The first step of the algorithm is to find the cohomology spaces whose dimensions are not equal to the number of A_1 singularities of f. This can be done with Linear Algebra and is implemented through the code in this document. Notation: Let $P_j\,\Omega^k$ be the set of all k-forms whose coefficients are homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3]$ of degree j. Let d be the degree of f, then A[f,n] is the matrix representation of the koszul differential $df\wedge : P_n\,\Omega^3 \to P_{n+d-1}\,\Omega^4$, i.e., A[f,n] returns a $4\binom{n+3}{3}$ by $\binom{n+d-1+3}{3}$ matrix, where n is some positive integer. Similarly the function B[f,n] is the matrix representation of $df\wedge : P_n\,\Omega^2 \to P_{n+d-1}\,\Omega^3$ and therefore B[f,n] returns a $6\binom{n+3}{3}$ by $4\binom{n+d-1+3}{3}$ matrix. Note that the 4 in $4\binom{n+3}{3}$ comes from the fact that there are four 3-forms $dx_0\wedge dx_1\wedge dx_2$, $dx_0\wedge dx_1\wedge dx_3$, $dx_0\wedge dx_2\wedge dx_3$, $dx_1\wedge dx_2\wedge dx_3$ and similarly the 6 in $6\binom{n+3}{3}$ is because there are six 2-forms $dx_0\wedge dx_1$, $dx_0\wedge dx_2$, $dx_0\wedge dx_3$, $dx_1\wedge dx_2$, $dx_1\wedge dx_3$, $dx_2\wedge dx_3$. Here we are using the lexicographic order and this is the order we stick with throughout all computations.

```
In[1]:= allExp[n_] := Module[{v = {}}, For[i = 1, i ≤ Length[IntegerPartitions[n, 4]], i++,
    v = Append[v, Permutations[PadRight[IntegerPartitions[n, 4][[i]], 4]]]];
   v = Reverse[Sort[ArrayFlatten[v, 1]]]]
polyToVec[f_, list_] := Module[{a = ConstantArray[0, Length[list]]}, If[
    Depth[MonomialList[f, {x0, x1, x2, x3}]] == 2, a = a, v = MonomialList[f, {x0, x1, x2, x3}];
    pos = Table[Position[list, Table[Exponent[v[[k]], xi], {i, 0, 3}]][[1, 1]],
      {k, 1, Length[v]}];
    coeff = v /. {x0 → 1, x1 → 1, x2 → 1, x3 → 1};
    For[i = 1, i ≤ Length[v], i++,
     a = ReplacePart[a, pos[[i]] → coeff[[i]]]
    ]]; a]
vecToPoly[v_, list_] := Table[Total[Table[(Product[xj^list[[Mod[k-1,Length[list]]+1,j+1]], {j, 0, 3}]) v[[k]],
     {k, Length[list] (m - 1) + 1, Length[list] m}]], {m, 1, 4}]
(* Assuming that f is homogeneous, this function finds the degree of f
 by finding the degree of the first monomial that appears in f. *)
homExp[f_] := Total[
  Table[Exponent[MonomialList[f[x0, x1, x2, x3], {x0, x1, x2, x3}][[1]], xi], {i, 0, 3}]]
```

In[5]:=
```
A[f_, n_] :=
  ArrayFlatten[Table[Table[polyToVec[(-1)^m (∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_m],
      allExp[n + homExp[f] - 1]], {k, 1, Length[allExp[n]]}], {m, 3, 0, -1}], 1];
B[f_, n_] := Module[{B = {}}, L = Length[allExp[n + homExp[f] - 1]];
  B = Join[(*df^dx_0^dx_1*) Table[Join[polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3],
          x_2], allExp[n + homExp[f] - 1]], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3],
          x_3], allExp[n + homExp[f] - 1]], ConstantArray[0, 2 L]], {k, 1, Length[allExp[n]]}],
    (*df^dx_0^dx_2*) Table[Join[polyToVec[-(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_1],
        allExp[n + homExp[f] - 1]], ConstantArray[0, L], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]])
         D[f[x_0, x_1, x_2, x_3], x_3], allExp[n + homExp[f] - 1]], ConstantArray[0, L]],
     {k, 1, Length[allExp[n]]}], (*df^dx_0^dx_3*) Table[Join[ConstantArray[0, L],
      polyToVec[-(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_1], allExp[n + homExp[f] - 1]],
