

Probability Notes

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Contents

| | | |
|----------|----------------------------------------------------------|----------|
| 1 | Probability Theorems | 1 |
| 1.1 | Set Theorems | 1 |
| 1.2 | Basic Probability Rules | 1 |
| 1.2.1 | Total Probability Theorem | 1 |
| 1.3 | Independence | 1 |
| 1.4 | Joint Probability Distributions | 1 |
| 1.5 | Expected Value | 2 |
| 1.5.1 | Total Expectation Theorem | 2 |
| 1.6 | Variance | 3 |
| 1.7 | Cumulative Probability Distribution | 3 |
| 2 | Binomial Random Variable | 3 |
| 2.1 | Mean and Variance | 3 |
| 3 | Continuous Uniform Random Variable | 4 |
| 3.1 | Mean and Variance | 4 |
| 4 | Gaussian Distribution | 4 |
| 5 | Covariance and Correlation | 4 |
| 6 | Iterated Expectation and Variance | 5 |
| 7 | Random number of Random Variables | 5 |
| 8 | Bernoulli Process | 5 |
| 8.1 | Mean and Variance | 6 |
| 8.2 | Interarrival Times (Geometric Random Variable) | 6 |
| 8.3 | Sum of Interarrival times | 6 |
| 9 | Poisson Process | 6 |
| 9.1 | Derivation from Bernoulli Process | 7 |
| 9.2 | Time till kth arrival | 7 |
| 9.3 | Time of 1 st arrival | 8 |
| 9.4 | Merging of Poisson Processes | 8 |
| 9.5 | Splitting of Poisson Process | 8 |
| 9.6 | Random Incidence for Poisson | 9 |

| | |
|-------------------------------------------------------|-----------|
| 10 Markov Process | 9 |
| 10.1 Recurring and Transient States | 9 |
| 10.2 Steady State Probabilities | 10 |
| 10.3 Birth Death Process | 10 |
| 10.4 Absorption Probabilities | 11 |
| 10.5 Weak Law of Large Numbers | 11 |
| 10.6 Markov Inequality/Chebychev Inequality | 12 |
| 11 Central Limit Theorem | 12 |
| 12 Bayesian Inference | 13 |
| 12.1 Maximum a Posteriori (MAP) | 13 |
| 12.2 Least Mean Square Estimate | 14 |
| 13 Exercises | 14 |
| 13.1 Problems | 14 |
| 13.2 Solutions | 17 |

1 Probability Theorems

1.1 Set Theorems

For any three sets, the following hold true

$$\begin{aligned}A &= (A \cap B) \cup (A \cap B^c) \text{ where } B \text{ and } B^c \text{ are disjoint} \\A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\end{aligned}$$

1.2 Basic Probability Rules

$$\begin{aligned}\text{If } A \cap B &= \phi, \text{ then } P(A \cup B) = P(A) + P(B) \\P(A|B)P(B) &= P(B|A)P(A) = P(A \cap B) && \text{Bayes' Theorem} \\P(A) &= P(A \cap B) + P(A \cap B^c) = P(A|B)P(B) + P(A|B^c)P(B^c) \\P(A \cap B \cap C) &= P(A)P(B|A)P(C|B, A) && \text{Chain Rule}\end{aligned}$$

1.2.1 Total Probability Theorem

Let A_1, A_2, \dots, A_n be n disjoint events that completely cover the event space, and B be another event, then

$$\begin{aligned}P(B) &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_n) \\ \text{or, } P(B) &= \sum_{i=1}^n P(B|A_i)P(A_i)\end{aligned}$$

1.3 Independence

Two events A and B are independent iff

$$P(A \cap B) = P(A)P(B)$$

Note that *independence* is not the same as *disjoint*

$$A \cap B = \phi \Rightarrow P(A \cap B) = 0 \text{ but } P(A) \neq P(B) \neq 0$$

Multiple events A_1, A_2, \dots, A_n are independent iff

$$P(A_i \cap A_j \cap \dots \cap A_k) = P(A_i)P(A_j) \dots P(A_k) \quad \forall i, j, \dots, k \mid i, j, \dots, k \in 1, 2, \dots, n$$

Conditional Independence is similar to the above equation. For an event C ,

$$P(A_i \cap A_j \cap \dots \cap A_k | C) = P(A_i | C)P(A_j | C) \dots P(A_k | C) \quad \forall i, j, \dots, k \mid i, j, \dots, k \in 1, 2, \dots, n$$

1.4 Joint Probability Distributions

Joint Probability Distributions are defined for two or more than two variables. In this section, we only consider two variables. The formal definition is

$$P_{XY}(x, y) = P(X = x \text{ and } Y = y)$$

Based on this definition, the following theorems follow

$$\sum_x \sum_y P_{XY}(x, y) = 1$$

$$P_X(x) = \sum_y P_{XY}(x, y) \quad \text{Marginal Probability}$$

$$P_{X|Y}(x|y) = P_{X|Y}(X = x|Y = y) = \frac{P_{XY}(x, y)}{P_Y y}$$

$$\sum_x P_{X|Y}(x|y) = 1 \quad \text{Since Y is fixed and we sum over all X's}$$

$$P_{XYZ}(x, y, z) = P_X(x)P_{Y|X}(y|x)P_{Z|X,Y}(z|x, y) \quad \text{Chain Rule}$$

1.5 Expected Value

Before going to expected value, let's define a Random Variable

Random Variable X is a linear map : $\mathbb{R} \rightarrow \mathbb{R}$. The value taken by the variable is denoted by x . X will have an associated probability distribution, i.e., $P_X(X = x)$. Using these quantities, we have

$$E[X] = \sum_x x P_X(X = x) \quad \text{Expected Value}$$

Based on this definition, the following theorems for expected value follow

$$E[\alpha] = \alpha E[X] \quad = \alpha E[X]$$

$$E[\alpha X + \beta] = \alpha E[X] + \beta$$

$$E[g(X)] = \sum_x g(x) P_X(X = x)$$

$$E[X^2] = \sum_x x^2 P_X(X = x) \quad \text{Also called Second Moment}$$

$$E[X|A] = \sum_x x P_{X|A}(X|A)$$

$$E[g(X)|A] = \sum_x g(x) P_{X|A}(X|A)$$

$$E[X + Y + Z] = E[X] + E[Y] + E[Z] \quad \text{Linearity of Expectation}$$

$$E[XY] = \sum_X \sum_Y xy P_{XY}(x, y)$$

$$E[g(X, Y)] = \sum_X \sum_Y g(xy) P_{XY}(x, y)$$

$$E[XY] = E[X]E[Y] \quad \text{if X and Y are independent}$$

where $\alpha, \beta \in \mathbb{R}$, $g(X) : \mathbb{R} \rightarrow \mathbb{R}$, and A is an event, X, Y, Z are Random Variables

1.5.1 Total Expectation Theorem

The *Total Expectation Theorem* is the natural extension of the *Total Probability Theorem*. Let A_1, A_2, \dots, A_n be n disjoint events that completely cover the event space, and X be random variable, then

$$E[X] = E[X|A_1]P(A_1) + E[X|A_2]P(A_2) + \dots + E[X|A_n]P(A_n)$$

$$\text{or, } E[X] = \sum_{i=1}^n E[X|A_i]P(A_i)$$

1.6 Variance

The formal definition of variance is

$$Var(X) = E[(X - \bar{X})^2] = E[X^2] - E[X]^2$$

Using this definition, the following theorems follow

$$E[X^2] = E[X]^2 + Var(X)$$

$$Var(\alpha) = 0$$

$$Var(\alpha X + \beta) = \alpha^2 Var(X)$$

$$Var(X + Y) = Var(X) + Var(Y) \text{ if } X \text{ and } Y \text{ are independent random variables}$$

1.7 Cumulative Probability Distribution

Cumulative probability distribution is defined for both discrete and continuous variables

$$F_x(X) = P(X \leq x) = \begin{cases} \int_{-\infty}^x p_X(t) dt & X \text{ is a discrete random variable} \\ \sum_{k \leq x} P_X(k) & X \text{ is a continuous random variable} \end{cases}$$

2 Binomial Random Variable

Binomial Random Variable X is defined as the number of successes in an experiment with n independent trials, where each trial can only have two outcomes, *success* or *failure*.

