

## Introduction

Consider a semi-infinite elastic solid with a thin strip peeled off, and the resulting crack filled with an incompressible fluid. The motion is driven by a bending moment applied to the “arm” of the solid. The aim is to be able to write down a set of equations governing the dynamics, in particular it is of interest to examine the relationship between the speed of traveling wave solutions  $c$ , the magnitude of the bending moment  $M$ , and the toughness of the solid  $K_I$ .

Relevant physical problems include both igneous intrusions beneath a volcano, and the formation of hydrofractures in an oil reservoir, since both involve the propagation of a crack through a brittle elastic solid driven by fluid injection.

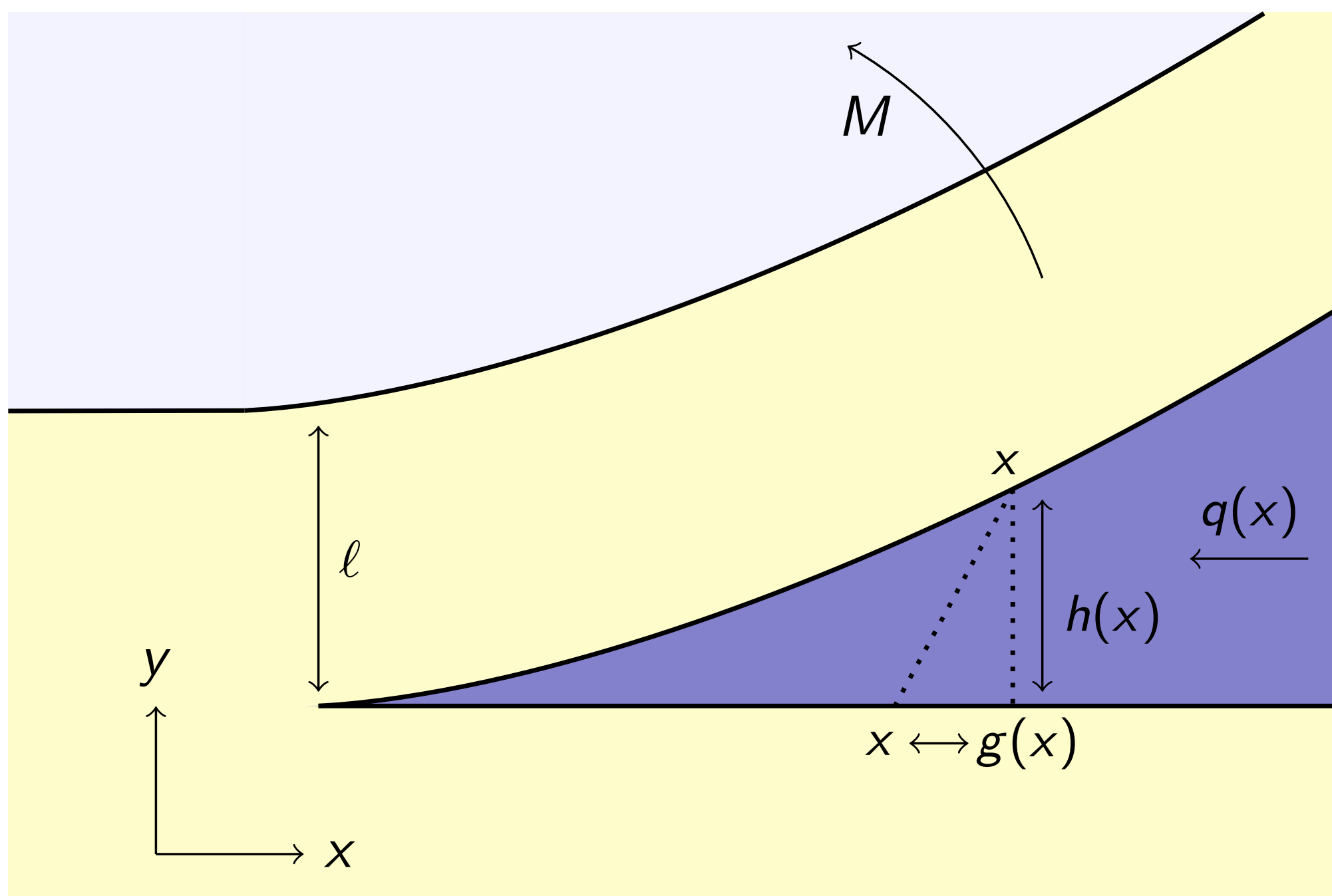


Figure 1 : Diagram to show the geometry of the problem.  $q(x)$  is the flux,  $g(x)$  the horizontal displacement,  $h(x)$  the vertical displacement, and  $\ell$  is the thickness of the arm.

