Initial summary of project and progress so far

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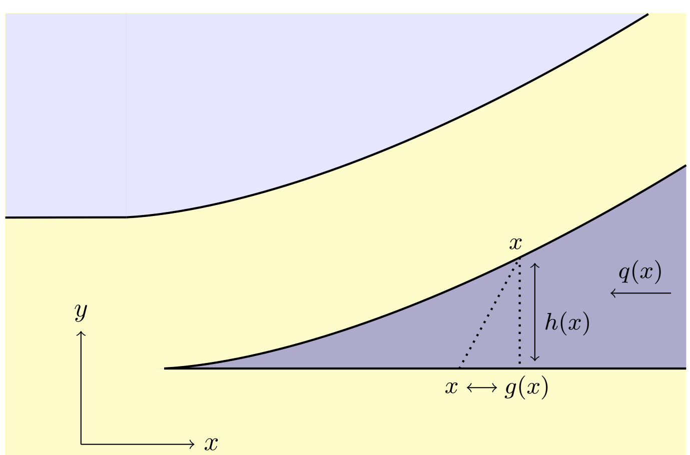
Introduction

Consider a semi-infinite crack in a solid with an incompressible fluid being driven into the crack. The solid above the crack is like a beam and we treat the solid as an elastic solid.

What we are interested in is the asymptotics as we approach the tip of the crack. This means that there are two asymptotic regimes that we are interested in. The first is the lubrication limit from fluid dynamics. The second is the elastic fracure region from solid mechanics.

1 Set up

Figure 1: Set up of problem



We look for a steadily propagating solution

$$h \equiv h(x - ct)$$

In the fluid we use standard lubrication results to find that

$$q = -\frac{1}{12\mu}\frac{dp}{dx}h^3$$

Use conservation of mass, i.e. $\dfrac{\partial q}{\partial x} + \dfrac{\partial h}{\partial t} = 0$ to get

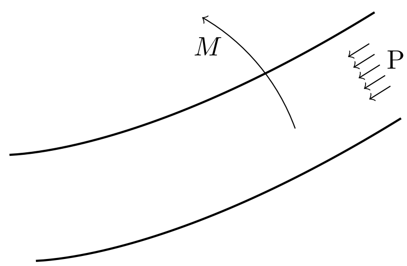
$$p' = \frac{\lambda}{h^2} \tag{1}$$

Where λ is a constant which essentially measures the speed of propagation. This governs the fluid. In the solid, we use results from elasticity. Let $P = \sigma_y$ be the stress and let $\tau_{xy} = 0$ i.e. no shear[1]. Then

$$\begin{pmatrix} P \\ M \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_y \\ \tau_{xy} \end{pmatrix} = C\int_0^\infty \begin{pmatrix} K_{11}(z-x) & K_{12}(z-x) \\ K_{21}(z-x) & K_{22}(z-x) \end{pmatrix} \begin{pmatrix} g'(z) \\ h'(z) \end{pmatrix} dz$$

Where C is a (known) constant, and the K_{ij} have (ugly) analytic expressions.

Boundary Conditions



At ∞ have from fracture mechanics

$$g' \to \frac{1}{4}(-P + 6M)$$

$$h'' \to 3M$$

But at 0 this depends on the fracture mechanics.

In the no fracture case (Think of two separate solids sitting on top of each other), the solution obeys lubrication right to 0. I.e.

$$h \sim x^{2/3} \qquad p' \sim x^{-4/3}$$
$$h' \sim x^{-1/3} \qquad p \sim x^{-1/3}$$

In the fracture case: "Axioms" of fracture mechanics imply that

$$h \approx Kx^{1/2}$$

Where K is the toughness. Possible "Dependent Variables"

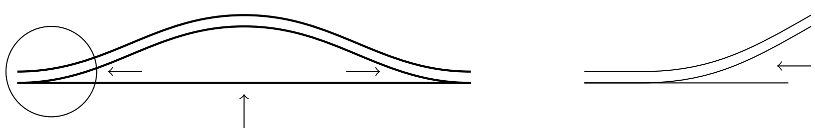
[1] It can be shown that the shear is an order of magnitude smaller than the pressure

- at ∞ have P, M. Where P is the only one we really care about, as we can set $M = 0$. After rescaling x, h it is possible to assume $P = 1$.
- At 0, have speed[2] λ and toughness K.

Given a fixed toughness K, propagation occurs at $\lambda(K)$ (and vica-versa).

