Viscous control of shallow elastic fracture

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This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/ relatedlink/jfm-keywords.pdf for the full list)

1. Introduction

Here we review the literature as well as describe the problem in more detail. We have the vertical displacement h, the horizontal displacement g, the thickness of the arm l, and the pressure p. We look for a travelling wave solution (propagating left), with speed c.

2. Formulation of problem

2.1. *Single tip*

From lubrication, we have Poiseulle flow in the crack. We obtain the flux, and conservation of mass as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3\,, \qquad \frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0\,, \tag{2.1}$$

which combined gives

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2\,. \tag{2.2}$$

Setting $p \to 0$ at $x \to \infty$, we can write this in integral form,

$$p(x) = -\int_x^\infty 12\mu c/h(\tilde{x})^2 \mathrm{d}\tilde{x}\,. \tag{2.3}$$

From the linear theory of elasticity, due to others who have studied this problem, we have

$$\begin{bmatrix} -\sigma_y \\ -\tau_{xy} \end{bmatrix} = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} = \frac{E}{4\pi l(1-\nu^2)}\int_0^\infty \boldsymbol{K}\left(\frac{\tilde{x}-x}{l}\right)\begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix}\mathrm{d}\tilde{x}\,, \tag{2.4}$$

where the integral kernel is

$$\boldsymbol{K}(\xi) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{(32-24\xi^2)}{(\xi^2+4)^3} & \frac{(48\xi^2-64)}{\xi(\xi^2+4)^3} \\ -\frac{(16\xi^4+16\xi^2+4)}{\xi(\xi^2+4)^3} & -\frac{(32-24\xi^2)}{(\xi^2+4)^3} \end{bmatrix}. \tag{2.5}$$

The boundary conditions near $x = 0$ are governed by fracture mechanics

$$K_I = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, h'(x)\,, \qquad K_{II} = \lim_{x\to 0} \frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, g'(x)\,. \tag{2.6a, b}$$

As we go to $x \gg l$, we are looking at the problem of peeling off a thin strip from an elastic half space. We can then use beam theory approximations, which give

$$M(x) = \frac{El^3}{12(1-\nu^2)}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = \frac{El^2}{6(1-\nu^2)}\frac{\mathrm{d}g}{\mathrm{d}x}, \qquad p = \frac{El^3}{12(1-\nu^2)}h^{(4)}(x) \tag{2.7a, b}$$

As $x \to \infty$, $M(x) \to M$, the applied bending moment, so this gives us boundary conditions on h'', g'.

2.2. *Double tip*

The equations are mostly unchanged from the single tip problem. From the geometry of the double tip problem, a second crack tip is situated at $x = -lL$, and $h(x), h'(x) = 0$ for $x < 0$. The boundary conditions at ∞ are unaffected, and the boundary conditions at the crack tip become

$$\lim_{x\to 0}\sqrt{x}\, h'(x) = L\,, \qquad \lim_{x\to -Ll}\frac{E}{1-\nu^2}\sqrt{\frac{\pi}{8}}\sqrt{x}\, g'(x) \;= K_{II}. \tag{2.8a, b}$$

The limits in the elasticity integral become $(-Ll, \infty)$.

2.3. *Rescaling*

We can define the following dimensionless variables

$$x = l\xi, \quad h(x) = \frac{12M(1-\nu^2)}{El}H(\xi), \quad g(x) = \frac{12M(1-\nu^2)}{El}G(\xi)\,, \tag{2.9}$$

$$p = \frac{3M}{\pi l^2}\Pi(\xi), \quad K_I = Ml^{-3/2}\kappa_I, \quad K_{II} = Ml^{-3/2}\kappa_{II}, \quad \lambda = \frac{4\pi\mu p^* l^3}{M^2}\,. \tag{2.10}$$

With these scalings, the equations become

$$\begin{bmatrix} \Pi \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{K}(\tilde{\xi}-\xi)\begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix}\mathrm{d}\tilde{\xi} \tag{2.11}$$

$$H^2\frac{\mathrm{d}\Pi}{\mathrm{d}\xi} = \lambda \quad \text{or} \quad \Pi(\xi) = -\int_\xi^\infty \lambda/H(\tilde{\xi})^2\mathrm{d}\tilde{\xi} \tag{2.12a, b}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2}, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}H' = \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi}G' = \kappa_{II}, \tag{2.13}$$

These shall be the governing equations for the rest of this paper.

2.4. *Beam theory asymptotics*

In the dimensionless variables, the outer asymptotics are of the form

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} = \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}\xi}, \qquad H^{(4)}(\xi) = \frac{3}{\pi}\Pi(\xi), \qquad \frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} \to 1 \tag{2.13a, b, c}$$

From integration by parts, we can write

$$H''(\xi) = 1 - \frac{1}{2}\int_{\xi}^{\infty} (\tilde{\xi}-\xi)^2 H^{(5)}(\tilde{\xi})\mathrm{d}\tilde{\xi}, \tag{2.14}$$

provided $\lim_{\xi\to\infty} \xi H^{(3)}(\xi) = \lim_{\xi\to\infty} \xi^2 H^{(4)}(\xi) = 0$. Then using equation 2.12a, we have that

$$H''(\xi) = 1 - \frac{3\lambda}{2\pi}\int_{\xi}^{\infty} \frac{(\tilde{\xi}-\xi)^2}{H(\tilde{\xi})^2}\mathrm{d}\tilde{\xi}. \tag{2.15}$$

Since $H(\xi) = \frac{1}{2}\xi^2 + o(\xi^2)$, as $\xi \to \infty$, we can use this to get a better estimate of H'';

$$H''(\xi) = 1 - \frac{2\lambda}{\pi}\frac{1}{\xi} + o(1/\xi). \tag{2.16}$$

This new expression can be used to refine the error estimate from $o(1/\xi)$, to $O(\log(\xi)/\xi^2)$.

