Constant d is the grid space Δx.
m is the map factor on mass grid.
m_c is the map factor on the cell corners.
m_u is the map factor on u grid.
m_v is the map factor on v grid.
f is the Coriolis parameter on mass grid.
f_c is the Coriolis parameter on the cell corners.
f_u is the Coriolis parameter on u grid.
f_v is the Coriolis parameter on v grid.
u is U-component wind on mass grid
U is U-component wind on U grid
v is V-component wind on mass grid
V is V-component wind on V grid

Grids

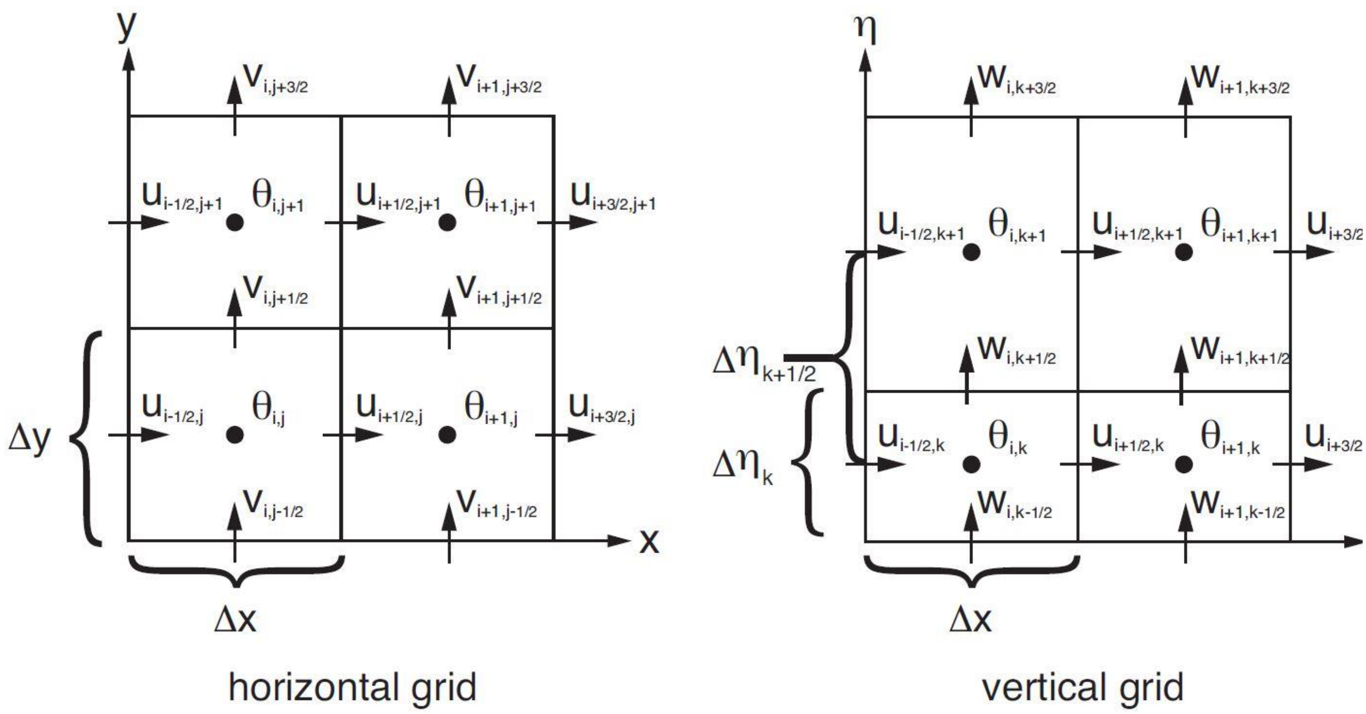

Figure 3.2: Horizontal and vertical grids of the ARW

Discrete Operators

$$
\begin{cases}
\overline{A} = \overline{A}^x = \frac{1}{2}\left(A_{i+\frac{1}{2},j} + A_{i-\frac{1}{2},j}\right) \\
\frac{\partial A}{\partial x} = A_x = \frac{1}{d}\left(A_{i+\frac{1}{2},j} - A_{i-\frac{1}{2},j}\right) \\
\overline{\overline{A}^x}^x = \overline{A}^{xx} = \frac{1}{4}\left(A_{i+1,j} + 2A_{i,j} + A_{i-1,j}\right) \\
\overline{\overline{A}^x}^y = \frac{1}{4}\left(A_{i+\frac{1}{2},+\frac{1}{2}} + A_{i-\frac{1}{2},j+\frac{1}{2}} + A_{i+\frac{1}{2},j-\frac{1}{2}} + A_{i-\frac{1}{2},j-\frac{1}{2}}\right) \\
\frac{\partial^2 A}{\partial x^2} = A_{xx} = \frac{1}{d^2}\left(A_{i+1,j} - 2A_{i,j} + A_{i-1,j}\right) \\
\overline{\frac{\partial A}{\partial x}} = \overline{A_x}^x = \frac{1}{2d}\left(A_{i+1,j} - A_{i-1,j}\right)
\end{cases}
$$

Quadratic Conservation Format

Governing Equations

$$
\begin{cases}
\frac{\partial u}{\partial t} = -m\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + f^* v - mg\frac{\partial z}{\partial x} \\
\frac{\partial v}{\partial t} = -m\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - f^* u - mg\frac{\partial z}{\partial y} \\
\frac{\partial z}{\partial t} = -m^2\left[\frac{\partial}{\partial x}\left(\frac{zu}{m}\right) + \frac{\partial}{\partial y}\left(\frac{zv}{m}\right)\right] = -m^2\left[u\frac{\partial}{\partial x}\left(\frac{z}{m}\right) + v\frac{\partial}{\partial y}\left(\frac{z}{m}\right) + \frac{z}{m}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\right]
\end{cases}
$$

Where $f^* = f + m^2\left[v\frac{\partial}{\partial x}\left(\frac{1}{m}\right) - u\frac{\partial}{\partial y}\left(\frac{1}{m}\right)\right] = f + u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}$

Discrete Equations

$$
\begin{cases}
\frac{\partial u}{\partial t} = -m_u\left(\overline{\overline{u}^x u_x}^x + \overline{\overline{v}^x u_y}^y + gz_x\right) + F_{U_{cor}} \\
\frac{\partial v}{\partial t} = -m_v\left(\overline{\overline{u}^y v_x}^x + \overline{\overline{v}^y v_y}^y + gz_y\right) + F_{V_{cor}} \\
\frac{\partial z}{\partial t} = -m^2\left[\overline{uz^*{}_x}^x + \overline{vz^*{}_y}^y + z^*\left(u_x + v_y\right)\right]
\end{cases}
$$

