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A CONTINGENT-CLAIMS VALUATION OF CONVERTIBLE SECURITIES*

Jonathan E. INGERSOLL, Jr.

University of Chicago, Chicago, IL 60637, USA

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This paper examines the pricing of convertible bonds and preferred stocks. The optimal policies for call and conversion of these securities are determined via the criterion of dominance. The techniques underlying the Black–Scholes Option Model are used to price convertible securities as contingent claims on the firm as a whole.

1. Introduction

A convertible security is one which, at the owner's option, may be exchanged for another security with different characteristics. Examples of convertible securities are warrants, puts and calls traded over-the-counter, and options traded on the CBOE and other exchanges. The most common convertible securities, however, are convertible bonds and preferred stocks.

Theoretical pricing models for convertible bonds first appeared in the 1960's. Their general valuation procedure was to set the price of the convertible equal to the maximum of its value as an ordinary bond or its value in common stock (after conversion) at some point in the future and then discount this value to the present. This method or a slight modification thereof was employed by Poensgen (1965), Baumol, Malkiel, and Quandt (1966), Weil, Segal, and Green (1968), Walter and Que (1973), Jennings (1974), and no doubt others.

Unfortunately all of these models are incomplete for one or more of the following reasons. If the future point of evaluation is prior to maturity, then the duration of the conversion privilege has effectively been limited. On the other hand, if only the date of maturity is considered, possible earlier conversion is ignored. Any such limiting of the conversion option may cause the bond to be

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undervalued. The appropriate discount rate is left unspecified in these theories except in the final two papers which employ the Capital Asset Pricing Model. The call provision is not explicitly considered except by Jennings. This will certainly cause the bond to be overvalued.

In many respects, these models were quite similar to the concurrent warrant pricing models developed by Samuelson (1965) and others, which predated the seminal Black and Scholes (1973) option model. This model has been extended by others and applied to other contingent claims such as bonds by Merton (1974) and dual purpose funds by Ingersoll (1976a). Nevertheless, there has been little attempt to apply the option model to convertible bonds or preferred stock.

That is the purpose of this paper. In section 2 dominance considerations are employed to determine the optimal conversion strategy for investors and the optimal call policy for the issuer of the convertibles. In sections 3 through 6 the Black–Scholes option pricing model is used to value the convertible as a contingent claim on the firm as a whole. Analytic solutions are determined both for discount bonds and convertible preferred stocks. Section 7 returns to consider the optimal conversion strategy under more general conditions.

Throughout this paper the term bond is intended to denote any security, senior to the common stock. It includes both debt issues and preferreds. Perpetual convertibles are specifically referred to as preferreds since consol bonds are extremely rare, and the author is aware of no convertible consol bonds.

2. Restrictions on convertibles pricing

Merton (1973) has established 'rational' restrictions on the pricing of call options and in a later paper (1974) has extended these results to the pricing of debt and equity claims on a levered firm. In this section the more important of these restrictions will be examined for their applicability to convertible securities.

We shall be concerned here with three varieties of senior securities: straight debt or preferred issues with a set of fixed claims against the firm's assets, non-callable convertibles entitled to a set of fixed claims and which may be exchanged for the common stock of the firm at the owner's option, and callable convertibles. Callable convertibles have the same provisions as their non-callable counterparts; however, they are subject to redemption by the company at the prevailing call price (plus accrued and unpaid interest). When the issue is called, the owners must surrender their claims for the redemption payment or exercise their option to convert.

The conversion terms of an issue are usually expressed in one of two ways. A *conversion ratio* states the number of shares of common stock which can be obtained upon the surrender of one convertible debenture or one share of convertible preferred stock. Alternatively, a conversion is brought about by 'purchasing' common stock at the stated *conversion price* using the convertible at its face value. Although the indenture will state the conversion terms in only one of these two fashions, they can be easily reexpressed in the other manner. If the face value of the convertible is b and the conversion price S, then $m = b/S$ shares of common can be 'purchased' by each debenture. Thus m is the conversion ratio. In this paper we will be dealing with convertible issues in aggregate; consequently it will be more convenient to express the terms of conversion by the *dilution factor*. The dilution factor, denoted γ, indicates the fraction of the common equity which would be held by the convertible issue's owners if the entire issue were converted. If there are N shares of common stock outstanding and the convertible issue can be exchanged, in aggregate, for n shares, then the dilution factor is $\gamma \equiv n/(n+N)$.

Other notation to be employed is: V is the market value of the company. $F(V, \tau; B, C)$ is the aggregate value of a straight senior issue of maturity τ with an aggregate balloon payment B making periodic fixed dividend or coupon payments of C dollars per year in aggregate. $G(V, \tau; B, C, \gamma)$ denotes the aggregate value of a non-callable convertible issue with the same provision as the straight issue, $F(\cdot)$, and which may be exchanged for the fraction γ of the common equity of the company as explained above. A straight issue is a pathological case of a convertible issue exchangeable for zero shares of common; i.e.,

$$G(V, \tau; B, C, \gamma = 0) = F(V, \tau; B, C). \tag{1}$$

$H[V, \tau; K(\tau), C, \gamma]$ denotes the aggregate value of a callable, convertible issue with the same provisions as the non-callable convertible but which is callable by the company upon the aggregate redemption payment of $K(\tau)$ dollars. $K(\tau)$ is the *effective* call price.[1] Again the representation has been only generalized. A non-callable issue can be interpreted as a callable issue with an infinite call price prior to maturity; i.e.,

$$H[V, \tau; K(\tau > 0) = \infty \quad K(0) = B, C, \gamma] = G(V, \tau; B, C, \gamma). \tag{2}$$

The assumptions employed in this paper are given below. Although not all of them are required for every proof or derivation, they are collected here for easy reference. These assumptions can be divided into three categories: behavioral assumptions concerning investor and management action, general market assumptions pertaining to the dominance criterion outlined in this section, and specific market assumptions permitting analytic solutions to the convertible problems later in the paper.

The behavioral assumptions are:

(A.1) Investors prefer more wealth to less. Prices are determined in the market place such that perfect substitutes are valued identically; dominant assets are presumed not to exist.

[1] $K(\tau)$ denotes the actual price paid at call, i.e., the nominal call price plus accrued interest. $K(0)$ is equal to the balloon payment, B, i.e., the face value plus interest accrued at maturity.

(A.2) The management of the company acts at all times to maximize common shareholder wealth subject to the restrictions placed upon it.

These assumptions are assumed to govern throughout the paper even if they are not explicitly stated. Assumption (A.2) concerning management action differs from assuming that management acts to maximize the value of the firm. This distinction has been made previously albeit under different conditions.[2]

The general market assumptions are:

(A.3) *Perfect markets:* The capital markets are perfect with no transactions costs, no taxes, and equal access to information for all investors.

(A.4) *No dividends:* There are no dividend payments or other disbursements to the common stockholders.[3]

(A.5) *Constant conversion terms:* The conversion terms of the convertible security are constant over time.[4]

(A.6) *Modigliani–Miller theorem:* There are no corporate taxes and the Modigliani–Miller proposition I obtains.

(A.7) *No call notice:* When the convertible is called, the owners must immediately surrender their claims for redemption or convert. No call notice is required.

(A.8) *Flat term structure:* The term structure of interest rates is flat and non-stochastic. The instantaneous compounding rate of interest is r.[5]

Of the above, the perfect markets, no dividends, and constant conversion terms assumptions are sufficient for the proofs of Theorem I and II defining the optimal conversion strategy for investors. The perfect markets, Modigliani–Miller theorem, and no call notice assumptions are sufficient to prove Theorem III, and the addition of the flat term structure assumption is sufficient for the proof of Theorem IV. These last two theorems define the company's optimal call policy.

[2]For example, see Stiglitz (1972, footnote 12).

[3]This restriction is lifted in section 7; however, there must always be some restriction on the size of the payments to the owners of the common stock. Otherwise in accordance with Assumption (A.2) governing management action they would pay to the shareholders a liquidating dividend equal to all the firm's assets.

[4]Broman (1963) reports that this was the case for approximately one-half of the convertible bonds issued during the 1950's. This restriction is also lifted in section 7.

[5]As shown in Theorem IV this assumption and a few other restrictions allow us to exclude the possibility of calls used to refinance when interest rates have fallen. The primary concern here is with calls announced to force conversion.

To obtain specific solutions for the convertibles we require, in addition to the above, the option model assumptions:

(A.9) *Frictionless markets:* Trading takes place continuously in time and there are no restrictions against borrowing or short sales.[6]

(A.10) *Ito dynamics:* The market value of the company follows an Ito diffusion process[7] with variance per unit time proportional to the square of the market value of the firm.

(A.11) *Capital structure:* The convertible issue is the only senior issue of the company. The only other claim is the common stock.

This last assumption is also employed in this section; however, this is merely a matter of convenience.[8] If the convertible issue and the common stock are the only two claims in the capital structure of the company, then after conversion only common stock will remain. Since the convertible issue is exchangeable for a fraction γ of the post-conversion equity, γV may be called the *conversion value* of the convertible.

We may now set up some preliminary pricing restrictions. Limited liability precludes a negative price for the issue. The limited liability of the equity together with the Modigliani–Miller theorem sets as an upper limit for a convertible the market value of the firm, V. From arbitrage considerations the convertible issue must sell for at least its conversion value γV. Otherwise convertibles could be purchased, exchanged for common stock and sold at the conversion value realizing (riskless) arbitrage profits. These three restrictions can be expressed compactly as

$$
\begin{aligned}
&0 \leqq \gamma V \leqq G(V, \tau; B, C, \gamma) \leqq V, \\
&0 \leqq \gamma V \leqq H[V, \tau; K(\tau), C, \gamma) \leqq V.
\end{aligned}
\tag{3}
$$

Upon call the owner of the convertibles will choose the more attractive of the

[6]The asymptotic relationship of the continuous-time trading solution to the discrete trading solution is discussed by Merton and Samuelson (1974, pp. 85–92). The assumption of unrestricted borrowing and short selling can be weakened by comparing returns on long positions in the convertibles and common stock using dominance rather than arbitrage. See Merton (1973, footnote 41) and Thorpe (1973) for discussions of this point in the context of option pricing.

[7]See McKean (1969) for the theory of Ito processes.