2.5. *Linear perturbation problem*

This section is problably better placed elsewhere... The equations of the linear perturbation problem:

$$\mathit{\Pi} = \mathit{\Pi}_0 + \mathcal{E}\mathit{\Pi}_1 + O(\mathcal{E}), \quad H = H_0 + \mathcal{E}H_1 + O(\mathcal{E}) \tag{2.17}$$

$$\begin{bmatrix} \mathit{\Pi}_1 \\ 0 \end{bmatrix} = \int_0^{\infty} \boldsymbol{K}(\xi-\tilde{\xi}) \begin{bmatrix} G_1'(\tilde{\xi}) \\ H_1'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H_0^2\mathit{\Pi}_1' + 2H_0H_1\mathit{\Pi}_0' = \lambda_1 \tag{2.18a,b}$$

$$H_1'' \to 0 \text{ as } \xi \to \infty, \qquad H_1 \sim \xi^s + \frac{\tilde{A}\lambda_1}{3\lambda_0^{2/3}}\xi^{2/3} + \ldots \text{ as } \xi \to 0 \tag{2.19a,b}$$

But these can be made into a more convenient form, by considering instead $\tilde{\mathit{\Pi}} = \mathit{\Pi}_0 - 3\lambda_0/\lambda_1\mathit{\Pi}_1$, and similar for $\tilde{H}$, $\tilde{G}$. The equations become

$$\begin{bmatrix} \tilde{\mathit{\Pi}} \\ 0 \end{bmatrix} = \int_0^{\infty} \boldsymbol{K}(\xi-\tilde{\xi}) \begin{bmatrix} \tilde{G}'(\tilde{\xi}) \\ \tilde{H}'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H_0^2\tilde{\mathit{\Pi}}' + 2H_0\tilde{H}\mathit{\Pi}_0' = 0 \tag{2.20a,b}$$

$$\tilde{H}'' \to 1 \text{ as } \xi \to \infty, \qquad \tilde{H} \sim -\frac{3\lambda_0}{\lambda_1}\xi^s + \ldots \text{ as } \xi \to 0 \tag{2.21a,b}$$

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking $n+1$ points $\boldsymbol{\xi} = (\xi_0 = 0, \xi_1 \ldots, \xi_n)$ at which we measure H', G', and n intermediate points $\boldsymbol{\zeta} = (\zeta_0, \ldots, \zeta_{n-1})$ at which to measure $\mathit{\Pi}$, so that $\xi_0 < \zeta_0 < \ldots < \zeta_{n-1} < \xi_n$. We work with $\sqrt{\xi}G'(\xi)$, $\sqrt{\xi}H'(\xi)$ near the tip to avoid singularities. We define $\boldsymbol{\theta}_G = [\sqrt{\xi_0}G'(\xi_0), \ldots, \sqrt{\xi_{t-1}}G'(\xi_{t-1}), G'(\xi_t), \ldots G'(\xi_n)]$, and $\boldsymbol{\theta}_H$ similarly, as well as $\boldsymbol{\theta} = [\boldsymbol{\theta}_G, \boldsymbol{\theta}_H]$, Typically $t \approx n/2$ was used. From the linearity of the elasticity integral (and the discretized integral) we may write

$$[\mathit{\Pi}(\zeta_1), \ldots, \mathit{\Pi}(\zeta_{n-1}), \underbrace{0, \ldots, 0}_{n-1}] = \boldsymbol{J}\boldsymbol{\theta}, \tag{3.1}$$

for some matrix $\boldsymbol{J}$. One can recover $H(\xi_i)$ from $\boldsymbol{\theta}_H$. Therefore, a discritized lubrication integral, yields an expression for $\Pi(\zeta_i)$ as a function of $\boldsymbol{\theta}_H$. So we can write

$$[\Pi(\zeta_1), \ldots, \Pi(\zeta_{n-1}), \underbrace{0\,, \,\ldots\,, \,0}_{n-1}] = \boldsymbol{J}\boldsymbol{\theta} = \boldsymbol{f}(\boldsymbol{\theta}_H)\,, \tag{3.2}$$

for some function $\boldsymbol{f}$.

Both $G'(\xi_n)$, and $H''(\xi_n)$ are known from our beam theory asymptotic expansion. But these are linear in $\boldsymbol{\theta}$, as $G'(\xi_n) = \theta_n$, and $H''(\xi_n) \approx (\theta_{2n} - \theta_{2n-1})/(\xi_n - \xi_{n-1})$, Therefore we can add another two rows to $\boldsymbol{J}$, so that

$$\boldsymbol{A}\boldsymbol{\theta} = [\boldsymbol{f}(\boldsymbol{\theta}), G'(\xi_n), H''(\xi_n)]\ . \tag{3.3}$$

Where the $\boldsymbol{A}$ is the enlarged matrix. This can be solved by Newton's method from quite arbitrary initial guesses.

