Viscous control of shallow elastic fracture

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(Received xx; revised xx; accepted xx)

This paper considers the problem of a semi-infinite crack parallel to the boundary of a half plane, with the crack filled by an incompressible viscous fluid. The dynamics are driven by a bending moment applied to the arm of the crack, and we look for travelling wave solutions. We examine two models of fracture; fracture with a single tip, and fracture with a wet tip proceded by a region of dry fracture.

Key words: Authors should not enter keywords on the manuscript, as these must be chosen by the author during the online submission process and will then be added during the typesetting process (see http://journals.cambridge.org/data/relatedlink/jfm-keywords.pdf for the full list)

1. Introduction

Here we review the literature as well as describe the problem in more detail. We have the vertical displacement h, the horizontal displacement g, the thickness of the arm l, and the pressure p. We look for a travelling wave solution (propagating left), with speed c.

2. Formulation of problem

From lubrication, we expect Poiselle flow in the crack. This gives us the flux, q, as

$$q = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3 \tag{2.1}$$

We also have the conservation equation

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.2}$$

Which combined gives us the equation

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 12\mu c/h^2 \tag{2.3}$$

From the linear theory of elasticity, due to others who have studied this problem, we have

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix} = \int_0^\infty \boldsymbol{K}(x-\tilde{x}) \begin{bmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{bmatrix} \mathrm{d}\tilde{x} \tag{2.4}$$

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Where the integral kernel is

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2.5}$$

Here we change into a set of dimensionless variables, and will spend the rest of the paper working with them. We have a length scale l, a pressure scale $p^* = E/12(1-\nu^2)$, and a time scale $t^* = 12\mu/p^*$. From these, we can define the following dimensionless parameters,

$$\mathcal{M} = \frac{M}{p^* l^2}, \qquad \mathcal{C} = \frac{c}{l/t^*} = \frac{12\mu c}{p^* l}, \qquad \mathcal{K}_I = \frac{K_I}{p^* l^{1/2}}, \qquad \mathcal{K}_{II} = \frac{K_{II}}{p^* l^{1/2}} \tag{2.6}$$

and variables

$$x = l\xi, \quad K_{ij} = U_{ij}/l, \quad h = \alpha l H(\xi), \quad g = \alpha l G(\xi), \quad p = \beta p^* \mathit{\Pi}(\xi) \tag{2.7}$$

The preferred scalings to be used in this paper are $\alpha = \pi\beta/3 = \mathcal{M}$, $\lambda = \pi\mathcal{C}/3\mathcal{M}^2$, which give

$$\begin{bmatrix} \mathit{\Pi} \\ 0 \end{bmatrix} = \int_0^\infty \boldsymbol{U}(\xi - \tilde{\xi}) \begin{bmatrix} G'(\tilde{\xi}) \\ H'(\tilde{\xi}) \end{bmatrix} \mathrm{d}\tilde{\xi}, \qquad H^2 \frac{\mathrm{d}\mathit{\Pi}}{\mathrm{d}\xi} = \lambda \tag{2.8}$$

$$\lim_{\xi\to\infty} H'' = 1, \quad \lim_{\xi\to\infty} G' = \frac{1}{2} \tag{2.9}$$

$$\lim_{\xi\to 0} 3\sqrt{2\pi\xi} H' = \frac{K_I}{Ml^{-3/2}} \equiv \kappa_I, \quad \lim_{\xi\to 0} 3\sqrt{2\pi\xi} G' = \frac{K_{II}}{Ml^{-3/2}} \equiv \kappa_{II}, \tag{2.10}$$

These shall be the governing equations for the rest of this paper.

3. Numerical scheme

3.1. *Single Tip*

We discretize the problem by taking n points $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ at which we measure H', G', and $n-1$ intermediate points $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{n-1})$ at which to measure $\mathit{\Pi}$, so that $\xi_1 < \zeta_1 < \ldots < \zeta_{n-1} < \xi_n$. We take $\xi_1 = 0$, where the crack tip is situated. Linear interpolation of G',H' would work poorly near the crack tip, since both functions are singular there. However, both $\sqrt{\xi}G'(\xi)$, and $\sqrt{\xi}H'(\xi)$ are regular functions, so we work with these instead, near the tip.

