In[17]:=
```
f[x0_, x1_, x2_, x3_] :=
  -3 x0^2 x1 x2 - 3 x0 x1^2 x2 - 3 x2^4 + 4 x0^2 x3^2 + 4 x0 x1 x3^2 + x1^2 x3^2 - 6 x1 x2 x3^2
(* koszul of a 3-form *)
koszul[v_] := v[[4]] * D[f[x_0, x_1, x_2, x_3], x_0] - v[[3]] * D[f[x_0, x_1, x_2, x_3], x_1] +
   v[[2]] * D[f[x_0, x_1, x_2, x_3], x_2] - v[[1]] * D[f[x_0, x_1, x_2, x_3], x_3]
(* de Rham of a 3-form *)
deRham[v_] := -D[v[[1]], x_3] + D[v[[2]], x_2] - D[v[[3]], x_1] + D[v[[4]], x_0]
homExp[f_] := Total[Exponent[MonomialList[f, {x_0, x_1, x_2, x_3}][[1]], {x_0, x_1, x_2, x_3}]]
allExp[n_] := Module[{v = {}}, For[i = 1, i ≤ Length[IntegerPartitions[n, 4]], i++,
    v = Append[v, Permutations[PadRight[IntegerPartitions[n, 4][[i]], 4]]]];
   v = Reverse[Sort[ArrayFlatten[v, 1]]]]
polyToVec[f_, list_] := Module[{a = ConstantArray[0, Length[list]]}, If[
    Depth[MonomialList[f, {x_0, x_1, x_2, x_3}]] == 2, a = a, v = MonomialList[f, {x_0, x_1, x_2, x_3}];
    pos = Table[Position[list, Table[Exponent[v[[k]], x_i], {i, 0, 3}]][[1, 1]],
      {k, 1, Length[v]}];
    coeff = v /. {x_0 → 1, x_1 → 1, x_2 → 1, x_3 → 1};
    For[i = 1, i ≤ Length[v], i++,
     a = ReplacePart[a, pos[[i]] → coeff[[i]]]
    ]]; a]
vecToPoly[v_, list_] := Table[Total[Table[(Product[x_j^list[[Mod[k-1,Length[list]]+1,j+1]], {j, 0, 3}]) v[[k]],
     {k, Length[list] (m - 1) + 1, Length[list] m}]], {m, 1, 4}]
A[f_, n_] := ArrayFlatten[Table[Table[polyToVec[(-1)^m (Product[x_j^allExp[n][[k,j+1]], {j, 0, 3}])
        D[f[x_0, x_1, x_2, x_3], x_m], allExp[n + homExp[f[x_0, x_1, x_2, x_3]] - 1]],
      {k, 1, Length[allExp[n]]}], {m, 3, 0, -1}], 1];
pExpand[x : (_Rational | _Integer), p_?PrimeQ, n_ /; Positive[n]] :=
  Module[{q = p^n, num, den, v, e},
   v = IntegerExponent[#, p] & /@ ({num, den} = #[x] & /@ {Numerator, Denominator});
   If[(e = v[[1]] - v[[2]]) > 0, num /= p^v[[1]], den /= p^v[[2]]];
   {e, Reverse@IntegerDigits[Mod[num PowerMod[den, -1, q], q], p, n]}];
```

In[•]:=
```
singPts = {{0, 0, 0, 1}, {1, 0, 0, 0},
   {0, 1, 0, 0}, {1, -1, 0, 0}, {-1/2, 1, 0, -Sqrt[2]/4}, {-1/2, 1, 0, Sqrt[2]/4}};
```

$$\mathrm{singPts} = \left\{\{0,0,0,1\},\{1,0,0,0\},\{0,1,0,0\},\{1,-1,0,0\},\left\{-\frac{1}{2},1,0,-\frac{\sqrt{2}}{4}\right\},\left\{-\frac{1}{2},1,0,\frac{\sqrt{2}}{4}\right\}\right\};$$

In[•]:=
```
G = GroebnerBasis[Table[D[f[x_0, x_1, x_2, x_3], x_k], {k, 0, 3}], {x_0, x_1, x_2, x_3}];
kosC =
```

