Valuation of Convertible Bonds with Credit Risk

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Bachelor of Science with Honours in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

A convertible bond is a complex derivative that cannot be priced as a simple combination of bond and stock components. Convertible bonds can be broken down as a bond with two embedded options (a put option for the investor and a call option for the issuer) and an option to convert the bond into stock. Due to the multiple continuous options, the pricing of the convertible bond is path dependent.

This research project explores and implements a binary tree and finite difference scheme to price the convertible bond, taking into account credit risk. Consideration is given to the sensitivities of the convertible bond and a possible hedging strategy.

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Chapter 1

Introduction

This research project aims to price convertible bonds based on the paper by Ayache et al. [4] and Milanov and Kounchev [22].

Convertible bonds are a hybrid instrument available on financial markets. The convertible bond is an instrument that is similar to a normal bond, except that the holder has the option to convert the bond into a specified number of shares. The convertible bond typically also has embedded options whereby the issuer may buy back the convertible bond for a specified price and whereby the investor can force the issuer to repurchase the convertible bond. In the event of a default by the issuer the bond could have partial recovery or total default.

1.1 Convertible Bonds with Credit Risk

A convertible bond is an annuity, which pays coupons C_i at t_i and pays R at maturity T, with the following embedded options:

- *A put option for the bond holder:* the bond holder may redeem the bond early for K_t^p, provided $t \in \Omega^p$. This option is not applicable at termination, $T \notin \Omega^p$.
- *A call option for the bond issuer:* the bond issuer may force early redemption of the bond for K_t^c, provided $t \in \Omega^c$. Normally $K_t^c > K_t^p$. This option is not applicable at termination, $T \notin \Omega^c$.
- *A conversion option:* the bond holder may convert the bond into κ_t shares, for a value of $\kappa_t S_t$, provided $t \in \Omega^v$. To optimise profits the bond holder would rather convert than exercise if $K_t^p < \kappa_t S_t$. This option of conversion supersedes the call option (i.e. if the bond issuer forces early redemption the bond holder may still convert the bond into shares). This option is applicable at termination, $T \in \Omega^v$.

In the event of default the bond holder may either recover the remains of the annuity or convert into the stock, whichever is worth more. Default is considered terminal.

Suppose the recovery of the bond is γ, and the drop in share price is η, then the value of the convertible bond in default is[1]:

$$X_t = \max(\gamma R, (1-\eta)\kappa_t S_t) \tag{1.1}$$

Table 1.1 illustrates the various choices made by the bond holder and the bond issuer and under what circumstances those choices are made [1]. Ayache et al. [4] uses an alternative method, a linear complementary problem, of describing the payoffs. Figure 4.1 illustrates the actions taken on the bond over the lifetime of a convertible bond. The impact of coupons is left as an implementation detail.

Action	**Payoff**	**Condition**
Put	K_t^p	$(V_t \leq K_t^p) \wedge (t \in \Omega^p) \wedge [(\kappa_t S_t < V_t) \vee (t \notin \Omega^v)]$
Call	K_t^c	$(V_t \geq K_t^c) \wedge (t \in \Omega^c) \wedge [(\kappa_t S_t < K_t^c) \vee (t \notin \Omega^v)]$
Conversion	$\kappa_t S_t$	$(\kappa_t S_t \geq V_t) \wedge (t \in \Omega^v)$
Forced conversion	$\kappa_t S_t$	$(V_t > \kappa_t S_t \geq K_t^c) \wedge (t \in \Omega^v) \wedge (t \in \Omega^c)$
Redemption	R	$(t = T) \wedge [(\kappa_t S_t \leq R) \vee (t \notin \Omega^v)]$
Hold		*otherwise*

Table 1.1: Payoff for the convertible bond. V_t is assumed to be the intrinsic value in this context only

1.2 Literature Review

Ingersoll [19] initially published on pricing a convertible bond with extensions from Brennan and Schwartz [10] and Brennan and Schwartz [9]. The original approach was to treat the bond and equity as components of the issuer's value and to treat default as when the issuer's value drops below a point where it can no longer meet its financial obligations. An overview of this type of approach is provided by Nyborg [23] and criticisms are addressed by Jarrow and Turnbull [20]. The main problems with this model is that the issuer's value is not directly observable, difficult to parameterise and all senior debt to the convertible bond also needs to be priced.

A second approach was to price the convertible bond based on the issuer's stock price. Ho and Pfeffer [17], Tsiveriotis and Fernandes [26] and McConnell

[1] This is assuming the bond recovery is based on the redemption value. Another possibility is to base the recovery on the bond value at time t

and Schwartz [21] are examples of those who implemented finite difference models. Davis and Lischka [12] first proposed using a trinomial tree, however did not implement one. Bardhan et al. [5] and Hull [18] both implemented binomial trees. A refined method, called “reduced form”, treats default as a discrete jump in time. Ayache et al. [4] was the first to allow stock jumps that did not go to zero. The probability of the loss jump over a short period of time is described by a hazard rate.

The Binomial Model with Credit Risk was first derived by Milanov and Kounchev [22], who showed that this model converges, in continuous time, to the Ayache et al. [4] model. Milanov and Kounchev [22] also showed the valuation method of this model is the same as done in the classical binomial model.

Also some Monte Carlo based pricing methods have been considered as proposed by Bossaerts [8] with Garcia [15] providing an optimisation approach to handle optimal early exercise of the American options. Further improvements to the pricing methods were done by Ammann et al. [1].

Other methods used for pricing include a finite element method by Barone-Adesi et al. [6] and a binomial tree method by Takahashi et al. [25] and Ayache et al. [3].

Chapter 2

Finite Difference Model

This section reproduces the derivation, using a different technique, of the stochastic model as described by Ayache et al. [4]. The parameters for the numerical solution, using a lattice construction, is explicitly derived.

2.1 Stochastic Process

Consider a stock with the following stochastic model for stock price movements:

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{2.1}$$

If we impose that the risk neutral expectation that the stock price evolution is the risk free rate, $\tilde{\mathbb{E}}[dS_t] = rS_t dt$, then $\mu = (r + \lambda\eta)$ and the differential equation becomes:

$$dS_t = (r + \lambda\eta) S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{2.2}$$

where:

μ	is the drift rate of the stock price
r	is the risk free rate
σ	is the log-volatility of the stock price
λ	is the hazard rate
η	is the percentage drop in stock price on a default event
$d\tilde{W}_t$	is a Wiener process
dq_t	is a Poisson jump process where the first jump is the default event

2.2 Derivation

Consider a portfolio, Π_t, that has a long position in the derivative, V_t, and a short position of Δ_t in the stock, S_t. The residual $\Pi_t = V_t - \Delta_t S_t$ is invested at the risk neutral rate. This yields a differential equation for Π_t:

$$\begin{aligned} d\Pi_t &= r\Pi_t dt \\ &= r(V_t - \Delta_t S_t)dt \end{aligned} \tag{2.3}$$

