Convertible bonds with market risk and credit risk

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Abstract

The incorporation of credit risk in the valuation of convertible bonds has mostly been rather ad-hoc in the literature. Here a model is introduced that attempts a consistent treatment of equity and credit risk.

It incorporates a Black-Scholes stock price (equity risk), Hull-White short rate (interest rate risk), and a hazard rate, depending on the asset price, which determines the probability of default (credit risk). The model can be calibrated to match the initial term structure of interest rates as well as the 'term structure of credit spreads'.

1 Introduction

A convertible bond is a coupon paying corporate bond that can be converted into company stock at the discretion of the holder. The purpose of this article is to describe a method to value a convertible bond, incorporating both interest rate and credit risk.

A convertible bond is a challenging instrument to value, because it is both an equity and an interest rate derivative. These two components are subject to different credit risk, because a company can always issue more of its stock, but not necessarily come up with sufficient cash to meet bond obligations. Furthermore, in reality, even the most basic convertible bonds often incorporate various additional features, such as call and put provisions, strike reset features, mandatory or restricted conversion, etc.

Three sources of randomness are at work: the stock price, the interest rate, and the credit spread. The probability of default over the next small period is given by the hazard rate. One could model all factors at the same time, but practitioners tend to eschew models with more than two factors.

There are various ways of reducing the problem to two factors: either, take the hazard rate to be a prespecified deterministic function of time, and model stock price and interest rate stochastically. Alternatively, assume the interest rate to be deterministic and the hazard rate to be stochastic.[1]

A third way will be explored here: A two dimensional trinomial lattice will be used to model the (rebased) stockprice on one dimension and the interest rate on the other. The hazard rate is assumed to be a deterministic function of the stock price: if the stock price falls below its (risk neutral) expectation, the hazard rate rises, and vice versa.

1.1 Convertible bonds

Convertible bonds are typically listed securities issued by companies and traded on secondary markets. The ratio of convertible to total debt was above 10% on average in the United States during this century [9, p. 233], and they accounted for above 5on the London Stock Exchange from 1973 to 1995 [11]. Companies issue them mainly because they enable them to lower their costs of debt funding ('debt sweetener') by implicitly selling an option on their stock (which will only be exercised if the company is doing well). This helps to resolve the problem of asymmetric information on the riskiness of the underlying assets, and reduces agency costs.[2] The other commonly cited reason for issuing convertibles is as an indirect and delayed issuing of equity ('delayed equity'), reducing dilution and circumventing regulatory hindrances.[3]

Investors on the other hand hold coonvertibles for their upside potential with limited downside. Sophisticated investors tend to consider convertibles a good deal, particularly considering the time value of the embedded option.[4]

As long as the bond holder does not convert, he regularly receives a coupon and is finally repaid his principal. If he chooses to convert during the lifetime of the bond, however, the bond is redeemed and the issuer receives some ordinary

[1] These two approaches seem very symmetrical, particularly since in the first case, a riskfree zero coupon bond is used as numeraire, and in the second a 'defaultable' dollar, hence the implementations are very similar. However, the treatment of the recovery value differs.

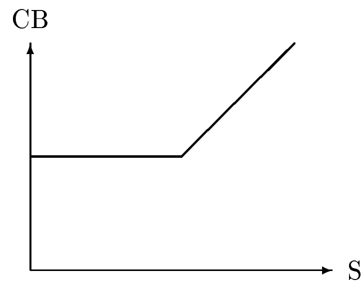
Both approaches have been explored and implemented by Davis [3].

[2] Equity holders have an incentive to increase the riskiness of the assets, which increases the value of equity and decreases the value of debt (holding firm value constant). On the other hand, the call component of a convertible increases in value, so the total value of the convertible can be made insensitive to changes in risk.

[3] Nyborg [11] explores in greater depth the motives and strategic issues involved in the issue of convertible debt.

[4] Kang and Lee [9] provide an empirical investigation of convertible debt offerings and report significant initial underpricing.

Figure 1: Payoff of a convertible bond (with $\kappa = 1$).



shares of the company at a previously agreed exchange price of EP per share.[5] Since the notional K of the bond is assumed to be converted upon redemption, the value received on conversion—parity—is $K\frac{S_t}{EP}$. This effectively constitutes a call option on the stock, since the value of the convertible bond CB at maturity is roughly

$$CB = \max(B, \kappa S) = B + (\kappa S - B)^+, \tag{1}$$

where $\kappa = K/EP$ is the conversion ratio, see figure 1.