$$\Big\{\Big\{-\frac{1}{256}x_1^2x_2^3+\frac{5x_2^5}{64}+\frac{5}{192}x_1x_2^2x_3^2-\frac{1}{96}x_2^3x_3^2+\frac{7}{288}x_1x_3^4,\ -\frac{5}{128}x_1^2x_2^2x_3+\frac{5}{64}x_1x_2^3x_3-\frac{1}{48}x_1x_2x_3^3+\frac{5}{48}x_2^2x_3^3,\ -\frac{5}{384}x_1^3x_2x_3+\frac{5}{192}x_1^2x_2^2x_3-\frac{1}{48}x_1x_2^3x_3+\frac{5}{24}x_2^4x_3-\frac{1}{144}x_1^2x_3^3+\frac{5}{48}x_1x_2x_3^3+\frac{x_3^5}{6},$$
$$\frac{5}{384}x_0x_1^2x_2x_3-\frac{1}{48}x_0x_1x_2^2x_3+\frac{1}{384}x_1^2x_2^2x_3-\frac{1}{96}x_0x_2^3x_3-\frac{1}{64}x_1x_2^3x_3+\frac{5}{48}x_2^4x_3+\frac{7}{288}x_0x_1x_3^3+\frac{5}{576}x_1^2x_3^3-\frac{1}{16}x_0x_2x_3^3+\frac{1}{48}x_1x_2x_3^3+\frac{x_3^5}{12}\Big\},$$
$$\Big\{-\frac{5}{384}x_1^2x_2^2-\frac{1}{192}x_1x_2^3-\frac{x_2^4}{96}+\frac{1}{36}x_1x_2x_3^2+\frac{5}{144}x_2^2x_3^2-\frac{x_3^4}{216},\ -\frac{1}{96}x_1x_2^2x_3-\frac{1}{72}x_2x_3^3,$$