[8]Although this assumption is employed for convenience in this section, it is not necessary in the proofs of the theorems. If there are multiple senior securities in the capital structure, we may interpret V as the sum of the market values of the common stock and the convertibles, rather than the value of the entire firm. The theorems would follow immediately. The solutions developed in sections 4, 5, and 6 would be correct if this value combination displayed lognormal returns; however, they would be inappropriate if the market value of the entire firm had lognormally distributed returns.

two options, converting and receiving γV or accepting the redemption payment $K(\tau)$. Since it has been assumed that no call notice is required, this decision must be made immediately upon the announcement of the call,

$$H[V, \tau; K(\tau), C, \gamma] = \text{Max}[K(\tau), \gamma V] \quad \text{at call.}^9 \tag{4}$$

A similar relationship holds at the maturity of a convertible issue, either callable or non-callable. If the firm does not default, then the owners of the convertible choose between converting or accepting the balloon payment B just as if the issue had been called. However, if payment cannot be made, the company will default. In the event of a default we employ the standard assumption that the assets of the firm revert immediately and entirely to the owners of the senior security. Since the value of the shareholders' claim after a default is zero, the management will not allow a default if one could possibly be avoided. A default will occur only when the balloon payment cannot be met by all the assets of the company; i.e., $V < B$. The payoff to the convertible issue at maturity is

$$\begin{aligned} H[V, 0; K(0) = B, C, \gamma] &= G(V, 0; B, C, \gamma) \\ &= \text{Min}[V, \text{Max}(B, \gamma V)]. \end{aligned} \tag{4'}$$

The proofs of the remaining restrictions make use of Merton's (1973) arguments against dominant assets where he defines dominance as follows: 'Security (portfolio) A is *dominant* over security (portfolio) B if on some known date in the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B in all possible states of the world.' In addition if there are cash payments, dividends for example, accruing to the securities, then dominance would require that they must be at least as large for A the dominant security as for B at all points of time. For strict comparability under dominance, it is required then that the dividend payments to the common and the dividend or coupon payments to the convertible occur on the same dates. This is often the case when comparing convertible preferred issues to the common. However it is not true for convertible bonds since the coupons are generally paid semiannually while common dividends are usually paid quarterly. If we assume that no dividends are paid (A.4), then it may be possible to show that a coupon-bearing bond dominates the equity. The reverse situation would be impossible to demonstrate. The proofs in the remainder of

[9] It is assumed here that a firm would not call an issue if it could not redeem it using all its assets in payment. If such an eventuality were to be specifically included, the payoff at call would be Min $[V, \text{Max}(K(\tau), \gamma V)]$ similar to the maturity payoff given in (4'). No generality is lost in excluding this case for such a call policy would result in a zero value for the common stock at the default. This clearly violates the assumption that the management acts to maximize shareholder wealth.

this section are of the former type. In section 7, when we wish to allow for the payment of dividends and avoid this problem of simultaneity, we can approximate both common dividends and coupons by equivalent continuously paid sums.

Returning to pricing restrictions on convertible securities it can be shown that the value of a convertible issue must be a non-decreasing function of the dilution factor γ since a convertible security cannot be worth less than another similar security exchangeable for fewer shares of common. A corollary which follows immediately from eq. (1) is that a convertible issue, $G(\cdot)$, must be at least as valuable as its straight security counterpart, $F(\cdot)$, since the straight security may be considered a convertible with a dilution factor of zero. The value of the straight security counterpart is often referred to as the *investment value* of the convertible. The relationships stated above may be expressed as

$$\begin{aligned} F(V,\tau;B,C) &\leqq G(V,\tau;B,C,\gamma_1) \\ &\leqq G(V,\tau;B,C,\gamma_2), \qquad 0<\gamma_1<\gamma_2, \qquad (5) \\ H[V,\tau;K(\tau),C,\gamma_1] &\leqq H[V,\tau;K(\tau),C,\gamma_2], \qquad \gamma_1<\gamma_2. \end{aligned}$$

Furthermore, these relationships may be strengthened to strict inequalities unless there is no possibility of an eventual conversion. The proofs of these restrictions are straightforward applications of the dominance criterion.

It is also straightforward to show that the price of a callable convertible issue, $H(\cdot)$, must be a non-decreasing function of its call price $K(\tau)$. Again, as a corollary from eq. (2), a callable convertible cannot be more valuable than a non-callable convertible,

$$H[V,\tau;K_1(\tau),C,\gamma] \leqq H[V,\tau;K_2(\tau),C,\gamma] \leqq G(V,\tau;B,C,\gamma), \quad (6)$$

for $K_1(\tau) \leqq K_2(\tau)$, all τ, and $K_1(0) = K_2(0) = B$.

With these restrictions established, we can now derive the optimal conversion strategy for investors owning convertibles.

Theorem I. If the perfect markets, no dividends, and constant conversion terms assumptions [(A.3), (A.4) *and* (A.5)] *are valid, then a convertible security will never be converted prior to maturity.*

The proof of this theorem is similar to Merton's (1973) proof that call options will never be exercised prior to maturity.

Proof. Consider the following two investment portfolios: (I) Purchase the entire convertible issue for $G(\cdot)$. (II) Purchase the fraction γ of both the convertibles and the common stock. Clearly the value of this second portfolio at any time in the future will be equal to the fraction γ of the market value of the firm at that time. Let V^* denote the value of the company at the maturity of the convertible. Per the preceding sentences the value of portfolio II will be γV^*. The value of portfolio I containing only the convertibles will depend upon the decisions made by their owners and the firm managers. From the previous discussion we know that the value of the convertibles at maturity will be $\mathrm{Min}[V^*, \mathrm{Max}(B, \gamma V^*)]$ as given in (4').

Table 1 compares the payoffs to these two portfolios. It is clear that portfolio I, the convertible issue, is worth more than portfolio II at maturity unless conversion takes place in which case they have the same value. Furthermore, at each payment date on the convertible issue, portfolio I will receive coupon or dividend payments of C while portfolio II will receive only γC. Both conditions for dominance exist; therefore, unless the current value of the convertibles exceeds the current value of portfolio II, the former will dominate the latter.

Table 1

Demonstration that the payoff at maturity to a convertible issue, $G(V, \tau; B, C, \gamma)$, will be at least as great as the payoff to the fraction γ of the firm considered as a whole.

Portfolio	Current value	Value of firm at maturity		
		$V^* \leqq B$	$B \leqq V^* < B/\gamma$	$B/\gamma \leqq V^*$
I	$G(V, \tau; B, C, \gamma)$	V	B	γV^*
II	γV	γV^*	γV^*	γV^*
Relationship between terminal values of portfolios I and II		$V_{\mathrm{I}}^* > V_{\mathrm{II}}^*$	$V_{\mathrm{I}}^* > V_{\mathrm{II}}^*$	$V_{\mathrm{I}}^* = V_{\mathrm{II}}^*$

Hence, $G(\cdot) > \gamma V$ to avoid dominance. Since the value of a non-callable, convertible issue always exceeds its conversion value prior to maturity, it is obviously never optimal to convert until the very instant of maturity. To do so would be to exchange one asset for another of lower value.

If an investor currently owns the convertibles of a particular company and wishes instead to have the common stock, it would be better to sell the convertibles for $G(\cdot)$ and buy γV worth of common rather than converting. The former method will represent a cash savings of $G(\cdot) - \gamma V > 0$ for the same final holding of common stock.

Corollary. If the conversion factor is dependent upon the time remaining before maturity, $\gamma = \gamma(\tau)$ *with* $\gamma(\tau) \leqq \gamma(0)$ *for all* τ, *and if the other conditions in Theorem I hold, then conversion will take place only at maturity.*[10]

[10]See Kahn and Kahn (1966) for a description of an unusual convertible preferred issue offered by Litton Industries whose conversion terms become more favorable over time. A more common reason for an increase in γ is a 'fixed terms delayed convertible', a bond which may not be converted until some time after the issue date. This corollary proves that the bondholders lose nothing in giving up the right to early conversion on these bonds.

Proof. The proof proceeds exactly like that for Theorem I. In this case the strategy of not converting prior to maturity will strictly dominate that of converting when $\gamma(\tau) < \gamma(0)$. That is, an investor who converts prematurely would have less wealth at maturity in all possible states of nature.

Theorem II. If the perfect markets, no dividends, and constant conversion terms assumptions [(A.3), (A.4) *and* (A.5)] *are valid, then a callable convertible will never be converted except at maturity or call.*

Proof. Consider the same two investment portfolios as in the proof of Theorem I, portfolio I holding only the convertibles and portfolio II invested in the fraction γ of both the convertibles and the common stock. If no call were to take place between now and maturity, then the final values of each of these two portfolios would be identical to those given in table 1, and the condition

Table 2

Demonstration that the payoff at call to a convertible, $H[V, \tau; K(\tau), C, \gamma]$, will be at least as great as the payoff to the fraction γ of the firm considered as a whole.

Portfolio	Current value	Value of firm at call, $\tau = T$	
		$V^* < K(T)/\gamma$	$K(T)/\gamma \leqq V^*$
I	$H[V, \tau; K(\tau), C, \gamma]$	$K(T)$	γV^*
II	γV	γV^*	γV^*
Relationship between values of portfolios I and II at call		$V_{\rm I}^* > V_{\rm II}^*$	$V_{\rm I}^* = V_{\rm II}^*$

for dominance would exist. Suppose on the other hand, a call is announced prior to maturity when the value of the company is V^*. In this case portfolio II will be worth, as always, the fraction γ of the company or γV^*. From (4) portfolio I, the convertibles, will be worth $\text{Max}[\gamma V^*, K(\tau)]$. Table 2 compares the payoffs at call to these two portfolios. The convertibles, portfolio I, will not be worth less than portfolio II and in some cases will be worth more. The condition for dominance exists both at maturity and at any intervening call. Furthermore, since portfolio I receives larger payments (C as compared to γC), both conditions for dominance exist. The current value of the convertible issue must exceed the current value of portfolio II, $H(\cdot) > \gamma V$. Since the value of the convertible issue exceeds its conversion value prior to any call or maturity, it can never be optimal to convert until such action is forced by a call or at maturity

The optimal conversion strategy defined by Theorems I and II may be summarized as: 'The right to exchange a convertible security for the underlying common stock should never be voluntarily exercised.' This advice is not revolutionary. In fact, it is well-known to professional investors.[11] However, in section 7, where the issue of voluntary conversion is examined in more detail, it will be found that there are situations in which it is advantageous to convert voluntarily.

In almost all cases, the indentures of senior securities contain provisions for retiring the issue prior to maturity by calling it at a specified price. The management of the firm will announce a call when it believes that the issue could be favorably refinanced. For example, if interest rates have declined, a debt issue might be refinanced with another issue at a lower coupon rate. Similarly if the company's prospects have improved and its financial leverage decreased, it might be possible to refinance at a lower coupon rate. Since a flat term structure has been assumed, only the latter example is relevant here. There is, however, an additional reason why a convertible might be called. If its conversion value exceeds the call price, management can call and *force* a conversion. In this manner the 'expense' of the future coupon or dividend payments can be avoided. Even if the convertible is a pure discount bond or has no remaining coupons, it can pay to force a conversion which denies the owner the right to further delay his decision on converting. The remainder of this section develops the optimal policy for calling convertible issues.