The bond pays coupons, the stock pays dividends. Hence there are intermediate cash flows, some of which are stochastic, though that is rarely modelled.

Features of convertible bonds

Very often, convertibles are callable: The issuer has the option to buy the bond back at a predetermined strike price (Often this strike price changes during the life of the bond). However, the holder typically has the right to convert the bond rather than deliver it once the call announcement has been made, hence the call provision is often used to force early conversion of the bond.

Early conversion of a convertible bond is never optimal under certain conditions[6], hence this call provision reduces the value of the convertible: It limits the investor's return if interest rates fall or the stock price rises. Often, convertible bonds are call-protected for some years and become callable only after that.

On the other hand, a put provision allows the holder to return the convertible to the issuer in exchange for cash at certain points in time for a predetermined price, and hence offer additional downside protection in case of rising interest rates.

With even more complex bonds, e.g. Japanese resettable convertibles, the holder's conversion options change according to the stock price and its history.[7]

[5] The exchange price is typically set a bit higher than the current stock price to avoid immediate conversion.

[6] See Ingersoll [7, Theorem I] for the exact conditions (essentially perfect market, no dividends, and constant conversion ratio).

[7] See RISK magazine [12] for these and more complications. These complications make the convertible path dependent and hence indicate Monte-Carlo simulation as the appropriate modelling tool. We ignore these possibilities, and since convertibles are American style instruments, modelling on a tree seems appropriate.

1.2 Valuation by arbitrage: The classical literature

Theoretical pricing models for convertible bonds first appeared in the 1960's,[8] though they were severely limited because they typically considered only one particular point in the future, found the maximum of conversion and bond value, and discounted at some rate.

Shortly after the breakthrough in option pricing by Black and Scholes in 1973 (and Merton's 'rational' option pricing theory same year), the new, theoretically sound method was applied to the case of convertible bonds.

Ingersoll's paper [7] from 1976 develops arbitrage arguments to derive several results concerning the optimal conversion strategy (for the holder) and call strategy (for the issuer) as well as closed form solutions for the value of convertible bonds in a variety of special cases. He assumes that the value of the firm as a whole is composed of equity and convertible bonds only and follows a geometric Brownian motion.

Brennan and Schwartz [1] publish similar results in 1977 under the same assumptions. They develop the PDE and boundary conditions for the value of convertible bonds under fairly general conditions and describe a finite difference method to solve it.

Credit risk in early approaches

In case the firm value falls under the debt notional, the firm defaults and the bond holders aquire the assets of the firm.

In this sense, credit risk is taken into account in these early papers. This approach considers the capital structure of the firm and views the convertible bond not as a derivative on the stock price, but a compound option on the physical assets underlying the financial securities, similar in spirit to Merton's 1977 model of debt as a portfolio of notional and a short put option on the total firm value. However, this theoretically appealing methodology suffers from problems in practice, as pointed out in 1995 by Jarrow and Turnbull [8]: The underlying physical assets are often not tradable, and their value and volatility not necessarily known. Second, all corporate liabilities senior to the convertible bond at hand must be valued at the same time. These complications make this approach unsuitable for pricing corporate liabilities[9] in practice.

1.3 Dealing with credit risk: The modern literature

Newer approaches therefore consider the convertible bond as a security contingent on the stock price and interest rate. The stock price is assumed to follow a geometric Brownian motion. A tree to value a convertible bond, even with call and put provisions, coupons, dividends, and other features, can then be constructed in the usual way.

The essential difficulty is the choice of the discount rate. Suppose the convertible is certain to remain a bond. The cash payoff is then subject to credit risk, and the appropriate interest rate incorporates a credit spread corresponding to the credit rating of the issuer.

[8] See [7] for references.

[9] It is more promising for areas where these problems are less prevalent, e.g. mortgages.

Suppose on the other hand the convertible is certain to be converted. The firm can always issue a share, which can then be sold and the proceeds invested risk free. The appropriate discounting rate is then the risk free rate.

Adjusting the credit spread

One way this issue has been dealt with is detailed in a research note [4] from Goldman Sachs (1994), and outlined in Hull [6, section 20.5]. They consider the probability of conversion at every node, and choose the discount rate to be an accordingly weighted arithmetic average between risk free rate and the risky rate (which is obtained by adding the issuer's credit spread).