Let X_i denote the Random Variable corresponding to the individual trials, with probability of success p . Then we have the following

$$X_i = \begin{cases} 1 & \text{if success in trial } i \\ 0 & \text{otherwise} \end{cases} \quad \text{indicator variable}$$

$$X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^n X_i$$

$$P(X = k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

2.1 Mean and Variance

First let's calculate the mean and variance for a single trial X_i

$$E[X_i] = 1 * p + 0 * (1 - p) = p$$

$$Var(X_i) = (1 - p)^2 p + (0 - p)^2 (1 - p) = p(1 - p)$$

We know that all X_i 's are independent. Hence, the mean and variance for X become

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = np$$

$$Var(X) = Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i) = np(1 - p)$$

3 Continuous Uniform Random Variable

A uniform random variable is defined as follows

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

3.1 Mean and Variance

$$\begin{aligned} E[X] &= \int_a^b x \frac{1}{b-a} dx = \left[\frac{x^2}{2(b-a)} \right]_a^b \\ &= \frac{a+b}{2} \\ \text{Var}(X) &= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

4 Gaussian Distribution

The gaussian distribution (or normal distribution) is defined between $-\infty$ and ∞ . It is parametrized by mean μ and variance σ , $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

As already described,

$$\begin{aligned} E[X] &= \mu \\ \text{Var}(X) &= \sigma^2 \end{aligned}$$

A *Standard Normal* is defined as a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Any normal distribution can be converted to a standard normal as $X = \frac{X-\mu}{\sigma}$. If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

5 Covariance and Correlation

For any two random variables X and Y ,

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - \bar{X})(Y - \bar{Y})] = E[XY] - E[X]E[Y] \\ \text{Cov}(X, X) &= \text{Var}(X) \\ \text{Corr}(X, Y) &= E\left[\left(\frac{X - \bar{X}}{\sigma_X}\right)\left(\frac{Y - \bar{Y}}{\sigma_Y}\right)\right] \\ &= \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \end{aligned}$$

Key points to note

- *Independence* $\Rightarrow \text{Cov}(X, Y) = \text{Corr}(X, Y) = 0$, but the converse is **not** true
- Correlation is dimensionless and $-1 \leq \text{Corr}(X, Y) \leq 1$ with value close to 0 implying minimal relation and values close to $-1, 1$ implying perfect relation

6 Iterated Expectation and Variance

The law of iterated expectation tells the following about expectation and variance

$$\begin{aligned} E[E[X|Y]] &= E[X] \\ \text{Var}(X) &= E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) \end{aligned}$$

Proof for Iterated Expectation

$$\begin{aligned} P(X) &= \sum_y P(X|Y)P(Y) \\ \Rightarrow E[X] &= \sum_x xP(X) = \sum_x \sum_y xyP(X|Y)P(Y) \\ &= \sum_y P(Y) \sum_x xP(X|Y) = \sum_y P(Y)E[X|Y] \\ \text{or, } E[X] &= E[E[X|Y]] \quad E[X|Y] \text{ is a function of } X \text{ and not } Y \end{aligned}$$

Proof for Variance

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ \text{Var}(X|Y) &= E[(X - \bar{X})^2|Y] = E[X^2|Y] - E[X|Y]^2 & 1 \\ \text{Var}[E(X|Y)] &= E[E(X|Y)^2] - E[E(X|Y)]^2 \\ &= E[E[(X|Y)^2]] - E[X]^2 & 2 \\ E[\text{Var}(X|Y)] &= E[E[X^2|Y]] - E[E[X|Y]^2] & \text{from 1} \\ &= E[X^2] - E[E[X|Y]^2] & 3 \\ E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) &= E[X^2] - E[X]^2 & \text{adding 2 and 3} \\ &= \text{Var}(X) \end{aligned}$$

7 Random number of Random Variables

Let X_i be independent identically distributed Random Variables and let $Y = \sum_{i=1}^N X_i$ be the sum of N such random variables where N itself is a random variable. Then,

$$\begin{aligned} Y &= X_1 + X_2 + \dots + X_N \\ E[Y|N=n] &= \sum_{i=1}^n E[X_i] \\ &= NE[X] \\ E[Y] &= E[E[Y|N]] = E[NE[X]] \\ &= E[N]E[X] & \text{since } E[X] \text{ will be a number} \\ \text{Var}(Y) &= E[\text{Var}(Y|N)] + \text{Var}(E[Y|N]) \\ &= E[N\text{Var}(X)] + \text{Var}(NE[X]) \\ &= E[N]\text{Var}(X) + E[X]^2\text{Var}(N) \end{aligned}$$

8 Bernoulli Process

Bernoulli process falls under the family of random processes, which are random variables continuously evolving over time. Bernoulli process can be described as a sequence of independent

Bernoulli trials, where each trial has only two outcomes : success with $P(\text{success}) = p$ and failure.

$$P_{X_t}(x_t) = \begin{cases} p & \text{if } X_t = 1 \\ 1 - p & \text{if } X_t = 0 \end{cases}$$

$$E[X_t] = p$$

$$Var(X_t) = p(1 - p)$$

8.1 Mean and Variance

Number of successes S in n time slots

$$P(S = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[S] = np$$

$$Var(S) = np(1 - p)$$

8.2 Interarrival Times (Geometric Random Variable)

Let T_1 denote the number of trials till the first success

$$P(T_1 = t) = (1 - p)^{t-1} p \quad t \in 1, 2, \dots$$

$$E[T_1] = \frac{1}{p}$$

$$Var(T_1) = \frac{1 - p}{p^2}$$

This process is memoryless as all future coin flips are independent of whatever has happened till now. Also, the distribution is a **Geometric Random Variable**.

8.3 Sum of Interarrival times

We are interested in the total time till k arrivals. Let this random variable be Y_k

$$Y_k = T_1 + T_2 + \dots + T_k \quad \text{where } T_i \text{'s are i.i.d geometric with parameter } p$$

$$P(Y_k = t) = P(k - 1 \text{ arrivals between } t = 1 \text{ to } t = t \text{ and last arrival at time } t)$$

$$= \binom{t-1}{k-1} p^k (1 - p)^{t-k} \quad \forall t \geq k$$

$$E[Y_k] = \sum_{i=1}^k k E[T_i]$$

$$= \frac{k}{p}$$

$$Var(Y_k) = \sum_{i=1}^k Var(T_i)$$

$$= \frac{k(1 - p)}{p^2}$$

9 Poisson Process

Poisson process also falls in the realm of random processes but is different from Bernoulli process as it is a continuous time process. This process is very commonly used to model arrival times

and number of arrivals in a given time interval.

$$P(k, \tau) = \text{Probability of } k \text{ arrivals in interval of duration } \tau$$

$$\sum_k P(k, \tau) = 1 \quad \text{for a given } \tau$$

Assumptions

- The Probability is dependent only on τ and not the *location* of the interval
- Number of arrivals in disjoint time intervals are *independent*

9.1 Derivation from Bernoulli Process

For a very small interval δ ,

$$P(k, \delta) = \begin{cases} 1 - \lambda\delta & k = 0 \\ \lambda\delta & k = 1 + O(\delta^2) \\ 0 & k > 2 \end{cases}$$

$$\lambda = \lim_{\delta \rightarrow 0} \frac{P(1, \delta)}{\delta} \quad \text{arrival rate per unit time}$$

$$E[k] = (\lambda\delta) * 1 + (1 - \lambda\delta) * 0$$

$$= \lambda\delta$$

$$\tau = n\delta$$

The last equation clearly implies that we can approximate the whole process as a bernoulli process where we have n miniscule time intervals with at most one arrival per interval.

$$P(k \text{ arrivals}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda\delta}{n}\right)^k \left(1 - \frac{\lambda\delta}{n}\right)^{n-k}$$

$$\lambda\tau = np \quad \text{or, arrival rate * time} = E[\text{arrivals}]$$

$$Poisson = \lim_{\delta \rightarrow 0, n \rightarrow \infty} Bernoulli$$

$$\text{or, } P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} \quad k = 0, 1, \dots, \text{ for a given } \tau$$

$$\text{where, } \sum_k P(k, \tau) = 1 \quad \text{for a given } \tau$$

Let N_t denote the no of arrivals till time t , then

$$E[N_t] = \lambda t$$

$$Var(N_t) = \lambda t$$

9.2 Time till kth arrival

Suppose the k^{th} arrival happens at a time t . Then we are saying that there have been $k - 1$ arrivals till time t and the k^{th} arrival happens at time t (precisely in an interval of $[t, t + \delta]$).

