Valuation of Convertible Bonds with Credit Risk

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Bachelor of Science with Honours in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

A convertible bond is a complex derivative that cannot be priced as a simple combination of bond and stock components. Convertible bonds can be broken down as a bond with two embedded options (a put option for the investor and a call option for the issuer) and an option to convert the bond into stock. Due to the multiple continuous options, the pricing of the convertible bond is path dependent.

This research project explores and implements a binary tree and finite difference scheme to price the convertible bond, taking into account credit risk.

Acknowledgements

Place acknowledgements here.

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Chapter 1

Introduction

What this paper shows and does.

1.1 Convertible Bonds with Credit Risk

Describe and define (in maths) what a convertible bond with credit risk is.

1.2 Literature Review

Describe the current state of literature w/ Convertible Bonds with Credit Risk, possibly split this and include it at the beginning of each relevant section?

1.3 Example

Describe the standard example that will be covered by each valuation method.

Chapter 2

Binomial Model with Credit Risk

The Binomial Model with Credit Risk was first derived by Milanov and Kounchev [2], who showed that this model converges, in continuous time, to the Ayache et al. [1] model. Milanov and Kounchev [2] also showed the valuation method of this model is the same as in the classical binomial model.

TODO: add further explanation and references.

2.1 Definition

Consider a stock with the following stochastic model[1]:

$$\frac{dS_t}{S_t} = (r + \lambda\eta)dt + \sigma dW_t - \eta dq_t \tag{2.1}$$

where:

r	is the risk free rate
σ	is the log-volatility of the stock price
λ	is the hazard rate
η	is the percentage drop in stock price on a default event
dW_t	is a Wiener process
dq_t	is a Poisson jump process where the first jump is the default event

[1] Of note $\mathbb{E}[\frac{dS_t}{S_t}] = rdt \Leftrightarrow \frac{dS_t}{S_t} = (r + \lambda\eta)dt + \sigma dW_t - \eta dq_t$

and

$$\mathbb{E}\left[\frac{dS_t}{S_t}\right] = rdt \tag{2.2}$$

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t+dt}}{S_t}\right] &= \mathbb{E}\left[\frac{S_t}{S_t} + \frac{dS_t}{S_t}\right] \\ &= 1 + \mathbb{E}\left[\frac{dS_t}{S_t}\right] \end{aligned}\tag{2.3}$$

$$\mathbb{V}\mathtt{ar}\left[\frac{dS_t}{S_t}\right] = (\sigma^2 + \lambda\eta^2)dt \tag{2.4}$$

$$\begin{aligned}\mathbb{V}\mathtt{ar}\left[\frac{S_{t+dt}}{S_t}\right] &= \mathbb{V}\mathtt{ar}\left[\frac{S_t}{S_t} + \frac{dS_t}{S_t}\right] \\ &= \mathbb{V}\mathtt{ar}\left[\frac{dS_t}{S_t}\right]\end{aligned} \tag{2.5}$$

and consider a binomial model that has time step δt, up and down steps, u and d with probability p_u and p_d[2] respectively, and where the probability of default is p_o[3] and (TODO: insert picture illustration)

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right] &= up_u + dp_d + (1-\eta)p_o \\ &= up_u + d(e^{-\lambda\delta t} - p_u) + (1-\eta)(1-e^{-\lambda\delta t})\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathbb{V}\mathtt{ar}\left[\frac{S_{t+\delta t}}{S_t}\right] &= u^2p_u + d^2p_d + (1-\eta)^2p_o - \mathbb{E}\left[\frac{\delta S_t}{S_t}\right]^2 \\ &= u^2p_u + d^2(e^{-\lambda\delta t} - p_u) + (1-\eta)^2(1-e^{-\lambda\delta t}) - \mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right]^2\end{aligned} \tag{2.7}$$

