# Valuation of Convertible Bonds with Credit Risk

David Naylor

Programme in Advanced Mathematics of Finance,
School of Computational and Applied Mathematics,
University of the Witwatersrand

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# Introduction

Project Objective:

- Understand and implement pricing of a convertible bond with credit risk:
    - Finite Difference Model
    - Binomial Model

# Literature

Finite Difference Model:

- Ho and Pfeffer (1996) and Tsiveriotis and Fernandes (1998) modelled the derivative using geometric Brownian motion
- Ayache et al. (2003) modelled non-total default jump of the stock

Binomial Model:

- Davis and Lischka (1999) first proposed a trinomial tree
- Bardhan et al. (1994), and Hull (2011) have implemented a binomial tree
- Milanov and Kounchev (2012) implemented a binomial model that converges to the stochastic model

# Outline of the Talk

# Outline

# Convertible Bond (1/3)

**Definition. (Convertible Bond)**

*A convertible bond has the following components:*

- *Annuity: a series of coupons with a final redemption payment*
- *Put: a put option on the bond for the bond holder*
- *Call: a call option on the bond for the bond issuer*
- *Conversion: an asset swap for stock by the bond holder*

# Convertible Bond (2/3)

## Definition. (Convertible Bond)

*A convertible bond has the following components*[a]*:*

- *Annuity: Coupons:* $C_i$ *at* $t_i$*; Redemption:* $R$ *at* $T$
- *Put: Strike:* $K_t^p$ *for* $t \in \Omega^p$*; Payoff:* $K_t^p 1_{(V_t \leq K_t^p)\wedge(t\in\Omega^p)}(V_t, t)$
- *Call: Strike:* $K_t^c$ *for* $t \in \Omega^c$*; Payoff:* $K_t^c 1_{(V_t \geq K_t^c)\wedge(t\in\Omega^c)}(V_t, t)$
- *Conversion: Stock:* $\kappa_t S_t$ *for* $t \in \Omega^v$*; Payoff:* $\kappa_t S_t 1_{(V_t \leq \kappa_t S_t)\wedge(t\in\Omega^v)}(V_t, t)$

[a] $V_t$ denoted the intrinsic value of the derivative

# Convertible Bond (3/3)

## Definition. (Convertible Bond)

*A convertible bond has the following components:*

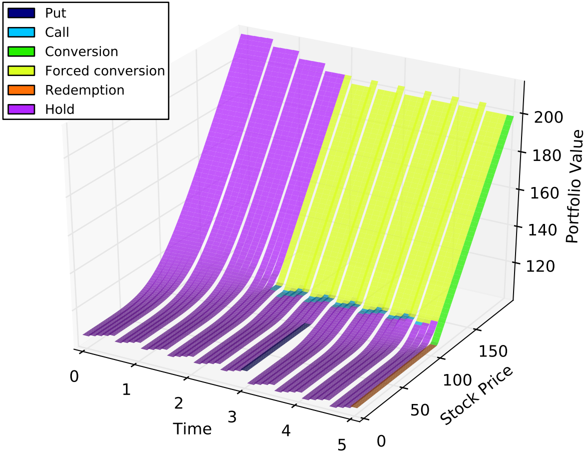
- *Annuity:* $t_i \neq T$
- *Put:* $T \notin \Omega^p$
- *Call:* $T \notin \Omega^v$; $K_t^c > K_t^p$
- *Conversion:* $\kappa > 0$; $T \in \Omega^v$; *supersedes the put and call option*

# Payoff and Actions

| Action | Payoff | Condition |
|---|---|---|
| Put | $K_t^p$ | $(V_t \leq K_t^p) \wedge (t \in \Omega^p) \wedge [(\kappa_t S_t < V_t) \vee (t \notin \Omega^v)]$ |
| Call | $K_t^c$ | $(V_t \geq K_t^c) \wedge (t \in \Omega^c) \wedge [(\kappa_t S_t < K_t^c) \vee (t \notin \Omega^v)]$ |
| Conversion | $\kappa_t S_t$ | $(\kappa_t S_t \geq V_t) \wedge (t \in \Omega^v)$ |
| Forced conversion | $\kappa_t S_t$ | $(V_t > \kappa_t S_t \geq K_t^c) \wedge (t \in \Omega^v) \wedge (t \in \Omega^c)$ |
| Redemption | $R$ | $(t = T) \wedge [(\kappa_t S_t \leq R) \vee (t \notin \Omega^v)]$ |
| Hold | | *otherwise* |