In the event of default the portfolio holdings Π_t^x has a long position in the derivative with residual value X_t and a short position of Δ_t in the stock with residual value of $(1-\eta)S_t$. Thus the difference of the portfolio holdings when going into default is:

$$\begin{aligned} d\Pi_t^x &= \Pi_t^x - \Pi_t \\ &= X_t - (V_t - \eta\Delta_t S_t) \end{aligned} \tag{2.4}$$

At any given instant the portfolio can either default or evolve in the normal stochastic manner. The probability of default in any given instant is λdt and the probability of non-default is $(1-\lambda)dt$. Taking the expectation of the differential of Π_t and applying Ito's lemma to the stochastic component:

$$\begin{aligned} \tilde{\mathbb{E}}_t[d\Pi_t] = (1-\lambda dt)[&\left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - \mu S_t \Delta_t\right) dt + \\ &\left(\sigma S_t \frac{\partial V}{\partial S} - \sigma S_t \Delta_t\right) d\tilde{W}_t] + \lambda dt(X_t - (V_t - \eta\Delta_t S_t)) \end{aligned} \tag{2.5}$$

By choosing Δ_t to eliminate the $d\tilde{W}_t$ term, $\Delta_t = \frac{\partial V}{\partial S}$, the differential simplifies to:

$$\tilde{\mathbb{E}}_t[d\Pi_t] = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \lambda dt(X_t - (V_t - \eta\Delta_t S_t)) \tag{2.6}$$

Equating the differential (2.3) and the expected differential (2.6) for Π_t the partial differential equation (PDE) for a derivative with credit risk is derived:

$$\begin{aligned} r(V_t - \Delta_t S_t)dt &= \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \lambda dt(X_t - (V_t - \eta\Delta_t S_t)) \\ (r+\lambda)V_t &= \frac{\partial V}{\partial t} + (r+\lambda\eta)S_t \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \lambda X_t \end{aligned} \tag{2.7}$$

2.3 Finite Difference

The derived PDE can be solved numerically by constructing a lattice of stock prices and time. V_j^n will denote the lattice value at the j$^{\text{th}}$ stock node and n$^{\text{th}}$ time node where the step between stock nodes is δS_t and the step between time nodes is δt. S_j will denote the stock value at the j$^{\text{th}}$ stock node and X_j will denote the defaulted value at the j$^{\text{th}}$ stock node. The stock prices go from a minimum of S_0 to a maximum of S_J.

All finite difference equations derived here can be expressed as functions of α, β and κ where:

$$\begin{aligned}
\alpha &= \tfrac{1}{2}\sigma^2 S_j^2 \frac{\delta t}{\delta S_t^2} - (r+\lambda\eta) S_j \frac{\delta t}{2\delta S_t} \\
\beta &= -(r+\lambda) - \sigma^2 S_j^2 \frac{\delta t}{\delta S_t^2} \\
\kappa &= \tfrac{1}{2}\sigma^2 S_j^2 \frac{\delta t}{\delta S_t^2} + (r+\lambda\eta) S_j \frac{\delta t}{2\delta S_t}
\end{aligned}$$

and it is assumed that for boundary nodes:

$$\begin{aligned}
\frac{V_0^n - V_{-1}^n}{\delta S_t} &= \frac{V_1^n - V_0^n}{\delta S_t} \\
V_{-1}^n &= 2V_0^n - V_1^n \\
\frac{V_J^n - V_{J-1}^n}{\delta S_t} &= \frac{V_{J+1}^n - V_J^n}{\delta S_t} \\
V_{J+1}^n &= 2V_J^n - V_{J-1}^n
\end{aligned}$$

2.3.1 Explicit Scheme

Using the backwards difference equation for the temporal derivative the finite difference equation is:

$$(r+\lambda)V_j^n = \frac{V_j^n - V_j^{n-1}}{\delta t} + (r+\lambda\eta)S_j \frac{V_{j+1}^n - V_{j-1}^n}{2\delta S_t} + \tfrac{1}{2}\sigma^2 S_j^2 \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\delta S_t^2} + \lambda X_j$$

$$V_j^{n-1} = \alpha V_{j-1}^n + (1+\beta)V_j^n + \kappa V_{j+1}^n + \lambda \delta t X_j \tag{2.8}$$

$$V_0^{n-1} = (1+\beta+2\alpha)V_0^n + (\kappa-\alpha)V_1^n + \lambda \delta t X_j \tag{2.9}$$

$$V_J^{n-1} = (\alpha-\kappa)V_{j-1}^n + (1+\beta-2\kappa)V_J^n + \lambda \delta t X_j \tag{2.10}$$

2.3.2 Implicit Scheme

Using the backwards difference equation for the temporal derivative and evaluating the spacial differentials at the previous time step the finite difference equation is:

$$(r+\lambda)V_j^{n-1} = \frac{V_j^n - V_j^{n-1}}{\delta t} + (r+\lambda\eta)S_j \frac{V_{j+1}^{n-1} - V_{j-1}^{n-1}}{2\delta S_t} + \tfrac{1}{2}\sigma^2 S_j^2 \frac{V_{j-1}^{n-1} - 2V_j^{n-1} + V_{j+1}^{n-1}}{\delta S_t^2} + \lambda X_j$$

$$-\alpha V_{j-1}^{n-1} + (1-\beta)V_j^{n-1} - \kappa V_{j+1}^{n-1} = V_j^n + \lambda \delta t X_t \tag{2.11}$$

$$(1-\beta-2\alpha)V_0^{n-1} - (\kappa-\alpha)V_1^{n-1} = V_0^n + \lambda \delta t X_t \tag{2.12}$$

$$-(\alpha-\kappa)V_{J-1}^{n-1} + (1-\beta-2\kappa)V_J^{n-1} = V_J^n + \lambda \delta t X_t \tag{2.13}$$

2.3.3 Crank-Nicolson Scheme

Using the backwards difference equation for the temporal derivative and taking the average of the spacial differentials at the current and previous time step the finite difference equation is:

$$(r+\lambda)\tfrac{1}{2}(V_j^n + V_j^{n-1}) = \frac{V_j^n - V_j^{n-1}}{\delta t} + (r+\lambda\eta)S_j \frac{1}{2}\left(\frac{V_{j+1}^n - V_{j-1}^n}{2\delta S_t} + \frac{V_{j+1}^{n-1} - V_{j-1}^{n-1}}{2\delta S_t}\right) + \tfrac{1}{2}\sigma^2 S_j^2 \frac{1}{2}\left(\frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\delta S_t^2} + \frac{V_{j-1}^{n-1} - 2V_j^{n-1} + V_{j+1}^{n-1}}{\delta S_t^2}\right) + \lambda X_j$$

$$-\alpha V_{j-1}^{n-1} + (2-\beta)V_j^{n-1} - \kappa V_{j+1}^{n-1} = \alpha V_{j-1}^n + (2+\beta)V_j^n + \kappa V_{j+1}^n + 2\lambda\delta t X_t \tag{2.14}$$

$$(2-\beta-2\alpha)V_0^{n-1} - (\kappa-\alpha)V_1^{n-1} = (2+\beta+2\alpha)V_0^n + (\kappa+\alpha)V_1^n + 2\lambda\delta t X_t \tag{2.15}$$

$$-(\alpha-\kappa)V_{J-1}^{n-1} + (2-\beta-2\kappa)V_J^{n-1} = (\alpha-\kappa)V_{J-1}^n + (2+\beta+2\kappa)V_J^n + 2\lambda\delta t X_t \tag{2.16}$$

Chapter 3

Binomial Model

This section reproduces the derivation, using a different technique, of the binomial model as described by Milanov and Kounchev [22]. An extension is added by considering the hedging strategy.