2.2 Derivation

If one equates the first moment of the stochastic model (2.3)[4], in δt time, with that of the binomial model (2.6) then:

$$\begin{aligned} e^{r\delta t} &= up_u + d(e^{-\lambda\delta t} - p_u) + (1-\eta)(1-e^{-\lambda\delta t}) \\ p_u(u-d) &= e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t}) \end{aligned}\tag{2.8}$$

$$p_u = \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{2.9}$$

$$\therefore \quad p_d = -\frac{e^{r\delta t} - ue^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{2.10}$$

[2] $p_d = 1 - p_u - p_o = e^{-\lambda\delta t} - p_u$

[3] The time till the first jump follows an exponential distribution with intensity λ and has probability $p_o = 1 - e^{-\lambda\delta t}$

[4] $\mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right] = e^{r\delta t}$

and if one equate the second moment about the mean of the stochastic model (2.5), in δt time, with that of the binomial model (2.7) then[5]:

$$\begin{aligned}(\sigma^2+\lambda\eta^2)\delta t &= u^2 p_u + d^2(e^{-\lambda\delta t}-p_u) + (1-\eta)^2(1-e^{-\lambda\delta t}) - \mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \\ &= (u^2-d^2)p_u + d^2 e^{-\lambda\delta t} + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t} && (2.11)\\ &= (u+d)(e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})) && (2.12)\\ &\quad + d^2 e^{-\lambda\delta t} + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t} \\ &= (u+d)(e^{r\delta t} - (1-\eta)(1-e^{-\lambda\delta t})) - ude^{-\lambda\delta t} && (2.13)\\ &\quad + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t}\end{aligned}$$

If one assumes $\delta t^2 = 0$, $ud = 1$ and $u = e^{\sqrt{A\delta t}}$ and the Taylor series expansion is taken for all exponential terms, then:

$$\begin{aligned}u &= 1 + \sqrt{A\delta t} + \frac{A\delta t}{2!} + \frac{(A\delta t)^{\frac{3}{2}}}{3!}\\ d &= 1 - \sqrt{A\delta t} + \frac{A\delta t}{2!} - \frac{(A\delta t)^{\frac{3}{2}}}{3!}\\ u + d &= 2 + A\delta t && (2.14)\end{aligned}$$

and substituting (2.14) into (2.13) with the appropriate expansions then:

$$\begin{aligned}(\sigma^2+\lambda\eta^2)\delta t &= (2+A\delta t)(1+r\delta t-\lambda\delta t(1-\eta)) - (1-\lambda\delta t)\\ &\quad + \lambda\delta t(1-\eta)^2 - (1-2r\delta t)\\ &= A\delta t + \lambda\delta t - \lambda\delta t(1-\eta)(1+\eta)\\ &= A\delta t + \lambda\eta^2\delta t\\ A\delta t &= \sigma^2\delta t && (2.15)\end{aligned}$$

2.3 Limits

The time step δt is assumed to be strictly positive ($\delta t > 0$), as time is monotonically increasing.

Although σ could be positive or negative, the formulæ use σ where, strictly, $|\sigma|$ should be used. If $\sigma = 0$ then the model is no longer useful. The restriction $\sigma > 0$ may be introduced for convenience and without loss of generality.

An implicit assumption this model makes is that p_u, p_d and p_o are a valid probability measure. This imposes the following conditions:

$$\min(p_u, p_d, p_o) \geq 0 \tag{2.16}$$

$$p_u + p_d + p_o = 1 \tag{2.17}$$

[5]Substituting equation (2.8) into (2.11) one gets (2.12)

Consider p_o, the probability of default. Its value is equal to the cumulative density function of an exponential distribution with intensity λ. As such p_o is already confined to the interval $[0,1]$ with the restriction that $\lambda > 0$. If $\lambda = 0$, no default is possible, then p_o will have a value of 0 and is a valid probability thus the restriction on λ can be relaxed to $\lambda \geq 0$.

Consider p_d, the probability of a down movement. Its value is defined in terms of p_u and p_o. If p_u and p_o are valid probabilities, within bounds, then so will p_d.

