Control & Guidance of Multiple Air-Vehicle Systems (MAS)

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Part IV

Task Assignment Problem for UAVs.

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Chapter 1

Introduction

1.0.1 Specific problem formulation

General description : To assign air-vehicles to perform as many simultaneous service requests as possible.

Starting conditions :

Given :

1. The MAS is performing coverage
2. Miltipble requests for service with the following inormation:
   - Number of air-vehicles required
   - Location where air vehicles need to visit
   - Earliest time of 1-st visit
   - Latest time of 1-st visit
   - Minimum duration per visit
   - Maximum interval between visits
   - Time of last visit

Mission Objective :

Find :

1. Air-vehicle(s) to be assigned to request and the corresponding paths to take to the service request
2. Variations to requests with minimal change if the request cannot be met

That :

1. Maximises the number of service requests that can be serviced

Constraints :

Subject to :

1. Air-vehicle performance and dynamics
2. Sensor performance
3. Air to Air datalink performance
4. LOS occlusion in area of operations
5. At least one air-vehicle being directly connected to GCS 90 procents of the time (for MAS to be provided feedback to GCS)

From mathematical point of view this problem can be reformulated as extremal problem which is includes nonlinear systems, PDE and ODE, conflicting situation, vector cost functions, incomplete information, constraints. It is clear that effective algorithms for the solution of this problem must be effective if we use them to solve:

- optimization problems with one costs function;
- optimization problem for ODEs;
- optimization problem for linear ODEs;
- optimization problems for linear discrete processes;
  Thus the logic in consideration inevitable leads to
- linear programming.

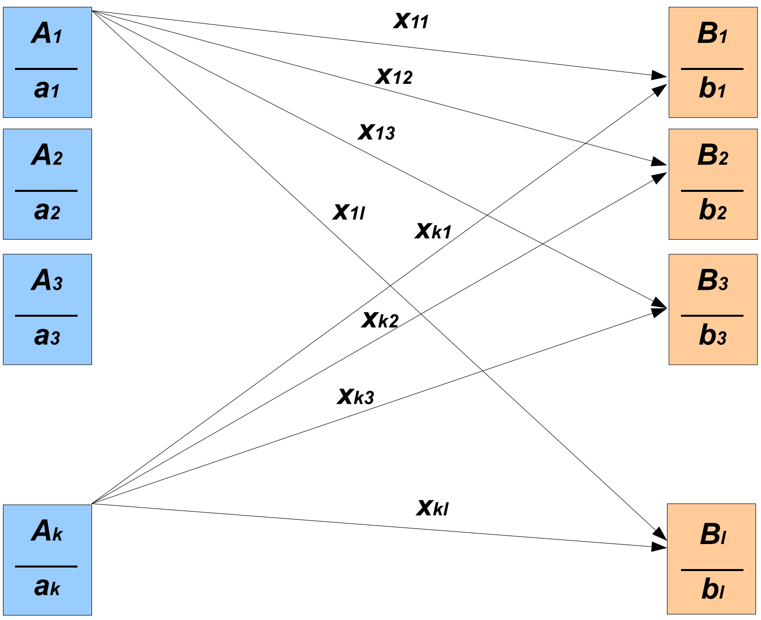
By this reason for the formulated problem above in a reduced form the special class of linear programming (LP) method will be considered in next chapter.

Chapter 2

LP assignment problem for group of UAVs

2.1 Formal problem statement

Assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$. Denote by $a_1, a_2, ..., a_i, ..., a_k$ their capacity, namely the maximal number of homogenous UAVs located in aerobase. Also we have l zones of area of operation $B_1, B_2, ..., B_j, ..., B_l$. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \ldots, l$.



Also assume that the sum of all requests are equal to the total number of available UAVs.

$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j \tag{2.1}$$

Next, for all i and j denote by c_{ij} the benefit of sending the UAV from i-th aerobase to j-th zone of area of operation. Note that the benefit of service can be given by different values, for example, by service time T_{ij}, fuel consumption, total number of UAVs involved etc. Then the problem is to define the plan of service for UAVs, in order to complete all incoming requests for service with maximal benefits. Denote by x_{ij} the number of UAVs from i-th aerobase which are send to service j-th zone. Then using our notation we can formulate the problem statement as the following integer programming problem: To find $x_{ij}, (i = 1, 2, ..., k; j = 1, 2, ..., l)$ such that, the total cost function for all services performed by all UAVs takes a maximal value

$$F = \sum_{i=1}^{k} \sum_{j=1}^{l} c_{ij} x_{ij} \to \min_{x_{ij}} \tag{2.2}$$

subject to

$$\begin{gathered}
\sum_{i=1}^{k} x_{ij} = b_j, \quad j = 1, 2, ..., l \\
\sum_{j=1}^{l} x_{ij} = a_i, \quad i = 1, 2, ..., k \\
\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j \\
x_{ij} \geq 0, \qquad x_{ij} \text{ are integer numbers.}
\end{gathered} \tag{2.3}$$

Here the first two constraints means that we are need to satisfy all service request from each area of operation and to use for that purpose all available UAVs.

Remark 1.

Also it should be noted that in case of inequalities constraints instead of 2.3, the problem can be reduced to the equality case by introducing an additional variables. For example, for $\sum_{i=1}^{k} a_i \geq \sum_{j=1}^{l} b_j$ we will need to introduce some "artificial" aerobase or area of operation with correspondent capacity(for aerobase) and requirements (for area of operation):

$$b_{l+1} = \sum_{i=1}^{k} a_i - \sum_{j=1}^{l} b_j$$
$$a_{k+1} = \sum_{j=1}^{l} b_j - \sum_{i=1}^{k} a_i$$

According to the given Remark 1 it is assumed below that the constraints of (2.3) hold as equalities.

Remark 2. There are a lot of methods to solve such kind of problem. In this report the formulated simple integer optimization problem will be extended to search service schedular for dynamical case of service requests. For this purpose a new supporting optimization method will be developed on the base of the so-called constructive approach by Gabasov R. and Kirillova F.M. [**?**]. For linear programming problem this method was compared with classical simplex method in the paper [**?**], also was applied in gas industry [**?**] The proposed method allows to use both initial information and current one produced during the solution process. This together with developed for this case duality theory leads to high efficiency of the associated numerical algorithms. Here we will present a specific case of simplex method developed for assignment problem and transportation problems. This method include the following basic steps:
1. To find initial plan,
2. Check optimality condition for that plan,
3. Construct the improved plan in case of nonoptimality.
Next we will demonstrate a key elements of this method.

2.1.1 Illustrative example

Assume that we have 3 airbases located at Changi A_1 with 3 UAVs ($a_1 = 3$) , Jurong West A_2 with 3 UAVs ($a_2 = 3$) , and Woodland A_3 with 1 UAV ($a_1 = 1$). Now 7 UAVs are requested from B_1-Raffles Place ($b_1 = 2$), B_2-Jurong Island ($b_2 = 2$), and B_3- Sentosa Island ($b_3 = 3$). Out task is to complete all requests in order to maximaze the total service time in the zones.

The mathematical problem statement of this task we will formulate using the following notations:

$A_i, i = 1, ..., k$- aerobases,
a_i - number of UAVs located in A_i,
$B_j, j = 1, ..., l$- areas of operations,

b_j- numbers of UAVs for service of B_j
d_{ij}- distance from A_i to B_j,
v_{ij}- UAVs speed
x_{ij}-number of UAVs from A_i to B_j,
h_i- UAVs endurance located on A_i aerobase

Also we assume that the following information are known: The distances between A_i and B_j given below in kilometers:

	A_1	A_2	A_3
A_1	0	32	22
A_2	32	0	17
A_3	22	17	0

(2.4)

Table 1: Distances between airbases A_i

	B_1	B_2	B_3
B_1	0	17	6
B_2	17	0	14
B_3	6	14	0

(2.5)

Table 2: Distances between zones B_j

	B_1	B_2	B_3
A_1	13	30	18
A_2	16	9	17
A_3	21	20	23

(2.6)

Table 3: Distances between aerobases A_i and area of operations B_j

The speed of UAVs are fixed $v_{ij} = 30\frac{m}{sec}$.

Next, for all i and j denote by $c_{ij} = \frac{d_{ij}}{v_{ij}}$ the benefit of sending the UAV from i-th aerobase to j-th zone of area of operation. The benefit means the flight time from $A_i \to B_j$.

	B_1	B_2	B_3
A_1	433	1000	600
A_2	533	300	566
A_3	700	666	766

(2.7)

Table 4: UAVs flight time from $A_i \to B_j$ (2.8)

Then using our notation we can formulate the problem statement as the following integer programming problem: To find $x_{ij}, (i = 1, 2, 3; j = 1, 2, 3)$ such that, the total service time performed by all UAVs takes a maximal value

$$T^{service} = \sum_{i=1}^{3} h_i - 2 \min_{x_{ij}} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{d_{ij}}{v_{ij}} x_{ij} \to \max_{x_{ij}} \tag{2.9}$$

Remark l:

The service time for each UAVs is equal to their endurance h_i minus the time needed to reach the preassigned zone and come back to the base. Thus the total service time of the group of UAVs involved in the mission is given by (2.9). Hence the total service time of the group of UAVs involved in the mission will be maximal if the total flight time to reach the preassigned zones is minimal

$$F = \sum_{i=1}^{3} \sum_{j=1}^{3} t_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} x_{ij} \to \min_{x_{ij}} \quad \text{where } c_{ij} = \frac{d_{ij}}{v_{ij}} \tag{2.10}$$

Then we can consider the following optimization problem:

$$F = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} \to \min_{x_{ij}} \tag{2.11}$$

$$\text{subject to} \tag{2.12}$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3$$

$$\sum_{j=1}^{3} x_{ij} = a_i, \quad i = 1,2,3 \tag{2.13}$$

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j \tag{2.14}$$

$$x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}. \tag{2.15}$$

where $A_i, i = 1,2,3$- number of aerobases,
$a_1 = 3, a_2 = 3, a_3 = 1$ - number of UAVs located in A_i,
$B_j, j = 1,2,3$- areas of operations,
$b_1 = 2, b_2 = 2, a_3 = 3$- numbers of UAVs for service of B_j
d_{ij}- distances from A_i to B_j given in table 3 , $v_{ij} = 30\frac{m}{sec}$
x_{ij}-number of UAVs from A_i to B_j
c_{ij}- given in table 4.

The first two constraints means that we are need to satisfy all service request from each area of operation and to use for that purpose all available UAVs. The last constraint means that the sum of all requests are equal to the total number of available UAVs.

The condition of that problem can be represented in table form:

	B_1	B_2	B_3	a_i
A_1	x_{11}	x_{12}	x_{13}	$a_1 = 3$
A_2	x_{21}	x_{22}	x_{23}	$a_2 = 3$
A_3	x_{31}	x_{32}	x_{33}	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

Below we give the detailed step-by-step procedure to determine the optimal solution.

2.1.2 Initial feasible solution

To construct the initial feasible solution we will use "North-West corner" method. The construction of the initial supporting feasible solution consist from the several steps on each of them are filled either a row or a table column. The procedure begins with the left top ("northwest") element $x_{11} = \min(a_1; b_1)$ of the plan. If $a_1 < b_1$, i.e. $x_{11} = a_1$, than from the further consideration we eliminate all elements from the first row. If $a_1 \geq b_1$, i.e. $x_{11} = b_1$, than all elements from the first column are eliminated. In the case $a_1 < b_1$ the next element of feasible solution will be chosen from the second row by the rule $x_{21} = \min(a_2; b_1 - a_1)$. Next, if $a_2 < b_1$, i.e. $x_{21} = a_2$, and in this case we eliminated from our further consideration all elements from the second row. If $a_2 \geq b_1 - a_1$, i.e. $x_{21} = b_1 - a_1$, and further we will not consider the elements from the first column. The next assignment will be made on the intersection of the second column and second row as follows: $x_{22} = \min(a_1 + a_2 - b_1; b_2)$. Then repeated this procedure we will find all elements of the initial supporting feasible solution.
In our case we have the following:

$$
\begin{aligned}
x_{11} &= \min(a_1; b_1) = \min(3; 2) = 2 \\
\Downarrow & \\
x_{12} &= \min(a_1 - b_1; a_2) = \min(1; 2) = 1 \\
\Downarrow & \\
x_{22} &= \min(a_2; b_2 + b_1 - a_!) = \min(3; 1) = 1 \\
\Downarrow & \\
x_{23} &= \min(a_2 + a_1 - b_1 - b_2; b_3) = \min(2; 3) = 2 \\
\Downarrow & \\
x_{33} &= \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2)) = \min(1; 1) = 1
\end{aligned}
\tag{2.16}
$$

	B_1	B_2	B_3	a_i
A_1	$x_{11} = 2$	$x_{12} = 1$		$a_1 = 3$
A_2		$x_{22} = 1$	$x_{23} = 2$	$a_2 = 3$
A_3			$x_{33} = 1$	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

Next calculate the value of our cost function:

$$F = 433 \times 2 + 1000 \times 1 + 300 \times 1 + 566 \times 2 + 766 \times 1 = 4061 \ seconds \approx 67.7 \ minutes$$

2.1.3 Optimality condition

We will use the so called method of potentials, also known as "$u - \nu$" method . Consider auxiliary numbers $u_1, u_2, ..., u_k$ and $\nu_1, \nu_2, ..., \nu_l$. For any admissible solution the value $\sum\limits_{i=1}^{k}\sum\limits_{j=l}^{l}(u_i + \nu_j)x_{ij}$ is the same and constant:

$$\sum_{i=1}^{k}\sum_{j=1}^{l}(u_i + \nu_j)x_{ij} = \sum_{i=1}^{k} u_i \sum_{j=1}^{l} x_{ij} + \sum_{j=1}^{l} \nu_j \sum_{i=1}^{k} x_{ij} = \sum_{i=1}^{k} u_i a_i + \sum_{j=1}^{l} \nu_j b_j = C$$

Next, assume that for some admissible solution we found the numbers u_i and ν_j such that the following conditions

$$\begin{aligned} u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0, \\ u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0 \end{aligned} \tag{2.17}$$

hold.