$$-\frac{1}{288}x_1^2x_2x_3-\frac{5}{72}x_1x_2^2x_3-\frac{1}{36}x_2^3x_3-\frac{1}{216}x_1x_3^3+\frac{1}{6}x_2x_3^3,\ \frac{1}{48}x_0x_1x_2x_3+$$
$$\frac{5}{576}x_1^2x_2x_3-\frac{1}{36}x_0x_2^2x_3-\frac{7}{144}x_1x_2^2x_3-\frac{1}{72}x_2^3x_3-\frac{1}{216}x_0x_3^3-\frac{1}{216}x_1x_3^3+\frac{1}{12}x_2x_3^3\Big\},$$
$$\Big\{\frac{9}{64}x_1^2x_2+\frac{3}{32}x_1x_2^2+\frac{3x_2^3}{16}-\frac{1}{4}x_1x_3^2-\frac{3}{8}x_2x_3^2,\ \frac{3}{16}x_1x_2x_3,\ \frac{1}{16}x_1^2x_3+\frac{3}{4}x_1x_2x_3+\frac{1}{2}x_2^2x_3-\frac{5x_3^3}{3},$$
$$-\frac{1}{4}x_0x_1x_3-\frac{3}{32}x_1^2x_3+\frac{1}{4}x_0x_2x_3+\frac{1}{2}x_1x_2x_3+\frac{1}{4}x_2^2x_3-\frac{5x_3^3}{6}\Big\},\ \Big\{-\frac{3}{16}x_1^2x_2+\frac{1}{4}x_1x_3^2+\frac{1}{4}x_2x_3^2,$$
$$-\frac{3}{4}x_2^2x_3,\ -\frac{5}{4}x_1x_2x_3+\frac{4x_3^3}{3},\ \frac{1}{4}x_0x_1x_3+\frac{1}{8}x_1^2x_3-\frac{1}{4}x_0x_2x_3-\frac{3}{4}x_1x_2x_3+\frac{2x_3^3}{3}\Big\},$$
$$\Big\{-\frac{1}{4}x_1x_2x_3-\frac{1}{4}x_2^2x_3+\frac{x_3^3}{6},\ -\frac{3x_2^3}{4}+\frac{1}{2}x_2x_3^2,\ -\frac{1}{4}x_1x_2^2+\frac{1}{6}x_1x_3^2-\frac{4}{3}x_2x_3^2,$$
$$\frac{1}{4}x_0x_2^2+\frac{1}{6}x_0x_3^2+\frac{1}{6}x_1x_3^2-\frac{2}{3}x_2x_3^2\Big\},\ \Big\{\frac{x_3}{12},\ -\frac{x_2}{4},\ -\frac{x_1}{12},\ \frac{x_0}{12}\Big\},$$
$$\Big\{\frac{x_1x_2}{8}+\frac{x_2^2}{4}-\frac{x_3^2}{12},\ -\frac{1}{4}x_2x_3,\ -\frac{1}{12}x_1x_3+\frac{2x_2x_3}{3},\ -\frac{1}{12}x_0x_3-\frac{x_1x_3}{12}+\frac{x_2x_3}{3}\Big\},$$
$$\Big\{\frac{x_1x_3}{4}+\frac{x_2x_3}{2},\ -\frac{3}{4}x_1x_2+\frac{3x_2^2}{2},\ -\frac{x_1^2}{4}+\frac{x_1x_2}{2}+\frac{8x_3^2}{3},\ \frac{x_0x_1}{4}-\frac{x_0x_2}{2}+\frac{4x_3^2}{3}\Big\},$$
$$\Big\{-\frac{x_2}{2},\ 0,\ -\frac{4x_3}{3},\ -\frac{2x_3}{3}\Big\},\ \Big\{-\frac{x_1}{2}-x_2,\ \frac{4x_3}{3},\ -\frac{8x_3}{3},\ -\frac{4x_3}{3}\Big\},\ \Big\{0,\ -\frac{2x_2}{3},\ -\frac{2x_1}{3},\ -\frac{x_1}{3}\Big\},$$
$$\Big\{\frac{1}{96}x_0x_1^2x_3-\frac{1}{48}x_0x_2^2x_3-\frac{1}{96}x_1x_2^2x_3-\frac{1}{24}x_2^3x_3+\frac{1}{18}x_0x_3^3+\frac{1}{36}x_1x_3^3,$$
$$\frac{1}{64}x_1^3x_2-\frac{1}{24}x_0x_1x_3^2-\frac{1}{48}x_1^2x_3^2-\frac{1}{12}x_1x_2x_3^2,$$
$$-\frac{1}{96}x_0x_1^3-\frac{1}{12}x_1x_2^3+\frac{1}{9}x_0x_1x_3^2-\frac{1}{36}x_1^2x_3^2-\frac{1}{18}x_0x_2x_3^2-\frac{1}{36}x_1x_2x_3^2,\ -\frac{1}{192}x_0^2x_1^2-\frac{1}{64}x_0x_1^3-$$
$$\frac{1}{24}x_0x_2^3-\frac{1}{16}x_1x_2^3+\frac{1}{18}x_0^2x_3^2+\frac{1}{9}x_0x_1x_3^2-\frac{1}{32}x_1^2x_3^2-\frac{1}{36}x_0x_2x_3^2+\frac{1}{144}x_1x_2x_3^2+\frac{1}{24}x_2^2x_3^2+\frac{x_3^4}{4}\Big\},$$
$$\Big\{\frac{1}{384}x_1^2x_2^2-\frac{1}{96}x_0x_2^3+\frac{1}{192}x_1x_2^3+\frac{1}{144}x_0x_1x_3^2+\frac{1}{36}x_0x_2x_3^2+\frac{1}{72}x_1x_2x_3^2-\frac{1}{144}x_2^2x_3^2-\frac{5x_3^4}{216},$$
$$\frac{1}{96}x_1^2x_2x_3-\frac{1}{36}x_0x_3^3-\frac{1}{72}x_1x_3^3,\ -\frac{1}{144}x_0x_1^2x_3-\frac{1}{36}x_0x_2^2x_3+\frac{2}{27}x_0x_3^3,$$
$$-\frac{1}{288}x_0^2x_1x_3-\frac{1}{96}x_0x_1^2x_3-\frac{1}{288}x_0x_1x_2x_3-\frac{1}{576}x_1^2x_2x_3-\frac{1}{48}x_0x_2^2x_3+\frac{1}{144}x_1x_2^2x_3-$$
$$\frac{1}{48}x_2^3x_3+\frac{1}{72}x_0x_3^3-\frac{7}{216}x_1x_3^3+\frac{5}{72}x_2x_3^3\Big\},\ \Big\{-\frac{3}{32}x_1^2x_2+\frac{3}{16}x_0x_2^2+\frac{3}{32}x_1x_2^2+\frac{1}{8}x_1x_3^2,$$
$$0,\ \frac{1}{2}x_0x_2x_3-\frac{1}{4}x_1x_2x_3+\frac{2x_3^3}{3},\ \frac{1}{8}x_0x_1x_3+\frac{1}{16}x_1^2x_3-\frac{7}{16}x_1x_2x_3+\frac{3}{8}x_2^2x_3+\frac{x_3^3}{12}\Big\},$$
$$\Big\{\frac{1}{8}x_0x_1x_3+\frac{1}{16}x_1^2x_3+\frac{1}{2}x_0x_2x_3+\frac{1}{8}x_1x_2x_3-\frac{1}{4}x_2^2x_3-\frac{x_3^3}{4},\ -\frac{1}{2}x_0x_3^2-\frac{1}{4}x_1x_3^2+\frac{5}{4}x_2x_3^2,$$
$$-\frac{1}{8}x_0x_1^2-\frac{x_1^3}{16}+\frac{4}{3}x_0x_3^2+\frac{13}{12}x_1x_3^2-\frac{2}{3}x_2x_3^2,\ -\frac{1}{16}x_0^2x_1-\frac{1}{8}x_0x_1^2+\frac{5}{12}x_0x_3^2+\frac{1}{24}x_1x_3^2+\frac{2}{3}x_2x_3^2\Big\},$$
$$\Big\{\frac{x_0x_3}{4}-\frac{x_2x_3}{4},\ -\frac{3}{4}x_0x_2-\frac{3x_2^2}{4},\ -\frac{1}{4}x_0x_1-\frac{x_1x_2}{4}-\frac{4x_3^2}{3},\ \frac{x_0^2}{4}+\frac{x_0x_2}{4}+\frac{x_3^2}{3}\Big\},$$