Where $z^* = \frac{z}{m}$

U grid

$$\overline{\overline{u}^x u_x}^x = \frac{1}{4d}\left[\left(u_{i+1,j} + u_{i,j}\right)\left(u_{i+1,j} - u_{i,j}\right) + \left(u_{i,j} + u_{i-1,j}\right)\left(u_{i,j} - u_{i-1,j}\right)\right]$$

$$\overline{\overline{v}^x u_y}^y = \frac{1}{4d}\left[\left(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i-\frac{1}{2},j+\frac{1}{2}}\right)\left(u_{i,j+1} - u_{i,j}\right) + \left(v_{i+\frac{1}{2},j-\frac{1}{2}} + v_{i-\frac{1}{2},j-\frac{1}{2}}\right)\left(u_{i,j} - u_{i,j-1}\right)\right]$$

$$z_x = \frac{1}{d}\left(z_{i+\frac{1}{2},j} - z_{i-\frac{1}{2},j}\right)$$

V grid

$$\overline{\overline{u}^{y} v_x}^{x} = \frac{1}{4d}\left[\left(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i+\frac{1}{2},j-\frac{1}{2}}\right)(v_{i+1,j} - v_{i,j}) + \left(u_{i-\frac{1}{2},j+\frac{1}{2}} + u_{i-\frac{1}{2},j-\frac{1}{2}}\right)(v_{i,j} - v_{i-1,j})\right]$$

$$\overline{\overline{v}^{y} v_y}^{y} = \frac{1}{4d}\left[(v_{i,j+1} + v_{i,j})(v_{i,j+1} - v_{i,j}) + (v_{i,j} + v_{i,j-1})(v_{i,j} - v_{i,j-1})\right]$$

$$z_y = \frac{1}{d}\left(z_{i,j+\frac{1}{2}} - z_{i,j-\frac{1}{2}}\right)$$

Mass grid

$$\overline{uz^{*}_{x}}^{x} = \frac{1}{2d}\left[u_{i+\frac{1}{2},j}(z^{*}_{i+1,j} - z^{*}_{i,j}) + u_{i-\frac{1}{2},j}(z^{*}_{i,j} - z^{*}_{i-1,j})\right]$$

$$\overline{vz^{*}_{y}}^{y} = \frac{1}{2d}\left[v_{i,j+\frac{1}{2}}(z^{*}_{i,j+1} - z^{*}_{i,j}) + v_{i,j-\frac{1}{2}}(z^{*}_{i,j} - z^{*}_{i,j-1})\right]$$

$$u_x = \frac{1}{d}\left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)$$

$$v_y = \frac{1}{d}\left(v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}\right)$$

On U grid, for all the variables on mass grid or V grid, the index i should minus $\frac{1}{2}$, and for the variables on V grid index j should plus $\frac{1}{2}$, thus

$$u_{i,j} = U_{i,j}$$

$$v_{i,j} = V_{i-\frac{1}{2},j+\frac{1}{2}}$$

$$z_{i,j} = Z_{i-\frac{1}{2},j}$$

$$\overline{\overline{u}^{x} u_x}^{x} = \frac{1}{4d}\left[(U_{i+1,j} + U_{i,j})(U_{i+1,j} - U_{i,j}) + (U_{i,j} + U_{i-1,j})(U_{i,j} - U_{i-1,j})\right]$$

$$\overline{\overline{v}^{x} u_y}^{y} = \frac{1}{4d}\left[(V_{i,j+1} + V_{i-1,j+1})(U_{i,j+1} - U_{i,j}) + (V_{i,j} + V_{i-1,j})(U_{i,j} - U_{i,j-1})\right]$$

$$z_x = \frac{1}{d}(Z_{i,j} - Z_{i-1,j})$$

$$\frac{\partial u}{\partial t} = -\frac{m_u}{4d}\left[(U_{i+1,j} + U_{i,j})(U_{i+1,j} - U_{i,j}) + (U_{i,j} + U_{i-1,j})(U_{i,j} - U_{i-1,j}) + (V_{i,j+1} + V_{i-1,j+1})(U_{i,j+1} - U_{i,j}) + (V_{i,j} + V_{i-1,j})(U_{i,j} - U_{i,j-1}) + 4g(Z_{i,j} - Z_{i-1,j})\right] + F_{U_{cor}}$$

On V grid, for all the variables on mass grid or U grid, the index j should minus $\frac{1}{2}$, and for the variables on U grid index i should plus $\frac{1}{2}$ thus

$$u_{i,j} = U_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$v_{i,j} = V_{i,j}$$

$$z_{i,j} = Z_{i.j-\frac{1}{2}}$$

$$\overline{\overline{u}^{y} v_x}^{x} = \frac{1}{4d}\left[(U_{i+1,j} + U_{i+1,j-1})(V_{i+1,j} - V_{i,j}) + (U_{i,j} + U_{i,j-1})(V_{i,j} - V_{i-1,j})\right]$$

$$\overline{\overline{v}^{y} v_y}^{y} = \frac{1}{4d}\left[(V_{i,j+1} + V_{i,j})(V_{i,j+1} - V_{i,j}) + (V_{i,j} + V_{i,j-1})(V_{i,j} - V_{i,j-1})\right]$$

$$z_y = \frac{1}{d}(z_{i,j} - z_{i,j-1})$$

$$\frac{\partial v}{\partial t} = -\frac{m_v}{4d}\left[(U_{i+1,j} + U_{i+1,j-1})(V_{i+1,j} - V_{i,j}) + (U_{i,j} + U_{i,j-1})(V_{i,j} - V_{i-1,j}) + (V_{i,j+1} + V_{i,j})(V_{i,j+1} - V_{i,j}) + (V_{i,j} + V_{i,j-1})(V_{i,j} - V_{i,j-1}) + 4g(z_{i,j} - z_{i,j-1}) \right] + F_{V_{cor}}$$

On Mass grid

$$u_{i,j} = U_{i+\frac{1}{2},j}$$

$$v_{i,j} = V_{i,j+\frac{1}{2}}$$

$$\overline{uz^*{}_x}^x = \frac{1}{2d}\left[u_{i+1,j}(z^*{}_{i+1,j} - z^*{}_{i,j}) + u_{i,j}(z^*{}_{i,j} - z^*{}_{i-1,j})\right]$$

$$\overline{vz^*{}_y}^y = \frac{1}{2d}\left[v_{i,j+1}(z^*{}_{i,j+1} - z^*{}_{i,j}) + v_{i,j}(z^*{}_{i,j} - z^*{}_{i,j-1})\right]$$

$$u_x = \frac{1}{d}(u_{i+1,j} - u_{i,j})$$

$$v_y = \frac{1}{d}(v_{i,j+1} - v_{i,j})$$

$$\frac{\partial z}{\partial t} = -\frac{m^2}{2d}\left[u_{i+1,j}(z^*{}_{i+1,j} - z^*{}_{i,j}) + u_{i,j}(z^*{}_{i,j} - z^*{}_{i-1,j}) + v_{i,j+1}(z^*{}_{i,j+1} - z^*{}_{i,j}) + v_{i,j}(z^*{}_{i,j} - z^*{}_{i,j-1}) + 2z^*{}_{i,j}(u_{i+1,j} - u_{i,j} + v_{i,j+1} - v_{i,j})\right]$$

Coriolis and Curvature Terms

$$F_{U_{cor}} = +\left(\overline{f}^x + \overline{\overline{u}^x \delta_y m - \overline{v}^y \delta_x m}^x\right)\overline{v}^{xy}$$

$$F_{V_{cor}} = -\left(\overline{f}^y + \overline{\overline{u}^x \delta_y m - \overline{v}^y \delta_x m}^y\right)\overline{u}^{xy}$$