The solution is called potential solution if it satisfies to condition (2.17) and the sum $u_i + \nu_j = \bar{c_{ij}}$ called pseudocost. Then the condition for potential solution can be rewritten (2.17) as

$$\begin{aligned} \bar{c_{ij}} - c_{ij} = 0, \ for \ x_{ij} > 0, \\ \bar{c_{ij}} - c_{ij} \leq 0, \ for \ x_{ij} = 0 \end{aligned} \tag{2.18}$$

Let us check the optimality condition for our problem. Consider the following table

	B_1	B_2	B_3	a_i	u_i
A_1	$\bar{c_{11}} = 433$ $c_{11} = 433$ $x_{11} = 2$	1000 1000 1		3	
A_2		330 330 1	566 566 2	3	
A_3			766 766 1	1	
b_j	2	2	3	F=4064	
ν_j					

Then we should find potential u_i and ν_j such that for $x_{ij} > 0$ the condition $c_{ij} = u_i + \nu_j$ hold. One of the potentials can be chosen arbitrary.

Let $\nu_1 = 0$, since $u_1 + \nu_1 = 433$ then $u_1 = 433$. Next following this logic we found step by step:

$$\nu_2 + u_1 = 1000 \longrightarrow \nu_2 = 1000 - 433 = 567,$$
$$\nu_2 + u_2 = 300 \longrightarrow u_2 = 300 - 567 = -267,$$
$$u_2 + \nu_3 = 566 \longrightarrow \nu_3 = 566 + 267 = 833,$$
$$\nu_3 + u_2 = 766 \longrightarrow u_3 = 766 - 833 = -67$$

Than we will have the following table:

	B_1	B_2	B_3	a_i	u_i
A_1	$\bar{c_{11}} = 433$ $c_{11} = 433$ $x_{11} = 2$	1000 1000 1		3	433
A_2		330 330 1	566 566 2	3	-267
A_3			766 766 1	1	-67
b_j	2	2	3	F=4064	
ν_j	0	567	833		

Now we are ready to check our initial supporting feasible solution for optimality. Namely to check the condition $\bar{c_{ij}} - c_{ij} \leq 0$ for $x_{ij} = 0$.

$\bar{c_{ij}} = u_i + \nu_j \longrightarrow \bar{c_{21}} = -267$
$\bar{c_{13}} = 433 + 833 = 1266,$

$\bar{c_{31}} = 0 - 67 = -67,$
$\bar{c_{32}} = 567 - 67 = 500$

Then in matrix of estimates $\Delta = c_{ij} - \bar{c_{ij}} = \begin{pmatrix} 0 & 0 & -666 \\ 800 & 0 & 0 \\ 767 & 166 & 0 \end{pmatrix}$ find a minimal element $\Delta_{13} = -666 = \min\limits_{i,j} \Delta_{ij}$.

In our case for one zero component of our feasible solution this conditions are not satisfied. Hence our solution is not optimal.

2.1.4 Improvement of the feasible solution

Change the initial feasible solution by adding the value *theta* to element x_{13} with some corrections of other elements too.

	B_1	B_2	B_3	a_i
A_1	2	$1-\theta$	θ	$a_1 = 3$
A_2		$1+\theta$	$2-\theta$	$a_2 = 3$
A_3			1	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum\limits_{i=1}^{3} a_i = \sum\limits_{j=1}^{3} b_j = 7$

Find the value $\theta = \min(2-\theta, 1-\theta) = 0 \longrightarrow \theta = 1$. Then we will have the following new feasible solution:

	B_1	B_2	B_3	a_i
A_1	2		1	$a_1 = 3$
A_2		2	1	$a_2 = 3$
A_3			1	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum\limits_{i=1}^{3} a_i = \sum\limits_{j=1}^{3} b_j = 7$

Next calculate the value of our cost function:

$$F = 433 \times 2 + 600 \times 1 + 300 \times 2 + 566 \times 1 + 766 \times 1 = 3398 \; seconds \approx 56.6 \; minutes$$

Now, we are need to repeat the described procedure again, namely we will need to calculate new potentials:

Let $\nu_1 = 0$, since $u_1 + \nu_1 = 433$ then $u_1 = 433$. Next following this logic we found step by step:

$$\nu_3 + u_1 = 600 \longrightarrow \nu_3 = 600 - 433 = 167,$$

$$\nu_3 + u_2 = 566 \longrightarrow u_2 = 567 - 167 = 389,$$

$$\nu_2 + u_2 = 300 \longrightarrow \nu_2 = 300 + 399 = -99,$$

$$\nu_3 + u_3 = 766 \longrightarrow u_3 = 766 - 167 = 599$$

Than we will have the following table:

	B_1	B_2	B_3	a_i	u_i
A_1	$\bar{c_{11}} = 433$ $c_{11} = 433$ $x_{11} = 2$		600 600 1	3	433
A_2		330 330 2	566 566 1	3	399
A_3			766 766 1	1	599
b_j	2	2	3	F=3398	
ν_j	0	-99	167		

Now we are ready to check our supporting feasible solution for optimality. Namely to check the condition $\bar{c_{ij}} - c_{ij} \leq 0$ for $x_{ij} = 0$.

$\bar{c_{ij}} = u_i + \nu_j \longrightarrow \bar{c_{12}} = u_1 + \nu_2 = 433 - 99 = 334$
$\bar{c_{21}} = 399 + 0 = 399,$
$\bar{c_{31}} = 599 + 0 = 599,$
$\bar{c_{32}} = 599 - 99 = 500$

Then in matrix of estimates $\Delta = c_{ij} - \bar{c_{ij}} = \begin{pmatrix} 0 & 666 & 0 \\ 134 & 0 & 0 \\ 101 & 166 & 0 \end{pmatrix}$

The optimality conditions are satisfied, since $\forall \Delta_{ij} \geq 0$.

Optimal solution are

$$\begin{aligned} & x_{11} = 2; \quad x_{13} = 1; \\ & x_{22} = 2; \quad x_{23} = 1; \\ & x_{33} = 1. \end{aligned} \tag{2.19}$$

Thus we will need to send our UAVs as follows: from Changi to Raffles Place: 2 UAVs;
from Changi to Sentosa Island: 1 UAV;
from Jurong West to Jurong Island: 2 UAVs;
from Jurong West to Sentosa Island : 1 UAV;
from Woodland to Sentosa Island: 1 UAV.

2.2 Assignment problem with timing constraints

2.2.1 A case with single UAVs at aerobases

To simplify at this stage our calculations we suppose that every aerobase has one UAV. Otherwise, the aerobases where there are several UAVs can be formally divided onto the collection of several aerobases with alone UAV at every one. In the next chapter we will consider the general case, too.

Notation

Introduce the following notations:
n— number of aerobases,
K — number of zones for service,
V_k— number of UAVs which are required for service of k-th zone, $k = 1, ..., K$
$[\underline{T}_k, \overline{T}_k]$— "time window" for k-th zone where $\underline{T}_k$ and $\overline{T}_k$ is the earliest and latest time for service of k-th zone),
r_{jk}— distance from j-th aerobase to k-th zone,
d_{ij}— distance from i-th zone to j-th zone.

Introduce the network of aerobases and zones as a pair (S, U). Here $S = \{1, 2, ..., n, n+1, ..., n + K\}$- the set of numbered nodes- aerobases and zones, such that to each node

corresponds aerobase or zone.
U-set of edges, which are connect the pair of nodes. The set S can be divided onto two subsets: S_A (set of aerobases) and S_Z (set of zones). Each node pair $(i, j), i \in S, j \in S$ corresponds the edge U_{ij} connecting the node i and node j. The edge U_{ij} have the characteristic ρ_{ij} — the distance between node i and j, i.e.
if $i \in S_A$ and $j \in S_Z$ then $\rho_{ij} = r_{ij}$;
if $i \in S_Z$ and $j \in S_Z$ then $\rho_{ij} = d_{ij}$.
Denote by
$\alpha_s, (s = 1, ..., n)$ — boolean variable where $\alpha_s = 1$ means that the $s-$ th aerobase (their UAV) involve into asked service, and $\alpha_s = 0$ — otherwise.
$\eta_i^{(s)}, (s = 1, ..., n;\ i = 1, ..., K$ — boolean variable where $\eta_i^{(s)} = 1$ means that the $s-$ th aerobase (their UAV) involve into service of $i-$ th zone, and $\eta_i^{(s)} = 0$ — otherwise.

2.2.2 Cost functions

Obviously, each assignment plan $\eta^{(s)} = \left(\eta_1^{(s)}, \eta_2^{(s)}, ..., \eta_K^{(s)}\right)$, $s = 1, ..., n$ of UAVs generates the boolean values α_s as follows

$$\alpha_s = \begin{cases} 1, \ if\ z^s > 0 \\ 0, \ if\ z^s = 0, \quad (s = 1, ..., n) \end{cases} \tag{2.20}$$

where $z^s = \sum\limits_{k=1}^{K} \eta_i^{(s)}$.

Then we can consider the cost functions

$$C_1(\eta) = \sum_{s=1}^{n} \alpha_s \tag{2.21}$$

that denotes the total number of UAVs used for service requests.

Next we introduce some other cost functions where it will be determined:
i) how many times each UAV is used in service
ii) total time service subject to constraints in the form of "time windows" for zone service.

To this aim we need to analyze some details of assignment plans in details.

2.2.3 Service logic and Constraints

Let $\eta^{(s)} = \left(\eta_1^{(s)}, \eta_2^{(s)}, ..., \eta_K^{(s)}\right)$ be an assignment plan for $s-th$ UAV (aerobase). Note, that the total number of all assignment plans for every aerobase is equal $K!$ (the number of all permutation of K elements). The value of $K!$ can be huge. By this reason, we can suppose that for each aerobase there exists some service order for considered zones. For example, this order can be determined in accordance with order of the assigned zone "time windows" such that the first for service is the zone with the smallest beginning of " window time". Some other ideas can be put to fix this order, also.

Next consider the time diagram of the considered flying route $\eta^{(s)}$.

Since in the considered route the zone-node $\eta_1^{(s)}$ is the first, and for this zone we have the time-window for service as $[\underline{T}_{\eta_1^{(s)}}, \overline{T}_{\eta_1^{(s)}}]$, then the time of the first departure from s-th base is:

$$t_1^{(s)} = \underline{T}_{\eta_1^{(s)}} - t_{fly}^{s \to \eta_1^{(s)}} \tag{2.22}$$

where $t_{fly}^{s \to \eta_1^{(s)}} = \dfrac{\rho_{s\eta_1^{(s)}}}{v_s}$ denotes the flying time from $s - th$ base to zone $\eta_1^{(s)}$.

Also it should be noted that it is not possible to start service of zone $\eta_1^{(s)}$ at the moment $\underline{T}_{\eta_1^{(s)}}$ if $t_1^{(s)} < 0$. But it is possible partially service if $h_s > t_{fly}^{s \to \eta_1^{(s)}} + t_{fly}^{\eta_1^{(s)} \to s}$, where h_s means the endurance of UAVs located at s-th base. If $t_1^{(s)} > 0$, then the service time of the first zone $\eta_1^{(s)}$ in the considered route $\eta^{(s)}$ is equal

$$T_{service}^{\eta_1^{(s)}} = \begin{cases} 0, \ if \ t_1^{(s)} < 0 \\ \overline{T}_{\eta_1}^{(s)} - \underline{T}_{\eta_1^{(s)}}, if \ t_1^{(s)} > 0 \ \text{ and } \ h_s > 2t_{fly}^{s \leftrightarrow \eta_1^{(s)}} + (\overline{T_{\eta_1}^{(s)}} - \underline{T_{\eta_1}^{(s)}}) \\ h_s - 2t_{fly}^{s \leftrightarrow \eta_1^{(s)}}, if \ t_0^{(s)} > 0 \quad \text{and} \ \ h_s < 2t_{fly}^{s \leftrightarrow \eta_1^{(s)}} + (\overline{T_{\eta_1}^{(s)}} - \underline{T_{\eta_1}^{(s)}}) \end{cases} \tag{2.23}$$

Thus, after analysis of the first node $\eta_1^{(s)}$ we can define the time of ending service for the first zone by s-th UAVs located at s-th base as follows:

$$t_{1,final}^{(s)} = \begin{cases} 0, \begin{cases} a) if \ t_1^{(s)} < 0 \text{(i.e. UAVs was not used for service of the first node)} \\ b) if \ t_1^{(s)} > 0 \ and \ h_s < 2t_{fly}^{s \leftrightarrow \eta_1^{(s)}} + (\overline{T_{\eta_1}^{(s)}} - \underline{T_{\eta_1}^{(s)}}) \\ \text{(i.e. UAVs was used at first zone and then} \\ \text{it returned to base due to restricted endurance)} \end{cases} \\ (t_{fly}^{s \to \eta_1^{(s)}} + T_{service}^{s \to \eta_1^{(s)}}), \text{(i.e. when endurance of UAV was more then} \\ \text{it is required for service zone } \eta_1^{(s)} \\ \text{and UAV can fly for service from zone } \eta_1^{(s)} \text{ to next zone } \eta_2^{(s)}) \end{cases} \tag{2.24}$$

Now consider how we can to start the service of the next zone from our route $\eta^{(s)}$ taking into account the previous analysis and (2.24). Find the starting moment

$$t_{start}^{\eta_2^{(s)}} = \begin{cases} \underline{T_{\eta_2}^{(s)}} - t_{fly}^{s \to \eta_2^{(s)}}, \text{ if } t_{1,final}^{(s)} = 0 (\text{i.e. when we are "start" from the base}) \\ t_{fly}^{s \to \eta_1^{(s)}} + T_{service}^{\eta_1^{(s)}}, \text{ otherwise (we are starting from the first zone } \eta_1^{(s)}) \end{cases} \tag{2.25}$$

It should be noted once again that, if $t_{start}^{\eta_2^{(s)}} < 0$, then this zone will be eliminated from further consideration, since the considered $s - th$ UAV does not reach this zone. In the case when $t_{start}^{\eta_2^{(s)}} > 0$ we can to continue the analysis of possibilities of servicing node (zone) $\eta_2^{(s)}$ taking into account the "time window" constraint $[\underline{T_{\eta_2}^{(s)}}, \overline{T_{\eta_2}^{(s)}}]$.

