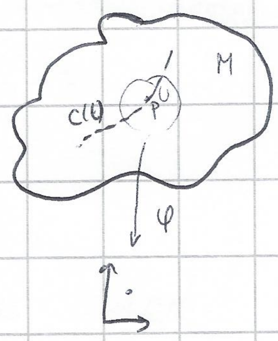
* Manifolds: $(U \subset M, \varphi)$



$$f : M \to \mathbb{R}$$

$$f'(c(t)) = \frac{df}{dx^\mu}\frac{dx^\mu}{dt} \overset{!}{=} X[f] \quad \text{w.} \quad X = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu}$$

Vectors on M.

– Vectors $\frac{dx^\mu}{dt} \rightsquigarrow$ components

$\frac{\partial}{\partial x^\mu} \rightsquigarrow$ basis.

Chain rule: $$\frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} \equiv \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu}$$

$$\Rightarrow \frac{\partial y^\mu}{\partial t} = \frac{\partial x^\nu}{\partial t}\frac{\partial y^\mu}{\partial x^\nu} \equiv \frac{\partial x^\nu}{\partial t}(\Lambda^{-1})_\nu{}^\mu$$

Vectors lie on T_pM

$$TM \equiv \bigsqcup_{p \in M} T_pM$$

- One-forms.

$$\omega : T_pM \to \mathbb{R}$$ form a vector space

$$V[f] = V^\mu \partial_\mu f \equiv \langle df, V \rangle \in \mathbb{R}$$

one-form

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu \quad \text{s.t.} \quad \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \delta^\mu_\nu$$

$\langle\, , \rangle$ ← inner prod.

$$\langle\, , \rangle : T_p^*M \times T_pM \to \mathbb{R}.$$

1-forms lie on T_p^*M

$$T^*M \equiv \bigsqcup_{p \in M} T_p^*M.$$

NOTE: forms transform with the inverse of vectors.

- <u>Tensors</u>

Multi-linear objects.

$$T = T^{a_1 \dots a_p}{}_{b_1 \dots b_q}\, \partial_{a_1} \dots \partial_{a_p}\, dx^{b_1} \dots dx^{b_q}$$

$$\cap$$

$$\otimes^p TM \otimes^q T^*M$$

{ How do the componenst transform? }

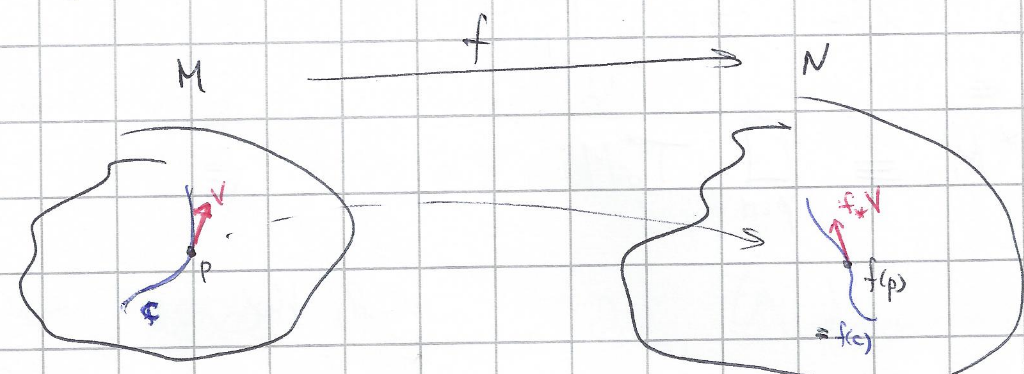
* <u>Induced Maps</u>

- Diff. Map

$$f : M \longrightarrow N$$

$$f_* : T_pM \longrightarrow T_{f(p)}N$$ ← diff. map.

