PHYS 133, Homework 1

Due: October 19 by midnight

The homework consists of three problems. In the first two, you will need to do some derivations. You can write them on paper and scan or type them up. For the third problem you will need to do some coding, produce graphs, and answer some questions. I recommend typing it up in the editor of your choice.The homework has to be submitted as a pdf file.

Problem 1

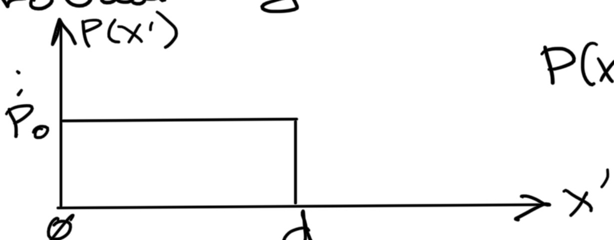
In experimental high-energy physics, the coordinate of a charged particle's track is determined by having an array of thin silicon stripe detectors arranged in a plane approximately perpendicular to the track. Thus, if a certain strip is hit by the particle, it produces an electric signal and we know that the particle traversed the detector somewhere between x_0 and $x_0 + d$, where x_0 is the coordinate of the beginning of the hit strip and d is the strip's width.

a) Write the formula for the probability function that describes the distribution of the x coordinates of the intersection between particle trajectories and the strip that spans the range from x_0 to $x_0 + d$. Make sure that the integral of your function over all possible values of x is equal to one.

b) Find the average of the distribution described by the function from (a). This will be your best estimate of the track's x coordinate if the strip that starts at x_0 produced a signal. You might be able to guess what the average is, but you need to derive the answer.

c) Find the standard deviation of the distribution described by the function from (a). This will be your best estimate of the uncertainty (Standard Deviation) of your position estimate from (b).

a) The probability of a particle hitting the strip in one location is the same as in any other $\Rightarrow$ it is a flat probability. For simplicity I will do the coordinate change: $x' = x - x_0 \Rightarrow$ $\Rightarrow$ The probability distribution will look like this:



$$P(x') = \begin{cases} 0 & \text{for } x' < 0 \\ P_0 & \text{for } 0 < x' < d \\ 0 & \text{for } d < x' \end{cases}$$

$$\int_{-\infty}^{+\infty} P(x')\,dx' = 1$$ if we want the probability of a particle to hit the strip anywhere be equal to 1.

$$\int_{-\infty}^{+\infty} P(x')\,dx' = \int_{0}^{d} P_0\,dx' = P_0 \cdot d = 1 \Rightarrow P_0 = \frac{1}{d}$$

cont'd

b) By definition $\mu = \int_{-\infty}^{+\infty} x \cdot P(x)\,dx$

In our case: $\mu = \int_{-\infty}^{+\infty} x' \cdot P(x')\,dx'$ with

$P(x')$ from (a) =>

$$\Rightarrow \mu = \frac{1}{d} \cdot \int_0^d x'\,dx' = \frac{1}{d} \cdot \frac{1}{2}\left(d^2 - \emptyset\right) = \boxed{\frac{d}{2} = \mu}$$

This result is not unexpected.

c) By definition: $\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot P(x)\,dx \Rightarrow$

$$\Rightarrow \sigma^2 = \frac{1}{d} \cdot \int_0^d \left(x - \frac{d}{2}\right)^2 dx = \frac{1}{d} \cdot \left\{ \int_0^d x^2 dx - d\int_0^d x\,dx + \frac{d^2}{4}\int_0^d dx \right\}$$

$$\Rightarrow \sigma^2 = \frac{1}{d} \cdot \left\{ \frac{1}{3}d^3 - \frac{1}{2}d^3 + \frac{1}{4}d^3 \right\} = \frac{d^2}{12} \Rightarrow \boxed{\sigma = \frac{d}{\sqrt{12}}}$$

Problem 2

The probability function of radioactive decay of an atom is:

$$P(t) = \frac{1}{\tau} e^{-t/\tau}$$

, where t is the time of the decay and τ is a parameter that characterizes how quickly decays occurs.

a) Check that the given probability function gives correct probability of the atom decaying if we don't care when the decay happens.

b) Find the average time at which an atom decays. The answer should tell you why τ is often called mean lifetime or just lifetime of an atom.

c) Find the stanadrd deviation of the decay times.

a) Probability of an atom decaying at all (no matter when) is 1. It is also given by $\int_0^\infty P(t)\,dt = \frac{1}{\tau} \cdot \int_0^\infty e^{-t/\tau} dt =$

$= \frac{1}{\tau}\left[-\tau \cdot e^{-\frac{t}{\tau}}\right]\Big|_0^\infty = 1$ => which is what we wanted to check.

b) $\mu = \frac{1}{\tau}\int_0^\infty t \cdot e^{-\frac{t}{\tau}} dt = \frac{1}{\tau}\left[-e^{-t/\tau}\left[\tau^2 + \tau \cdot t\right]\right]\Big|_0^\infty \ominus$

