Valuation of Convertible Bonds with Credit Risk

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Bachelor of Science with Honours in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Place abstract here. Based on paper [1].

Acknowledgements

Place acknowledgements here.

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Chapter 1

Introduction

What this paper shows and does.

1.1 Convertible Bonds with Credit Risk

Describe and define (in maths) what a convertible bond with credit risk is.

1.2 Literature Review

Describe the current state of literature w/ Convertible Bonds with Credit Risk, possibly split this and include it at the beginning of each relevant section?

1.3 Example

Describe the standard example that will be covered by each valuation method.

Chapter 2

Binomial Model with Credit Risk

The Binomial Model with Credit Risk was first derived by Milanov and Kounchev [2], who showed that this model converges, in continuous time, to the Ayache et al. [1] model. Milanov and Kounchev [2] also showed the valuation method of this model is the same as in the classical binomial model.

TODO: add further explanation and references.

2.1 Definition

Consider a stock with the following stochastic model[1]:

$$\frac{dS_t}{S_t} = (r + \lambda\eta)dt + \sigma dW_t - \eta dq_t \tag{2.1}$$

where:

r	is the risk free rate
σ	is the log-volatility of the stock price
λ	is the hazard rate
η	is the percentage drop in stock price on a default event
dW_t	is a Wiener process
dq_t	is a Poisson jump process where the first jump is the default event

[1] Of note $\mathbb{E}[\frac{dS_t}{S_t}] = rdt \Leftrightarrow \frac{dS_t}{S_t} = (r + \lambda\eta)dt + \sigma dW_t - \eta dq_t$

and

$$\mathbb{E}\left[\frac{dS_t}{S_t}\right] = rdt \tag{2.2}$$

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t+dt}}{S_t}\right] &= \mathbb{E}\left[\frac{S_t}{S_t} + \frac{dS_t}{S_t}\right] \\ &= 1 + \mathbb{E}\left[\frac{dS_t}{S_t}\right]\end{aligned} \tag{2.3}$$

$$\mathbb{V}\mathtt{ar}\left[\frac{dS_t}{S_t}\right] = (\sigma^2 + \lambda\eta^2)dt \tag{2.4}$$

$$\begin{aligned}\mathbb{V}\mathtt{ar}\left[\frac{S_{t+dt}}{S_t}\right] &= \mathbb{V}\mathtt{ar}\left[\frac{S_t}{S_t} + \frac{dS_t}{S_t}\right] \\ &= \mathbb{V}\mathtt{ar}\left[\frac{dS_t}{S_t}\right]\end{aligned} \tag{2.5}$$

and consider a binomial model that has time step δt, up and down steps, u and d with probability p_u and p_d[2] respectively, and where the probability of default is p_o[3] and (TODO: insert picture illustration)

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right] &= up_u + dp_d + (1-\eta)p_o \\ &= up_u + d(e^{-\lambda\delta t} - p_u) + (1-\eta)(1-e^{-\lambda\delta t})\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathbb{V}\mathtt{ar}\left[\frac{S_{t+\delta t}}{S_t}\right] &= u^2p_u + d^2p_d + (1-\eta)^2p_o - \mathbb{E}\left[\frac{\delta S_t}{S_t}\right]^2 \\ &= u^2p_u + d^2(e^{-\lambda\delta t} - p_u) + (1-\eta)^2(1-e^{-\lambda\delta t}) - \mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right]^2\end{aligned} \tag{2.7}$$

2.2 Derivation

If one equates the first moment of the stochastic model (2.3)[4], in δt time, with that of the binomial model (2.6) then:

$$\begin{aligned}e^{r\delta t} &= up_u + d(e^{-\lambda\delta t} - p_u) + (1-\eta)(1-e^{-\lambda\delta t}) \\ p_u(u-d) &= e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})\end{aligned} \tag{2.8}$$

$$p_u = \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{2.9}$$

$$\Rightarrow p_d = -\frac{e^{r\delta t} - ue^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \tag{2.10}$$

[2] $p_d = 1 - p_u - p_o = e^{-\lambda\delta t} - p_u$

[3] The time till the first jump follows an exponential distribution and has probability $1 - e^{-\lambda\delta t}$

[4] $\mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right] = e^{r\delta t}$

and if one equate the second moment about the mean of the stochastic model (2.5), in δt time, with that of the binomial model (2.7) then[5]:

$$
\begin{aligned}
(\sigma^2+\lambda\eta^2)\delta t &= u^2 p_u + d^2(e^{-\lambda\delta t}-p_u) + (1-\eta)^2(1-e^{-\lambda\delta t}) - \mathbb{E}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \\
&= (u^2-d^2)p_u + d^2 e^{-\lambda\delta t} + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t} && (2.11)\\
&= (u+d)(e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})) && (2.12)\\
&\quad + d^2 e^{-\lambda\delta t} + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t} \\
&= (u+d)(e^{r\delta t} - (1-\eta)(1-e^{-\lambda\delta t})) - ude^{-\lambda\delta t} && (2.13)\\
&\quad + (1-\eta)^2(1-e^{-\lambda\delta t}) - e^{-2r\delta t}
\end{aligned}
$$