The probability of conversion at the final layer (t=T) of the tree is either 1 or 0, depending on whether the convertible is converted or not. At previous nodes, the conversion probability is calculated as the average of the conversion probabilities of the successor nodes. If the convertible is converted (or put) at a node, the conversion probability is reset to 1 (or 0). Goldman Sachs describe a tree using this method in greater detail, including call and put provisions.

Separating cash and equity

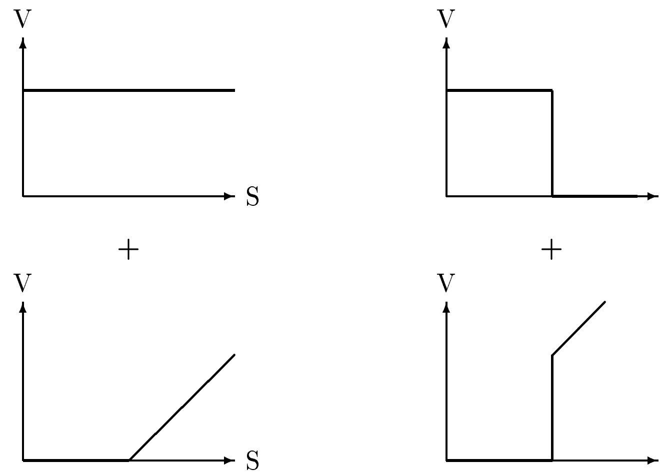
A different approach was introduced by Tsiveriotis and Fernandes [13] in 1998. Rather then considering the convertible bond value as a portfolio of normal bond and call option (see figure 2, left column), they decompose it in a cash account and stock account (right column). Both are maintained and discounted separately.[10] This is done by introducing an artificial security, namely the cash only part of the convertible bond, that pays all the cash payments, but no equity that an optimally behaving holder of a convertible would receive. They derive the joint PDE and boundary conditions for the artificial security and the normal convertible and describe the numerical solution using the explicit finite difference method.

All models mentioned so far do not consider interest rate risk, which for convertible bonds is evidently more important than for pure equity options. Brennan and Schwartz (1980, [2]) and Longstaff and Schwartz (1993, [10]) introduce two factor models that include interest rate risk, but they are not necessarily consistent with the initial term structure of interest rates.

Ho and Pfeffer [5] analysed a two factor model in 1996 that adapts to the initial term structure. When the stock movement is ignored, their two dimensional binomial lattice produces the same interest rates and arbitrage free bond prices as in the Ho and Lee model (1986). They analyse a sample of bonds and place special emphasis on hedging strategies and calculating the greeks. However, they consider bond value ('investment value') and call value ('Latent Warrant') separately, and capture the credit risk by a constant credit spread that is added to the Treasury rate at each point to discount the bond's cash flow. Hence, their approach might be very valuable for investors, but is not very satisfying theoretically.

[10] This seems similar to the previously mentioned method, in which the probability of conversion is calculated and an accordingly adjusted discount rate used. However, that fails to account correctly for intermediate (risky) cash flows, such as coupons or contingent flows due to call and put provisions.

Figure 2: Decomposing the payoff



CB = Bond + Call CB = digital + asset-or-nothing

The present approach: hazard rate depending on stock price

The approach described here incorporates a Black-Scholes asset price, Hull-White short rate, and a hazard rate that determines the probability of default. The model can be calibrated to match the initial term structure of interest rates as well as the 'term structure of credit spreads', i.e. the implied (risk neutral) survival probability of a company.[11]

Hence, the model is consistent with observed market data, poses no unsurmountable problems concerning parameter estimation, is theoretically appealing, can be calculated straightforwardly on a two dimensional lattice, and there seem to be no theoretical reasons precluding the incorporation of more realistic features of convertible bonds.

2 Underlyings: Stock price, short rate, hazard rate

Default behaviour

The default is modelled by a point process $N(t)$, which starts out at zero but jumps to one when default occurs. When default occurs, the stock price jumps to zero[12], while the price of the coupon bearing convertible bond jumps to a recovery value ℓK, a predetermined fraction of the notional.

The probability of default over a small period is proportional (to first order) to the time varying intensity $h(t)$, the hazard rate. Then the expectation of

[11] It can—under certain assumptions—be recovered from comparing prices of company bonds with Treasury bonds, see [8].

