Integration of diff. forms.

On an n-dimensional manifold M, the integrand is properly understood as an n-form. In other words, an integral over an n-dimension region $\Sigma \subset M$ is a map from an n-form field ω to real numbers:

$$\int_{\Sigma} : \omega \longrightarrow \mathbb{R}$$

Definition: Let M be a connected manifold covered by $\{U_i\}$. The manifold M is orientable if, for any overlaping charts U_i and U_j, there exist local coordinates $\{X^\mu\}$ for U_i and $\{Y^\nu\}$ for U_j, such that:

$$J \equiv \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0$$

If an n-dimentional manifold M is orientable, there exist an n-form ω which vanishes nowhere. This n-form ω is called volume element, which play the role of a measure when we integrate a function $f \in \mathcal{F}(M)$ over M. Now if we consider an n-form:

$$\omega_{(n)} = f(p)\, dx^1 \wedge \ldots \wedge dx^n$$

with positive-definite $f(p)$ on a chart (U, φ) whose coordinates are $x = \varphi(p)$. Let's consider $p \in U_i \cap U_j \neq 0$, and:

$$dx^1 \wedge \cdots \wedge dx^m = \frac{1}{m!} \epsilon_{\mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$$

Ex: 3-dim case:

$$dx^1 \wedge dx^2 \wedge dx^3 \overset{?}{=} \frac{1}{3!} \epsilon_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3}$$

$$= \frac{1}{3!} (\epsilon_{123} dx^1 \wedge dx^2 \wedge dx^3 + \epsilon_{132} dx^1 \wedge dx^3 \wedge dx^2 + \epsilon_{213} dx^2 \wedge dx^1 \wedge dx^3$$

$$\ldots + \epsilon_{231} dx^2 \wedge dx^3 \wedge dx^1 + \epsilon_{312} dx^3 \wedge dx^1 \wedge dx^2 + \epsilon_{321} dx^3 \wedge dx^2 \wedge dx^1)$$

$$= \frac{6}{3!} \epsilon_{123} dx^1 \wedge dx^2 \wedge dx^3$$

for a general case: $\boxed{dx^1 \wedge \ldots \wedge dx^p = \frac{1}{p!} \epsilon_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}}$

Now, for a coordinate transformation:

$$\omega_{(m)} = f(p) dx^1 \wedge \ldots \wedge dx^n = f(p) \frac{1}{m!} \epsilon_{\mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$$

$$= f(p) \frac{1}{M!} \epsilon_{\mu_1 \ldots \mu_N} \frac{\partial x^{\mu_1}}{\partial y^{\nu_1}} dy^{\nu_1} \wedge \ldots \wedge \frac{\partial x^{\mu_N}}{\partial y^{\nu_N}} dy^{\nu_N}$$

but, we also know the property:

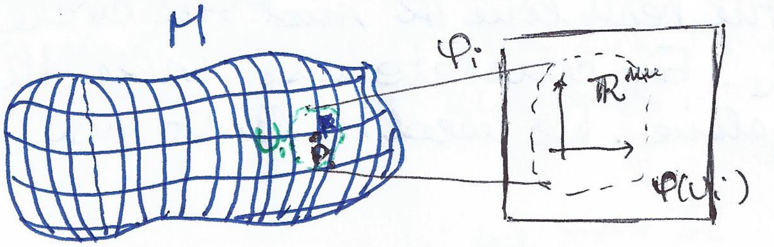
$$\epsilon_{\mu_1' \ldots \mu_N'} \det(M) = \epsilon_{\mu_1 \ldots \mu_N} M^{\mu_1}{}_{\mu_1'} \ldots M^{\mu_N}{}_{\mu_N'}$$

$$\rightarrow \omega_{(M)} = f(p) \frac{1}{M!} \epsilon_{\nu_1 \ldots \nu_N} \det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) dy^{\nu_1} \wedge \ldots \wedge dy^{\nu_N}$$

$$= f(p) \det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) dy^1 \wedge \ldots \wedge dy^m$$

And, if ω is assumed orientable, $\det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) > 0$. #

...w, we are ready to define an integration of a function $f: M \to \mathbb{R}$ over an orientable manifold M. In a coordinate neighbourhood U_i with the coordinate x, we define the integration of an m-form $f\omega$ by.



$$\omega = g(p)\, dx^1 \wedge \dots \wedge dx^m$$

$$\int_{U_i} f\omega \equiv \int_{\varphi(U_i)} f(\varphi_i^{-1}(x))\, g(\varphi_i^{-1}(x))\, dx^1 \dots dx^m \quad (*)$$

Where the RHS is an ordinary multiple integration of a function of m-variables. Once the integral of f over U_i is defined, the integral over the whole M is given with the help of the "partition of unity".