Let Y_k be the required time,

$$\begin{aligned}
f_{Y_k}(t)\delta &= P(t \leq Y_k \leq t + \delta) \\
&= P(k-1 \text{ arrivals in } [0, t])(\lambda\delta) \\
&= \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} (\lambda\delta) \\
f_{Y_k}(t) &= \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}
\end{aligned}$$

Erlang Distribution

9.3 Time of 1st arrival

Using the Erlang Distribution described above, we have

$$f_{Y_1}(t) = \lambda e^{-\lambda t}$$

$Y_k = T_1 + T_2 + \dots + T_k$ where all T_i are independent and exponential distributions.

9.4 Merging of Poisson Processes

Merging of two Poisson processes is also a Poisson process. Consider two flasbulbs of Red and Green colours, flashing as Poisson processes with rates λ_1 and λ_2 . Then the process denoting the combined flashing of the two bulbs is also Poisson.

Consider a very small interval of time δ . In this small interval, any of the individual bulbs can have at most one flashes (since we ignore higher order terms). Thus, the following four possibilities arise

| 0 | <i>Red</i> | \overline{Red} |
|--------------------|-------------------------------------------|------------------------------------------------|
| <i>Green</i> | $\lambda_1 \delta \lambda_2 \delta$ | $(1 - \lambda_1 \delta) \lambda_2 \delta$ |
| \overline{Green} | $\lambda_1 \delta (1 - \lambda_2 \delta)$ | $(1 - \lambda_1 \delta)(1 - \lambda_2 \delta)$ |

Table 1: Base Probabilities for flashes

| 0 | <i>Red</i> | \overline{Red} |
|--------------------|--------------------|----------------------------------------|
| <i>Green</i> | 0 | $\lambda_2 \delta$ |
| \overline{Green} | $\lambda_1 \delta$ | $(1 - (\lambda_1 + \lambda_2) \delta)$ |

Table 2: Probabilities after ignoring δ^2 terms

Thus, the combined process is Poisson with parameter $\lambda_1 + \lambda_2$

$$P(\text{arrival happened from first process}) = \frac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

9.5 Splitting of Poisson Process

Suppose we have a Poisson process with parameter λ which we split into two processes up and down, with probabilities p and $1 - p$. The two resulting processes are also Poisson with different parameters.

Consider a small time slot of length δ . Then,

$$\begin{aligned}
P(\text{arrival in this time slot}) &= \lambda \delta \\
P(\text{arrival in up slot}) &= \lambda \delta p \\
P(\text{arrival in down slot}) &= \lambda \delta (1 - p)
\end{aligned}$$

Thus, up and down are themselves Poisson with parameters λp and $\lambda(1 - p)$ respectively.

9.6 Random Incidence for Poisson

Suppose we have a Poisson process with parameter λ running forever. We show up at a random time instant. What is the length of the chosen interarrival time (the total of the time from the last arrival to the next arrival).

Let T'_1 denote the time that has elapsed since the last arrival and T_1 be the time till the next arrival. Note that the reverse process is also Poisson with the same parameter. Thus,

$$E[\text{interarrival time}] = E[T'_1 + T_1] = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$$

This may seem paradoxical since the time difference between any two arrivals in a Poisson process is same and it's expected length is $\frac{1}{\lambda}$, whereas we got an interval twice this length. The paradox is resolved by considering the fact that when we choose a random point in time, it is more likely to fall in an interval of larger size than the smaller ones (since probability will be proportional to the length of the interval).

Consider a separate example where we want to compare two values $E[\text{size of a family}]$ and $E[\text{size of a family of a given person}]$.

The two value will be different due to the underlying nature of the way experiment is conducted. For the first, we randomly choose families and average their sizes. Here, family of any size is equally likely to be picked. In the second case, we first pick a person from the population, get their family size, and then average the sizes of the families. Note that, this experiment is biased since the we are more likely to select people from larger families (or equivalently, it is more likely that we pick a person from a large family since the probability of picking is proportional to the family size). Hence, the second value will likely be larger and the two quantities are not equal.

10 Markov Process

Markov Process is a discrete time process that is not memoryless. Here the random variable takes several possible states, and the probability distribution is defined in such a way that $P(\text{transition from state 1 to state 2})$ is dependent on state 1.

Let X_n be the random variable denoting the state after n transitions and X_0 will represent the starting state (which can be given or random). Markov assumption states that *Given the current state, past does not matter*. Armed with these,

$$\begin{aligned} p_{ij} &= P(\text{next state } j \mid \text{current state } i) \\ p_{ij} &= P(X_{n+1} = j \mid X_n = i) = P(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0) \\ r_{ij}(n) &= P(X_n = j \mid X_0 = i) && \text{or, in state } j \text{ after } n \text{ steps} \\ r_{ij}(n) &= \sum_{k=1}^m r_{ik}(n-1)p_{kj} \end{aligned}$$

10.1 Recurring and Transient States

A state i is called *recurrent* if, starting from i , and travelling anywhere, it is always possible to return to i . If a state is not recurrent, it is *transient*. States in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.

10.2 Steady State Probabilities

Do $r_{ij}(n)$ converge to some π_j (independent of i) ?

Yes if,

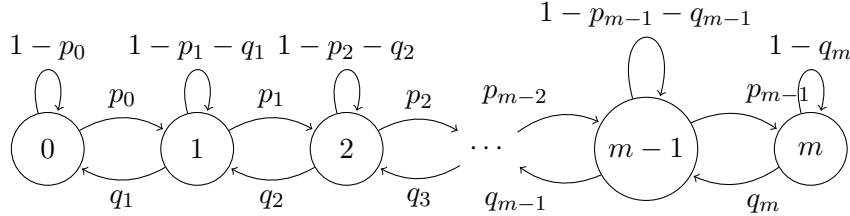
- recurrent states are all in a single class
- single recurrent class is not periodic (otherwise oscillations are possible)

Assuming yes,

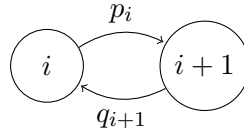
$$\begin{aligned}
 r_{ij}(n) &= \sum_k r_{ik}(n-1)p_{kj} \\
 \lim_{n \rightarrow \infty} r_{ij}(n) &= \sum_k r_{ik}(n-1)p_{kj} \\
 \pi_{ij} &= \sum_k \pi_{ik}p_{kj} && \text{balance equations} \\
 \sum_i \pi_i &= 1 \\
 \text{frequency of transitions } k \rightarrow j &= \pi_k p_{kj} && \text{in one step} \\
 \text{frequency of transitions into } j &= \sum_k \pi_k p_{kj} && \text{influx from all connected states}
 \end{aligned}$$

10.3 Birth Death Process

Consider the checkout counter example. The states are represented by the number of people currently being processed, and we always move n to $[n-1, n, n+1]$, i.e., either the people in the queue decrease by one, remain same or increase by one. Let the probability for moving up be p and moving down be q .