For $\xi_i < \xi < \xi_{i+1}$, we interpolate as

$$G'(\xi) = \begin{cases} \xi^{-1/2}(a_i\xi + b_i) \\ a_i\xi + b_i \end{cases}, \quad H'(\xi) = \begin{cases} \xi^{-1/2}(c_i\xi^{1/2} + d_i) \\ c_i\xi + d_i \end{cases}, \quad \text{for} \quad \begin{cases} i < t \\ i \geqslant t \end{cases} \tag{3.4}$$

The choice of interpolating function was based on the appearance of the relevant functions. We will also define a_n, b_n, c_n, d_n for interpolation beyond ξ_n. With this choice of interpolation, there exist exact closed form expressions for both the lubrication integral, and the elasticity integral, in terms of the $a_i - d_i$ coefficients.

It therefore remains to determine $a_i - d_i$ in terms of $\boldsymbol{\theta}$. Continuity of G', H' imposes $2(n-1)$ linear equations. We also have the $2n$ equations following from the definition of $\boldsymbol{\theta}$, (such as $a_i\xi_i + b_i = \theta_i$ for $t \leqslant i \leqslant n$).

From our asymptotic expansion (via beam theory) we know $\theta_n = G'(\xi_n)$ and $a_n = G''(\xi_n)$. Therefore we can write

$$a_n = \frac{G''(\xi_n)}{G'(\xi_n)}\theta_n, \qquad b_n = \theta_n - a_n\xi_n = \left(1 - \frac{G''(\xi_n)}{G'(\xi_n)}\xi_n\right)\theta_n \tag{3.5}$$

With H, we know that $c_n = H''(\xi_n)$, $c_{n-1} = H''(\xi_{n-1})$, and so we have that

$$c_n = \frac{H''(\xi_n)}{H''(\xi_{n-1})}c_{n-1}, \qquad d_n = -c_n\xi_n + c_{n-1}\xi_n + d_{n-1} \tag{3.6}$$

Therefore, we have enough equations to know the $a_i - d_i$ in terms of $\boldsymbol{\theta}$.

Note that we choose a value of λ, fix the boundary conditions at $\xi \to \infty$, then solve the problem and subsequently recover the boundary conditions at $\xi = 0$ (κ_I, κ_{II}). This can then be inverted, so that we think of $\lambda = \lambda(\kappa_I)$. Physically, we know κ_I, and want to find λ, but in numerically solving the problem, it makes more sense to choose λ and recover κ_I.

The spacing of the points should reflect that the important part of the problem is happening near the tip, and this is where the points should be concentrated. The spacing that was typically used in numerical calculations was

$$\xi_i = \tan^2(\chi\, i/m), \quad i = 1, \ldots, m < n \tag{3.7}$$

where χ is chosen so that $\tan^2(\chi) = O(10)$, and the remaining points are added in a geometric progression, so that

$$\xi_{i+1} = (\xi_m/\xi_{m-1})\xi_i, \quad i = m, \ldots, n-1 \tag{3.8}$$

3.2. *Linear Perturbation Problem*

From equation 2.21b, we anticipate a singularity of the form ξ^{s-1} in $\tilde{H}'$, (we still expect a $\xi^{-1/2}$ singularity in $\tilde{G}'$). Therefore, the interpolation was changed to reflect this. Some of the integrals no longer have exact expressions. In this case, they are calculated by a numerical integration routine.

The lubrication equation for the linear perturbation problem (2.20b), is linear in $\tilde{H}$. Therefore, we can obtain two expressions for $\tilde{\Pi}(\zeta_i)$ that are linear in $\tilde{G}'(\xi_j)$, $\tilde{H}'(\xi_j)$. Together with the boundary conditions and beam theory asymptotics, (we haven't changed the integral kernel, so the asymptotics remain the same) there are enough equations to numerically solve the linear perturbation problem. There is no need to use Newton's method, as we can simply solve the linear set of equations.

3.3. *Double Tip*

In solving the problem of two tips situated at $-L$ and 0, an additional r points are taken to cover $-L \leqslant \xi < 0$. The spacing of points for $\xi < 0$ was chosen so that there was a concentration of points near $-L$ and near 0.

We interpolate G' expecting a $\xi^{-1/2}$ singularity at $\xi = -L$, and H' expecting a $\xi^{-1/2}$ singularity at $\xi = 0$. We do not calculate Π for $\xi < 0$ (although it is easily done), but just require that $\sigma_{xy} = 0$ for $\xi < 0$. This provides enough equations for the problem to be solved as before, with Newton's method.

Note that we input $-L$ and λ and recover κ_I, κ_{II}, where κ_I is measured at 0. Physically, for $L > 0$, we must have $\kappa_I = 0$. Numerically we solve for some λ, L, find $\kappa_I > 0$ and extrapolate to $\kappa_I = 0$.