      polyToVec[-(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_2], allExp[n + homExp[f] - 1]],
      ConstantArray[0, L]], {k, 1, Length[allExp[n]]}], (*df^dx_1^dx_2*) Table[
     Join[polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_0], allExp[n + homExp[f] - 1]],
      ConstantArray[0, 2 L], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_3],
       allExp[n + homExp[f] - 1]]], {k, 1, Length[allExp[n]]}], (*df^dx_1^dx_3*)
    Table[Join[ConstantArray[0, L], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]]) D[f[x_0, x_1, x_2, x_3], x_0],
       allExp[n + homExp[f] - 1]], ConstantArray[0, L], polyToVec[-(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]])
        D[f[x_0, x_1, x_2, x_3], x_2], allExp[n + homExp[f] - 1]]], {k, 1, Length[allExp[n]]}],
    (*df^dx_2^dx_3*) Table[Join[ConstantArray[0, 2 L], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]])
         D[f[x_0, x_1, x_2, x_3], x_0], allExp[n + homExp[f] - 1]], polyToVec[(∏_{j=0}^{3} x_j^allExp[n][[k,j+1]])
         D[f[x_0, x_1, x_2, x_3], x_1], allExp[n + homExp[f] - 1]]], {k, 1, Length[allExp[n]]}]];
  B]
```

In[7]:=
```
deRham[v_] := -D[v[[1]], x3] + D[v[[2]], x2] - D[v[[3]], x1] + D[v[[4]], x0]
(* above is the de Rham differential for 3-forms ONLY. *)
(* koszul of a 3-form *)
koszul[v_] := v[[4]] * D[f[x0, x1, x2, x3], x0] - v[[3]] * D[f[x0, x1, x2, x3], x1] +
  v[[2]] * D[f[x0, x1, x2, x3], x2] - v[[1]] * D[f[x0, x1, x2, x3], x3]
f[x0_, x1_, x2_, x3_] := -3 x0^2 x1 x2 + 4 x0^2 x3^2 - 3 x0 x1^2 x2 +
  4 x0 x1 x3^2 + x1^2 x3^2 - 6 x1 x2 x3^2 - 3 x2^4
```

To get a better picture of the functions defined above let us consider the following example.

In[10]:=
```
A[f, 0] // MatrixForm
Dimensions[A[f, 0]]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -8 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -12 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 6 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 8 & 0 & -3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[11]= {4, 20}

The matrix A[f,0] represents the map df ^ : $P_0\,\Omega^3 \to P_3\,\Omega^4$ and has $4\binom{0+3}{3} = 4$ rows and $\binom{0+4-1+3}{3} = 6!/(3!\times 3!) = 20$ columns. The first row represents $df \wedge dx_0 \wedge dx_1 \wedge dx_2 = -f_3\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ where $f_3 = \partial f / \partial x_3$. The partial $-f_3$ is calculated below.

In[12]:=
```
-D[f[x0, x1, x2, x3], x3]
```

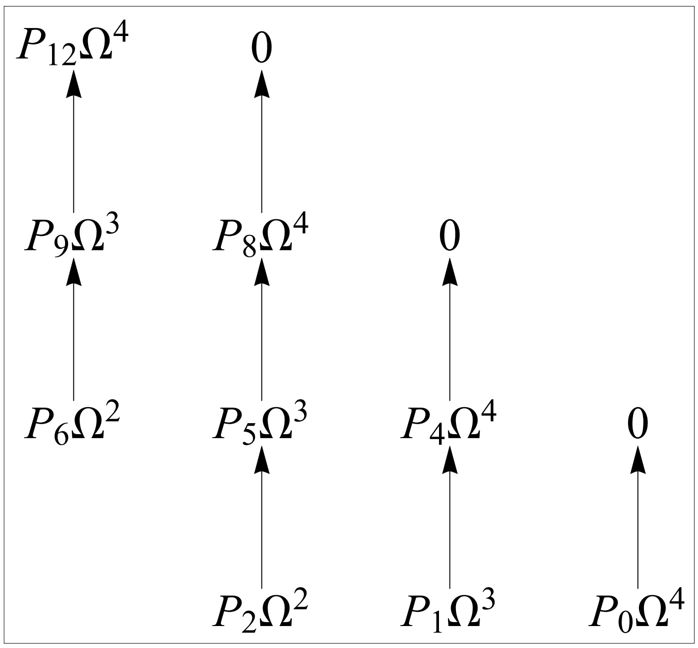
Out[12]= $-8\, x_0^2\, x_3 - 8\, x_0\, x_1\, x_3 - 2\, x_1^2\, x_3 + 12\, x_1\, x_2\, x_3$

The non-zero coefficients of $-f_3$ are seen in the first row of A[f,0] and what column they are in is based off of the lexicographic ordering of monomials $x_0 > x_1 > x_2 > x_3$. With this ordering the first 4 monomials of degree 3 are x_0^3, $x_0^2\, x_1$, $x_0^2\, x_2$, $x_0^2\, x_3$. That is why there is a -8 in the first row, fourth column of A[f,0], to represent the term $-8\, x_0^2\, x_3$. The second row of A[f,0] is $df \wedge dx_0 \wedge dx_1 \wedge dx_3 = f_2\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ and so forth. For A[f,1] the first row is $df \wedge (x_0\, dx_0 \wedge dx_1 \wedge dx_2)$ and the matrix B[f,n] is constructed in the same way, however the first $\binom{n+d-1+3}{3}$ columns represent the 3-form $dx_0 \wedge dx_1 \wedge dx_2$, the next $\binom{n+d-1+3}{3}$ columns represent the 3-form $dx_0 \wedge dx_1 \wedge dx_3$ and so on (the lexicographical ordering is always used).

