} \tag{2.12a, b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2}, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' \leqslant \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}G' \leqslant \kappa_{II}, \tag{2.13}$$

These shall be the governing equations for the rest of this paper.

2.4. *Beam theory asymptotics*

In the dimensionless variables, the outer asymptotics are of the form

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} = \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}\xi}, \qquad H^{(4)}(\xi) = \frac{3}{\pi}\Pi(\xi), \qquad \frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} \to 1 \tag{2.13a, b, c}$$

From integration by parts, we can write

$$H''(\xi) = 1 - \frac{1}{2}\int_\xi^\infty (\tilde{\xi} - \xi)^2 H^{(5)}(\tilde{\xi})\mathrm{d}\tilde{\xi}, \tag{2.14}$$

provided $\lim_{\xi\to\infty} \xi H^{(3)}(\xi) = \lim_{\xi\to\infty} \xi^2 H^{(4)}(\xi) = 0$. Then using equation 2.12a, we have that

$$H''(\xi) = 1 - \frac{3\lambda}{2\pi}\int_\xi^\infty \frac{(\tilde{\xi} - \xi)^2}{H(\tilde{\xi})^2}\mathrm{d}\tilde{\xi}. \tag{2.15}$$

Since $H(\xi) = \frac{1}{2}\xi^2 + o(\xi^2)$, as $\xi \to \infty$, we can use this to get a better estimate of H'';

$$H''(\xi) = 1 - \frac{2\lambda}{\pi}\frac{1}{\xi} + o(1/\xi). \tag{2.16}$$

This new expression can be used to refine the error estimate from $o(1/\xi)$, to $O(\log(\xi)/\xi^2)$.

2.5. *Linear perturbation problem*

This section is problably better placed elsewhere... The equations of the linear perturbation problem:

$$\Pi = \Pi_0 + \mathcal{E}\Pi_1 + O(\mathcal{E}), \quad H = H_0 + \mathcal{E}H_1 + O(\mathcal{E}) \tag{2.17}$$

$$\begin{bmatrix} \Pi_1 \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\xi - \tilde{\xi}) \begin{bmatrix} G_1'(\tilde{\xi}) \\ H_1'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H_0^2 \Pi_1' + 2H_0 H_1 \Pi_0' = \lambda_1 \tag{2.18a, b}$$

$$H_1'' \to 0 \text{ as } \xi \to \infty, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}} \xi^{2/3} + \ldots \text{ as } \xi \to 0 \tag{2.19a, b}$$

But these can be made into a more convenient form, by considering instead $\tilde{\Pi} = \Pi_0 - 3\lambda_0/\lambda_1 \Pi_1$, and similar for $\tilde{H}$, $\tilde{G}$. The equations become

$$\begin{bmatrix} \tilde{\Pi} \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\xi - \tilde{\xi}) \begin{bmatrix} \tilde{G}'(\tilde{\xi}) \\ \tilde{H}'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H_0^2 \tilde{\Pi}' + 2H_0 \tilde{H} \Pi_0' = 0 \tag{2.20a, b}$$

$$\tilde{H}'' \to 1 \text{ as } \xi \to \infty, \qquad \tilde{H} \sim -\frac{3\lambda_0}{\lambda_1} \xi^s + \ldots \text{ as } \xi \to 0 \tag{2.21a, b}$$

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking $n+1$ points $\boldsymbol{\xi} = (\xi_0 = 0, \xi_1 \ldots, \xi_n)$ at which we measure H', G', and n intermediate points $\boldsymbol{\zeta} = (\zeta_0, \ldots, \zeta_{n-1})$ at which to measure Π, so that $\xi_0 < \zeta_0 < \ldots < \zeta_{n-1} < \xi_n$. We work with $\sqrt{\xi}G'(\xi)$, $\sqrt{\xi}H'(\xi)$ near the tip to avoid singularities. We define $\boldsymbol{\theta}_G = [\sqrt{\xi_0}G'(\xi_0), \ldots, \sqrt{\xi_{t-1}}G'(\xi_{t-1}), G'(\xi_t), \ldots G'(\xi_n)]$, and $\boldsymbol{\theta}_H$ similarly, as well as $\boldsymbol{\theta} = [\boldsymbol{\theta}_G, \boldsymbol{\theta}_H]$, Typically $t \approx n/2$ was used. From the linearity of the elasticity integral (and the discretized integral) we may write