What we really want is to find λ for $K = 0$, and then the "small toughness" solution for $K \approx 0$.

Figure 2: Why we are interested in this problem

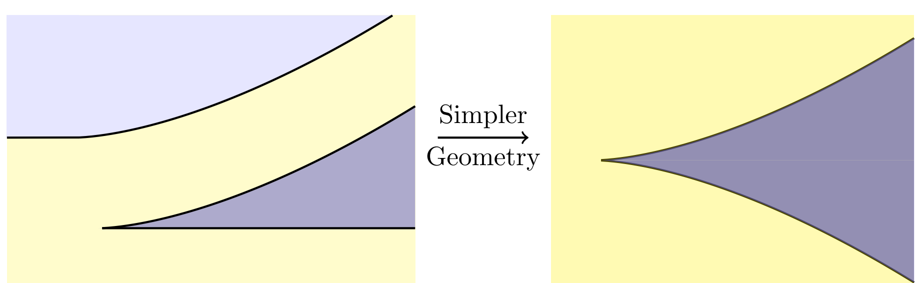


2 Zero toughness solution

Recall we have

$$\begin{pmatrix} \sigma_y \\ \tau_{xy} \end{pmatrix} = C \int_0^\infty \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix} dz$$

In the complicated geometry. Looking close to the crack tip can move into a much simpler geometry where the solid extends infinitely in the y direction. $K(\lambda_0) = 0$. In this simpler geometry, $\sigma = p$ and this depends only on h', since



we can ignore non singular terms near the crack tip.

$$\implies p(x) = \int_0^\infty \frac{h'(z)}{z - x} dz$$

i.e. p is the Hilbert transform of h'. Also have

$$p' = \frac{\lambda}{h^2}$$

[2]Well, a parameter which essentially is a scaled measure of the speed

so try $h = x^\alpha \implies$

$$p \sim x^{\alpha-1}$$
$$p' \sim x^{-2\alpha}$$

$$\implies -2\alpha = \alpha - 2$$

or $\alpha = 2/3$ as claimed earlier, where we have made use of the Hilbert transform

$$\int_0^\infty \frac{z^{s-1}}{z-x} dz = -\pi \cot(\pi s) z^{s-1}$$

So starting with a solution $h_0 = A_0 x^{2/3} + \dots$ near the crack tip, have that

$$p_0(x) = \int_0^\infty \frac{48(z-x)^2 - 64}{(z-x)((z-x)^2+4)^3} \, \frac{2A_0}{3} z^{-1/3} dz = \dots \approx \frac{2A_0}{3} \int_0^\infty \frac{z^{-1/3}}{z-x} dz$$

and therefore get that

$$p_0 = -\frac{3\lambda_0}{A_0^2 x^{1/3}} + \dots$$

3 Perturbing the Zero toughness solution

Suppose that we have a solution for λ_0, $K(\lambda_0) = 0$ (and h_0, g_0, p_0). We want to now consider when $K \approx 0$, $\lambda \approx \lambda_0$ i.e. perturb this solution.

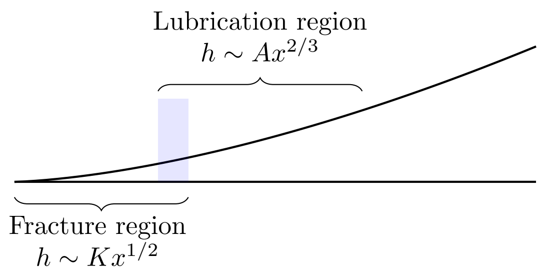
$K = 0$ solution:

$$h_0 = A_0 x^{2/3} + \dots$$
$$p_0' = \frac{\lambda_0}{A_0^2 x^{4/3}} + \dots$$
$$p_0 = -\frac{3\lambda_0}{A_0^2 x^{1/3}} + \dots$$

Which are the leading order terms as $x \to 0$. Now, given (g_0, h_0) perturb:

$$g = g_0 + \mathcal{E}(K)\; g_1 + \dots$$
$$h = h_0 + \mathcal{E}(K)\; h_1 + \dots$$
$$p = p_0 + \mathcal{E}(K)\; p_1 + \dots$$
$$\lambda = \lambda_0 + \mathcal{E}(K)\; \lambda_1 + \dots$$

Where we assume $\mathcal{E}(K)$ is small, for example $\mathcal{E}(K) = K$, but we need not have this in general. In this perturbation, we get two regions close to the fracture, the lubrication region and the fracture region.



