Linear perturbation problem

Dominic Skinner

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Here we consider the linear perturbation problem, and how it can be solved numerically. First we rescale into dimensionless parameters. Recall the full equations are:

$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \underline{\underline{K}}(x-z) \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \tag{1}$$

$$12\mu c = h^2 p' \tag{2}$$

$$\begin{cases} \lim\limits_{x\to\infty} h''(x) & = & \frac{12(1-\nu^2)}{E\ell^3} M \\ \lim\limits_{x\to\infty} g'(x) & = & \frac{6(1-\nu^2)}{E\ell^3} M \end{cases} \tag{3}$$

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x) \tag{4}$$

1 Rescaling

Let us use a length scale ℓ, pressure scale $p^* = \frac{E}{12(1-\nu^2)}$, and a time scale $t^* = 12\mu/p*$. We define the following dimensionless parameters.

$$\mathcal{M} = \frac{M}{p^*\ell^2}, \qquad \mathcal{C} = \frac{c}{\ell/t^*} = \frac{12\mu c}{p^*\ell}, \qquad \mathcal{K} = \frac{K_I}{p^*\ell^{1/2}}$$

We also define the variables (with α and β dimensionless rescalings to be determined)

$$x = \ell\xi, \qquad K_{ij} = \Lambda_{ij}/\ell, \qquad h = \alpha\ell H(\xi), \qquad p = \beta p^* \Pi(\xi)$$

So that

$$\begin{pmatrix} \Pi \\ 0 \end{pmatrix} = \frac{3\alpha}{\pi\beta} \int \Lambda \begin{pmatrix} G' \\ H' \end{pmatrix} d\xi, \qquad H^2\Pi' = \frac{\mathcal{C}}{\alpha^2\beta},$$

$$H'' \to \mathcal{M}/\alpha, \qquad 3\sqrt{2\pi\xi}H' \sim \frac{K_I}{(4\pi\mu x p^{*2}\ell^{1/2})^{1/3}}$$

Choosing $\alpha = \pi\beta/3 = \mathcal{M}$, $\lambda = \dfrac{\pi\mathcal{C}}{3\mathcal{M}^3} = \dfrac{4\pi\mu c p^{*2}\ell^5}{M^3}$ gets Tim's scalings.

$$\begin{pmatrix} \Pi \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G' \\ H' \end{pmatrix} d\xi, \qquad H^2\Pi' = \lambda,$$

$$H'' \to 1, \qquad 3\sqrt{2\pi\xi}H' \sim \frac{K_I}{M\ell^{-3/2}} \equiv \kappa$$

Now suppose that $(G_0, H_0, \Pi_0, \lambda_0)$ gives the solution for $\kappa = 0$. The outer limit of the LEFM solution is

$$H \sim H_0 + \mathcal{E}(\kappa)\left(\frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3} + \xi^s + \dots\right)$$

where $\mathcal{E} = C\kappa^{4-6s}\lambda_0^{2s-1}$, $s = 0.138673$, and $C = \beta_1(2/9\pi)^{2-3s}(1/4\pi)^{2-3s} = 8.99 \times 10^{-5}$. Working to first order in $\mathcal{E}$ we have the linear outer problem

$$\begin{pmatrix} \Pi_1 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_1' \\ H_1' \end{pmatrix} d\xi, \qquad H_0^2\Pi_1' + 2H_0H_1\Pi_0' = \lambda_1,$$

$$H_1'' \to 0, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3}$$

But we also have to zeroth order

$$\begin{pmatrix} \Pi_0 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_0' \\ H_0' \end{pmatrix} d\xi, \qquad H_0^2\Pi_0' = \lambda_0,$$

$$H_0'' \to 1, \qquad H_0 \sim \tilde{A}\lambda_0^{1/3}\xi^{2/3}$$

Subtracting a scaled version of the first order solution from the zeroth order solution, we get that

$$\begin{pmatrix} \Pi_0 - a\Pi_1 \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} G_0' - aG_1' \\ H_0' - aH_1' \end{pmatrix} d\xi, \quad H_0^2(\Pi_0 - a\Pi_1)' + 2H_0(H_0 - aH_1)\Pi_0' = 3\lambda_0 - a\lambda_1,$$

$$(H_0 - aH_1)'' \to 1, \qquad H_0 - aH_1 \sim \frac{\tilde{A}}{3\lambda_0^{2/3}}\xi^{2/3}(3\lambda_0 - a\lambda_1) - a\xi^s$$