At any moment of time, the management of a firm with a callable convertible outstanding must decide whether or not to announce a call. Calling the issue implies an immediate payment of the call price or a conversion by the investors. Not calling implies a continuation of the decision process and, possibly, the payment of coupons or dividends. A call policy is a decision rule which renders a 'yes' or 'no' answer to the question, 'Should the convertible be called now?' By assumption two, the optimal call policy is that policy which maximizes the value of the common equity or, what is the same thing under the Modigliani–Miller theorem, minimizes the value of the convertible issue. From the paragraph above it is clear that the optimal call policy will entail a call only when the value of the firm is 'large'.

The call policy decision will yield a number $\bar{V}$ such that the convertible should be called if an only if the current market value of the firm is equal to or greater than $\bar{V}$. The value of the call policy variable $\bar{V}$ can be a function of all the state variables of the problem, the time remaining until maturity, the current and expected future interest rates, the risk associated with the firm's assets, the size of the future coupon and dividend payments, the conversion terms, etc. Due to the economics of the problem and the assumptions employed in this paper it is sufficient to consider the call policy a function solely of the time remaining before maturity with the other variables entering as parameters of the function, $\bar{V} = \bar{V}(\tau)$.

[11]See Graham (1973) for example.

Theorem III. If the perfect markets, Modigliani–Miller theorem and no call notice assumptions [(A.3), (A.6) *and* (A.7)] *are valid, then the optimal call policy* $\bar{V}(\tau)$, *for a convertible satisfies* $\bar{V}(\tau) \leqq K(\tau)/\gamma$ *for all* τ *where* $K(\tau)$ *is the call price plus accrued interest.*

Proof. Suppose that this is not the case. Assume that $V'(\tau)$ is the optimal call policy and that for some maturity T, $V'(T) > K(T)/\gamma$. Now suppose that the value of the company is $V^* \equiv K(T)/\gamma$ when the maturity of the convertible is T. Since $V^* < V'(T)$ the convertible would not be called under the V' policy. From Theorem II we know that this convertible is worth more than its conversion value when 'alive', hence $H[V^*, T; K(T), C, \gamma] > \gamma V^* = K(\tau)$. On the other hand if the convertible were called at this point, the investors would be indifferent between accepting the call price or converting. The convertible would be worth only its conversion value or call price, $H[V^*, T; K(T), C, \gamma] = \gamma V^* = K(T)$. Since the $V'(\tau)$ strategy does not lead to the minimal convertible price it cannot be the optimal call policy, contradicting the original assumption.

The implication of Theorem III is that conversion should be forced as soon as it is possible to do so. The management of the company should never allow the conversion value of a convertible issue to exceed the call price. The economic rationale behind this policy is clear from Theorem II. Since it is in the best interests of the convertible owners to delay the decision to convert for as long as possible, it must be in the best interest of the company to force a conversion as soon as it is feasible. When the conversion value of a convertible issue exceeds its call price, a conversion can be forced immediately. When the call price is larger a conversion cannot be forced.

If a company is levered with senior debt as well as convertible subordinated debentures or convertible preferred stock, then Theorem III still obtains; however, its interpretation must be changed slightly. The conversion value is no longer a constant fraction of the market value of the firm. Nevertheless, the company should force conversion as soon as possible.

While the assumptions underlying Theorem III appear straightforward, the conclusion is perplexing. Companies rarely call a convertible issue until its conversion value is substantially in excess of the call price. Typically this excess can be as large as thirty percent. Possible causes for this discrepancy include the required call notice, the effect of corporate taxes, and underwriting costs – market imperfections which have all been abstracted from in this paper.

Theorem III serves only to establish an upper limit on the call policy decision. We would also like to know under what conditions the call policy can be determined exactly. From Theorem III we know that a convertible should be called only when the owner of the convertible would prefer to receive the cash repayment to converting or when he is indifferent between these two options since the conversion value at call is less than or equal to the call price plus accrued interest. Should the optimal call policy for a convertible entail calling in the former situation (i.e., preference for the cash settlement), then conversion will not take place. In this case determining the optimal call policy would be similar to solving for the call policy on an ordinary bond, an as yet unsolved problem involving a time dependent boundary condition. However, it is possible to establish conditions sufficient to ensure that a call will occur only at the point of indifference, i.e., $\bar{V}(\tau) = K(\tau)/\gamma$.

Merton (1973) has established that the only reasons for the premature exercise of a call option are cash payments to the basis security or an increase in the exercise (call) price. Although he explicitly considers only options on common stock, the results are applicable to call options on non-convertible bonds as well. Applying his rules, a sufficient condition for no early call of a non-convertible bond is that the promised coupon yield be less than the interest rate.[12] A verification of this sufficient condition is obvious. The fair rate of interest on any risky loan must be in excess of the pure rate of interest. If a loan is outstanding at a lower rate, it certainly will never be called.

The same condition will guarantee that a convertible security is never called for refunding purposes, but only to force conversion. The following lemma will be used in the proof of Theorem IV.

Lemma. If the perfect markets, Modigliani–Miller theorem, no call notice, and flat term structure assumptions [(A.3), (A.6), (A.7) *and* (A.8)] *are valid, if the effective call price satisfies* $K(\tau) \geqq C/r$, *and if the firm's management follows the call policy* $\bar{V}(\tau) = K(\tau)/\gamma$, *then the value of a callable convertible must be strictly less than the call price prior to call,* $H(\cdot) < K(\tau)$ *for* $V < \bar{V}(\tau)$.

Proof. Consider the following two investments: portfolio I holding the convertibles and portfolio II consisting of $K(\tau)$ dollars invested in a bank account growing at the rate r compounded continuously. On each coupon (dividend) date of the convertible C dollars are withdrawn from the bank account. The payment streams of the two portfolios are exactly matched. However after the initial date, the bank account balance will always be greater than the prevailing effective call price. It will be larger for two reasons. First the balance will grow at a compound rate r while the call price grows at the simple interest rate $C/K(\tau) \leqq r$. Second the nominal call price may decrease at certain times. Note that on coupon dates both the bank account and the effective call price will decrease by the amount of the coupon. Suppose the convertible is called prior

[12] If we denote the face value of the bond by B, then the condition given is $C/B < r$. Since the effective call price is never less than the face value, we have $K(\tau) \geqq B > C/r$ which is identical to Merton's (1973) condition (13). Furthermore, the nominal call price never increases with time; thus, the effective call price can increase only as fast as interest payments accrue. Under the given condition on the interest payments this rate of increase must be less than the interest rate. This corresponds to condition (11) in Merton (1973). This sufficient condition can be weakened to $K(\tau) \geqq C/r$, all τ; however, for most issues the call price is equal to the face value after some time.

to maturity. Under the stated call policy, the owners will be indifferent between converting or accepting payment. With either option they will receive value equal to the current effective call price which is less than the bank balance. On the other hand suppose that the convertible is not called prior to maturity. Under the stated call policy the value of the company at that time, V^*, would have to be less than B/γ. From (4′) the value of the convertible would be $\text{Min}(V^*, B)$. The bank balance would be worth at least $K(0) \equiv B$. Portfolio II, the bank account, dominates portfolio I, the convertibles. Unless the current price of the convertibles is less than the bank balance $K(\tau)$ a dominant asset exists. Consequently $H(\cdot) < K(\tau)$.

Theorem IV. If the conditions for the above lemma hold, then the optimal call strategy for a convertible is to call when the firm value reaches $V = \bar{V}(\tau) \equiv K(\tau)/\gamma$.

Proof. Suppose that this is not the case. Assume that $V''(\tau)$ is the optimal call policy and that for some maturity T, $V''(T) < \bar{V}(\tau) \equiv K(\tau)/\gamma$. Thus if the value of the company reaches $V''(T)$ when the maturity of the convertible is T, it would be called. From (4), the owners would accept the cash payment hence $H[V''(T), T; K(T), C, \gamma] = K(T)$. On the other hand, if the $\bar{V}$ policy were followed, the convertible would not be called. From the lemma above $H[V''(T), T; K(T), C, \gamma] < K(T)$. The call policy $V''(\tau)$ does not result in the minimum price for the convertible; hence, it can not be the optimal call policy. The proper call policy must then satisfy $\bar{V}(\tau) \gtrless K(\tau)/\gamma$ which together with Theorem III implies that the optimal call policy must be $\bar{V}(\tau) = K(\tau)/\gamma$.

As with Theorem III the general rule for a company with senior debt as well as convertibles is to call when the conversion value of the convertible is equal to the effective call price.

3. Bond pricing by Black–Scholes methodology

If trading in assets takes place continuously in time and the company's market value follows a 'lognormal' Ito process, then Merton (1974) has demonstrated that any contingent claim whose value can be written as a function solely of market value and maturity, $f(V, \tau)$, must satisfy the basic partial differential equation,

$$\tfrac{1}{2}\sigma^2 V^2 f_{vv} + (rV - C) f_v - rf - f_\tau + c = 0, \tag{7}$$

where C denotes the cash payments to all claims on the firm, c is that portion of the disbursements payable to the claim under consideration, σ^2 is the instantaneous variance of return (the variance rate), and subscripts denote partial differentiation. Eq. (7) is an extension of the original Black–Scholes (1973) equation allowing for the possibility of dividends, coupons, and the like. The exact value for the contingent claim depends only upon two boundary conditions, one initial condition, and the functional forms of the two payment streams C and c.

For ordinary bonds, which have an initial condition $F(V, 0; B, C) = \operatorname{Min}(V, B)$ and the 'limited liability' boundary conditions $0 \leq F(\cdot) \leq V$, solutions to (7) are known in two particular cases. If the bond is a discount bond, then the solution is simply a transformation of the original Black–Scholes option solution

$$F(V, \tau; B, 0) = B\mathrm{e}^{-r\tau}\,[\Phi(h_2)+d^{-1}\,\Phi(h_1)], \tag{8}$$

where

$$d \equiv B\mathrm{e}^{-r\tau}/V,$$

$$h_1 \equiv (\log d-\tfrac{1}{2}\sigma^2\tau)/\sigma\sqrt{\tau},$$

$$h_2 \equiv -(\log d+\tfrac{1}{2}\sigma^2\tau)/\sigma\sqrt{\tau},$$

$$\Phi(x) \equiv (\sqrt{(2\pi)})^{-1}\textstyle\int_{-\infty}^{x}\exp(-t^2/2)\mathrm{d}t,$$ the cumulative normal distribution.