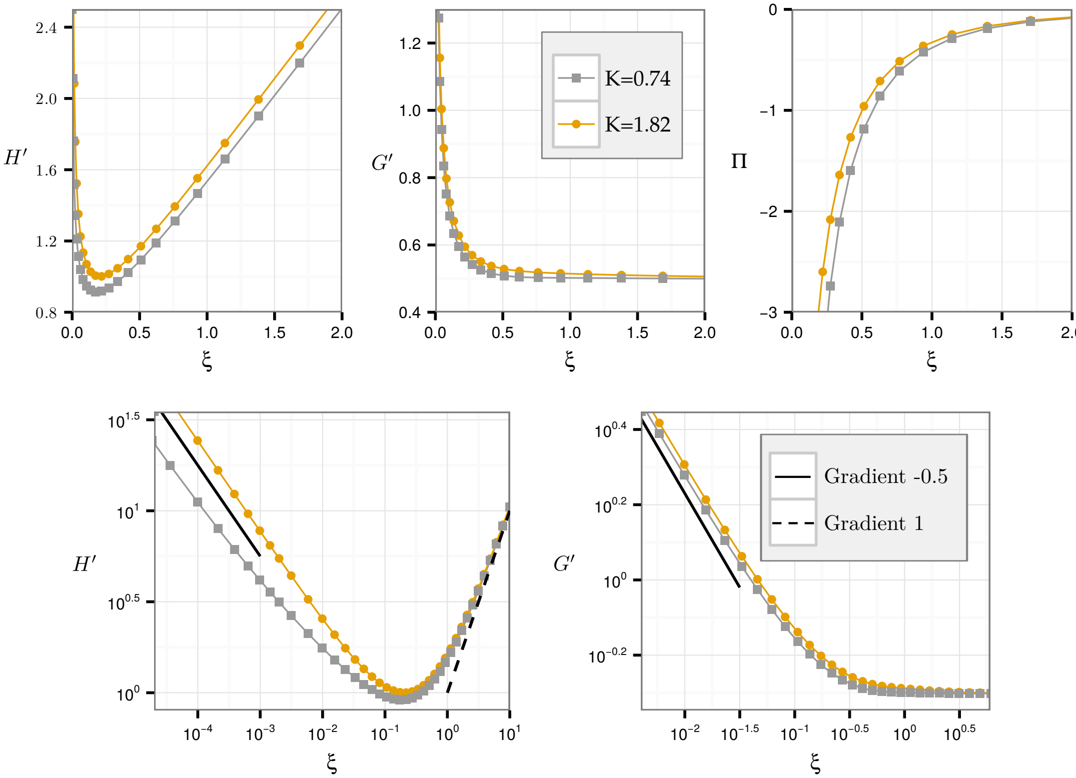

FIGURE 1. Numerical solutions for two typical κ_I values. log-log plots are shown for H',G', with solid lines indicating the predicted asymptotics; $\xi^{-1/2}$ singularities near $\xi = 0$, and $H' \sim x$ as $\xi \to \infty$. Figure produced with $n = 465$, $x_n = 819$.

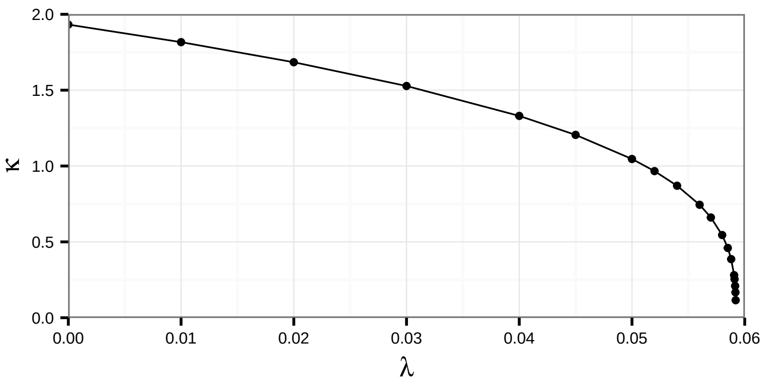

FIGURE 2. Here we vary the parameter λ and plot the change in κ_I. Figure produced with $n = 465$, $x_n = 819$.

4. Results

4.1. *Single tip*

Start off with some of the basic graphs showing H',G', and Π against ξ.

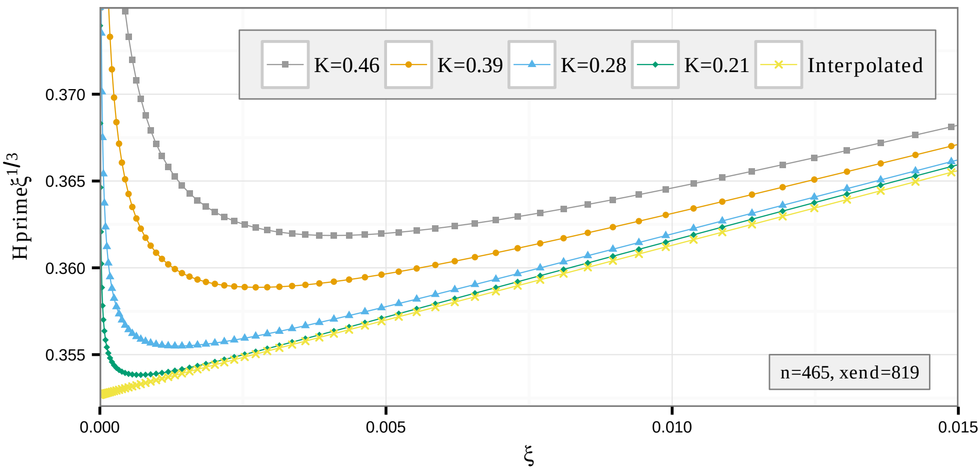

FIGURE 3. As $\kappa_I \to 0$, H' moves from a $\xi^{-1/2}$ singularity to a $\xi^{-1/3}$ singularity. We can not calculate $\kappa_I = 0$, but the extrapolation to it is shown.