## Governing Equations

We assume that the flow everywhere satisfies the lubrication equations. From fluid mechanics, we then get the equation

$$12\mu c = h(x)^2 \frac{dp}{dx}$$

Where  $p(x)$  is the pressure, and  $\mu$  the viscosity.

From elasticity, using Muskhelishvili methods, we can derive the equation

$$\begin{pmatrix} p \\ 0 \end{pmatrix} = \frac{E}{4\pi(1-\nu^2)} \int_0^\infty \begin{pmatrix} K_{11}(x-\tilde{x}) & K_{12}(x-\tilde{x}) \\ K_{21}(x-\tilde{x}) & K_{22}(x-\tilde{x}) \end{pmatrix} \begin{pmatrix} g'(\tilde{x}) \\ h'(\tilde{x}) \end{pmatrix} d\tilde{x}$$

Where  $K_{ij}$  is the integral kernel specific to this geometry,  $E$  is the Young's modulus,  $\nu$  is Poisson's ratio.

- Boundary conditions as  $x \rightarrow \infty$  are governed by the bending moment. For large  $x$  the geometry is well approximated by beam theory. This gives the equation

$$M(x) = \frac{E\ell^3}{12(1-\nu^2)} \frac{d^2h}{dx^2}$$

Where  $M(x)$  tends to a constant bending moment as  $x \rightarrow \infty$ .

- The boundary conditions as  $x \rightarrow 0$  are governed by “*Linear Elastic Fracture Mechanics*”, (LEFM). This gives the condition

$$K_I = \lim_{x \rightarrow 0} \frac{E}{1-\nu^2} \sqrt{\frac{\pi}{8}} \sqrt{x} h'(x)$$

We move into dimensionless variables now

$$(x, h, g, p, K_I, K_{ij}) \rightarrow (\xi, H, G, \Pi, \kappa, \Lambda_{ij})$$

Where the new equations and boundary conditions become

$$(\Pi, 0) = \int \Lambda \cdot (G', H') d\xi, \quad H^2 \Pi' = \lambda$$

$$\lim_{\xi \rightarrow \infty} H'' = 1, \quad \lim_{\xi \rightarrow 0} 3\sqrt{2\pi} H' = \kappa$$

## Zero Toughness Solution

Instead of tackling the general problem, (which we expect to not have an analytic solution) we investigate the case where  $\kappa \ll 1$ , the “*small toughness solution*.” Perhaps an even simpler problem to consider is the “*zero toughness solution*” for  $\kappa = 0$ . However, we have the following dichotomy,

- For  $\kappa = 0$ , one can show that the leading order behaviour as  $\xi \rightarrow 0$  is  $H(\xi) \sim \xi^{2/3}$
- For any  $\kappa > 0$ , no matter how small, near  $\xi = 0$ ,  $H(\xi) \sim \xi^{1/2}$

## Small Toughness Solution

Here we take after Garagash and Detournay [1]. They consider a similar problem of fluid driven fracture in a different geometry, with the propagation being driven by fluid injection. They construct a small toughness solution in the following way:

- Near the tip there is the “LEFM boundary layer” which accounts for the  $H \sim \xi^{1/2}$  behaviour, and does not resemble the zero toughness solution.
- Away from the tip, the solution behaves as

$$H(\xi) = H_0(x) + \mathcal{E}(\kappa)H_1(\xi) + o(\mathcal{E})$$

where  $H_0$  is the zero toughness solution, and  $\mathcal{E}(\kappa)$  is an as yet unknown function of  $\kappa$ . (Similar for  $G, \Pi, \lambda$ ).

By matching the outer asymptotics of the LEFM boundary layer solution, and the inner asymptotics of the expansion in  $\mathcal{E}$ , one can show that  $\mathcal{E} = C\kappa^{4-6s}\lambda_0^{2s-1}$ . Where  $s \approx 0.1386$  comes from solving a transcendental equation,  $C$  can be determined numerically, and  $\lambda_0 = \lambda(\kappa = 0)$  is also determined numerically.