For a consol bond with coupons paid continuously at the rate C dollars per unit time, Merton (1974) has provided the solution

$$F(V, \infty; \cdot, C) = (C/r)\left[1-\frac{(ad)^a}{\Gamma(a+2)}M(a, a+2, -ad)\right], \tag{9}$$

where

$$d \equiv (C/r)/V,$$

$$a \equiv 2r/\sigma^2,$$

and $M(\cdot, \cdot, \cdot)$ is the confluent hypergeometric function of Kummer.[13] In both cases d represents the 'quasi' debt to firm value ratio when the debt is valued as if it were riskless.

For our purposes it will be more convenient to write the consol solution as

$$F(V, \infty; \cdot, C) = (C/r)[1-P(a, ad)+d^{-1}\,P(a+1, ad)], \tag{9'}$$

where

$$P(a, z) \equiv (1/\Gamma(a))\textstyle\int_0^z \mathrm{e}^{-t}t^{a-1}\mathrm{d}t$$

is the incomplete gamma function,[14] i.e., the fraction of the gamma function $\Gamma(a)$ up through z, and a and d are defined in (9).[15]

[13]See Slater (1966) for a description and tables of the confluent hypergeometric function.

[14]See Davis (1966) for a description of the incomplete gamma function.

[15]This solution can be verified by direct substitution into (7) or by using Merton's (1974) solution and the following identities from Slater (1966)

$$M(a, a+2, X) = (a+1)\,M(a, a+1, X)-aM(a+1, a+2, X),$$

$$M(a, a+1, -X) = \Gamma(a+1)\,X^{-a}\,P(a, X).$$

We shall also employ the Black–Scholes option function,

$$W(S,\tau;E) \equiv S\Phi(x_1) - Ee^{-r\tau}\,\Phi(x_2), \tag{10}$$

where

$$x_1 \equiv [\log(S/E)+(r+\tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau},$$

$$x_2 \equiv x_1 - \sigma\sqrt{\tau},$$

As is well known, the value of the equity of a company with one senior discount bond in the firm's capital structure is also given by the option function.

4. Non-callable convertible discount bonds

The first convertible we shall consider is a non-callable convertible, discount bond. The price function for this bond, $G(V,\tau;B,0,\gamma)$, will satisfy the standard contingent claims equation (7). In this case both cash payment terms, C and c, will be zero since the bond is a discount bond and, by assumption, there are no dividends paid to the common stock. The modified contingent claims equation is

$$\tfrac{1}{2}\sigma^2 V^2 G_{vv} + rVG - rG_v - G_\tau = 0, \tag{7'}$$

subject to $G(0,\tau)=0$, $G(V,\tau) \leqq V$, and $G(V,0) = \mathrm{Max}[\gamma V, \mathrm{Min}(V,B)]$. The first two constraints are the boundary conditions implied by limited liability as expressed in (3). The latter is the payoff at maturity described by (4′). From Theorem I we know that the arbitrage condition which requires the bond to be at least as valuable as its conversion value, $G(\cdot) \geqq \gamma V$, need not concern us for it is never optimal to convert the bond prior to maturity, and the solution will automatically satisfy this condition.

The solution to (7′) could be determined by transformation to the heat flow equation following Black and Scholes (1973) and Merton (1973). Alternately the Cox–Ross (1976) technique of discounting the expected payoff to the bond in a risk-neutral setting could be employed. However, we can avoid both of these techniques by examining a related problem. Let $g(V,\tau)$ denote the premium of the convertible bond over its straight discount bond value. I.e., $G(V,\tau;B,0,\gamma) \equiv F(V,\tau;B,0)+g(V,\tau)$. Substituting into (7′) we have

$$\tfrac{1}{2}\sigma^2 V^2 g_{vv} + rVg_v - rg - g_\tau = 0. \tag{7''}$$

All terms relating to the discount bond $F(\cdot)$ disappear since this function itself satisfies the contingent claims equation. The boundary conditions remain unchanged $g(0,\tau)=0$, $g(V,\tau) \leqq V$. The initial condition becomes $g(V,0) = \mathrm{Max}(\gamma V - B, 0) = \gamma \mathrm{Max}(V - B/\gamma, 0)$. Eq. (7″) and its boundary conditions are those of the Black–Scholes option function. The initial condition is virtually identical. The two must be closely related. In fact we can readily see that the value of the bond premium must be proportional to that of an option on the firm with an exercise price of B/γ. I.e.,

$$\begin{aligned} g(V,\tau) &= \gamma W(V,\tau;B/\gamma) \\ &= W(\gamma V,\tau;B). \end{aligned} \tag{11}$$

The last step follows from the homogeneity properties of the option function.

The discount convertible function is therefore

$$G(V,\tau;B,0,\gamma) = F(V,\tau;B,0) + W(\gamma V,\tau;B). \tag{12}$$

It is clear from (12) that this convertible bond is equal in value to a portfolio consisting of an ordinary discount bond plus a warrant entitling the owner to purchase the same fraction of the equity of the company upon an exercise payment equal to the principal on the bond.[16] In section 6 we shall prove that this is always the case under the stated assumptions which preclude an early exercise of the conversion right.

A graph of this convertible bond function is provided in fig. 1 in the next section together with that for a callable convertible. The comparative statics are given below. With reference to (12) these can be readily verified from the known properties of the discount bond and option functions,

$$\begin{array}{llll} \text{(a)} & G_v > 0, & \text{(e)} & G_{vv} \lesseqgtr 0, \\ \text{(b)} & G_B > 0, & \text{(f)} & G_{\sigma^2} \lesseqgtr 0, \\ \text{(c)} & G_r < 0, & \text{(g)} & G_\tau \lesseqgtr 0. \\ \text{(d)} & G_\gamma > 0, & & \end{array} \tag{13}$$

(13a) through (13d) give the results expected from analysis of straight debt issues and common sense. (13e) through (13g) indicate the dual debt–equity character of the convertible. In each case when firm value is low and there is

[16] Strictly $W(\gamma V,\tau;B)$ is the value of an option rather than a warrant. Using the same technique as that used to derive the value of the conversion premium $g(\cdot)$ in this section, it can be demonstrated that the aggregate value of warrants, exchangeable for the fraction γ of an all equity company upon payment of an aggregate exercise price of E, would be $W[\gamma V,\tau;(1-\gamma)E]$. The exercise price is effectively lowered since the fraction γ of the payment reverts to the warrant holders through the common stock they receive. If the common stock is levered by a discount bond promising B dollars at the maturity of the warrant, the warrant will then be worth $W[\gamma V,\tau;(1-\gamma)E+\gamma B]$. This time the effective exercise price is raised since the warrant holders will be paying the fraction γ of the balloon payment if they choose to exercise. If the aggregate exercise price E and the balloon payment B are equal, the warrant will have the value $W(\gamma V,\tau;E)$ equal to the premium on the convertible bond in (12).

little likelihood of eventual conversion, the bond behaves in the manner of straight debt (upper inequalities). For large asset values, it behaves like levered equity (lower inequalities).

5. Callable convertible discount bonds

Since most convertible bonds are issued with a call provision, the analysis would be far from complete if this feature were ignored. In light of our continuous time analysis, the natural functional form to assume for the call price of a discount bond is the exponential, $K(\tau) = Be^{-\rho\tau}$. If the rate of change in the call price ρ is zero or negative, then Theorem IV will be applicable. With a slight alteration in the lemma Theorem IV can also be proved for positive values for ρ less than the interest rate. With the above restriction on the call price, we know that the optimal call policy entails calling when the firm value reaches $\bar{V}(\tau) \equiv K(\tau)/\gamma$.

A callable, convertible, discount bond with call policy $\bar{V}(\tau)$ described above satisfies the standard contingent claims equation (7). As for the non-callable bond, the cash payment terms are zero. This convertible $H[V, \tau; K(\tau), 0, \gamma]$ is the solution to

$$\tfrac{1}{2}\sigma^2 V^2 H_{vv} + rVH_v - rH - H_\tau = 0, \qquad 0 \leqq V \leqq \bar{V}(\tau), \tag{7'''}$$

subject to $H(0, \tau) = 0$, $H[\bar{V}(\tau), \tau] = K(\tau)$, and $H(V, 0) = \mathrm{Min}(V, B)$. The first boundary condition is an expression of limited liability as implied in (3). The second condition indicates the receipt of the call price at a call.[17] The initial condition is the usual expression for the final payment to a bond. The opportunity to convert (i.e., γ) does not enter into the initial condition because the bond will never be converted at maturity if the company follows the optimal call strategy. A bond would be converted at maturity only if its coversion value exceeded the balloon payment, $\gamma V > B$. However, this would imply that the company had not followed the optimal call policy since $V > B/\gamma = K(0)/\gamma \equiv \bar{V}(0)$ in violation of Theorem III.

While eq. (7''') could be solved by transformation to the semi-infinite heat flow problem in physics, the economics of the solution are clearer if we use the method suggested by Cox and Ross (1976). In this procedure the value of the contingent claim is determined as if all investors were risk-neutral and the expected rate of return on all assets were the riskless rate. It is then verified that this solution is correct for any set of risk preferences by substituting it into the partial differential equation.[18] For a financial claim whose present worth

[17]Since, under the optimal policy, a call occurs at the point where call price and conversion value are equal, we may assume that the bondholders accept the call price with no loss in generality.

[18]For a description and justification of this solution method see Cox and Ross (1976, pp. 153–155).

derives solely from its value at some known date in the future, such as a European option, the price is simply the expectation of this future value discounted to the present.

For convertibles the process is conceptually the same. However, we now must include the possibility of a payoff prior to maturity if the bond is callable,[19]

$$H[V_0, \tau; K(\tau), \gamma] = \mathrm{e}^{-r\tau} \int_0^{B/\gamma} \mathrm{Min}(V, B)\, p(V, \tau; V_0)\, \mathrm{d}V + \int_0^{\tau} \mathrm{e}^{-rt} K(\tau - t)\, q(t; V_0)\, \mathrm{d}t; \tag{14}$$

$p(\cdot)\mathrm{d}V$ is the (defective) probability density of the market value of the company at maturity conditional on a current value of V_0 and conditional on no call having been issued.[20] The first integral in (14) gives the value of the convertible arising from the balloon payment or possible default at maturity. $q(\cdot)$ is the probability density function of a call being issued at time t conditional on a current value of V_0 for the company. Thus, $q(\cdot)$ is the probability that at time t the value of the company is $\bar{V}(\tau - t)$ and no call has previously been issued. The second integral in (14) gives the value of the convertible arising from the possibility of a call prior to maturity.