Then

$$T_{service}^{\eta_2^{(s)}} = \begin{cases} a) 0, if\ t_{start}^{\eta_2^{(s)}} < 0 \\ b) \overline{T_{\eta_2}^{(s)}} - \underline{T_{\eta_2}^{(s)}}, if\ t_{start}^{\eta_2^{(s)}} > 0,\ t_{1,final}^{(s)} = 0,\ h_s > 2t_{fly}^{s \leftrightarrow \eta_2^{(s)}} + (\overline{T_{\eta_2}^{(s)}} - \underline{T_{\eta_2}^{(s)}}) \\ \text{i.e the case, when we will start from the base and we have sufficient} \\ \text{ endurance to serve the node } \eta_2^{(s)} \text{ and coming back to base} \\ c) h_s - 2t_{fly}^{s \to \eta_1^{(s)}}, if\ t_{start}^{\eta_2^{(s)}} > 0,\ t_{1,final}^{(s)} = 0,\ h_s < 2t_{fly}^{s \to \eta_2^{(s)}} + (\overline{T_{\eta_2}^{(s)}} - \underline{T_{\eta_2}^{(s)}}) \\ \text{(i.e. the case when the window does not completely "close")} \\ d) h_s - t_{fly}^{s \to \eta_1^{(s)}} - t_{fly}^{\eta_1 \to \eta_2} - t_{fly}^{\eta_2 \to s}, \text{if } \eta_2 \text{ is served from } \eta_1 \text{ and then it comes} \\ \text{ back base } s, \text{ since there was not sufficient endurance to continue service} \\ e) T_{service}^{\eta_1^{(s)}} + (\overline{T_{\eta_2}^{(s)}} - \underline{T_{\eta_2}^{(s)}}), \text{if the node } \eta_2 \text{ is served from } \eta_1 \text{ and} \\ \text{it have sufficient endurance.} \end{cases} \tag{2.26}$$

Continue by analogy with above the given analysis for the remainder zones from the considered route $\eta^{(s)}$ we find the sequence

$$T_{service}^{\eta_1^{(s)}}, T_{service}^{\eta_2^{(s)}}, ..., T_{service}^{\eta_K^{(s)}}$$

of time services of each zones. Then the total service time which generates the considered route $\eta^{(s)}$ is

$$T_{service}(\eta^{(s)}) = T_{service}^{\eta_1^{(s)}} + T_{service}^{\eta_2^{(s)}} + ... + T_{service}^{\eta_K^{(s)}} \tag{2.27}$$

and, hence, the total service time of the required zones is

$$T_{service} = \sum_{s=1}^{n} T_{service}(\eta^{(s)}) \tag{2.28}$$

To guarantee the needed number V_k of pre-assigned UAVs for $k - th$ zone we should set the following constraints for the introduced boolean variables

$$\sum_{s=1}^{n} \eta_k^{(s)} = V_k, \ k = 1.2, ..., K \tag{2.29}$$

Finally, the assignment problem with timing constraints can be formulated as the following boolean optimization problem; Maximize the total service time

$$\sum_{s=1}^{n} T_{service}(\eta^{(s)}) \rightarrow \max_{\eta^{(s)} \in \text{boolean}} \tag{2.30}$$

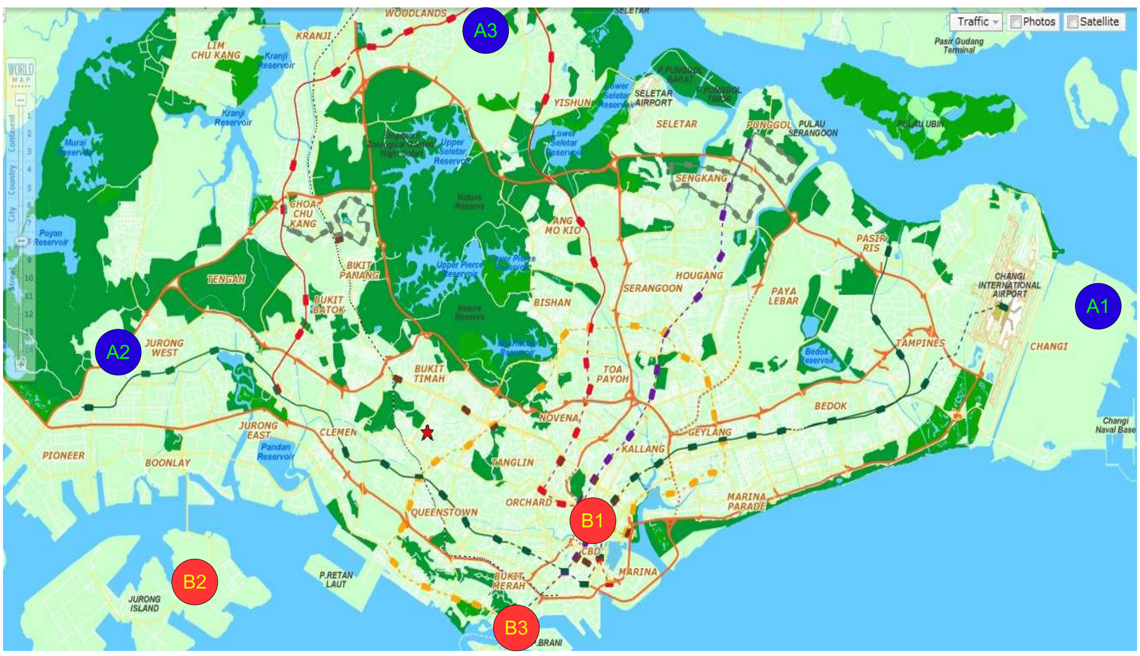
subject to constraints

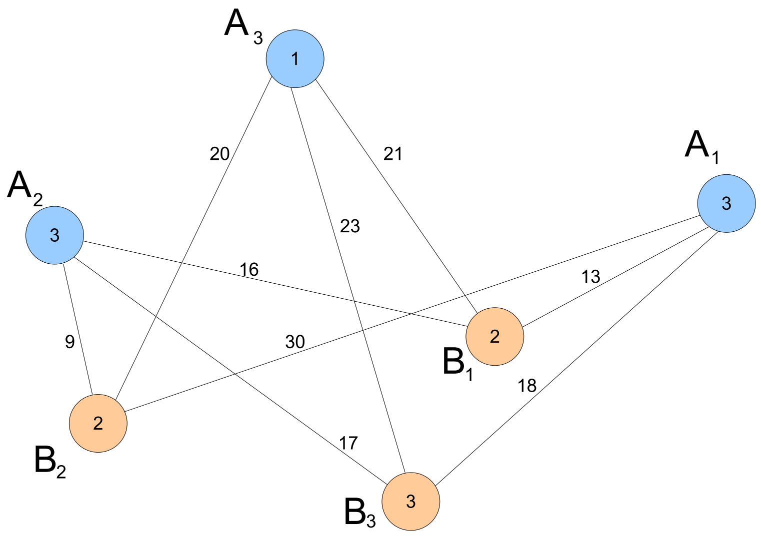
$$\sum_{s=1}^{n} \eta_k^{(s)} = V_k, \ k = 1.2, ..., K \tag{2.31}$$

2.2.4 Illustrative examples

Consider first the example from previous section where some "time window" requirements are added.

Namely, assume that we have 3 airbases located at Changi A_1 with 3 UAVs ($a_1 = 3$), Jurong West A_2 with 3 UAVs ($a_2 = 3$), and Woodland A_3 with 1 UAV ($a_1 = 1$). Now 7 UAVs are requested from B_1-Raffles Place ($b_1 = 2$), B_2-Jurong Island ($b_2 = 2$), and B_3-Sentosa Island ($b_3 = 3$). Out task is to complete all requests in order to maximaze the total service time in the zones and satisfies all timing constraints.

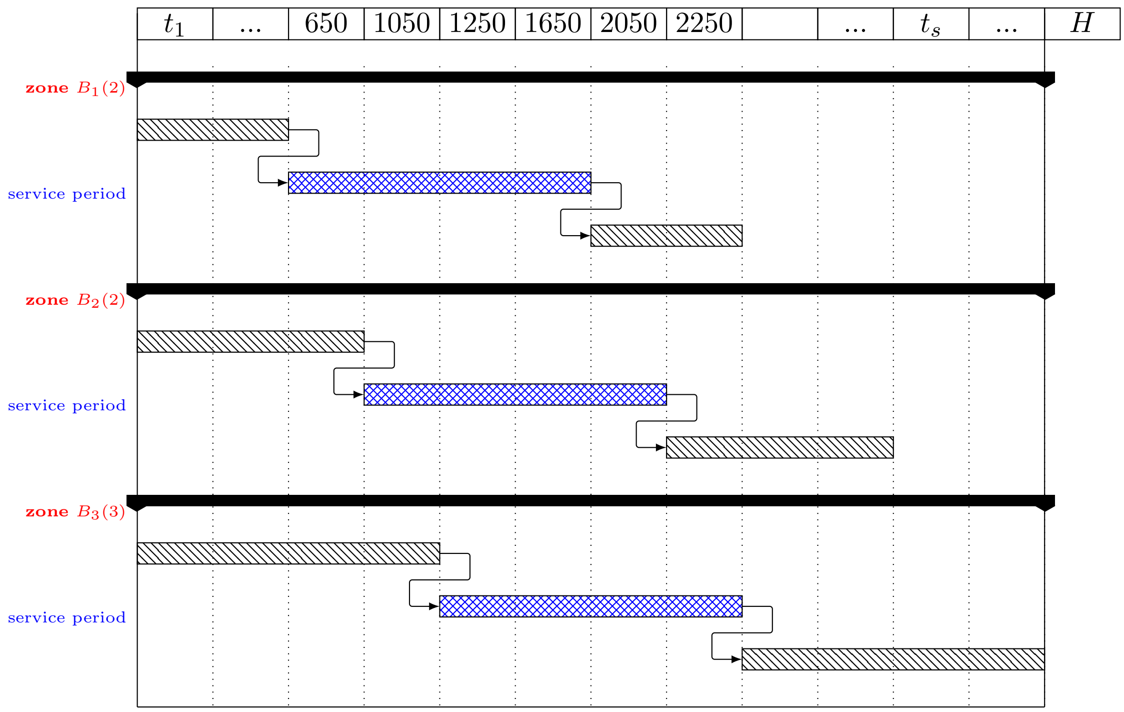




We will use the following notation for this problem:

$A_i, i = 1, 2, 3$- number of aerobases,
$a_1 = 3, a_2 = 3, a_3 = 1$ - number of UAVs located in A_i,
$B_j, j = 1, 2, 3$- areas of operations,
$b_1 = 2, b_2 = 2, a_3 = 3$- numbers of UAVs for service of B_j
d_{ij}- distances from A_i to B_j given in table 3;
x_{ij}-number of UAVs from A_i to B_j
c_{ij}- given in table 4. $h_i = 3600sec$- UAVs endurance located on A_i aerobase;
$v_{ij} = 30\frac{m}{sec}$ - speed of UAVs;
$t^f_{B_i}$- earliest time for visit zone $B_i, i = 1, 2, 3$;
$t^l_{B_i}$- latest time for visit zone $B_i, i = 1, 2, 3$
Let the following value for "time windows":

$$\begin{aligned} t^f_{B_1} &= 650sec, \quad t^l_{B_1} = 1650sec; \\ t^f_{B_2} &= 1050sec, \quad t^l_{B_2} = 2050sec; \\ t^f_{B_3} &= 1250sec, \quad t^l_{B_3} = 2250sec; \end{aligned} \tag{2.32}$$



From this diagram clear that we can divide our problem by considering the assignments problem on the following 5 periods:
Period 1: [650,1050] - 1 problem for zone B_1 to assign 2 UAVs (i.e. $B_1(2)$);
Period 2: [1050,1250] - 2 problems for zones B_1 and B_2 (to assign 4 UAVs – $B_1(2)$, $B_2(2)$);
Period 3: [1250,1650] - 3 problems for zones $B_1(2)$, $B_2(2)$ and $B_3(3)$;
Period 4: [1650,2050] - 2 problems for zones $B_2(2)$ and $B_3(3)$;
Period 5: [2050,2250] - 1 problem for zone $B_3(3)$.

2.2.5 Analysis and modification of the assignment solution for the ”time windows” cases

In the previous section in the absence of ”time windows” constraints we obtained the solution of the assignment problem of the form:

	B_1	B_2	B_3	a_i
A_1	2	0	1	$a_1 = 3$
A_2	0	2	1	$a_2 = 3$
A_3	0	0	1	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

And the value of our cost function is

$$F = 3398 \; seconds \approx 56.6 \; minutes$$

Now we will check the feasibility of this solution for the same problem with the given above timing constraints.

Period 1 - [650,1050]:
In this period we are need to send 2 UAVs from the base A_1 to zone B_1. The total endurance of each UAVs are 3600$sec.$ (see initial data). The flying time from A_1 to B_1 is 433$sec.$ (see Table 4). Then both UAVs must take off from A_1 at $t_{11}^{departure} = 650 - 433 = 217 \; sec.$ in order to arrive to B_1 at required time $\underline{T}_1 = 650$. Also we are known "working time" in the zone B_1 is 1000$sec.$ Hence UAVS must return to base A_1 at $t_{11}^{arrival} = 1650 + 433 = 2083 sec.$. The total flying time for each UAVs are $\tau_{11}^{fly} = 433 + 1000 + 433 = 1866 sec.$ (or only for period 1: $\tau_{11}^{fly_1} = 433 + (1050 - 650) + 433 = 1266 sec.$). And this time are not more than their endurance $\tau_{11}^{fly} < h = 3600 sec.$. Thus, our old solution satisfies all requirements of the problem.

Period 2 -[1050,1250]:
a) We have 2 UAVs in $B_1(2)$ (their remind endurance $h_r = 3600 - 433 + (1250 - 650) = 3600 - 1033 = 2567$). Then $\tau_{11}^{fly_2} = 1033 + 433 = 1466$. Clear that $\tau_{11}^{fly_2} < h_r = 2567$. Then we can continue to serve the zone B_1 by 2 UAVs which was sent from A_1.

b) From $A_2(3)$ we send 2 UAVs to zone B_2. Calculate the take off time (departure) from A_2 and also possible arrival time to airbase A_2 from B_2 after the end of the period 2:
$t_{22}^{departure} = 1050 - 300 = 750 \; sec.$
$t_{22}^{arrival} = 1250 + 300 = 1550 \; sec.$
$\tau_{22}^{fly} = 300 + 1000 + 300 < h = 3600 sec.$ (i.e. total flying time are not more than available endurance (in the case when this inequality is not hold, we will need to consider the flying time for particular period)). Thus our old solution is satisfies all requirements of the problem.