3.2. *Double Tip*

Here we mention details specific to the problem with both a fluid and a dry tip. Perhaps details of the $\sin^2$ spacing or the interpolation to $K_I = 0$ should be mentioned here.

4. Results

Here we put the majority of the graphs which show the results of the numerics.

5. Discussion

This is where we discuss the figures, possibly include more figures, and draw the results and conclusions of this paper.

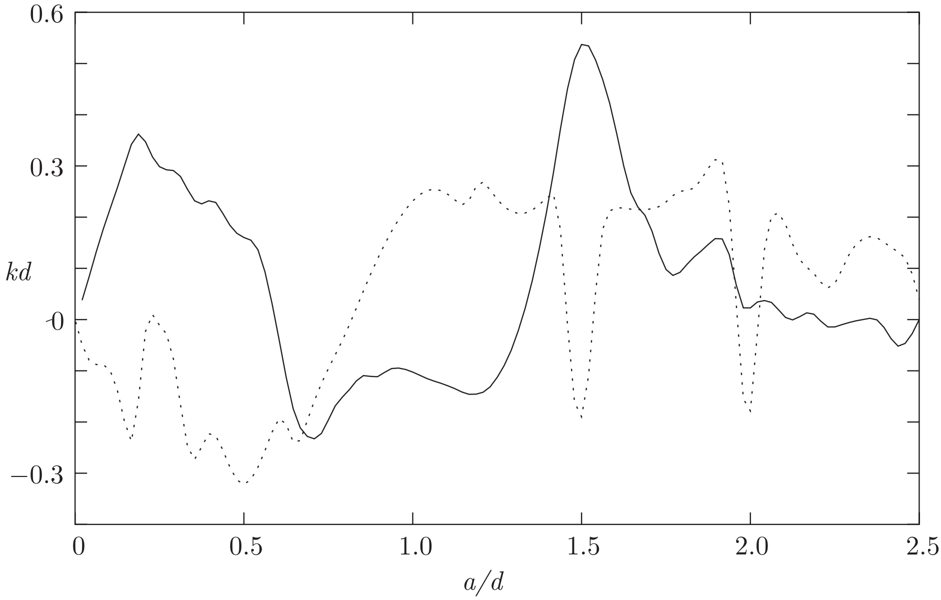

FIGURE 1. Trapped-mode wavenumbers, kd, plotted against a/d for three ellipses: ——, $b/a = 1$; $\cdots\cdots$, $b/a = 1.5$.

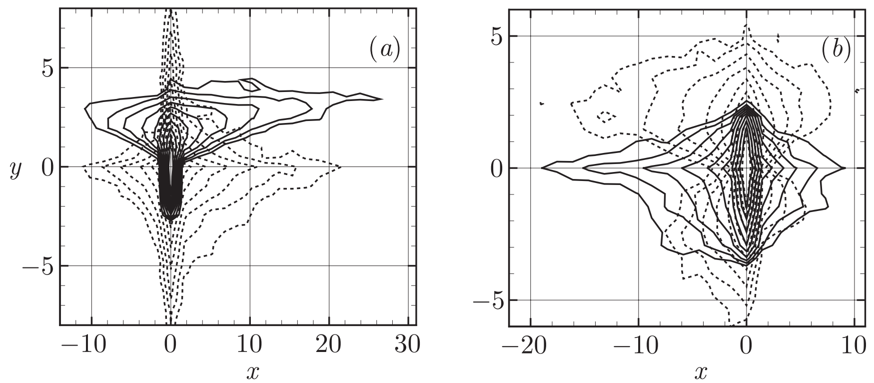

FIGURE 2. The features of the four possible modes corresponding to (a) periodic and (b) half-periodic solutions.

5.1. *Figures*

5.2. *Tables*

Tables, however small, must be numbered sequentially in the order in which they are mentioned in the text. The word *table* is only capitalized at the start of a sentence. See table 1 for an example.