Definition: Take an open covering $\{U_i\}$ of M such that each point of M is covered with a finite number of U_i. (if this is always posible, M is called paracompact, which we assume to be the case). If a family of differentiable function $\epsilon_i(p)$ satisfies:

i) $0 \leq \epsilon_i(p) \leq 1$

ii) $\epsilon_i(p) = 0$ if $p \notin U_i$

iii) $\epsilon_1(p) + \epsilon_2(p) + \dots = 1$ for any point $p \in M$

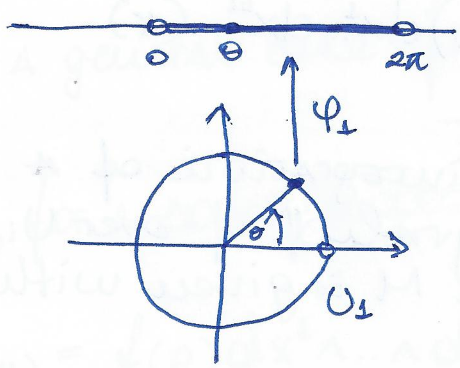
the family $\{\epsilon_i(p)\}$ is called partition of unity subordinate to the covering $\{U_i\}$.

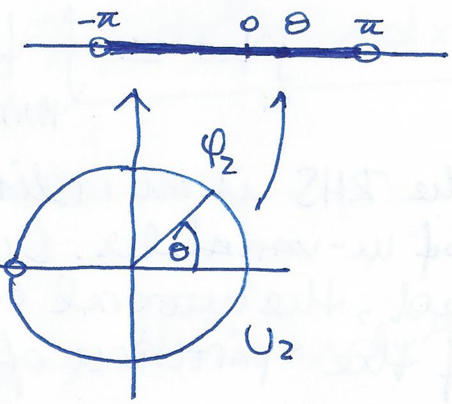
From (iii): $\quad f(p) = \sum_i f(p)\epsilon_i(p) \equiv \sum_i f_i(p)$

that vanishes outside U_i by (ii). Now, given a point $p \in M$, assume of paracompactness ensure that only finite terms in the summation over i. For each $f_i(p)$, we may define the integral over U_i according to $(*)$. For f over M, the integral is given by:

$$\int_M f\omega \equiv \sum_i{}' \int f_i \omega .$$

EXAMPLE: Let's consider a $n=1$ dimension manifold. There are two possible manifold: the real line $\mathbb{R}$ and the circle S^1. Let us work out an atlas of S_1. For concreteness take the circle $x^2+y^2=1$ in the xy-plane. We need at least two charts:





$\varphi_1^{-1} : \theta \longrightarrow (\cos\theta, \sin\theta)$

(whose image is $S^1 - \{(1,0)\}$)

$\varphi_2^{-1} : \theta \longrightarrow (\cos\theta, \sin\theta)$

(whose image is $S^1 - \{(-1,0)\}$)

And consider $\epsilon_1(\theta) = \cos^2\theta/2$ and $\epsilon_2(\theta) = \sin^2\theta/2$. Lets us integrate the funcion $f(\theta) = \cos^2\theta$ for example.

$$\int_{S^1} f\omega = \sum_i{}' \int f_i(\theta)\omega = \int_{U_1} f(\theta)\epsilon_1(\theta)d\theta + \int_{U_2} f(\theta)\epsilon_2(\theta)d\theta .$$

$$= \int_0^{2\pi} \cos^2\theta \cos^2\theta/2 \, d\theta + \int_{-\pi}^{\pi} \cos^2\theta \sin^2\theta/2 \, d\theta$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$