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The action of Lie groups on manifolds

It is a generalisation of flows

Def

Let G be a Lie group & M be a manifold. The action of G on M is a diff. map

$$\sigma : G \times M \to M$$

satisfying

$$\sigma(e,p) = p \qquad \forall \ p \in M$$

$$\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p) .$$

{ Usually will be denoted gp or $g_1 g_2 p$ }

The action σ is said to be

1) transitive: $\forall \ p_1, p_2 \in M$ $\exists!$ g s.t. $\sigma(g, p_1) = p_2$

2) free: for $p \in M$ $\sigma(g,p) = p \Rightarrow g = e$

no fixed points

3) effective: $\forall \ p \in M$ $\sigma(g,p) = p \Rightarrow g = e$

Other concepts

The <u>orbit</u> of p under the action σ is a subset of M

$$G_p = \{\sigma(g,p) \mid g \in G\}$$

The <u>isotropy group</u> of $p \in M$ is a subgroup of G defined by

$$H(p) = \{g \in G \mid \sigma(g,p) = p\}$$

also called stabilizer or little group of p.

Let G be a Lie group and H a subgroup of G. the coset G/H admits a differentiable structure and becomes a manifold, called homogeneous manifold

Matrix valued diff. forms

Can be thought as matrices whose entries are diff. forms on M

$$R = (R_{ij}) \qquad R(v) = (R_{ij}(v))$$

$$dR = (dR_{ij}) \qquad (R \wedge S)_{ik} = R_{ij} \wedge S_{jk}$$

In general $\mathrm{Mat}(n) \simeq n^2$ dim. space.

In the restriction to $GL(n)$, one might define

R^{-1} ~~K~~ ~~R~~ matrix valued diff. form.

finally, for a map $f: M \longrightarrow N$

$$(f^*R)_{ij} = f^*(R_{ij})$$

Calculate dA^{-1}, where A^{-1} is the matrix valued function on $GL(n)$ assigning to each invertible matrix its inverse

Action of groups on themselves.

- Left action:
$$L_a : G \to G \qquad L_a m = am$$

- Right action:
$$R_a : G \to G \qquad R_a m = ma^{-1}$$

They induce maps on TG

$$L_{a*} : T_m G \longrightarrow T_{am} G$$
$$R_{a*} : T_m G \longrightarrow T_{ma^{-1}} G$$

The Maurer-Cartan form

Let A be a matrix of functions on G, & dA a matrix of differential forms.
Consider
$$A^{-1} dA \qquad (\equiv \omega)$$

* Show that for a fixed $g \in G$
$$L_g^* (A^{-1} dA) = A^{-1} dA$$

* Show that
$$R_g^* (A^{-1} dA) = g (A^{-1} dA) g^{-1}$$

* Show that
$$d\omega + \omega \wedge \omega = 0$$

Restriction to a subgroup of $GL(n)$.

If H is a Lie subgroup of G, then H is a submanifold

$$i : H \longrightarrow G \quad \text{is an embedding.}$$

Thus $\quad i^*(A^{-1}dA) = \omega \quad . \in \Lambda^1(\mathfrak{g})$

Example

If $H = O(n)$, then $A(t)$ a curve on $O(n)$ satisfy

$$A(t)\, A^T(t) = \mathbb{1} \qquad A(t=0) = \mathbb{1}$$

$$\rightarrow \quad A'(0) + (A^T)'(0) = 0$$

or $\quad A'(0)$ is antisymmetric.

~~Since any 1-param. subgroup can be written as $e^{t\xi}$~~

Then $\mathfrak{h}$ consists of anti-symmetric matrices

$$\omega = \{ \Omega \in \mathfrak{gl}(n) \mid \Omega = -\Omega^T \}$$

FRAMES.

Let V be a n-dimensional vector space. A frame on V
is an isomorphism

$$f : \mathbb{R}^n \rightarrow V$$

$$f = (f_1, \_\_\_, f_n).$$

Two different frames are related by a $GL(n)$ tra

Then, $F \equiv \{f\} \simeq GL(n)$

$$\therefore \quad df = f\omega$$

linear combination

s.t.

$$d(df) = df \wedge \omega + f \wedge d\omega$$

$$0 = f \wedge (\omega \wedge \omega + d\omega)$$

nilpotency

$$\therefore \quad d\omega + \omega \wedge \omega = 0$$

Euclidean frames

Let $V = \mathbb{R}^n$ with $n = d+1$, be a vector space, the set of frames is identified with $GL(n)$.

One might restrict to Euclidean motion

$$\left(\begin{array}{c|c} R & v \\ \hline 0 & 1 \end{array}\right) \quad \text{for} \quad R \in O(d) \ \& \ v \in \mathbb{R}^d$$

s.t. acting on $\begin{pmatrix} w \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} Rw+v \\ 1 \end{pmatrix}$

Define a frame

$$f_i = \left(\begin{array}{c} e_i \\ \hline 0 \end{array}\right)_{i=1,\dots,d} \quad \& \quad f_n = \left(\begin{array}{c} v \\ \hline 1 \end{array}\right)$$

or $f = \left(\begin{array}{c|c} e & v \\ \hline & 1 \end{array}\right)$.

Denote

$$i^*\omega = \left(\begin{array}{c|c} \Omega & \theta \\ \hline 0 & 0 \end{array}\right) \quad \text{with} \quad \Omega_{ij} = -\Omega_{ji}$$

Thus $d\left(\begin{array}{c|c} e & v \\ \hline & 1 \end{array}\right) = \left(\begin{array}{c|c} e\Omega & e\theta \\ \hline 0 & 0 \end{array}\right)$

or $\begin{cases} de_j = \Omega_{ij}\, e_i = -(\Omega \wedge e)_j \\ dv = \theta_i\, e_i \end{cases}$

Using orthogonality of e_i

$$\Omega_{ij} = (de_j, e_i) \quad \& \quad \theta_i = (dv, e_i)$$

$$d(dv) = 0 \implies d\theta_i + (\Omega \wedge e)_i = 0$$
$$d(\Omega)$$

In conclusion,

$$d\Omega^i{}_j + \Omega^i{}_k \wedge \Omega^k{}_j = 0$$

$$d\theta^i + \Omega^i{}_j \wedge e^j = 0$$

structure equations for Euclidean geometry

- Derive the last equations.
- Show that the general M.C. equation is
$$d\omega + \tfrac{1}{2}[\omega \wedge, \omega] = 0 \qquad \omega \in \Lambda^1(\mathfrak{g})$$
- Guided problem:

Structure eqs. for a $S^2 \subset \mathbb{R}^3$.