3.1 Binomial Process

Consider a binomial model that has time step δt, up and down steps, u and d with probability p_u and p_d[1] respectively, and where the probability of default is p_o[2]. The mean and variance of this model is:

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right] &= up_u + dp_d + (1-\eta)p_o \\
&= up_u + d(e^{-\lambda\delta t} - p_u) + (1-\eta)(1-e^{-\lambda\delta t}) \qquad (3.1)\\
\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+\delta t}}{S_t}\right] &= u^2p_u + d^2p_d + (1-\eta)^2p_o - \tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \\
&= u^2p_u + d^2(e^{-\lambda\delta t} - p_u) + (1-\eta)^2(1-e^{-\lambda\delta t}) - \tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \qquad (3.2)
\end{aligned}$$

[1] $p_d = 1 - p_u - p_o = e^{-\lambda\delta t} - p_u$

[2] The time till the first jump follows an exponential distribution with intensity λ and has probability $\tilde{\mathbb{P}}(q_{t+dt} = 1 | q_t = 0) = p_o = 1 - e^{-\lambda\delta t}$

The mean and variance of the stock's stochastic differential equation is:

$$\tilde{\mathbb{E}}\left[\frac{dS_t}{S_t}\right] = rdt \tag{3.3}$$

$$\begin{aligned}\tilde{\mathbb{E}}\left[\frac{S_{t+dt}}{S_t}\right] &= \tilde{\mathbb{E}}\left[\frac{S_t}{S_t}+\frac{dS_t}{S_t}\right]\\ &= 1+\tilde{\mathbb{E}}\left[\frac{dS_t}{S_t}\right]\end{aligned} \tag{3.4}$$

$$\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{dS_t}{S_t}\right] = (\sigma^2+\lambda\eta^2)dt \tag{3.5}$$

$$\begin{aligned}\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+dt}}{S_t}\right] &= \tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_t}{S_t}+\frac{dS_t}{S_t}\right]\\ &= \tilde{\mathbb{V}}\mathtt{ar}\left[\frac{dS_t}{S_t}\right]\end{aligned} \tag{3.6}$$

3.2 Derivation

If one equates the first moment of the stochastic model (3.4)[3], in δt time, with that of the binomial model (3.1) then:

$$e^{r\delta t} = up_u + d(e^{-\lambda\delta t}-p_u)+(1-\eta)(1-e^{-\lambda\delta t})$$

$$p_u(u-d) = e^{r\delta t}-de^{-\lambda\delta t}-(1-\eta)(1-e^{-\lambda\delta t}) \tag{3.7}$$

$$p_u = \frac{e^{r\delta t}-de^{-\lambda\delta t}-(1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{3.8}$$

$$\therefore \quad p_d = -\frac{e^{r\delta t}-ue^{-\lambda\delta t}-(1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{3.9}$$

and if one equates the second moment about the mean of the stochastic model (3.6), in δt time, with that of the binomial model (3.2) then[4]:

$$(\sigma^2+\lambda\eta^2)\delta t = u^2p_u + d^2(e^{-\lambda\delta t}-p_u)+(1-\eta)^2(1-e^{-\lambda\delta t}) - \tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]^2$$

$$= (u^2-d^2)p_u + d^2e^{-\lambda\delta t}+(1-\eta)^2(1-e^{-\lambda\delta t})-e^{-2r\delta t} \tag{3.10}$$

$$\begin{aligned}&= (u+d)(e^{r\delta t}-de^{-\lambda\delta t}-(1-\eta)(1-e^{-\lambda\delta t}))\\ &\quad + d^2e^{-\lambda\delta t}+(1-\eta)^2(1-e^{-\lambda\delta t})-e^{-2r\delta t}\end{aligned} \tag{3.11}$$

$$\begin{aligned}&= (u+d)(e^{r\delta t}-(1-\eta)(1-e^{-\lambda\delta t}))-ude^{-\lambda\delta t}\\ &\quad +(1-\eta)^2(1-e^{-\lambda\delta t})-e^{-2r\delta t}\end{aligned} \tag{3.12}$$

[3] $\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right] = e^{r\delta t}$

[4] Substituting equation (3.7) into (3.10) one gets (3.11)

If one assumes $\delta t^2 = 0$, $ud = 1$ and $u = e^{\sqrt{A\delta t}}$ and the Taylor series expansion is taken for all exponential terms, then:

$$\begin{aligned}
u &= 1 + \sqrt{A\delta t} + \frac{A\delta t}{2!} + \frac{(A\delta t)^{\frac{3}{2}}}{3!} \\
d &= 1 - \sqrt{A\delta t} + \frac{A\delta t}{2!} - \frac{(A\delta t)^{\frac{3}{2}}}{3!} \\
u + d &= 2 + A\delta t
\end{aligned} \tag{3.13}$$

and substituting (3.13) into (3.12) with the appropriate expansions then:

$$\begin{aligned}
(\sigma^2 + \lambda\eta^2)\delta t &= (2 + A\delta t)(1 + r\delta t - \lambda\delta t(1-\eta)) - (1 - \lambda\delta t) \\
&\quad + \lambda\delta t(1-\eta)^2 - (1 - 2r\delta t) \\
&= A\delta t + \lambda\delta t - \lambda\delta t(1-\eta)(1+\eta) \\
&= A\delta t + \lambda\eta^2\delta t \\
A\delta t &= \sigma^2\delta t
\end{aligned} \tag{3.14}$$

3.3 Restrictions

The time step δt is assumed to be strictly positive ($\delta t > 0$), as time is monotonically increasing.

Although σ could be positive or negative, the formulæ use σ where, strictly, $|\sigma|$ should be used. If $\sigma = 0$ then the model is no longer useful. The restriction $\sigma > 0$ may be introduced for convenience and without loss of generality.

An implicit assumption this model makes is that p_u, p_d and p_o are a valid probability measure. This imposes the following conditions:

$$\min(p_u, p_d, p_o) \geq 0 \tag{3.15}$$

$$p_u + p_d + p_o = 1 \tag{3.16}$$

Consider p_o, the probability of default. Its value is equal to the cumulative density function of an exponential distribution with intensity λ. As such p_o is already confined to the interval $[0, 1]$ with the restriction that $\lambda > 0$. If $\lambda = 0$, no default is possible, then p_o will have a value of 0 and is a valid probability thus the restriction on λ can be relaxed to $\lambda \geq 0$.