$$\left\{-\frac{1}{4}x_0x_2-\frac{x_1x_2}{8}+\frac{x_3^2}{6},\ -\frac{1}{2}x_2x_3,\ -\frac{2}{3}x_0x_3-\frac{x_1x_3}{2},\ -\frac{1}{6}x_0x_3+\frac{x_1x_3}{12}-\frac{x_2x_3}{2}\right\},\ \{0,0,0,-1\},$$
$$\left\{-\frac{1}{2}x_0x_3+\frac{x_1x_3}{2}+\frac{3x_2x_3}{4},\ \frac{9x_2^2}{4}-2x_3^2,\ -x_0x_1-\frac{x_1^2}{2}+\frac{3x_1x_2}{4}+4x_3^2,\ -\frac{x_0^2}{2}-x_0x_1-\frac{3x_0x_2}{4}\right\},$$
$$\left\{-\frac{1}{2},0,0,0\right\},\ \{0,0,1,1\},\ \left\{0,-\frac{1}{3},0,0\right\}\};$$

In[ ]:=
```
Expand[Table[G[[k]] - koszul[kosC[[k]]], {k, 1, Length[kosC]}]]
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[ ]:=
```
n1 = {0, 0, 0, -2, 0, 0, -2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4, 0, 0, -4, 6, 0, 0, 0, -8, -1, 3, 0, 0,
   0, 4, 0, 0, 0, 0, -2, -5, 3, 0, -2, 6, 0, 0, 0, -8, 0, 0, 0, 0, 0, 0, 0, 0, 6, 0};
vecToPoly[n1, allExp[3]]
```

Out[ ]=
$$\{-2x_0^2x_3-2x_0x_1x_3+x_1^2x_3-4x_3^3,\ 0,\ -4x_0^2x_1-4x_0x_1^2-x_1^3+6x_0x_1x_2+3x_1^2x_2-8x_0x_3^2+4x_1x_3^2,$$
$$-2x_0^3-5x_0^2x_1-2x_0x_1^2+3x_0^2x_2+6x_0x_1x_2-8x_0x_3^2+6x_2x_3^2\}$$

In[ ]:=
```
shiftBy[remainder_] := Module[{shift}, d = homExp[remainder];
  basis = {x0^(d-2), x0^(d-3) x1, x0^(d-3) x3, x0^(d-4) x1^2, x1^(d-2), x3^(d-2)};
  allDeR =
   Expand[Total[Table[s_k deRham[basis[[k]] * vecToPoly[n1, allExp[3]]], {k, 1, 6}]]];
  eqns = Table[Expand[allDeR - (remainder) /. {x0 → singPts[[k, 1]],
       x1 → singPts[[k, 2]], x2 → singPts[[k, 3]], x3 → singPts[[k, 4]]}], {k, 1, 6}];
  For[k = 1, k ≤ 6, k++,
   eqns[[k]] = eqns[[k]] == 0];
  sol = FullSimplify[Solve[eqns, Table[s_k, {k, 1, 6}]]];
  shift = Dot[Table[sol[[1, k, 2]], {k, 1, 6}],
    Table[Expand[deRham[basis[[k]] * vecToPoly[n1, allExp[3]]]], {k, 1, 6}]];
  shift]
```

In[ ]:=
```
b = {x0^4, x0^3 x1, x0^3 x2, x0^3 x3, x0^2 x1^2, x0^2 x1 x3,
   x0^2 x2^2, x0^2 x2 x3, x0^2 x3^2, x0 x1^3, x0 x1^2 x3, x0 x1 x2^2, x0 x2^2 x3, x0 x3^3};
sub = {-12 x0^2 x1^2 - 12 x0 x1^3 - 3 x1^4 - 36 x0 x1^2 x2 + 48 x0 x1 x3^2 + 128 x3^4,
   16 x0^4 + 72 x0^2 x1^2 + 56 x0 x1^3 + 9 x1^4 + 144 x0 x1^2 x2 - 576 x3^4, 4 x0^3 x1 - 6 x0^2 x1^2 - 6 x0 x1^3 - x1^4 -
    18 x0 x1^2 x2 + 64 x3^4, 8 x0^3 x3 - 6 x0 x1^2 x3 + x1^3 x3 - 16 x0 x3^3, 9 x0 x1^2 x2 + 12 x0^2 x3^2 - 8 x3^4};
```

In[26]:=
```
p = 5;
```

In[ ]:=
```
a = 2;
```

In[∘]:=
```
StringJoin[DateString["Hour12"], ":", DateString["Minute"],
 ":", DateString["Second"], " ", DateString["AMPM"]]
```

$$\omega = \text{Numerator}\left[p^2 \frac{\left(b[[1]]\right)^p (x_0 x_1 x_2 x_3)^{p-1}}{f[x_0, x_1, x_2, x_3]^{2p}} \sum_{k=a}^{a}\left(\frac{\left(k+1\right)\left(f[x_0, x_1, x_2, x_3]^p - f[x_0{}^p, x_1{}^p, x_2{}^p, x_3{}^p]\right)^k}{f[x_0, x_1, x_2, x_3]^{p*k}}\right)\right];$$

$$\text{deg} = 4\,p + 4\left(p - 1\right) + 4\,p * a;$$

$$c = \frac{1}{\left(\frac{\text{deg}}{4}\right)!};$$