[12] Note that the jump size is previsible, so it is possible to hedge against default. This enables the arbitrage argument.

$N(t)$ under $\mathbb{Q}$ is[13] $E[N(t)] = \int_{s=0}^{t} h(s)\,ds$, so

$$dM(t) = dN(t) - \big(1 - N(t)\big)h(t)\,dt$$

defines a martingale.

The hazard rate is modelled as deterministic or stochastic, depending on the model, and can be calibrated to reproduce the implied survival probabilities[14].

Stock price in general

The stock price follows

$$\begin{aligned} dS(t) &= \big(r(t) - y(t)\big)S(t)\,dt + \sigma_1 S(t)\,dW_1(t) \\ &\qquad - S(t_-)\big(dN(t) - h(t)\,dt\big) \\ &= \big[r(t) + h(t) - y(t)\big]S(t)\,dt + \sigma_1 S(t)\,dW_1(t) - S(t_-)\,dN(t). \end{aligned}$$

It is a martingale under $\mathbb{Q}$ when rebased by the money market account. When default occurs, the stock price immediately before default $S(t_-)$ is subtracted, i.e. the stock price jumps to zero and stays there. Prior to default, the stock price is given by

$$S(t) = S_0 \exp\left[\int_0^t \Big(r(s) + h(s) - y(s)\Big)\,ds - \frac{1}{2}\sigma_1^2 t + \sigma_1 W_1(t)\right]. \qquad (2)$$

The risk neutral stock price return is increased by $h(t)$ to compensate for the possibility of default and reduced by $y(t)$ to compensate for dividend payments.

2.1 One factor

Short rate and hazard rate are first assumed to be deterministic functions of time. Then one can write (2) as

$$S(t) = F_A(t) \cdot \mathcal{E}[\sigma_1 W_1](t),$$

where $F_A(t) = S_0 \exp\left[\int_0^t \big(r(s) + h(s) - y(s)\big)\,ds\right]$ is a deterministic function of time which can be interpreted as the forward price, and

$$\mathcal{E}[\sigma_1 W_1](t) = \frac{e^{\sigma_1 W_1(t)}}{E\left[e^{\sigma_1 W_1(t)}\right]} = e^{\sigma_1 W_1(t) - \frac{1}{2}\sigma_1^2 t}$$

is the exponential martingale of the martingale $\sigma_1 W_1$.

Hence, for calculations, one just has to calculate the forward $F_A(t)$, model $\mathcal{E}[\sigma W_1]$ on a one-dimensional tree, and can then immediately calculate the stock price at every node and hence derivatives prices.

[13] Questions of existence and uniqueness of the equivalent martingale measure are deferred for the time being. Note that the survival probability is not a tradable asset and need not grow at the riskless rate.

One could introduce a defaultable zero coupon bond (issued by the issuer of the shares) that would need to be a martingale when appropriately rebased, and develop hedging strategies along that line. However, a defaultable bond turns out not to be log-normal, hence a different approach is used here.

[14] They can be estimated from market data (namely the credit spread of the issuing company over Treasury), once an assumption concerning the recovery value has been made, see Jarrow and Turnbull (1995,[8]).

2.2 Two factors: Stochastic interest rate

Short rate dynamics

The hazard rate is still assumed to be deterministic, while the short rate dynamics are now modelled by a mean reverting [15] process with time varying mean (extended Vasicek):

$$dr(t) = \big[\theta(t) - \lambda\, r(t)\big]\, dt + \sigma_2\, dW_2(t), \tag{3}$$

where $W_2(t)$ is a Brownian motion with instantaneous correlation ρ to the first Brownian motion, all under the risk neutral measure $\mathbb{Q}$.

The mean reversion parameter λ and the volatility σ_2 are assumed to be constant, while the $\theta(t)$ is assumed to be a (locally bounded) deterministic function of time, used to calibrate to the observed term structure of interest rates.

