Numerically calculating dimensionless coefficients

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Recall that

$$\gamma = 1/\lambda^{1/3}$$

$$\mathfrak{G} = \frac{1}{\lambda^{1/3}}\mathscr{G} \qquad \mathfrak{H} = \frac{1}{\lambda^{1/3}}\mathscr{H}$$

$$\mathfrak{P} = \frac{1}{\lambda^{1/3}}\mathscr{P} \qquad \mathfrak{K} = \frac{1}{\lambda^{1/3}}\mathscr{K}$$

The relevant asymptotic form is

$$\lambda = \lambda_0 + \mathcal{E}(\mathscr{K})\lambda_1 + \dots$$

$$\mathscr{H}(\xi) = (A_0\xi^{2/3} + \dots) + \mathcal{E}(\mathscr{K})\left(\frac{A_0\lambda_1}{3\lambda_0}\xi^{2/3} + \xi^s + \dots\right) + \dots$$

$$\mathscr{P}(\xi) = (-\frac{3\lambda_0}{A_0^2}\xi^{-1/3} + \dots) + \mathcal{E}(\mathscr{K})\left(\frac{2\pi A_0\lambda_1}{9\lambda_0\sqrt{3}}\xi^{-1/3} + \frac{4\pi}{9\sqrt{3}(1-s)}\xi^{s-1} + \dots\right) + \dots$$

$$\mathcal{E}(\mathscr{K}) = C\mathscr{K}^u\lambda_0^{2s-1}$$

Or, in the alternative scaling

$$\mathfrak{H}(\xi) = \left(\left(\frac{243}{4\pi^2}\right)^{1/6}\xi^{2/3} + \dots\right) + \mathcal{E}(\mathfrak{K})\left(\xi^s + \dots\right) + \dots$$

$$\mathfrak{P}(\xi) = \left(-\left(\frac{2\pi}{3}\right)^{2/3}\xi^{-1/3} + \dots\right) + \mathcal{E}(\mathfrak{K})\left(\frac{4\pi}{9\sqrt{3}(1-s)}\xi^{s-1} + \dots\right) + \dots$$

$$\mathcal{E}(\mathfrak{K}) = C\mathfrak{K}^u$$

We wish to calculate the parameters λ_0, λ_1, C. To do this, we will also introduce the parameter D,

$$D = C\lambda_1\lambda_0^{2s-1}$$

so that $\lambda = \lambda_0 + D\mathscr{K}^u + \dots$.

The good news

Very rough estimates give that $\lambda_0 \approx 0.06$, $D \approx -0.01$, $\lambda_1 \approx -0.2$. For values of $\mathscr{K} < 1$, this gives $|\mathcal{E}(\mathscr{K})| < 0.05$. Further, for $\mathscr{K} < 0.5$ (we can go this low numerically, but not much further) $|\mathcal{E}(\mathscr{K})| < 0.005$. So for moderately low values of $\mathscr{K}$, ignoring $\mathcal{E}^2$ terms is an excellent approximation.

Plotting λ against $\mathscr{K}^u$ yields a very good approximation to a straight line. Using simple linear regression we can calculate λ_0, and D, to around 2.s.f. accuracy.

$$\lambda_0 \approx 0.0059$$

$$D \approx -0.0074$$

Given this value of D, we need only calculate C, to have the full range of constants.

The bad news

Let us move into the alternative scalings.

$$\mathfrak{H}(\xi) = \mathfrak{H}_0(\xi) + \mathcal{E}(\mathfrak{K})\mathfrak{H}_1(\xi)$$

$$\mathfrak{P}(\xi) = \mathfrak{P}_0(\xi) + \mathcal{E}(\mathfrak{K})\mathfrak{P}_1(\xi)$$

$$\mathcal{E}(\mathfrak{K}) = C\mathfrak{K}^u$$

Where this holds away from the LEFM boundary layer. We know that $\mathfrak{H}_1(\xi) \to \xi^s$ near $\xi = 0$. Something we can do is to fix a point ξ, and consider $\mathfrak{H}(\xi; \mathfrak{K}^u)$. As $\mathfrak{K}^u \to 0$, plotting $\mathfrak{H}(\xi; \mathfrak{K}^u)$ against $\mathfrak{K}^u$ yields a straight line. From this line, one calculates $\mathfrak{H}_0(\xi)$ and $C\mathfrak{H}_1(\xi)$.

The suspected cause of the problem is that $\mathfrak{H}_1 = \xi^s$ is never that good of an approximation. In particular, plotting $C\mathfrak{H}_1$ against ξ^s should yield a straight line through the origin with gradient C, at least for small ξ. The problem is that it doesn't. I have not yet resolved this issue.

An alternative means of progress, would be to recall that C arises as the coefficient of an asymptotic expansion. Recall the problem:

$$\Pi(\zeta) = -\int_0^\infty \frac{\eta'(\xi)}{\xi - \zeta} d\xi, \qquad \eta^2 \Pi' = 1, \qquad \eta(\xi) \to \frac{2}{3\sqrt{2\pi}} \xi^{1/2}$$

The solution to this problem has the far field asymptotics,

$$\eta(\xi) = \left(\frac{243}{4\pi^2}\right)^{1/6} \xi^{2/3} + C\xi^s + \dots$$

This may well turn out to be a much more convenient way to find C, as the problem is certainly simpler.