If one assumes $\delta t^2 = 0$, $ud = 1$ and $u = e^{\sqrt{A\delta t}}$ and the Taylor series expansion is taken for all exponential terms, then:

$$
\begin{aligned}
u &= 1 + \sqrt{A\delta t} + \frac{A\delta t}{2!} + \frac{(A\delta t)^{\frac{3}{2}}}{3!} \\
d &= 1 - \sqrt{A\delta t} + \frac{A\delta t}{2!} - \frac{(A\delta t)^{\frac{3}{2}}}{3!} \\
u + d &= 2 + A\delta t && (2.14)
\end{aligned}
$$

and substituting (2.14) into (2.13) with the appropriate expansions then:

$$
\begin{aligned}
(\sigma^2+\lambda\eta^2)\delta t &= (2 + A\delta t)(1 + r\delta t - \lambda\delta t(1-\eta)) - (1-\lambda\delta t) \\
&\quad + \lambda\delta t(1-\eta)^2 - (1-2r\delta t) \\
&= A\delta t + \lambda\delta t - \lambda\delta t(1-\eta)(1+\eta) \\
&= A\delta t + \lambda\eta^2\delta t \\
A\delta t &= \sigma^2\delta t && (2.15)
\end{aligned}
$$

2.3 Valuation

Consider a portfolio with value V_t at time t, Δ_t invested in the stock S_t and $\Pi_t = V_t - \Delta_t S_t$ invested at the risk neutral rate. At time $t+\delta t$ one should have:

$$
\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d & \text{with probability } p_d \\ X_t - (1-\eta)S_t & \text{with probability } p_o \end{cases}
$$

where

[5]Substituting equation (2.8) into (2.11) one gets (2.12)

V_t^u	is the value of the portfolio an up-step at time $t+\delta t$
V_t^d	is the value of the portfolio a down-step at time $t+\delta t$
X_t	is the value of default at time t

If one wishes to hedge against up and down movements of the stock then:

$$\begin{aligned}
V_t^u - \Delta_t S_t u &= V_t^d - \Delta_t S_t d \\
\Delta_t(d-u) &= V_t^d - V_t^u \\
\Delta_t &= \frac{V_t^u - V_t^d}{u-d} \qquad (2.16)
\end{aligned}$$

The required value of Π_t at time $t+\delta t$ is:

$$\begin{aligned}
\Pi_{t+\delta t} &= \Pi_t e^{r\delta t} \\
&= \left(V_t - \frac{V_t^u - V_t^d}{u-d}\right) e^{r\delta t} \qquad (2.17)
\end{aligned}$$

as Π_t is invested at the risk free rate, and taking the expected value of $\Pi_{t+\delta t}$, which must equal (2.17), one gets:

$$\begin{aligned}
(V_t - \frac{V_t^u - V_t^d}{u-d})e^{r\delta t} &= \mathbb{E}\left[\Pi_{t+\delta t}\right] \\
&= \left(V_t^u - \frac{V_t^u - V_t^d}{u-d}u\right) e^{-\lambda\delta t} \\
&\quad + \left(X_t - \frac{V_t^u - V_t^d}{u-d}(1-\eta)\right)(1-e^{-\lambda\delta t}) \\
&= V_t^u\left(e^{-\lambda\delta t} - \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d}\right) \\
&\quad + V_t^d\left(\frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t})}{u-d}\right) + X_t(1-e^{-\lambda\delta t}) \\
V_t e^{r\delta t} &= V_t^u \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \\
&\quad + V_t^d \frac{ue^{-\lambda\delta t} + (1-\eta)(1-e^{-\lambda\delta t}) - e^{r\delta t}}{u-d} + X_t(1-e^{-\lambda\delta t}) \\
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) \qquad (2.18)
\end{aligned}$$

thus the pricing method of this binomial model is the same as the pricing method of the classical binomial model:

$$\begin{aligned}
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) \\
&= e^{-r\delta t}\mathbb{E}\left[V_{t+\delta t}\right] \qquad (2.19)
\end{aligned}$$

2.4 Parameters and Formulæof Model

Based on the above the following tables specifies a binomial model with credit risk:

Table 2.1: Parameters of the Binomial Model with Credit Risk

Parameter	**Description**
r	Risk free rate
σ	Log-volatility of the stock price
λ	Hazard rate of default
η	Percentage drop of stock price in a default event
δt	Time-step
V_t^u	Value of portfolio for an up-step at $t + \delta t$
V_t^d	Value of portfolio for a down-step at $t + \delta t$
X_t	Value of defaulted portfolio at t

Table 2.2: Formalæ for the Binomial Model with Credit Risk

Formulæ	**Description**
$u = e^{\sigma\sqrt{\delta t}}$	Multiplier for an up-step
$d = e^{-\sigma\sqrt{\delta t}}$	Multiplier for a down-step
$p_u = \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d}$	Probability of an up-step
$p_d = e^{-\lambda\delta t} - p_u$	Probability of a down-step
$p_o = 1 - e^{-\lambda\delta t}$	Probability of default
$V_t = e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o)$	Value of portfolio at t

2.5 Example and Hedging Strategy

Provide a numerical example (with diagrams) and provide example of the hedging strategy.

Chapter 3

Finite Difference Model

Describe the finite difference model based on the paper by Ayache et al. [1].

Chapter 4

Numerical Example

Implement above models, explain implementation and results, by example.

Chapter 5

Conclusion

Summarise what this paper shows and does (i.e. Introduction + Conclusion = Executive Summary).

Bibliography

[1] E. Ayache, P. A. Forsyth, and K. R. Vertzal, *The valuation of convertible bonds with credit risk*, Journal of Derivatives **11** (2003), 9–44.

[2] K. Milanov and O. Kounchev, *Binomial tree model for convertible bond pricing within equity to credit risk framework*, 2012.