The unique solution to (3) is given by the Gaussian process

$$r(t) = e^{-\lambda t} r_0 + e^{-\lambda t} \int_0^t e^{\lambda s} \theta(s)\, ds + e^{-\lambda t} \sigma_2 \int_0^t e^{\lambda s}\, dW_2(s) \tag{4}$$

with mean and covariance function

$$\begin{aligned} \alpha(t) &:= E_{\mathbb{Q}}[r(t)] = e^{-\lambda t} r_0 + e^{-\lambda t} \int_0^t e^{\lambda s} \theta(s)\, ds \\ \operatorname{Cov}[r(s), r(t)] &= \frac{\sigma_2^2}{2\lambda} \left(e^{-\lambda |s-t|} - e^{-\lambda (s+t)} \right). \end{aligned}$$

Observe that the first two terms in (4) (the expectation of $r(t)$) are just a deterministic function of time, while the remaining term is a Gaussian martingale. Hence write

$$r(t) = \alpha(t) + e^{-\lambda t} X_2(t), \tag{5}$$

where

$$X_2(t) := \sigma_2 \int_0^t e^{\lambda s}\, dW_2(s). \tag{6}$$

One does not need to bother with $\theta(t)$ any further, but can immediately choose $\alpha(t)$ to fit the current zero coupon bond prices.

Bond prices

Under the risk neutral measure $\mathbb{Q}$, the price at time t of the bond maturing at time T is given by $P^T(t) = E_{\mathbb{Q}} \left[e^{-\int_t^T r(s)\, ds} \middle| \mathcal{F}_t \right]$, and Feynman-Kac yields the usual PDE with solution

$$P^T(t) = e^{A^T(t) - B^T(t) r(t)},$$

$A^T(t)$ is not further needed. Solving for $B^T(t)$ gives

$$B^T(t) = \frac{1}{\lambda} \left(1 - e^{-\lambda (T-t)} \right) \tag{7}$$

[15] The term 'mean reverting' is slightly misleading, since the specification (3) below is under the riskneutral, not the real world measure. How the process behaves under the change of drift concomitant with the change of measure is unspecified. However, it is sure to say that the short rate has finite unconditional variance.

Finally, plugging the obtained values in Itô's lemma yields the process for the T-bond price under $\mathbb{Q}$:

$$\frac{dP^T(t)}{P^T(t)} = r(t)\,dt - \sigma_2\,B^T(t)\,dW_2(t),$$

The stock price with stochastic interest rates

Again, the stock price prior to default is

$$S(t) = S_0 \exp\left[\int_0^t \Big(r(s) + h(s) - y(s)\Big)\,ds - \frac{1}{2}\sigma_1^2 t + \sigma_1 W_1(t)\right].$$

However, now the integral over $r(t)$ is stochastic and path dependent. This complicates numerical calculations because it precludes filling out the tree from the last $(t = T)$ layer. Hence, consider the T-forward price: rebase the asset with respect to a zero coupon bond with maturity time T and consider $Y_B(t) = S(t)/P^T(t)$. Itô's Lemma yields

$$\begin{aligned}\frac{dY_B(t)}{Y_B(t)} &= \Big(\big(\sigma_2 B^T(t)\big)^2 + \rho\sigma_1\sigma_2 B^T(t) - y(t)\Big)\,dt \\ &\qquad + \sigma_1\,dW_1(t) + \sigma_2 B^T(t)\,dW_2(t)\; - dN(t) \qquad\qquad (8)\\ &=: \big(m(t) - y(t)\big)\,dt + \tilde{\sigma}(t)\,d\tilde{W}(t)\; - dN(t).\end{aligned}$$

The stochastic short rate $r(t)$ canceled out, and the remaining drift term

$$m(t) = \big(\sigma_2 B^T(t)\big)^2 + \rho\sigma_1\sigma_2 B^T(t)$$

is a deterministic function of time.

The two random parts in line (8) stemming from asset and interest risk respectively can be aggregated into

$$\begin{aligned}dX_1(t) &= \sigma_1\,dW_1(t) + \sigma_2 B^T(t)\,dW_2(t)\\ X_1(t) &= \int_0^t \tilde{\sigma}(s)d\tilde{W}(s),\end{aligned}$$

where the aggregated instantaneous volatility $\tilde{\sigma}(t)$ is

$$\tilde{\sigma}(t) = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 B^T(t) + \big(\sigma_2 B^T(t)\big)^2},$$

and the new Brownian motion $\tilde{W}(t)$ is defined by

$$\tilde{\sigma}(t)d\tilde{W}(t) = \sigma_1\,dW_1(t) + \sigma_2 B^T(t)\,dW_2(t).$$

With these definitions, $\frac{dY_B(t)}{Y_B(t)} = \big(m(t) - y(t)\big)\,dt + dX_1(t)$, so Y_B is log-normal and

$$Y_B(t) = F_B(t) \cdot \mathcal{E}[X_1](t)$$

with $F_B(t) = Y_1(0)\exp\left[\int_0^t \big(m(s) - y(s)\big)\,ds\right]$. Hence $Y_B(t)$ is again the product of a deterministic forward and an exponential martingale $\mathcal{E}[X_1]$.