$\cancel{V[g]}$ ~~N~~



$\cancel{V^\mu \frac{\partial g}{\partial x^\mu}}$ $\cancel{(f_* V)^\nu \frac{\partial}{\partial y^\nu}}$

$\mathcal{F}(N) : N \to \mathbb{R}$

$$(f_* V)[g] = V[\underbrace{g \circ f}_{M \to N \to \mathbb{R}}]$$

— Pullback

$$f : M \to N$$

$$f^* : T^*_{f(p)} N \to T^*_p M$$

$\omega \in T^* N$

$$\cancel{(f^*\omega)(g)}$$

$\in TM$

$$\langle f^*\omega, V \rangle \equiv \langle \omega, f_* V \rangle .$$

Let M, N, P be manifolds and
$M \xrightarrow{f} N \xrightarrow{g} P$, Show.

$$(g \circ f)_* = g_* \circ f_*$$
$$(g \circ f)^* = f^* \circ g^*$$

*. <u>Submanifolds.</u>

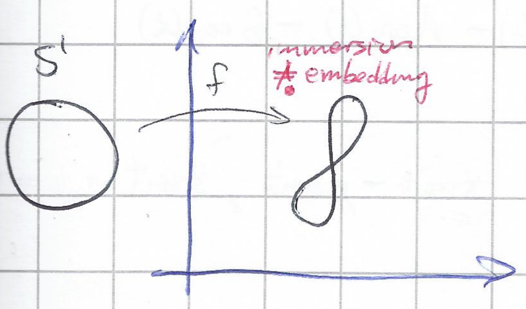
Let $f : M \to N$ be a smooth map and $\dim(M) \leq \dim(N)$

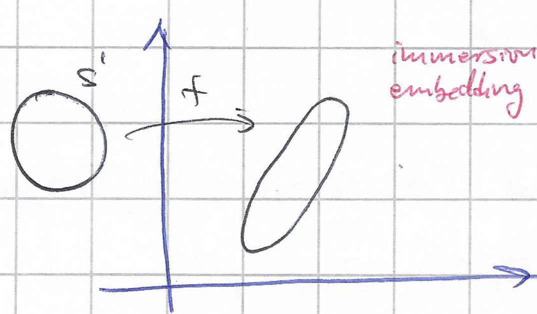
a) f is an <u>immersion</u> of M into N if f_* is an injection

b) f is an embedding of M into N if f is an injection and immersion

$\in$

If (b) $\Rightarrow$ $\mathrm{Im}(f(M))$ is a submanifold of N.





Flows & Lie derivatives

$X^\mu \rightarrow$ vector field

$x(t) \rightarrow$ curve

Integral curve : $\dfrac{dx^\mu}{dt} = X^\mu(x(t))$

$\sigma : \mathbb{R} \times M \rightarrow M$ is a flow generated by X if

$$\frac{d\sigma^\mu(t,x_0)}{dt} = X^\mu(\sigma(t,x_0))$$ $\leftarrow$ σ is an integral curve of X

& $\sigma(t, \sigma(s,x_0)) = \sigma(t+s, x_0)$

Ex. $X(x,y) = -y\,\partial_x + x\,\partial_y$

a generic vector has coord (x,y) or $x(t)\partial_x + y(t)\partial_y$

$\therefore$ $\sigma(t,(x,y)) = x(t)\,\partial_x + y(t)\,\partial_y$.

From the flow eq.

$$\dot{x}\,\partial_x + \dot{y}\,\partial_y = -y\partial_x + x\partial_y .$$

It must be satisfied comp. by comp.

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned} \Rightarrow \ddot{x} + x = 0 \Rightarrow \begin{aligned} x(t) &= A\cos(t) + B\sin(t) \\ y(t) &= A\sin(t) - B\cos(t) \end{aligned}$$

if the init. cond $\vec{x} = (x_0, y_0)$ $\quad \sigma(t,(x_0,y_0)) = (x\cos t - y\sin t,\ x\sin t + y\cos t$

Show $\sigma_t(\sigma_s) = \sigma_{t+s}$ for this case.

{ Find the flow generated by $X = y\partial_x + x\partial_y$ }

For a fixed t, $\sigma_t : M \to M$ is an Abelian group.