Where,

$$\overline{f}^x = \frac{f_{i+\frac{1}{2},j} + f_{i-\frac{1}{2},j}}{2}$$

$$\overline{f}^y = \frac{f_{i,j+\frac{1}{2}} + f_{i,j-\frac{1}{2}}}{2}$$

$$\overline{\overline{u}^x \delta_y m - \overline{v}^y \delta_x m}^x = \overline{\overline{u}^x \delta_y m}^x - \overline{\overline{v}^y \delta_x m}^x$$

$$\overline{\overline{u}^x \delta_y m - \overline{v}^y \delta_x m}^y = \overline{\overline{u}^x \delta_y m}^y - \overline{\overline{v}^y \delta_x m}^y$$

$$\overline{\overline{u}^x \delta_y m}^x = \frac{1}{4d}\left[(u_{i+1,j} + u_{i,j})\left(m_{i+\frac{1}{2},j+\frac{1}{2}} - m_{i+\frac{1}{2},j-\frac{1}{2}}\right) + (u_{i,j} + u_{i-1,j})\left(m_{i-\frac{1}{2},j+\frac{1}{2}} - m_{i-\frac{1}{2},j-\frac{1}{2}}\right)\right]$$

$$\overline{\overline{v}^y \delta_x m}^x = \frac{1}{4d}\left[\left(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i+\frac{1}{2},j-\frac{1}{2}}\right)(m_{i+1,j} - m_{i,j}) + \left(v_{i-\frac{1}{2},j+\frac{1}{2}} + v_{i-\frac{1}{2},j-\frac{1}{2}}\right)(m_{i,j} - m_{i-1,j})\right]$$

$$\overline{\overline{u}^x \delta_y m}^y = \frac{1}{4d}\left[\left(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}\right)(m_{i,j+1} - m_{i,j}) + \left(u_{i+\frac{1}{2},j-\frac{1}{2}} + u_{i-\frac{1}{2},j-\frac{1}{2}}\right)(m_{i,j} - m_{i,j-1})\right]$$

$$\overline{\overline{v}^y \delta_x m}^y = \frac{1}{4d}\left[(v_{i,j+1} + v_{i,j})\left(m_{i+\frac{1}{2},j+\frac{1}{2}} - m_{i-\frac{1}{2},j+\frac{1}{2}}\right) + (v_{i,j} + v_{i,j-1})\left(m_{i+\frac{1}{2},j-\frac{1}{2}} - m_{i-\frac{1}{2},j-\frac{1}{2}}\right)\right]$$

$$\overline{v}^{xy} = \frac{1}{4}\left(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i-\frac{1}{2},j+\frac{1}{2}} + v_{i+\frac{1}{2},j-\frac{1}{2}} + v_{i-\frac{1}{2},j-\frac{1}{2}}\right)$$

$$\overline{u}^{xy} = \frac{1}{4}\left(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}} + u_{i+\frac{1}{2},j-\frac{1}{2}} + u_{i-\frac{1}{2},j-\frac{1}{2}}\right)$$

On U grid, for all the variables on mass grid or V grid, the index i should minus $\frac{1}{2}$, thus

$$u_{i,j} = U_{i,j}$$

$$v_{i,j} = V_{i-\frac{1}{2},j+\frac{1}{2}}$$

$$m_{i,j} = m_{u_{i,j}}$$

$$m_{i,j} = m_{v_{i-\frac{1}{2},j+\frac{1}{2}}}$$

$$\overline{f}^{x} = \frac{f_{i,j} + f_{i-1,j}}{2} = f_{u_{i,j}}$$

$$\overline{\overline{u}^{x}\delta_y m}^{x} = \frac{1}{4d}\left[\left(U_{i+1,j} + U_{i,j}\right)\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + \left(U_{i,j} + U_{i-1,j}\right)\left(m_{v\,i-1,j+1} - m_{v_{i-1,j}}\right)\right]$$

$$\overline{\overline{v}^{y}\delta_x m}^{x} = \frac{1}{4d}\left[\left(V_{i,j+1} + V_{i,j}\right)\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) + \left(V_{i-1,j+1} + V_{i-1,j}\right)\left(m_{u_{i,j}} - m_{u_{i-1,j}}\right)\right]$$

$$\overline{v}^{xy} = \frac{1}{4}\left(V_{i,j+1} + V_{i-1,j+1} + V_{i,j} + V_{i-1,j}\right)$$

$$F_{U_{cor}} = +\frac{1}{4}\Big\{f_{u_{i,j}} + \frac{1}{4d}\Big[\left(U_{i+1,j} + U_{i,j}\right)\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + \left(U_{i,j} + U_{i-1,j}\right)\left(m_{v\,i-1,j+1} - m_{v_{i,j}}\right) - \left(V_{i,j+1} + V_{i,j}\right)\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) - \left(V_{i-1,j+1} + V_{i-1,j}\right)\left(m_{u_{i,j}} - m_{u_{i-1,j}}\right)\Big]\Big\}\left(V_{i,j+1} + V_{i-1,j+1} + V_{i,j} + V_{i-1,j}\right)$$

On V grid, for all the variables on mass grid or U grid, the index j should minus $\frac{1}{2}$, thus

$$u_{i,j} = U_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$v_{i,j} = V_{i,j}$$

$$m_{i,j} = m_{v_{i,j}}$$

$$m_{i,j} = m_{u_{i+\frac{1}{2},j-\frac{1}{2}}}$$

$$\overline{f}^{y} = \frac{f_{i,j} + f_{i,j-1}}{2} = f_{v_{i,j}}$$

$$\overline{\overline{u}^{x}\delta_y m}^{y} = \frac{1}{4d}\left[\left(U_{i+1,j} + U_{i,j}\right)\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + \left(U_{i+1,j-1} + U_{i,j-1}\right)\left(m_{v_{i,j}} - m_{v_{i,j-1}}\right)\right]$$

$$\overline{\overline{v}^{y}\delta_x m}^{y} = \frac{1}{4d}\left[\left(V_{i,j+1} + V_{i,j}\right)\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) + \left(V_{i,j} + V_{i,j-1}\right)\left(m_{u_{i+1,j-1}} - m_{u_{i,j-1}}\right)\right]$$

$$\overline{u}^{xy} = \frac{1}{4}\left(U_{i+1,j} + U_{i,j} + U_{i+1,j-1} + U_{i,j-1}\right)$$

$$F_{V_{cor}} = -\frac{1}{4}\Big\{f_{v_{i,j}} + \frac{1}{4d}\Big[\left(U_{i+1,j} + U_{i,j}\right)\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + \left(U_{i+1,j-1} + U_{i,j-1}\right)\left(m_{v_{i,j}} - m_{v_{i,j-1}}\right) - \left(V_{i,j+1} + V_{i,j}\right)\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) - \left(V_{i,j} + V_{i,j-1}\right)\left(m_{u_{i+1,j-1}} - m_{u_{i,j-1}}\right)\Big]\Big\}\left(U_{i+1,j} + U_{i,j} + U_{i+1,j-1} + U_{i,j-1}\right)$$