Setting $a = 3\lambda_0/\lambda_1$ and defining $\tilde{H} = H_0 - aH_1$ etc. gets the equations

$$\begin{pmatrix} \tilde{\Pi} \\ 0 \end{pmatrix} = \int \Lambda \begin{pmatrix} \tilde{G}' \\ \tilde{H}' \end{pmatrix} d\xi, \quad H_0^2\tilde{\Pi}' + 2H_0\tilde{H}\Pi_0' = 0,$$

$$\tilde{H}'' \to 1, \qquad \tilde{H} \sim -\frac{3\lambda_0}{\lambda_1}\xi^s$$

Note that this is a slightly different scaling than proposed before, but this has the same boundary conditions at ∞ as before, and the elasticity integral equation is exactly the same as before. This means the old code can hopefully be reused, as well as the fact that we can easily impose conditions at ∞, whereas it is non-obvious how to easily impose them at 0.

2 Numerical strategy

The old method of linearising via $\tilde{H}'(\xi) \approx \frac{1}{\sqrt{\xi}}(a\xi + b)$ may no longer be too helpful, since we do not predict such a $\xi^{-1/2}$ singularity. It is possible (and even likely) that $\tilde{G}'$ still has such a singularity, but we predict $\tilde{H}'$ to have a ξ^{s-1} singularity near the origin. Therefore, we shall use the $\tilde{H}'(\xi) \approx \xi^{s-1}(a\xi + b)$ approximation.

Recall that,

$$\begin{pmatrix} \tilde{\Pi}(z_1) \\ \vdots \\ \tilde{\Pi}(z_{n-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} B_{1,1} & \cdots & B_{1,2n} \\ \vdots & \ddots & \vdots \\ B_{2(n-1),1} & \cdots & B_{2(n-1),2n} \end{pmatrix} \boldsymbol{\gamma} = BT\boldsymbol{\theta}$$

Where B is in lieu of the Kernel integral, T is the interpolation matrix, and recall that

$$\boldsymbol{\gamma} = (a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots d_n)$$

$$\boldsymbol{\theta} = (a_1\xi_1 + b_1, \ldots, a_n\xi_n + b_n, c_1\xi_1 + d_1, \ldots, c_n\xi_n + d_n)$$

Now, the difference from before is that the second equation for $\tilde{\Pi}$ is linear in $\tilde{H}$.

$$\tilde{\Pi} = \int_z^\infty \frac{2\tilde{H}\Pi_0'}{H_0} d\xi$$

So (after a struggle) one might be able to alternatively represent $\tilde{\Pi}$ via

$$\begin{pmatrix} \tilde{\Pi}(z_1) \\ \vdots \\ \tilde{\Pi}(z_{n-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} R_{1,1} & \cdots & R_{1,2n} \\ \vdots & \ddots & \vdots \\ R_{n-1,1} & \cdots & R_{n-1,2n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \boldsymbol{\theta}$$

To get that $(BT - R)\boldsymbol{\theta} = 0$. We can add another two rows to the matrix by demanding $\theta_n = 1/2$ and $\dfrac{\theta_{2n} - \theta_{2n-1}}{x_{2n} - x_{2n-1}} = 1$. This gives us a matrix equation to solve: $A\boldsymbol{\theta} = \boldsymbol{c}$, where $\boldsymbol{c} = 0$ except $c_n = 1/2$, $c_{2n} = 1$. Inverting A should get the required answer, no iteration needed.

3 Numerical Details

3.1 Lubrication matrix

We need to figure out a way to work out the matrix R. Our best possible representation of H_0 and $\tilde{H}$ look something like

$$H_0(\xi) = \begin{cases} \xi^{2/3}(w_i^0 \xi + e_i^0) + r_i^0 & i < t \\ w_i^0 \xi^2 + e_i^0 \xi + r_i^0 & i \geq t \end{cases}$$

$$\tilde{H}(\xi) = \begin{cases} \xi^{s}(w_i \xi + e_i) + r_i & i < t \\ w_i \xi^2 + e_i \xi + r_i & i \geq t \end{cases}$$

For $\xi \in [\xi_i, \xi_{i+1}]$.