$\sigma_0 = \mathbb{1}$. + etc

$\sigma^\mu_\varepsilon(x) = x^\mu + \varepsilon X^\mu$ generator of the flow.

Iterating:

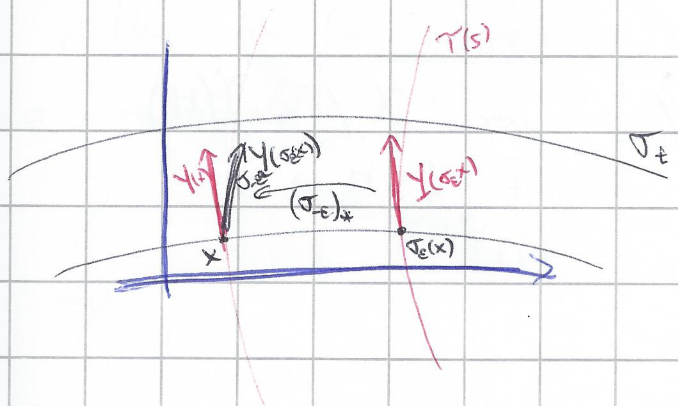
$$\sigma^\mu(t,x) = e^{tX} x^\mu$$

similar to Lie group !!!

Lie derivative

Let X, Y be vector fields on M, s.t.,

$$\frac{d}{dt}\sigma^\mu(t,x) = X(\sigma(t,x))$$
$$\frac{d}{ds}\tau^\mu(s,x) = Y(\tau(s,x)).$$



$$\mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[(\sigma_{-\varepsilon})_* Y(\sigma_\varepsilon(x)) - Y(x)\right]$$

$$Y^\mu(\sigma_\varepsilon(x)) = Y^\mu(x + \varepsilon X)\partial_\mu\Big|_{x+\varepsilon X} = \left[Y^\mu(x) + \varepsilon X^\nu \partial_\nu Y^\mu(x)\right]\partial_\mu\Big|_{x+\varepsilon X}$$

$$(\sigma_{-\varepsilon})_* Y(\sigma_\varepsilon(x)) = \left[\;^\circ\!/\!_\circ\;\right] \partial_\mu (x^\nu - \varepsilon X^\nu)\, \partial_\nu\Big|_{(x+\varepsilon X)-\varepsilon X}$$

$$= \left[\;^\circ\!/\!_\circ\;\right]\left(\partial_\mu - \varepsilon\, \partial_\mu X^\nu \partial_\nu\right)$$

$$= Y^\mu(x)\partial_\mu + \varepsilon\left[X^\nu \partial_\nu Y^\mu \partial_\mu - Y^\mu \partial_\mu X^\nu \partial_\nu\right] + \mathcal{O}(\varepsilon^2)$$

rename indices

$$\mathcal{L}_X Y = X[Y] - Y[X] \equiv [X, Y]$$ ← Lie bracket

<u>NOTE.</u> $[\,,\,]$ is anti-symmetric, bilinear & satisfy the Jacobi identity.

- Calculate:
  $$\mathcal{L}_{fX} Y \,,\quad \mathcal{L}_X (fY)$$
- Show $f_*[X,Y] = [f_*X, f_*Y]$ for $X, Y \in \mathfrak{X}(M)$ & $f: M \to N$.
- $\mathcal{L}_X f = X[f]$
- $\mathcal{L}_X \omega = (X^\nu \partial_\nu \omega_\mu + \partial_\mu X^\nu \omega_\nu)\, dx^\mu$

Differential Forms.

Def:

A diff. form of order p or p-form is a totally anti-symmetric tensor of ~~type~~ $\binom{0}{p}$

Basis.

for contra-variant tensor ($\{\partial_\mu\}$, $\{\partial_\mu \otimes \partial_\nu\}$, $\{\partial_\mu \otimes \partial_\nu \otimes \partial_\lambda\}$, ...

for covariant tensors $\{dx^\mu\}$, $\{dx^\mu \otimes dx^\nu\}$, ...