$$[\Pi(\zeta_1), \ldots, \Pi(\zeta_{n-1}), \underbrace{0, \ldots, 0}_{n-1}] = \boldsymbol{J}\boldsymbol{\theta}\,, \tag{3.1}$$

for some matrix $\boldsymbol{J}$. One can recover $H(\xi_i)$ from $\boldsymbol{\theta}_H$. Therefore, a discritized lubrication integral, yields an expression for $\Pi(\zeta_i)$ as a function of $\boldsymbol{\theta}_H$. So we can write

$$[\Pi(\zeta_1), \ldots, \Pi(\zeta_{n-1}), \underbrace{0, \ldots, 0}_{n-1}] = \boldsymbol{J}\boldsymbol{\theta} = \boldsymbol{f}(\boldsymbol{\theta}_H)\,, \tag{3.2}$$

for some function $\boldsymbol{f}$.

The values of both $G'(\xi_n)$, and $H''(\xi_n)$ are known from our beam theory asymptotic expansion. But these are linear in $\boldsymbol{\theta}$, since $G'(\xi_n) = \theta_n$, and $H''(\xi_n) \approx (\theta_{2n} - \theta_{2n-1})/(\xi_n - \xi_{n-1})$, Therefore we can add another two rows to $\boldsymbol{J}$, so that

$$\boldsymbol{A}\boldsymbol{\theta} = [\boldsymbol{f}(\boldsymbol{\theta}), G'(\xi_n), H''(\xi_n)]\,. \tag{3.3}$$

Where the $\boldsymbol{A}$ is the enlarged matrix. This can be solved by Newton's method from quite arbitrary initial guesses.

For $\xi_i < \xi < \xi_{i+1}$, we interpolate as

$$G'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad H'(\xi) = \begin{cases} \xi^{-1/2}(c_i\xi^{1/2} + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for} \quad \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.4}$$

The choice of interpolating function was based on the appearance of the relevant functions. We will also define a_n, b_n, c_n, d_n for interpolation beyond ξ_n. With this choice of interpolation, there exist exact closed form expressions for both the lubrication integral, and the elasticity integral, in terms of the $a_i - d_i$ coefficients.

It therefore remains to determine $a_i - d_i$ in terms of $\boldsymbol{\theta}$. Continuity of G', H' imposes $2(n-1)$ linear equations. We also have the $2n$ equations following from the definition of $\boldsymbol{\theta}$, (such as $a_i\xi_i + b_i = \theta_i$ for $t \leqslant i \leqslant n$).

From our asymptotic expansion (via beam theory) we know $\theta_n = G'(\xi_n)$ and $a_n = G''(\xi_n)$. Therefore we can write

$$a_n = \frac{G''(\xi_n)}{G'(\xi_n)}\theta_n, \qquad b_n = \theta_n - a_n\xi_n = \left(1 - \frac{G''(\xi_n)}{G'(\xi_n)}\xi_n\right)\theta_n. \tag{3.5}$$

With H, we know that $c_n = H''(\xi_n)$, $c_{n-1} = H''(\xi_{n-1})$, and so we have that

$$c_n = \frac{H''(\xi_n)}{H''(\xi_{n-1})}c_{n-1}, \qquad d_n = -c_n\xi_n + c_{n-1}\xi_n + d_{n-1}. \tag{3.6}$$

Therefore, we have enough equations to know the $a_i - d_i$ in terms of $\boldsymbol{\theta}$.

Note that numerically, we choose a value of λ, solve the problem and subsequently recover the boundary conditions at $\xi = 0$ (κ_I, κ_{II}). This can then be inverted, so that we think of $\lambda = \lambda(\kappa_I)$, since this is the physical interpretation.