Consider p_d, the probability of a down movement. Its value is defined in terms of p_u and p_o. If p_u and p_o are valid probabilities, within bounds, then so will p_d.

Consider p_u, the probability of an up movement. Taking into account the definition of p_d, this requires that p_u satisfies:

$$
\begin{aligned}
0 &\leq p_u \leq e^{-\lambda\delta t} && (3.17)\\
0 &\leq \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \leq e^{-\lambda\delta t} \\
0 &\leq e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t}) \leq e^{-\lambda\delta t}(u-d) \\
e^{-\lambda\delta t}(d-(1-\eta)) &\leq e^{r\delta t} - (1-\eta) \leq e^{-\lambda\delta t}(u-(1-\eta)) \\
\ln\left(\frac{d-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) &\leq \lambda\delta t \leq \ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) && (3.18)\\
\delta t &\leq \frac{1}{\lambda}\ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) && (3.19)
\end{aligned}
$$

Using the limit on both δt and λ the inequalities around the $\lambda\delta t$ are also limited thus:

$$
\begin{aligned}
0 &\leq \ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \\
e^{r\delta t} - (1-\eta) &\leq u - (1-\eta) \\
r\delta t &\leq \sigma\sqrt{\delta t} \\
\delta t &\leq \frac{\sigma^2}{r^2} && (3.20)
\end{aligned}
$$

The inequality of $\ln\left(\frac{d-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right) \leq 0$ results in the same inequality as above.

Based on the probability limits (3.16) no restriction is placed on r, the risk free rate, or η, the percentage drop in stock price.

3.4 Valuation

3.4.1 Valuation without Coupons

Consider a portfolio, Π_t, that has a long position in the derivative, V_t, and a short position of Δ_t in the stock, S_t. The residual $\Pi_t = V_t - \Delta_t S_t$ is invested at the risk neutral rate. At time $t + \delta t$ one should have:

$$
\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d & \text{with probability } p_d \\ X_t - (1-\eta)\Delta_t S_t & \text{with probability } p_o \end{cases}
$$

where

- V_t^u is the value of the portfolio on an up-step at time $t + \delta t$
- V_t^d is the value of the portfolio on a down-step at time $t + \delta t$
- X_t is the value of default at time t

If one wishes to hedge against up and down movements of the stock then:

$$\begin{aligned}
V_t^u - \Delta_t S_t u &= V_t^d - \Delta_t S_t d \\
\Delta_t S_t (d-u) &= V_t^d - V_t^u \\
\Delta_t &= \frac{V_t^u - V_t^d}{S_t(u-d)}
\end{aligned} \tag{3.21}$$

The required value of Π_t at time $t + \delta t$ is:

$$\begin{aligned}
\Pi_{t+\delta t} &= \Pi_t e^{r\delta t} \\
&= \left(V_t - \frac{V_t^u - V_t^d}{u-d} \right) e^{r\delta t}
\end{aligned} \tag{3.22}$$

as Π_t is invested at the risk free rate, and taking the expected value of $\Pi_{t+\delta t}$, which must equal (3.22), one gets:

$$\begin{aligned}
\left(V_t - \frac{V_t^u - V_t^d}{u-d} \right) e^{r\delta t} &= \tilde{\mathbb{E}}_t[\Pi_{t+\delta t}] \\
&= \left(V_t^u - \frac{V_t^u - V_t^d}{u-d} u \right) e^{-\lambda\delta t} \\
&\quad + \left(X_t - \frac{V_t^u - V_t^d}{u-d}(1-\eta) \right)(1 - e^{-\lambda\delta t}) \\
&= V_t^u \left(e^{-\lambda\delta t} - \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \right) \\
&\quad + V_t^d \left(\frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \right) + X_t(1-e^{-\lambda\delta t}) \\
V_t e^{r\delta t} &= V_t^u \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \\
&\quad + V_t^d \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t}) - e^{r\delta t}}{u-d} + X_t(1-e^{-\lambda\delta t}) \\
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o)
\end{aligned} \tag{3.23}$$

thus the pricing method of this binomial model is the same as the pricing method of the classical binomial model:

$$\begin{aligned}
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) \\
&= e^{-r\delta t} \tilde{\mathbb{E}}_t[V_{t+\delta t}]
\end{aligned} \tag{3.24}$$

3.4.2 Valuation with Coupons

Consider a portfolio, as above, with the add of coupon cash flows. At time $t + \delta t$ one should have:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u + c_i e^{t+\delta t - t_i^c} & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d + c_i e^{t+\delta t - t_i^c} & \text{with probability } p_d \\ X_t - (1-\eta)S_t + c_i(1 - q_{t_i^c})e^{t+\delta t - t_i^c} & \text{with probability } p_o \end{cases}$$

where

c_i is the i^th coupon value

t_i^c is the time of the i^th coupon, with $t_i^c \in (t, t+\delta t)$

$q_{t_i^c}$ is an indicator variable that default happens after coupon payment[5]

Using the above method it is trivial to show that:

$$\begin{aligned} V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i e^{-(r+\lambda)(t_i^c - t)} \\ &= e^{-r\delta t}\tilde{\mathbb{E}}_t[V_{t+\delta t}] + c_i e^{-(r+\lambda)(t_i^c - t)} \end{aligned} \tag{3.25}$$

if, however, $t_i^c = t$, thus c_i arrives with certainty at t, it is trivially shown that:

$$\begin{aligned} V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i \\ &= e^{-r\delta t}\tilde{\mathbb{E}}_t[V_{t+\delta t}] + c_i \end{aligned} \tag{3.26}$$

3.5 Synthesising and Hedging Strategy

Consider the strategy where $V_t - \Delta_t S_t$ is invested at the risk free rate and $\Delta_t S_t$ is held in stock. This creates the portfolio $\Pi_t = \Delta_t S_t + (V_t - \Delta_t S_t) = V_t$ and in one time step the portfolio is worth:

$$\Pi_{t+\delta t} = \begin{cases} \Delta_t S_t u + e^{r\delta t}(V_t - \Delta_t S_t) & \text{with probability } p_u \\ \Delta_t S_t d + e^{r\delta t}(V_t - \Delta_t S_t) & \text{with probability } p_d \\ (1-\eta)\Delta_t S_t + e^{r\delta t}(V_t - \Delta_t S_t) & \text{with probability } p_o \end{cases} \tag{3.27}$$

[5] $\tilde{\mathbb{P}}(q_{t_i^c} = 0 | q_s = 1, s \in (t, t+\delta t]) = \frac{e^{-\lambda(t_i^c - t)} - e^{-\lambda\delta t}}{1 - e^{-\lambda\delta t}}$, the probability that the coupon arrives before default occurs, given that default occurs within the interval