Now since $f = -3\, x_0^2\, x_1\, x_2 + 4\, x_0^2\, x_3^2 - 3\, x_0\, x_1^2\, x_2 + 4\, x_0\, x_1\, x_3^2 + x_1^2\, x_3^2 - 6\, x_1\, x_2\, x_3^2 - 3\, x_2^4$ is a quartic we are only interested in the modules $P_j\,\Omega^k$ where $j+k$ is a multiple of 4. The picture below shows these modules, where the map is the Koszul differential, df^.

In[13]:=
```
coor = {{1, 0}, {0, 0}, {-1, 0}, {0, 1}, {-1, 1},
   {-2, 1}, {-1, 2}, {-2, 2}, {-2, 3.02}, {1, 1}, {0, 2}, {-1, 3}};
str = {"P_0Ω^4", "P_1Ω^3", "P_2Ω^2", "P_4Ω^4", "P_5Ω^3", "P_6Ω^2", "P_8Ω^4", "P_9Ω^3", "P_12Ω^4", "0", "0", "0"};
Graphics[
 {Table[Inset[Style[str[[k]], FontSize → Scaled[1/15], FontFamily → "Times New Roman"],
     coor[[k]]], {k, 12}],
  Table[Arrow[{coor[[k]] + {0, 0.12}, coor[[k]] + {0, 0.88}}], {k, 8}]},
 PlotRange → {{-2.37, 1.3}, {-0.17, 3.22}}, Frame → True, FrameTicks → None]
```

Out[15]=



By inspection $\dim(\ker(P_0\,\Omega^4 \to 0)) = 0$ since the one basis element $dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ is sent to zero. For $\dim(\ker(P_1\,\Omega^3 \to P_4\,\Omega^4))$ we must call on Mathematica's NullSpace[] and Transpose[] functions. Note that by the definition of A[f,n] and B[f,n] multiplication is defined on the left so this is why we need the Transpose[] function.

In[16]:=
```
Length[NullSpace[Transpose[A[f, 1]]]]
```

Out[16]= 0

Hence $\dim(\ker(P_1\,\Omega^3 \to P_4\,\Omega^4)) = 0$. Now for the dimensions of the other spaces.

In[17]:=
```
{Length[NullSpace[Transpose[A[f, 5]]]] - MatrixRank[B[f, 2]],
 Binomial[3 + 4, 3] - MatrixRank[A[f, 1]]}
{Length[NullSpace[Transpose[A[f, 9]]]] - MatrixRank[B[f, 6]],
 Binomial[3 + 8, 3] - MatrixRank[A[f, 5]]}
```

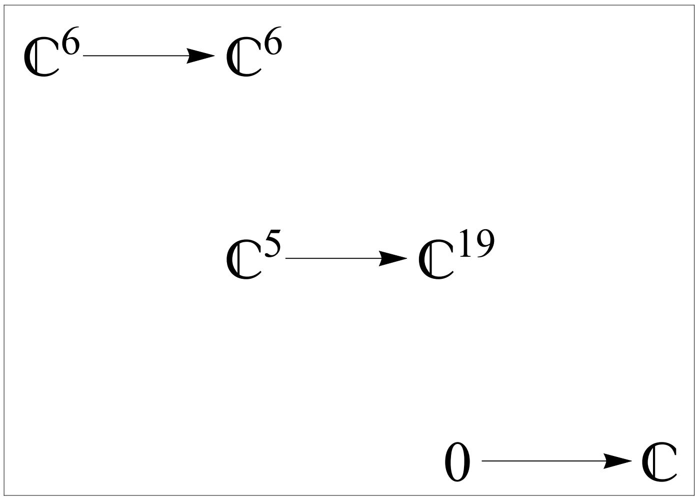
Out[17]= {5, 19}

Out[18]= {6, 6}

We now have the dimensions of all six spaces mentioned in step 1 of the algorithm and display them in the E_1 page below.