The spacing of the points should reflect that the important part of the problem is happening near the tip, and this is where the points should be concentrated. The spacing that was typically used in numerical calculations was

$$\xi_i = \tan^2(\chi\, i/m), \quad i = 1, \ldots, m < n \tag{3.7}$$

where χ is chosen so that $\tan^2(\chi) = O(10)$, and the remaining points are added in a geometric progression, so that

$$\xi_{i+1} = (\xi_m/\xi_{m-1})\xi_i, \quad i = m, \ldots, n-1 \tag{3.8}$$

3.2. *Linear Perturbation Problem*

From equation 2.21b, we anticipate a singularity of the form ξ^{s-1} in $\tilde{H}'$, (we still expect a $\xi^{-1/2}$ singularity in $\tilde{G}'$). Therefore, the interpolation was changed to reflect this. Some of the integrals no longer have exact expressions. In this case, they are calculated by a numerical integration routine.

The lubrication equation for the linear perturbation problem (2.20b), is linear in $\tilde{H}$. Therefore, we can obtain two expressions for $\tilde{\Pi}(\zeta_i)$ that are linear in $\tilde{G}'(\xi_j)$, $\tilde{H}'(\xi_j)$. Together with the boundary conditions and beam theory asymptotics, (we haven't changed the integral kernel, so the asymptotics remain the same) there are enough equations to numerically solve the linear perturbation problem. There is no need to use Newton's method, as we can simply solve the linear set of equations.

3.3. *Double Tip*

In solving the problem of two tips situated at $-L$ and 0, an additional r points are taken to cover $-L \leqslant \xi < 0$. The spacing of points for $\xi < 0$ was chosen so that there was a concentration of points near $-L$ and near 0.

We interpolate G' expecting a $\xi^{-1/2}$ singularity at $\xi = -L$, and H' expecting a $\xi^{-1/2}$ singularity at $\xi = 0$. We do not calculate Π for $\xi < 0$ (although it is easily done), but just require that $\sigma_{xy} = 0$ for $\xi < 0$. This provides enough equations for the problem to be solved as before, with Newton's method.