One might define a basis for antisymmetric tensors by defining an antisymmetric tensor product or wedge product

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

and so on

The vector space of diff. forms of order p on a manifold M is denoted

$$\Omega^p(M) \quad \text{or} \quad \Lambda^p(M),$$

moreover, if $\dim(M) = m$

$$\dim(\Lambda^p(M)) = \binom{m}{p} = \frac{m!}{p!\,(m-p)!}$$

In general $\Lambda^p(M) \subset \bigotimes^p T^*M$

- <u>Exterior product</u>

$$\wedge : \Lambda^p(M) \times \Lambda^q(M) \longrightarrow \Lambda^{p+q}(M)$$

$$(\omega \wedge \xi)(V_1, \dots, V_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma} \cancel{\mathrm{sgn}}\,(-1)^{|\sigma|}\, \omega(V_{\sigma_1}, \dots, V_{\sigma_p})\, \xi(V_{\sigma_{p+1}}, \dots, V_{\sigma_{p+q}})$$

This exterior prod. endow an algebra structure to

$$\Lambda^{\circ}(M) \equiv \Lambda^0(M) \oplus \Lambda^1(M) \oplus \dots \oplus \Lambda^m(M)$$

Clearly

$$\dim(\Lambda^{\circ}(M)) = 2^m$$

- <u>Exterior derivative</u>

heuristically: is a derivative operation which acts on p-forms and results into a (p+1)-form

$$d : \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M)$$

Properties

$$d^2 = 0 \longleftarrow \text{nilpotency}$$

Diff. forms in $\mathbb{R}^3$

$\Lambda^0(\mathbb{R}^3) = \text{functions} \equiv \mathcal{F}(\mathbb{R})$; $\Lambda^1(\mathbb{R}^3) = T^*(\mathbb{R}^3) \simeq \mathbb{R}^3$

$\Lambda^2(\mathbb{R}^3) = T^*(\mathbb{R}^3) \wedge T^*(\mathbb{R}^3) \simeq \mathbb{R}^3$

$\Lambda^3(\mathbb{R}^3) = T^*(\mathbb{R}^3) \wedge T^*(\mathbb{R}^3) \wedge T^*(\mathbb{R}^3) \simeq \text{functions} - \mathcal{F}(\mathbb{R})$

Let $f \in \Lambda^0(\mathbb{R}^3)$

$$df = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz$$

comp

grad. of f

Let $\omega_1 \in \Lambda^1(\mathbb{R}^3) \Rightarrow \omega_1 = \omega_x\, dx + \omega_y\, dy + \omega_z\, dz$

& $$d\omega_1 = (\partial_x \omega_y - \partial_y \omega_x)\, dx \wedge dy + (\partial_y \omega_z - \partial_z \omega_y)\, dy \wedge dz + (\partial_z \omega_x - \partial_x \omega_z)\, dz \wedge dx$$

components of curl of ω_1

Let $\omega_2 \in \Lambda^2(\mathbb{R}^3) \Rightarrow \omega_2 = \omega_{xy}\, dx \wedge dy + \omega_{yz}\, dy \wedge dz + \omega_{zx}\, dz \wedge dx$

$$\therefore d\omega_2 = (\partial_x \omega_{yz} + \partial_y \omega_{zx} + \partial_z \omega_{xy})\, dx \wedge dy \wedge dz$$ div of ω_2

$d\omega_3 = 0$

Show by explicit calculation that

$$d\omega_1(X,Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y]).$$

In general

$$d\omega_p(X_1 \ldots X_{p+1}) = \sum_{i=1}^{p} (-1)^{i+1} X_i[\omega_p(X_1 \ldots \hat{X}_i \ldots X_{p+1})]$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots \hat{X}_i \ldots \hat{X}_j \ldots X_{p+1})$$

A map $f: M \to N$ induces the pullback

$$f^*: T^*_{f(p)} N \to T^*_p M,$$

which is extended naturally to tensors of type $\binom{0}{p}$ (or forms)

$$(f^* \omega)(V_1 \ldots V_p) = \omega(f_* V_1, \ldots, f_* V_p)$$

Show $d(f^*\omega) = f^*(d\omega)$
$f^*(\omega \wedge \xi) = (f^*\omega) \wedge f^*\xi)$

Cohomology

The exterior derivative induces a sequence

$$0 \xrightarrow{i} \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \rightarrow \dots \xrightarrow{d} \Lambda^{m-1}(M) \xrightarrow{d} \Lambda^m(M) \xrightarrow{d} 0$$

called the de Rham complex.