The short rate $r(t) = \alpha(t) + e^{-\lambda t} X_2(t)$, on the other hand, is a deterministic function of time and the other martingale X_2.

To model the twodimensional Gaussian martingale on a two dimensional tree, one needs the (time varying) covariance structure.

Write

$$\operatorname{Var}\left[\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}\right] = \int_0^t Q(s)ds = t \cdot \bar{Q}(t),$$

where $Q(t)$ contains 'forward' or 'instantaneous' covariances while $\bar{Q}(t)$ contains 'term' or 'average' covariances. Then

$$\begin{aligned} Q_{11}(t) &= \sigma_1^2 + 2\rho\sigma_1\sigma_2 B^T(t) + \left(\sigma_2 B^T(t)\right)^2 \\ Q_{22}(t) &= \sigma_2^2 e^{2\lambda t} \\ Q_{12}(t) &= \rho\sigma_1\sigma_2 e^{\lambda t} + \sigma_2^2 B^T(t) e^{\lambda t}. \end{aligned}$$

2.3 Two factors: Stochastic hazard rate

Now, take the short rate to be deterministic, and model the hazard rate as a mean-reverting Ornstein-Uhlenbeck process.

The situation then is symmetric to the model before, the same tree can be used. However, the short rate and hazard rate are exchanged, which results in different treatment of the recovery value in case of default.

The stock price

Again, the stock price prior to default is given by (2). However, now the integral over $h(t)$ is stochastic and pathdependent, hence rebase[16] by the survival probability to maturity $\eta^T(t)$.

2.4 Two-and-a-half factors

Model the short rate as before as extended Vasicek, equation (3), and the hazard rate as[17]

$$\begin{aligned} h(t) &= \gamma(t) - \sigma_3 W_1(t), \text{ so} \\ dh(t) &= \gamma'(t) - \sigma_3\, dW_1(t), \end{aligned} \tag{9}$$

where W_1 is the Brownian motion that is driving the asset price. The summand $\gamma(t)$ is used to calibrate the hazard rate so that is reproduces the implied survival probabilities[18].

[16] This corresponds to the idea of an exchange rate between dollars and 'defaultable dollars' as proposed in 1995 by Jarrow and Turnbull [8].

[17] With a reasonable choice of σ_3 one can prevent the hazard rate from turning negative while the stock return stays in, say, five standard deviations from expectation.

[18] Jarrow and Turnbull [8] show how to estimate the survival probabilities under the risk neutral measure from market data (namely the credit spread of the issuing company over Treasury), once an assumption concerning the recovery value has been made.

Survival probability

The survival probability $\eta^T(t)$ (i.e. the probability that $N(T) = 0$ given $N(t) = 0$) is[19]

$$\eta^T(t) = E\left[e^{-\int_t^T h(s)\,ds}\Big|\mathcal{F}_t\right],$$

so, for the choice of hazard rate above,

$$\eta^T(t) = \exp\left[-\int_t^T \gamma(s)\,ds + \sigma_3 W_1(t)(T-t) + \frac{\sigma_3^2}{6}(T-t)^3\right],$$

and applying Itô's lemma, one obtains the SDE for the survival probability:

$$\frac{d\eta^T(t)}{\eta^T(t)} = h(t)\,dt + \sigma_3 \cdot (T-t)\,dW_1(t). \tag{10}$$

The survival probability grows with rate $h(t)$, analogue to the bond.

Again, the stock price prior to default is given by (2). However, now the integral over *both* $r(t)$ and $h(t)$ is stochastic and pathdependent, hence rebase[20] by both bond and survival probability, $Y_D(t) = S(t)/\big(P^T(t)\cdot\eta^T(t)\big)$.

$$\begin{aligned}\frac{dY_D(t)}{Y_D(t)} = \Big[&\big(\sigma_2 B^T(t)\big)^2 + \sigma_3^2(T-t)^2 + \rho\sigma_1\sigma_2 B^T(t)\\ &- \rho\sigma_2\sigma_3 B^T(t)(T-t) - \sigma_1\sigma_3(T-t) - y(t)\Big]\,dt\\ &+ [\sigma_1 - \sigma_3(T-t)]\,dW_1(t) + \sigma_2 B^T(t)\,dW_2(t)\; - dN(t). \end{aligned}\tag{11}$$