```
While[deg > 6,
 remdr = PolynomialReduce[ω, G, {x0, x1, x2, x3}];
 If[Depth[remdr[[2]]] == 1, ω = deRham[remdr[[1]].kosC],
  ω = deRham[PolynomialReduce[ω - shiftBy[remdr[[2]]], G, {x0, x1, x2, x3}][[1]].kosC]];
 If[Mod[deg, 20] == 0, Print[deg]];
 deg = deg - 4;
]
ans = Transpose[
   RowReduce[Transpose[Join[Table[polyToVec[b[[k]], allExp[4]], {k, 1, Length[b]}],
      Table[polyToVec[sub[[k]], allExp[4]], {k, 1, Length[sub]}],
      A[f, 1], {polyToVec[c * ω, allExp[4]]}]]]];
```

$$\text{Join}\left[\text{ans}[[36, 1 ;; 14]], \left\{\text{deRham}\left[\text{ArrayFlatten}\left[\text{Partition}[\text{ans}[[36, 20 ;; 35]], 4].\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, 1\right]\right]\right\}\right]$$

```
StringJoin[DateString["Hour12"], ":", DateString["Minute"], ":",
 DateString["Second"], " ", DateString["AMPM"]]
(*SendMail[<|"To"→"sstetson@uci.edu","Subject"→"Code","Body"->"Finished upstairs."|>]*)
```

Out[∘]= 12:07:58 AM

60

40

20

Out[ ]=

$$\left\{-\frac{75\,701\,289\,037\,258\,558\,629\,452\,744\,721\,420\,085\,395\,789\,184\,099\,136\,852\,566\,075}{41\,372\,859\,244\,332\,903\,863\,537\,751\,593\,937\,596\,117\,941\,359\,828\,403\,224\,576},\right.$$
$$-\frac{57\,614\,172\,991\,729\,027\,558\,315\,828\,455\,007\,611\,560\,539\,469\,899\,204\,919\,257\,775}{13\,790\,953\,081\,444\,301\,287\,845\,917\,197\,979\,198\,705\,980\,453\,276\,134\,408\,192},$$
$$\frac{560\,488\,916\,637\,902\,126\,435\,558\,617\,458\,866\,147\,057\,680\,649\,236\,433\,989\,675}{2\,298\,492\,180\,240\,716\,881\,307\,652\,866\,329\,866\,450\,996\,742\,212\,689\,068\,032}, 0,$$
$$\frac{4\,569\,372\,889\,633\,778\,723\,200\,661\,361\,956\,967\,760\,980\,082\,284\,005\,764\,412\,225}{2\,398\,426\,622\,859\,878\,484\,842\,768\,208\,344\,208\,470\,605\,296\,221\,936\,418\,816},$$
$$0, -\frac{181\,143\,908\,267\,257\,620\,471\,625}{5\,181\,967\,342\,768\,180\,804\,190\,208}, 0,$$
$$-\frac{1\,910\,511\,310\,285\,188\,284\,370\,488\,017\,623\,264\,827\,278\,495\,260\,526\,047\,107\,875}{985\,068\,077\,246\,021\,520\,560\,422\,656\,998\,514\,193\,284\,318\,091\,152\,457\,728},$$
$$-\frac{1\,214\,427\,223\,571\,600\,507\,589\,449\,448\,917\,148\,158\,726\,447\,980\,206\,344\,404\,225}{1\,157\,282\,776\,065\,256\,052\,126\,930\,114\,515\,736\,954\,348\,010\,065\,130\,160\,128}, 0,$$
$$-\frac{4\,265\,895\,301\,292\,562\,949\,255\,144\,858\,454\,526\,859\,546\,372\,577\,247\,947\,266\,325}{1\,532\,328\,120\,160\,477\,920\,871\,768\,577\,553\,244\,300\,664\,494\,808\,459\,378\,688}, 0,$$
$$\left.0, -\frac{111\,784\,888\,645\,624\,356\,847\,847\,005\,486\,206\,502\,226\,215\,166\,935\,902\,108\,125}{811\,232\,534\,202\,605\,958\,108\,583\,364\,587\,011\,688\,587\,085\,486\,831\,435\,776}\right\}$$

Out[ ]= 12:08:20 AM

In[ ]:=
```
Table[
 Column[Table[pExpand[Sum[r9[[k, l]], {k, 1, j}], p, 12], {j, 1, Length[r9]}]], {l, 1, 15}]
```

Out[ ]=
```
{{0, {3, 2, 3, 0, 4, 1, 0, 3, 1, 1, 4, 2}}
 {0, {3, 1, 0, 2, 3, 3, 1, 3, 1, 4, 3, 0}}
 {0, {3, 1, 2, 4, 0, 3, 4, 2, 3, 0, 2, 1}}
 {0, {3, 1, 2, 2, 3, 2, 4, 3, 3, 4, 0, 0}}
 {0, {3, 1, 2, 3, 1, 1, 0, 1, 3, 4, 3, 1}} ,
 {0, {3, 1, 2, 3, 3, 1, 2, 0, 3, 2, 2, 0}}
 {0, {3, 1, 2, 3, 3, 2, 1, 0, 2, 4, 0, 4}}
 {0, {3, 1, 2, 3, 3, 2, 4, 3, 1, 2, 2, 4}}