In[19]:=
```
strE1 = {"ℂ", "0", "ℂ19", "ℂ5", "ℂ6", "ℂ6"};
coorE1 = {{1, 0}, {0, 0}, {0, 1}, {-1, 1}, {-1, 2}, {-2, 2}};
Graphics[
 {Table[Inset[Style[strE1[[k]], FontSize → Scaled[1/12], FontFamily → "Times New Roman"],
    coorE1[[k]]], {k, 6}], {Arrow[{{0.12, 0}, {0.86, 0}}]},
  {Arrow[{{-0.85, 1}, {-0.25, 1}}]}, {Arrow[{{-1.85, 2}, {-1.2, 2}}]}},
 PlotRange → {{-2.24, 1.17}, {-0.17, 2.25}}, Frame → True, FrameTicks → None]
```

Out[21]=



Since $\dim(H^3(K_f^\bullet)_9) = \dim(H^4(K_f^\bullet)_8) = 6 = \#\mathrm{Sing}(f)$ we do not have to find bases for these spaces. Moving down one "level" we have $\dim(H^3(K_f^\bullet)_5) = 5$ and $\dim(H^4(K_f^\bullet)_4) = 19$ so we must find bases for these spaces. The simplest way to do this is to add the rows of NullSpace[Transpose[A[f,5]]] to B[f,2] and see if they increase the rank of the resulting matrix. The following code shows how this can be done.

In[22]:=
```
MatrixRank[B[f, 2]]
N5 = NullSpace[Transpose[A[f, 5]]];
MatrixRank[Join[{N5[[1]], N5[[2]], N5[[3]], N5[[5]], N5[[6]]}, B[f, 2]]]
```

Out[22]= 60

Out[24]= 65

These 5 elements give us a basis for $H^3(K_f^\bullet)_5$ but computationally we can do better because these 3-forms are too "big." To see why, let
$\{h_1, h_2, h_3, h_4\} = h_1\, dx_0 \wedge dx_1 \wedge dx_2 + h_2\, dx_0 \wedge dx_1 \wedge dx_3 + h_3\, dx_0 \wedge dx_2 \wedge dx_3 + h_4\, dx_1 \wedge dx_2 \wedge dx_3$, then the following 3-form is an element of $H^3(K_f^\bullet)_5 = \ker(P_5\,\Omega^3 \to P_8\,\Omega^4)/\mathrm{im}(P_2\,\Omega^2 \to P_5\,\Omega^3)$.

In[25]:=
```
vecToPoly[N5[[1]], allExp[5]]
```

Out[25]=
$$\{-4160\, x_0^4 x_3 - 5244\, x_0^3 x_1 x_3 + 988\, x_0^2 x_1^2 x_3 + 534\, x_0 x_1^3 x_3 + 4\, x_1^4 x_3 - 354\, x_0^3 x_2 x_3 - 297\, x_0^2 x_1 x_2 x_3 + 72\, x_0 x_1^2 x_2 x_3 + 12\, x_1^3 x_2 x_3 - 144\, x_0^2 x_2^2 x_3 - 144\, x_0 x_1 x_2^2 x_3 + 72\, x_1^2 x_2^2 x_3 - 8752\, x_0^2 x_3^3 - 2384\, x_0 x_1 x_3^3 + 146\, x_1^2 x_3^3 - 708\, x_0 x_2 x_3^3 - 210\, x_1 x_2 x_3^3 - 288\, x_2^2 x_3^3 - 432\, x_3^5,$$
$$-288\, x_0^3 x_3^2 - 432\, x_0^2 x_1 x_3^2 + 72\, x_1^3 x_3^2 - 576\, x_0 x_3^4 - 288\, x_1 x_3^4,\; -8320\, x_0^4 x_1 - 10\,704\, x_0^3 x_1^2 - 4588\, x_0^2 x_1^3 - 666\, x_0 x_1^4 - 4\, x_1^5 + 900\, x_0^3 x_1 x_2 + 450\, x_0^2 x_1^2 x_2 + 576\, x_0^2 x_1 x_2^2 + 288\, x_0 x_1^2 x_2^2 - 2144\, x_0^3 x_3^2 + 14\,384\, x_0^2 x_1 x_3^2 + 2956\, x_0 x_1^2 x_3^2 - 362\, x_1^3 x_3^2 - 1152\, x_0^2 x_2 x_3^2 - 1152\, x_0 x_2^2 x_3^2 + 864\, x_0 x_3^4 + 1296\, x_1 x_3^4,$$
$$-4160\, x_0^5 - 11\,592\, x_0^4 x_1 - 7202\, x_0^3 x_1^2 - 1320\, x_0^2 x_1^3 - 8\, x_0 x_1^4 + 450\, x_0^4 x_2 + 900\, x_0^3 x_1 x_2 + 288\, x_0^3 x_2^2 + 576\, x_0^2 x_1 x_2^2 - 9824\, x_0^3 x_3^2 - 648\, x_2 x_3^4\}$$

The following 3-form is also in $H^3(K_f^{\bullet})_5$ and it has fewer terms with smaller coefficients (in $\mathbb{Z}$) than the 3-form above.