Now consider the portfolio value for an up-movement in stock:

$$\begin{aligned}
\Pi^u_{t+\delta t} &= \Delta_t S_t u + e^{r\delta t}(V_t - \Delta_t S_t) \\
&= \Delta_t S_t(u - e^{r\delta t}) + e^{r\delta t} V_t \\
&= (V^u_t - V^d_t)\left(\frac{u - e^{r\delta t}}{u-d}\right) + (V^u_t p_u + V^d_t p_d + X_t p_o) \\
&\quad + V^d_t \left(\frac{ue^{-\lambda\delta t} + (1-\eta)p_o - e^{r\delta t}}{u-d}\right) + X_t p_o \Big) \\
&= V^u_t + \left(V^u_t\left(\frac{d - de^{-\lambda\delta t} - (1-\eta)p_o}{u-d}\right)\right. \\
&\quad \left. + V^d_t \left(\frac{ue^{-\lambda\delta t} + (1-\eta)p_o - u}{u-d}\right) + X_t p_o\right) \\
&= V^u_t + \left(\left(\frac{V^u_t d - V^d_t u}{u-d}\right) - (1-\eta)\left(\frac{V^u_t - V^d_t}{u-d}\right) + X_t\right) p_o \\
&= V^u_t + \left(-V^u_t + \left(\frac{V^u_t - V^d_t}{u-d}\right) u - (1-\eta)\Delta_t S_t + X_t\right) p_o \\
&= V^u_t + (X_t - (V^u_t - \Delta_t S_t u) - (1-\eta)\Delta_t S_t) p_o
\end{aligned} \tag{3.28}$$

Similarly the portfolio value for a down-movement in stock:

$$\Pi^d_{t+\delta t} = V^d_t + (X_t - (V^d_t - \Delta_t S_t d) - (1-\eta)\Delta_t S_t) p_o \tag{3.29}$$

The portfolio value for a default in stock:

$$\begin{aligned}
\Pi^o_{t+\delta t} &= (1-\eta)\Delta_t S_t + e^{r\delta t}(V_t - \Delta_t S_t) \\
&= \Delta_t S_t((1-\eta) - e^{r\delta t}) + e^{r\delta t} V_t \\
&= (V^u_t - V^d_t)\left(\frac{(1-\eta) - e^{r\delta t}}{u-d}\right) + (V^u_t p_u + V^d_t p_d + X_t p_o) \\
&= X_t p_o - V^u_t \left(\frac{de^{-\lambda\delta t} - (1-\eta)e^{-\lambda\delta t}}{u-d}\right) + V^d_t \left(\frac{ue^{-\lambda\delta t} - (1-\eta)e^{-\lambda\delta t}}{u-d}\right) \\
&= X_t p_o - \left(\frac{V^u_t d - V^d_t u}{u-d}\right) e^{-\lambda\delta t} + (1-\eta)\left(\frac{V^u_t - V^d_t}{u-d}\right) e^{-\lambda\delta t} \\
&= X_t p_o + ((V^u_t - \Delta_t S_t u) + (1-\eta)\Delta_t S_t) e^{-\lambda\delta t} \\
&= X_t - (X_t - (V^u_t - \Delta_t S_t u) - (1-\eta)\Delta_t S_t) e^{-\lambda\delta t}
\end{aligned} \tag{3.30}$$

Consider $H^c_t = X_t - (V^u_t - \Delta_t S_t u) - (1-\eta)\Delta_t S_t$, which could be considered the un-adjusted[6] default cost after hedging. $H^c_t p_o$ is received for both an up- and down-step in stock price, so $H^c_t p_o$ is received $e^{-\lambda\delta t}$ of the time and $H^c_t e^{-\lambda\delta t}$ is paid p_o of the time, on balance.

[6] Adjustments need to be made for the probability of default

An important note about using the self-financing property of this model in practise is that it will require a large pool of uncorrelated portfolios with the same adjusted default cost after hedging and actuarial consideration to effectively handle the "premiums" and reserves.

Equation 3.27 can be rewritten as:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u + H_t^c p_o & \text{with probability } p_u \\ V_t^d + H_t^c p_o & \text{with probability } p_d \\ X_t - H_t^c e^{-\lambda \delta t} & \text{with probability } p_o \end{cases} \tag{3.31}$$

It is trivial to confirm that the expected value of $\Pi_{t+\delta t}$ is $e^{r\delta t}\Pi_t$.

3.6 Parameters and Formulæ of Model

Based on the above, Table 3.1 lists the parameters required for this binomial tree, Table 3.2 lists the limits of this binomial tree, and Table 3.3 lists the formulæ to price this binomial tree.

Parameter	**Description**
r	Risk free rate
σ	Log-volatility of the stock price
λ	Hazard rate of default
η	Percentage drop of stock price in a default event
δt	Time-step
V_t^u	Value of portfolio for an up-step at $t + \delta t$
V_t^d	Value of portfolio for a down-step at $t + \delta t$
X_t	Value of defaulted portfolio at t
c_i	Value of the i^{th} coupon payment
t_i^c	Time of the i^{th} coupon payment

Table 3.1: Parameters of the Binomial Model with Credit Risk

Limit	**Description**
$0 < \sigma$	Volatility must be positive
$0 \leq \lambda$	Hazard rate must be non-negative
$0 < \delta t$	Time step must be positive
$\delta t \leq \frac{\sigma^2}{r^2}$	Time step must be small enough to handle volatility[7]
$\delta t \leq \frac{1}{\lambda} \ln\left(\frac{u-(1-\eta)}{e^{r\delta t}-(1-\eta)}\right)$	Time step must be small enough to handle hazard rate[8]

Table 3.2: Limits of the Binomial Model with Credit Risk

Formulæ	**Description**
$u = e^{\sigma\sqrt{\delta t}}$	Multiplier for an up-step
$d = e^{-\sigma\sqrt{\delta t}}$	Multiplier for a down-step
$p_u = \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d}$	Probability of an up-step
$p_d = e^{-\lambda\delta t} - p_u$	Probability of a down-step
$p_o = 1 - e^{-\lambda\delta t}$	Probability of default
$V_t = e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i$	Value of portfolio at t for $t_i^c = t$
$\Delta_t = \frac{V_t^u - V_t^d}{S(u-d)}$	Delta ratio for hedging
$H_t^c = X_t - (V_t^u - \Delta_t S_t u) - (1-\eta)\Delta_t S_t$	Unadjusted default cost after hedging

Table 3.3: Formalæ for the Binomial Model with Credit Risk

[7] If $r = 0$ then this inequality is not applicable.

[8] If $\lambda = 0$ then this inequality is not applicable.

Chapter 4

Numerical Example

In the following analysis of the numerical results Table 4.1, the bond parameters, and Table 4.2, the stock parameters, are used as a basis. There are three different types of stock price decreases that are used: total default when the stock losses all value, typical default when the stock losses 30% of its value and partial default where the stock price does not lose any value.

If t^+ is the time immediately after a coupon payment and t^- is the immediately before a coupon payment then the coupon payment is handled as follows:

$$V_{t_i^-} = V_{t_i^+} + C_i \tag{4.1}$$

and it is assumed that embedded options can only be exercised after receiving the coupon payment (i.e. at t_i^+).