 {0, {3, 1, 4, 4, 3, 4, 0, 0, 0, 0, 3, 3}}   {0, {4, 4, 1, 0, 0, 3, 4, 0, 0, 0, 0, 1}}
 {0, {3, 0, 3, 0, 3, 3, 2, 0, 2, 3, 0, 1}}   {0, {4, 1, 2, 3, 1, 3, 0, 2, 3, 1, 3, 1}}
 {0, {3, 0, 0, 3, 1, 2, 4, 2, 2, 1, 0, 2}}   {0, {4, 1, 3, 4, 3, 0, 4, 4, 2, 4, 3, 3}}
 {0, {3, 0, 0, 1, 3, 1, 3, 0, 0, 0, 1, 2}}   {0, {4, 1, 3, 3, 4, 2, 3, 1, 0, 4, 3, 1}}
 {0, {3, 0, 0, 2, 3, 1, 1, 1, 2, 1, 2, 2}} , {0, {4, 1, 3, 1, 3, 4, 3, 0, 3, 2, 4, 4}} ,
 {0, {3, 0, 0, 2, 0, 3, 1, 2, 3, 0, 4, 0}}   {0, {4, 1, 3, 1, 4, 3, 4, 3, 1, 2, 1, 1}}
 {0, {3, 0, 0, 2, 0, 4, 3, 2, 4, 0, 1, 1}}   {0, {4, 1, 3, 1, 4, 1, 2, 2, 1, 0, 1, 0}}
 {0, {3, 0, 0, 2, 0, 4, 1, 4, 3, 3, 3, 0}}   {0, {4, 1, 3, 1, 4, 1, 1, 0, 3, 3, 2, 0}}

 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 4, 0, 0, 0, 2, 3, 0, 1, 4, 2, 2}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 4, 0, 2, 3, 1, 0, 4, 0, 3, 1}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 3, 0, 2, 0, 4, 1, 3, 0, 3, 0}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 3, 1, 2, 2, 4, 3, 3, 1, 4, 2}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} , {0, {1, 2, 3, 3, 1, 2, 4, 2, 4, 3, 0, 0}} ,
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 3, 3, 0, 4, 2, 1, 3, 2, 0, 2}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 3, 3, 0, 1, 1, 2, 1, 1, 4, 2}}
 {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {0, {1, 2, 3, 3, 0, 1, 2, 1, 1, 3, 2, 3}}
```