5.3. *Mathematical notation*

5.3.1. *Setting variables, functions, vectors, matrices etc*

Italic font should be used for denoting variables, with multiple-letter symbols avoided except in the case of dimensionless numbers such as Re, Pr and Pe (Reynolds, Prandtl, and Péclet numbers respectively, which are defined as `\Rey`, `\Pran` and `\Pen` in the template).

a/d	$M=4$	$M=8$	Callan *et al.*
0.1	1.56905	1.56	1.56904
0.3	1.50484	1.504	1.50484
0.55	1.39128	1.391	1.39131
0.7	1.32281	10.322	1.32288
0.913	1.34479	100.351	1.35185

TABLE 1. Values of kd at which trapped modes occur when $\rho(\theta)=a$

Upright Roman font (or upright Greek where appropriate) should be used for:
Operators: sin, log, d, $\varDelta$, e etc.
Constants: i ($\sqrt{-1}$), π (defined as `\upi`), etc.
Functions: Ai, Bi (Airy functions, defined as `\Ai` and `\Bi`), Re (real part, defined as `\Real`), Im (imaginary part, defined as `\Imag`), etc.
Physical units: cm, s, etc
Abbreviations: c.c. (complex conjugate), h.o.t. (higher-order terms), DNS, etc.
Bold italic font (or bold sloping Greek) should be used for:
Vectors (with the centred dot for a scalar product also in bold): $\boldsymbol{i\cdot j}$
Bold sloping sans serif font, defined by the `\mathsfbi` macro, should be used for:
Tensors and matrices: $\boldsymbol{\mathsf{D}}$
Script font (for example $\mathcal{G}$, $\mathcal{R}$) can be used as an alternative to italic when the same letter denotes a different quantity (use `\mathcal` in LaTeX)
The product symbol ($\times$) should only be used to denote multiplication where an equation is broken over more than one line, to denote a cross product, or between numbers (the $\cdot$ symbol should not be used, except to denote a scalar product specifically).

5.3.2. *Other symbols*

A centred point should be used only for the scalar product of vectors. Large numbers that are not scientific powers should not include commas, but have the form 1600 or 16 000 or 160 000. Use O to denote 'of the order of', not the LaTeX $\mathcal{O}$.

6. Citations and references

All papers included in the References section must be cited in the article, and vice versa. Citations should be included as, for example "It has been shown (Rogallo 1981) that..." (using the `\citep` command, part of the natbib package) "recent work by Dennis (1985)..." (using `\citet`). The natbib package can be used to generate citation variations, as shown below.
`\citet[pp. 2-4]{Hwang70}`:
Hwang & Tuck (1970, pp. 2-4)
`\citep[p. 6]{Worster92}`:
(Worster 1992, p. 6)
`\citep[see][]{Koch83, Lee71, Linton92}`:
(see Koch 1983; Lee 1971; Linton & Evans 1992)
`\citep[see][p. 18]{Martin80}`:
(see Martin 1980, p. 18)
`\citep{Brownell04,Brownell07,Ursell50,Wijngaarden68,Miller91}`:
(Brownell & Su 2004, 2007; Ursell 1950; van Wijngaarden 1968; Miller 1991)

The References section can either be built from individual `\bibitem` commands, or can be built using BibTex. The BibTex files used to generate the references in this document can be found in the zip file at http://journals.cambridge.org/data/relatedlink/jfm-ifc.zip.
Where there are up to ten authors, all authors' names should be given in the reference list. Where there are more than ten authors, only the first name should appear, followed by et al.

Acknowledgements should be included at the end of the paper, before the References section or any appendicies, and should be a separate paragraph without a heading. Several anonymous individuals are thanked for contributions to these instructions.

Appendix A

This appendix contains sample equations in the JFM style. Please refer to the LaTeX source file for examples of how to display such equations in your manuscript.

$$(\nabla^2 + k^2)G_s = (\nabla^2 + k^2)G_a = 0 \tag{A 1}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{v} = 0, \quad \nabla^2 P = \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{v} \times \boldsymbol{w}). \tag{A 2}$$

$$G_s, G_a \sim 1/(2\pi) \ln r \quad \text{as} \qquad r \equiv |P - Q| \rightarrow 0, \tag{A 3}$$

$$\left. \begin{aligned} \frac{\partial G_s}{\partial y} = 0 \quad \text{on} \qquad y = 0, \\ G_a = 0 \quad \text{on} \qquad y = 0, \end{aligned} \right\} \tag{A 4}$$