Let's estimate the steady state probabilities. Consider the following diagram splitting the chain into two parts through the two adjacent states



In this case, to maintain steady state, long term frequency of left-right transition should be same as right left transition, i.e., $\pi_i p_i = \pi_{i+1} q_i$

In the special case of $p_i = p$ and $q_i = q \forall i$,

$$\begin{aligned}\rho &= \frac{p}{q} && \text{load factor} \\ \pi_{i+1} &= \pi_i \frac{p}{q} = \pi_i \rho \\ \pi_i &= \pi_0 \rho^i && i = 0, \dots, m \\ \text{Using } \sum_{i=0}^m \pi_0 \rho^i &= 1, \\ \pi_0 &= \frac{1}{\sum_{i=0}^m \rho^i} \\ \text{if } p < q \text{ and } m &\rightarrow \infty, \\ \pi_0 &= 1 - \rho \\ \pi_i &= (1 - \rho) \rho^i \\ E[X_n] &= \frac{\rho}{1 - \rho} && \text{Exponential Distribution}\end{aligned}$$

When $\rho = 1$ or $p = q$, then all states are equally likely - symmetric random walk.

10.4 Absorption Probabilities

let a_i denote the probability of absorption and μ_i denote the expected no of steps until absorption starting from state i . Then,

$$\begin{aligned}a_i &= \sum_j a_j p_{ij} && \text{outflux to the possible states} \\ \mu_i &= 1 + \sum_j \mu_j p_{ij}\end{aligned}$$

For multiple absorption states, we can possibly consider them together as a group and calculate the relevant quantities.

For a given state s ,

$$\begin{aligned}E[\text{steps to first time reach } s \text{ from } i] &= t_i \\ t_i &= E[\min\{n \geq 0 \text{ such that } X_n = s\}] \\ t_s &= 0 \\ t_i &= 1 + \sum_j t_j p_{ij} && \text{outflux to all possible states}\end{aligned}$$

Mean recurrence time (mean time to reach back a state) for s

$$\begin{aligned}t_s^* &= E[\min\{n \geq 1 \text{ such that } X_n = s\} | X_0 = s] \\ t_s^* &= 1 + \sum_j t_j p_{sj}\end{aligned}$$

10.5 Weak Law of Large Numbers

Suppose we want to know the mean height of penguins in the world. The absolutely correct answer can be obtained by taking the average of the entire population. But this is not practical, and often we will have to resort to estimating the quantity through a sample. Let there be

n penguins in the sample and X_1, X_2, \dots, X_n be the random variables denoting their heights. Then,

$$M_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$\lim_{n \rightarrow \infty} E[M_n] = E[X] = \text{The true mean}$$

10.6 Markov Inequality/Chebychev Inequality

For nonnegative random variable X ,

$$E[X] = \sum_x xp_X(x) \geq \sum_{x \geq a} xp_X(x) \quad \text{discrete case}$$

$$= \int_x xp_X(x) \geq \int_{x \geq a} xp_X(x) \quad \text{continuous case}$$

Applying the above set of inequalities to the variable $X - \mu$

$$E[(X - \mu)^2] \geq a^2 P((X - \mu)^2 \geq a^2)$$

$$\text{or, } Var(X) \geq a^2 P(|X - \mu| \geq a)$$

For continuous case,

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx \\ &\geq \int_{-\infty}^{\mu - c} (x - \mu)^2 f_X(x) dx + \int_{\mu + c}^{\infty} (x - \mu)^2 f_X(x) dx \\ &\geq c^2 P(|X - \mu| \geq c) \end{aligned}$$

Hence,

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

or,

$$\boxed{P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}} \quad \text{where } c = k\sigma$$

Going back to the problem of estimating the mean,

$$M_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$E[M_n] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \mu \quad \text{expectation of expectation}$$

$$Var(M_n) = \sum_{i=1}^n Var\left(\frac{X_i}{n}\right) = \frac{\sigma^2}{n} \quad \text{since } X_i \text{ are independent}$$

$$\boxed{P(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}}$$

or, as $n \rightarrow \infty$, $M_n - \mu \rightarrow 0$, ϵ is the error bound/confidence.

11 Central Limit Theorem

Chebychev's inequality gives a loose bound. We can do better with CLT. Let X be a random variable with mean μ and variance σ^2 , and let X_i be independent identically distributed random

variables with the same distribution as X . Then,

$$\begin{aligned}
S_n &= X_1 + X_2 + \dots + X_n \\
Z_n &= \frac{S_n - E[S_n]}{\sigma_n} \text{ random variable with mean 0 and variance 1} \\
&= \frac{S_n - nE[X]}{\sqrt{n}\sigma} \\
\text{or, } S_n &= \sqrt{n}\sigma Z_n + nE[X] \\
\text{In } \lim_{n \rightarrow \infty} Z_n &\rightarrow Z \text{ (standard normal)} \\
\text{or, } \boxed{Z = \frac{S_n - nE[X]}{\sqrt{n}\sigma}} &\text{ only for CDF (no comment on PDF/PMF)} \\
\text{Thus, } \boxed{P(Z > c) = P\left(\frac{S_n - nE[X]}{\sqrt{n}\sigma} > c\right)}
\end{aligned}$$

By defining the confidence on how close we desire S_n to the actual mean, we can calculate the required value of the n using standard normal CDF tables. However, we need to have an estimate of variance of the distribution in order to do the estimate of n .

12 Bayesian Inference

We have a signal S that goes through a "model" a through which we observe X (with sum added noise N). The aim of Bayesian Inference is to try to infer S given the observed X .

Hypothesis testing is done on an unknown that takes some possible values, and the aim is to arrive at a value that gives a small probability of incorrect decision (e.g. - Radar)

Estimation is aimed at finding the value of a quantity with a small estimation error (e.g. poll estimation)

Bayes Rule

$$\begin{aligned}
p_{\Theta|X}(\theta|x) &= \frac{p_{\Theta}(\theta)p_{X|\Theta}(x|\theta)}{p_X(x)} && \theta \text{ and } X \text{ are both discrete} \\
\text{or, } Posterior &= \frac{Prior * Model}{Data} \\
p_{\Theta|X}(\theta|x) &= \frac{p_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{f_X(x)} && \theta \text{ is discrete and } X \text{ is continuous}
\end{aligned}$$

Note that Bayesian inference will give us a distribution over the possible values, but it is often desirable to get an estimate.

12.1 Maximum a Posteriori (MAP)

MAP is a point estimate of the unknown quantity and is defined as follows

$$p_{\Theta|X}(\theta|x) = \max_{\theta} p_{\Theta|X}(\theta|x) \quad \theta \text{ with maximum posterior probability}$$

In continuous case, expected value can be a better estimate

12.2 Least Mean Square Estimate

Here, we aim to find an estimate such that

$$\theta^* = \min_c E[(\Theta - c)^2]$$

$$E[(\Theta - c)^2] = E[\Theta^2] - 2cE[\Theta] + c^2$$

Taking derivative, $\frac{dE}{dc} = 0$

$$\boxed{c = E[\Theta]}$$

In general, $c = E[\Theta|X]$ minimizes $E[(\Theta - g(X))^2]$ over all estimators $g(X)$

$E[\Theta]$ minimizes the least squares estimate

When X is observed, the best estimate simply becomes $E[\Theta|X]$.