- ω is said to be ~~comple~~ closed if $d\omega = 0$.
- ω is said to be exact if $\omega = d\xi$.

Due to nilpotency $(d^2 = 0)$

$$\mathrm{Im}\left(d : C^\infty(\Lambda^{p-1}(M)) \to C^\infty(\Lambda^p(M))\right) \subset \mathrm{Ker}\left(d : C^\infty(\Lambda^p(M)) \to C^\infty(\Lambda^{p+1}(M))\right)$$

So. $$H^p(M) \equiv \frac{\mathrm{Ker}\left(d : C^\infty(\Lambda^p(M)) \to C^\infty(\Lambda^{p+1}(M))\right)}{\mathrm{Im}\left(d : C^\infty(\Lambda^{p-1}(M)) \to C^\infty(\Lambda^p(M))\right)}$$

is the p^{th} de Rham cohomology group.

- Interior product & Lie derivative

$$i_X : \Lambda^p(M) \longrightarrow \Lambda^{p-1}(M)$$

$$i_X\omega(V_1 - V_{p-1}) \equiv \omega(X, V_1, - V_{p-1})$$

Let $\omega \in \Lambda^p(M)$

$$i_X \omega = \frac{1}{(p-1)!} X^\nu \omega_{\nu \mu_2 \dots \mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$

$$= \frac{1}{p!} \sum_{s=1}^{p} X^{\mu_s} \omega_{\mu_1 \dots \mu_s \dots \mu_p} (-1)^{s-1} dx^{\mu_1} \wedge \dots \wedge \widehat{dx^{\mu_s}} \wedge \dots \wedge dx^{\mu_p}$$

$$(d i_X + i_X d)\, \omega_1 = d(X^\mu \omega_\mu) + i_X\left(\tfrac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)\, dx^\mu \wedge dx^\nu\right)$$

$$= (\partial_\nu X^\mu)\, \omega_\mu\, dx^\nu + X^\mu (\partial_\nu \omega_\mu)\, dx^\nu$$

$$+ X^\mu (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)\, dx^\nu$$

$$= \left[\omega_\mu (\partial_\nu X^\mu) + X^\mu (\partial_\mu \omega_\nu)\right] dx^\nu$$

compre with $\mathcal{L}_X \omega$

$$\mathcal{L}_X \omega = (d i_X + i_X d)\, \omega$$

One might prove the last expr. generalises to p-forms.

- derivation: $i_X(\omega \wedge \eta) \quad i_X \omega \wedge \eta + (-1)^{|\omega|} \omega \wedge i_X \eta$
- nilpotency: $i_X^2 = 0$
- $\mathcal{L}_X i_X \omega = i_X \mathcal{L}_X \omega$

adding a zero

$$(d i_X + i_X d)\, i_X \omega = i_X d\, i_X \omega + i_X^2 d\omega = i_X \mathcal{L}_X \omega$$

MECHANICS.

Let H be a Hamiltonian, (q^μ, p_μ) $\mu = 1 — n$ denote coordinates on the phase space Φ^n $(\dim(\Phi^n) = 2n)$

$$\Lambda^2(\Phi^n) \ni \omega = dp^*_\mu \wedge dq^\mu$$

symplectic form $\begin{cases} \text{- closed} \\ \text{- non-degenerated} \end{cases}$

Let $\theta = q^\mu dp_\mu \in \Lambda^1(\Phi^n)$ s.t. $\omega = -d\theta$.