Since the firm dynamics are geometric Brownian motion in continuous time, the probability density $p(\cdot)$ must satisfy the well-known 'forward' or Fokker–Planck equation. For our purposes it will be easier to work with the log of firm value whose dynamics are ordinary or arithmetic Brownian motion. Let $X \equiv \log[V/\bar{V}(\tau)]$, then by Ito's lemma the 'risk neutral' dynamics for X are[21]

$$\mathrm{d}X = (r - \tfrac{1}{2}\sigma^2 - \rho)\, \mathrm{d}t + \sigma \mathrm{d}Z \equiv \mu \mathrm{d}t + \sigma \mathrm{d}Z. \tag{15}$$

At any time in the future X will be normally distributed. It has an expected drift of μ and variance of σ^2 per unit time. The associated Fokker–Planck equation for $p'(\cdot)$, the probability density for X, is

$$\frac{\partial p'}{\partial t} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 p'}{\partial X^2} - \mu \frac{\partial p'}{\partial X}, \qquad X < 0. \tag{16}$$

[19] Black and Cox (1976) show that any claim on a firm can be decomposed into four components each of which contributes to the claim's value. The four components represent the value of receiving the terminal distribution at maturity, the value of any interim disbursements such as coupons, and the values arising from any reorganization of the firm at low and high levels of market value. The first component has been the primary concern of most contingent-claims problems. The call payment is of the last type.

[20] This probability density is 'defective' since it does not integrate to one in the interval $(0, B/\gamma)$. Rather it integrates only to the probability of no call having occurred. I.e.,

$$\int_0^{B/\gamma} p(\cdot)\, \mathrm{d}V = \int_0^{\tau} q(\cdot)\, \mathrm{d}t < 1.$$

[21] The firm's 'risk neutral' dynamics are given by $\mathrm{d}V = rV\mathrm{d}t + \sigma V \mathrm{d}Z$. Using Ito's lemma the dynamics for X are thus $\mathrm{d}X = (r - \tfrac{1}{2}\sigma^2 - \rho)\, \mathrm{d}t + \sigma\, \mathrm{d}Z$. For a further discussion of Ito's lemma see, for example, Merton (1976).

We are concerned only with negative values for X since if X ever reaches zero in value, then $V = \bar{V}(\tau)$ and the convertible will be called.

The desired probability densities can now be determined as the solutions to two classical problems in mathematics. The probability density for X is[22]

$$p'(X,\tau;X_0) = \Phi'\left[\frac{X-X_0-\mu\tau}{\sigma\sqrt{\tau}}\right] - \exp\left[-\frac{2\mu X_0}{\sigma^2}\right]\Phi'\left[\frac{X+X_0-\mu\tau}{\sigma\sqrt{\tau}}\right], \qquad X<0, \tag{17}$$

where $\Phi'(\cdot)$ is the normal density function. The probability density of the time of call, $q(\cdot)$, is the first passage time density of X through the origin. This is [23]

$$q(t;X_0) = \frac{-X_0}{\sigma\sqrt{(2\pi t^3)}}\exp\left[-\frac{(X_0+\mu t)^2}{2\sigma^2 t}\right]. \tag{18}$$

Making the substitution $V = K(\tau)e^X/\gamma$ into (14) and using (17) and (18) we can perform the indicated integrations.[24] The solution for the callable convertible is

$$H(V,\tau) = F(V,\tau;B,0) + W(\gamma V,\tau;B) + Z^{2(r-\rho)/\sigma^2}\left[F(\gamma V',\tau;B',0) - F(\gamma V',\tau;B'/\gamma,0)\right],$$

where

$$B' \equiv B\exp(r-\rho)\tau,$$

$$V' \equiv V\exp(\rho-r)\tau,$$

$$Z \equiv \bar{V}(\tau)/V = K(\tau)/\gamma V. \tag{19}$$

Inspection of (19) reveals that the first two terms are the value of a non-callable, convertible bond as given in (12). Hence, the last term represents the discount due to the call feature. Alternately we may view the last term as (the negative of) the value of the call privilege to the equity owners, and think of the bond as a non-callable, convertible bond with a call option on it sold back to the company.[25] Since the discount bond function $F(\cdot)$ is an increasing function of the payment promised, the call privilege always has positive value to the equity owners.

A case of special interest, since it is the limiting case for which the call policy is known, is when the rate of growth in the call price just equals the interest rate.

[22] See, for example, Cox and Miller (1965, p. 221, eq. (71)).

[23] See, for example, Cox and Miller (1965, p. 221, eq. (73)).

[24] The author will furnish a detailed derivation upon request.

[25] The call option allows the company to reacquire the bond or to force a conversion at the choice of the bond holders.

In this case ($\rho = r$) the bond has the value

$$H(V, \tau) = F(V, \tau; B, 0) + W(\gamma V, \tau; B/\gamma). \tag{20}$$

In this special case the convertible bond can be viewed as a portfolio of an ordinary bond plus a warrant. This is reminiscent of the non-callable convertible although the associated warrants have different exercise prices.

The comparative statistics for the callable convertibles are

$$\begin{array}{llll} \text{(a)} & H_v > 0, & \text{(e)} & H_r < 0, \\ \text{(b)} & H_B > 0, & \text{(f)} & H_{vv} \lesseqgtr 0, \\ \text{(c)} & H_\gamma > 0, & \text{(g)} & H_{\sigma^2} \lesseqgtr 0, \\ \text{(d)} & H_\rho < 0, & \text{(h)} & H_\tau \lesseqgtr 0. \end{array} \tag{21}$$

(21a) through (21d) give us the common sense results. (21f) through (21h) repeat the results of the non-callable bond. Again in each case when asset value is low and there is little likelihood of an eventual call, the bond behaves as if it

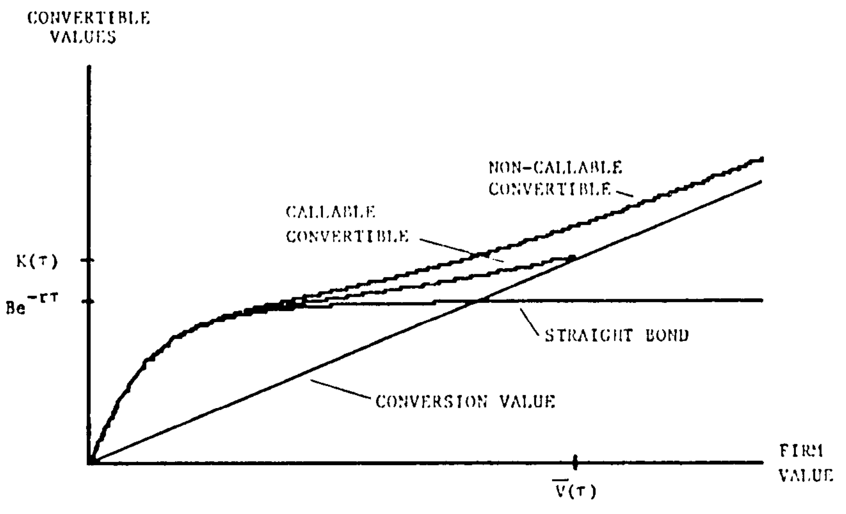

Fig. 1. Plot of discount convertible bond functions for different firm values, with a given maturity, promised payment, call price and dilution factor on the bonds and a given variance rate on the firm and riskless market rate of interest. Unlike the straight bond, the convertible values lie above the conversion value, and the functions are convex at large firm values. The callable convertible is called at the point where its call price and conversion value are equal. These plots were constructed with the following parameter values: maturity $\tau = 5$, promised payment $B = 100$, call price $K(\tau) = B \cdot \exp(-\rho\tau)$ for $\rho = 0.02$, dilution factor $\gamma = 0.2$, variance rate $\sigma^2 = 0.05$, and interest rate $r = 0.07$. Firm value range is 0 to 625. Convertible bond range is 0 to 175.

were an ordinary discount bond (upper inequality). For high asset values it behaves like levered equity (lower inequality). The response of this bond to static interest rate changes can best be understood by realizing that holding the bond is, as was previously pointed out, similar to being long a non-callable, convertible bond and short a call option on that bond. We know from (13c) that the convertible bond part is a decreasing function of the interest rate. We would suspect from Merton's (1973) analysis of options that the call option held by the management would be an increasing function of the interest rate.

Plots of a regular bond's price and prices of non-callable and callable convertible bonds are presented in fig. 1.[26] The plot shows the callable bond for a particular value of ρ. Since the bond is a decreasing function of call price, other curves can be easily imagined for other values of this parameter. As the call price becomes indefinitely large ($\rho \to -\infty$), the callable bond curve would approach that of the non-callable bond. All three bond types are increasing functions of asset value as fig. 1 shows. The convertible bonds are concave in asset value for small values and convex for larger values. Despite the convexity neither is a levered security since their elasticities with respect to firm value are always less than one.[27] Thus, the bonds are always less risky than the common stock or the company as a whole just as ordinary bonds are.[28] The convexity of the function at large firm values is a result of the 'bond floor value' (i.e., the value of the convertible as a straight bond) noted by Brigham (1966) and others. The 'bond floor value' is somewhat of an illusion since, as we can easily see in fig. 1, it is also subject to erosion by a declining firm value.

6. Coupon payments and convertible preferred stocks

A convertible bond receiving coupon payments continuously at the rate C dollars per unit time will satisfy the contingent claims partial differential equation as given in (7),

$$\tfrac{1}{2}\sigma^2 V^2 G_{vv} + (rV - C)G_v - rG - G_\tau + C = 0, \tag{22}$$

subject to the same boundary conditions applicable in the case of discount bonds namely $G(0, \tau) = 0$, $G(V, \tau) \leqq V$, and $G(V, 0) = \operatorname{Max}(\gamma V, \operatorname{Min}(V, B))$.

[26] The reader is also referred to the recent paper by Brennan and Schwartz (1975) which has graphs of convertible bonds under a variety of conditions. The authors employ finite difference approximations to solve equations similar to (7).

[27] Generally, securities which are convex functions are levered securities, for example call options. Another security which violates this general rule is the 'down and out' option. See Merton (1973) for details.

[28] 'Less risky' is used here in the Rothschild–Stiglitz (1970) sense. The standard deviation of returns on the bonds are $(VGv/G)\sigma$ and $(VHv/H)\sigma$ or the elasticity multiplied by the standard deviation on the company's returns. Since all contingent claims are (instantaneously) perfectly correlated with the value of the company, intuition tells us that either standard deviation or covariance are equivalent measures of relative risk. See Merton (1973, app. 2) for a rigorous proof of the validity of this measure of risk.

Theorem V. Given the assumptions of this paper, a non-callable, coupon-bearing, convertible bond has the same value as an ordinary bond with the same coupons, principal, and maturity plus an attached stock purchase warrant of identical maturity, exchangeable for as many shares as is the convertible, and whose gross exercise payment is equal to the face value of the bond.