Component	**Parameter**	**Value**
Annuity	Notional[1]	100
	Coupon[2]	8%
	Coupon frequency[3]	Semi-annually
	Maturity	$T := 5$
	Recovery	$\gamma := 0\%$
Put	Strike	$K_t^p := 105$
	Period(s)	$\Omega^p := \{3\}$
Call	Strike[4]	$K_t^c := 110 + C_i \frac{t \pmod{0.5}}{0.5}$
	Period(s)	$\Omega^c := [2, 5)$
Conversion	Quantity of stocks	$\kappa_t := 1$
	Period(s)	$\Omega^v := [0, 5]$

Table 4.1: Convertible Bond Parameters

[1] $R := 104$

Parameter	Value
Risk free rate	$r := 5\%$
Volatility	$\sigma := 20\%$
Hazard rate	$\lambda := 2\%$
Default (total)	$\eta := 100\%$
Default (typical[5])	$\eta := 30\%$
Default (partial)	$\eta := 0\%$

Table 4.2: Stock Parameters

4.1 Sensitivity

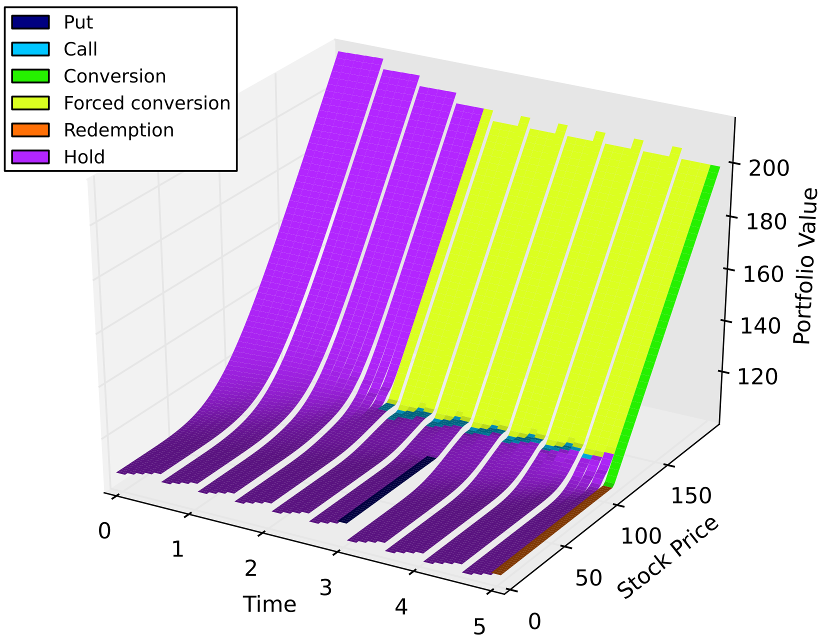

Figure 4.1: Payoff of the convertible bond, with total default, and colours indicating the action taken for that payoff. $\delta t = 2^{-3}$, $\delta S_t = 2^1$ and $S_t \in [0, 250]$.

[2] $C_i := 4$

[3] $t_i := \{0.5, 1, \ldots, 4.5\}$

[4] The strike of 110 is the clean price and needs to be adjusted for accrued interest, on a simple interest basis

[5] Beneish and Press [7] found that stock prices typically drop 30% on announcement of default

Figure 4.1 illustrates the payoff of the convertible bond over its lifetime and the actions that led to the payoff (see Table 1.1 for the list of actions, the associated payoff and the conditions required for the payoff).

The figure has clear jumps at each coupon payment, producing a saw effect. The other influencing factors are shown below:

- *Put:* the "Put" action is taken at time $t = 3$ at stock prices from $S_t = 0$ till $S_t \approxeq 75$. This option causes the value of the convertible bond prior to time $t = 3$ to be elevated due to the higher payoff than the intrinsic value.

- *Call:* the "Call" action is taken from time $t = 2$ till almost $t = 5$ with a narrow plateau around stock prices around $S_t = 110$. The call option does reduce the intrinsic value as apposed to being just the base for the forced conversion.

- *Conversion:* on termination the "Conversion" action is taken for $S_t \geq 104$. At no other time is there voluntary conversion as, at time $t = 2$ the upper payoff is similar to that of a forward and for similar reasons the value of the convertible bond increases. Due to the increasing value the intrinsic value is never below the conversion threshold.

- *Forced conversion:* the "Forced conversion" action is taken from time $t = 2$ until almost $t = 5$ above stock price $S_t \approxeq 110$.

- *Redemption:* on termination the "Redemption" action is taken for $S_t < 104$.

- *Hold:* the convertible bond is held for all other areas. Of particular note is at time $t = 3$ there is a narrow "window" from stock price $S_t \approxeq 75$ to $S_t \approxeq 110$ where the bond holder will not redeem, or be forced to redeem, the bond. It is apparent that the changes of holding the bond beyond time $t = 3$ is quite small. Due to this the convertible bond value is dominated by the put, call and conversion options with the redemption value, and coupons after time $t = 3$ have minimal impact.

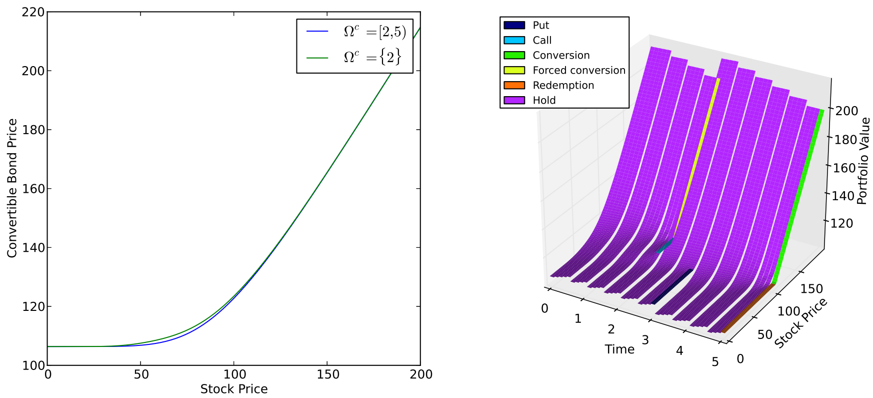

Figure 4.2: Comparison of initial value of the standard convertible bond, with total default, compared to one with a singular time for the call option. Also the payoff of the "simple call" and colours indicating the action taken for the payoff.

The above analysis suggests that the majority of impact that the call option has on the convertible bond's value is the leading edge at time $t = 2$ and Figure 4.2 illustrates the minimal change in price due to having a call with a single exercise time of $\Omega^c = \{2\}$.

In considering the sensitivity of the convertible bond to changes in each of the four components consider the following figures.