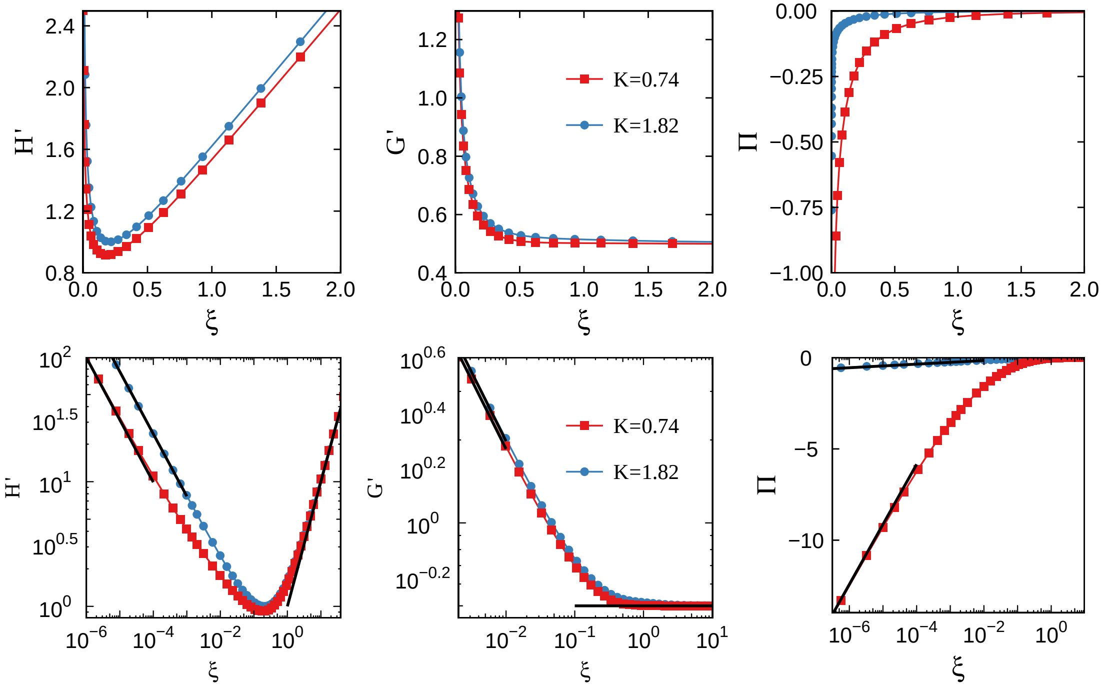

FIGURE 2. Numerical solutions for two typical values of κ_I. Logarithmic scales are shown, with solid lines indicating the predicted asymptotics; $H', G' \propto \xi^{-1/2}$, $\Pi \propto \ln(\xi)$, near $\xi = 0$, $H' \to \xi$, $G' \to 1/2$, $\Pi \to 0$ as $\xi \to \infty$. Figure produced with $n = 465$, $\xi_n = 819$.

Note that we input $-L$ and λ and recover κ_I, κ_{II}, where κ_I is measured at 0. From this, we extrapolate to $\kappa_I = 0$, and invert the relations so that $\lambda = \lambda(\kappa_{II})$, $L = L(\kappa_{II})$, to reflect the physical interpretation.

4. Results

4.1. *Single tip*

The single problem was solved numerically for the full range of λ values, $0 \leqslant \lambda < 0.059$, which corresponds to the values $0 < \kappa_I \leqslant 1.9$. We have arbitrarily chosen κ_I as the parameter determining the speed, although it would have been equally valid to consider κ_{II} as the independant parameter. The results for H, G, and Π are shown in figure 2, where the predicted asymptotics are shown as solid lines. These predicted asymptotics are namely that near $\xi = 0$, $H' \sim \kappa_I \xi^{-1/2}/(3\sqrt{2\pi})$, $G' \sim \kappa_{II}\xi^{-1/2}/(3\sqrt{2\pi})$, $\Pi \sim 9\pi\lambda/(2\kappa_I^2)\ln(\xi)$, and that as $\xi \to \infty$, $G' \to 1/2$, $H' \sim \xi$, $\Pi \to 0$.

4.1.1. *Calculating H for $\kappa_I = 0$*

$H(\xi; \kappa_I = 0)$ will be needed for the linear pertubation problem. Numerically, for each ξ_i, $H'(\xi_i; \kappa_I = 0)$ is extrapolated from $H'(\xi_i, \kappa_j)$ from two κ_j values. Figure 4.1 shows that $H'(\xi; 0.21)$ is a good approximation to $H'(\xi; 0)$, away from a boundary layer near $\xi = 0$. The size of the boundary layer becomes smaller as κ_I decreases, but to avoid using very small values of κ_I, the effects of the boundary layer are removed by simply extending the linear trend present in $0.002 < \xi < 0.003$ all the way to $\xi = 0$.

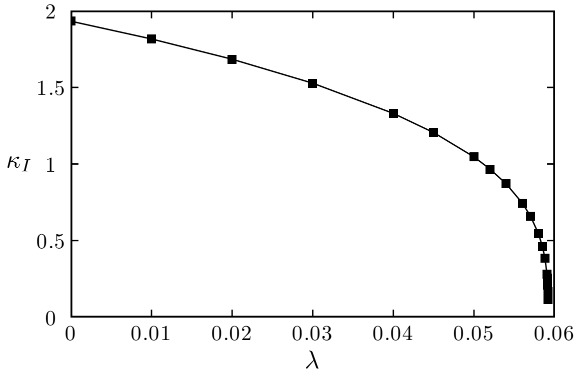

FIGURE 3. Here we vary the parameter λ and plot the change in κ_I. Figure produced with $n = 465$, $\xi_n = 819$.