Proof. We know from Theorem I that the right to convert will never be used prior to maturity. Also from Merton's (1973) analysis the warrant will never be prematurely exercised. Thus, we need only be concerned with the payoffs at maturity. Denote by V^* the value of the company at maturity. If $V^* < B$, the company would default under either bond. If $B \leqq V^* < B/\gamma$, both sets of bond holders would receive the principal in payment. The equity would be worth $V^* - B$ which is not large enough to merit an exercise of the warrants. If $V^* \geqq B/\gamma$ then the convertible bond holders would convert, and the holders of the regular bonds would receive B dollars which they would then use to pay the exercise fee on the warrants. In both cases they will own the same number of shares of the company. Table 3 shows the payoffs in detail.

Table 3

Demonstration that the payoffs to a convertible coupon-bearing bond, $G(V,\tau;B,C,\gamma)$, and a portfolio consisting of an identical non-convertible, coupon-bearing bond $F(V,\tau;B,C)$ plus a warrant of identical maturity, exchangable for the same fraction γ of the equity of the firm upon a gross exercise payment equal to the face value of the bond, $\gamma W(V,\tau;B/\gamma)$ are equal.

		Value of firm at maturity		
Portfolio	Current value	$V^* \leqq B$	$B < V^* \leqq B/\gamma$	$B/\gamma < V^*$
I	$G(V,\tau;B,C,\gamma)$	V^*	B^*	γV^*
II	$F(V,\tau;B,C)+\gamma W(V,\tau;B/\gamma)$	V^*+0	$B+0$	$B+\gamma V^*-B$
		$V_{\mathrm{I}}^* = V_{\mathrm{II}}^*$	$V_{\mathrm{I}}^* = V_{\mathrm{II}}^*$	$V_{\mathrm{I}}^* = V_{\mathrm{II}}^*$

Theorem V is a generalization of the result found in section 6 for discount bonds. The bond–warrant combination can differ in value from the convertible bond only when an early exercise of the warrant or conversion of the bond may be optimal. For example if the common stock is paying large dividends, exercising the warrant might be optimal. On the other hand, the dividends may not be large enough to warrant giving up the remaining coupon and principal payments owed the convertible bond. The alternate case when it would be desirable to convert the bond but not exercise the warrant is also possible if the exercise price of the latter is higher than the investment value of the bond. The question of early conversion is examined in section 7.

The only type of coupon bond for which a closed form solution is known is the perpetuity or consol bond whose solution was given in section 3. Using Theorem V and eq. (9), we can write

$$G(V, \infty; C) = (1-\gamma)\, F(V, \infty; C)+\gamma V. \tag{23}$$

While perpetual bonds are uncommon financial instruments, (23) is also the correct pricing function for non-callable, convertible preferred stocks.

Results as general as those presented in Theorem V are not readily obtainable for callable convertible bonds. In general only the perpetual solutions can be obtained since the general contingent claims equation (7) then becomes an ordinary differential equation. Consequently, the remainder of this section will be devoted to convertible preferred stocks which are perpetual claims. One special case which does have a solution is a convertible preferred with a constant call price K. We shall denote its value by $H(V; \infty, C)$. The condition of Theorem IV will be met if $K \geqq C/r$, in which case the call policy will be $\bar{V} = K/\gamma$.

The convertible preferred satisfies the differential equation,

$$\tfrac{1}{2}\sigma^2 V^2 H'' + (rV - C)H' - rH + C = 0, \tag{24}$$

subject to $H(0) = 0$ and $H(K/\gamma) = K$. Two independent homogeneous solutions satisfying (24) are $V - C/r$ and $F(V, \infty; C) - C/r$, the consol bond solution given in (9) less the riskless value of the dividends. A particular solution is C/r. Using the lower boundary condition we determine that the arbitrary constants must sum to one, and the upper boundary condition gives us

$$H(V, \infty) = V + (1-\gamma)[1-(\gamma/K)F(K/\gamma, \infty)]^{-1}[F(V, \infty) - V]. \tag{25}$$

When the condition given in Theorem IV is not valid (i.e., $K < C/r$), the optimal call policy can not be determined a priori. It must be determined as a part of the solution. This may be accomplished in the manner suggested by Merton (1973). For a convertible preferred with a constant call price K, the optimal call point will be independent of time, hence $\bar{V}(\tau) = \bar{V}$ a constant. To solve for the preferred's price we assume that the strategy $\bar{V}$ is known and let $h(V, \bar{V})$ be the solution to (24) subject to $h(0, \bar{V}) = 0$ and $h(\bar{V}, \bar{V}) = K$, for a given value of $\bar{V}$. Since the firm's managers who determine the call strategy are presumed to act in the best interests of the equity owners, they will choose that policy which maximizes the value of the equity or, what is the same thing, minimizes the value of the preferred,

$$H(V, \infty) = \min_{\bar{V} \leqq K/\gamma} [h(V, \bar{V})]. \tag{26}$$

Proceeding as before we derive a valuation formula now consistent with any call price K. The differential equation and the lower bondary condition are the same; hence, the homogeneous solutions do not change and the arbitrary constants still sum to one,

$$H(V,\infty) = bF(V,\infty;C)+(1-b)V. \tag{27}$$

b may be determined as a function of $\bar{V}$ by invoking the upper boundary condition,

$$b \equiv -\frac{K-\bar{V}}{\bar{V}-F(\bar{V},\infty;C)}. \tag{28}$$

Finally $\bar{V}$ is the solution to the constrained minimization problem (26). If $\bar{V}$ has an interior solution it is given by

$$1-P(a+1,2C/\sigma^2\bar{V}) = [1-P(a,2C/\sigma^2\bar{V})](C/rK), \tag{29}$$

where

$$a \equiv 2r/\sigma^2,$$

and $P(\cdot)$ is the incomplete gamma function introduced earlier.[29]

If an interior solution to (26) is obtained, then the value of the convertible is

$$H(V,\infty) = V-[1-P(a+1,2C/\sigma^2\bar{V})]^{-1}[V-F(V,\infty;C)]. \tag{30}$$

This solution is independent of the conversion terms of the convertible since γ does not appear. At first this independence may seem counter-intuitive; however, it is easily explained. For an interior solution, $\bar{V} \leq K/\gamma$ which means that investors will not choose to convert when the preferred is called. Instead they will accept the call price. Furthermore, from Theorem II they will never voluntarily convert. Hence, we have a preferred stock which, although nominally convertible, will actually never be converted. It should not be surprising therefore that the terms of conversion do not affect its pricing. Consequently the function given in (30) is also valid for pricing ordinary (i.e., non-convertible) preferred stock with a call price K.[30]

If an interior solution to (26) is not obtained, then the constraint of Theorem III is binding and the preferred stock will be called at the point where conversion just marginally would become profitable. In this case the price of the preferred

[29] $\bar{V} = 0$ is always a solution to (29). If a positive solution for $\bar{V}$ exists to (29), that is the desired solution. If this solution is less than K/γ, then it is an 'interior solution'.

[30] A non-convertible preferred stock or bond may always be viewed as a convertible preferred stock or bond with a conversion factor of zero, $\gamma = 0$. In this case a solution to (29) will exist if $K < C/r$. A fortiori, it will be an 'interior solution'. Thus, we have the common sense result that a non-convertible preferred stock will be called only if its call price is less than its maximum market value, the riskless present value of all future dividends.

is again given by (25). This is valid despite the fact that the sufficient condition in Theorem IV does not hold.

Fig. 2 illustrates the convertible preferred stock functions. They appear much like the discount bonds in fig. 1. There are only two qualitative differences. The non-callable convertible approaches a limiting value in excess of its conversion value unlike the discount convertible. For large asset values the premium above conversion value disappears for the latter since the probability of eventual

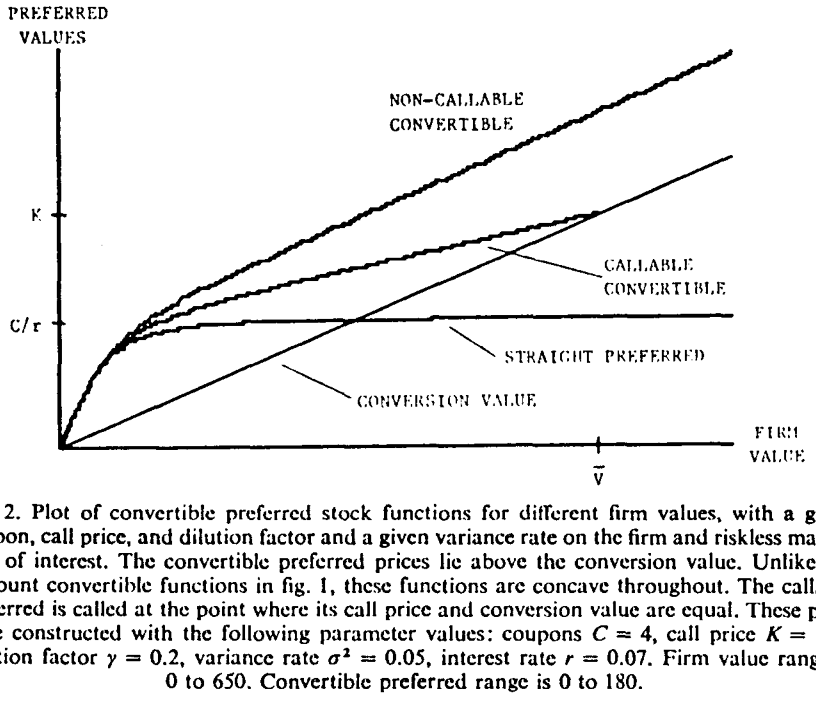

Fig. 2. Plot of convertible preferred stock functions for different firm values, with a given coupon, call price, and dilution factor and a given variance rate on the firm and riskless market rate of interest. The convertible preferred prices lie above the conversion value. Unlike the discount convertible functions in fig. 1, these functions are concave throughout. The callable preferred is called at the point where its call price and conversion value are equal. These plots were constructed with the following parameter values: coupons $C = 4$, call price $K = 104$, dilution factor $\gamma = 0.2$, variance rate $\sigma^2 = 0.05$, interest rate $r = 0.07$. Firm value range is 0 to 650. Convertible preferred range is 0 to 180.

conversion becomes unity. For the preferred stock the premium remains since it is due to the present value of the future dividends. Also the convertible preferred functions are concave throughout. The convexity of the discount bonds derives not from their finite maturity but from the final balloon payment which can be a more attractive payoff at maturity than is converting. As the value of the firm is decreased the value of the bond does not fall as fast as does its conversion value since the insurance value of the balloon payment becomes larger.