13 Exercises

13.1 Problems

1. Independence in Complements

Given $A \perp B$, show $A \perp B^c$ and $A^c \perp B^c$. [Solution](#)

2. Conditional Independence

A, B , and C are independent with $P(C) > 0$. Show that $A \perp B|C$. [Solution](#)

3. Geometry of Meeting

R and J have to meet at a given place and each will arrive at the given place independent of each other with a delay of 0 to 1hr uniformly distributed. The pairs of delays are all equally likely. The first to arrive waits for 15 minutes and leaves. What is the probability of meeting ? [Solution](#)

4. Expectation of Function

Let X and Y be random variables with $Y = g(X)$. Show $E[Y] = \sum_x g(x)p_X(x)$. [Solution](#)

5. Cumulative Distribution Function

A random variable X is a combination of a continuous and discrete distribution as follows

$$f_X(x) = \begin{cases} 0.5 & a \leq x \leq b \\ 0.5 & x = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Find the Cumulative Distribution of X . [Solution](#)

6. Number of tosses till first head

When tossing a fair coin, what is the $E[\# \text{ tosses till the first H}]$. [Solution](#)

7. Iterated Expectation Proof

For discrete variables, show $E[X] = E[E[X|Y]]$. [Solution](#)

8. Iterated Expectation for three variables

For three random variables X, Y and Z , show $E[Z|X] = E[E[Z|X, Y]|X]$. [Solution](#)

9. Iterated Expectation practice

A class has two sections denoted by the random variable Y . Let X denote the quiz score of a student. Given that section 1 has 10 students, section 2 has 20 students, $E[X|Y=1] = 90, E[X|Y=2] = 60, Var(X|Y=1) = 10, Var(X|Y=2) = 20$, find $E[X]$ and $Var(X)$. [Solution](#)

10. **Hat Problem**

n people throw their hats in a box and then pick a hat at random. What is the expected number of people who pick their own hat ? [Solution](#)

11. **Breaking a stick**

A stick of length l is broken first at X uniformly chosen between $[0, l]$, and then at Y , uniformly chosen between $[0, X]$. Find the expected length of the shorter part. [Solution](#)

12. **PMF of $g(X)$**

Let X be uniform in $[0, 2]$, then find the PMF of $Y = X^3$. [Solution](#)

13. **Waiting for Taxi**

A taxi stand and bus stop near Al's home are at the same location. Al goes there and if a taxi is waiting $P = \frac{2}{3}$, he boards it. Otherwise, he waits for a taxi or bus to come, whichever is first. Taxi takes anywhere between 0 to 10 mins (uniform) while a bus arrives in exactly 5 mins. He boards whichever is first. Find CDF and $E[\text{wait time}]$. [Solution](#)

14. **Bayes Theorem**

Let Q be a continuous random variable with PDF

$$f_Q(q) = \begin{cases} 6q(1-q) & 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where Q represents $P(\text{success})$ for a Bernoulli X , i.e., $P(X = 1|Q = q) = q$. Find $f_{Q|X}(q|x) \forall x \in [0, 1]$ and q . [Solution](#)

15. **A Normal Transformation**

Let X be $N(0, 1)$ and $Y = g(X)$. Find $p_Y(y)$.

$$g(t) = \begin{cases} -t & t \leq 0 \\ \sqrt{t} & t > 0 \end{cases}$$

[Solution](#)

16. **Binomial Shooter**

A shooter takes 10 hits in a shooting range and each shot has $p = 0.2$ of hitting target independent of each other. Let X = number of hits. Find

- (a) PMF of X
- (b) $P(\text{no hits})$
- (c) $P(\text{scoring more than misses})$
- (d) $E[X]$ and $Var(X)$
- (e) Suppose the entry is \$3 and each shot fetches \$2. Let Y = profit. Find $E[Y]$ and $Var(Y)$.
- (f) Suppose entry is free and total reward is square of number of hits. Let Z be profit. Find $E[Z]$.

[Solution](#)

17. **Mosquito and Tick**

Every second, a mosquito lands with $P = 0.5$. Once it lands, it bites with $P = 0.2$. Let X be the time between successive mosquito bites. Find $E[X]$ and $Var(X)$.

Now suppose a tick comes into play independent of mosquito. It lands with $P = 0.1$ and once landed, bites with $P = 0.7$. Let Y be the time between successive bug bites. Find $E[Y]$ and $Var(Y)$. [Solution](#)

18. **HH or TT**

Given a coin with $P(H) = p$, find the $E[\text{number of tosses till } HH \text{ or } TT]$. [Solution](#)

19. **A Three Coin Game**

Let 3 fair coins be tossed at every turn. Given all coins and turns are independent, calculate the following (assuming success is defined as all three coins landing the same side up))

- (a) PMF of K , no of trials upto but not including the 2^{nd} success
- (b) E and Var of M , the $E[\text{number of tails}]$ before first success.

[Solution](#)

20. **Linear Expectations**

Bob conducts trials in a similar manner to Problem 19, but with four coins. He repeatedly removes a coin at success until just a single coin remains. Calculate the Expected number of tosses till the finish of experiment. [Solution](#)

21. **Papers Drawn with Replacement**

Suppose there are n papers in a drawer. We take one paper, sign it, and then put it back into the drawer. We take one more paper out and if it is not signed, we sign it and put it back in the drawer. If the paper is already signed, we simply put it back in the drawer. We repeat this process until all the papers are signed. Find the $E[\text{papers drawn till all papers are signed}]$. What is the value of this quantity as $n \rightarrow \text{large}$. [Solution](#)

22. **A Three Variable Inequality**

Let X, Y, Z be three exponentially distributed random variables with parameters λ, μ , and ν respectively. Find $P(X < Y < Z)$. [Solution](#)

23. **Poisson Emails**

You get emails according to a Poisson process at the rate of 5 messages/hour. You check email every 30 minutes. Find

- $P(\text{no new message})$
- $P(\text{one new message})$

[Solution](#)

24. **Poisson Fishing**

We go fishing where we catch fishes at the rate of 0.6/hour. We fish for two hours. If we do not catch a fish in the first two hours, we fish until the first catch. Find the following

- $P(\text{fish for } > 2 \text{ hours})$
- $P(\text{fish for } > 2 \text{ but } < 5 \text{ hours})$
- $P(\text{catch at least two fish})$
- $E[\text{fish}]$
- $E[\text{Total fishing time}]$
- $E[\text{future fishing time—fished for two hours}]$

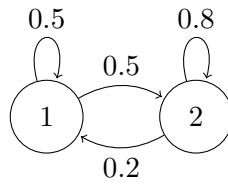
[Solution](#)

25. **Poisson Lightbulbs**

We have three identical but independent lightbulbs whose lifetimes are modelled by a Poisson process with parameter λ . Given that we start all the three bulbs together, find the $E[\text{time until last bulb dies out}]$. [Solution](#)

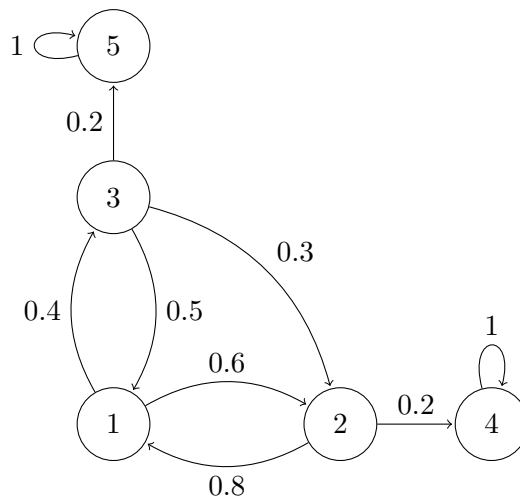
26. Steady State Markov Process

Find the steady state probabilities of the following Markov Process



[Solution](#)

27. Absorption Probabilities



Calculate the absorption probabilities for state 4 and expected time to absorption from all states. (for absorption time, assume $p_{35} = 0$ and $p_{32} = 0.5$) [Solution](#)

28. Estimating Binomial with CLT, 1/2 correction

Given a Bernoulli Process with $n = 36$ and $p = 0.5$, find $P(S_n \leq 21)$. [Solution](#)