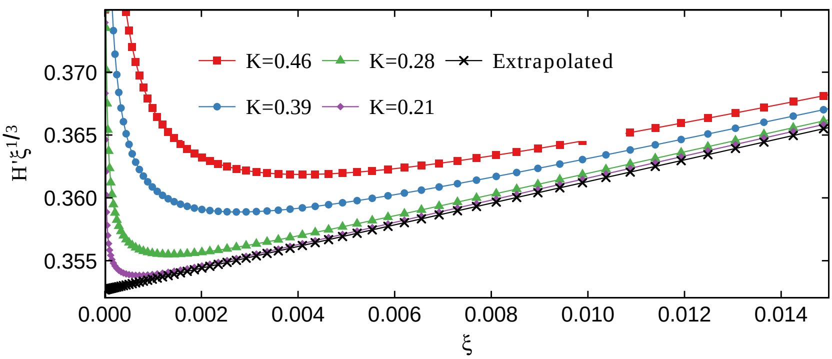

FIGURE 4. As $\kappa_I \to 0$, H' moves from a $\xi^{-1/2}$ singularity to a $\xi^{-1/3}$ singularity. We can not calculate $\kappa_I = 0$, but the extrapolation to it is shown. Figure produced with $n = 465$, $\xi_n = 819$.

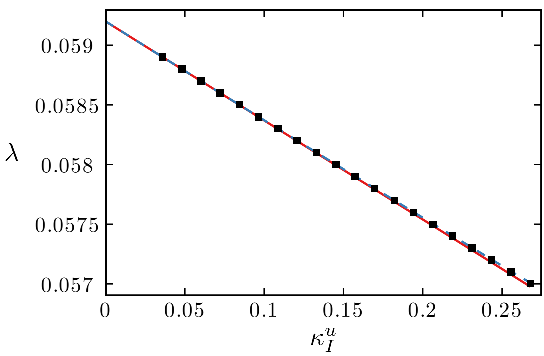

FIGURE 5. The numerical values of κ_I^u are plotted as points against the values of λ. A linear fit from the two smallest κ_I values is plotted as a solid line, a quadratic fit from the three smallest κ_I values is plotted as a dashed line. They are almost indistiguishable at this scale. The difference between the two extrapolations to $\kappa_I = 0$, provides an estimate of the error in calculating λ_0, (not accounting for the error due to n), which in this instance is $\approx 0.002\%$. This figure was made with $n = 524$, $\xi_n = 846$.

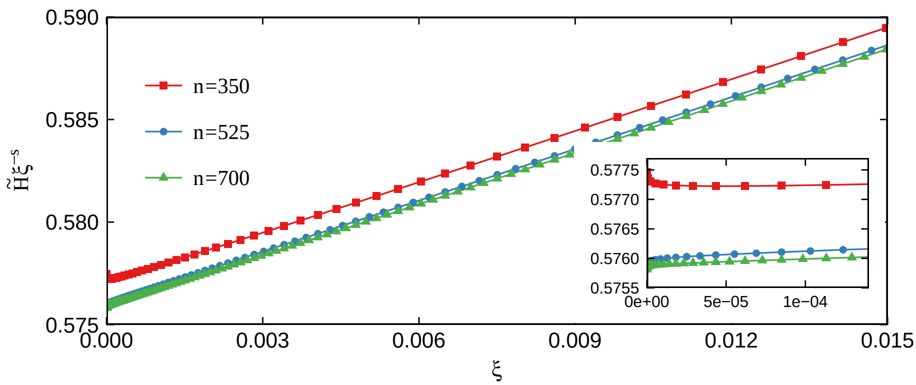

FIGURE 6. The numerical solution of the linear perturbation problem near $\xi = 0$ for a selection of resolutions, all with $\xi_n = 875$. Of interest, is the value of the intercept, which as shown is dependent on the resolution. Also shown is the numerical divergence near the tip, due to the difficulty in calculating H_0 for $\xi \ll 1$.

4.2. *Linear perturbation problem*

We solve the linear perturbation problem. All that we really want to know is that we see the $\tilde{H} \sim \xi^s$ behaviour that we expect, and we ask what the intercept of $\tilde{H}$ is. It is perhaps worth mentioning the difficulties in measuring the intercept and perhaps a notion of the sensitivity of the result on the estimate provided for H_0. Illustrating that is the next figure

4.3. *Two tips*

After the linear perturbation problem, we move on to the two tip problem. Perhaps some graphs that show an outline of the full numerical problem with non-zero κ_I and κ_{II}, although these are not physical.

We now move on to the $\kappa_I = 0$ set of relations.