```
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} , {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} ,
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 0, 0, 4, 3, 0, 0, 4, 1, 1, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 1, 1, 4, 3, 3, 3, 3, 0, 2, 2}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 3, 2, 3, 0, 3, 4, 4, 4, 0, 2}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 3, 2, 2, 0, 1, 0, 1, 4, 0, 4}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} , {1, {3, 1, 3, 2, 0, 2, 1, 3, 3, 1, 1, 2}} ,
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 3, 2, 2, 0, 2, 1, 2, 2, 3, 4}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 3, 2, 2, 0, 4, 3, 4, 0, 4, 3}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {1, {3, 1, 3, 2, 2, 0, 2, 3, 4, 0, 2, 2}}

{0, {3, 2, 0, 2, 3, 3, 3, 3, 4, 3, 0, 4}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 3, 0, 3, 0, 3, 3, 3, 4, 4, 4}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 0, 4, 1, 4, 3, 3, 1, 0, 3, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 0, 2, 1, 4, 2, 1, 1, 2, 3, 1}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 0, 3, 2, 2, 1, 2, 2, 2, 0, 2}} , {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} ,
{0, {3, 1, 0, 3, 4, 2, 0, 0, 2, 2, 4, 3}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 0, 3, 4, 3, 3, 0, 3, 4, 3, 3}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {3, 1, 0, 3, 4, 3, 1, 0, 2, 2, 2, 1}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

{0, {1, 4, 0, 4, 2, 4, 4, 4, 4, 0, 4, 4}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 0, 2, 0, 0, 4, 4, 2, 3, 3, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 0, 3, 0, 0, 3, 3, 4, 0, 1, 0}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 0, 0, 1, 2, 3, 4, 3, 3, 2, 0}} , {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} ,
{0, {1, 2, 0, 0, 0, 4, 1, 2, 3, 0, 4, 3}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 0, 0, 0, 1, 3, 2, 2, 0, 1, 4}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
{0, {1, 2, 0, 0, 0, 1, 4, 1, 3, 1, 1, 3}}   {∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {1, 4, 2, 0, 1, 2, 2, 1, 4, 3, 0, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 0, 0, 1, 0, 0, 2, 1, 1, 3, 3}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 2, 2, 0, 3, 3, 3, 4, 0, 0, 4}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 0, 1, 2, 3, 4, 2, 3, 0, 2, 4}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} , {2, {4, 3, 2, 2, 0, 4, 1, 0, 4, 2, 0, 4}} }
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 2, 2, 3, 3, 1, 2, 1, 0, 4, 3}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 2, 2, 0, 3, 3, 1, 2, 2, 1, 0}}
{∞, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}   {2, {4, 3, 2, 2, 0, 1, 0, 2, 0, 1, 2, 4}}
```

In[ ]:=
```
Manipulate[
 pExpand[Sum[r1[[k, 1]] + r2[[k, 2]] + r3[[k, 3]] + r4[[k, 4]] + r5[[k, 5]] + r6[[k, 6]] +
    r7[[k, 7]] + r8[[k, 8]] + r9[[k, 9]] + r10[[k, 10]] + r11[[k, 11]] + r12[[k, 12]] +
    r13[[k, 13]] + r14[[k, 14]] + r15[[k, 15]], {k, 1, t}], p, 10], {t, 1, 8, 1}]
```

Out[ ]=

```
t
{0, {4, 4, 4, 4, 4, 4, 4, 1, 3, 4}}
```

In[ ]:=

```
pExpand[-1, p, 10]
```

Out[ ]=

```
{0, {4, 4, 4, 4, 4, 4, 4, 4, 4, 4}}
```

In[ ]:=

```
FrobandRed[bas_, a_] := Module[{reduce},
  ω = Numerator[
     p^2 ((bas)^p (x0 x1 x2 x3)^(p-1))/(f[x0, x1, x2, x3]^((*2*)p)) Sum[((*(k+1)*) (f[x0, x1, x2, x3]^p - f[x0^p, x1^p, x2^p, x3^p])^k)/(f[x0, x1, x2, x3]^(p*k)), {k, a, a}]];
  deg = (*4p*)+4 (p - 1) + 4 p * a;
  c = 1/((deg/4)!);
  While[deg > 6,
   remdr = PolynomialReduce[ω, G, {x0, x1, x2, x3}];
   If[Depth[remdr[[2]]] == 1, ω = deRham[remdr[[1]].kosC],
    ω = deRham[PolynomialReduce[ω - shiftBy[remdr[[2]]], G, {x0, x1, x2, x3}][[1]].kosC]];
   If[Mod[deg, 20] == 0, Print[deg]];
   deg = deg - 4;
  ];
  ans = Transpose[
    RowReduce[Transpose[Join[Table[polyToVec[b[[k]], allExp[4]], {k, 1, Length[b]}],
       Table[polyToVec[sub[[k]], allExp[4]], {k, 1, Length[sub]}],
       A[f, 1], {polyToVec[c * ω, allExp[4]]}]]]];
  reduce = Join[ans[[36, 1 ;; 14]], {deRham[ArrayFlatten[
       Partition[ans[[36, 20 ;; 35]], 4].{{x0}, {x1}, {x2}, {x3}}, 1]]}];
  reduce]
```

In[ ]:=