Table : Payoff for the convertible bond. $V_t$ is the intrinsic value of the derivative

# Payoff and Actions



# Credit Risk

## Definition. (Convertible Bond on Default)

*In default the convertible bond has the following components:*

- *Annuity: residual value*[a] *of* $\gamma R$
- *Conversion: stock price of* $(1-\eta)\kappa_t S_t$

*The value (payoff) the convertible bond is:*

$$X_t = \max(\gamma R, (1-\eta)\kappa_t S_t) \tag{1}$$

*Default is considered terminal.*

[a] This is assuming the bond recovery is based on the redemption value. Another possibility is to base the recovery on the bond value at time $t$.

# Outline

# Stock Process with Credit Risk (1/3)

**Definition. (Geometric Brownian with Credit Risk)**

*A stock process with geometric Brownian motion, with a jump component to model the default event, has the following stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{2}$$

*with $q_t$ being a Poison process with intensity $\lambda$ where the first event $q_t = 1$ is the default event.*

# Stock Process with Credit Risk (2/3)

**Corollary.**

*The stochastic process $q_t$, when representing terminal default, follows an exponential distribution with:*

$$q_t \sim \exp(\lambda) \tag{3}$$

$$\tilde{\mathbb{P}}(q_{t+dt} = 1 | q_t = 0) = \lambda dt \tag{4}$$

$$\tilde{\mathbb{E}}[dq_t] = \lambda dt \tag{5}$$

$$\tilde{\mathbb{V}}\mathrm{ar}[dq_t] = \lambda dt \tag{6}$$

# Stock Process with Credit Risk (3/3)

**Corollary.**

*Under risk neutral probability space the drift rate of $S_t$ is $\mu = (r + \lambda\eta)$ and the stochastic differential equation is:*

$$dS_t = (r + \lambda\eta)S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{7}$$

# Partial Differential Equation

**Theorem. (Black-Scholes with Default)**

*The price of an option, on an underlying $S_t$, has the following partial differential equation:*

$$(r+\lambda)V_t = \frac{\partial V}{\partial t} + (r+\lambda\eta)S_t\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} + \lambda X_t \tag{8}$$

# Derivation

The following steps are used in the derivation for Equation 8:

1. Create a portfolio $\Pi_t = V_t - \Delta_t S_t$
2. Invest the residual of $\Pi_t$ at the risk free rate: $d\Pi_t = r\Pi_t dt$
3. Equate the expected value of the derivative of $\Pi_t$ with the risk free rate:

$$\begin{aligned} r\Pi_t dt = \tilde{\mathbb{E}}[d\Pi_t] = \tilde{\mathbb{P}}(q_{t+dt} = 0)\,\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 0] + \\ \tilde{\mathbb{P}}(q_{t+dt} = 1)\,\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 1] \end{aligned} \quad (9)$$

4. Note that the derivative of $\Pi_t$ when going into default is:

$$\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 1] = X_t - (1-\eta)\Delta_t S_t - (V_t - \Delta_t S_t) \quad (10)$$

5. Choose $\Delta_t = \frac{\partial V}{\partial S}$ to eliminate the $d\tilde{W}_t$ terms

# Outline

# Binomial Tree (1/2)

**Definition. (Binomial Process with Default)**

*A binomial process with up movement $u$ and down movement $d$ and a terminal default event $(1-\eta)$ has the following mass distribution function:*

$$f(S_{t+\delta t}) = \begin{cases} p_u & \text{if } S_{t+\delta t} = uS_t \\ p_d & \text{if } S_{t+\delta t} = dS_t \\ p_o & \text{if } S_{t+\delta t} = (1-\eta)S_t \end{cases} \tag{11}$$

# Binomial Tree (2/2)

**Corollary.**

*The expected value and variance of the Binomial Process with Default is:*

$$\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right] = up_u + dp_d + (1-\eta)p_o \tag{12}$$

$$\tilde{\mathbb{V}}\mathrm{ar}\left[\frac{S_{t+\delta t}}{S_t}\right] = u^2 p_u + d^2 p_d + (1-\eta)^2 p_o - \tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \tag{13}$$

# Binomial Process (1/2)

## Theorem. (Binomial Process with Default)

*A Binomial Process with Default has the following parameters:*

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\delta t}} \\
d &= e^{-\sigma\sqrt{\delta t}} \\
p_u &= \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \\
p_d &= e^{-\lambda\delta t} - p_u \\
p_o &= 1 - e^{-\lambda\delta t} \\
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i
\end{aligned}
$$