Period 3 -[1250,1650]:
a) From previous periods we are know the following information: 2 UAVs was sent from A_1 to B_1: $A_1(1) \xrightarrow{2} B_1(2)$; (status- active)
2 UAVs was sent from A_2 to B_2: $A_2(1) \xrightarrow{2} B_2(2)$; (status- active)

b) And now we need to send :
1 UAV: $A_1(0) \underset{\rightarrow}{1} B_3(3)$;
$t_{13}^{departure} = 1250 - 600 = 650\ sec.$; $t_{31}^{arrival} = 2250 + 600 = 2850\ sec.; \Rightarrow$
$\tau_{131}^{fly} = t_{31}^{arrival} - t_{13}^{departure} = 2200 < h$
1 UAV: $A_2(0) \underset{\rightarrow}{1} B_3(3)$;
$t_{23}^{departure} = 1250 - 566 = 684\ sec.$; $t_{32}^{arrival} = 2250 + 566 = 2816\ sec.; \Rightarrow$
$\tau_{232}^{fly} = t_{32}^{arrival} - t_{23}^{departure} = 2132 < h$
1 UAV: $A_3(0) \underset{\rightarrow}{1} B_3(3)$;
$t_{33}^{departure} = 1250 - 766 = 484\ sec.$; $t_{33}^{arrival} = 2250 + 766 = 3016\ sec.; \Rightarrow$
$\tau_{33}^{fly} = t_{33}^{arrival} - t_{33}^{departure} = 2532 < h$

Thus our old solution is satisfies all requirements of the problem for that period.

Period 4 -[1650,2050]:
a) From previous periods we are know the following information: 2 UAVs was sent from A_1 to B_1: $A_1(0) \xrightarrow{2} B_1(2)$; (current status- coming back)
2 UAVs was sent from A_2 to B_2: $A_2(0) \xrightarrow{2} B_2(2)$; (current status- active)
1 UAVs was sent from A_1 to B_3: $A_1(0) \xrightarrow{1} B_3(3)$; (current status- active)
1 UAVs was sent from A_2 to B_3: $A_2(0) \xrightarrow{1} B_3(3)$; (current status- active)
1 UAVs was sent from A_3 to B_3: $A_3(0) \xrightarrow{1} B_3(3)$; (current status- active)
b) No new requests.

Period 5 -[2050,2250]:
a) From previous periods we are know the following information: 2 UAVs was sent from A_1 to B_1: $A_1(0) \xrightarrow{2} B_1(2)$; (current status- coming back)
2 UAVs was sent from A_2 to B_2: $A_2(0) \xrightarrow{2} B_2(2)$; (current status- coming back)
1 UAVs was sent from A_1 to B_3: $A_1(0) \xrightarrow{1} B_3(3)$; (current status- active)
1 UAVs was sent from A_2 to B_3: $A_2(0) \xrightarrow{1} B_3(3)$; (current status- active)
1 UAVs was sent from A_3 to B_3: $A_3(0) \xrightarrow{1} B_3(3)$; (current status- active)
b) No new requests.

Then, we can make a conclusion that the old solution is optimal solution for our problem with timing constraints.

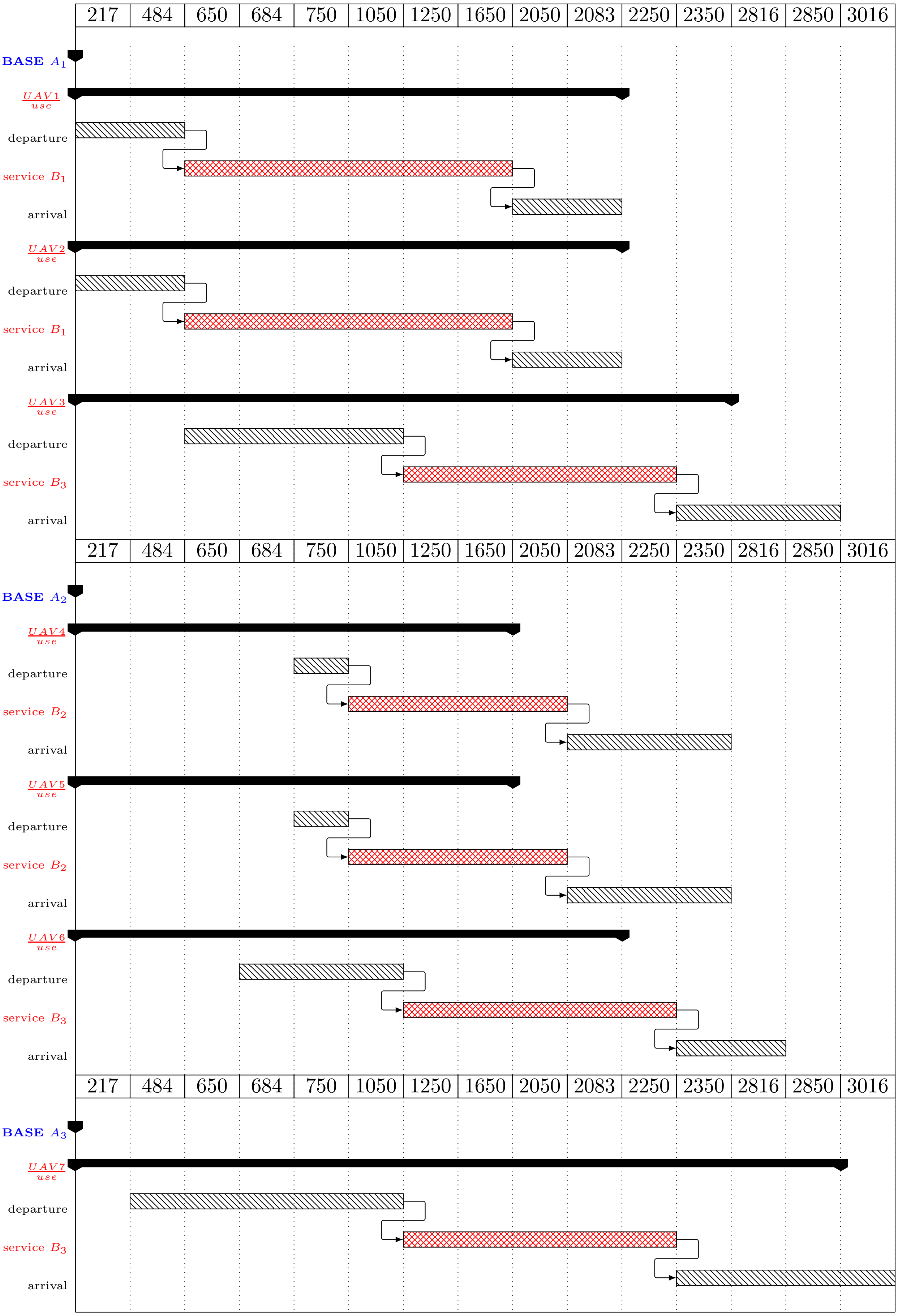
And we have the following schedular plan:

$$A_1(3) \xrightarrow{2} B_1(2)\ ,\ A_1(3) \xrightarrow{1} B_3(3)\ ,\ A_2(3) \xrightarrow{2} B_2(2)\ ,\ A_2(3) \xrightarrow{1} B_3(3)\ ,\ A_3(1) \xrightarrow{1} B_3(3)$$

or in table form:

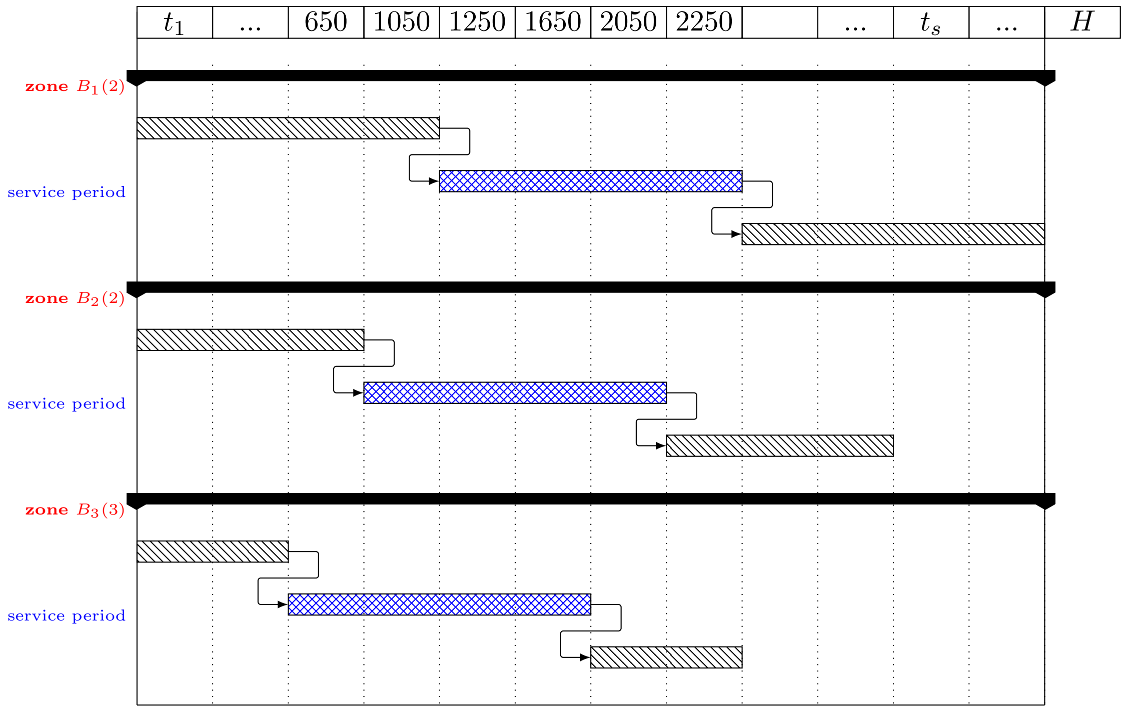
	B_1	B_2	B_3	a_i
A_1	2	0	1	$a_1 = 3$
A_2	0	2	1	$a_2 = 3$
A_3	0	0	1	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

		B_1		B_2		B_3	
		Departure time (D/T)	Arrival time (A/T)	D/T	A/T	D/T	A/T
A_1	UAV 1	217	2083	-	-	-	-
	UAV 2	217	2083	-	-	-	-
	UAV 3	-	-	-	-	650	2850
A_2	UAV 4	-	-	750	2350	-	-
	UAV 5	-	-	750	2350	-	-
	UAV 6	-	-	-	-	684	2816
A_3	UAV 7	-	-	-	-	484	3016


217 484 650 684 750 1050 1250 1650 2050 2083 2250 2350 2816 2850 3016
BASE A_1
UAV1 use
departure
service B_1
arrival
UAV2 use
departure
service B_1
arrival
UAV3 use
departure
service B_3
arrival
217 484 650 684 750 1050 1250 1650 2050 2083 2250 2350 2816 2850 3016
BASE A_2
UAV4 use
departure
service B_2
arrival
UAV5 use
departure
service B_2
arrival
UAV6 use
departure
service B_3
arrival
217 484 650 684 750 1050 1250 1650 2050 2083 2250 2350 2816 2850 3016
BASE A_3
UAV7 use
departure
service B_3
arrival

2.2.6 Example with time window restriction

Consider another case of strong timing constraints for the problem above when the old solution does not satisfy the "time windows" requirements. Namely let us to interchange the "time windows" for zone B_1 and B_3 as it is shown on the diagram:



Also we assume that initial data are the same as in the previous case. Namely, the distances between A_i and B_j are given as follows (in kilometers):

	A_1	A_2	A_3
A_1	0	32	22
A_2	32	0	17
A_3	22	17	0

Distances between A_i

	B_1	B_2	B_3
B_1	0	17	6
B_2	17	0	14
B_3	6	14	0

Distances between B_j

	B_1	B_2	B_3
A_1	13	30	18
A_2	16	9	17
A_3	21	20	23

Distances between A_i and B_j

The speed of UAVs is fixed $v_{ij} = 30\frac{m}{sec}$.

Next, for all i and j denote by c_{ij} the benefit of sending the UAV from i-th aerobase to j-th zone of area of operation. In particular, this benefit can be given in the form $c_{ij} = \frac{d_{ij}}{v_{ij}}$ that means the flight time from $A_i \to B_j$ (see Table):

	B_1	B_2	B_3
A_1	433	1000	600
A_2	533	300	566
A_3	700	666	766

(2.33)

UAVs flight time from $A_i \to B_j$

The optimization problem is to maximize the total service time of the group of UAVs. It should be noted that the total service time of the group of UAVs involved in the mission will be maximal if the total flight time to reach the preassigned zones is minimal.

Now we will test the old solution of the assignment problem obtained in the previous section without "time window" constraints. This solution is presented by the following matrix:

$$X^{flight} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.34}$$

Now we will analyze this plan in accordance with the new time constraints. For the zone B_3 the "time window" is $[650, 1650]$. In according with the plan (2.34) we need to send to the zone B_3 one UAV from each base. Clear that from base A_1 and A_2 the UAVs are able to reach the zone B_3 to the moment of opening "time window ", namely at the time moment 650 (i.e. the flight time $A_1 \xrightarrow{600} B_3$ and $A_2 \xrightarrow{566} B_3$). But from the base A_3 the UAV not able to reach the zone B_3 at required time moment, since $A_3 \xrightarrow{766} B_3$. Thus the old solution (2.34) is not satisfactory.

Find a new flight plan X.

Consider again the partition of the pre-assigned zone service into five periods. We will solve optimization assignment problem for each period on the base of the step-by-step procedure. Also, to realize a modification of the old solution we can extend the number of bases by adding some UAVs from zones sent at the previous periods. In particular, for the second period $[1050, 2050]$ in addition to the given base A_1, A_2, and A_3 we will add the zone B_3 also as a base, namely we can use UAVs from this zone in order to complete requests from remaining zones. This logic can be used after each period (i.e. the number of A_i and B_j will be changed on each period).