$$-x_0^2 x_3(2 x_0^2 + 2 x_0 x_1 - x_1^2 + 4 x_3)^2\, dx_0 \wedge dx_1 \wedge dx_2 -$$
$$x_0^2(4 x_0^2 x_1 + x_0(4 x_1^2 - 6 x_1 x_2 + 8 x_3^2) + x_1(x_1^2 - 3 x_1 x_2 - 4 x_3^2))\, dx_0 \wedge dx_2 \wedge dx_3) +$$
$$x_0^2(-2 x_0^3 + x_0^2(-5 x_1 + 3 x_2) + 6 x_2 x_3^2 - 2 x_0(x_1^2 - 3 x_1 x_2 + 4 x_3^2))\, dx_0 \wedge dx_2 \wedge dx_3$$

We now explain where this 3-form comes from. First let us find the smallest j such that $\ker(P_j\, \Omega^3 \to P_{j+3}\, \Omega^4) \neq 0$.

In[26]:=
```
Table[Length[NullSpace[Transpose[A[f, k]]]], {k, 0, 3}]
```

Out[26]= {0, 0, 0, 7}

Hence $\ker(P_j\, \Omega^3 \to P_{j+3}\, \Omega^4) = 0$ for $j = 0, 1, 2$ and $\ker(P_3\, \Omega^3 \to P_6\, \Omega^4) \neq 0$.

In[27]:=
```
n1 = NullSpace[Transpose[A[f, 3]]][[1]];
vecToPoly[n1, allExp[3]]
MatrixRank[B[f, 0]]
MatrixRank[Join[{n1}, B[f, 0]]]
```

Out[28]=
$$\{-2\, x_0^2 x_3 - 2\, x_0 x_1 x_3 + x_1^2 x_3 - 4\, x_3^3,\; 0,\; -4\, x_0^2 x_1 - 4\, x_0 x_1^2 - x_1^3 + 6\, x_0 x_1 x_2 + 3\, x_1^2 x_2 - 8\, x_0 x_3^2 + 4\, x_1 x_3^2,\; -2\, x_0^3 - 5\, x_0^2 x_1 - 2\, x_0 x_1^2 + 3\, x_0^2 x_2 + 6\, x_0 x_1 x_2 - 8\, x_0 x_3^2 + 6\, x_2 x_3^2\}$$

Out[29]= 6

Out[30]= 7

From the computations above we can see that $n_1 \in H^3(K_f^{\bullet})_3 = \ker(P_3\, \Omega^3 \to P_6\, \Omega^4) / \operatorname{im}(P_3\, \Omega^3 \to P_6\, \Omega^4)$. However we are not interested in $H^3(K_f^{\bullet})_3$, but in $H^3(K_f^{\bullet})_5$. Recall that the Koszul differential is $\mathbb{C}[x_0, x_1, x_2, x_3]$ –linear so if $df \wedge \omega = 0$, then $df \wedge (g(x_0, x_1, x_2, x_3)\, \omega) = g(x_0, x_1, x_2, x_3)\, (df \wedge \omega) = 0$. Hence $x_0^2\, n_1 \in \ker(P_5\, \Omega^3 \to P_8\, \Omega^4)$ and now we must check if it is in $\operatorname{im}(P_2\, \Omega^2 \to P_5\, \Omega^3)$.