$$-\frac{1}{2\pi} \int_0^\infty \gamma^{-1} [exp(-k\gamma|y-\eta|) + exp(-k\gamma(2d-y-\eta))] \cos k(x-\xi)t \, \mathrm{d}t, \qquad 0 < y, \quad \eta < d, \tag{A 5}$$

$$\gamma(t) = \left\{ \begin{array}{ll} -\mathrm{i}(1-t^2)^{1/2}, & t \leqslant 1 \\ (t^2-1)^{1/2}, & t > 1. \end{array} \right. \tag{A 6}$$

$$-\frac{1}{2\pi} \int_0^\infty B(t) \frac{\cosh k\gamma(d-y)}{\gamma \sinh k\gamma d} \cos k(x-\xi)t \, \mathrm{d}t$$

$$G = -\frac{1}{4}\mathrm{i}(H_0(kr) + H_0(kr_1)) - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-k\gamma d}}{\gamma \sinh k\gamma d} \cosh k\gamma(d-y) \cosh k\gamma(d-\eta) \tag{A 7}$$

Note that when equations are included in definitions, it may be suitable to render them in line, rather than in the equation environment: $\boldsymbol{n}_q = (-y'(\theta), x'(\theta))/w(\theta)$. Now $G_a = \frac{1}{4}Y_0(kr) + \widetilde{G_a}$ where $r = \{[x(\theta) - x(\psi)]^2 + [y(\theta) - y(\psi)]^2\}^{1/2}$ and $\widetilde{G_a}$ is regular as $kr \rightarrow 0$. However, any fractions displayed like this, other than $\frac{1}{2}$ or $\frac{1}{4}$, must be written on the line, and not stacked (ie 1/3).

$$\frac{\partial}{\partial n_q} \left(\frac{1}{4} Y_0(kr) \right) \sim \frac{1}{4\pi w^3(\theta)} [x''(\theta)y'(\theta) - y''(\theta)x'(\theta)]$$

$$= \frac{1}{4\pi w^3(\theta)}[\rho'(\theta)\rho''(\theta) - \rho^2(\theta) - 2\rho'^2(\theta)] \quad \text{as} \quad kr \to 0. \tag{A 8}$$

$$\frac{1}{2}\phi_i = \frac{\pi}{M}\sum_{j=1}^{M}\phi_j K_{ij}^a w_j, \qquad i = 1, \ldots, M, \tag{A 9}$$

where

$$K_{ij}^a = \begin{cases} \partial G_a(\theta_i, \theta_j)/\partial n_q, & i \neq j \\ \partial \widetilde{G_a}(\theta_i, \theta_i)/\partial n_q + [\rho_i'\rho_i'' - \rho_i^2 - 2\rho_i'^2]/4\pi w_i^3, & i = j. \end{cases} \tag{A 10}$$

$$\rho_l = \lim_{\zeta \to Z_l^-(x)} \rho(x,\zeta), \quad \rho_u = \lim_{\zeta \to Z_u^+(x)} \rho(x,\zeta) \tag{$A\,11a,b$}$$

$$(\rho(x,\zeta), \phi_{\zeta\zeta}(x,\zeta)) = (\rho_0, N_0) \quad \text{for} \quad Z_l(x) < \zeta < Z_u(x). \tag{A 12}$$

$$\tau_{ij} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + (\overline{\overline{u}_i u_j^{SGS} + u_i^{SGS}\overline{u}_j}) + \overline{u_i^{SGS}u_j^{SGS}}, \tag{A 13a}$$

$$\tau_j^\theta = (\overline{\overline{u}_j\overline{\theta}} - \overline{u}_j\overline{\theta}) + (\overline{\overline{u}_j\theta^{SGS} + u_j^{SGS}\overline{\theta}}) + \overline{u_j^{SGS}\theta^{SGS}}. \tag{A 13b}$$

$$\boldsymbol{Q}_C = \begin{bmatrix} -\omega^{-2}V_w' & -(\alpha^t\omega)^{-1} & 0 & 0 & 0 \\ \dfrac{\beta}{\alpha\omega^2}V_w' & 0 & 0 & 0 & \mathrm{i}\omega^{-1} \\ \mathrm{i}\omega^{-1} & 0 & 0 & 0 & 0 \\ \mathrm{i}R_\delta^{-1}(\alpha^t + \omega^{-1}V_w'') & 0 & -(\mathrm{i}\alpha^t R_\delta)^{-1} & 0 & 0 \\ \dfrac{\mathrm{i}\beta}{\alpha\omega}R_\delta^{-1}V_w'' & 0 & 0 & 0 & 0 \\ (\mathrm{i}\alpha^t)^{-1}V_w' & (3R_\delta^{-1} + c^t(\mathrm{i}\alpha^t)^{-1}) & 0 & -(\alpha^t)^{-2}R_\delta^{-1} & 0 \end{bmatrix}. \tag{A 14}$$