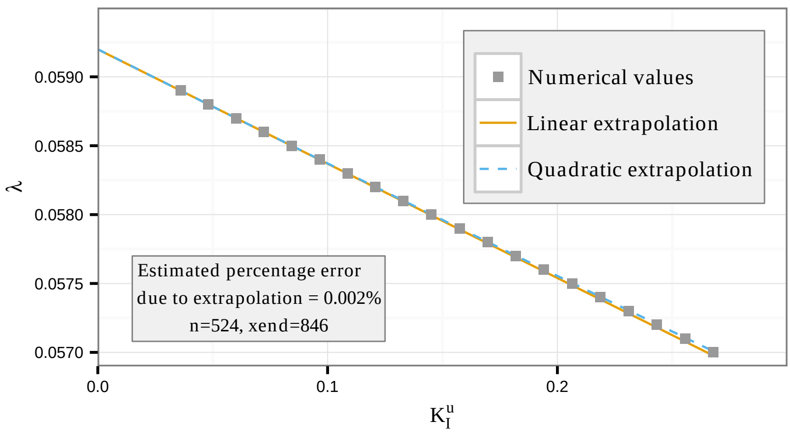

FIGURE 4. A linear fit from the two smallest κ_I values is plotted, as is a quadratic fit from the three smallest κ_I values. They are almost indistiguishable at this scale. The difference between the two extrapolations to $\kappa_I = 0$, provides an estimate of the error in calculating λ_0, (not accounting for the error due to n.)

4.2. *Linear perturbation problem*

We solve the linear perturbation problem. All that we really want to know is that we see the ξ^{s-1} behaviour that we expect, and we ask what the intercept of $\tilde{H}_1$ is. It is perhaps worth mentioning the difficulties in measuring the intercept and perhaps a notion of the sensitivity of the result on the estimate provided for H_0. Illustrating that is the next figure

4.3. *Two tips*

After the linear perturbation problem, we move on to the two tip problem. Perhaps some graphs that show an outline of the full numerical problem with non-zero κ_I and κ_{II}, although these are not physical.

We now move on to the $\kappa_I = 0$ set of relations.

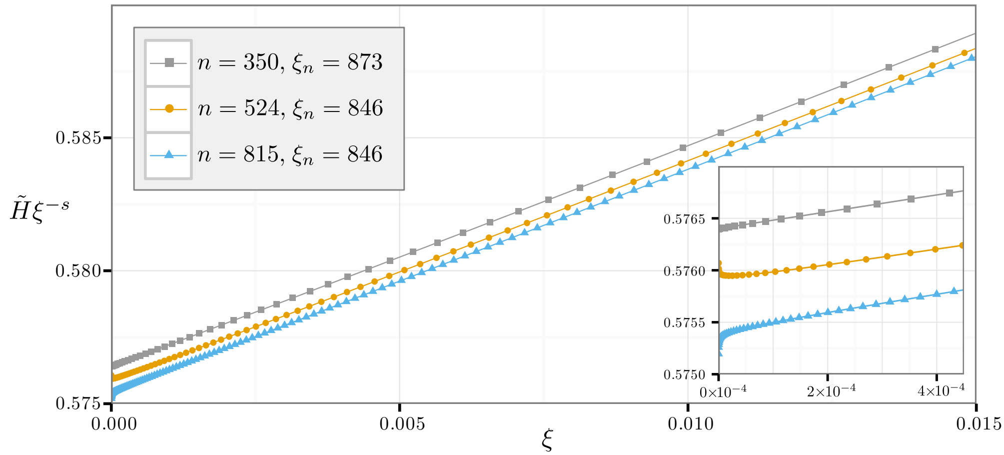

FIGURE 5. The numerical solution of the linear perturbation problem for a selection of resolutions. Of interest is the value of the intercept, which as shown is dependent on the resolution. Also shown is the numerical instability near the tip, due to our difficulty in calculating H_0 for $\xi \ll 1$.