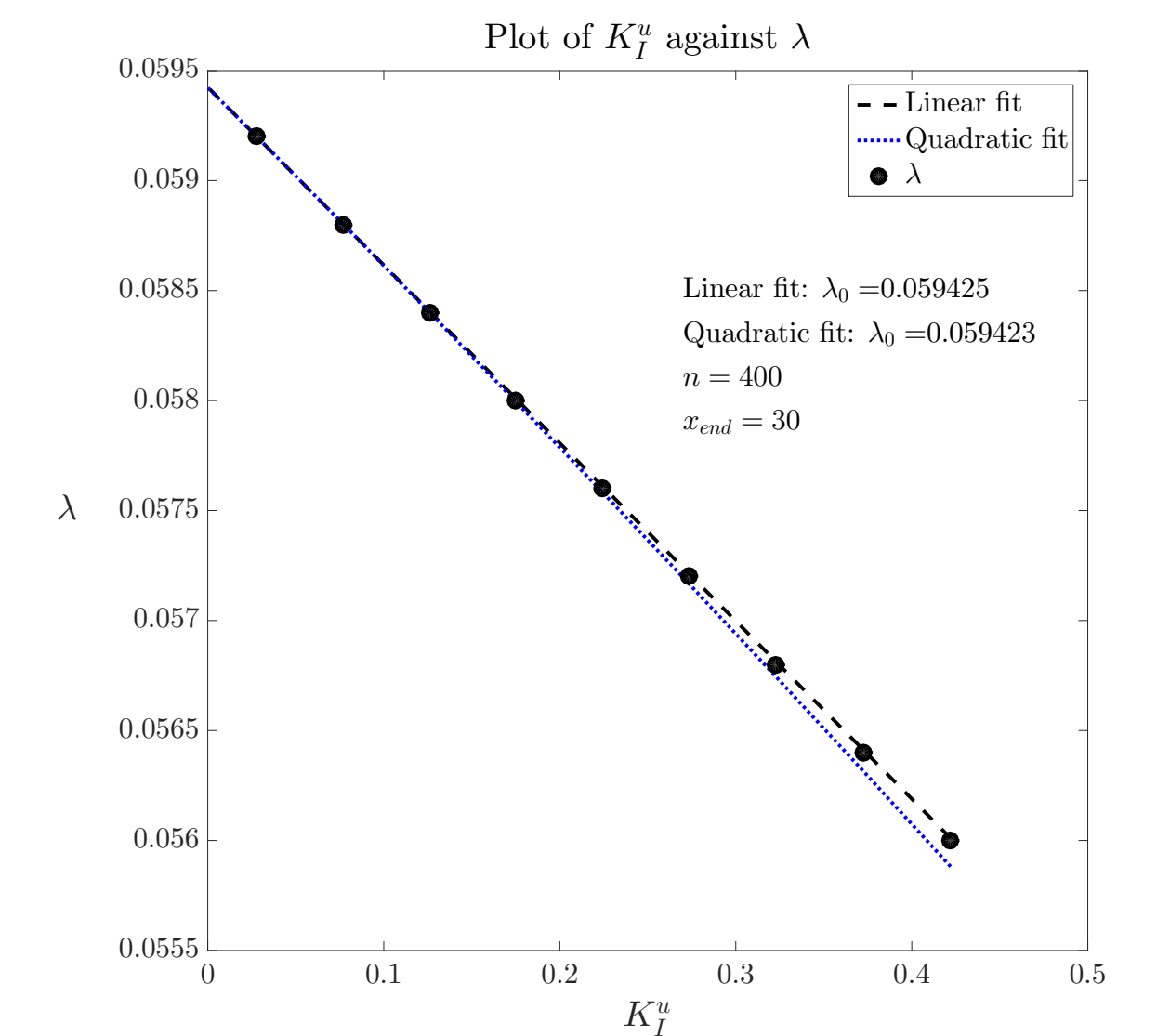


Figure 2 : Close numerical agreement with the  $\lambda = \lambda_0 + \mathcal{E}\lambda_1$  approximation.