Recall the equation we are trying to solve:

$$\tilde{\Pi} = \int_z^\infty \frac{2\tilde{H}\Pi_0'}{H_0} d\xi$$

We want to solve for $(\tilde{\Pi}(z_1), \ldots \tilde{\Pi}(z_{n-1}))$. We will also use the lubrication equation $\Pi_0' H_0^2 = \lambda_0$ to remove Π_0'.

$$\tilde{\Pi}(z_k) = \int_{z_k}^{\xi_{k+1}} \frac{2\lambda_0 \tilde{H}}{H_0^3} d\xi + \sum_{r=k+1}^{n-1} \int_{\xi_r}^{\xi_{r+1}} \frac{2\lambda_0 \tilde{H}}{H_0^3} d\xi + \int_{\xi_n}^{\infty} \frac{2\lambda_0 \tilde{H}}{H_0^3} d\xi$$

It becomes a question of approximating integrals of the form

$$\int_{\tau_1}^{\tau_2} \frac{2\lambda_0 \tilde{H}}{H_0^3} d\xi$$

Where $\tau_2 = \xi_{r+1}$, $\tau_1 = \xi_r$ or z_r. The way chosen to approximate the integrand was

$$2\lambda_0 \tilde{H}/H_0^3 = \begin{cases} \xi^{s-4/3}(a_i \xi + b_i) & i < t \\ \xi^{-5}(a_i \xi + b_i) & i \geq t \end{cases}$$

To reflect the suspected asymptotics of the integrand, near 0 and near ∞. The general way to think about this is if $2\lambda_0 \tilde{H}/H_0^3 = \xi^\alpha (a_i \xi + b_i)$, then

$$\int_{\tau_1}^{\tau_2} \frac{2\lambda_0 \tilde{H}}{H_0^3} d\xi = a_i \left[\frac{\tau_2^{\alpha+2} - \tau_1^{\alpha+2}}{\alpha+2} \right] + b_i \left[\frac{\tau_2^{\alpha+1} - \tau_1^{\alpha+1}}{\alpha+1} \right]$$

By matching at the endpoints, one can determine the coefficients, a, b linearly in terms of the w, e, r, coeffs. Further, since the integral is clearly linear in a, b, one can find formulae to make the integral linear in the w, e, r.

3.2 Integral Kernel

We wish to recalculate the integral kernel, given that near the origin, $\tilde{H}' = \xi^{s-1}(a_i \xi + b_i)$. We assume the same $\tilde{G}'$ spacing as always. This raises the problem of calculating such integrals as

$$\int_{\ell_\xi}^{u_\xi} \frac{48(\xi - z)^2 - 64}{(\xi - z)((\xi - z)^2 + 4)^3} \xi^\alpha d\xi$$

$$\int_{\ell_\xi}^{u_\xi} \frac{24(\xi - z)^2 - 32}{((\xi - z)^2 + 4)^3} \xi^\alpha d\xi$$

Where $\alpha = s, s-1$ and $u_\xi = \xi_{k+1}$, $\ell_\xi = \xi_k$. In a deviation from what was done previously, there is no nice analytic solution to this equation in for general α. There is a solution in terms of hypergeometric functions, but these take a long time to compute, and worse, don't seem to give a very good answer. That is to say that they don't capture the nature of the singularity, removed by using a Cauchy principal value. We can rewrite the first integral as

$$\int_{\ell_\xi}^{u_\xi} \frac{48(\xi - z)^2 - 64}{(\xi - z)((\xi - z)^2 + 4)^3} \xi^\alpha d\xi = \int_{\ell_\xi}^{u_\xi} \frac{(\xi - z)^5 + 12(\xi - z)^3 + 96(\xi - z)}{((\xi - z)^2 + 4)^3} \xi^\alpha d\xi - \int_{\ell_\xi}^{u_\xi} \frac{\xi^\alpha}{\xi - z} d\xi$$

Which splits the integrals into singular and non singular parts. The non singular parts are mostly well behaved and can be integrated with matlabs inbuilt routine. However, when $\alpha = s - 1$, they have an integrable singularity near the origin that numerical integrators can't handle. This can be removed manually via integration by parts. For instance, consider a well behaved function f. Then

$$\int_{u_\xi}^{\ell_\xi} f(\xi - z)\xi^\alpha d\xi = \left[\frac{\xi^{\alpha+1}}{\alpha + 1} f(\xi - z)\right]_{\ell_\xi}^{u_\xi} - \int_{\ell_\xi}^{u_\xi} \frac{\xi^{\alpha+1}}{\alpha + 1} f'(\xi - z) d\xi$$

Where for $-1 < \xi < 0$ the left hand side is numerically suspect, whereas the right hand side is numerically tractable.