Consider p_u, the probability of an up movement. Taking into account the definition of p_d, this requires that p_u satisfies:

$$0 \leq p_u \leq e^{-\lambda\delta t} \tag{2.18}$$

$$0 \leq \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \leq e^{-\lambda\delta t}$$

$$0 \leq e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t}) \leq e^{-\lambda\delta t}(u-d)$$

$$e^{-\lambda\delta t}(d-(1-\eta)) \leq e^{r\delta t} - (1-\eta) \leq e^{-\lambda\delta t}(u-(1-\eta))$$

$$\ln\left(\frac{d-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \leq \lambda\delta t \leq \ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \tag{2.19}$$

$$\delta t \leq \frac{1}{\lambda}\ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \tag{2.20}$$

Using the limit on both δt and λ the inequalities around the $\lambda\delta t$ are also limited thus:

$$0 \leq \ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right)$$

$$e^{r\delta t} - (1-\eta) \leq u - (1-\eta)$$

$$r\delta t \leq \sigma\sqrt{\delta t}$$

$$\delta t \leq \frac{\sigma^2}{r^2} \tag{2.21}$$

The inequality of $\ln\left(\frac{d-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \leq 0$ results in the same inequality as above.

Based on the probability limits (2.17) no restriction is placed on r, the risk free rate, or η, the percentage drop in stock price.

2.4 Valuation

Consider a portfolio with value V_t at time t, Δ_t invested in the stock S_t and $\Pi_t = V_t - \Delta_t S_t$ invested at the risk neutral rate. At time $t + \delta t$ one should have:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d & \text{with probability } p_d \\ X_t - (1-\eta)S_t & \text{with probability } p_o \end{cases}$$

where

V_t^u is the value of the portfolio on an up-step at time $t + \delta t$

V_t^d is the value of the portfolio on a down-step at time $t + \delta t$

X_t is the value of default at time t

If one wishes to hedge against up and down movements of the stock then:

$$\begin{aligned} V_t^u - \Delta_t S_t u &= V_t^d - \Delta_t S_t d \\ \Delta_t(d-u) &= V_t^d - V_t^u \\ \Delta_t &= \frac{V_t^u - V_t^d}{u-d} \end{aligned} \tag{2.22}$$

The required value of Π_t at time $t + \delta t$ is:

$$\begin{aligned} \Pi_{t+\delta t} &= \Pi_t e^{r\delta t} \\ &= \left(V_t - \frac{V_t^u - V_t^d}{u-d}\right) e^{r\delta t} \end{aligned} \tag{2.23}$$

as Π_t is invested at the risk free rate, and taking the expected value of $\Pi_{t+\delta t}$, which must equal (2.23), one gets:

$$\begin{aligned} (V_t - \frac{V_t^u - V_t^d}{u-d})e^{r\delta t} &= \mathbb{E}\left[\Pi_{t+\delta t}\right] \\ &= \left(V_t^u - \frac{V_t^u - V_t^d}{u-d}u\right) e^{-\lambda\delta t} \\ &\quad + \left(X_t - \frac{V_t^u - V_t^d}{u-d}(1-\eta)\right)(1-e^{-\lambda\delta t}) \\ &= V_t^u\left(e^{-\lambda\delta t} - \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d}\right) \\ &\quad + V_t^d\left(\frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d}\right) + X_t(1-e^{-\lambda\delta t}) \\ V_t e^{r\delta t} &= V_t^u \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \\ &\quad + V_t^d \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t}) - e^{r\delta t}}{u-d} + X_t(1-e^{-\lambda\delta t}) \\ V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) \end{aligned} \tag{2.24}$$

thus the pricing method of this binomial model is the same as the pricing method of the classical binomial model:

$$\begin{aligned} V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) \\ &= e^{-r\delta t}\mathbb{E}_t\left[V_{t+\delta t}\right] \end{aligned} \tag{2.25}$$

2.5 Valuation with Coupon Cash-flow

Consider a portfolio with value V_t at time t, as above, with an additional coupon cash flow. At time $t + \delta t$ one should have:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u + c_i e^{t+\delta t - t_i^c} & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d + c_i e^{t+\delta t - t_i^c} & \text{with probability } p_d \\ X_t - (1-\eta) S_t + c_i q_{t_i^c} e^{t+\delta t - t_i^c} & \text{with probability } p_o \end{cases}$$

where

c_i	is the i$^{\text{th}}$ coupon value
t_i^c	is the time of the i$^{\text{th}}$ coupon, with $t_i^c \in (t, t+\delta t)$
$q_{t_i^c}$	is an indicator variable that default happens after coupon payment[6]