Approximate formula in the single tip case:

$$\lambda \approx 0.059 - 0.0083\kappa_I^u + 0.00033\kappa_I^{3u/2} \tag{4.1}$$

$$\lambda \approx -1.5 + 2.7\kappa_{II} - 1.1\kappa_{II}^2 \tag{4.2}$$

In the double tip case:

$$\lambda \approx 0.10 - 0.022\kappa_{II}^2 \tag{4.3}$$

These equations provide a fairly good approximation to the data, see figure 10.

From dry fracture mechanics and conservation of energy, one expects a relationship of the form $\lambda = \alpha + \beta\kappa_{II}^2$. However, in this case, α and β should depend on the geometry, H, and so should be functions of κ_{II}. However, numerical evidence shows them as being approximately constant. Part of the reason for this, is the decoupling between the fluid problem and the dry tip. Suppose one solves the two tip problem, for some L. This gives a geometry, the reference H'. From this, one can choose any λ, and find G', and value of κ_{II}. The effect of this is shown in figure 11, a changing H' has little effect on the λ, κ_{II} relationship.

5. Discussion

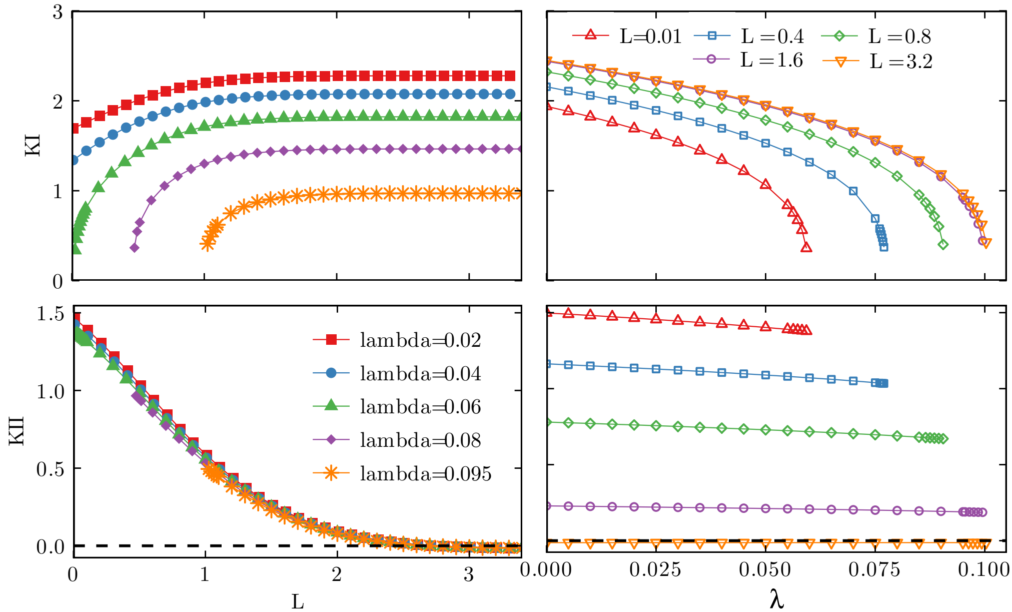

FIGURE 7. Some of the numerical results for the two tip problem. Having $\kappa_I \neq 0$ at $\xi = 0$ and $L \neq 0$ is unphysical, but is what is found numerically. We can recover the physical solution by increasing λ for fixed L until $\kappa_I = 0$. Figure made with $n = 995$, $\xi_n = 846$.

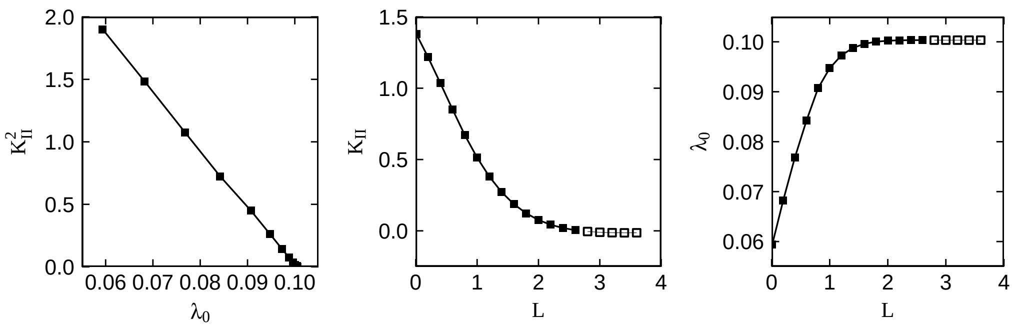

FIGURE 8. The results of extrapolating to $\kappa_I = 0$. Hollow squares indicate a value of $\kappa_{II} < 0$. Figure made with $n = 995$, $\xi_n = 846$.