$\ominus \frac{1}{\tau} \cdot \tau^2 = \boxed{\tau = \mu}$ this is why τ is called lifetime.

c) $\sigma^2 = \frac{1}{\tau}\int_0^\infty (t-\tau)^2 \cdot e^{-\frac{t}{\tau}} dt \ominus$

$\ominus \frac{1}{\tau}\left[\underbrace{\int_0^\infty t^2 \cdot e^{-t/\tau} dt}_{=2\tau^3} - 2\tau \underbrace{\int_0^\infty t \cdot e^{-\frac{t}{\tau}} dt}_{=\tau^2} + \tau^2 \underbrace{\int_0^\infty e^{-\frac{t}{\tau}} dt}_{=\tau}\right]$

$\int_0^\infty t^2 e^{-t/\tau} dt = -\left[e^{-t/\tau}(2\tau^3 + 2\tau^2 \cdot t + \tau \cdot t^2)\right]\Big|_0^\infty = 2\tau^3$ =>

=> $\sigma^2 = \frac{1}{\tau}\left[2\tau^3 - 2\tau^3 + \tau^3\right] = \tau^2$ => $\boxed{\sigma = \tau}$

In other words, the larger the lifetime, the larger the spread of decay times.

cont’d

Problem 3

For this problem you will need to use computer to generate and plot data. You can use a programming tool of your choice to do that. Python is recommended, but not required.

a) Generate 10 random numbers using a standard random number generator that produces numbers in the range between 0 and 1 with any number within this range having equal probability to appear (flat probability distribution). Calculate the average of your ten numbers.

b) Repeat (a) 100 times. It will give you 100 averages.

c) Calculate the mean and the standard deviation of your 100 averages. Are they what you expected them to be? Explain your answer. You will need to use Central Limit Theorem to figure out what you expect to get.

d) Plot the distribution of your 100 averages as a histogram. Use bin width and the histogram range appropriate for showing the distribution. Overlay the histogram with the probability function you expect for the averages. Does it look like the two agree?

Hint: You will need to change the normalization factor of the probability function so that the area under it is equal to the area under the histogram in order to be able to compare them.

e) Now instead of generating 10 random numbers for each of the 100 averages, generate 40 random numbers for each of the 100 averages. Repeat c) and d).

Note: When making plots, make sure to label the axis and put in the units for each axis. Let's say that the initial numbers you were simulating were in the range from 0 to 1m.

Solution

For (a) and (b) I simulated 100 sets of 10 random numbers (lengths). Each simulated length was drawn from a random flat distribution with the range from 0m to 1m. In other words, the probability of getting any length within this range was the same as getting any other length. The flat distribution like that is the same as we put together in Problem 1. That means that we know that $\mu_x = 0.5m$ and $\sigma_x = 1/\sqrt{12}m$ (the width of the range of possible values of length was equal to d in Problem 1; here it is equal to 1m). I calculated the average of each of the 100 sets using the formula:

$$\bar{x}_j = \frac{1}{n}\sum x_{ij}$$

, where n=10 is the number of simulated lengths in each of the 100 sets and x_{ij} is the i-th simulated length from the j-th dataset. j has values from 1 to N=100. $\bar{x}_j$ is the calculated average of the j-th dataset.

(c) Now I can calculate the average: $\bar{x}$ and the standard deviation (SD): S of my 100 averages x_j. I do that by using the following two formulas.

$$\bar{x} = \frac{1}{N}\sum x_j$$

$$S^2 = \frac{1}{N-1}\sum (x_j - \bar{x})^2$$

Note that both $\bar{x}$ and S are estimated of the true μ and σ of the distribution of the averages. What are those true values? We find them through the Central Limit Theorem. It states that if have an average of n random numbers drawn from a probability distribution with mean μ_x and SD σ_x, this average will itself be distributed according to Gaussian distribution with the mean $\mu = \mu_x$ and the SD $\sigma = \sigma_x/\sqrt{n}$. For our case $\mu_x = 0.5m$ and $\sigma_x = 1/\sqrt{12}m$. From this we can find the true (or expected) values for the mean and SD of the distribution of averages. The values for the mean and the SD calculated based on our finite simulated data (estimated values) and the true (expected) values are given in Table 1. The expected SD becomes two times smaller when we go from $n = 10$ to $n = 40$. The estimated mean and SD are close to the expected ones for $n = 10$ and $n = 40$.