Completed Discrete Equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} = -\frac{m_u}{4d}\big[&(U_{i+1,j}+U_{i,j})(U_{i+1,j}-U_{i,j})+(U_{i,j}+U_{i-1,j})(U_{i,j}-U_{i-1,j})+(V_{i,j+1}+V_{i-1,j+1})(U_{i,j+1}-U_{i,j})\\
&+(V_{i,j}+V_{i-1,j})(U_{i,j}-U_{i,j-1})+4g(Z_{i,j}-Z_{i-1,j})\big]\\
&+\frac{1}{4}\Big\{f_{u_{i,j}}\\
&+\frac{1}{4d}\Big[(U_{i+1,j}+U_{i,j})\left(m_{v_{i,j+1}}-m_{v_{i,j}}\right)+(U_{i,j}+U_{i-1,j})\left(m_{v\ i-1,j+1}-m_{v_{i,j}}\right)\\
&-(V_{i,j+1}+V_{i,j})\left(m_{u_{i+1,j}}-m_{u_{i,j}}\right)-(V_{i-1,j+1}+V_{i-1,j})\left(m_{u_{i,j}}-m_{u_{i-1,j}}\right)\Big]\Big\}(V_{i,j+1}+V_{i-1,j+1}+V_{i,j}\\
&+V_{i-1,j})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} = -\frac{m_v}{4d}\big[&(U_{i+1,j}+U_{i+1,j-1})(V_{i+1,j}-V_{i,j})+(U_{i,j}+U_{i,j-1})(V_{i,j}-V_{i-1,j})+(V_{i,j+1}+V_{i,j})(V_{i,j+1}-V_{i,j})\\
&+(V_{i,j}+V_{i,j-1})(V_{i,j}-V_{i,j-1})+4g(z_{i,j}-z_{i,j-1})\big]\\
&-\frac{1}{4}\Big\{f_{v_{i,j}}\\
&+\frac{1}{4d}\Big[(U_{i+1,j}+U_{i,j})\left(m_{v_{i,j+1}}-m_{v_{i,j}}\right)+(U_{i+1,j-1}+U_{i,j-1})\left(m_{v_{i,j}}-m_{v_{i,j-1}}\right)\\
&-(V_{i,j+1}+V_{i,j})\left(m_{u_{i+1,j}}-m_{u_{i,j}}\right)-(V_{i,j}+V_{i,j-1})\left(m_{u_{i+1,j-1}}-m_{u_{i,j-1}}\right)\Big]\Big\}(U_{i+1,j}+U_{i,j}+U_{i+1,j-1}\\
&+U_{i,j-1})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial t} = -\frac{m^2}{2d}\big[&u_{i+1,j}(z^*{}_{i+1,j}-z^*{}_{i,j})+u_{i,j}(z^*{}_{i,j}-z^*{}_{i-1,j})+v_{i,j+1}(z^*{}_{i,j+1}-z^*{}_{i,j})+v_{i,j}(z^*{}_{i,j}-z^*{}_{i,j-1})\\
&+2z^*{}_{i,j}(u_{i+1,j}-u_{i,j}+v_{i,j+1}-v_{i,j})\big]
\end{aligned}
$$

Diffusion terms

High order diffusion term could be calculate by using Shuman Operator, therefore

$$\nabla^2 h = h_{xx}+h_{yy} = \frac{h_{i+1,j}+h_{i-1,j}-2h_{i,j}}{(\Delta x)^2}+\frac{h_{i,j+1}+h_{i,j-1}-2h_{i,j}}{(\Delta y)^2}$$

$$
\begin{aligned}
\nabla^3 h &= h_{xxx}+h_{yyy}\\
&= \frac{\left[\frac{h_{i+1,j}+h_{i-1,j}-2h_{i,j}}{(\Delta x)^2}\right]_{i+\frac{1}{2},j}-\left[\frac{h_{i+1,j}+h_{i-1,j}-2h_{i,j}}{(\Delta x)^2}\right]_{i-\frac{1}{2},j}}{\Delta x}
+\frac{\left[\frac{h_{i,j+1}+h_{i,j-1}-2h_{i,j}}{(\Delta y)^2}\right]_{i,j+\frac{1}{2}}-\left[\frac{h_{i,j+1}+h_{i,j-1}-2h_{i,j}}{(\Delta y)^2}\right]_{i,j-\frac{1}{2}}}{\Delta y}\\
&= \frac{\frac{h_{i+\frac{3}{2},j}+h_{i-\frac{1}{2},j}-2h_{i+\frac{1}{2},j}}{(\Delta x)^2}-\frac{h_{i+\frac{1}{2},j}+h_{i-\frac{3}{2},j}-2h_{i-\frac{1}{2},j}}{(\Delta x)^2}}{\Delta x}+\frac{\frac{h_{i,j+\frac{3}{2}}+h_{i,j-\frac{1}{2}}-2h_{i,j+\frac{1}{2}}}{(\Delta y)^2}-\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{3}{2}}-2h_{i,j-\frac{1}{2}}}{(\Delta y)^2}}{\Delta y}
\end{aligned}
$$

$$= \frac{h_{i+\frac{3}{2},j} - h_{i-\frac{3}{2},j} + 3h_{i-\frac{1}{2},j} - 3h_{i+\frac{1}{2},j}}{(\Delta x)^3} + \frac{h_{i,j+\frac{3}{2}} - h_{i,j-\frac{3}{2}} + 3h_{i,j-\frac{1}{2}} - 3h_{i,j+\frac{1}{2}}}{(\Delta y)^3}$$

For the same reason:

While $\Delta x = \Delta y$

$$\nabla^6 h = h_{(6x)} + h_{(6y)}$$

$$= \frac{h_{i+3,j} + h_{i-3,j} + h_{i,j+3} + h_{i,j-3} - 6(h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2}) + 15(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}) - 40h_{i,j}}{(\Delta x)^6}$$

$$\nabla^7 h = h_{(7x)} + h_{(7y)}$$

$$= \frac{1}{(\Delta x)^7}\Big[h_{i+\frac{7}{2},j} - h_{i-\frac{7}{2},j} + h_{i,j+\frac{7}{2}} - h_{i,j-\frac{7}{2}} - 7\left(h_{i+\frac{5}{2},j} - h_{i-\frac{5}{2},j} + h_{i,j+\frac{5}{2}} - h_{i,j-\frac{5}{2}}\right)$$
$$+ 21\left(h_{i+\frac{3}{2},j} + h_{i-\frac{3}{2},j} + h_{i,j+\frac{3}{2}} + h_{i,j-\frac{3}{2}}\right) - 35\left(h_{i+\frac{1}{2},j} + h_{i,j+\frac{1}{2}}\right)\Big]$$

$$\nabla^8 h = h_{(8x)} + h_{(8y)}$$

$$= \frac{1}{(\Delta x)^8}\big[h_{i+4,j} - h_{i-4,j} + h_{i,j+4} - h_{i,j-4} - 8(h_{i+3,j} + h_{i-3,j} + h_{i,j+3} + h_{i,j-3})$$
$$+ 28(h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2}) - 56(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}) + 140h_{i,j}\big]$$

$$\frac{\partial h}{\partial t} = (-1)^{\frac{n}{2}+1} K_n \nabla^n h$$

2nd order

$$\frac{\partial h}{\partial t} = -K_2 \nabla^2 h$$
$$= K_2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)$$
$$= K_2\left[\frac{h_{i+1,j} + h_{i-1,j} - 2h_{i,j}}{(\Delta x)^2} + \frac{h_{i,j+1} + h_{i,j-1} - 2h_{i,j}}{(\Delta y)^2}\right]$$
$$K_2 = \frac{(\Delta x)^2}{8\Delta t}$$

While $\Delta x = \Delta y$,

$$\frac{\partial h}{\partial t} = \frac{(\Delta x)^2}{8\Delta t} \cdot \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j}}{(\Delta x)^2}$$
$$\int h dt = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j}}{8}$$

4th order

$$\frac{\partial h}{\partial t} = -K_4 \nabla^4 h$$

$$= -K_4\left(\frac{\partial^4 h}{\partial x^4} + \frac{\partial^4 h}{\partial y^4}\right)$$

$$= -K_4\left[\frac{(h_{i+2,j} + h_{i-2,j}) - 4\left(h_{i+1,j} + h_{i-1,j}\right) + 6h_{i,j}}{(\Delta x)^4} + \frac{(h_{i,j+2} + h_{i,j-2}) - 4\left(h_{i,j+1} + h_{i,j-1}\right) + 6h_{i,j}}{(\Delta y)^4}\right]$$

$$K_4 = \frac{(\Delta x)^4}{32\Delta t}$$

While $\Delta x = \Delta y$,

$$\int hdt = -\frac{h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2} - 4\left(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}\right) + 12h_{i,j}}{32}$$