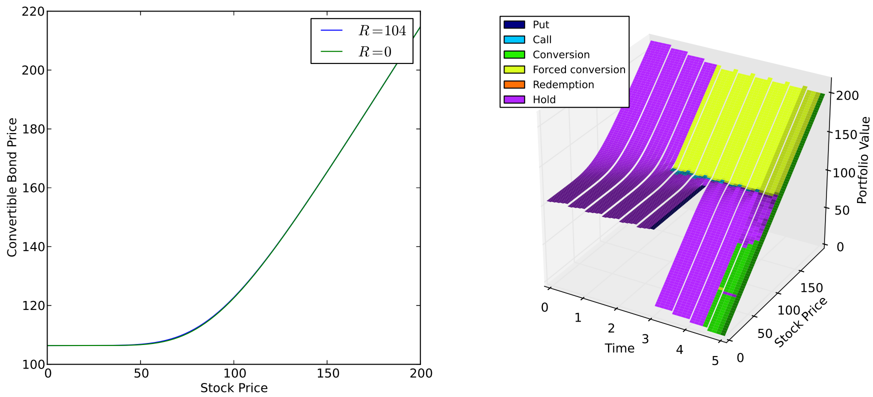

Figure 4.3: Comparison of initial value of standard convertible bonds, with total default, to one with no redemption value. Also the payoff of the "no redemption" and colours indicating the action taken for the payoff.

Annuity: Figure 4.3 compares the convertible bond, with total default, with normal and no redemption value. The payoff surface shows that at time $t = 3$ the put effectively "resets" the payoff profile to that of a normal redemption value.

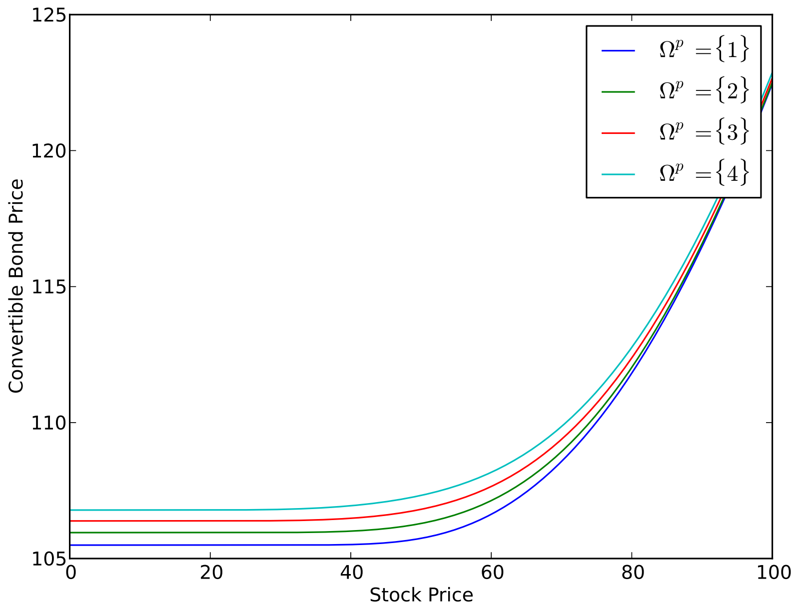

Figure 4.4: Comparison of initial value of convertible bonds, with total default, with different times for the put option.

Put: Figure 4.4 compares the price of a convertible bond with differing put times. The figure illustrates that the price of the bond is correlated with the put exercise time. This effect happens because the coupon rate (converted to NACC) is greater than the risk free rate. If the coupon rate were lower that an inverse correlation would be observed. The effect dominant for $S_t \in [0, 50]$ and tapers of until almost no impact by $S_t = 100$.

Call: Figure 4.5 compares the price of a convertible bond with differing call times. The figure illustrates that the price of the bond is correlated with the call exercise time. This effect happens because the payoff behaves similarly to a forward on the stock and thus is increasing with time.

Conversion: Figure 4.6 compares the price of a convertible bond with differing conversion times. The figure illustrates the impact the conversion option, in conjunction with the call option, has on the price. For conversion intervals that overlap with the leading edge of the call option (i.e. $\Omega^v = [1, 5]$ and $\Omega^v = [2, 4]$) the value is unchanged however when there is no overlap (i.e. $\Omega^v = [3, 5]$ and $\Omega^v = [4, 4]$) the upper section of the convertible bond's value is significantly suppressed limiting around the strike price of the call option.

As seen above the convertible bond is insensitive to changes in the redemption value and the majority of the value is determined by the leading edge of the call and put options (i.e. $\min(\Omega^c)$ and $\min(\Omega^p)$). The call option influences the upper section, and the put option influences the lower section, of the pricing curve of the convertible bond. Another important criteria of the sensitivity is whether the

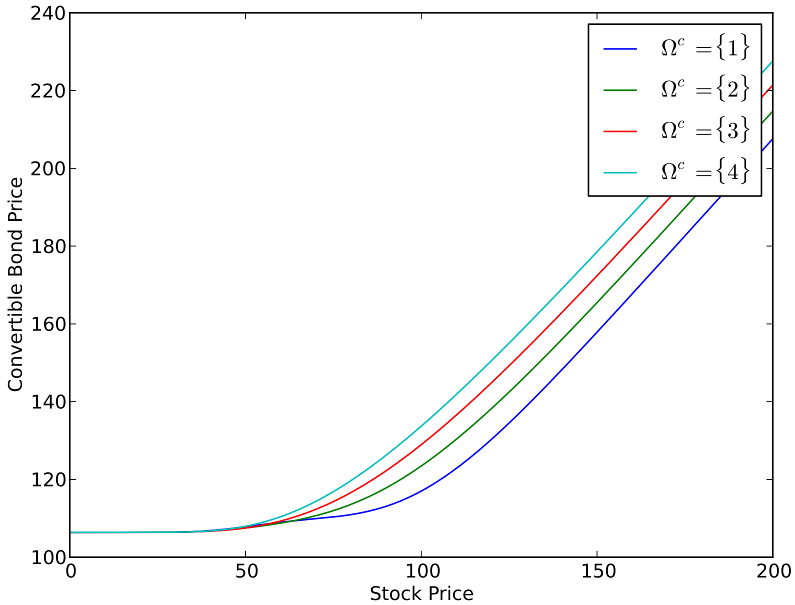

Figure 4.5: Comparison of initial value of convertible bonds, with total default, with different times for the call option.

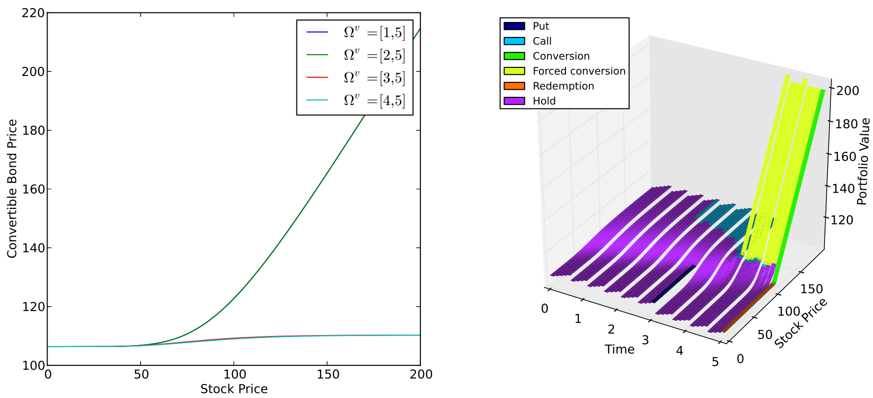

Figure 4.6: Comparison of initial value of convertible bonds, with total default, with different times for the conversion option. Also the payoff of for $\Omega^v = [4,5]$ and colours indicating the action taken for the payoff.

conversion option happens on or before the leading edge of the call option or after the leading edge of the call option.

