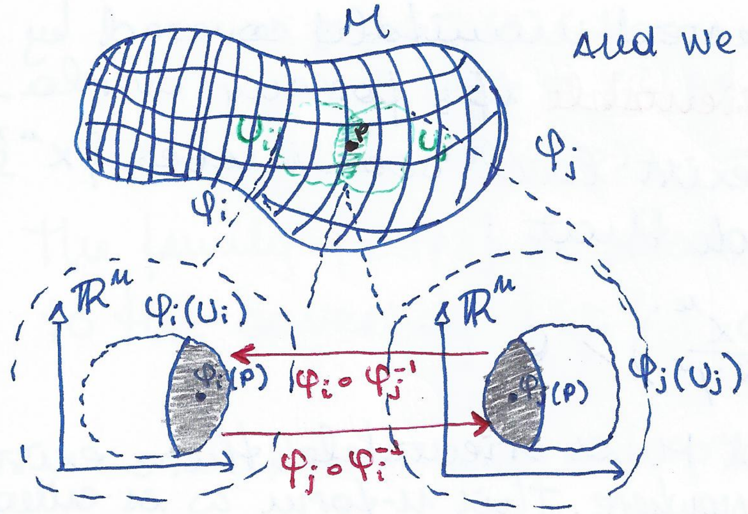
The integral of forms:

$$\int_{\Sigma_i} : \omega \longrightarrow \mathbb{R}.$$



And we have two overlapping charts. (U_i, φ_i) and (U_j, φ_j)

Where, chart U_i have coordinates $\varphi_i(p) = x^\mu$ and U_j have coordinates $\varphi_j(p) = y^\nu$

Now: $\omega_{(n)} = h(p)\, dx^1 \wedge \ldots \wedge dx^n$

but $dx^1 \wedge .. \wedge dx^n = \frac{1}{n!} \epsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge .. \wedge dx^{\mu_3}$

EX: 3-dim case: $dx^1 \wedge dx^2 \wedge dx^3 \overset{?}{=} \frac{1}{3!} \epsilon_{abc}\, dx^a \wedge dx^b \wedge dx^c$

$= \frac{1}{3!}\big(\epsilon_{123}\, dx^1 \wedge dx^2 \wedge dx^3 + \epsilon_{132}\, dx^1 \wedge dx^3 \wedge dx^2 + ..$

$.. + \epsilon_{213}\, dx^2 \wedge dx^1 \wedge dx^3 + \epsilon_{231}\, dx^2 \wedge dx^3 \wedge dx^1 + ..$

$.. + \epsilon_{312}\, dx^3 \wedge dx^1 \wedge dx^2 + \epsilon_{321}\, dx^3 \wedge dx^2 \wedge dx^1 \big)$

$= \cancel{\frac{1}{3!}}\big(\epsilon_{123}\, dx^1 \wedge dx^2 \wedge dx^3 \cdot \cancel{(6)} \big) = + dx^1 \wedge dx^2 \wedge dx^3 .$

Now: $\omega_{(n)} = h(p) \frac{1}{n!} \epsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$

And, a change of coordinates from: $x^\mu \rightarrow y^\nu$

$$\omega_{(n)} = h(p) \frac{1}{n!} \epsilon_{\mu_1 \ldots \mu_n} \frac{\partial x^{\mu_1}}{\partial y^{\nu_1}} dy^{\nu_1} \wedge \ldots \wedge \frac{\partial x^{\mu_n}}{\partial y^{\nu_n}} dy^{\nu_n}$$

but, we also know: $\epsilon_{\mu_1' \ldots \mu_n'} \det M = \epsilon_{\mu_1 \ldots \mu_n} M^{\mu_1}{}_{\mu_1'} \ldots M^{\mu_n}{}_{\mu_n'}$

$$\omega_{(m)} = \frac{h(p)}{m!} \epsilon_{\nu_1 \dots \nu_m} \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^{\nu_1} \wedge \dots \wedge dy^{\nu_m}$$

$$\omega_{(m)} = h(p) \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^1 \wedge \dots \wedge dy^m$$

Definition: Let M be a connected manifold covered by $\{U_i\}$. The manifold M is orientable if, for any overlapping charts U_i and U_j, there exist local coordinates $\{x^\mu\}$ for U_i and $\{y^\nu\}$ for U_j, such that:

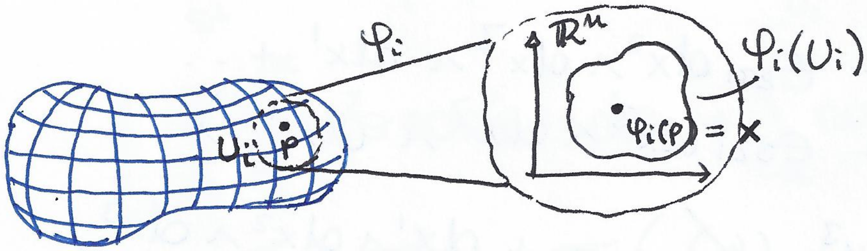
$$J \equiv \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0$$

→ If an m-dimensional manifold M is orientable, there exist an m-form ω which vanishes nowhere. this m-form ω is called volume element, which plays the role of a measure when we integrate a function f over M.