## Numerical Solution of Equations

The set of scaled equations can be discretized, and then solved numerically as follows. We choose a set of points  $\xi$  to measure  $G, H$ , and an intermediate set of points  $z$  to measure  $\Pi$ , so  $\xi_1 < z_1 < \xi_2 < \dots < z_{n-1} < \xi_n$ . Since both  $H', G'$  are singular near the origin, they are badly approximated by linear functions. The solution is to approximate  $G'(\xi) = \frac{1}{\sqrt{\xi}}(a_i\xi + b_i)$  near the tip and to approximate  $G'(\xi) = a_i\xi + b_i$  away from the tip for  $\xi_i < \xi < \xi_{i+1}$  (similar with  $H'$ ). We store the values  $\theta = (a_1\xi_1 + b_1, \dots, a_n\xi_n + b_n)$ .

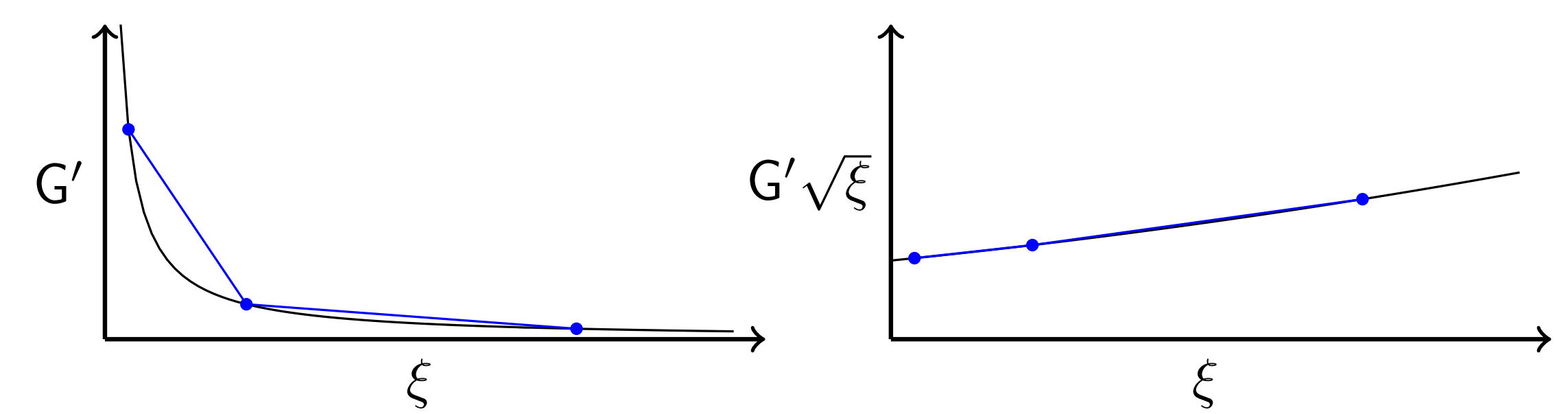


Figure 3 : Relative improvement in interpolation for a given number of points, once known singular behaviour is accounted for.