The finite maturity callable convertible function will be the solution to the standard partial differential equation (7) subject to the boundary conditions, $H(0, \tau) = 0$, $H(K(\tau)/\gamma, \tau) = K(\tau)$, and $H(V, 0) = \text{Min}(V, B)$. Although this problem has no closed form analytic solution, it can readily be solved numerically by a finite difference equation approximation.

7. Voluntary conversion

We have seen previously that, under the assumptions maintained so far, convertible bonds will never be voluntarily converted. Conversion of the bonds into equity will take place only when forced by a call or by expiration of the privilege at the maturity of the bond. We now examine two causes of voluntary conversion. To emphasize the voluntary nature of the conversion we shall confine our attention at first to non-callable bonds.

One reason that a bond might be voluntarily converted is an adverse change in the conversion rights (i.e., a decrease in γ). We are considering here only contractural decreases in the convertibility of the bond (i.e., from an increase in the conversion price) not those caused by new equity financing.[31] If a fair price is received for the new financing issue, it could not cause a voluntary conversion. On the contrary, new financing would increase the debt value of the convertible since the leverage would become less.

If assumptions one through five hold otherwise, it is easy to show that a voluntary conversion will take place only at those points of time just prior to an increase in the conversion price. To see that a voluntary conversion may be optimal at such points note that if the value of the company is quite high, it is virtually certain that the bonds will eventually be converted. Consequently the convertible's price will be close to its conversion value γV. If the bondholders chose to convert, they would receive the value γV in stock. On the other hand, if they chose not to convert, their bonds would suddenly fall in value to a price near the new conversion value $\gamma' V$ when the conversion factor changed to its lower value γ'.

Let us consider a general example when there are discrete changes in the conversion price of the bond over its lifetime according to the following program:

Conversion factor	Maturity
γ_1	$0 \leqq \tau < \tau_1$
γ_2	$\tau_1 \leqq \tau < \tau_2$
$\vdots$	$\vdots$
γ_n	$\tau_{n-1} \leqq \tau$

[31]Since $\gamma = n/(n+N)$ where N is the number of shares of equity outstanding, new financing will decrease γ. We also do not consider the very rare type of convertible for which a change in conversion price takes place not at a specified date, but after a stated fraction of the bonds have been converted. This involves the bondholders in a game of 'chicken' in which each tries to be the last to convert under the more favorable terms. Pricing these bonds would require game theory.

Furthermore we assume that the γ_i are strictly increasing in size $\gamma_1 < \gamma_2 \dots < \gamma_n$.[32] During the last interval, $0 \leqq \tau < \tau_1$, there are no changes in the conversion price; hence, this is the standard problem with solution $G_1(V, \tau) = G(V, \tau; B, \gamma)$ as given by (12).

During the second to last interval, there are again no changes in the conversion price. The bond price at this time will be the solution to the standard partial differential equation (7′) with the initial condition, $G_2(V, \tau_1) = \text{Max}[\gamma_2 V, G_1(V, \tau_1; B, \gamma_1)]$. Proceeding in the dynamic programming-like method suggested by Merton (1973), we find during each interval of time the bond price is the solution of (7′) with the initial condition $G_{i+1}(V, \tau_i) = \text{Max}[\gamma_{i+1} V, G_i(V, \tau_i)]$. Problems of this type can easily be solved with a finite difference approximation to the partial differential equation. Brennan and Schwartz (1975) employ this technique.

Fig. 3 provides a graphic solution to this problem. If the bond is not converted at the time indicated by maturity τ_i, it will have the value denoted by $G_i(\cdot)$.

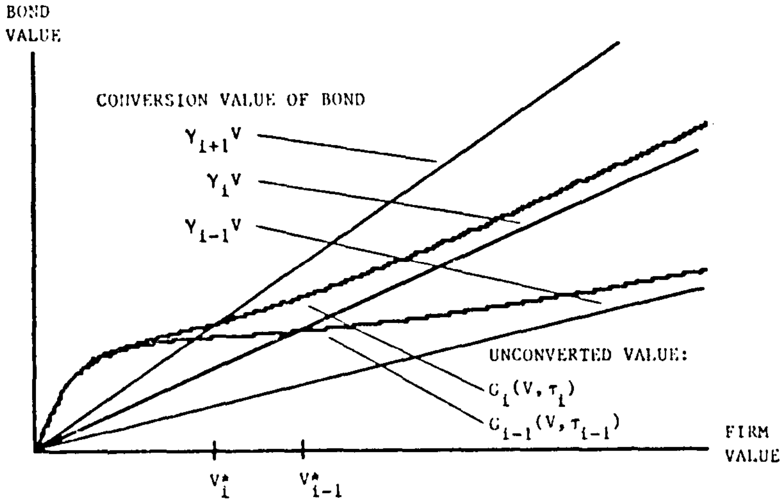

Fig. 3. Graphic determination of the optimal conversion points, V_i^*, for a convertible bond whose conversion terms become less favorable over time. The optimal conversion points are indicated by the intersections of the curves denoting the value of the bond if not converted and the lines denoting the conversion value of the bond. This plot is constructed for a discount bond with a promised payment $B = 100$, and which may be exchanged for $\gamma = 10\%$ of the firm's common stock during the last 5 years before maturity, $\gamma = 20\%$ of the common stock with maturity between 5 and 10 years, and $\gamma = 30\%$ with a maturity of greater than 10 years. The firm's variance rate is $\sigma^2 = 0.10$. The interest rate is $r = 0.07$. With 5 years remaining to maturity conversion will occur if the firm value is greater than $V^*_{i-1} = 386$. With 10 years remaining to maturity conversion will occur if the firm value is greater than $V_i^* = 276$.

[32]This would correspond to a set of increasing conversion prices. If there were a decrease from one interval to the next, the two intervals could be combined and the larger γ assumed throughout the combined interval by application of the corollary to Theorem I.

If it is converted, the value of common stock received in exchange will be $\gamma_{i+1} V$. The cost of not converting is a result of the decrease in conversion value as marked by the vertical distance between the two lines $\gamma_{i+1} V$ and $\gamma_i V$. This cost is proportional to firm value. The cost of converting is a result of the loss of the insurance protection provided by the promised balloon payment on the bond.[33] This cost is the vertical distance between the curve $G_i(\cdot)$ and the line $\gamma_i V$. The rational bond holder will accept only the smaller of the two costs. He will convert if and only if the value of the firm exceeds the critical value marked V_i^*.

The determination of these critical values can not be separated from the pricing of the bond since they depend upon the valuation formula for the next interval. We can, however, assess the comparative statics. V_i^* is the solution to

$$\gamma_{i+1} V_i^* = G_i(V_i^*, \tau_i). \tag{31}$$

Hence, for any argument of $G_i(\cdot)$ denoted X,

$$\frac{\partial V_i^*}{\partial X} = \frac{\partial G_i(V_i^*, \tau_i)}{\partial X}\left[\gamma_{i+1} - \frac{\partial G_i(V_i^*, \tau_i)}{\partial V}\right]^{-1}. \tag{32}$$

The sign of the term in brackets is positive as is apparent from fig. 3. Thus the sign of many of the comparative statics can be determined by common sense. Any change which would increase the value of the bond if it were not converted will increase V_i^*. In particular,[34]

$$\begin{aligned} &\text{(a)} \quad \partial V_i^*/\partial \gamma_j > 0, \qquad j \leqq i, \\ &\text{(b)} \quad \partial V_i^*/\partial \tau_j < 0, \qquad j < i, \\ &\text{(c)} \quad \partial V_i^*/\partial B > 0. \end{aligned} \tag{33}$$

It should be noted that the critical values need not form a monotonic sequence. If the conversion factor γ_{i+1} is reduced, the first line in fig. 3 will be lowered. The remaining lines and curves will not be affected since they deal with the pricing of the bond after conversion at γ_{i+1} is no longer possible. Thus as γ_{i+1} is reduced, the intersection determining V_i^* moves to the right, and V_i^* may exceed V_{i-1}^*.

Dividend payments to the common shareholder may also make an early conversion optimal. This problem may be handled in exactly the same manner

[33] Plus the future coupons for coupon bonds.

[34] It is clear by dominance that $G_i(\cdot)$ must be an increasing function of the promised payment B and of the current and future conversion factors. By dominance it is also obvious that $G_i(\cdot)$ must be a decreasing function of the future change points, τ_i, since the smaller is any particular τ_i the longer into the future will the higher conversion factors prevail.

as changes in the convertibility provided the dividend policy is non-stochastic.[35] After the last dividend payment of size $\delta(V, \tau)$ before maturation of the bonds, the price of the convertibles will be given by the standard formulation (12). On the ex-dividend date of this last dividend, the bondholders will choose to convert if $\gamma(V+\delta) > G(V, \tau)$ where V is the ex-dividend value of the firm. Prices at all previous dates are determined by backward induction as previously.

If the dividend date corresponds to a coupon date, then the bondholders will clearly not convert if the coupon they will get is larger than the dividend payable to the shares they would receive. Usually the coupon payments are larger than the post-conversion dividends on the bond. Unfortunately dividends are generally paid quarterly while coupon payments are semi-annual. Thus, there are at least two dates a year when dividends alone are paid.

If we treat the problem of dividends and coupons in continuous times we can get additional results and avoid this simultaneity problem. The results derived will no longer be exactly true in the real world; however, the qualitative behavior should not be too far wrong. If dividends are paid continuously, it is now possible that conversion will take place at any time prior to maturity, and the critical values must be replaced by a continuous function $V^*(\tau)$. This function is the optimal conversion strategy. If the value of the firm exceeds $V^*(\tau)$ when the maturity of the bond is τ, then the bond should be converted. The following lemma and theorem are the continuous time generalization of the reasoning in the above paragraph.

Lemma. If there are continuously paid coupons to the bonds and dividends to the common shares at C and $\delta(V, \tau)$ dollars per unit time respectively, then the bonds will be converted only at those times prior to maturity when $C < \gamma(C+\delta)$.

Proof. Consider the two investments in Theorem I: (I) purchase a fraction γ of the convertibles and equity, and (II) purchase the convertible bonds. The cash payments to (I) will be $\gamma(C+\delta)$, the cash payments to (II) will be C. From Theorem I the latter will be a dominant asset unless the cash payments of the former are larger or $C < \gamma(C+\delta)$. If the convertible is dominant, $G(\cdot) > \gamma V$ and it will not pay to convert.

If the total cash payments by the firm are not changed following a voluntary conversion, then the former bondholders will receive dividends in the amount $\gamma(C+\delta)$. Thus the condition given in the lemma above is that the potential dividends to be received exceed the coupons currently being received.

