Bundles & Connections

So far, tangent and cotangent bundles have been considered, and other generalisations have been slightly appearing.

Instead of attack the problem formally, in the following a heuristic introduction to more general bundles is shown, and also the definition of connections on these bundles.

Bundles

A bundle is a geometric structure composed by two manifolds (E, M) ← $E \supset M$ in some sense together with a "projection" map $\pi : E \longrightarrow M$.

locally $E \simeq M \times F$ ← F is a manifold in general

and F is called fibre of E

Examples

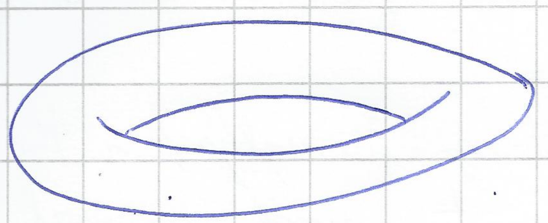
$E =$ Cylinder ; $M = \mathbb{R}$; $F = S^1$; $\pi : E \longrightarrow \mathbb{R}$

$\pi : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}$

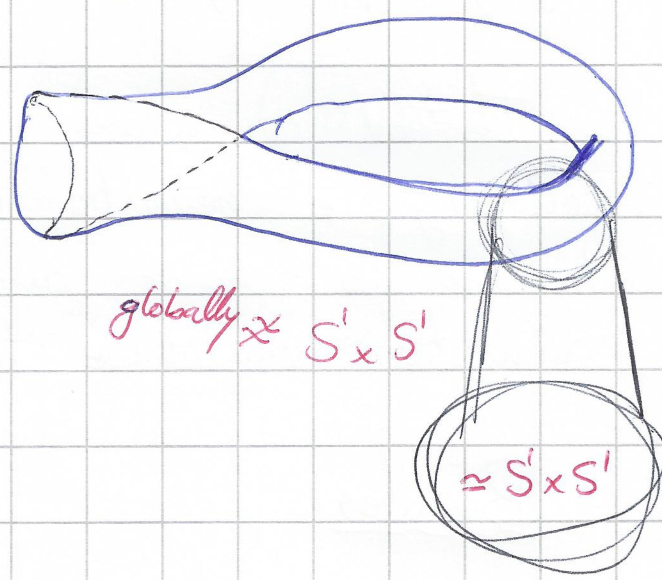
In this case $E = M \times F$ globally, then the bundle is called trivial.

Consider the 2-torus and Klein's bottle. both are bundles on S^1 with fibre S^1.

They differ in the "junction" of the fibre globally.



$T^2 \simeq S^1 \times S^1$



Types of bundles

They are classified according to the fibre structure.

- Fibre : general
- Vector : F is a vector space
- Principal: F is has a Lie group structure

Sections on bundles

A section on E is a smooth ~~app~~ map

$$\psi : M \longrightarrow E$$

Usually $\left(\psi \circ \pi \overset{?}{=} id_E \right) \quad \left(\pi \circ \psi = id_M \right)$ ✓

{ Sections are the geometrical objects associated to physical fields }

Principal Bundles

Their fibre is a Lie Group. Therefore, there is an action of G on the fibre F.

Associated bundles

$P(M,G)$ a principal bundle
F the fibre

$P \times F / G$ is a bundle

$$(u, f) \sim (ug, g^{-1} f)$$

$P \times_\rho V / G \qquad (u, v) \sim (ug, \rho(g)^{-1} v)$

Connections in tangent bundle

$$\nabla : TM \times TM \longrightarrow TM$$

$$\nabla : (X, Y) = \nabla_X Y$$

s.t.

$$\nabla_{fX} Y = f \nabla_X Y \qquad \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$$

&

$$\nabla_X fY = f \nabla_X Y + Y . X[f]$$

On a vector basis

$$\nabla_i \partial_k = \Gamma_i{}^j{}_k \, \partial_j$$

In general

space of sections on E

$$\nabla : TM \otimes \mathcal{E} \longrightarrow \mathcal{E}$$

$$\nabla_i Y = \underbrace{\left(\partial_i Y^j + \Gamma_i{}^j{}_k Y^k \right)}_{\text{covariant derivative}} \partial_j$$

Parallel transport.

Let V be a tangent vector ~~field~~ ~~to~~ to a curve, i.e.,

$$V = \frac{dx^\mu(c(t))}{dt} \partial_\mu \Big|_{c(t)} ,$$

then $X \in TM$ is said to be parallel transported along $c(t)$ if

$$\nabla_V X = 0 \quad \forall t \in I$$

If the tangent vector to the curve (V) ~~is~~ satisfies the parallel transport condition

$$\nabla_V V = 0$$

the $c(t)$ is called a geodesic

- Use that $\nabla_i \partial_k = \Gamma_i{}^j{}_k \partial_j$
  $\nabla'_\ell \partial'_n = \Gamma'_\ell{}^m{}_n \partial'_m$
  to find the transf. rule of Γ's
- Find the action of ∇ on 1-forms and rank 2 tensors.
- Write in coord. the cond. of parallel trans. & geodesic curve

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

- Find coordinate expresions for T & R
- Show that they are multilinear object (i.e. they are tensors)

NOTE:

In general the concept of curvature can be extended to sections on a bundle, where $\nabla : TM \otimes \mathcal{E} \to \mathcal{E}$ while torsion cannot.

(since X & Y are necessarily the same kind of objects)