Lubrication region

$$h(x) = A(K)x^{2/3} + \dots$$

$$A(K) = \left(\frac{243\lambda(K)^2}{4\pi^2} \right)^{1/6} = \underbrace{A_0}_{\text{in } h_0} + \underbrace{\frac{A_0 \lambda_1}{3\lambda_0} \mathcal{E}(K)}_{\text{in } h_1 \text{ term}} + \dots$$

and so

$$h = \underbrace{A_0 x^{2/3}}_{h_0} + \underbrace{\left(\frac{A_0 \lambda_1}{3\lambda_0} x^{2/3} + x^s \right)}_{h_1} \mathcal{E}(K) + \dots$$

Solve for h_1 via $p' = \lambda / h^2$

$$(p_0 + \mathcal{E} p_1)' = \frac{\lambda_0 + \mathcal{E}\lambda_1}{(h_0 + \mathcal{E} h_1)^2}$$

At order $\mathcal{E}$

$$\lambda_1 = h_0^2 p_1' + 2h_0 h_1 p_0'$$

So

$$p_1' = \frac{\lambda_1}{3A_0^2} x^{-4/3} - \frac{4\pi}{9\sqrt{3}} x^{s-2}$$

Then do the Hilbert transform

$$p_1 = \int_0^\infty \begin{pmatrix} K_{11} & K_{12} \end{pmatrix} \begin{pmatrix} g_1' \\ h_1' \end{pmatrix} dz$$

Plugging into the earlier expression, find that $s \approx 0.138773$. So in the lubrication region, h scales like $Ax^{2/3} + Bx^{0.13867\dots}$

Fracture region

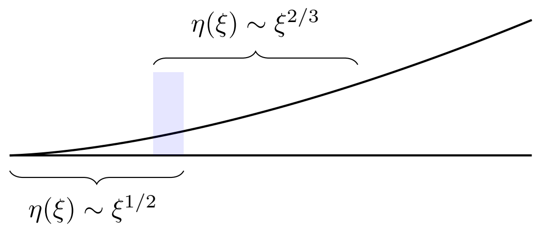
Here pretend no free surface as we are zoomed in far enough that we see the solid in every direction. $h \sim Kx^{1/2}$ at $x \to 0$. We can remove λ, K by making the problem dimensionless.

$$p \to \Pi \qquad h \to \eta \qquad x \to \xi$$

$(\lambda = K = 1)$

$$\eta(\xi) = \tilde{A}\xi^{2/3} + C\xi^t$$

t some constant. But doing the same thing as earlier, compare results of $\Pi' = \frac{1}{\eta^2}$ and $\Pi =$ Hilbert transform of η'.



From this, we find that $t = s = 0.1386\ldots$. Redimensionalising, get that

$$h(x) = \mathcal{O}x^{2/3} + K^{4-6s}x^s$$

i.e. $\mathcal{E}(K) = K^{4-6s}$. This tells us that

$$\lambda = \lambda_0 + K^{4-6s}\lambda_1$$

But we don't know λ_1. Well, get it out of linearized problem:

$$\begin{cases} \lambda_1 & = & h_0^2 p_1' + 2h_0 p_0' h_1 \\ \begin{pmatrix} p_1 \\ 0 \end{pmatrix} & = & \int \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} g_1' \\ h_1' \end{pmatrix} \end{cases}$$

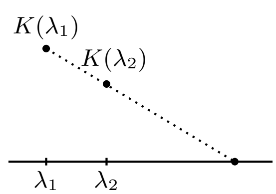
What are the boundary conditions?

$$h_1'' \to 0, \qquad g_1' \to 0 \qquad \text{at } x \to \infty \quad (??)$$

$$\mathcal{E}(K)h_1 \to Kx^{1/2} \text{ at } x \to 0$$

4 Numerical Methods

Given λ you can find $K(\lambda)$ via running a numerical simulation. Then it is essentially Newtons method to find λ_0 s.t. $K(\lambda_0) = 0$.



What is observed is $\lambda - \lambda_0 \approx K^3$ (or is it $K^{3.14\cdots}$ very difficult to tell numerically).

Discretize: First try to represent h_1', g_1' as piecewise linear functions. Recall that $K_{ij}(z-x)\,(az+b)$ has an analytic expression. This can cause issues near the tip where $h' \sim x^{-1/2}$. So for half the panels (near tip) use scheme $g', h' \sim ax^{1/2} + bx^{-1/2}$ and for the rest of the panels use $g', h' \sim ax + b$.