Period 1: [650,1050]. Here we have 1 problem for zone B_3 to assign 3 UAVs (i.e. $B_3(3)$);

In according to "time windows" requirements (see Diagram), for the first period we have three airbase $A_1(3), A_2(3), A_3(1)$ and we are need to assign 3 UAVs to serve zone B_3. From analysis of the (2.33) follows that we can assign 3 UAVs from base A_2 or A_1. But the distance table follows that A_2 is closer to B_3 then A_1. In this case it is obviously, that we will assign 3 UAVs from base A_2 to serve B_3: $A_2(0) \xrightarrow{3} B_3(3)$ with minimal cost function value.

Period 2: [1050,2050] - 2 problems for zone B_3 to assign 3 UAVs (i.e. $B_3(3)$) and 2 UAVs for zone $B_2(2)$. Also we have four base: $A_1(3), A_2(0), A_3(1)$ and "new" $B^a_{31}(3)$ and two zone for service $B_3(3)$ and $B_2(2)$. This requirements can be written as follows:

	B_1	B_2	B_3	a_i
A_1		x_{12}	x_{13}	$a_1 = 3$
A_2				$a_2 = 0$
A_3		x_{32}	x_{33}	$a_3 = 1$
B^a_{31}		x_{42}	x_{43}	$b_{31} = 3$
b_j	$b_1 = 0$	$b_2 = 2$	$b_3 = 3$	

$\Downarrow$ We can reduce problem by eliminating from our consideration the base A_2 and zone B_1, since $a_2 = 0$ and $b_1 = 0$ at this period:

	B_2	B_3	a_i
A_1	x_{12}	x_{13}	$a_1 = 3$
A_3	x_{32}	x_{33}	$a_3 = 1$
B^a_{31}	x_{42}	x_{43}	$b_{31} = 3$
b_j	$b_2 = 2$	$b_3 = 3$	

To construct the initial feasible solution we will use "North-West corner" method. In our case we have the following:

$$\begin{aligned} x_{12} &= \min(b_2; a_1) = \min(2; 3) = 2 \\ \Downarrow & \\ x_{13} &= \min(a_1 - b_2; b_3) = \min(1; 3) = 1 \\ \Downarrow & \\ x_{33} &= \min(a_3; b_3 - a_1 - b_2) = \min(1; 2) = 1 \\ \Downarrow & \\ x_{43} &= \min(b_{31}; b_3 - a_1 - b_2 - a_3) = \min(3; 1) = 1 \end{aligned} \tag{2.35}$$

	B_2	B_3	a_i
A_1	2	1	$a_1 = 3$
A_3		1	$a_3 = 1$
B_{31}^a		1	$b_{31} = 3$
b_j	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = 7 > \sum_{j=1}^{2} b_j = 5$

The solution (2.35) is not feasible since balance condition are not hold

$$\sum_{i=1}^{3} a_i = 7 > \sum_{j=1}^{2} b_j = 5.$$

By this reason will use the information from the period 1, namely that we sent from A_2 3 UAVs to zone B_3 and in the second we still need at the zone B_3 3 UAVs. Then we can fixed the number of UAVs at the cell $(B_{31}^a; B_3)$ and denote this value as $z_{33}^* = 3$:

	B_2	B_3	a_i
A_1			$a_1 = 3$
A_3			$a_3 = 1$
B_{31}^a		3	$b_{31} = 3$
b_j	$b_2 = 2$	$b_3 = 3$	

Then we are eliminate this value from further consideration by introducing a new artificial variable as:

$$\tilde{b_3} = b_3 - z_{33}^* = 3 - 3 = 0; \quad \tilde{a_3} = a_3 - z_{33}^* = 3 - 3 = 0. \tag{2.36}$$

Than we have a new table:

	B_2	a_i
A_1		$a_1 = 3$
A_3		$a_3 = 1$
b_j	$b_2 = 2$	$\sum_{i=1}^{2} a_i = 4 > \sum_{j=1}^{1} b_j = 2$

The model above is "open model" while the balance condition are not hold. Other words we have more UAVs that we are need. In order to reduce to the problem, where balance condition are hold, introduce artificial base $\tilde{A}_4$ and zone $\tilde{B}_4$ in order to send there the rest amount of UAVs $\tilde{b} = \sum_{i=1}^{2} a_i - \sum_{j=1}^{1} b_j = 4 - 2 = 2$. Also we assume that the cost of usage these UAVs such that it will be not possible to use this UAVs in other zones. For example, set the flight time as $\tilde{A}_4 \xrightarrow{\infty} \tilde{B}_4$ or $\tilde{A}_4 \xrightarrow{M} \tilde{B}_4$, where M is huge (this fact blokes these zones and bases to involve them in the next consideration since the flying time will be not optimal). Then we can find initial feasible plan using Northwest corner method:

	B_2	$\tilde{B}_4$	a_i
A_1	2	1	$a_1 = 3$
A_3		1	$a_3 = 1$
$\tilde{A}_4$		0	$\tilde{a}_4 = 0$
b_j	$b_2 = 2$	$\tilde{b} = 2$	

Let us check the optimality condition for our problem. Consider the following table

	B_2	$\tilde{B}_4$	a_i	u_i
A_1	$c_{12} = 1000$ $x_{12} = 2$	$c_{14} = M$ $x_{14} = 1$	$a_1 = 3$	
A_3	$c_{32} = 666$	$c_{34} = M$ $x_{34} = 1$	$a_3 = 1$	
$\tilde{A}_4$	$c_{42} = M$	$c_{44} = M$ $x_{44} = 0$	$\tilde{a}_4 = 0$	
b_j	$b_2 = 2$	$\tilde{b}_4 = 2$		
ν_j				

Then we should find potential u_i and ν_j such that for $x_{ij} > 0$ the condition $c_{ij} = u_i + \nu_j$ hold. One of the potentials can be chosen arbitrary.

Let $\nu_2 = 0$, since $u_1 + \nu_2 = 1000$ then $u_1 = 1000$. Next following this logic we found step by step:

$$\nu_4 + u_1 = M \longrightarrow \nu_4 = M - 1000,$$

$$\nu_4 + u_2 = M \longrightarrow u_2 = M - M + 1000 = 1000,$$

$$u_4 + \nu_4 = M \longrightarrow u_4 = 1000,$$

Than we will have the following table:

	B_2	$\tilde{B_4}$	a_i	u_i
A_1	$c_{12} = 1000$ $x_{12} = 2$	$c_{14} = M$ $x_{14} = 1$	$a_1 = 3$	1000
A_3	$c_{32} = 666$	$c_{34} = M$ $x_{34} = 1$	$a_3 = 1$	1000
$\tilde{A_4}$	$c_{42} = M$	$c_{44} = M$ $x_{44} = 0$	$\tilde{a_4} = 0$	1000
b_j	$b_2 = 2$	$\tilde{b_4} = 2$		
ν_j	0	M-1000		

Now we are ready to check our initial supporting feasible solution for optimality. Namely to check the condition

$$\Delta_{ij} = u_i + v_j - c_{ij} = \bar{c_{ij}} - c_{ij} \le 0 \;\text{ for } x_{ij} = 0; \tag{2.37}$$

$$\Delta_{ij} = u_i + v_j - c_{ij} = \bar{c_{ij}} - c_{ij} = 0 \;\text{ for } x_{ij} > 0. \tag{2.38}$$

Then $\bar{c_{ij}} = u_i + \nu_j \longrightarrow \bar{c_{32}} = 1000$
$\bar{c_{34}} = 0 + 1000 = 1000$

Now in the so-called matrix of estimates

$$\Delta = (\Delta_{ij}) = (\bar{c_{ij}} - c_{ij}) = (u_i + v_j - c_{ij}) = \begin{pmatrix} 0 & 0 \\ 334 & 0 \\ 1000 - M & 0 \end{pmatrix}$$

find the maximal element $\Delta_{34} = 334 = \max\limits_{i,j} \Delta_{ij}$ (since $\Delta_{42} = 1000 - M < 0$ if M is large).

In our case for one zero component of our feasible solution this conditions are not satisfied. Hence our solution is not optimal. Change the initial feasible solution by adding the value θ to element x_{34} with some corrections of other elements too.

	B_2	$\tilde{B}_4$	a_i	u_i
A_1	$2-\theta$	$1+\theta$	$a_1 = 3$	
A_3	θ	$1-\theta$	$a_3 = 1$	
$\tilde{A}_4$			$\tilde{a_4} = 0$	
b_j	$b_2 = 2$	$\tilde{b_4} = 2$		
ν_j				

In order to find θ consider the following inequality $\min\{2-\theta, 1-\theta\} = 0 \Rightarrow \theta = 1$. Then new plan is :

	B_2	$\tilde{B}_4$	a_i	u_i
A_1	1	2	$a_1 = 3$	
A_3	1		$a_3 = 1$	
$\tilde{A}_4$			$\tilde{a_4} = 0$	
b_j	$b_2 = 2$	$\tilde{b_4} = 2$		
ν_j				

Let us check the optimality condition for our problem. Consider the following table

	B_2	$\tilde{B}_4$	a_i	u_i
A_1	$c_{12} = 1000$ $x_{12} = 1$	$c_{14} = M$ $x_{14} = 2$	$a_1 = 3$	
A_3	$c_{32} = 666$ $x_{32} = 1$	$c_{34} = M$	$a_3 = 1$	
$\tilde{A}_4$	$c_{42} = M$	$c_{44} = M$	$\tilde{a_4} = 0$	
b_j	$b_2 = 2$	$\tilde{b_4} = 2$		
ν_j				

Then find potential u_i and ν_j such that for $x_{ij} > 0$ the condition $c_{ij} = u_i + \nu_j$ hold.

One of the potentials can be chosen arbitrary.

Let $\nu_2 = 0$, since $u_1 + \nu_2 = 1000$ then $u_1 = 1000$. Next following this logic we found step by step:

$$\nu_2 + u_3 = 666 \longrightarrow u_3 = 666,$$

$$\nu_4 + u_3 = M \longrightarrow \nu_4 = M - 666,$$

$$u_4 + \nu_4 = M \longrightarrow u_4 = 666,$$

Then we will have the following table:

	B_2	$\tilde{B}_4$	a_i	u_i
A_1	$c_{12} = 1000$ $x_{12} = 1$	$c_{14} = M$ $x_{14} = 2$	$a_1 = 3$	1000
A_3	$c_{32} = 666$ $x_{32} = 1$	$c_{34} = M$	$a_3 = 1$	666
$\tilde{A}_4$	$c_{42} = M$	$c_{44} = M$	$\tilde{a_4} = 0$	666
b_j	$b_2 = 2$	$\tilde{b_4} = 2$		
ν_j	0	$M - 666$		

Now we are ready to check our initial supporting feasible solution for optimality. For that we are need to calculate $\bar{c_{ij}} = u_i + \nu_j$ and construct the matrix of estimates $\Delta_{ij} = \bar{c_{ij}} - c_{ij}$:

Then in matrix of estimates $\Delta = \bar{c_{ij}} - c_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 666 - M & 0 \end{pmatrix} \leq 0$ Thus, optimality conditions are fulfieled and, hence, the obtained new plan $x_{12} = 1$, $x_{32} = 1$ is optimal on the second period $[1050, 1250]$. The component $x_{14} = 2$ of optimal plan we are not consider, since zone $\tilde{B}_4$ is artificial. But we should add to this plan the fixed component $z^*_{33} = x^*_{23} = 3$

Thus on the second period we have the following situation:
1) $A_2(0) \xrightarrow{3} B_3(3)$- three UAVs continue their work in zone B_3;
2) $A_1 \xrightarrow{1} B_2$ - From base A_1 one UAV should be sent to zone B_2 at time moment $t_{12} = 1050 - 1000 = 50$;
3) $A_3 \xrightarrow{1} B_2$ - From base A_3 one UAV should be sent to zone B_2 at time moment $t_{32} = 1050 - 666 = 334$.

Period 3: [1250,1650].

We have three zones for service $(B_3(3), B_2(2), B_1(2))$ and one base $A_1(2)$, while another base we are not consider at this moment since they are empty, namely $A_2(0)$ and $A_3(0)$.

From assignment diagram for second period it is clear that zones $B_3(3), B_2(2)$ are already provided required number of UAVs. Thus we have to assign 2 UAVs only for one zone $B_1(2)$ and we have only one "free" base $A_1(2)$ with two UAVs. Thus on the third period we have the following situation:
1) $A_2(0) \xrightarrow{3} B_3(3)$- three UAVs continue their work in zone B_3;
2) $A_1 \xrightarrow{1} B_2$ - continue their work in zone B_2;
3) $A_3 \xrightarrow{1} B_2$ - continue their work in zone B_2;
4) $A_1 \xrightarrow{2} B_1$ - from base A_1 two UAVs should be sent to zone B_1 at time moment $t_{11} = 1250 - 1000 = 250 sec.$

Period 4: [1650,2050]- All bases are empty. Required to continue work at the zones $B_3(3), B_2(2)$ and at the zone $B_3(3)$ the service are already completed. Estimate resources (remain endurance) of UAVs which was sent to the zones $B_3(3), B_2(2)$. In case if they are have sufficient resources to flight back then we will continue to service corresponding zones. 1) $A_3 \xrightarrow{1} B_2$ - from base A_3 one UAV was sent to zone B_2 at time moment $t_{32} = 1050 - 666 = 334$. Calculate the time which this UAV was in air. $\tau^7 = 1650 - 334 = 1316 sec.$, where 7 means the ID-number of UAV. Also we should take into account the flight time ($666 sec.$) form the zone B_2 to base A_3. Thus the UAV (ID-number 7) will be in the air $\tau^7_{fly} = 1316 + 666 = 1982 sec.$, but the endurance for this UAV is $3600 sec.$ by initial condition. Then this UAV can continue to service zone B_2 2) $A_1 \xrightarrow{1} B_2$ - from base A_1 one UAV was sent to zone B_2 at time moment $t_{32} = 50$. Calculate the time which this UAV was in air. $\tau^1 = 1650 - 50 = 1600 sec.$, where 1 means the ID-number of UAV. Also we should take into account the flight time ($1000 sec.$) form the zone B_2 to base A_1. Thus the UAV (ID-number 1) will be in the air $\tau^7_{fly} = 1600 + 1000 = 2600 sec.$, but the endurance for this UAV is $3600 sec.$ by initial condition. Then this UAV can also continue to service zone B_2.