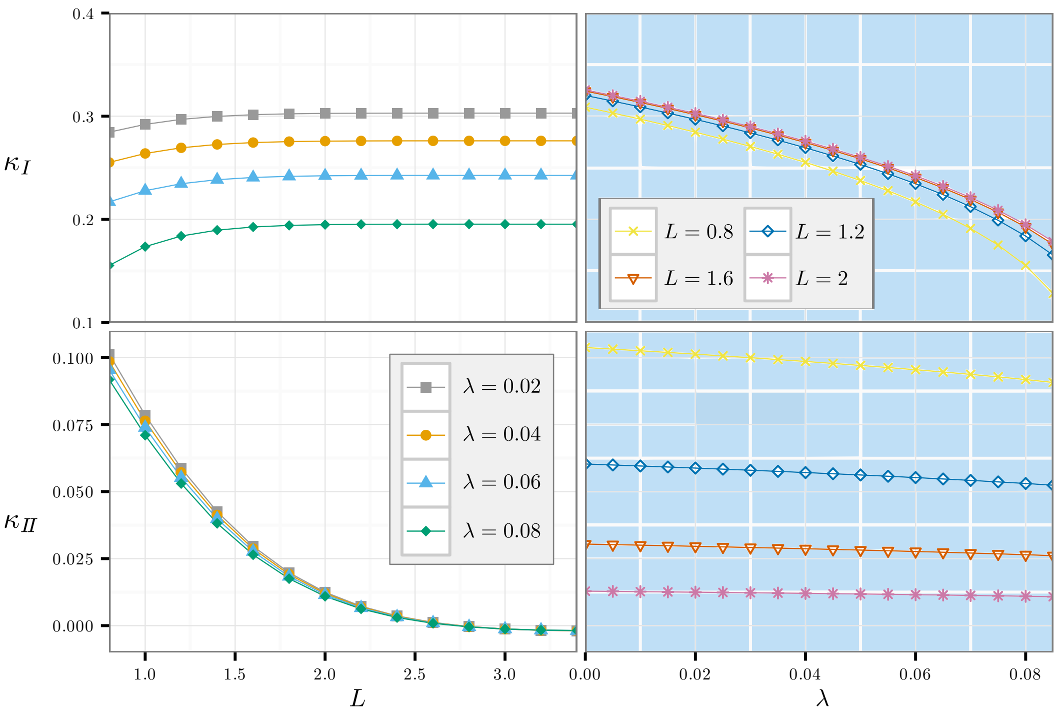

FIGURE 6. Some of the numerical results for the two tip problem. Having $\kappa_I \neq 0$ at $\xi = 0$ and $L \neq 0$ is unphysical, but is what is found numerically. We can recover the physical solution by increasing λ for fixed L until $\kappa_I = 0$. Figure made with $n = 995$, $\xi_n = 846$.

Then we could move on to talk about the decoupling between the fluid problem and the dry fracture problem. Relavent graphs to include would show that H really doesn't vary much with λ_0, and that given a reference H', one can construct G' with relative ease.

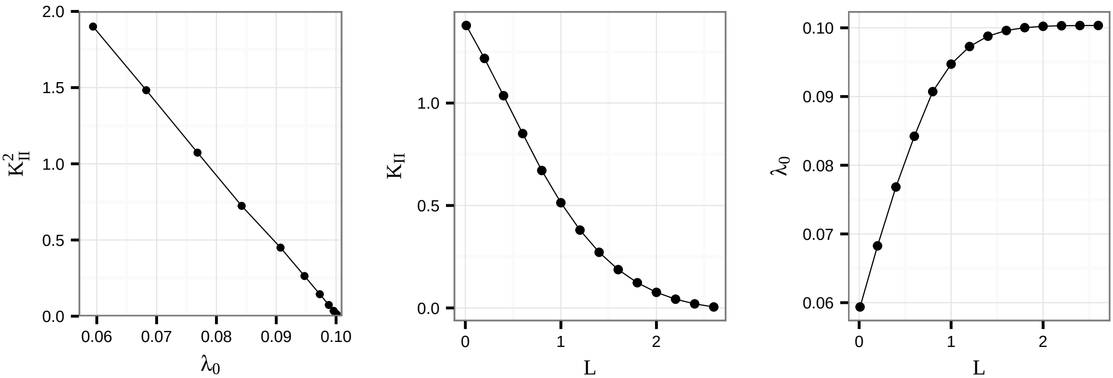

FIGURE 7. The results of extrapolating to $\kappa_I = 0$. Figure made with $n = 995$, $\xi_n = 846$.

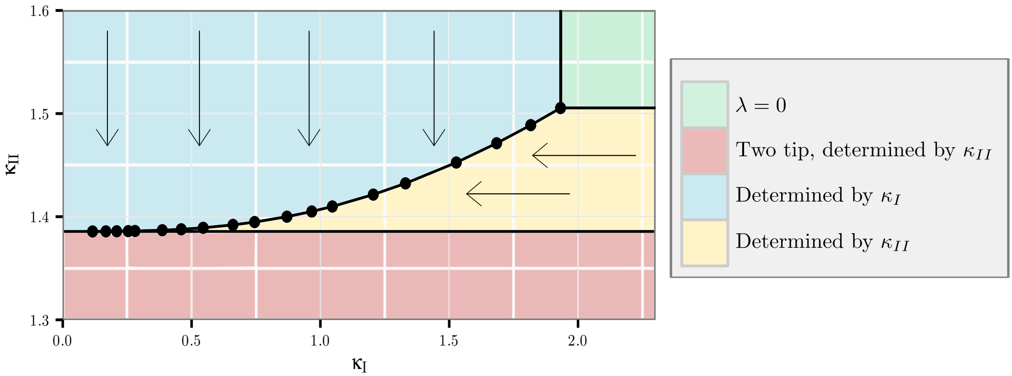

FIGURE 8. Given values (κ_I, κ_{II}), this graph determines which frature regime occurs and so how λ and/or L should be calculated. Figure made with $n = 465$, $\xi_n = 819$.