Given a function $f(q,p)$, one define the Hamiltonian vector field.

$$X_f = \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

Then

$$i_{X_f}\omega = -\frac{\partial f}{\partial q^\mu}dq^\mu - \frac{\partial f}{\partial p_\mu}dp_\mu = -df$$

Define $Q = (q^\mu, p_\mu)$ the coord., ~~&~~ since the Hamilton eq.

$$\frac{dq^\mu}{dt} = \frac{\partial H}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}$$

One might write then as

$$\cancel{\dot{Q} = i_{X_H}}$$

$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

$$= -\frac{dq^\mu}{dt}\frac{\partial}{\partial q} - \frac{dp_\mu}{\partial t}\frac{\partial}{\partial p_\mu}$$

$$= -\frac{d}{dt}$$

$$\pm dH = i_{X_H}\omega$$

$$i_{X_H}\omega = -\frac{\partial H}{\partial q^\mu}dq^\mu - \frac{\partial H}{\partial p_\mu}dp_\mu = -dH$$

Also

$$i_{X_f}(i_{X_g}\omega) = -i_{X_f}(dg) = -i_{X_f}\left(\partial_{q^\mu}g\, dq^\mu + \partial_{p_\mu}g\, dp_\mu\right)$$

$$= -\frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q^\mu} + \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu}$$

$$= \{f, g\}_{P.B.}$$

GAUGE THEORY (Abelian)

Let $A_\mu = (-\phi, A)$ be a 4-vector. $\quad g = \text{diag}(-,+,+,+)$
Construct,

$$A = -\phi\, dt + A_i\, dx^i, \quad \text{a 1-form.}$$

$$\begin{aligned} F &= dA \\ &= -\partial_i \phi\, dx^i \wedge dt + \partial_t A_i\, dt \wedge dx^i + \partial_i A_j\, dx^i \wedge dx^j \\ &= (\partial_i \phi + \partial_t A_i)\, dt \wedge dx^i + B_{(2)} \\ &= dt \wedge E_{(1)} + B_{(2)} \end{aligned}$$

Nilpotency of d $\Rightarrow$ $dF = d(dA) = 0$

On the other hand,

$$dF = dt \wedge d_s E_{(1)} + \partial_t B_{(2)} + d_s B_{(2)} = 0$$

space exterior derivative

time exterior derivative

term w/time: $\quad \partial_t \vec{B} + \nabla \times \vec{E} = 0$
term w. only/space: $\quad \vec{\nabla} \cdot \vec{B} = 0$

Bianchi identities

- Gauge invariance: $A \mapsto A + df$

$\Rightarrow \quad F \to F' = dA' = dA + d^2 f = F$

$\vec{E}$ & $\vec{B}$ do not vary under this transformation

Prepare a 20 min. session about integration of diff. forms

Lie groups & algebras

Group.

A set of elements $\{g\}$ together with an operation $\cdot : G \times G -$ satisfying 1) $\exists$ identity 2) closure 3) associativity 4) $\exists$ $a^{-1}_{\in G}$ $\forall a \in G$

Def 2:

A Lie group, G, is a dif. manifold endowed with a group structure

$$\cdot : G \times G \to G \qquad \& \qquad (\ )^{-1} : G \to G$$
$$(g_1, g_2) \mapsto g_1 \cdot g_2 \qquad\qquad g_1 \to g_1^{-1}$$

Examples

$(\mathbb{R}^3, \cdot)$; $(\mathbb{R}^n, +)$; $GL(n; \mathbb{C})$

Thm: Every closed subgroup H of $\overset{\text{a Lie group}}{G}$ is a Lie group.

$$GL \subset SL \subset SU \subset SO$$

$$\cancel{O \subset SO}$$

If G is a Lie group and $H \subset G$, one might define a equivalence relation $\sim$ s.t.

$$g' \sim g \quad \text{if } \exists\, h \in H \text{ s.t. } g' = gh$$

$[g]$ is called an equivalence class

G/H is also a manifold called coset manifold

If H is a normal subgroup of G then G/H has a Lie group structure

$\{ (hg' = g'h'') \ \forall g' \in G$ for some h, h'' in H

One can define the action of an element $a \in G$ on the manifold in two ways.