# Binomial Process (2/2)

## Theorem. (Binomial Process with Default)

*A Binomial Process with Default has the following restrictions on the parameters:*

$$
\begin{aligned}
0 &< \sigma \\
0 &\leq \lambda \\
0 &< \delta t \\
\delta t &\leq \frac{\sigma^2}{r^2} \\
\delta t &\leq \frac{1}{\lambda} \ln \left( \frac{u - (1 - \eta)}{e^{r\delta t} - (1 - \eta)} \right)
\end{aligned}
$$

# Derivation (1/3)

The following steps are used in the derivation for the parameters of the Binomial Process with Default.

1. Note that the arrival time of a default event follows an exponential distribution with hazard rate $\lambda$:

$$p_o = 1 - e^{-\lambda\delta t} \tag{14}$$

$$p_d = e^{-\lambda\delta t} - p_u \tag{15}$$

2. Equate $\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]$ with $\tilde{\mathbb{E}}\left[\frac{S_{t+dt}}{S_t}\right] = e^{r\delta t}$, where $dt \approx \delta t$
3. Equate $\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+\delta t}}{S_t}\right]$ with $\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+dt}}{S_t}\right] = (\sigma^2 + \lambda\eta^2)\delta t$, where $dt \approx \delta t$
   1. Assume $\delta t^2 = 0$
   2. $ud = 1$
   3. $u = e^{\sqrt{A\delta t}}$
   4. Use Taylor series expansion for all exponential terms

# Derivation (2/3)

The following steps are used in the derivation for the valuation of the Binomial Process with Default.

1. Construct a portfolio $\Pi_t = V_t - \Delta_t S_t$ with:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d & \text{with probability } p_u \\ X_t - (1-\eta) S_t & \text{with probability } p_o \end{cases} \quad (16)$$

2. Invest the residual of $\Pi_t$ at the risk free rate: $\Pi_{t+\delta t} = \Pi_t e^{r\delta t}$
3. Equate the expected value of $\Pi_{t+\delta t}$ with the risk free rate:

$$\Pi_t e^{-r\delta t} = \tilde{\mathbb{E}}[\Pi_{t+\delta t}] \quad (17)$$

4. Choose $\Delta_t = \frac{V_t^u - V_t^d}{S_t(u-d)}$ to hedge against up and down movements of the stock

# Derivation (3/3)

The following steps are used in the derivation for the limits of the Binomial Process with Default.

1. Note that $\{p_u, p_d, p_o\}$ is required to be a valid probability set:

$$\min(p_u, p_d, p_o) \geq 0 \tag{18}$$

$$p_u + p_d + p_o = 1 \tag{19}$$

2. Using the relationship between $p_u$, $p_d$ and $p_o$ the following inequality needs to hold:

$$0 \leq p_u \leq e^{-\lambda \delta t} \tag{20}$$

3. Impose the requirements that $\lambda \delta t \geq 0$

# Outline

# Parameters for the Convertible Bond

Consider the following parameters of a Convertible Bond:

| **Component** | **Parameter** | **Value** |
|---|---|---|
| Annuity | Notional | 100 |
| | Coupon | 8% |
| | Coupon frequency | Semi-annually |
| | Maturity | $T := 5$ |
| | Recovery | $\gamma := 0\%$ |
| Put | Strike | $K_t^p := 105$ |
| | Period(s) | $\Omega^p := \{3\}$ |
| Call | Strike | $K_t^c := 110 + C_i \frac{t \pmod{0.5}}{0.5}$ |
| | Period(s) | $\Omega^c := [2, 5)$ |
| Conversion | Quantity of stocks | $\kappa_t := 1$ |
| | Period(s) | $\Omega^v := [0, 5]$ |

Table : Convertible Bond Parameters

# Parameters for the Stock

| Parameter | Value |
|---|---|
| Risk free rate | $r := 5\%$ |
| Volatility | $\sigma := 20\%$ |
| Hazard rate | $\lambda := 2\%$ |
| Default (total) | $\eta := 100\%$ |
| Default (typical[1]) | $\eta := 30\%$ |
| Default (partial) | $\eta := 0\%$ |

Table : Stock Parameters

[1] Beneish and Press (1995) found that stock prices typically drop 30% on announcement of default