5. Discussion

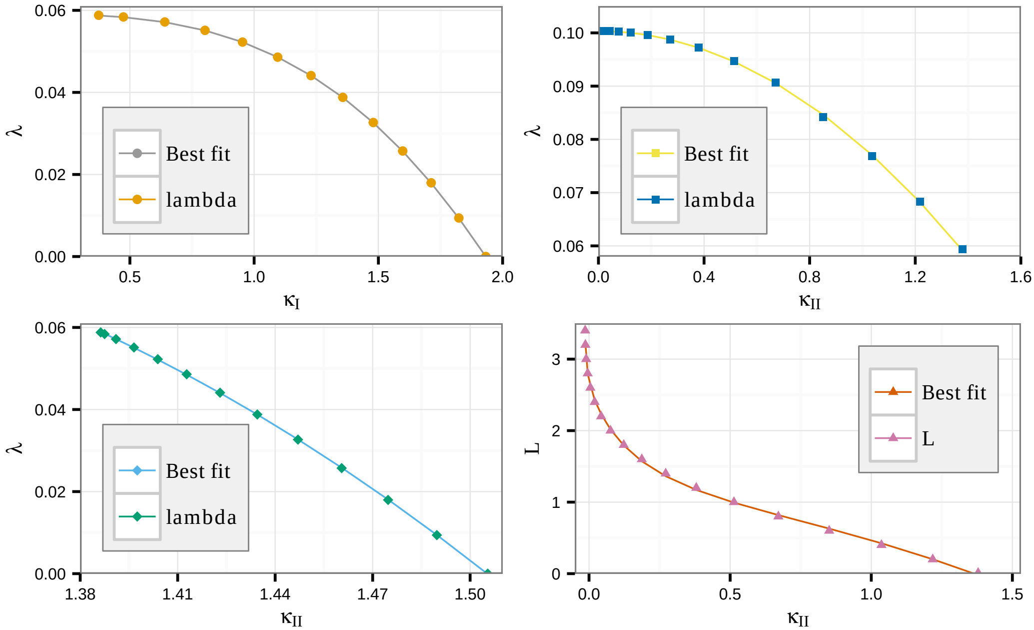

FIGURE 9. The formula valid for all κ_I. For the single tip calculations, $n = 815$ was used, for the double tip $n = 995$. $\xi_n = 846$ in both cases.

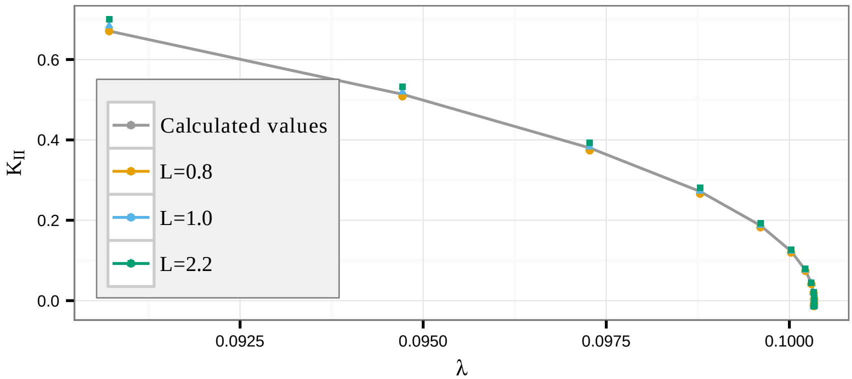

FIGURE 10. Reconstructing the full solution given a reference H'. Figure made with $n = 995$, $\xi_n = 846$.

a/d	$M=4$	$M=8$	Callan *et al.*
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

TABLE 1. Values of kd at which trapped modes occur when $\rho(\theta) = a$

6. Citations and references

All papers included in the References section must be cited in the article, and vice versa. Citations should be included as, for example "It has been shown (Rogallo 1981) that..." (using the `\citep` command, part of the natbib package) "recent work by Dennis (1985)..." (using `\citet`). The natbib package can be used to generate citation variations, as shown below.
`\citet[pp. 2-4]{Hwang70}`:
Hwang & Tuck (1970, pp. 2-4)
`\citep[p. 6]{Worster92}`:
(Worster 1992, p. 6)
`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
`\citep{Brownell04,Brownell07,Ursell50,Wijngaarden68,Miller91}`:
(Brownell & Su 2004, 2007; Ursell 1950; van Wijngaarden 1968; Miller 1991)
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Where there are up to ten authors, all authors' names should be given in the reference list. Where there are more than ten authors, only the first name should appear, followed by et al.

Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

Appendix A

This appendix contains sample equations in the JFM style. Please refer to the LaTeX source file for examples of how to display such equations in your manuscript.

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \tag{A 1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \nabla^2 P = \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{w}). \tag{A 2}$$

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \quad r \equiv |P - Q| \to 0, \tag{A 3}$$

$$\left.\begin{aligned}\frac{\partial G_s}{\partial y} = 0 \quad \text{on} \quad y = 0,\\ G_a = 0 \quad \text{on} \quad y = 0,\end{aligned}\right\} \tag{A 4}$$

$$-\frac{1}{2\pi}\int_0^\infty \gamma^{-1}[\exp(-k\gamma|y-\eta|)+\exp(-k\gamma(2d-y-\eta))]\cos k(x-\xi)t\,\mathrm{d}t, \qquad 0<y, \quad \eta<d, \tag{A 5}$$

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{1/2}, & t \leqslant 1 \\ (t^2-1)^{1/2}, & t > 1. \end{cases} \tag{A 6}$$

$$-\frac{1}{2\pi}\int_0^\infty B(t)\frac{\cosh k\gamma(d-y)}{\gamma \sinh k\gamma d}\cos k(x-\xi)t\,\mathrm{d}t$$

$$G = -\frac{1}{4}\mathrm{i}(H_0(kr)+H_0(kr_1)) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-k\gamma d}}{\gamma \sinh k\gamma d}\cosh k\gamma(d-y)\cosh k\gamma(d-\eta) \tag{A 7}$$

Note that when equations are included in definitions, it may be suitable to render them in line, rather than in the equation environment: $\boldsymbol{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G_a}$ where $r = \{[x(\theta)-x(\psi)]^2 + [y(\theta)-y(\psi)]^2\}^{1/2}$ and $\widetilde{G_a}$ is regular as $kr \to 0$. However, any fractions displayed like this, other than $\frac{1}{2}$ or $\frac{1}{4}$, must be written on the line, and not stacked (ie 1/3).