Both the stochastic short rate $r(t)$ and hazard rate $h(t)$ cancelled out, and the drift term

$$\begin{aligned}m(t) = \big(\sigma_2 B^T(t)\big)^2 + \sigma_3^2(T-t)^2 + \rho\sigma_1\sigma_2 B^T(t)\\ - \rho\sigma_2\sigma_3 B^T(t)(T-t) - \sigma_1\sigma_3(T-t)\end{aligned}$$

is a deterministic function of time that can be calculated right away.

The random parts in line (11) stemming from asset, default and interest risk respectively can be aggregated into

$$dX_1(t) = [\sigma_1 - \sigma_3(T-t)]\,dW_1(t) + \sigma_2 B^T(t)\,dW_2(t).$$

With this definition, $\frac{dY_D(t)}{Y_D(t)} = \big[m(t) - y(t)\big]\,dt + dX_1(t)\; - dN(t)$, so Y_D is log-normal and

$$Y_D(t) = F_D(t)\cdot\mathcal{E}[X_1](t) \tag{12}$$

with $F_D(t) = Y_D(0)\,\exp\Big[\int_0^t\big(m(s) - y(s)\big)\,ds\Big]$. Hence $Y_D(t)$ is again the product of a deterministic forward and an exponential martingale $\mathcal{E}[X_1]$.

[19] Analogue to zero coupon prices as a function of the short rate.

[20] It might seem more natural to use a defaultable zero coupon bond as a numeraire. Assuming that its value jumps to a predetermined fraction of a non defaultable zero coupon bond upon default, however, it turns out that the defaultable zero is not log-normal.

Volatilities and Covariances

For modelling, the covariance structure is needed. Write

$$\operatorname{Var}\begin{pmatrix} X_1(t) \\ X_2(t) \\ W_1(t) \end{pmatrix} = \int_0^t Q(s)ds = t \cdot \bar{Q}(t). \tag{13}$$

With

$$\begin{aligned} dX_1(t) &= [\sigma_1 - \sigma_3(T-t)]\, dW_1(t) + \sigma_2 B^T(t)\, dW_2(t) \\ dX_2(t) &= \sigma_2\, e^{\lambda t}\, dW_2(t), \end{aligned}$$

we have

$$\begin{aligned} Q_{11}(t) &= \big(\sigma_1 - (T-t)\sigma_3\big)^2 + 2\rho\big(\sigma_1 - (T-t)\sigma_3\big)\sigma_2 B^T(t) + \big(\sigma_2 B^T(t)\big)^2 \\ Q_{22}(t) &= \sigma_2^2 e^{2\lambda t} \\ Q_{33}(t) &= 1 \\ Q_{12}(t) &= \rho\big(\sigma_1 - (T-t)\sigma_3\big)\sigma_2 e^{\lambda t} + \sigma_2^2 B^T(t) e^{\lambda t} \\ Q_{13}(t) &= \sigma_1 - (T-t)\sigma_3 + \rho\sigma_2 B^T(t) \\ Q_{23}(t) &= \rho\sigma_2 e^{\lambda t}. \end{aligned}$$

and, with the abbreviations $C(T,t) = \frac{e^{-\lambda(T-t)} - e^{-\lambda T}}{\lambda t}$ and $c(T) = C(T,T) = \frac{1-e^{-\lambda T}}{\lambda T}$:

$$\begin{aligned} \bar{Q}_{11}(t) &= \sigma_1^2 + \frac{2\rho\sigma_1\sigma_2 + \sigma_2^2}{\lambda}\big(1 - C(T,t)\big) \\ &\quad - \frac{2\rho\sigma_2\sigma_3}{\lambda^2}\Big(e^{-\lambda(T-t)} + \tfrac{\lambda}{2}(2T-t) - (1+\lambda T)C(T,t)\Big) \\ &\quad - \sigma_1\sigma_3(2T-t) + \frac{\sigma_3^2}{3}(3T^2 - 3tT + t^2) \\ \bar{Q}_{22}(t) &= \sigma_2^2 \cdot c(-2t) \\ \bar{Q}_{33}(t) &= 1 \\ \bar{Q}_{12}(t) &= \frac{\sigma_2^2}{\lambda}\big(c(-t) - C(T,2t)\big) + \rho\sigma_1\sigma_2 \cdot c(-t) \\ &\quad - \frac{\rho\sigma_2\sigma_3}{\lambda}\left((1+\lambda T)c(-t) - e^{\lambda t}\right) \\ \bar{Q}_{13}(t) &= \sigma_1 + \frac{\rho\sigma_2}{\lambda}\big(1 - C(T,t)\big) - \frac{\sigma_3(2T-t)}{2} \\ \bar{Q}_{23}(t) &= \rho\sigma_2 \cdot c(-t) \end{aligned}$$