In[31]:=
```
B2 = B[f, 2];
MatrixRank[B2]
MatrixRank[Join[
  {ArrayFlatten[Table[polyToVec[x0^2 vecToPoly[n1, allExp[3]][[k]], allExp[5]], {k, 4}], 1],
   ArrayFlatten[Table[polyToVec[x0 x1 vecToPoly[n1, allExp[3]][[k]], allExp[5]], {k, 4}],
    1], ArrayFlatten[
    Table[polyToVec[x0 x3 vecToPoly[n1, allExp[3]][[k]], allExp[5]], {k, 4}], 1],
   ArrayFlatten[Table[polyToVec[x1^2 vecToPoly[n1, allExp[3]][[k]], allExp[5]], {k, 4}], 1],
   ArrayFlatten[
    Table[polyToVec[x3^2 vecToPoly[n1, allExp[3]][[k]], allExp[5]], {k, 4}], 1]}, B2]]
```

Out[32]= 60

Out[33]= 65

Therefore $\{x_0{}^2\, n_1, x_0\, x_1\, n_1, x_0\, x_3\, n_1, x_1{}^2\, n_1, x_3{}^2\, n_1\}$ is a basis for $H^3(K_f{}^\bullet)_5$. Now we find a basis for $H^4(K_f{}^\bullet)_4$.

In[34]:=
```
bTop = {};
mon = {};
k = 1;
While[Length[bTop] < 19,
 If[MatrixRank[Join[Table[polyToVec[bTop[[j]], allExp[4]], {j, 1, Length[bTop]}],
     A[f, 1], {polyToVec[Product[x_z^allExp[4][[k,z+1]], {z, 0, 3}], allExp[4]]}]] ==
   16 + Length[bTop] + 1, bTop = Append[bTop, Product[x_z^allExp[4][[k,z+1]], {z, 0, 3}]];
  mon = Append[mon, k]];
 k++]
bTop
```

Out[38]= $\{x_0^4, x_0^3\, x_1, x_0^3\, x_2, x_0^3\, x_3, x_0^2\, x_1^2, x_0^2\, x_1\, x_2, x_0^2\, x_1\, x_3, x_0^2\, x_2^2, x_0^2\, x_2\, x_3, x_0^2\, x_3^2, x_0\, x_1^3, x_0\, x_1^2\, x_2, x_0\, x_1^2\, x_3, x_0\, x_1\, x_2^2, x_0\, x_2^2\, x_3, x_0\, x_3^3, x_1^4, x_1^3\, x_3, x_3^4\}$

We now have bases for $H^3(K_f{}^\bullet)_5$ and $H^4(K_f{}^\bullet)_4$ but these leave something to be desired since de Rham of some 3-forms (in $H^3(K_f{}^\bullet)_5$) is not always a linear combination of the 19 monomials above. For instance $d(x_0\, x_3\, n_1)$ contains the monomials $x_0\, x_1\, x_2\, x_3$ and $x_2\, x_3{}^3$ which are not one of the 19. We show $d(x_0\, x_3\, n_1)$ below.

In[39]:=
```
Expand[deRham[x0 x3 vecToPoly[n1, allExp[3]]]]
PolynomialReduce[Expand[deRham[x0 x3 vecToPoly[n1, allExp[3]]]], bTop, {x0, x1, x2, x3}]
```

Out[39]= $-3\, x_0^2\, x_1\, x_3 - 3\, x_0\, x_1^2\, x_3 + 3\, x_0^2\, x_2\, x_3 + 6\, x_0\, x_1\, x_2\, x_3 - 4\, x_0\, x_3^3 + 6\, x_2\, x_3^3$

Out[40]= $\{\{0, 0, 0, 0, 0, 0, -3, 0, 3, 0, 0, 0, -3, 0, 0, -4, 0, 0, 0\}, 6\, x_0\, x_1\, x_2\, x_3 + 6\, x_2\, x_3^3\}$

Note that this is not a problem since we can write $6\, x_0\, x_1\, x_2\, x_3 + 6\, x_2\, x_3{}^3$ as a combination of the 19 monomial plus df^ some 3-form.

In[41]:=
```
Expand[2 x0^3 x3 + 3 x0^2 x1 x3 + 3/2 x0 x1^2 x3 + 1/4 x1^3 x3 - 3 x0^2 x2 x3 + koszul[{1/8 (2 x0 + x1), 0, x3, 1/2 x3}]]
```

Out[41]= $6\, x_0\, x_1\, x_2\, x_3 + 6\, x_2\, x_3^3$

Note that we have used fractions and had to incorporate the Koszul differential. It would be nice if there was a basis where the image of de Rham always landed in the basis for H^4 and for this particular example such a basis exists. Again this step is not necessary, but I (Scott Stetson) like it when I can just read off the integer coefficients of the image of de Rham to create the matrix representation of $d : H^3(K_f^{\bullet})_5 \to H^4(K_f^{\bullet})_4$. However I don't know if this is always possible so that's one of the reasons why I haven't fully automated this code. Anyway, in order to find this nice basis I had to guess certain basis elements to start off with and then I modded out by the image of Koszul in order to eliminate certain monomials. I will not explain all of this coded since this step is not necessary, but here it is below.