Theorem VI. Voluntary conversion of a convertible bond will occur only if the current dividend yield on the stock exceeds the current yield on the bond.

[35]By non-stochastic it is meant that the dividend policy $\delta(V, \tau)$ is a deterministic function of asset value and maturity, i.e., it adds no additional uncertainties beyond those already formulated within the problem.

Proof. Let i and k denote the current yields and B and E the values of the bond and stock. The condition given in the lemma can then be written $iB < \gamma(iB+kE)$. When a voluntary conversion occurs, the bond price equals its conversion value, $B = \gamma V$, hence $E = (1-\gamma)V$. Substituting we find $i < k$.

The following example illustrates the qualities of a voluntary conversion. The example is a convertible preferred with dividend rate C. The dividend policy for the common is chosen to make the total payout proportional to firm value with proportionality constant b (i.e., $\delta = bV-C$).[36] Since both the dividend payments are independent of time and the preferred is perpetual, the optimal conversion strategy will be a constant $V^*(\tau) = V^*$. The conversion strategy V^* is to be determined as a part of the solution. The strategy chosen will be the one that maximizes the convertible's value. From (7) the convertible bond is the solution to

$$\tfrac{1}{2}\sigma^2 V^2 G'' + (r-b)VG' - rG + C = 0, \tag{34}$$

subject to the conditions $G(0) = C/r$,[37] $G(V^*) = \gamma V^*$, and V^* maximizes the function $G(\cdot)$. The solution to (34) is

$$G(V) = C/r + [\gamma V^* - C/r](V/V^*)^\eta, \tag{35}$$

where

$$V^* = C\eta/(\eta-1)r\gamma,$$

$$\eta \equiv [-r+b+\tfrac{1}{2}\sigma^2 + \sqrt{((r-b-\tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2)}]\,\sigma^{-2}.$$

An ordinary preferred stock with the same dividend payments would have a value of C/r since the financing policy would guarantee its dividends in perpetuity. The premium due to the conversion option is the last term in (35), $[\gamma V^* - C/r](V/V^*)^\eta$. If the company had instead issued ordinary preferred stock and warrants with a claim on the same fraction γ and a total exercise payment of E, the warrants would be worth

$$(\gamma V' - X)(V/V')^\eta, \tag{36}$$

where

$$X \equiv E + \gamma(C/r - E),$$

$$V' = \eta X/(\eta-1)\gamma.$$

V' is the value of the company when the warrants are exercised under the optimal strategy.

[36] Under this dividend policy, which necessitates new financing, it is in the best interests of the equity to default on the bonds at some positive firm value rather than to have the owners continually pouring money into the firm which only is paid out to the bondholders. However, this is not the issue under consideration here.

[37] The new financing will pay the coupons in perpetuity with no chance of default. See also the previous footnote.

If the exercise price on the warrant, E, is less than the 'investment value' of the convertible preferred, C/r, then $X < C/r$ and the warrants would be exercised before the preferreds were converted. Furthermore, the preferred–warrant combination is more valuable than the convertible preferred. If the exercise price on the warrant is larger than the 'investment value' of the convertible preferred, then the latter would be converted first and be more valuable. The optimal early exercise negates the equality between the convertible and the straight instrument–warrant combination which was demonstrated in section 6.

We shall now examine the voluntary conversion of callable convertibles. Two points may be noted to start:

Theorem VII. Whenever it is optimal to voluntarily convert a non-callable convertible bond, it will also be optimal to convert a callable, convertible bond which is otherwise identical.

Proof. By assumption it is optimal to convert the non-callable bond; thus its value if it were not converted must be less than the fraction γ of the firm. From (5) the callable bond is no more valuable than the non-callable bond; hence it, too, must be less valuable than γV (i.e., $H(\cdot) \leqq G(\cdot) < \gamma V$). Consequently it too should be converted.

Theorem VIII. The possibility of a voluntary conversion does not affect the optimal call policy.[38] *Thus, Theorem III is still valid and $\bar{V}(\tau) \leqq K(\tau)/\gamma$. Furthermore, if the conditions for Theorem IV hold, then $\bar{V}(\tau) = K(\tau)/\gamma$.*

Proof. Since the conditions that might cause a voluntary conversion can not increase the value of the convertible, the reasoning in the proofs of both Theorems is still valid.

The optimal call policy can be affected by voluntary conversion only if the convertible owners are behaving irrationally. If they should be converting, because of large common dividends, for example, but are not, the management should not call the bonds and 'force' the bondholders into the correct action. This would only hurt the common stock owners who are benefitting from the convertible owners' mistake. The independence of the call policy from the conversion strategy guaranteed by Theorem VIII allows us to price callable convertibles by the same dynamic programming method used previously on non-callable convertibles.

[38]Of course the bond may no longer be 'alive' when the firm value reaches this point due to an earlier voluntary conversion.

8. Conclusions

We have developed a method for the determination of the optimal conversion and call policies for convertible securities. Analytic solutions for the prices have been determined in specific cases under somewhat restrictive assumptions. However, the general option pricing method is quite robust. Merton (1973) has analyzed the option model with a stochastic interest rate. Ingersoll (1976a) has included a differential tax structure. Cox and Ross (1976) have priced options for alternative stochastic processes, and Merton (1976) has attacked the problem of a discontinuous return structure. All of these models are directly applicable to non-callable convertible bonds by a straightforward application of Theorem V. Callable convertibles can be priced, at least numerically, with only slight modifications to the model since the proper call strategy has been already determined through Theorem IV.

Under the conditions of Theorem IV, it has been established that the optimal call strategy for the management to follow is to call a convertible bond at the point when its conversion value is equal to the call price. This is a strong result since the implementation of the implied strategy requires the knowledge of only the contractual items call price and conversion price and the market price per share of the common stock. Even if the term structure is stochastic or Theorem IV is otherwise inapplicable, Theorem III sets an upper limit on the optimal call point requiring only the same, observable inputs.

Unfortunately even casual empiricism indicates that the implied strategy of Theorem III is completely at odds with the observed practice of firms. Almost without exception calls are delayed until the conversion value of the bond greatly exceeds the call price. The assumptions underlying Theorem III, no transactions costs, no corporate taxes, and no required notice of call, all market imperfections not considered here, are possible explanations for this discrepancy between theory and practice. These issues have been examined by Ingersoll (forthcoming), and they cannot account for the observed call policies. Further study may reveal whether investors fully discount the effects of improper call policies in the pricing of convertibles or if the owners of convertibles receive excess returns as a result of the suboptimal calls.

References

Baumol, W.J., B.G. Malkiel and R.E. Quandt, 1966, The valuation of convertible securities, Quarterly Journal of Economics 80, 48–59.

Black, F. and J. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, Journal of Finance 31, 351–367.

Black, F. and M. Scholes, 1972, The valuation of option contracts and a test of market efficiency, Journal of Finance 27, 399–417.

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 83, 637–654.

Brennan, M.J. and E.S. Schwartz, 1975, Convertible bonds: Valuation and optimal strategies for call and conversion, Faculty of Commerce and Business Administration Working Paper no. 336 (University of British Columbia, Vancouver).

Brigham, E.F., 1966, An analysis of convertible debentures: Theory and some empirical evidence, Journal of Finance 21, 35–54.

Broman, K.I., 1963, The use of convertible subordinated debentures by industrial firms 1949–1959, Quarterly Review of Economics and Business 3, 65–75.

Cox, D.R. and H.D. Miller, 1965, The theory of stochastic processes (Wiley, New York).

Cox, J.C. and S.A. Ross, 1976, The valuation of options for alternate stochastic processes, Journal of Financial Economics 3, 145–166.

Davis, P.J., 1966, Gamma function and related functions, in: Handbook of mathematical functions, National Bureau of Standards Applied Math Series 55 (US Government Printing Office, Washington, DC) ch. 6.

Graham, B., 1973, The intelligent investor (Harper and Row, New York).

Ingersoll, J., 1976a, A theoretical and empirical investigation of the dual purpose funds: An application of contingent-claims analysis, Journal of Financial Economics 3, 83–123.

Ingersoll, J., 1976b, Using the Black–Scholes option model in investment decision making: Designing a convertible preferred issue, Proceedings Seminar on the Analysis of Security Prices, May 1976 (Center for Research in Security Prices, University of Chicago, Chicago, IL).

Ingersoll, J., forthcoming, An examination of corporate call policies on convertible securities, Journal of Finance.

Jennings, E.H., 1974, An estimate of convertible bond premiums, Journal of Financial and Quantitative Analysis 9, 33–56.

Kahn, I. and A.R. Kahn, 1966, Financial analysts digest, Financial Analysts Journal 22, July–Aug., 119–124.

McKean, H.P., Jr., 1969, Stochastic integrals (Academic Press, New York).

Merton, R.C., 1973, Theory of rational option pricing, Bell Journal of Economics and Management Science 4, 141–183.

Merton, R.C., 1974, On the pricing of corporate debt: The risk structure of interest rates, Journal of Finance 29, 449–470.

Merton, R.C., 1976, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics 3, 125–144.

Merton, R.C. and P.A. Samuelson, 1974, Fallacy of the log-normal approximation to optimal decision-making over many periods, Journal of Financial Economics 1, 67–94.

Noddings, T.C., 1973, The Dow Jones-Irwin guide to convertible securities (Richard D. Irwin, Homewood, IL).

Poensgen, O.H., 1965, The valuation of convertible bonds, Part 1, Industrial Management Review 7, Fall, 77–92.

Poensgen, O.H., 1966, The valuation of convertible bonds, Part 2, Industrial Management Review 7, Spring, 83–98.

Pye, G., 1966, The value of the call option on a bond, Journal of Political Economy 74, 200–205.

Rothschild, M. and J.E. Stiglitz, 1970, Increasing risk: I. A definition, Journal of Economic Theory 2, 225–243.

Samuelson, P.A., 1965, Rational theory of warrant pricing, Industrial Management Review 6, Spring, 13–31.

Slater, L.J., 1966, Confluent hypergeometric function, in: Handbook of Mathematical Functions, National Bureau of Standards Applied Math Series 55 (US Government Printing Office, Washington, DC) ch. 13.

Stiglitz, J.E., 1972, Some aspects of the pure theory of corporate finance: Bankruptcies and take-overs, Bell Journal of Economics and Management Science 3, 458–482.

Thorpe, E.O., 1973, Extensions of the Black–Scholes option model, 39th Session of the International Statistical Institute (Vienna, Austria).

Walter, J.E. and A.V. Que, 1973, The valuation of convertible bonds, Journal of Finance 28, 713–732.

Weil, R., J.E. Segall and D. Green, Jr., 1968, Premiums on convertible bonds, Journal of Finance 23, 445–463.