6th order

$$\frac{\partial h}{\partial t} = -K_6 \nabla^6 h$$

$$= -K_6\left(\frac{\partial^6 h}{\partial x^6} + \frac{\partial^6 h}{\partial y^6}\right)$$

$$= K_6\left[\frac{(h_{i+3,j} + h_{i-3,j}) - 6\left(h_{i+2,j} + h_{i-2,j}\right) + 15\left(h_{i+1,j} + h_{i-1,j}\right) - 20h_{i,j}}{(\Delta x)^6} + \frac{(h_{i,j+3} + h_{i,j-3}) - 6\left(h_{i,j+2} + h_{i,j-2}\right) + 15\left(h_{i,j+1} + h_{i,j-1}\right) - 20h_{i,j}}{(\Delta y)^6}\right]$$

$$K_6 = \frac{(\Delta x)^6}{128\Delta t}$$

While $\Delta x = \Delta y$,

$$\int hdt = \frac{h_{i+3,j} + h_{i-3,j} + h_{i,j+3} + h_{i,j-3} - 6\left(h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2}\right) + 15\left(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}\right) - 40h_{i,j}}{128}$$

8th order

$$\frac{\partial h}{\partial t} = -K_8 \nabla^8 h$$

$$= -K_8\left[h_{i+4,j} - h_{i-4,j} + h_{i,j+4} - h_{i,j-4} - 8\left(h_{i+3,j} + h_{i-3,j} + h_{i,j+3} + h_{i,j-3}\right) + 28\left(h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2}\right) - 56\left(h_{i+1,j} + h_{i,j+1}\right) + 70h_{i,j}\right]$$

$$K_8 = \frac{(\Delta x)^8}{512\Delta t}$$

While $\Delta x = \Delta y$,

$$\int hdt = -\frac{1}{512}\left[h_{i+4,j} - h_{i-4,j} + h_{i,j+4} - h_{i,j-4} - 8\left(h_{i+3,j} + h_{i-3,j} + h_{i,j+3} + h_{i,j-3}\right) + 28\left(h_{i+2,j} + h_{i-2,j} + h_{i,j+2} + h_{i,j-2}\right) - 56\left(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}\right) + 140h_{i,j}\right]$$

Advection Format Equation to Flux Format Equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = -m\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) + f^{*}v - mg\dfrac{\partial z}{\partial x} \\ \dfrac{\partial v}{\partial t} = -m\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) - f^{*}u - mg\dfrac{\partial z}{\partial y} \\ \dfrac{\partial z}{\partial t} = -m^{2}\left[\dfrac{\partial}{\partial x}\left(\dfrac{zu}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zv}{m}\right)\right] \end{cases}$$

Where $f^{*} = f + m^{2}\left[v\frac{\partial}{\partial x}\left(\frac{1}{m}\right) - u\frac{\partial}{\partial y}\left(\frac{1}{m}\right)\right] = f + u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}$

For u direction,

$$\frac{\partial zu}{\partial t} = u\frac{\partial z}{\partial t} + z\frac{\partial u}{\partial t}$$

Therefore,

$$\frac{\partial u}{\partial t} = \frac{1}{z}\left(\frac{\partial zu}{\partial t} - u\frac{\partial z}{\partial t}\right)$$

$$\frac{1}{z}\left(\frac{\partial zu}{\partial t} - u\frac{\partial z}{\partial t}\right) = -m\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + f^{*}v - mg\frac{\partial z}{\partial x}$$

$$\frac{\partial zu}{\partial t} = u\frac{\partial z}{\partial t} - mz\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

Because,

$$\frac{\partial z}{\partial t} = -m^{2}\left[\frac{\partial}{\partial x}\left(\frac{zu}{m}\right) + \frac{\partial}{\partial y}\left(\frac{zv}{m}\right)\right]$$

Therefore,

$$\frac{\partial zu}{\partial t} = u\left\{-m^{2}\left[\frac{\partial}{\partial x}\left(\frac{zu}{m}\right) + \frac{\partial}{\partial y}\left(\frac{zv}{m}\right)\right]\right\} - mz\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

$$\frac{\partial zu}{\partial t} = -m^{2}\left[u\frac{\partial}{\partial x}\left(\frac{zu}{m}\right) + u\frac{\partial}{\partial y}\left(\frac{zv}{m}\right) + \frac{z}{m}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right] + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

Let $z^{*} = \frac{z}{m}$

$$\frac{\partial zu}{\partial t} = -m^{2}\left[u\frac{\partial}{\partial x}(z^{*}u) + u\frac{\partial}{\partial y}(z^{*}v) + z^{*}u\frac{\partial u}{\partial x} + z^{*}v\frac{\partial u}{\partial y}\right] + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

Let $p = z^{*}u, q = z^{*}v$

$$\frac{\partial zu}{\partial t} = -m^{2}\left(u\frac{\partial p}{\partial x} + p\frac{\partial u}{\partial x} + u\frac{\partial q}{\partial y} + q\frac{\partial u}{\partial y}\right) + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

$$\frac{\partial zu}{\partial t} = -m^{2}\left(\frac{\partial pu}{\partial x} + \frac{\partial qu}{\partial y}\right) + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

Therefore,

$$\frac{\partial zu}{\partial t} = -m^{2}\left[\frac{\partial}{\partial x}\left(\frac{zu^{2}}{m}\right) + \frac{\partial}{\partial y}\left(\frac{zuv}{m}\right)\right] + zf^{*}v - mgz\frac{\partial z}{\partial x}$$

By using the same method on v direction, we can get the flux format equation group

$$\begin{cases} \dfrac{\partial zu}{\partial t} = -m^{2}\left[\dfrac{\partial}{\partial x}\left(\dfrac{zu^{2}}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zuv}{m}\right)\right] + zf^{*}v - mgz\dfrac{\partial z}{\partial x} \\ \dfrac{\partial zv}{\partial t} = -m^{2}\left[\dfrac{\partial}{\partial x}\left(\dfrac{zuv}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zv^{2}}{m}\right)\right] - zf^{*}u - mgz\dfrac{\partial z}{\partial y} \\ \dfrac{\partial z}{\partial t} = -m^{2}\left[\dfrac{\partial}{\partial x}\left(\dfrac{zu}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zv}{m}\right)\right] \end{cases}$$

Energy Conservation Format

Governing Equations

$$
\begin{cases}
\dfrac{\partial zu}{\partial t} = -m^2\left[\dfrac{\partial}{\partial x}\left(\dfrac{zu^2}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zuv}{m}\right) + g\dfrac{z}{m}\dfrac{\partial z}{\partial x}\right] + \left(f + u\dfrac{\partial m}{\partial y} - v\dfrac{\partial m}{\partial x}\right) zv \\
\dfrac{\partial zv}{\partial t} = -m^2\left[\dfrac{\partial}{\partial x}\left(\dfrac{zuv}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zv^2}{m}\right) + g\dfrac{z}{m}\dfrac{\partial z}{\partial y}\right] - \left(f + u\dfrac{\partial m}{\partial y} - v\dfrac{\partial m}{\partial x}\right) zu \\
\dfrac{\partial z}{\partial t} = -m^2\left[\dfrac{\partial}{\partial x}\left(\dfrac{zu}{m}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{zv}{m}\right)\right]
\end{cases}
$$