```
In[42]:= bJun = {};
indexB2 = {};
k = 1;
While[Length[bJun] < 15,
 If[
  MatrixRank[Join[bJun, {polyToVec[deRham[vecToPoly[B2[[k]], allExp[5]]], allExp[4]]}]] ==
   Length[bJun] + 1,
  bJun = Append[bJun, polyToVec[deRham[vecToPoly[B2[[k]], allExp[5]]], allExp[4]]];
  indexB2 = Append[indexB2, k]];
 k++]
indexB2

Out[46]= {3, 4, 6, 7, 8, 9, 10, 14, 17, 19, 20, 34, 37, 39, 40}
```

In[47]:=
```
m = Join[Table[polyToVec[
     deRham[vecToPoly[B2[[indexB2[[k]]]], allExp[5]]], allExp[4]], {k, 15}], Table[
    polyToVec[deRham[(Product[Subscript[x, z]^allExp[2][[k,z+1]], {z, 0, 3}]) vecToPoly[n1, allExp[3]]], allExp[4]], {k, 2}],
   Table[polyToVec[deRham[(Product[Subscript[x, z]^allExp[2][[k,z+1]], {z, 0, 3}]) vecToPoly[n1, allExp[3]]], allExp[4]],
    {k, 4, 5}], {polyToVec[deRham[Subscript[x, 3]^2 vecToPoly[n1, allExp[3]]], allExp[4]]}];
Manipulate[Join[{Range[35]}, Transpose[
     Permute[Transpose[RowReduce[Transpose[Permute[Transpose[m], InversePermutation[
            PermutationCycles[Join[RotateLeft[Complement[Range[35], mon], a], mon]]]]]]],
      PermutationCycles[Join[RotateLeft[Complement[Range[35], mon], a], mon]]]][[
    16 ;; 20]]] // MatrixForm, {{a, 2}, 1, 16, 1}]
mon
```

a

Out[48]=

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{3}{4} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & \frac{7}{2} & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -\frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[49]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 20, 21, 23, 35}

In[50]:=
```
Transpose[Permute[Transpose[RowReduce[Transpose[Permute[Transpose[m], InversePermutation[
      PermutationCycles[Join[RotateLeft[Complement[Range[35], mon], 2], mon]]]]]]],
   PermutationCycles[Join[RotateLeft[Complement[Range[35], mon], 2], mon]]]][[16 ;; 20]]
```

Out[50]= $\left\{\left\{0, 0, 0, 0, -\frac{1}{4}, 0, 0, 0, 0, 0, -\frac{1}{4}, -\frac{3}{4}, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{1}{16}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{8}{3}\right\}, \left\{1, 0, 0, 0, \frac{9}{2}, 0, 0, 0, 0, 0, \frac{7}{2}, 9, 0, 0, 0, 0, 0, 0, 0, 0, \frac{9}{16}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -36\right\}, \left\{0, 1, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, -\frac{3}{2}, -\frac{9}{2}, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16\right\}, \left\{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{4}, 0, 0, 0, 0, 0, 0, -2, 0, 0, \frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \frac{3}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{3}\right\}\right\}$

In[51]:= $\text{Expand}\left[\text{Dot}\left[12 \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \frac{3}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{3}\right\}, \text{Table}\left[\prod_{z=0}^{3} x_z^{\text{allExp[4][[k,z+1]]}}, \{k, 35\}\right]\right]\right]$

Out[51]= $9 x_0 x_1^2 x_2 + 12 x_0^2 x_3^2 - 8 x_3^4$