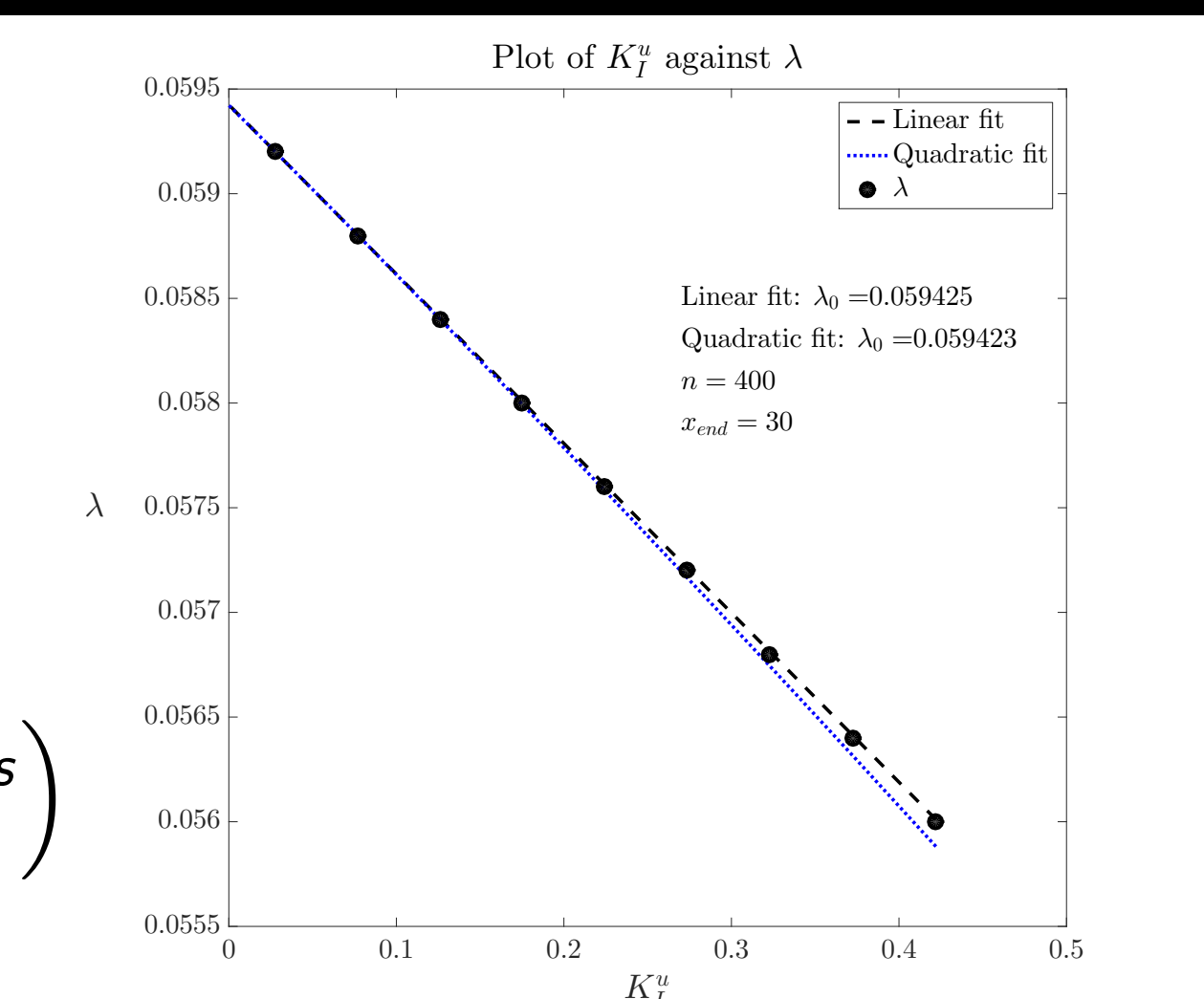
Once we have the values  $\theta$ , it is possible to linearly recover the coefficients, i.e.  $(a_1, \dots, a_n, b_1, \dots, b_n) = T\theta$  where  $T$  is some matrix. It is clear, given this interpolation, the elasticity integral depends linearly on the  $\theta$  values. We can rewrite the lubrication integral as  $\Pi(z) = \int_z^\infty \lambda/H^2 d\xi$ . This depends non-linearly on the  $\theta$  values.

There are now two different expressions for  $(\Pi(z_1), \dots, \Pi(z_{n-1}))$ , so  $n-1$  equations. There are an additional  $n-1$  equations from the elasticity integral. We can get another two equations from the boundary conditions as  $\xi \rightarrow \infty$ . This gives  $2n$  equations for  $2n$  unknowns ( $\theta$  and  $H'$  equivalent). This is enough to solve the problem using Newton's method, to give  $G', H'$ . We use  $\lambda$  as an input parameter and solve for  $\kappa$ , although in the physical problem we think of  $\kappa$  as the independent variable.

## Results

It has been calculated that  $\lambda_0 \approx 0.0591$ . To calculate  $\lambda_1$ ,  $C$  is harder, since the linear perturbation problem must be solved (linearise and work only to first order in  $\mathcal{E}$ ). We found  $C \approx 5.8 \times 10^{-3}$ ,  $\lambda_1 \approx -0.31$ . Redimensionalising

$$c = \frac{36(1-\nu^2)^2 M^3}{\pi \mu E^2 \ell^5} \left( \lambda_0 + C \lambda_0^{2s-1} \lambda_1 (\ell^{3/2} K_I / M)^{4-6s} \right)$$



## References

- [1] Garagash, D.I., Detournay, E., *Plane-Strain Propagation of a Fluid-Driven Fracture: Small Toughness Solution*, Journal of Applied Mechanics, 2005.