29. LMS estimate

Given the prior $f_{\Theta|(\theta)}$, uniform in $[4, 10]$, and $f_{X|\Theta}(x|\theta)$ is uniform in $[\theta - 1, \theta + 1]$, estimate the posterior of θ . [Solution](#)

13.2 Solutions

1. Question

(a)

$$P(A \cap B) = P(A)P(B)$$

$$\begin{aligned} P(A) &= P((A \cap B) \cup (A \cap B^c)) \\ &= P(A \cap B) + P(A \cap B^c) \end{aligned}$$

since disjoint

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A)(1 - P(B)) = P(A)P(B^c) \end{aligned}$$

(b)

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c)\end{aligned}$$

2. Question

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = P(A)P(B) = P(A|C)P(B|C) \quad \text{Due to independence}$$

3. Question

Suppose R arrives at x hours. J has to arrive between x hrs to x hrs + 15 mins. Similarly if J arrives at y hours, R has to arrive between y hours to y hours + 15 mins. These are regions enclosed by the regions $x \leq 1, y \leq 1, y \leq x + \frac{1}{4}$ and $y \geq x - \frac{1}{4}$. The probability is then $1 - P(\text{not meeting}) = 1 - 2(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}) = \frac{7}{16}$.

4. Question

$$\begin{aligned}E[Y] &= \sum_y y p_Y(y) = \sum_y \sum_{x:g(x)=y} p_X(x) = \sum_y \sum_{x:g(x)=y} y p_X(x) \\ &= \sum_y \sum_{x:g(x)=y} g(x) p_X(x) = \sum_x g(x) p_X(x)\end{aligned}$$

5. Question

Cumulative Distribution of X can be found by integration and is as follows

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 0.5x & 0 \leq x < 0.5 \\ 0.75 & x = 0.5 \\ 0.75 + 0.5(x - 0.5) & 0.5 < x \leq 1 \\ 1 & 1 < x \end{cases}$$

6. Question

Let X be the # of tosses till first H . Then, $(X = 1) \cap (X > 1) = \phi$. Using *Total Expectation Theorem*

$$\begin{aligned}E[X] &= P(X = 1)E[X|X = 1] + P(X > 1)E[X|X > 1] \\ &= 0.5 * 1 + 0.5E[X] \\ \Rightarrow E[X] &= 2\end{aligned}$$

$P(X = 1) = 0.5$ because then we get the head in the first toss itself. Since $P(X = 1) + P(X > 1) = 1$, we have $P(X > 1) = 0.5$. $E[X] = E[X|X > 1]$ because the tosses are *independent* and thus memoryless.

7. Question

Note that $E[X|Y]$ is a function of y .

$$\begin{aligned}
E[E[X|Y]] &= \sum_y E[X|Y]p_Y(y) \\
&= \sum_y \sum_x xp_{X|Y}p_Y \\
&= \sum_y \sum_x xp_{X,Y}(x, y) \\
&= \sum_x x \sum_y p_{X,Y}(x, y) \\
&= \sum_x xp_X(x) \\
&= E[X]
\end{aligned}$$

8. [Question](#)

Note that $E[Z|X, Y]$ will be a function of both X and Y .

$$\begin{aligned}
E[Z|X, Y] &= \sum_z zp_{Z|X,Y}(z|x, y) \\
E[E[Z|X, Y]|X] &= \sum_y E[Z|X, Y]p_{X,Y|X}(x, y|x) \\
&= \sum_y \sum_z zp_{Z|X,Y}(z|x, y)p_{Y|X}(y|x) \\
&= \sum_y \sum_z z \frac{p_{X,Y,Z}(x, y, z)}{p_X(x)} \\
&= \sum_z z \sum_y \frac{p_{X,Y,Z}(x, y, z)}{p_X(x)} \\
&= \sum_z z \frac{p_{X,Z}(x, z)}{p_X(x)} \\
&= \sum_z zp_{Z|X}(z|x) \\
&= E[Z|X]
\end{aligned}$$

9. [Question](#)

We use the formulae from iterated expectation to calculate these.

$$\begin{aligned}
P_Y(y) &= \begin{cases} \frac{1}{3} & y = 1 \\ \frac{2}{3} & y = 2 \end{cases} \\
E[X] &= E[E[X|Y]] = \sum_y E[X|Y]P(Y) \\
&= 90 * \frac{1}{3} + 60 * \frac{2}{3} \\
Var(X) &= E[Var(X|Y)] + Var(E[X|Y]) \\
&= \sum_y Var(X|Y)P(Y) + ((90 - E[E[X|Y]])^2 \frac{1}{3} + (60 - E[E[X|Y]])^2 \frac{2}{3}) \\
&= \frac{650}{3}
\end{aligned}$$

10. [Question](#)

Let X denote the number of people who pick their own hat. We have been asked $E[X]$. Let X_i be a binary random variable denoting whether the i^{th} person picked their own hat, i.e.,

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person picks their own hat} \\ 0 & \text{otherwise} \end{cases}$$

$$P(X_i = 1) = \frac{1}{n}$$

$$E[X_i] = 1 * \frac{1}{n} + 0 * (1 - \frac{1}{n}) = \frac{1}{n}$$

Consequently

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = 1$$

It is interesting to see the variance of X . Note that the formula for variance is $E[X^2] - E[X]^2$. Thus,

$$X^2 = \left(\sum_{i=1}^n X_i\right)^2 = \sum_{i=1}^n X_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n X_i X_j$$

$$E[X^2] = \sum_{i=1}^n E[X_i^2] + \sum_{i=1}^n \sum_{j=1, j \neq i}^n E[X_i X_j]$$

Note that X_i and X_j are not independent since after the first person has picked the hat, only $n - 1$ hats remain

$$X_i X_j = \begin{cases} 1 & \text{if } X_i = X_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X_i X_j = 1) = P(X_i = 1)P(X_j = 1 | X_i = 1) = \frac{1}{n} * \frac{1}{n-1}$$

$$E[X_i X_j] = 1 * \left(\frac{1}{n} * \frac{1}{n-1}\right) + 0 * \left(1 - \frac{1}{n} * \frac{1}{n-1}\right) = \frac{1}{n(n-1)}$$

$$E[X_i^2] = 1^2 \frac{1}{n} + 0^2 \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Putting these values in the original equation for variance

$$E[X^2] = n \frac{1}{n} + \frac{1}{n} \frac{1}{n-1} \left(\frac{n(n-1)}{2} * 2\right) = 2$$

$$Var(X) = 2 - 1^2 = 1$$

11. [Question](#)

The following is the joint probability distribution of X and Y

$$f_{XY}(x, y) = f_X(x)f_{Y|X}(y|x) = \frac{1}{l} \frac{1}{x} = \frac{1}{xl} \quad \forall 0 \leq y \leq x \leq 1$$

Using marginal probabilities, we can calculate $f_Y(y)$ and $E[Y]$ as

$$f_Y(y) = \int f_{XY}(x, y) dx = \int_y^l \frac{1}{xl} dx = \frac{1}{l} \log \frac{l}{y} \quad \text{Note that for any } y, y \leq x \leq l$$

$$E[Y] = \int y f_Y(y) = \int_0^l l y \frac{1}{l} \log \frac{l}{y} = \frac{l}{4}$$

This problem can also be approached using iterated expectation

$$\begin{aligned} E[Y] &= E[E[Y|X]] = E[\text{uniform random variable between 0 and } x] \\ &= E\left[\frac{X}{2}\right] = \frac{1}{2}E[X] \\ &= \frac{l}{4} \end{aligned}$$

12. [Question](#)

Always solve such questions using the cumulative distribution approach.

