Numerical methods

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Consider the govering equations

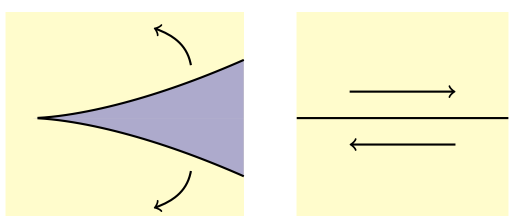
$$\begin{pmatrix} p(z) \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_y \\ \tau_{xy} \end{pmatrix} = \int_0^\infty \begin{pmatrix} K_{11}(x-z) & K_{12}(x-z) \\ K_{21}(x-z) & K_{22}(x-z) \end{pmatrix} \begin{pmatrix} g'(x) \\ h'(x) \end{pmatrix} dx \quad (1)$$

$$h^2 p' = \lambda \quad (2)$$

Have the "input" parameters as

- BC's P, M (or equivalently g', h'' at $x \to \infty$)
- λ , the speed

Want to solve for the toughness K_I and K_{II}. In this project, we have so far focused on K_I.



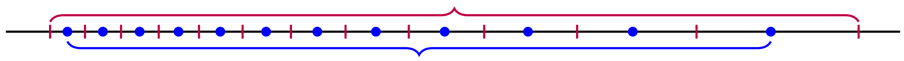
K_I, Bending moment $\qquad$ K_{II}, Shear

Goal: Find λ such that $K_I(\lambda) = 0$, "Zero toughness solution". Given this we then want to investigate the behaviour for small $K_I \approx 0$. To do this, take some given value of λ and then solve equations 1, 2.

Discretization of problem

The method chosen to discretize the problem is to take a vector x of n points at which we measure g', h' and have a vector z of $n-1$ intermediate points at which p is measured. (The spacing chosen is a tan spacing)

x, the n points at which h', g' are measured



z, the $n-1$ points at which p is measured

The discrete version of $\begin{pmatrix} p \\ 0 \end{pmatrix} = \int \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix}$ becomes

$$\begin{pmatrix} p(z_1) \\ \vdots \\ p(z_{n-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} B_{1,1} & \cdots & B_{1,2n} \\ \vdots & \ddots & \vdots \\ B_{2(n-1),1} & \cdots & B_{2(n-1),2n} \end{pmatrix} \begin{pmatrix} g'(x_1) \\ \vdots \\ g'(x_n) \\ h'(x_1) \\ \vdots \\ h'(x_n) \end{pmatrix}$$

Where the matrix B depends on the choice of spacings for x, z but does not depend on the values that g', h' take. We can go further and incorporate the boundary conditions into this equation. The discretized versions of the boundary conditions, become[1] $g'(x_n) = 1/2$ and $\frac{h'(x_n) - h'(x_{n-1})}{x_n - x_{n-1}} = 1$. These conditions are linear in terms of g', h', and so by adding another two rows onto the matrix B, get that

$$\begin{pmatrix} p(z_1) \\ \vdots \\ p(z_{n-1}) \\ 0 \\ \vdots \\ 0 \\ g'(\infty) \\ h''(\infty) \end{pmatrix} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,2n} \\ \vdots & \ddots & \vdots \\ A_{2n,1} & \cdots & A_{2n,2n} \end{pmatrix} \begin{pmatrix} g'(x_1) \\ \vdots \\ g'(x_n) \\ h'(x_1) \\ \vdots \\ h'(x_n) \end{pmatrix}$$

Where $g'(\infty), h''(\infty)$ are some constants that are the boundary conditions.

Now we use the second equation for p, namely $p = \int_z^\infty \lambda/h^2 dx$. This depends on h' in a very much non linear way. It will however provide an expression for the $p(z_i)$ in terms of the $h'(x_j)$. Thus, switching to the notation $\boldsymbol{h'} = (g', h')$ where the first n coords of $\boldsymbol{h'}$ are the coords of g' and the second n are the

[1] Or something similar depending on the exact scalings

coords of h'. We have that

$$f(\boldsymbol{h}') = \begin{pmatrix} p(z_1) \\ \vdots \\ p(z_{n-1}) \\ 0 \\ \vdots \\ 0 \\ g'(\infty) \\ h''(\infty) \end{pmatrix} = A\boldsymbol{h}'$$

Where we now just need to solve for $\boldsymbol{h}'$.

Newton's method

Suppose $\boldsymbol{h}'$ is iterate 1. To get the next iterate you need to solve (to first order)

$$f(\boldsymbol{h}' + \delta\boldsymbol{h}') = A(\boldsymbol{h}' + \delta\boldsymbol{h}')$$

$$f(\boldsymbol{h}') + (Df|_{\boldsymbol{h}'})(\delta\boldsymbol{h}') = A\boldsymbol{h}' + A\delta\boldsymbol{h}'$$

Where $Df|_{\boldsymbol{h}'}$ is a matrix of partial derivatives. Therefore, get to first order that

$$\delta\boldsymbol{h}' = (A - Df|_{\boldsymbol{h}'})^{-1}(f(\boldsymbol{h}') - A\boldsymbol{h}')$$

Ingredients:

- Matrix A itself (of which the $2(n-1) \times 2n$ part is the integral kernel)
- The function $f(\boldsymbol{h}')$. I.e. given $\boldsymbol{h}'$ you need to calculate $\int_z^\infty \lambda/h^2 dx$ (Key functions "hprime_to_h" and "hprime_to_p").
- Need to calculate Df which involves calculating $\dfrac{\partial}{\partial\boldsymbol{h}'}\displaystyle\int_z^\infty \frac{\lambda}{h(x)^2}dx$

So we have worked out numerically $K(\lambda)$, now we want to solve $K(\lambda_0) = 0$ for λ_0. We do a "march". Sublety in that $K < 0$ is unphysical, so a guess of $\lambda > \lambda_0$ where $K(\lambda_0) = 0$ does not make any physical sense (& will get bad numerical results). To get around this difficulty, take the next iterate of λ as smaller than predicted.

For example in Figure 1, the obvious choice for λ_4 (the light blue circle) is larger than the true value of λ_0, and therefore the naive extrapolation method won't quite work.

Figure 1: March to find λ_0