$$L_a g = ag \leftarrow \text{left action}$$
$$R_a g = ga \leftarrow \text{right action}$$

these maps induce actions on $T_g G$

$$L_{a*} : T_g G \longrightarrow T_{ag} G$$
$$R_{a*} : T_g G \longrightarrow T_{ga} G$$

Def

A vector X on G is said to be a left invariant vector field if

$$L_{a*} X|_g = X|_{ag}$$

Examples

Let $x \mapsto x+a$ ~~is generated by~~ $\cancel{X =}$ be the transformation L_a, and $X = \partial_x$

$$L_{a*} X|_x = \underbrace{\frac{\partial (x+a)}{\partial x}}_{1} \frac{\partial}{\partial (x+a)} = \frac{\partial}{\partial (x+a)} = X|_{L_a x} \checkmark$$

Then X is a L.I.V.

NOTE:

Since the manifold and the group are equivalents, the group action on a point of the manifold yields another point. Therefore, Left invariant vector field are somehow vector which are defined on the whole manifold

Since $f_*[X,Y] = [f_*X, f_*Y]$,
it follows that

$$L_{a*}[X,Y]\big|_g = [X,Y]\big|_{ag} \qquad (*)$$

if X & Y are left-invariant vector fields.

~~It implies [,] is closed under say left-differen.~~

Let $\mathfrak{g}$ be the set of l.i.v.f. on G.

The map $T_eG \to \mathfrak{g}$ is an isomorphism

$$\therefore \quad \dim(\mathfrak{g}) = \dim(G)$$

$(*) \to \mathfrak{g}$ is closed under $[\,,\,]$ action.

Def.
The set of left invariant vector field $\mathfrak{g}$ with the Lie bracket $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called Lie algebra of the lie group G

Relation between 1-param. subgroup - flow

A ~~1-param.~~ curve $\phi : \mathbb{R} \to G$ is called a 1p subg.

if

$$\phi(t)\,\phi(s) = \phi(t+s)$$

Given ϕ, $\exists$ a vector field X s.t.

$$\frac{d}{dt}\phi^\mu(t) = X^\mu(\phi(t))$$ ← ϕ is a flow

and X is left-invariant vector field.

It follows from the flow definition that

$$\phi_A(t) = e^{tA}$$

group element

left-invariant vector field (algebra element)

Frames & structure equation

Given a basis of left-invariant vector fields $\mathfrak{g} = \{X_i|_g\}$,

since

$$[X_a, X_b]|_g \in \mathfrak{g} \Rightarrow [X_a, X_b]|_g = C_{ab}{}^c(g)\, X_c|_g$$

$C_{ab}{}^c(g)$ are in fact constants $C_{ab}{}^c(g) = C_{ab}{}^c$ and are

called structure constants of the Lie group G

One might introduce a dual basis to $\{X_a\}$ denoted by $\{\theta^a\}$ s.t.

$$\langle \theta^a | X_b \rangle = \delta^a{}_b$$

θ's are ~~called~~ said to be left-invariant oneforms.

Maurer-Cartan's structure equation

$$d\theta^a = -\frac{1}{2} C_{bc}{}^a \, \theta^b \wedge \theta^c$$

{ Show the above eq. explicitly by letting act on (X_m^a, X_n^a) }

One defines a Lie-algebra valuated 1-form $\theta: T_gG \longrightarrow T_eG$ by

$$\theta : X \longrightarrow (L_{g^{-1}})_* X \qquad \text{for} \quad X \in T_gG$$

θ is called canonical one-form or

Maurer-Cartan form

Thm a) The canonical 1-form is expanded as
$$\theta = V_\mu \otimes \theta^\mu \quad \text{where} \quad V_\mu \overset{\text{basis}}{\in} T_eG \ \& \ \theta^\mu \in T_e^*G \text{ are their dual}$$

b) θ satisfies $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$

$$d\theta = V_\mu \otimes d\theta^\mu$$
$$[\theta \wedge \theta] = [V_\mu, V_\nu] \otimes \theta^\mu \wedge \theta^\nu$$

{ Prove the THM!!! }