Discrete Equations

$$
\begin{cases}
\dfrac{\partial \overline{z}^x u}{\partial t} = -m_u^2\left[\left(\overline{\overline{z^*}^x u}^x\ \overline{u}^x\right)_x + \left(\overline{\overline{z^*}^y v}^x\ \overline{u}^y\right)_y + g\overline{z^*}^x z_x\right] + \left(f_u + \overline{\overline{u}^x \delta_y m}^x - \overline{\overline{v}^y \delta_x m}^x\right)\overline{\overline{z}^y v}^{xy} \\
\dfrac{\partial \overline{z}^y v}{\partial t} = -m_v^2\left[\left(\overline{\overline{z^*}^x u}^y\ \overline{v}^x\right)_x + \left(\overline{\overline{z^*}^y v}^y\ \overline{v}^y\right)_y + g\overline{z^*}^y z_y\right] - \left(f_v + \overline{\overline{u}^x \delta_y m}^y - \overline{\overline{v}^y \delta_x m}^y\right)\overline{\overline{z}^x u}^{xy} \\
\dfrac{\partial z}{\partial t} = -m^2\left[\left(\overline{z^*}^x u\right)_x + \left(\overline{z^*}^y v\right)_y\right]
\end{cases}
$$

Discrete Terms

U-direction momentum

$$
\overline{\overline{z^*}^x u}^x\ \overline{u}^x = \frac{1}{8}\left[\left(z^*_{i+1,j} + z^*_{i,j}\right)u_{i+\frac{1}{2},j} + \left(z^*_{i,j} + z^*_{i-1,j}\right)u_{i-\frac{1}{2},j}\right]\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)
$$

$$
\begin{aligned}
\left(\overline{\overline{z^*}^x u}^x\ \overline{u}^x\right)_x = \frac{1}{8d}\Big\{&\left[\left(z^*{}_{i+\frac{3}{2},j} + z^*{}_{i+\frac{1}{2},j}\right)u_{i+1,j} + \left(z^*{}_{i+\frac{1}{2},j} + z^*{}_{i-\frac{1}{2},j}\right)u_{i,j}\right]\left(u_{i+1,j} + u_{i,j}\right) \\
&- \left[\left(z^*{}_{i+\frac{1}{2},j} + z^*{}_{i-\frac{1}{2},j}\right)u_{i,j} + \left(z^*{}_{i-\frac{1}{2},j} + z^*{}_{i-\frac{3}{2},j}\right)u_{i-1,j}\right]\left(u_{i,j} + u_{i-1,j}\right)\Big\}
\end{aligned}
$$

$$
\overline{\overline{z^*}^y v}^x\ \overline{u}^y = \frac{1}{8}\left[\left(z^*_{i+\frac{1}{2},j+\frac{1}{2}} + z^*_{i+\frac{1}{2},j-\frac{1}{2}}\right)v_{i+\frac{1}{2},j} + \left(z^*_{i-\frac{1}{2},j+\frac{1}{2}} + z^*_{i-\frac{1}{2},j-\frac{1}{2}}\right)v_{i-\frac{1}{2},j}\right]\left(u_{i,j+\frac{1}{2}} + u_{i,j-\frac{1}{2}}\right)
$$

$$
\begin{aligned}
\left(\overline{\overline{z^*}^y v}^x\ \overline{u}^y\right)_y = \frac{1}{8d}\Big\{&\left[\left(z^*{}_{i+\frac{1}{2},j+1} + z^*{}_{i+\frac{1}{2},j}\right)v_{i+\frac{1}{2},j+\frac{1}{2}} + \left(z^*{}_{i-\frac{1}{2},j+1} + z^*{}_{i-\frac{1}{2},j}\right)v_{i-\frac{1}{2},j+\frac{1}{2}}\right]\left(u_{i,j+1} + u_{i,j}\right) \\
&- \left[\left(z^*{}_{i+\frac{1}{2},j} + z^*{}_{i+\frac{1}{2},j-1}\right)v_{i+\frac{1}{2},j-\frac{1}{2}} + \left(z^*{}_{i-\frac{1}{2},j} + z^*{}_{i-\frac{1}{2},j-1}\right)v_{i-\frac{1}{2},j-\frac{1}{2}}\right]\left(u_{i,j} + u_{i,j-1}\right)\Big\}
\end{aligned}
$$

$$
g\overline{z^*}^x z_x = \frac{g}{2d}\left(z^*_{i+\frac{1}{2},j} + z^*_{i-\frac{1}{2},j}\right)\left(z_{i+\frac{1}{2},j} - z_{i-\frac{1}{2},j}\right)
$$

$$
\overline{\overline{u}^x \delta_y m}^x = \frac{1}{4d}\left[\left(u_{i+1,j} + u_{i,j}\right)\left(m_{i+\frac{1}{2},j+\frac{1}{2}} - m_{i+\frac{1}{2},j-\frac{1}{2}}\right) + \left(u_{i,j} + u_{i-1,j}\right)\left(m_{i-\frac{1}{2},j+\frac{1}{2}} - m_{i-\frac{1}{2},j-\frac{1}{2}}\right)\right]
$$

$$
\overline{\overline{v}^y \delta_x m}^x = \frac{1}{4d}\left[\left(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i+\frac{1}{2},j-\frac{1}{2}}\right)\left(m_{i+1,j} - m_{i,j}\right) + \left(v_{i-\frac{1}{2},j+\frac{1}{2}} + v_{i-\frac{1}{2},j-\frac{1}{2}}\right)\left(m_{i,j} - m_{i-1,j}\right)\right]
$$

$$\overline{\overline{z}^y v}^x = \frac{1}{4}\left[\left(z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}}\right)v_{i+\frac{1}{2},j} + \left(z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{1}{2}}\right)v_{i-\frac{1}{2},j}\right]$$

$$\overline{\overline{z}^y v}^{xy} = \frac{1}{8}\left[\left(z_{i+\frac{1}{2},j+1} + z_{i+\frac{1}{2},j}\right)v_{i+\frac{1}{2},j+\frac{1}{2}} + \left(z_{i-\frac{1}{2},j+1} + z_{i-\frac{1}{2},j}\right)v_{i-\frac{1}{2},j+\frac{1}{2}} + \left(z_{i+\frac{1}{2},j} + z_{i+\frac{1}{2},j-1}\right)v_{i+\frac{1}{2},j-\frac{1}{2}} + \left(z_{i-\frac{1}{2},j} + z_{i-\frac{1}{2},j-1}\right)v_{i-\frac{1}{2},j-\frac{1}{2}}\right]$$