From this analysis follows that we can consider zone B_2 after finishing period 4 as a "new" base $B^a_{25}(2)$. 3)Besides this, after fours period we also can consider zone $B_3(3)$ with 3 UAVs as a new base B^a_{35}, since the resource of the UAVS worked there are sufficient to continue another mission requests (i.e. $\tau^4 = \tau^5 = \tau^6 = 1650 - 84 = 1566 sec.$). Their flight time to come back to base A_2 from the zone B_3 only $566 sec.$.

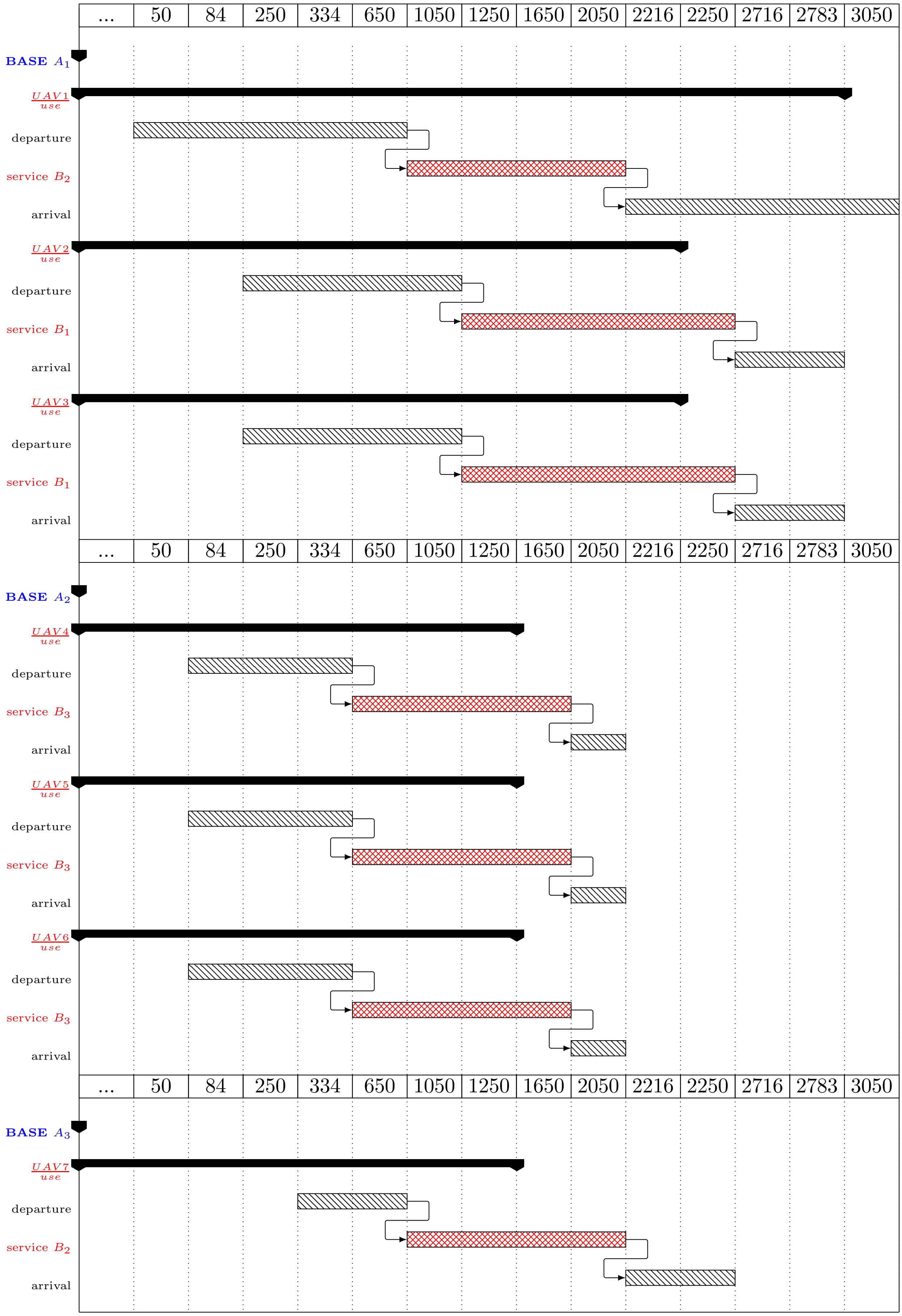
Since there is no new request the UAVs from B_2 can come back to their bases after the finishing period 4.

Period 5: [2050,2250]- It is required to continue work only at the zone $B_1(2)$. To this zone we was sent 2 UAVs from the base A_1. Calculate the remaining resources of these UAVs in order to estimate their ability to continue the service in B_1 during period 5. Calculate the time when these UAVs was in air. $\tau^2 = \tau_3 = 2050 - 250 = 1800 sec.$, where 2 and 3 means the ID-numbers of UAVs. Also we should take into account the flight time ($433 sec.$) form the zone B_1 to base A_1. Thus the UAVs (ID-number 2 and 3) will be in the air $\tau^2_{fly} = \tau^3_{fly} = 1800 + 433 = 2233 sec.$, but the endurance for this UAV is $3600 sec.$ by initial condition. Then this UAVs can continue to service zone B_1.

Thus all requests are completed. And our solution presented in next table:

	B_1	B_2	B_3	a_i
A_1	2	1	0	$a_1 = 3$
A_2	0	0	3	$a_2 = 3$
A_3	0	1	0	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

		B_1		B_2		B_3	
		Departure time (D/T)	Arrival time (A/T)	D/T	A/T	D/T	A/T
A_1	UAV 1	-	-	50	3050	-	-
	UAV 2	250	2783	-	-	-	-
	UAV 3	250	2783	-	-	-	-
A_2	UAV 4	-	-	-	-	84	2050
	UAV 5	-	-	-	-	84	2050
	UAV 6	-	-	-	-	84	2050
A_3	UAV 7	-	-	334	2250	-	-


... 50 84 250 334 650 1050 1250 1650 2050 2216 2250 2716 2783 3050
BASE A_1
UAV1 use
departure
service B_2
arrival
UAV2 use
departure
service B_1
arrival
UAV3 use
departure
service B_1
arrival
BASE A_2
UAV4 use
departure
service B_3
arrival
UAV5 use
departure
service B_3
arrival
UAV6 use
departure
service B_3
arrival
BASE A_3
UAV7 use
departure
service B_2
arrival

Calculate now our expenses. 1) We are used all available UAVs in order to complete all service request, since in Period 3 $[1250, 1650]$ we should use all UAVs simultaneously in order to serve three zones $(B_3(3), B_2(2), B_1(2))$. 2) Calculate the total flying time of all UAVs. For this reason calculate first the arrival time to the base after finishing the service in the corresponding zones:

UAV 1: $t^1_{arrival} = 2050 + 1000 = 3050$;
UAV 2: $t^2_{arrival} = 2250 + 533 = 2783$;
UAV 3: $t^3_{arrival} = 2250 + 533 = 2783$;
UAV 4: $t^4_{arrival} = 1650 + 566 = 2216$;
UAV 5: $t^5_{arrival} = 1650 + 566 = 2216$;
UAV 6: $t^6_{arrival} = 1650 + 566 = 2216$;
UAV 7: $t^7_{arrival} = 2050 + 666 = 2716$.

Then total flying time of all UAVs are : $1 * (3050 - 50) + 2 * (2783 - 250) + 3 * (2216 - 84) + 1 * (2716 - 334) = 16394 sec. \approx 4,954 hours$

2.2.7 An modification of the standard assignment problem solution for the time constraints

To realize a modification of the old solution we can extend the number of bases by adding some UAVs from zones sent at the previous periods. In particular, in addition to the base A_1, A_2, and A_3 we will consider the zone B_3 also as a base, namely we can use UAVs from this zone in order to complete requests from remaining zones. This logic can be used after each period (i.e. the number of A_i and B_j will be changed on each period).

Now we try to modify the old solution of the assignment problem obtained in the previous section without "time window" constraints. This solution is presented by the following matrix:

$$X^{flight} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.39}$$

It was shown above that for the zone B_3 in "time window" $[650, 1650]$ this solution is not satisfactory. Namely, in according with the plan (2.39) we need to send to the zone B_3 one UAV from each base. Clear that from base A_1 and A_2 the UAVs are able to reach the zone B_3 to the moment of opening "time window ", namely at the time moment 650 (i.e.

the flight time $A_1 \xrightarrow{600} B_3$ and $A_2 \xrightarrow{566} B_3$). But from the base A_3 the UAV not able to reach the zone B_3 at required time moment, since $A_3 \xrightarrow{766} B_3$.

Thus we are need to find another plan for "time window" $[650, 1650]$. In order to find alternative for UAV from A_3 find a nearest base to B_3. From the table below follows that nearest base is B_1. And the flight time from B_3 to B_1: $t_{fly}^{B_3 \to B_1} = \frac{d_{ij}}{v_i j} = \frac{6000}{30} = 200$ ($B_3 \xrightarrow{200} B_1$).

	B_1	B_2	B_3
B_1	0	17	6
B_2	17	0	14
B_3	6	14	0

(2.40)

Distances between zones B_j

From (2.39) follows that we will send 2 UAVs to the zone B_1 from base A_1 to time windows [1250,2250]. Thus we can use another UAV form base A_1 in order to serve on first period [650,1650] base B_3, since flight time $A_1 \xrightarrow{600} B_3$ allows us to do this. Then we modify the initial plan as follows:

$$X^{flight^1} = \begin{pmatrix} 2-1 & 0 & 1+1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.41}$$

Thus on the first period [650;1650] we have the following situation:
1) $A_1 \xrightarrow{2} B_3$- from base A_1 two UAVs should be sent to zone B_3 at time moment $t_{13}^{departure} = 650 - 600 = 50sec.$ 2) $A_2 \xrightarrow{1} B_3$ - from base A_2 one UAV should be sent to zone B_3 at time moment $t_{23}^{departure} = 650 - 566 = 84sec.$

Also we are know that according to old plan (2.39) $A_3 \xrightarrow{766} B_3$. It means that at time moment $t = 766$ the UAV from A_3 can reach the zone B_3 and we can come back to old plan after that moment. Namely if we will send from B_3 one of UAV belonging to the base A_1 to B_1 ($B_3 \xrightarrow{1} B_1$) at time moment $t = 766$ then this UAV reach the zone B_1 at time moment $t = 766 + 200 = 966$, but the zone B_1 will be closed for service and will be open only at time moment $t = 1250$. From this logic clear that we will send UAV from A_3 to B_3 tacking into account the the time window for base B_1 and flight time $B_3 \xrightarrow{200} B_1$. Namely, UAV from A_3 to reach the zone B_3 at the moment $t = 1250 - 200 = 1050$. For this reason it should departure form A_3 at $t_{33}^{departure} = 1050 - 766 = 284sec.$. And UAV beloning to A_1 will move from B_3 to B_1 at time moment $t_{31}^{b} = 1050$. And after that moment we will come back to old plan (2.39).

Thus we make the following modification:
1) Switch departure time of UAV (ID number 7) from base A_3: $t_{33}^{departure} = 284sec$
2) For UAV (ID number 1) from base A_1: a) $t_{13}^{departure} = 50$; b) service zone B_3 during period [650,1050]; c) then fly $B_3 \xrightarrow{200} B_1$; d) service zone B_1 during period [1250,2250]; e) then fly $B_1 \xrightarrow{433} A_1$; f) arrive to base A_1 at time moment $t_{11}^{arrival} = 2250 + 433 = 2683sec.$
For UAV (ID number 2) from base A_1: a) $t_{11}^{departure} = 1250 - 433 = 817$; b) service zone B_1 during period [1250,2250]; c) then fly $B_1 \xrightarrow{433} A_1$; d) arrive to base A_1 at time moment $t_{11}^{arrival} = 2250 + 433 = 2683sec.$

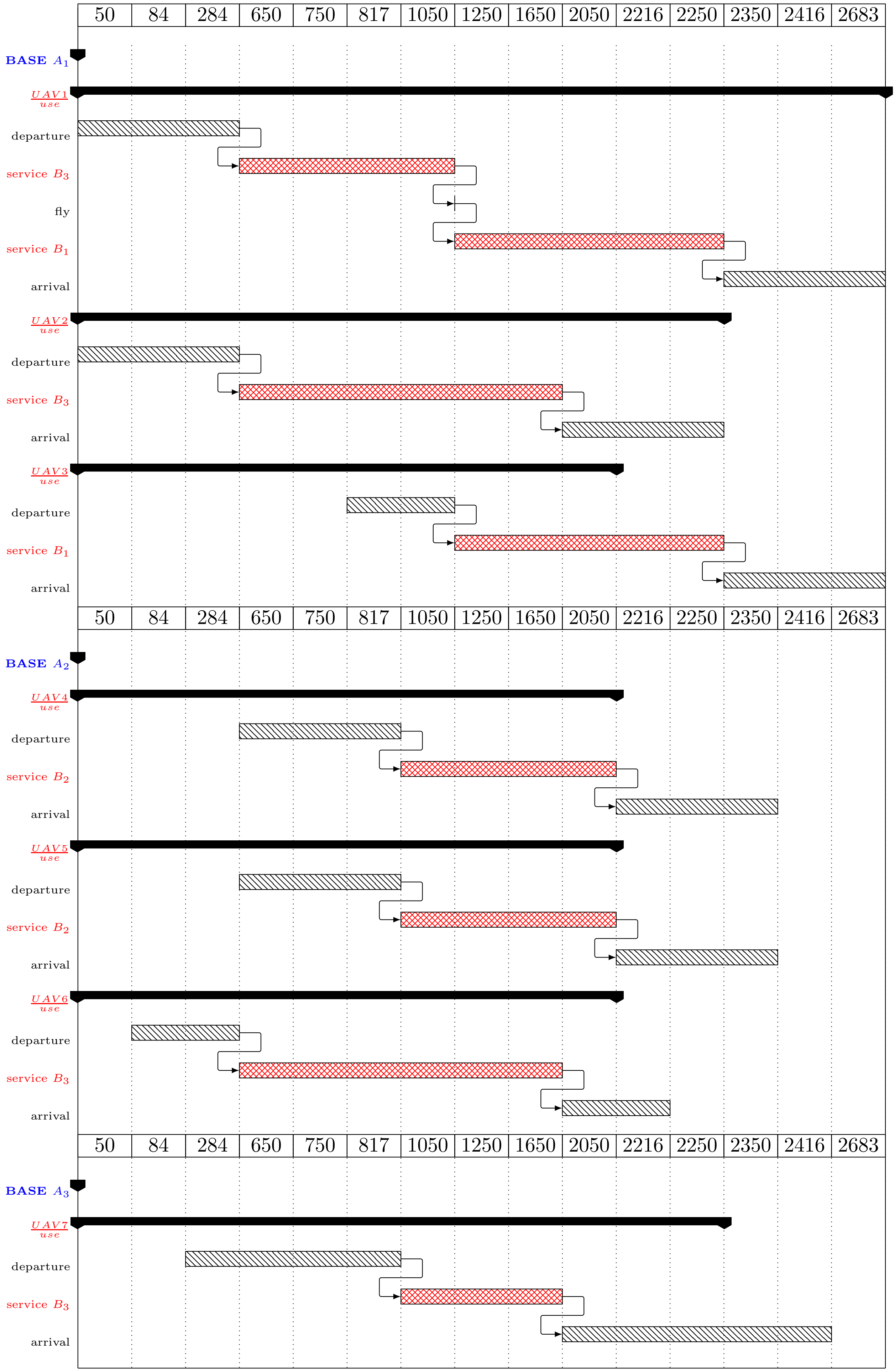
And we have the following schedular plan:

$$A_1(3) \xrightarrow{2} B_1(2)\ ,\ A_1(3) \xrightarrow{1} B_3(3)\ ,\ A_2(3) \xrightarrow{2} B_2(2)\ ,\ A_2(3) \xrightarrow{1} B_3(3)\ ,\ A_3(1) \xrightarrow{1} B_3(3)$$

or in table form:

	B_1	B_2	B_3	a_i
A_1	2	0	1	$a_1 = 3$
A_2	0	2	1	$a_2 = 3$
A_3	0	0	1	$a_3 = 1$
b_j	$b_1 = 2$	$b_2 = 2$	$b_3 = 3$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 7$

		B_1		B_2		B_3	
		Departure time (D/T)	Arrival time (A/T)	D/T	A/T	D/T	A/T
A_1	UAV 1	-	2683	-	-	50	-
	UAV 2	-	-	-	-	50	2250
	UAV 3	817	2683	-	-	-	-
A_2	UAV 4	-	-	650	2350	-	-
	UAV 5	-	-	650	2350	-	-
	UAV 6	-	-	-	-	84	2216
A_3	UAV 7	-	-	-	-	284	2416


50 84 284 650 750 817 1050 1250 1650 2050 2216 2250 2350 2416 2683
BASE A_1
UAV1 use
departure
service B_3
fly
service B_1
arrival
UAV2 use
departure
service B_3
arrival
UAV3 use
departure
service B_1
arrival
BASE A_2
UAV4 use
departure
service B_2
arrival
UAV5 use
departure
service B_2
arrival
UAV6 use
departure
service B_3
arrival
BASE A_3
UAV7 use
departure
service B_3
arrival

Calculate now our expenses. 1) We are used all available UAVs in order to complete all service request, since in Period 3 $[1250, 1650]$ we should use all UAVs simultaneously in order to serve three zones $(B_3(3), B_2(2), B_1(2))$. 2) Calculate the total flying time of all UAVs. For this reason calculate first the arrival time to the base after finishing the service in the corresponding zones:

UAV 1: $t^1_{arrival} = 2683;$
UAV 2: $t^2_{arrival} = 1650 + 600 = 2250;$
UAV 3: $t^3_{arrival} = 2683;$
UAV 4: $t^4_{arrival} = 2050 + 300 = 2350;$
UAV 5: $t^5_{arrival} = 2050 + 300 = 2350;;$
UAV 6: $t^6_{arrival} = 1650 + 566 = 2216;$
UAV 7: $t^7_{arrival} = 1650 + 766 = 2416.$

Then total flying time of all UAVs are : $1*(2683-50)+1*(2250-50)+1*(2683-817)+2*(2350-750)+1*(2216-84)+1*(2416-284) = 2633-2200+1866+3200+2132+2132 = 12163 sec. \approx 3,39 hours$

Chapter 3

Dynamical assignment of UAVs for multiple areas of operation

3.1 Problem statement

Let $[0, H]$ is the given period for service of $B_1, B_2, ..., B_j, ..., B_l$ zones of area of operation. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \dots, l$. Also, assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$ with $a_1, a_2, ..., a_i, ..., a_k$ number of homogenous UAVs, respectively. The problem is to assign UAVs between areas of operations $B_j, j = 1, \dots, l$ in a such way that the total service time will be maximal.

3.2 Variables and constants

Divide the interval $[0, H]$ by the moments $t = i\Delta,\ i = 1, 2, ..., \nu$ where $\nu = \left[\dfrac{H}{\Delta}\right]$ denotes the integer part of the fraction $\dfrac{H}{\Delta}$, and Δ is a small number the concrete value of that depends on efficiency of numerical algorithms and will be determined later. Hence, we have the time interval partition

$$0 < \Delta < 2\Delta < ... < i\Delta < (i+1)\Delta < .... < H.$$

For each discrete moment $t = i\Delta$, $i = 1, 2, ..., \nu$, introduce the following variables:

1. $x_{ij}(t)$ is the number of UAVs from i-th aerobase send to j-th zone at the moment t;

2. $a_i(t)$ is the number of UAVs at $i-th$ aerobase at the moment t;

3. $b_j(t)$ is the number of UAVs that are serving the $j-th$ zone at the moment t;

4. t_{ij} is the flight time from $i-th$ aerobase to $j-th$ zone;

5. k and l are the number of aerobases and zones for service, respectively;

6. h_i is the flight endurance for UAVs from $i-th$ aerobase.

Obviously, at the initial moment $t = 0$ we have $b_j(0) = b_j,\ j = 1, 2, ..., l;\quad a_i(0) = a_i,\ i = 1, 2, ..., k.$

Now, we obtain the relation describing the dynamic of introduced variables.

3.3 Constraints

1) The number of UAVs at $i-th$ aerobase at the next moment $t+\Delta$ is composed of UAVs that are being at the previous moment t, plus UAVs that are returned during the period $[t, t+\Delta]$, and minus UAVs that were send to zones at the moment t. These facts give the following equalities

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t+\Delta-h_i),\ i = 1, ..., k. \tag{3.1}$$

The term $\sum_{j=1}^{l} x_{ij}(t+\Delta-h_i)$ denotes UAVs that were send early, and that should come back due to their flight endurance

Here we consider those objects where argument $t+\Delta-h_i > 0$. Otherwise, the term $x_{ij}(t+\Delta-h_i)$ means that $i-th$ UAV has sufficient endurance to continue service of $j-th$ zone, and hence, it can not come back to aerobase.

The initial conditions are $a_i(0) = a_i,\ i = 1, 2, ..., k.$

2) The number of UAVs that will serve the $j-th$ zone at the next moment $t+\Delta$ is composed of UAVs that are serving this zone at the previous moment t and having sufficient flight endurance, plus UAVs that reach this zone during the period $(t, t+\Delta]$, and

minus UAVs that are out-of-fuel to the moment t. These facts lead the following equalities

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}),\ j = 1, ..., l. \tag{3.2}$$

The term $\sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij})$ denotes UAVs that should leave the $j - th$ zone due to their out-of-fuel. The term $\sum_{i=1}^{k} x_{ij}(t - t_{ij})$ denotes UAVs that were send early and should reach the $j - th$ zone during the period $(t, t + \Delta]$.

Here we consider those objects where arguments $t - h_i + t_{ij} > 0$ and $t - t_{ij} > 0$.

The initial conditions are $b_j(0) = b_j,\ j = 1, 2, ..., l$.

3) The variables $x_{ij}(t)$ at each moment t satisfy the following conditions

$$\begin{aligned} a_i(t) + \sum_{j=1}^{l} x_{ij}(t) &= a_i,\ i = 1, ..., k. \\ b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) &= b_j,\ (t - t_{ij} > 0)\ j = 1, ..., l. \end{aligned} \tag{3.3}$$

The first equation images the fact that the being UAVs can be allocated among zones. The second equation means that at each moment the service request should be satisfied.

3.4 Types of objective function

4) The cost value function can be determined as follows:

a) the total service time for multiple zones

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}). \tag{3.4}$$

b) the total number of UAVs "circles"

$$J_2(x) = \sum_{t=0}^{\nu} x_{ij}(t) \tag{3.5}$$

c) the total unobservable time for multiple zones

$$J_3(x) = \sum_{t=0}^{\nu} x_{ij}(t)(H - h_i - 2t_{ij}) \tag{3.6}$$

Thus, the optimal schedule problem of UAVs for multiple zones can be formulated as, for example, the following special integer dynamical linear programming problem:

maximize the cost value function

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu} \tag{3.7}$$

subject to

$$\begin{aligned} a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t+\Delta-h_i),\ i = 1, ..., k. \\ b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}),\ j = 1, ..., l. \\ a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k. \\ b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1, ..., l. \end{aligned} \tag{3.8}$$

where $\nu = \left[\frac{H}{\Delta}\right]$ denotes the integer part of the fraction $\frac{H}{\Delta}$.

Here we consider those objects where arguments $t - h_i + t_{ij} > 0$ and $t - t_{ij} > 0$.

Remark 2.

The proposed partition of the planing horizon $[0, H]$ with small step Δ yields an ability to produce optimal schedule for UAVs, in fact, in regime of real time. The realization of this idea demands the development of some fast numerical algorithms for solution of the special classes of linear programming problems. Some new approaches to accelerate the solution of general linear programming problem is discussed in the paper [**?**]

Remark 3. The proposed dynamical transportation problem (3.21)—(3.22) for allocation of MAS can be presented as a statistic problem given in the previous paragraph. But this way leads to the huge dimensions of the variables involved, and this together the specific structure of the considered problem are a serious obstacle for suitable solution for reasonable time. By this reason the development of special methods and design on this base of fast numerical methods for assignment problems of MAS with next their realization in the corresponding computer chips are actual.

3.5 Dynamical assignment of UAVs for multiple areas of operation

3.5.1 Problem statement

Let $[0, H]$ is the given period for service of $B_1, B_2, ..., B_j, ..., B_l$ zones of area of operation. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \ldots, l$. Also, assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$ with $a_1, a_2, ..., a_i, ..., a_k$ number of homogenous UAVs, respectively.

The problem is to assign UAVs between areas of operations $B_j, j = 1, \ldots, l$ in a such way that the total service "profit" will be maximal. The different notions of "profit" will be introduced later.

3.5.2 Variables and constants

Divide the interval $[0, H]$ by the moments $t_s = s\Delta,\ s = 1, 2, ..., \nu$ where $\nu = \left[\dfrac{H}{\Delta}\right]$ denotes the integer part of the fraction $\dfrac{H}{\Delta}$, and Δ is a small number. The concrete value of Δ can be stated experimentally and depends on efficiency of the used numerical algorithms.

Hence, we have the time interval partition

$$0 < \Delta < 2\Delta < ... < s\Delta < (s + 1)\Delta < .... < H.$$

For each discrete moment $t_s = s\Delta$, $s = 1, 2, ..., \nu$, introduce the following variables:

1. $x_{ij}(t_s)$ is the number of UAVs from i-th aerobase send to j-th zone at the moment t_s;

2. $a_i(t_s)$ is the number of UAVs at $i-th$ aerobase at the moment t_s;
3. $b_j(t_s)$ is the number of UAVs that are serving the $j-th$ zone at the moment t_s;
4. t_{ij} is the flight time from $i-th$ aerobase to $j-th$ zone;
5. k and l are the number of aerobases and zones for service, respectively;
6. h_i is the flight endurance for UAVs from $i-th$ aerobase.

Note that homogeneous of UAVs in each aerobase is not restricted since UAVs can be classified or, in final, in the simplest case we can consider the position when each aerobase is complicated by a single UAV. Obviously, at the initial moment $t = 0$ we have $b_j(0) = b_j,\ j = 1, 2, ..., l;\quad a_i(0) = a_i,\ i = 1, 2, ..., k.$

Now, we state the relation describing the dynamic of introduced variables.

3.5.3 Constraints

1) The number of UAVs at $i-th$ aerobase at the next moment $t_s+\Delta$ is composed of UAVs that are being at the previous moment t_s, plus UAVs that are returned during the period $[t_s, t_s + \Delta]$, and minus UAVs that were send to zones at the moment t_s. These facts give the following equalities

$$a_i(t_s + \Delta) = a_i(t_s) - \sum_{j=1}^{l} x_{ij}(t_s) + \sum_{j=1}^{l} x_{ij}(t_s + \Delta - h_i),\ i = 1, ..., k. \tag{3.9}$$

The term $\sum_{j=1}^{l} x_{ij}(t_s + \Delta - h_i)$ denotes UAVs that were send early, and that should come back due to their flight endurance. Otherwise, the term $x_{ij}(t_s + \Delta - h_i)$ means that $i - th$ UAV has sufficient endurance to continue service of $j-th$ zone, and hence, it can not come back to aerobase. Here we consider those objects where argument $t_s + \Delta - h_i > 0$. The initial conditions are $a_i(0) = a_i,\ i = 1, 2, ..., k.$

2) The number of UAVs that will serve the $j - th$ zone at the next moment $t_s + \Delta$ is composed of UAVs that are serving this zone at the previous moment t_s and having sufficient flight endurance, plus UAVs that reach this zone during the period $(t_s, t_s + \Delta]$, and minus UAVs that are out-of-fuel to the moment t. These facts lead the following equalities

$$b_j(t_s + \Delta) = b_j(t_s) - \sum_{i=1}^{k} x_{ij}(t_s - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t_s - t_{ij}),\ j = 1, ..., l. \tag{3.10}$$

The term $\sum_{i=1}^{k} x_{ij}(t_s - h_i + t_{ij})$ denotes UAVs that should leave the $j-th$ zone due to their out-of-fuel. The term $\sum_{i=1}^{k} x_{ij}(t_s - t_{ij})$ denotes UAVs that were send early and should reach the $j-th$ zone during the period $(t_s, t_s + \Delta]$. Here we consider those objects where arguments $t_s - h_i + t_{ij} > 0$ and $t_s - t_{ij} > 0$. The initial conditions are $b_j(0) = b_j,\ j = 1, 2, ..., l$.