4.2 Default Cost after Hedging

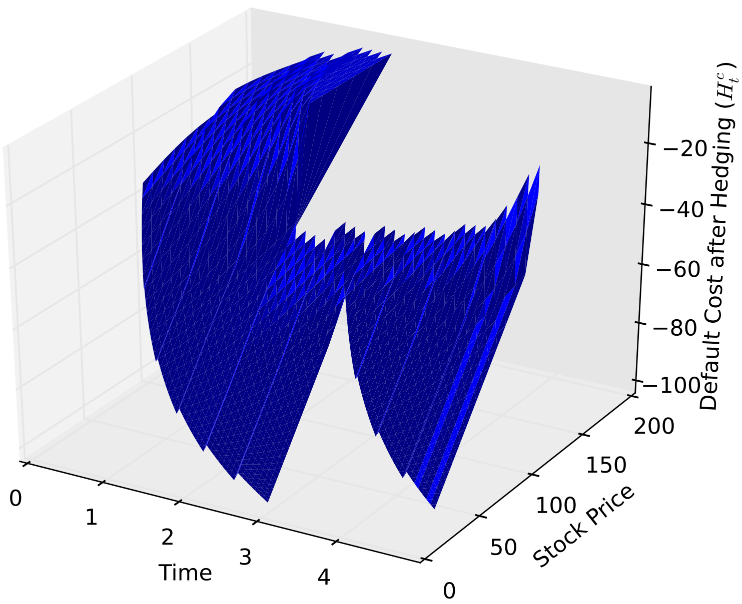

Figure 4.7: Default Cost after Hedging, H_t^c, for a convertible bond with typical default.

Figure 4.7 shows the un-adjusted default cost after hedging. The shape shows areas where hedging was not possible: the upper right where the call and conversion (forced) options applied and the lower right where the put option limits the stock's ability to extend all the way down. The hedging strategy is described in Equations 3.27 and 3.31.

The figure shows that the convertible bond has negative default cost after hedging. If one were to synthesise the convertible bond then default events would be profitable due to a surplus (indicated by the negative in the graph) after paying the recovery component X_t.

If one were to hedge a long position in the convertible bond then one would loose money in default due to the hedging strategy[6]. However, the hedging will be profitable until default due to the "insurance premium" component of the hedging strategy.

The hedging strategy is not perfect. However, if a large enough "population" of uncorrelated convertible bonds are used in the hedging strategy then protection from hedging losses in default is possible.

4.3 Convergence

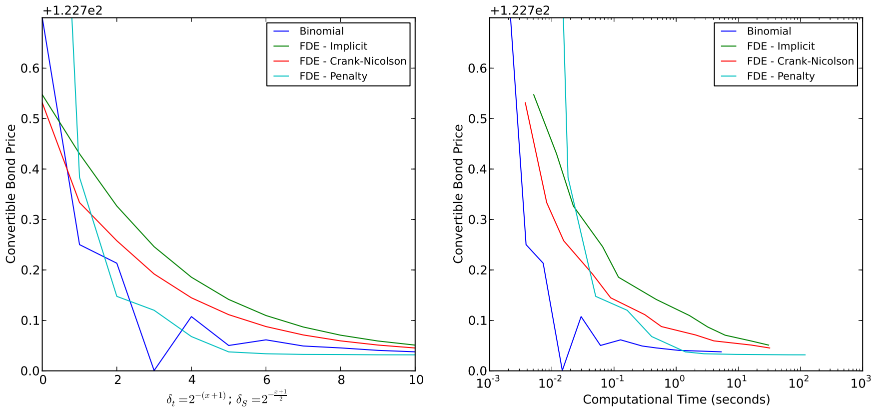

Figure 4.8: Comparison of the convergence of different pricing methods, as a function of step size and a function of time

Figure 4.8 looks at the convergence of the different pricing methods. The explicit finite difference scheme was not used due to stability issues. The penalty scheme is an extension of the Crank-Nicolson scheme as described by Aoubiza and Laydi [2] and Christara and Dang [11]. The tolerance for the penalty method is based on the step sizes for stock and time, namely $\min(\delta t, \delta S_t)^{-2}$, as recommended by Forsyth and Vetzal [14].

The Crank-Nicolson scheme converges faster, both for step size and time, than the implicit scheme. The binomial model converges faster than the Crank-Nicolson in time and in step size except for the biggest step sizes. In the limit[7] the penalty method converges the fastest, both in time and step size.

If time is a limiting resource then the penalty method is the worst option with the binomial model offering better convergence than the other finite difference schemes, otherwise the penalty method is the best method.

[6]In addition to the original loss due to default

[7]As the step sizes approach zero and as computational time approaches infinity

Chapter 5

Conclusion

This project derived and implemented two pricing methods of convertible bonds that allowed for differing levels of default. Parameters for three schemes, the explicit, implicit and Crank-Nicolson, were derived.

The binomial model was extended by considering the hedging strategy and showed that the binomial model does not provide for perfect hedging, requiring a cross financing of default and non-default events. The cross financing requirement was shown to be H_t^c, the unadjusted default cost after hedging, and a numerical analysis showed the convertible bond has a negative H_t^c.

The sensitivities of the convertible bond was considered, with the put and call options influencing the lower and upper sections of the price curve respectively. The conversion option could have impact on the price if it does not coincide with the leading edge of the call option. The redemption value was shown to have little impact on the price.

Possible avenues of further research include:

- consideration of the impact the bond and stock recovery has on the convertible bond price,
- exploration of better hedging strategies,
- alternate finite difference schemes for better convergence,
- Monte Carlo pricing, and
- simulations of "insuring" the convertible bond with respect to H_t^c.

Appendix A

Appendix

A.1 Finite Difference Improvements

Other improvements to the finite difference schemes were implemented although not used in the analysis.

Rannacher [24] originally described a method of handling discrete jumps in a finite difference equation, such as those introduced by coupons. The method is, in this case, two implicit half steps instead of the a the normal Crank-Nicolson step following a discrete jump. Giles and Carter [16] recommends using four quarter steps instead of a half steps. This recommendation was used and it demonstrated better smoothing capability of the price delta and gamma.

Duffy [13] describes a class of exponential fitted schemes that are able to handle small drift and volatility parameters.

A.2 Software

All software using in this research project is open source.

Software	**Version**
TeXLive	2012
Python	2.7.3
NumPy	1.6.2
SciPy	0.10.1
Matplotlib	1.2.0
Git	1.7.11.5
pmake[1]	

Table A.1: Software

A.3 Hardware

Component	Specification
CPU	Intel Core i7 2600
Memory	16GiB

Table A.2: Hardware

[1]As distributed with FreeBSD 9

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