V-direction momentum

$$\overline{\overline{z^*}^x u}^y \overline{v}^x = \frac{1}{8}\left[\left(z^*_{i+\frac{1}{2},j+\frac{1}{2}} + z^*_{i-\frac{1}{2},j+\frac{1}{2}}\right)u_{i,j+\frac{1}{2}} + \left(z^*_{i+\frac{1}{2},j-\frac{1}{2}} + z^*_{i-\frac{1}{2},j-\frac{1}{2}}\right)u_{i,j-\frac{1}{2}}\right]\left(v_{i+\frac{1}{2},j} + v_{i-\frac{1}{2},j}\right)$$

$$\left(\overline{\overline{z^*}^x u}^y \overline{v}^x\right)_x = \frac{1}{8d}\left\{\left[\left(z^*_{i+1,j+\frac{1}{2}} + z^*_{i,j+\frac{1}{2}}\right)u_{i+\frac{1}{2},j+\frac{1}{2}} + \left(z^*_{i+1,j-\frac{1}{2}} + z^*_{i,j-\frac{1}{2}}\right)u_{i+\frac{1}{2},j-\frac{1}{2}}\right]\left(v_{i+1,j} + v_{i,j}\right) - \left[\left(z^*_{i,j+\frac{1}{2}} + z^*_{i-1,j+\frac{1}{2}}\right)u_{i-\frac{1}{2},j+\frac{1}{2}} + \left(z^*_{i,j-\frac{1}{2}} + z^*_{i-1,j-\frac{1}{2}}\right)u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\left(v_{i,j} + v_{i-1,j}\right)\right\}$$

$$\overline{\overline{z^*}^y v}^y \overline{v}^y = \frac{1}{8}\left[\left(z^*_{i,j+1} + z^*_{i,j}\right)v_{i,j+\frac{1}{2}} + \left(z^*_{i,j} + z^*_{i,j-1}\right)v_{i,j-\frac{1}{2}}\right]\left(v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}\right)$$

$$\left(\overline{\overline{z^*}^y v}^y \overline{v}^y\right)_y = \frac{1}{8d}\left\{\left[\left(z^*_{i,j+\frac{3}{2}} + z^*_{i,j+\frac{1}{2}}\right)v_{i,j+1} + \left(z^*_{i,j+\frac{1}{2}} + z^*_{i,j-\frac{1}{2}}\right)v_{i,j}\right]\left(v_{i,j+1} + v_{i,j}\right) - \left[\left(z^*_{i,j+\frac{1}{2}} + z^*_{i,j-\frac{1}{2}}\right)v_{i,j} + \left(z^*_{i,j-\frac{1}{2}} + z^*_{i,j-\frac{3}{2}}\right)v_{i,j-1}\right]\left(v_{i,j} + v_{i,j-1}\right)\right\}$$

$$g\overline{z^*}^y z_y = \frac{g}{2d}\left(z^*_{i,j+\frac{1}{2}} + z^*_{i,j-\frac{1}{2}}\right)\left(z_{i,j+\frac{1}{2}} - z_{i,j-\frac{1}{2}}\right)$$

$$\overline{\overline{u}^x \delta_y m}^y = \frac{1}{4d}\left[\left(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i-\frac{1}{2},j+\frac{1}{2}}\right)\left(m_{i,j+1} - m_{i,j}\right) + \left(u_{i+\frac{1}{2},j-\frac{1}{2}} + u_{i-\frac{1}{2},j-\frac{1}{2}}\right)\left(m_{i,j} - m_{i,j-1}\right)\right]$$

$$\overline{\overline{v}^y \delta_x m}^y = \frac{1}{4d}\left[\left(v_{i,j+1} + v_{i,j}\right)\left(m_{i+\frac{1}{2},j+\frac{1}{2}} - m_{i-\frac{1}{2},j+\frac{1}{2}}\right) + \left(v_{i,j} + v_{i,j-1}\right)\left(m_{i+\frac{1}{2},j-\frac{1}{2}} - m_{i-\frac{1}{2},j-\frac{1}{2}}\right)\right]$$

$$\overline{\overline{z}^x u}^{xy} = \frac{1}{8}\left[\left(z_{i+1,j+\frac{1}{2}} + z_{i,j+\frac{1}{2}}\right)u_{i+\frac{1}{2},j+\frac{1}{2}} + \left(z_{i+1,j-\frac{1}{2}} + z_{i,j-\frac{1}{2}}\right)u_{i+\frac{1}{2},j-\frac{1}{2}} + \left(z_{i,j+\frac{1}{2}} + z_{i-1,j+\frac{1}{2}}\right)u_{i-\frac{1}{2},j+\frac{1}{2}} + \left(z_{i,j-\frac{1}{2}} + z_{i-1,j-\frac{1}{2}}\right)u_{i-\frac{1}{2},j-\frac{1}{2}}\right]$$

Z equation

$$\left(\overline{z^*}^x u\right)_x = \frac{1}{2d}\left[\left(z^*_{i+1,j} + z^*_{i,j}\right)u_{i+\frac{1}{2},j} - \left(z^*_{i,j} + z^*_{i-1,j}\right)u_{i-\frac{1}{2},j}\right]$$

$$\left(\overline{z^*}^y v\right)_y = \frac{1}{2d}\left[\left(z^*_{i,j+1} + z^*_{i,j}\right)v_{i,j+\frac{1}{2}} - \left(z^*_{i,j} + z^*_{i,j-1}\right)v_{i,j-\frac{1}{2}}\right]$$

On U grid

$$\left(\overline{\overline{z^*}^x u}^x \overline{u}^x\right)_x = \frac{1}{8d}\left\{\left[\left(z^*_{i+1,j} + z^*_{i,j}\right)U_{i+1,j} + \left(z^*_{i,j} + z^*_{i-1,j}\right)U_{i,j}\right]\left(U_{i+1,j} + U_{i,j}\right) - \left[\left(z^*_{i,j} + z^*_{i-1,j}\right)U_{i,j} + \left(z^*_{i-1,j} + z^*_{i-2,j}\right)U_{i-1,j}\right]\left(U_{i,j} + U_{i-1,j}\right)\right\}$$

$$\left(\overline{\overline{z^*}^y v}^x \overline{u}^y\right)_y = \frac{1}{8d}\{[(z^*{}_{i,j+1} + z^*{}_{i,j})V_{i,j+1} + (z^*{}_{i-1,j+1} + z^*{}_{i-1,j})V_{i-1,j+1}](U_{i,j+1} + U_{i,j})$$

$$- [(z^*{}_{i,j} + z^*{}_{i,j-1})V_{i,j} + (z^*{}_{i-1,j} + z^*{}_{i-1,j-1})V_{i-1,j}](U_{i,j} + U_{i,j-1})\}$$

$$g\overline{z^*}^x z_x = \frac{g}{2d}(z^*_{i,j} + z^*_{i-1,j})(z_{i,j} - z_{i-1,j})$$

$$\overline{\overline{u}^x \delta_y m}^x = \frac{1}{4d}\left[(U_{i+1,j} + U_{i,j})\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + (U_{i,j} + U_{i-1,j})\left(m_{v\,i-1,j+1} - m_{v_{i-1,j}}\right)\right]$$

$$\overline{\overline{v}^y \delta_x m}^x = \frac{1}{4d}\left[(V_{i,j+1} + V_{i,j})\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) + (V_{i-1,j+1} + V_{i-1,j})\left(m_{u_{i,j}} - m_{u_{i-1,j}}\right)\right]$$

$$\overline{\overline{z}^y v}^{xy} = \frac{1}{8}\{[(z_{i,j+1} + z_{i,j})V_{i,j+1} + (z_{i-1,j+1} + z_{i-1,j})V_{i-1,j+1}] - [(z_{i,j} + z_{i,j-1})V_{i,j} + (z_{i-1,j} + z_{i-1,j-1})V_{i-1,j}]\}$$