```
StringJoin[DateString["Hour12"], ":", DateString["Minute"],
 ":", DateString["Second"], " ", DateString["AMPM"]]
r10 = ConstantArray[0, 8];
For[j = 1, j ≤ Length[r10], j++,
 r10[[j]] = FrobandRed[b[[10]], j - 1];
 Print["k=", j - 1, " is done.  Finished at ", StringJoin[DateString["Hour12"], ":",
   DateString["Minute"], ":", DateString["Second"], " ", DateString["AMPM"], "."]]
]
```

In[27]:=

```
F = {Total[r1], Total[r2], Total[r3], Total[r4], Total[r5], Total[r6], Total[r7], Total[r8],
   Total[r9], Total[r10], Total[r11], Total[r12], Total[r13], Total[r14], Total[r15]};
```

In[28]:=

```
adic = MonomialList[CharacteristicPolynomial[F, T], T] /. {T → 1};
```

In[29]:=
```
Column[Table[pExpand[adic[[k]], p, 6], {k, Length[adic]}]]
```

Out[29]=
```
{0, {4, 4, 4, 4, 4, 4}}
{0, {4, 4, 4, 4, 4, 4}}
{1, {2, 1, 0, 0, 0, 0}}
{2, {3, 0, 0, 0, 0, 0}}
{3, {2, 1, 0, 0, 0, 0}}
{5, {4, 4, 4, 4, 4, 4}}
{5, {2, 2, 4, 4, 4, 4}}
{6, {4, 4, 4, 4, 4, 4}}
{7, {1, 0, 0, 0, 0, 0}}
{8, {3, 2, 0, 0, 0, 0}}
{10, {1, 0, 0, 0, 0, 0}}
{10, {3, 3, 4, 4, 4, 4}}
{11, {2, 4, 4, 4, 4, 4}}
{12, {3, 3, 4, 4, 4, 4}}
{13, {1, 0, 0, 0, 0, 0}}
{15, {1, 0, 0, 0, 0, 0}}
```

In[30]:= **Factor**$\left[-T^{15}-T^{14}+35\,T^{13}+75\,T^{12}+875\,T^{11}-3125\,T^{10}-40\,625\,T^{9}-15\,625\,T^{8}+78\,125\,T^{7}+5\,078\,125\,T^{6}+9\,765\,625\,T^{5}-68\,359\,375\,T^{4}-146\,484\,375\,T^{3}-1\,708\,984\,375\,T^{2}+1\,220\,703\,125\,T+30\,517\,578\,125\right]$

Out[30]= $-(-5+T)^3\,(5+T)^4\,\left(25+T^2\right)^2\,\left(625-100\,T+10\,T^2-4\,T^3+T^4\right)$

In[31]:= **Expand**$\left[\left(T-\left(1+\sqrt{11}+i\sqrt{13-2\sqrt{11}}\right)\right)\left(T-\left(1+\sqrt{11}-i\sqrt{13-2\sqrt{11}}\right)\right)\left(T-\left(1-\sqrt{11}+i\sqrt{13+2\sqrt{11}}\right)\right)\left(T-\left(1-\sqrt{11}-i\sqrt{13+2\sqrt{11}}\right)\right)\right]$

Out[31]= $625-100\,T+10\,T^2-4\,T^3+T^4$

Point count.

In[32]:= **Column**$\left[\mathbf{Expand}\left[\mathbf{Table}\left[1+5^n+5^{2n}+3*5^n+4\,(-5)^n+2\,(5\,i)^n+2\,(-5\,i)^n+\left(1+\sqrt{11}+i\sqrt{13-2\sqrt{11}}\right)^n+\left(1+\sqrt{11}-i\sqrt{13-2\sqrt{11}}\right)^n+\left(1-\sqrt{11}+i\sqrt{13+2\sqrt{11}}\right)^n+\left(1-\sqrt{11}-i\sqrt{13+2\sqrt{11}}\right)^n,\{n,10\}\right]\right]\right]$

Out[32]=
```
30
722
15 870
397 042
9 755 950
244 202 162
6 103 347 630
152 591 625 682
3 814 701 086 910
95 367 479 315 602
```