$$\boldsymbol{\eta}^t = \hat{\boldsymbol{\eta}}^t \exp[\mathrm{i}(\alpha^t x_1^t - \omega t)], \tag{A 15}$$

where $\hat{\boldsymbol{\eta}}^t = \boldsymbol{b}\exp(\mathrm{i}\gamma x_3^t)$.

$$\mathrm{Det}[\rho\omega^2\delta_{ps} - C_{pqrs}^t k_q^t k_r^t] = 0, \tag{A 16}$$

$$\langle k_1^t, k_2^t, k_3^t \rangle = \langle \alpha^t, 0, \gamma \rangle \tag{A 17}$$

$$\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{A 18}$$

$$f(\psi_1) = \frac{3b}{\pi[2(a + b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\frac{(\sin\psi_1 - \sin\psi)(a + b\cos\psi)^{1/2}}{[1 - \cos(\psi_1 - \psi)](2 + \alpha)^{1/2}}\mathrm{d}x, \tag{A 19}$$

$$\begin{aligned} g(\psi_1) = {} & \frac{3}{\pi[2(a + b\cos\psi_1)]^{3/2}}\int_0^{2\pi}\left(\frac{a + b\cos\psi}{2 + \alpha}\right)^{1/2}\left\{f(\psi)[(\cos\psi_1 - b\beta_1)S + \beta_1 P]\right. \\ & \times \frac{\sin\psi_1 - \sin\psi}{1 - \cos(\psi_1 - \psi)} + g(\psi)\left[\left(2 + \alpha - \frac{(\sin\psi_1 - \sin\psi)^2}{1 - \cos(\psi - \psi_1)} - b^2\gamma\right)S\right. \\ & \left.\left. + \left(b^2\cos\psi_1\gamma - \frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi, \delta) - (2 + \alpha)\cos\psi_1 E(\tfrac{1}{2}\pi, \delta)\right]\right\}\mathrm{d}\psi, \end{aligned} \tag{A 20}$$

$$\alpha = \alpha(\psi, \psi_1) = \frac{b^2[1-\cos(\psi-\psi_1)]}{(a+b\cos\psi)(a+b\cos\psi_1)}, \quad \beta - \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}. \tag{A 21}$$

$$\left.\begin{aligned} H(0) &= \frac{\epsilon \overline{C}_v}{\tilde{v}_T^{1/2}(1-\beta)}, \quad H'(0) = -1 + \epsilon^{2/3}\overline{C}_u + \epsilon \hat{C}_u'; \\ H''(0) &= \frac{\epsilon u_*^2}{\tilde{v}_T^{1/2} u_P^2}, \quad H'(\infty) = 0. \end{aligned}\right\} \tag{A 22}$$

Lemma 1. *Let $f(z)$ be a trial Batchelor (1971, pp. 231–232) function defined on* $[0,1]$. *Let Λ_1 denote the ground-state eigenvalue for $-\mathrm{d}^2 g/\mathrm{d}z^2 = \Lambda g$, where g must satisfy $\pm \mathrm{d}g/\mathrm{d}z + \alpha g = 0$ at $z = 0, 1$ for some non-negative constant α. Then for any f that is not identically zero we have*

$$\frac{\alpha(f^2(0)+f^2(1)) + \displaystyle\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z}{\displaystyle\int_0^1 f^2 \mathrm{d}z} \geqslant \Lambda_1 \geqslant \left(\frac{-\alpha + (\alpha^2 + 8\pi^2\alpha)^{1/2}}{4\pi}\right)^2. \tag{A 23}$$

Corollary 1. *Any non-zero trial function f which satisfies the boundary condition $f(0) = f(1) = 0$ always satisfies*

$$\int_0^1 \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{A 24}$$

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