Recovering W_1

To keep the computation tractable, only $X_1(t)$ and $X_1(t)$ are modelled directly on the tree. Hence, $W_1(t)$ is inaccessible.[21] Therefore, a linear estimator is used to obtain its conditional expectation given $\mathbf{X}(t)$:

[21] Even if all nodes were stored while going forward (which is unneccessary for this approach), the tree including W_1 would not recombine.

$$\hat{W}_1(t) = E[W_1(t)|\mathbf{X}(t)] = \beta'(t)\mathbf{X}(t) \tag{14}$$

with

$$\beta'(t) = \begin{pmatrix} \bar{Q}_{13}(t) \\ \bar{Q}_{23}(t) \end{pmatrix}' \cdot \left[\begin{pmatrix} \bar{Q}_{11}(t & \bar{Q}_{12}(t) \\ \bar{Q}_{12}(t) & \bar{Q}_{22}(t) \end{pmatrix} \right]^{-1}$$

On the other hand, from (2), one obtains

$$\begin{aligned}
\sigma W_1(t) &= \ln \frac{S(t)}{F_A(t)} + \tfrac{1}{2}\sigma_1^2 t \\
&= \ln \frac{S(t)}{S_0} - \int_0^t \big(r(s) + h(s) - y(s)\big)\, ds + \tfrac{1}{2}\sigma_1^2 t \\
&= \ln \frac{P^T(t)\eta^T(t)}{P^T(0)\eta^T(0)} - \ln E[e^{X_1(t)}] \\
&\quad + \int_0^t \big(m(s) - r(s) - h(s)\big)\, ds + \tfrac{1}{2}\sigma_1^2 t + X_1(t)
\end{aligned}$$

This seems to give $W_1(t)$ as a function of the factor $X_1(t)$, given that one knows the zero coupon value and survival probability at a node. However, the integral over $r(t)$ and $h(t)$ is pathdependent.

Summary of the two-and-a-half factor model

So far, we have expressed the rebased asset price $Y_D(t)$, interest rate and hazard rate as some deterministic (time dependent) functions of the Gaussian martingale $\mathbf{X}(t)$ and $W_1(t)$, the distribution of which is known.

$$\begin{aligned}
Y_D(t) &= f_1(X_1(t), t) && \text{Equation (12)} \\
r(t) &= f_2(X_2(t), t) && \text{Equation (5)} \\
h(t) &= f_3(W_1(t), t) && \text{Equation (9)}
\end{aligned}$$

$$\left[\begin{pmatrix} \mathbf{X}(t) - \mathbf{X}(s) \\ W_1(t) - W_1(s) \end{pmatrix} \middle| \mathcal{F}_s; N(s) = 0 \right] \sim N\left[\mathbf{0}, \int_s^t Q(u)\, du \right]$$

3 Empirical results

If the price of the convertible as a function of current stock price is plotted for different hazard rate volatilities, there is virtually no difference for very small and very large stock prices, due to the calibration to the term structure of interest rates and the term structure of credit spreads respectively.

However, for stock prices close to par, the higher the hazard rate volatility, the lower the convertible value. Also, the value of the opportunity to convert prior to maturity ('American' convertible, as opposed to a theoretical 'European') increases as the hazard rate volatility increases.

The hazard rate as estimated using $\hat{W}_1(t)$ from equation (14) behaves as intuitively expected, it approaches zero[22] as the stock price increases, and increases rapidly as the stock price falls to zero. Plotting the hazard rate against the logarithm of the stock price reveals the ultimately linear relationship. The hazard rate depends on W_1 not X_1 and X_2, W_1 is a linear estimator using X_1 and X_2, and the deviations from the line come from the information X_2 provides.

[22] Only if σ_3 is carefully chosen, otherwise it might turn negative.

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