Using the above method it is trivial to show that:

$$\begin{aligned} V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i e^{-(r+\lambda)(t_i^c - t)} \\ &= e^{-r\delta t}\mathbb{E}\left[V_{t+\delta t}\right] + c_i e^{-(r+\lambda)(t_i^c - t)} \end{aligned} \tag{2.26}$$

if, however, $t_i^c = t$, thus c_i arrives with certainty at t, it is trivially shown that:

$$\begin{aligned} V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i \\ &= e^{-r\delta t}\mathbb{E}_t\left[V_{t+\delta t}\right] + c_i \end{aligned} \tag{2.27}$$

2.6 Parameters and Formulæ of Model

Based on the above Table 2.1 lists the parameters required for, Table 2.2 lists the limits of, and Table 2.3 lists the formulæ to price, this binomial table.

2.7 Example and Hedging Strategy

Provide a numerical example (with diagrams) and provide example of the hedging strategy.

[6] $\mathbb{P}\left(q_{t_i^c} = 1 | q \in [t, t+\delta t]\right) = \frac{e^{-\lambda(t_i^c - t)} - e^{-\lambda \delta t}}{1 - e^{-\lambda \delta t}}$, the probability that the coupon arrives before default occurs, given that default occurs within the interval

[7] If $r = 0$ then this inequality is not applicable.

[8] If $\lambda = 0$ then this inequality is not applicable.

Table 2.1: Parameters of the Binomial Model with Credit Risk

Parameter	**Description**
r	Risk free rate
σ	Log-volatility of the stock price
λ	Hazard rate of default
η	Percentage drop of stock price in a default event
δt	Time-step
V_t^u	Value of portfolio for an up-step at $t + \delta t$
V_t^d	Value of portfolio for a down-step at $t + \delta t$
X_t	Value of defaulted portfolio at t
c_i	Value of the i^{th} coupon payment
t_i^c	Time of the i^{th} coupon payment

Table 2.2: Limits of the Binomial Model with Credit Risk

Limit	**Description**
$0 < \sigma$	Volatility must be positive
$0 \leq \lambda$	Hazard rate must be non-negative
$0 < \delta t$	Time step must be positive
$\delta t \leq \frac{\sigma^2}{r^2}$	Time step must be small enough to handle volatility[7]
$\delta t \leq \frac{1}{\lambda} \ln \left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)} \right)$	Time step must be small enough to handle hazard rate[8]

Table 2.3: Formalæ for the Binomial Model with Credit Risk

Formulæ	**Description**
$u = e^{\sigma\sqrt{\delta t}}$	Multiplier for an up-step
$d = e^{-\sigma\sqrt{\delta t}}$	Multiplier for a down-step
$p_u = \frac{e^{r\delta t}-de^{-\lambda\delta t}-(1-\eta)(1-e^{-\lambda\delta t})}{u-d}$	Probability of an up-step
$p_d = e^{-\lambda\delta t} - p_u$	Probability of a down-step
$p_o = 1 - e^{-\lambda\delta t}$	Probability of default
$V_t = e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i$	Value of portfolio at t for $t_i^c = t$

Chapter 3

Finite Difference Model

Describe the finite difference model based on the paper by Ayache et al. [1].

Chapter 4

Numerical Example

Implement above models, explain implementation and results, by example.

Chapter 5

Conclusion

Summarise what this paper shows and does (i.e. Introduction + Conclusion = Executive Summary).

Bibliography

[1] E. Ayache, P. A. Forsyth, and K. R. Vertzal, *The valuation of convertible bonds with credit risk*, Journal of Derivatives **11** (2003), 9–44.

[2] K. Milanov and O. Kounchev, *Binomial tree model for convertible bond pricing within equity to credit risk framework*, 2012.