In[52]:= $\text{sub} = \{-12 x_0^2 x_1^2 - 12 x_0 x_1^3 - 3 x_1^4 - 36 x_0 x_1^2 x_2 + 48 x_0 x_1 x_3^2 + 128 x_3^4, 16 x_0^4 + 72 x_0^2 x_1^2 + 56 x_0 x_1^3 + 9 x_1^4 + 144 x_0 x_1^2 x_2 - 576 x_3^4, 4 x_0^3 x_1 - 6 x_0^2 x_1^2 - 6 x_0 x_1^3 - x_1^4 - 18 x_0 x_1^2 x_2 + 64 x_3^4, 8 x_0^3 x_3 - 6 x_0 x_1^2 x_3 + x_1^3 x_3 - 16 x_0 x_3^3, 9 x_0 x_1^2 x_2 + 12 x_0^2 x_3^2 - 8 x_3^4\}$

Out[52]= $\{-12 x_0^2 x_1^2 - 12 x_0 x_1^3 - 3 x_1^4 - 36 x_0 x_1^2 x_2 + 48 x_0 x_1 x_3^2 + 128 x_3^4, 16 x_0^4 + 72 x_0^2 x_1^2 + 56 x_0 x_1^3 + 9 x_1^4 + 144 x_0 x_1^2 x_2 - 576 x_3^4, 4 x_0^3 x_1 - 6 x_0^2 x_1^2 - 6 x_0 x_1^3 - x_1^4 - 18 x_0 x_1^2 x_2 + 64 x_3^4, 8 x_0^3 x_3 - 6 x_0 x_1^2 x_3 + x_1^3 x_3 - 16 x_0 x_3^3, 9 x_0 x_1^2 x_2 + 12 x_0^2 x_3^2 - 8 x_3^4\}$

In[53]:=
```
test = {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 20, 21, 23, 35};
Length[test] + 16
MatrixRank[
```
$\text{Join}\left[\text{Table}\left[\text{polyToVec}\left[\prod_{z=0}^{3} x_z^{\text{allExp[4][[test[[k]],z+1]]}}, \text{allExp[4]}\right], \{k, \text{Length[test]}\}\right], \text{A[f, 1]}\right]\right]$

Out[54]= 35

Out[55]= 35

In[56]:= $\text{bTop} = \text{Table}\left[\prod_{z=0}^{3} x_z^{\text{allExp[4][[test[[k]],z+1]]}}, \{k, \text{Length[test]}\}\right]$

Out[56]= $\{x_0^4, x_0^3 x_1, x_0^3 x_2, x_0^3 x_3, x_0^2 x_1^2, x_0^2 x_1 x_3, x_0^2 x_2^2, x_0^2 x_2 x_3, x_0^2 x_3^2, x_0 x_1^3, x_0 x_1^2 x_2, x_0 x_1^2 x_3, x_0 x_1 x_2^2, x_0 x_1 x_3^2, x_0 x_2^2 x_3, x_0 x_3^3, x_1^4, x_1^3 x_3, x_3^4\}$

In[57]:=
```
Table[PolynomialReduce[sub[[k]], bTop, {x0, x1, x2, x3}][[2]], {k, Length[sub]}]
```

Out[57]= {0, 0, 0, 0, 0}

The following is the matrix representation of $d : H^3(K_f^{\bullet})_5 \to H^4(K_f^{\bullet})_4$.

In[58]:=
```
E2 = Table[PolynomialReduce[sub[[k]], bTop, {x0, x1, x2, x3}][[1]], {k, Length[sub]}];
E2 // MatrixForm
```

Out[59]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 & -12 & -36 & 0 & 0 & 48 & 0 & 0 & -3 & 0 & 128 \\ 16 & 0 & 0 & 0 & 72 & 0 & 0 & 0 & 0 & 56 & 144 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & -576 \\ 0 & 4 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & -6 & -18 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 64 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -16 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 \end{pmatrix}$$

It has rank 5, so we must add 14 linearly independent rows to make it full rank.

In[60]:=
```
MatrixRank[E2]
```

Out[60]= 5

In[61]:=
```
MatrixRank[
 Join[Table[PolynomialReduce[sub[[k]], bTop, {x0, x1, x2, x3}][[1]], {k, Length[sub]}],
  IdentityMatrix[19][[1 ;; 10]], {IdentityMatrix[19][[12]],
   IdentityMatrix[19][[13]], IdentityMatrix[19][[15]], IdentityMatrix[19][[16]]}]]
```

Out[61]= 19

In[62]:=
```
Join[bTop[[1 ;; 10]], {bTop[[12]], bTop[[13]], bTop[[15]], bTop[[16]]}]
```

Out[62]= $\{x_0^4, x_0^3 x_1, x_0^3 x_2, x_0^3 x_3, x_0^2 x_1^2, x_0^2 x_1 x_3, x_0^2 x_2^2, x_0^2 x_2 x_3, x_0^2 x_3^2, x_0 x_1^3, x_0 x_1^2 x_3, x_0 x_1 x_2^2, x_0 x_2^2 x_3, x_0 x_3^3\}$

The 14 monomials above are our basis for ${E_2}^{4,4}$.