$$\begin{aligned}\frac{\partial}{\partial n_q}\left(\frac{1}{4}Y_0(kr)\right) &\sim \frac{1}{4\pi w^3(\theta)}[x''(\theta)y'(\theta) - y''(\theta)x'(\theta)] \\ &= \frac{1}{4\pi w^3(\theta)}[\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as} \quad kr \to 0.\end{aligned} \tag{A 8}$$

$$\frac{1}{2}\phi_i = \frac{\pi}{M}\sum_{j=1}^{M}\phi_j K_{ij}^a w_j, \qquad i = 1, \ldots, M, \tag{A 9}$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i,\theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G}_a(\theta_i,\theta_i)/\partial n_q + [\rho_i'\rho_i'' - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \tag{A 10}$$

$$\rho_l = \lim_{\zeta \to Z_l^-(x)} \rho(x,\zeta), \quad \rho_u = \lim_{\zeta \to Z_u^+(x)} \rho(x,\zeta) \tag{$A\,11a,b$}$$

$$(\rho(x,\zeta), \phi_{\zeta\zeta}(x,\zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \tag{A 12}$$

$$\tau_{ij} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + (\overline{\overline{u}_i u_j^{SGS} + u_i^{SGS}\overline{u}_j}) + \overline{u_i^{SGS}u_j^{SGS}}, \tag{A 13a}$$

$$\tau_j^\theta = (\overline{\overline{u}_j\overline{\theta}} - \overline{u}_j\overline{\theta}) + (\overline{\overline{u}_j\theta^{SGS} + u_j^{SGS}\overline{\theta}}) + \overline{u_j^{SGS}\theta^{SGS}}. \tag{A 13b}$$

$$
\boldsymbol{Q}_C = \begin{bmatrix}
-\omega^{-2}V_w' & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\
\dfrac{\beta}{\alpha\omega^2}V_w' & 0 & 0 & 0 & \mathrm{i}\omega^{-1} \\
\mathrm{i}\omega^{-1} & 0 & 0 & 0 & 0 \\
\mathrm{i}R_\delta^{-1}(\alpha^t + \omega^{-1}V_w'') & 0 & -(\mathrm{i}\alpha^t R_\delta)^{-1} & 0 & 0 \\
\dfrac{\mathrm{i}\beta}{\alpha\omega}R_\delta^{-1}V_w'' & 0 & 0 & 0 & 0 \\
(\mathrm{i}\alpha^t)^{-1}V_w' & (3R_\delta^{-1} + c^t(\mathrm{i}\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0
\end{bmatrix}. \tag{A 14}
$$

$$
\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[\mathrm{i}(\alpha^t x_1^t - \omega t)], \tag{A 15}
$$

where $\hat{\boldsymbol{\eta}}^t = \boldsymbol{b}\exp(\mathrm{i}\gamma x_3^t)$.

$$
\mathrm{Det}[\rho\omega^2\delta_{ps} - C^t_{pqrs}k^t_q k^t_r] = 0, \tag{A 16}
$$

$$
\langle k_1^t, k_2^t, k_3^t\rangle = \langle \alpha^t, 0, \gamma\rangle \tag{A 17}
$$

$$
\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{A 18}
$$

$$
f(\psi_1) = \frac{3b}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\frac{(\sin\psi_1 - \sin\psi)(a+b\cos\psi)^{1/2}}{[1-\cos(\psi_1-\psi)](2+\alpha)^{1/2}}\mathrm{d}x, \tag{A 19}
$$

$$
\begin{aligned}
g(\psi_1) = {} & \frac{3}{\pi[2(a+b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\left(\frac{a+b\cos\psi}{2+\alpha}\right)^{1/2}\Bigg\{f(\psi)[(\cos\psi_1 - b\beta_1)S + \beta_1 P] \\
& \times \frac{\sin\psi_1 - \sin\psi}{1-\cos(\psi_1-\psi)} + g(\psi)\left[\left(2+\alpha - \frac{(\sin\psi_1-\sin\psi)^2}{1-\cos(\psi-\psi_1)} - b^2\gamma\right)S\right. \\
& \left. + \left(b^2\cos\psi_1\gamma - \frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi,\delta) - (2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi,\delta)\right]\Bigg\}\mathrm{d}\psi,
\end{aligned} \tag{A 20}
$$

$$
\alpha = \alpha(\psi,\psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}, \quad \beta - \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \tag{A 21}
$$

$$
\left.
\begin{aligned}
& H(0) = \frac{\epsilon\overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}, \quad H'(0) = -1 + \epsilon^{2/3}\overline{C}_u + \epsilon\hat{C}_u'; \\
& H''(0) = \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2}u_P^2}, \quad H'(\infty) = 0.
\end{aligned}
\right\} \tag{A 22}
$$

LEMMA 1. *Let $f(z)$ be a trial Batchelor (1971, pp. 231–232) function defined on* $[0,1]$. *Let Λ_1 denote the ground-state eigenvalue for $-\mathrm{d}^2g/\mathrm{d}z^2 = \Lambda g$, where g must satisfy $\pm\mathrm{d}g/\mathrm{d}z + \alpha g = 0$ at $z = 0, 1$ for some non-negative constant α. Then for any f that is not identically zero we have*

$$
\frac{\alpha(f^2(0)+f^2(1)) + \displaystyle\int_0^1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\mathrm{d}z}{\displaystyle\int_0^1 f^2\mathrm{d}z} \geqslant \Lambda_1 \geqslant \left(\frac{-\alpha + (\alpha^2 + 8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \tag{A 23}
$$

Corollary 1. *Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies*

$$\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{A 24}$$

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