This leaves the important issue of the singularity, and the Cauchy principal value, ie. the integral

$$\int_{\ell_\xi}^{u_\xi} \frac{\xi^\alpha}{\xi - z} d\xi$$

We will do some substutions and some reshuffling until this takes a form that is convenient to work with.

$$\int_{\ell_\xi}^{u_\xi} \frac{\xi^\alpha}{\xi - z} d\xi = z^\alpha \int_{\ell_\xi/z}^{u_\xi/z} \frac{u^\alpha}{u - 1} du = z^\alpha \left\{ \int_{\ell_\xi/z-1}^{u_\xi/z-1} \frac{\theta^\alpha - 1}{\theta} d\theta + \int_{\ell_\xi/z-1}^{u_\xi/z-1} 1/\theta d\theta \right\}$$

Where $\frac{\theta^\alpha - 1}{\theta}$ is a well behaved function near 0. We also can show that

$$\int_{\ell_\xi/z-1}^{u_\xi/z-1} 1/\theta \; d\theta = \log \left| \frac{u_\xi - z}{z - \ell_\xi} \right|$$

Which conveniently holds whether or not the interval contains the origin. The algorithm goes something like:

$$\int_{\ell_\xi}^{u_\xi} \frac{48(\xi - z)^2 - 64}{(\xi - z)((\xi - z)^2 + 4)^3} \xi^\alpha d\xi = \underbrace{\int_{\ell_\xi}^{u_\xi} \frac{(\xi - z)((\xi - z)^4 + 12(\xi - z)^2 - 96)}{((\xi - z)^2 + 4)^3} \xi^\alpha d\xi}_{I_1} + \underbrace{\int_{\ell_x}^{u_x} \frac{\xi^\alpha}{z - \xi} d\xi}_{I_2}$$

$\mathbf{I}_1$

Is $\ell_x < 0.5$ and $\alpha < 0$?

- Yes. The integral as it stands will have trouble at 0 Write

$$I_1 = \left[\frac{\xi^{\alpha+1}}{\alpha+1} \frac{(\xi - z)((\xi - z)^4 + 12(\xi - z)^2 - 96)}{((\xi - z)^2 + 4)^3} \right]_{\ell_\xi}^{u_\xi}$$
$$- \int_{\ell_x}^{u_x} \frac{\xi^{\alpha+1}}{\alpha+1} \frac{d}{dx} \left\{ \frac{(\xi - z)((\xi - z)^4 + 12(\xi - z)^2 - 96)}{((\xi - z)^2 + 4)^3} \right\} d\xi$$

  Where both parts of the expression are numerically tractable.

- No. Just evaluate the integral numerically, as is.

$\mathbf{I}_2$

- Case: $\alpha > 0$ or $\ell_\xi / z > 1/2$.

$$I_2 = -z^\alpha \left\{ \int_{\ell_\xi/z-1}^{u_\xi/z-1} \frac{(1+\theta)^\alpha - 1}{\theta} d\theta + \log \left| \frac{u_\xi - z}{z - \ell_\xi} \right| \right\}$$

- Case: $\alpha < 0$, $u_\xi/z < 1/2$. No problems with Cauchy PV, but might have some issues with integrable singularity at 0.

$$I_2 = -z^\alpha \left\{ \left[\frac{u^{\alpha+1}}{\alpha+1} \frac{1}{u-1} \right]_{\ell_\xi/z}^{u_\xi/z} + \int_{\ell_\xi/z}^{u_\xi/z} \frac{u^{\alpha+1}}{\alpha+1} \frac{1}{(u-1)^2} du \right\}$$

- Case: $\alpha < 0$, $\ell_\xi/z < 1/2 < u_\xi/z$. Trouble from both poles. We split the integral

$$\begin{aligned} I_2 &= - z^\alpha \left\{ \int_{\ell_\xi/z}^{1/2} \frac{u^\alpha}{u-1} du + \int_{1/2}^{u_\xi/z} \frac{u^\alpha}{u-1} du \right\} \\ &= - z^\alpha \left\{ \left[\frac{u^{\alpha+1}}{\alpha+1} \frac{1}{u-1} \right]_{\ell_\xi/z}^{1/2} + \int_{\ell_\xi/z}^{1/2} \frac{u^{\alpha+1}}{\alpha+1} \frac{1}{(u-1)^2} du \right. \\ &\qquad \left. + \int_{-1/2}^{u_\xi/z-1} \frac{(1+\theta)^\alpha - 1}{\theta} d\theta + \log |2u_\xi/z - 1| \right\} \end{aligned}$$