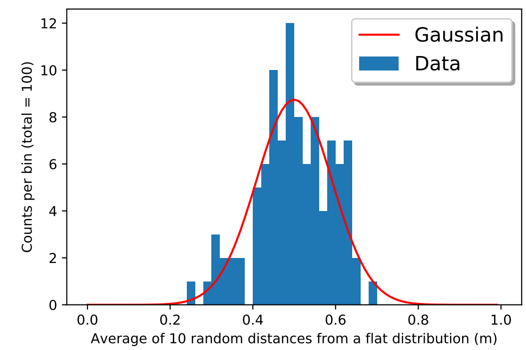
I wanted to test a common sense assumption that the estimated values based on a finite sample become closer to the true values as the sample size is increasing. To do that, I increased the number of generated random sets from 100 to 10^5. As expected, the estimated mean and SD became closer to the expected values (Table 1).

	n=10; N=100	n=10; N=10^5	n=40; N=10	n=40; N=10^5
Estimated Mean	0.501	0.500	0.501	0.500
Expected Mean	0.5	0.5	0.5	0.5
Estimated SD	0.0904	0.0914	0.0471	0.0457
Expected SD	0.0913	0.0913	0.0456	0.0456

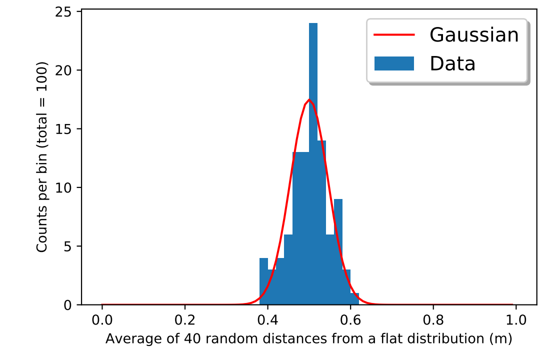
TABLE 1. *Expected and estimated average and SD values for different simulation parameters.* Each column corresponds to a specific number (n) of lengths simulated to calculate an individual average. The lengths were simulated based on a flat distribution with the range from $0m$ to $1m$. The number (N) of the individual averages were then used to calculate the mean of the averages and the SD of the averages. The two are denoted as Estimated Mean and SD because they are calculated based on a finite sample. Expected Mean and SD were calculated based on the properties of the flat distribution and the Central Limit Theorem.

Now I know that the Central Limit Theorem makes a correct prediction for the mean and SD of the distribution of my averages. What about the shape of the distribution? The prediction is that the shape should be Gaussian. To test this I plotted histogram of the simulated averages for the four combinations of n and N in Table Table 1. I then overlaid each histogram with a Gaussian function that had the corresponding Expected Mean and SD from Table 1. The resulting plots can be found in Figure 1. The distribution of data representing N=100 agrees with the Gaussian prediction in general, but it is hard to draw the definitive conclusion about how well. At the same time, the data sample with N=10^5 shows very nice agreement with the Gaussian prediction. This agreement supports Central Limit Theorem.

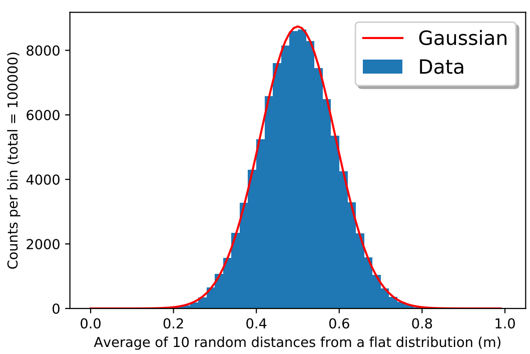
For comparison purposes I normalized the Gaussian function to have the same integral as the corresponding histogram. This was done by multiplying the standard Gaussain probability function by ab, where a is the number of averages in the histogram (100 or 10^5 *for my data) and b is the bin width of the histograms.*



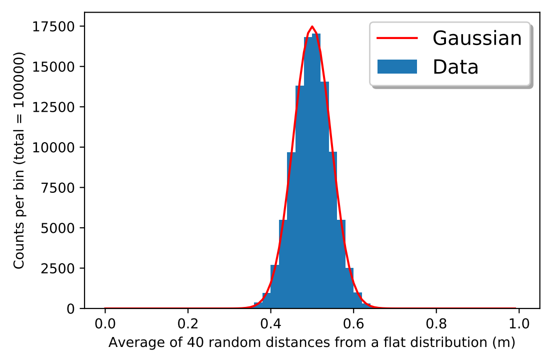
(A) 10^2 averages of 10 lengths



(B) 10^2 averages of 40 lengths



(C) 10^5 averages of 10 lengths



(D) 10^5 averages of 40 lengths

FIGURE 1. *Comparison of distributions of the simulated data and Central Limit Theorem predictions.* Blue histograms show distributions of averages of 100 sets of 10 (A) and 40 (B) random lengths generated from flat distribution with the range of $0m$ to $1m$. Panels (C) and (D) show distributions of averages of 100000 sets of 10 and 40 randoms lengths respectively. In each panel the red curve corresponds to the Gaussian distribution of the averages with the mean and standard deviation predicted by the Central Limit Theorem. The Gaussian probability function was normalized in such a way that the area under the function is equal to the area of the corresponding data histogram.