ASTR 119 Final Project Option #2
Logistic Maps

Consider a population of organisms whose rate of reproduction depends somehow upon the number density of individuals.

Clearly, the rate of reproduction depends upon the number of individuals already present as these are the ultimate means of reproduction.

The rate at which these individuals reproduce is also frequently dependent on the availability of resources, which in turn is proportional to the number of individuals using up those resources.

We can represent this behavior schematically as

$$P_{n+1} = P_n(a - bP_n)$$

where P_n is the population of generation n, a is the coefficient describing how the population would growth with time if no other constraints (such as limited resources) applied, and the bP_n term describes how the present population limits available resources.

For b = 0, you get unchecked geometrical growth

$$P_n = P_0 a^n$$

where n is the number of generations evolved.

With b!=0, interesting behavior can result. First, let's re-write things. Let

$$P_n = (a/b)x_n$$

and let

$$r = a/4$$

then we can write:

$$x_{n+1} = 4rx_n(1 - x_n)$$

If 0<= r <= 1, then if 0 <= x_0 <= 1 we will have 0 <= x_n <=1 for all n.

Therefore, we have mapped the initial x_0 to another value of x between 0 and 1, and we call this kind of mapping a *logistic* map.

Picking a value for x_0 and r, we can evolve this equation over n generations and study the behavior.

Divergence

Depending on the value of *r*, trajectories for x_n will begin to diverge. Can we quantify how fast the trajectories diverge? Yes, try this:

$$|\Delta x_n| = |\Delta x_0| e^{\lambda n}$$

Here $|\Delta x_0|$ is the ratio of initial conditions and $|\Delta x_n|$ is the ratio of the value of the trajectories after n iterations.

If $\lambda < 0$, the two trajectories will converge to the same limit cycle.

If $\lambda > 0$, the trajectories will diverge exponentially.

For the logistics map, the magnitude of the separation can only increase to 0.5 (because we map from 0 to 1).

We can write

$$\lambda = \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|$$

Note that

$$\frac{\Delta x_n}{\Delta x_0} = \frac{\Delta x_1}{\Delta x_0} \frac{\Delta x_2}{\Delta x_1} \frac{\Delta x_3}{\Delta x_2} \cdots$$

Therefore

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right|$$

where the natural log changes the product to a sum.

If we take Δx_1 -> 0, then we can write

$$\frac{dx_{i+1}}{dx_i} = f'(x_i)$$

In our case,

$$x_{n+1} = f(x_i) = 4rx_n(1 - x_n)$$

so we can write

$$\frac{dx_{i+1}}{dx_i} = f'(x_i) = 4r(1 - 2x_i)$$

We can then define the *Lyapunov* exponent as

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| = \frac{1}{n} \sum_{i=0}^{n-1} \ln[4r(1 - 2x_i)]$$

Final Project #2 Requirements

1) Write a scheme to evolve a logistic map x_n for a specified number *n,* initial conditions x_0, and ratio *r*. Be sure to remap to the domain x=[0,1] if needed.

2) Starting with initial conditions x_0= 0.65 in each case, compute the logistic map for *n* = [0,100] (integer values), repeating for all values r = [0.7,1.00] in increments of 0.01.

3) Record the values of x_n and plot them as small red dots for each value of *r*. Compute the *Lyapunov* exponent λ for each map, and plot it as a blue line as a function of *r*.

4) For approximately what values of *r* does λ = 0?

5) Plot in separate panels all values of x_n and λ for each value of r.

6) Extra credit:

Produce an animation of x_n and λ as a function of r, with r progressing with each frame in 0.01 increments, and for each animation frame plot in a second panel x_n vs. n.