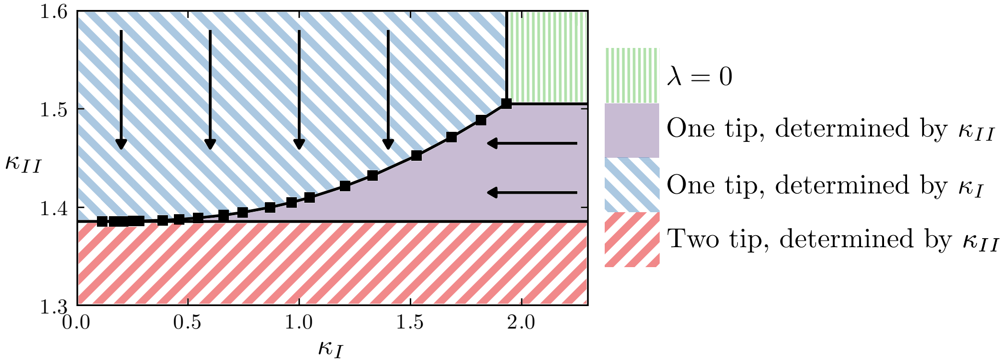

FIGURE 9. Given values (κ_I, κ_{II}), this graph determines which frature regime occurs and so how λ and/or L should be calculated. Figure made with $n = 465$, $\xi_n = 819$.

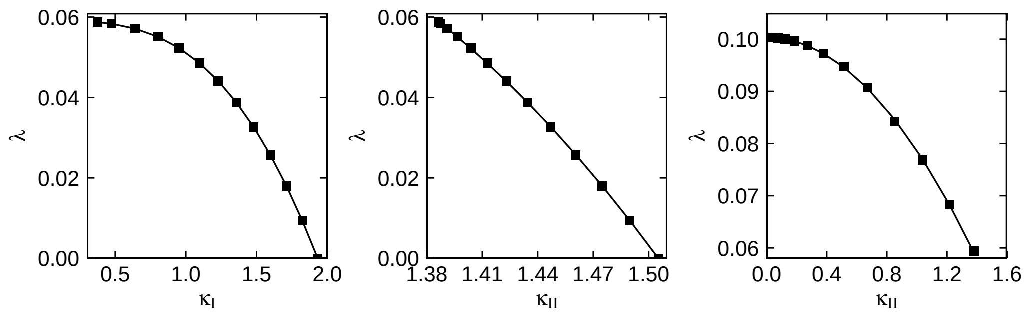

FIGURE 10. The formula (solid lines) giving good approximation to the calculated values (symbols). For the single tip calculations, $n = 815$ was used, for the double tip $n = 995$. $\xi_n = 846$ in both cases.

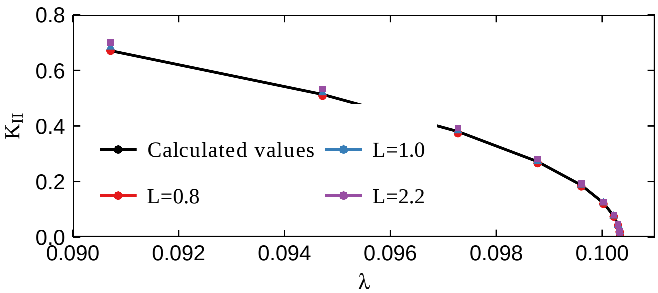

FIGURE 11. Reconstructing the full solution given a reference H'. Figure made with $n = 995$, $\xi_n = 846$.

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`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
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Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

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