Now;

In a coordinate neighbourhood U_i with coordinate x, we define the integration of an m-form ω.

$\omega_{(m)} = h(p)\, dx^1 \wedge \dots \wedge dx^m$ on U_i where $\omega(x) \neq 0$.



$$\int_{U_i} \omega = \int_{\varphi(U_i)} h(\underbrace{\varphi_i^{-1}(x)}_{p})\, dx^1 \dots dx^m$$

Where RHS is an ordinary multiple integration of m-variable function. Once the integral of ω over U_i is defined. Now the integral over whole M is given by partition of unity.

...inition: take a covering $\{U_i\}$ of M such that each point of M is covered with a finite number of U_i (if this always posible, M is called paracompact). If a family of differentiable function $\epsilon_i(p)$ satisfies:

i) $0 \leqslant \epsilon_i(p) \leqslant 1$

ii) $\epsilon_i(p) = 0$ si $p \notin U_i$

iii) $\epsilon_1(p) + \epsilon_2(p) + .. = 1$ for any $p \in M$.

The family $\{\epsilon_i(p)\}$ is called partition of unity subordinate to the covering $\{U_i\}$

$$\text{Now}: \quad \omega_{(m)} = h(p)\, dx^1 \wedge .. \wedge dx^m$$

$$\rightarrow \omega_{(m)} = \sum_i{}' \omega_{(m)} \cdot \epsilon_i(p)$$

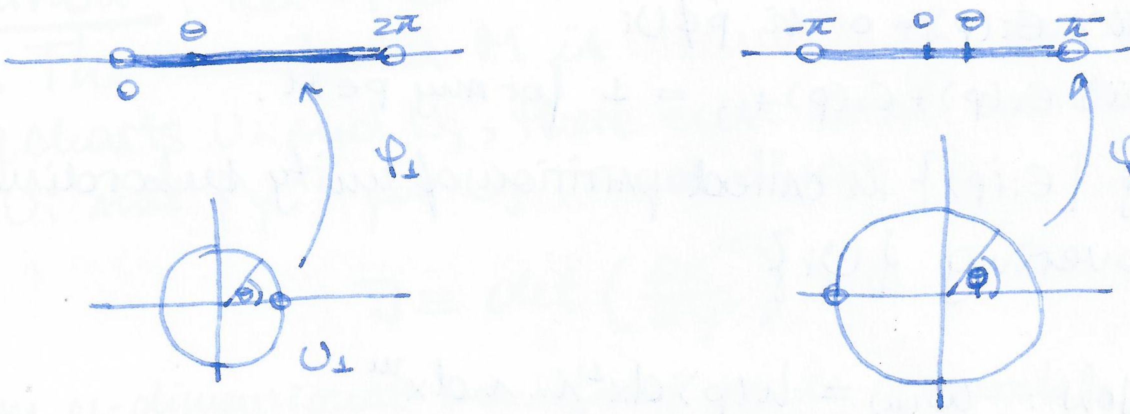
$$= \sum_i{}' h(p)\epsilon_i(p)\, dx^1 \wedge .. \wedge dx^m$$

$$\equiv \sum_i{}' h_i(p)\, dx^1 \wedge .. \wedge dx^m$$

$$\text{with} \quad h_i(p) \equiv h(p)\epsilon_i(p)$$

$$\text{Now}: \quad \boxed{\sum_i{}' \int_{\varphi_i(U_i)} h_i(\varphi_i^{-1}(x))\, dx^1 .. dx^m = \int_M \omega_{(m)}}$$

Example: Let's consider a $m=1$ dimension manifold. There are two possible manifolds: the $\mathbb{R}$-line and the circle S^1. Let's work out an atlas of S^1. For concreteness, take the circle $x^2+y^2=1$ in the xy-plane. We need at least two charts.



With $\varphi_1^{-1}: \theta \longrightarrow (\cos\theta, \sin\theta)$ whose image is $S^1 - \{(1,0)\}$
$\varphi_2^{-1}: \theta \longrightarrow (\cos\theta, \sin\theta)$ whose image is $S^1 - \{(-1,0)\}$

And consider $\epsilon_1(\theta) = \cos^2\theta/2$ and $\epsilon_2(\theta) = \sin^2\theta/2$. Let us integrate the function: $f(\theta) = \cos^2\theta$ for example

$$\int_{S^1} \omega = \sum_i \int f(\theta)\,d\theta = \int_{U_1} f(\theta)\epsilon_1(\theta)\,d\theta + \int_{U_2} f(\theta)\epsilon_2(\theta)\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \cos^2\theta/2\,d\theta + \int_{-\pi}^{\pi} \cos^2\theta \sin^2\theta/2\,d\theta$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

And we also know by direct integration:

$$\boxed{\int_0^{2\pi} \cos^2\theta\,d\theta = \pi}$$