$$P(X \leq x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x & 0 \leq x \leq 2 \\ 1 & 2 < x \end{cases}$$

$$P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{\frac{1}{3}})$$

$$= \begin{cases} 0 & y < 0 \\ \frac{1}{2}y^{\frac{1}{3}} & 0 \leq y^{\frac{1}{3}} \leq 2 \\ 1 & 2 < y^{\frac{1}{3}} \end{cases}$$

$$f_Y(y) = \frac{dP(Y \leq y)}{dy}(y)$$

$$= \begin{cases} 0 & y < 0 \\ \frac{1}{6}y^{-\frac{2}{3}} & 0 \leq y \leq 8 \\ 0 & 8 < y \end{cases}$$

13. [Question](#)

Let X be the waiting time and $F_X(x)$ be the CDF. Then,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{3} & x = 0 \\ \frac{2}{3} + \frac{1}{30}x & 0 < x < 5 \\ 1 & 5 \leq x \end{cases}$$

The PDF is simply the derivate of the CDF. Thus, expectation is

$$E[X] = \frac{2}{3}(0) + \int_0^5 \frac{1}{30}x dx + \frac{1}{6}(5) = \frac{5}{4} \text{ mins}$$

14. [Question](#)

From Bayes' theorem

$$\begin{aligned} f_{Q|X}(q|x) &= \frac{f_{X|Q}(x|q)f_Q(q)}{f_X(x)} \\ &= \frac{f_{X|Q}(x|q)f_Q(q)}{\int_0^1 f_{X|Q}(x|q)f_Q(q) dq} \end{aligned}$$

We will need to solve separately for $x = 0$ and $x = 1$ as x is discrete.

$$f_{Q|X=0}(q|x=0) = \frac{(1-q) * 6q(1-q)}{\int_0^1 (1-q) * 6q(1-q) dq} = 12q(1-q)^2$$

$$f_{Q|X=1}(q|x=1) = \frac{q * 6q(1-q)}{\int_0^1 q * 6q(1-q) dq} = 12q^2(1-q)$$

15. [Question](#)

Questions of this type must only be approached through CDF. First find the CDF of Y and then it's PDF.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) \\ &= P(X \in [-y, 0] \cup X \in [0, y^2]) \\ &= (F_X(0) - F_X(-y)) + (F_X(y^2) - F_X(0)) \\ &= F_X(y^2) - F_X(-y) \\ p_Y(y) &= \frac{F_Y(y)}{dy} \\ &= \frac{dF_X(y^2)}{dx} \frac{d(y^2)}{dy} - \frac{dF_X(-y)}{dx} \frac{d(-y)}{dy} \\ &= 2yp_X(y^2) + p_X(-y) \\ &= 2y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^4}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \end{aligned}$$

16. [Question](#)

- (a) $P(X = k) = \binom{10}{k} 0.2^k 0.8^{10-k}$
- (b) $P(\text{no hits}) = 0.8^{10}$
- (c) $P(X \geq 6) = \sum_{k=6}^{10} \binom{10}{k} 0.2^k 0.8^{10-k}$
- (d) $E[X] = np = 2, \text{Var}(X) = np(1-p) = 1.6$ for Bernoulli distribution
- (e) $Y = 2X - 3, E[Y] = 2E[X] - 3 = 1, \text{Var}(Y) = 4\text{Var}(X) = 6.4$
- (f) $Z = X^2, E[Z] = E[X^2] = \text{Var}(X) + E[X]^2 = 5.6$

17. [Question](#)

For the mosquito, $P(\text{bite}) = P(\text{land})P(\text{bite}|\text{land}) = 0.1$. X is a geometric random variable. $E[X] = 1/p = 10$ and $\text{Var}(X) = \frac{1-p}{p^2} = 90$.

For the mosquito and tick combined, $P(\text{mosquito and tick}) = 0.1 + 0.1 * 0.7 - 0.1 * 0.1 * 0.7 = 0.163$. This is again a geometric random variable with $E[Y] = 1/0.163$ and $\text{Var}(Y) = (1 - 0.163)/(0.163^2)$.

18. [Question](#)

This quantity can be calculated using the law of total expectation

$$E[X] = E[X|A_1]P(A_1) + E[X|A_2]P(A_2) + \dots + E[X|A_n]P(A_n) \quad \text{where } A_i \text{ are disjoint}$$

Let H_1 denote heads at first toss, H_2 denote heads at the second toss, T_1 denote tails at

first toss and T_2 denote tails at the second toss. Then,

$$\begin{aligned}
E[X] &= E[X|H_1]P(H_1) + E[X|T_1]P(T_1) \\
E[X|H_1] &= E[X|H_1H_2]P(H_2|H_1) + E[X|H_1T_2]P(T_2|H_1) \\
&= 2p + (1 + E[X|T_1])(1 - p) \\
E[X|T_1] &= E[X|T_1T_2]P(T_2|T_1) + E[X|T_1H_2]P(H_2|T_1) \\
&= 2(1 - p) + (1 + E[X|H_1])p
\end{aligned}$$

$E[X|H_1T_2] = 1 + E[X|T_1]$ because the tails after the first heads implies the first heads is now irrelevant and we have wasted one toss on the heads. The remaining process is same as starting from the first coin toss as tails.

Solving for the conditional expectations,

$$\begin{aligned}
E[X|H_1] &= \frac{3 - 2p + p^2}{1 - p + p^2} \\
E[X|T_1] &= \frac{2 + p^2}{1 - p + p^2} \\
E[X] &= \frac{2 + p - p^2}{1 - p + p^2}
\end{aligned}$$

19. Question

Define X as the following random variable

$$X = \begin{cases} 1, p = \frac{1}{4} & HHH \text{ or } TTT \\ 0, p = \frac{3}{4} & \text{otherwise} \end{cases}$$

- (a) K is simply a binomial distribution, where we want the 2^{nd} success to happen at the $K + 1$ th trial.

$$p_K(k) = \binom{k}{1} \frac{1}{4} \frac{3}{4}^{k-1} \quad \text{since the last trial is success}$$

- (b) M = number of tails before first success. Let the success be at $N + 1$. Defin Y as

$$\begin{aligned}
Y &= \begin{cases} 1 & p = \frac{1}{2} \quad HHT, HTH, \text{ or } THH \\ 2 & p = \frac{1}{2} \quad HTT, THT, \text{ or } TTH \end{cases} \\
E[Y] &= 1 * \frac{1}{2} + 2 * \frac{1}{2} \\
Var(Y) &= (1 - \frac{3}{2})^2 * \frac{1}{2} + (2 - \frac{3}{2})^2 * \frac{1}{2} \\
E[N + 1] &= \frac{1}{p} = 4 \\
Var(N + 1) &= Var(N) = \frac{1 - p}{p^2} = \frac{1 - \frac{1}{4}}{\frac{1}{4}^2} \\
M &= Y_1 + Y_2 + \cdots Y_N \\
E[M] &= E[Y_1 + Y_2 + \cdots Y_N] \\
Var(M) &= Var(Y_1 + Y_2 + \cdots Y_N)
\end{aligned}$$

Note that both Y and N are random variables here. Using the formulae for random number of random variables,

$$\begin{aligned} E[M] &= E[E[M|N]] = E[NE[Y]] = E[N]E[Y] = (4-1) * \frac{3}{2} = \frac{9}{2} \\ Var(M) &= Var(E[M|N]) + E[Var(M|N)] = Var(NE[Y]) + E[NVar(Y)] \\ &= E[Y]^2 Var(N) + E[N]Var(Y) = \frac{9}{4} * 12 + 3 * \frac{1}{4} = \frac{111}{4} \end{aligned}$$

20. [Question](#)

Let X be the number of tosses till the first coin is removed. This is a geometric random variable with $P(\text{success}) = \frac{1}{8}$. then $E[X] = 1/p = 8$. Now Y be the number of tosses till the second coin is removed (counting tosses after removal of first coin). Note that geometric random variables are memory less and what happened before the start of the "experiment" will not matter. Thus, $E[Y] = 1/(1/4) = 4$. Similarly, Z is the tosses till the last coin is removed and $E[Z] = 1/(1/2) = 2$. Note that the number of tosses till the end of experiment is simply $X + Y + Z$. $E[X + Y + Z] = E[X] + E[Y] + E[Z] = 14$.