On V grid

$$\left(\overline{\overline{z^*}^x u}^y \overline{v}^x\right)_x = \frac{1}{8d}\{[(z^*{}_{i+1,j} + z^*{}_{i,j})U_{i+1,j} + (z^*{}_{i+1,j-1} + z^*{}_{i,j-1})U_{i+1,j-1}](V_{i+1,j} + V_{i,j})$$

$$- [(z^*{}_{i,j} + z^*{}_{i-1,j})U_{i,j} + (z^*{}_{i,j-1} + z^*{}_{i-1,j-1})U_{i,j-1}](V_{i,j} + V_{i-1,j})\}$$

$$\left(\overline{\overline{z^*}^y v}^y \overline{v}^y\right)_y = \frac{1}{8d}\{[(z^*{}_{i,j+1} + z^*{}_{i,j})V_{i,j+1} + (z^*{}_{i,j} + z^*{}_{i,j-1})V_{i,j}](V_{i,j+1} + V_{i,j})$$

$$- [(z^*{}_{i,j} + z^*{}_{i,j-1})V_{i,j} + (z^*{}_{i,j-1} + z^*{}_{i,j-2})V_{i,j-1}](V_{i,j} + V_{i,j-1})\}$$

$$g\overline{z^*}^y z_y = \frac{g}{2d}(z^*_{i,j} + z^*_{i,j-1})(z_{i,j} - z_{i,j-1})$$

$$\overline{\overline{u}^x \delta_y m}^y = \frac{1}{4d}\left[(U_{i+1,j} + U_{i,j})\left(m_{v_{i,j+1}} - m_{v_{i,j}}\right) + (U_{i+1,j-1} + U_{i,j-1})\left(m_{v_{i,j}} - m_{v_{i,j-1}}\right)\right]$$

$$\overline{\overline{v}^y \delta_x m}^y = \frac{1}{4d}\left[(V_{i,j+1} + V_{i,j})\left(m_{u_{i+1,j}} - m_{u_{i,j}}\right) + (V_{i,j} + V_{i,j-1})\left(m_{u_{i+1,j-1}} - m_{u_{i,j-1}}\right)\right]$$

$$\overline{\overline{z}^x u}^{xy} = \frac{1}{8}[(z_{i+1,j} + z_{i,j})U_{i+1,j} + (z_{i+1,j-1} + z_{i,j-1})U_{i+1j-1} + (z_{i,j} + z_{i-1,j})U_{i,j} + (z_{i,j-1} + z_{i-1,j-1})U_{i,j-1}]$$

On Z grid

$$\left(\overline{z^*}^x u\right)_x = \frac{1}{2d}[(z^*{}_{i+1,j} + z^*{}_{i,j})U_{i+1,j} - (z^*{}_{i,j} + z^*{}_{i-1,j})U_{i,j}]$$

$$\left(\overline{z^*}^y v\right)_y = \frac{1}{2d}[(z^*{}_{i,j+1} + z^*{}_{i,j})V_{i,j+1} - (z^*{}_{i,j} + z^*{}_{i,j-1})V_{i,j}]$$

Completed Discrete Equations

$$\frac{\partial \overline{z}^x u}{\partial t} = -m_{u_{i,j}}^2 \langle \frac{1}{8d} \{ [(z^*{}_{i+1,j} + z^*{}_{i,j}) U_{i+1,j} + (z^*{}_{i,j} + z^*{}_{i-1,j}) U_{i,j}] (U_{i+1,j} + U_{i,j})$$
$$- [(z^*{}_{i,j} + z^*{}_{i-1,j}) U_{i,j} + (z^*{}_{i-1,j} + z^*{}_{i-2,j}) U_{i-1,j}] (U_{i,j} + U_{i-1,j})$$
$$+ [(z^*{}_{i,j+1} + z^*{}_{i,j}) V_{i,j+1} + (z^*{}_{i-1,j+1} + z^*{}_{i-1,j}) V_{i-1,j+1}] (U_{i,j+1} + U_{i,j})$$
$$- [(z^*{}_{i,j} + z^*{}_{i,j-1}) V_{i,j} + (z^*{}_{i-1,j} + z^*{}_{i-1,j-1}) V_{i-1,j}] (U_{i,j} + U_{i,j-1}) \} + \frac{g}{2d} (z^*_{i,j} + z^*_{i-1,j}) (z_{i,j} - z_{i-1,j}) \rangle$$
$$+ \frac{1}{8} \Big\{ f_{u_{i,j}}$$
$$+ \frac{1}{4d} \Big[(U_{i+1,j} + U_{i,j}) \left(m_{v_{i,j+1}} - m_{v_{i,j}} \right) + (U_{i,j} + U_{i-1,j}) \left(m_{v\ i-1,j+1} - m_{v_{i-1,j}} \right)$$
$$- (V_{i,j+1} + V_{i,j}) \left(m_{u_{i+1,j}} - m_{u_{i,j}} \right) - (V_{i-1,j+1} + V_{i-1,j}) \left(m_{u_{i,j}} - m_{u_{i-1,j}} \right) \Big] \Big\} \Big[(z_{i,j+1} + z_{i,j}) V_{i,j+1}$$
$$+ (z_{i-1,j+1} + z_{i-1,j}) V_{i-1,j+1} + (z_{i,j} + z_{i,j-1}) V_{i,j} + (z_{i-1,j} + z_{i-1,j-1}) V_{i-1,j} \Big]$$

$$\frac{\partial \overline{z}^y v}{\partial t} = -m_{v_{i,j}}^2 \langle \frac{1}{8d} \{ [(z^*{}_{i+1,j} + z^*{}_{i,j}) U_{i+1,j} + (z^*{}_{i+1,j-1} + z^*{}_{i,j-1}) U_{i+1,j-1}] (V_{i+1,j} + V_{i,j})$$
$$- [(z^*{}_{i,j} + z^*{}_{i-1,j}) U_{i,j} + (z^*{}_{i,j-1} + z^*{}_{i-1,j-1}) U_{i,j-1}] (V_{i,j} + V_{i-1,j})$$
$$+ [(z^*{}_{i,j+1} + z^*{}_{i,j}) V_{i,j+1} + (z^*{}_{i,j} + z^*{}_{i,j-1}) V_{i,j}] (V_{i,j+1} + V_{i,j})$$
$$- [(z^*{}_{i,j} + z^*{}_{i,j-1}) V_{i,j} + (z^*{}_{i,j-1} + z^*{}_{i,j-2}) V_{i,j-1}] (V_{i,j} + V_{i,j-1}) \} + \frac{g}{2d} (z^*_{i,j} + z^*_{i,j-1}) (z_{i,j} - z_{i,j-1}) \rangle$$
$$- \frac{1}{8} \Big\{ f_{v_{i,j}}$$
$$+ \frac{1}{4d} \Big[(U_{i+1,j} + U_{i,j}) \left(m_{v_{i,j+1}} - m_{v_{i,j}} \right) + (U_{i+1,j-1} + U_{i,j-1}) \left(m_{v_{i,j}} - m_{v_{i,j-1}} \right)$$
$$- (V_{i,j+1} + V_{i,j}) \left(m_{u_{i+1,j}} - m_{u_{i,j}} \right) - (V_{i,j} + V_{i,j-1}) \left(m_{u_{i+1,j-1}} - m_{u_{i,j-1}} \right) \Big] \Big\} \Big[(z_{i+1,j} + z_{i,j}) U_{i+1,j}$$
$$+ (z_{i+1,j-1} + z_{i,j-1}) U_{i+1j-1} + (z_{i,j} + z_{i-1,j}) U_{i,j} + (z_{i,j-1} + z_{i-1,j-1}) U_{i,j-1} \Big]$$

$$\frac{\partial z}{\partial t} = -\frac{m_{i,j}^2}{2d} [(z^*{}_{i+1,j} + z^*{}_{i,j}) U_{i+1,j} - (z^*{}_{i,j} + z^*{}_{i-1,j}) U_{i,j} + (z^*{}_{i,j+1} + z^*{}_{i,j}) V_{i,j+1} - (z^*{}_{i,j} + z^*{}_{i,j-1}) V_{i,j}]$$