# Payoff and Action

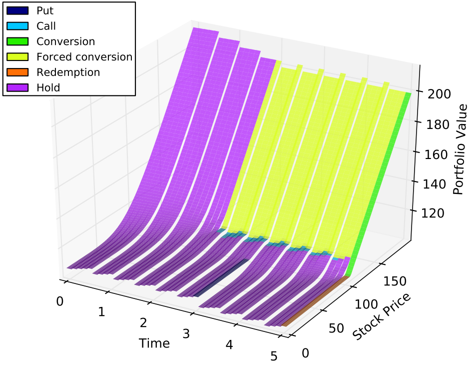



Figure : Payoff of the convertible bond with total default and colours indicating the action taken for that payoff. $\delta t = 2^{-3}$, $\delta S_t = 2^1$ and $S_t \in [0, 250]$.

# Simplified Call

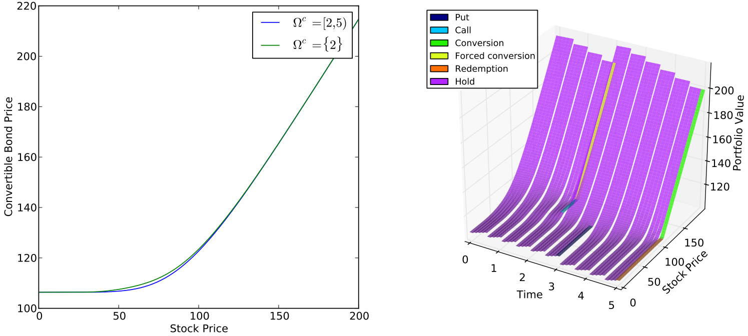



Figure : Comparison of initial value of the standard convertible bond compared to one with a singular time for the call provision. Also the payoff of the "simple call" and colours indicating the action taken for the payoff.

# Varying Call Time

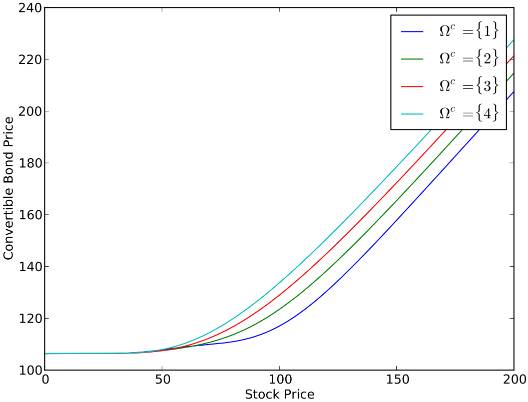



Figure : Comparison of initial value of convertible bonds with different times for the call provision.

# Varying Put Time

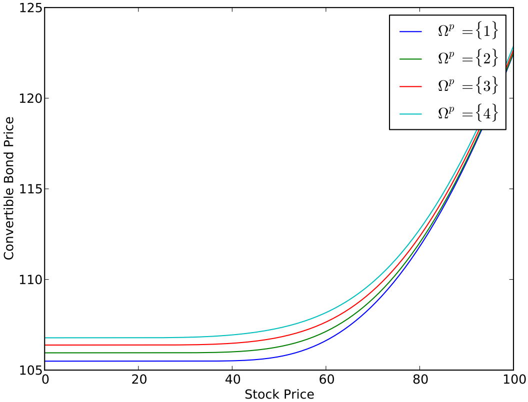



Figure : Comparison of initial value of convertible bonds with different times for the put provision.

# Varying Redemption Value

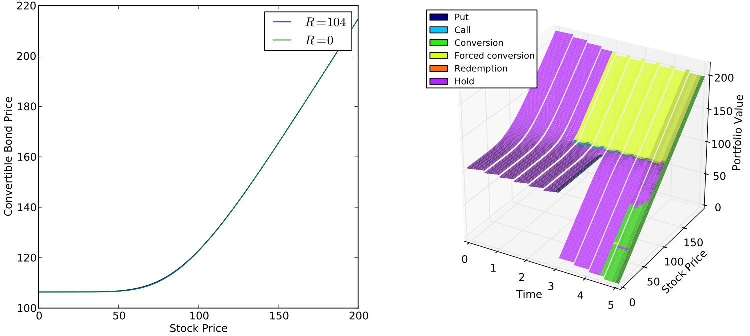



Figure : Comparison of initial value of standard convertible bonds to one with no redemption value. Also the payoff of the "no redemption" and colours indicating the action taken for the payoff.

# Outline

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