21. [Question](#)

Note that the process till the end is a combination of multiple binomial process, such that any process lasts till the first success. Suppose we sign a paper and keep this in the drawer. Now the total signed papers in the drawer is k out of n and the $P(\text{success}) = \frac{n-k}{n}$ and $E[\text{draws till next unsigned paper}] = \frac{1}{p} = \frac{n}{n-k}$. Total draws

$$\begin{aligned} E &= \frac{n}{1} + \frac{n}{2} + \dots + \frac{n}{n} \\ &= n(1 + \frac{1}{2} + \dots + \frac{1}{n}) \\ \lim_{n \rightarrow \text{large}} E &= n \log(n) \end{aligned}$$

22. [Question](#)

A very straightforward way is to use a triple integral

$$P(X < Y < Z) = \int_0^{\inf} \int_0^z \int_0^y \lambda e^{-\lambda x} \mu e^{-\mu y} \nu e^{-\nu z} dx dy dz = \frac{\lambda \mu}{(\lambda + \mu + \nu)(\mu + \nu)}$$

$P(X < Y < Z)$ can be broken down as $P(X < \min(Y, Z))P(Y < Z)$. Consider just $P(Y < Z)$

$$P(Y < Z) = \int_0^{\inf} \int_0^z \mu e^{-\mu y} \nu e^{-\nu z} dy dz = \frac{\mu}{\mu + \nu}$$

Thus, when two exponential processes are considered, probability of arrival of 1st before 2nd is simply the percentage ratio of parameters. Thus,

$$\begin{aligned} P(X < \min(Y, Z)) &= \frac{\lambda}{\lambda + (\mu + \nu)} && Y \text{ and } Z \text{ can be combined as a single process} \\ P(Y < Z) &= \frac{\mu}{\mu + \nu} \\ P(X < Y < Z) &= P(X < \min(Y, Z))P(Y < Z) \\ &= \frac{\lambda \mu}{(\lambda + \mu + \nu)(\mu + \nu)} \end{aligned}$$

23. **Question** We can model the arrival process like a Poisson process. $\lambda = 5$ and $\tau = \frac{1}{2}$

$$P(\lambda, \tau, k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$P(5, \frac{1}{2}, 0) = \frac{(5 * \frac{1}{2})^0 e^{-5 * \frac{1}{2}}}{0!}$$

$$P(5, \frac{1}{2}, 1) = \frac{(5 * \frac{1}{2})^1 e^{-5 * \frac{1}{2}}}{1!}$$

24. **Question**

- $P(\text{fish for } > 2 \text{ hours}) = P(k = 0, \tau = 2) = e^{-0.6 * 2}$
- $P(\text{fish for } > 2 \text{ but } < 5 \text{ hours}) = P(\text{first catch in } [2, 5] \text{ hours}) = P(k = 0, \tau = 2)(1 - P(k = 0, \tau = 3))$ which is no fish in $[0, 2]$ but at least 1 fish in the next 3 hours (which will be independent of first 2 hours)
- $P(\text{catch at least two fish}) = P(\text{at least 2 catches before 2 hours}) = 1 - P(k = 0, \tau = 2) - P(k = 1, \tau = 2)$
- $E[\text{fish}]$ has two possibilities, either single fish after 2 hours, or many fish before 2 hours. $E[\text{fish}] = E[\text{fish} | \tau \leq 2](1 - P(\tau > 2)) + E[\text{fish} | \tau > 2]P(\tau > 2) = (0.6 * 2) * (1 - P(k = 0, \tau = 2)) + 1 * P(k = 0, \tau = 2)$
- $E[\text{Total fishing time}] = 2 + P(k = 0, \tau = 2)\frac{1}{\lambda}$, since we fish for atleast 2 hours
- $E[\text{future fishing time—fished for two hours}]$ can be obtained using the memoryless property of Poisson process. The expected time till first arrival is independent of what has happened till now. Thus, $E[T_1] = \frac{1}{\lambda}$

25. **Question**

Start with the merged Poisson process which will denote the time till the first bulb will fail. For this process, $\lambda' = 3\lambda$. Hence, $E[\text{first bulb fails}] = \frac{1}{3\lambda}$. After the first bulb dies out, we are left with a process with $\lambda' = 3\lambda$. Due to memoryless property, $E[\text{second bulb fails}] = \frac{1}{2\lambda}$ and consequently $E[\text{last bulb fails}] = \frac{1}{\lambda}$. Note the above two times denote the time difference, i.e. the time taken for the bulb to die out after the last bulb died out. Thus, $E[\text{time until last bulb dies out}] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$

26. **Question**

Using balance equations, we have

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$

$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22}$$

$$\pi_1 + \pi_2 = 1$$

Solving, $\pi_1 = \frac{2}{7}$ and $\pi_2 = \frac{5}{7}$

27. **Question**

Let a_i denote the absorption probabilities into state 4 starting from i

$$a_5 = 0, a_4 = 1$$

$$a_i = \sum_j a_j p_{ij}$$

$$a_2 = a_1 p_{21} + a_4 p_{24}$$

$$a_3 = a_1 p_{31} + a_2 p_{32} + a_5 p_{35}$$

$$a_1 = a_2 p_{12} + a_3 p_{13}$$

Solving, $a_1 = \frac{9}{14}, a_2 = \frac{5}{7}$ and $a_3 = \frac{15}{28}$

Let μ_i denote the expected time till absorption starting from i , then

$$\begin{aligned}\mu_4 &= 0 \\ \mu_1 &= 1 + \mu_2 p_{12} + \mu_3 p_{13} \\ \mu_2 &= 1 + \mu_1 p_{21} + \mu_4 p_{24} \\ \mu_3 &= 1 + \mu_1 p_{31} + \mu_2 p_{32}\end{aligned}$$

Solving, $\mu_1 = \frac{55}{4}, \mu_2 = 12$ and $\mu_3 = \frac{111}{8}$

28. **Question** The exact answer will be

$$\sum_{k=0}^{21} \binom{36}{k} \left(\frac{1}{2}\right)^{36} = 0.8785$$

But the same can be estimated using the CLT as follows

$$\begin{aligned}\mu &= np = 18 \\ \sigma^2 &= np(1-p) = 9 \\ P(S_n \leq 21) &\approx P\left(\frac{S_n - 18}{3} \leq \frac{21 - 18}{3}\right) \approx 0.843\end{aligned}$$

Our estimate is in the rough range of the answer but not quite close. We can do better using the $\frac{1}{2}$ correction

$$\begin{aligned}P(S_n \leq 21) &= P(S_n < 22) \text{ since } S_n \text{ is an integer} \\ \text{Consider } P(S_n \leq 21.5) &\text{ as a compromise between the two} \\ P(S_n \leq 21.5) &= P\left(\frac{S_n - 18}{3} \leq \frac{21.5 - 18}{3}\right) \approx 0.879\end{aligned}$$

In a similar manner, $P(S_n = 19) = P(18.5 \leq S_n \leq 19.5)$ using $\frac{1}{2}$ correction.

29. **Question**

We need to evaluate $f_{\Theta|X}(\theta|x)$ in order to get $E[\Theta|X]$.

$f_{X,\Theta}(x, \theta) = f_X(x)f_{\Theta|X}(\theta|x)$ which is a parallelogram on the $\theta - x$ plane at the points (3, 4), (5, 4), (9, 10) and (11, 10). Then $E[\Theta|X]$ can be obtained by drawing vertical lines on the planes and calculating the $E[\theta]$ over that line. It is a line which bends at two points.