3) The variables $x_{ij}(t_s)$ at each moment $t_s, s = 1, ..., \nu$ satisfy the following conditions

$$\begin{aligned} a_i(t_s) + \sum_{j=1}^{l} x_{ij}(t_s) &= a_i,\ i = 1, ..., k. \\ b_j(t_s) + \sum_{i=1}^{k} x_{ij}(t_s - t_{ij}) &= b_j,\ (t_s - t_{ij} > 0)\ j = 1, ..., l. \end{aligned} \tag{3.11}$$

The first equation images the fact that the being UAVs can be allocated among zones. The second equation means that at each moment the service request should be satisfied.

The given above maim body of the problem constraints can be completed by additional conditions (constraints) followed from description of the Task 5.

4) Let τ_j^{first} is the given earliest time of 1-st visit to j zone for each $j,\ 1 \leq j \leq l$. Then the constraints (3.9)—(3.11) can be supplemented by the following:

$$\sum_{i=1}^{k} x_{ij}(s_j^{first}) \neq 0,\ 1 \leq j \leq l \tag{3.12}$$

where s_j^{first} is the discrete moment from the set $s = 1, 2, ..., \nu$ satisfying the following conditions:

$$s_j^{first}\Delta \leq \tau_j^{first} \leq (s_j^{first} + 1)\Delta \qquad \text{for some} \ \ s_j^{first} \in \{1, 2, ..., \nu\}.$$

The inequality (3.12) means that there exist at least one aerobase such that their UAVs will start with 1-st service visit to j zone no later on the preassigned moment τ_j^{first}.

Remark. If the τ_j^{first} is treated as the moment before of which the service of j zone is prohibited, then the constraints (3.9)—(3.11) can be supplemented by the following:

$$x_{ij}(s\Delta) = 0,\ 1 \leq j \leq l,\ \forall\ s\Delta \leq \tau_j^{first} \text{ and } \ \forall\ i = 1, 2, ..., k \tag{3.13}$$

5) Let τ_j^{latest} is the given latest time of 1-st visit to j zone for each $j,\ 1 \leq j \leq l$. Then the constraints (3.9)—(3.11) can be supplemented by the following:

$$\sum_{i=1}^{k} x_{ij}(s_j^{first}) \neq 0, j,\ 1 \leq j \leq l \tag{3.14}$$

where s_j^{first} is the discrete moment from the set $s = 1, 2, ..., \nu$ satisfying the following conditions:

$$s_j^{first}\Delta \leq \tau_j^{first} \leq (s_j^{first} + 1)\Delta \qquad \text{for some} \quad s_j^{first} \in \{1, 2, ..., \nu\}$$

and such that

$$s_j^{first} + h_i \leq \tau_j^{latest}, \quad 1 \leq j \leq l, \; \forall \; i \in I_j^{first} \tag{3.15}$$

where

$$I_j^{first} = \{i, \; 1 \leq i \leq k : x_{ij}(s_j^{first}) \neq 0\}.$$

The couple of inequalities (3.14) — (3.15) means that there exist at least one aerobase such that their UAV the 1-st visit to j zone will begin no later the pre-assigned earliest time τ_j^{first}, and the ending this 1-st visit to j zone will be no later the pre-assigned the latest time τ_j^{latest} .

6) Let τ_j^{last} is the given last time of visits to j zone for each $j, \; 1 \leq j \leq l$. Then the constraints (3.9)—(3.11) can be supplemented by the following:

$$x_{ij}(s_j^{last}) = 0, \quad \text{for all} \quad s_j^{last} \leq t_s \leq \nu, \quad 1 \leq i \leq k \quad \text{and} \quad 1 \leq j \leq l \tag{3.16}$$

where s_j^{last} is the discrete moment from the set $s = 1, 2, ..., \nu$ satisfying the following conditions:

$$(s_j^{last} - 1)\Delta \leq \tau_j^{last} \leq s_j^{last}\Delta \qquad \text{for some} \quad s_j^{last} \in \{1, 2, ..., \nu\}.$$

The equalities (3.16) denotes that all visits to j zone after the preassigned moment τ_j^{last} are prohibited.

3.5.4 Types of objective function

The cost value function used for optimization problem can be determined as follows:

a) the total service time for multiple zones

$$J_1(x) = \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{s=0}^{\nu} x_{ij}(t_s)(h_i - 2t_{ij}). \tag{3.17}$$

b) the total number of UAVs "circles"

$$J_2(x) = \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{s=0}^{\nu} x_{ij}(t_s) \tag{3.18}$$

c) the total unobservable time for multiple zones

$$J_3(x) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(t_s)(H - h_i - 2t_{ij}) \tag{3.19}$$

d) time of the first visit in the worst-case zone

$$J_4(x) = \max_{1\le j\le k} t_j^{first}, \quad \text{where } t_j^{first} = \min_{1\le i\le k}\min_{1\le s\le \nu}\{t_s : x_{ij}(t_s) \neq 0.\} \tag{3.20}$$

Hence, the optimization problem can be equipped by any of the proposed cost functions. In addition, some combinations of these function with the proper weighting coefficients can be used as a new cost function.

Thus, the optimal schedule problem of UAVs for multiple zones can be formulated, for example in the case of maximization of the total service time for multiple zones, as the following special integer dynamical linear programming problem (we change t_s by $t_s = s\Delta,\ s = 1, 2, ..., \nu$): maximize the cost value function

$$J_1(x) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(s\Delta)(h_i - 2t_{ij}) \to \max_{x_{ij}(s\Delta)\in\mathbb{N}, s=0,1,....,\nu} \tag{3.21}$$

subject to

$$\begin{aligned} a_i(s\Delta + \Delta) &= a_i(s\Delta) - \sum_{j=1}^{l} x_{ij}(s\Delta) + \sum_{j=1}^{l} x_{ij}(s\Delta + \Delta - h_i),\ i = 1, ..., k. \\ b_j(s\Delta + \Delta) &= b_j(s\Delta) - \sum_{i=1}^{k} x_{ij}(s\Delta - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(s\Delta - t_{ij}),\ j = 1, ..., l. \\ & a_i(s\Delta) + \sum_{j=1}^{l} x_{ij}(s\Delta) = a_i,\ i = 1, ..., k. \\ & b_j(s\Delta) + \sum_{i=1}^{k} x_{ij}(s\Delta - t_{ij}) = b_j,\ j = 1, ..., l, \\ & s = 1, 2, ..., \nu, \end{aligned} \tag{3.22}$$

and

$$\sum_{i=1}^{k} x_{ij}(s_j^{first}) \neq 0,\ 1 \le j \le l \tag{3.23}$$

$$s_j^{first} + h_i \le \tau_j^{latest},\quad 1 \le j \le l,\ \forall\, i \in I_j^{first}, \tag{3.24}$$

$$x_{ij}(s_j^{last}) = 0, \quad \text{for all}\ \ s_j^{last} \le t_s \le \nu, \quad 1 \le i \le k \ \text{ and } \ 1 \le j \le l \tag{3.25}$$

where

s_j^{first} is the discrete moment from the set $s = 1, 2, ..., \nu$ satisfying the conditions

$$s_j^{first}\Delta \le \tau_j^{first} \le (s_j^{first} + 1)\Delta \qquad \text{for some}\ \ s_j^{first} \in \{1, 2, ..., \nu\},$$

s_j^{last} is the discrete moment from the set $s = 1, 2, ..., \nu$ satisfying the conditions:

$$(s_j^{last} - 1)\Delta \leq \tau_j^{last} \leq s_j^{last}\Delta \qquad \text{for some} \;\; s_j^{last} \in \{1, 2, ..., \nu\}$$

and

$$I_j^{first} = \{i, \; 1 \leq i \leq k : x_{ij}(s_j^{first}) \neq 0\}.$$

Again note that $\nu = \left[\frac{H}{\Delta}\right]$ means the integer part of the fraction $\frac{H}{\Delta}$.

In (3.22) we consider those terms and elements where arguments $s\Delta - h_i + t_{ij} > 0$ and $s\Delta - t_{ij} > 0$.

Other optimization problem with another cost function mentioned above can be formulated by similar manner.

Remark 2.

The proposed partition of the planing horizon $[0, H]$ with small step Δ yields an ability to produce optimal schedule for UAVs, in fact, in regime of real time. The realization of this idea demands the development of some fast numerical algorithms for solution of the special classes of linear programming problems. Some new approaches to accelerate the solution of general linear programming problem is discussed in the paper [**?**]

Remark 3. In order to take into account the other request followed from description of the Task 5, the proposed model can be reformulated with the corresponding cost function. For example, the request to organize the zone service with maximum intervals between visits can be presented by maximization of the total unobservable time

$$J_3(x) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(t_s)(H - h_i - 2t_{ij}) \rightarrow \max_{x_{ij}(s\Delta)\in\mathbb{N}, s=0,1,...,\nu} \tag{3.26}$$

subject to constraints of (3.22) and (3.23).

The request to organize the zone service with minimum duration per visits can be presented by minimization of the total service time

$$J_1(x) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(t_s)(h_i - 2t_{ij}) \rightarrow \min_{x_{ij}(s\Delta)\in\mathbb{N}, s=0,1,...,\nu}. \tag{3.27}$$

subject to constraints of (3.22)and (3.23).

Remark 4. Another way to satisfy the multiple requests for zone service can be realized by optimization of one of the selected cost function and including the remained cost function into the main body of constraints (3.22)and (3.23) as follows, for example:

minimize the total number of UAVs "circles"

$$J_2(x) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(t_s) \to \min_{x_{ij}(s\Delta)\in\mathbb{N}, s=0,1,\dots,\nu}. \tag{3.28}$$

subject to constraints of (3.22)and (3.23) and the following new constraints

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(s\Delta)(h_i - 2t_{ij}) \leq A, \tag{3.29}$$

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{s=0}^{\nu} x_{ij}(t_s)(H - h_i - 2t_{ij}) \leq B \tag{3.30}$$

where A and B are the known numbers those values can be given by specialists or can be determined experimentally.

Solution result representation

The obtained solution of optimization problem (3.21)—(3.23) can be presented in the form convenient for the practical uses. Such presentation can be done, for example, by Diagram or Schedule Table of time and duration visits by each MAS for the chosen zones.

Let $x_{ij}^0(t_s),\ s = 1,...,\nu,\ i = 1,...,k,\ j = 1,...,l$ is optimal solution of (3.21)— (3.23) where $t_s = s\Delta$ and Δ is the sampling (discretisation) step.

First, we indicate resulting information concerning history of zone observation due to the obtained solution. For each zone j, where $1 \leq j \leq l$, define the following characteristics:

- $I_j = \{i \in \{1, 2, ..., k\} : x_{ij}^0(t_s) \neq 0, s = 1, 2..., \nu\}$ — the indexes of aerobase used for observation of j zone;
- $k_j = |I_j|$—the total number of aerobases involved in observation of j zone. Here $|I_j|$ denotes the number of elements of the set I_j);
- $m_j = |S_j|,$ where $S_j = \{s \in \{1, 2, ..., \nu\} : x_{ij}^0(t_s) \neq 0, i = 1, 2..., k\}$ — the number of the discrete intervals of the form $[t_s, t_s + \Delta]$ during of which the observation by UAVs is realized for j zone. (Here $|S_j|$ denotes the number of elements of the set S_j);
- $T_j^{zone} = m_j\Delta$— the total duration of observation time for the j zone;
- $N_j^{zone} = \sum\limits_{i=1}^{k}\sum\limits_{s=1}^{\nu} x_{ij}^0(t_s)$—total number of UAVs used for observation of j zone;
- $t_j^{first} = \min\limits_{1\leq s\leq\nu}\{t_s : x_{ij}^0(t_s) \neq 0, i = 1, 2..., k\}$—the time of first visit to j zone;

- $t_j^{last} = \max\limits_{1\leq s\leq\nu}\{t_s : x_{ij}^0(t_s) \neq 0, i = 1, 2..., k\} + \Delta$—the time of the ending of observation of j zone;
- i_j^{first} —- those aerobases the UAVs of which were the first visitors of j zone, where i_j^{first} is the indexes from the set $\{1, 2, ..., k\}$ where the minimum for t_j^{first} is reached ;
- i_j^{last} — those aerobases the UAVs of which were the last visitors of j zone, where i_j^{last} is the indexes from the set $\{1, 2, ..., k\}$ where the maximum for the value t_j^{last} is reached.

It is possible, also, to restore more detail information concerning the visits of j zone between the first moment t_j^{first} of visit and the last moment t_j^{last} with detailed history for each observation intervals and UAVs involved in this observation.

On other hand, for UAVs addressed to observation mission can be useful information concerning their visits schedule for all zones for observation of which they are used.

For each i aerobase, where $1 \leq i \leq k$, define the following characteristics:

- $J_i = \{j \in \{1, 2, ..., l\} : x_{ij}^0(t_s) \neq 0, s = 1, 2..., \nu\}$ — the indexes of zone for observation of which the i aerobase is used;

  $M_i = \sum\limits_{i=1}^{k}\sum\limits_{s=1}^{\nu} x_{ij}^0(t_s)$—total number of UAVs used for observation of j zone;
- $\tau_i^{first} = \min\limits_{1\leq s\leq\nu}\{t_s : x_{ij}^0(t_s) \neq 0, j = 1, 2..., l\}$—the time of the first mission fly of UAVs of i aerobase;
- j_i^{first} —- those zones, for which the UAVs of i aerobase are used firstly for observation mission, where j_i^{first} is the indexes from the set $\{1, 2, ..., l\}$ where the minimum for τ_i^{first} is reached ;

This list of characteristics for UAVs of i aerobase can be continued by obviously manner.

3.5.5 Reformulation in matrix form

The proposed dynamical transportation problem (3.21)—(3.23) for allocation of MAS can be presented as a statistic problem given in the previous paragraph. But this way leads to the huge dimensions of the variables involved, and this together the specific structure of the considered problem are a serious obstacle for suitable solution for reasonable time. By this reason the development of special methods and design on this base of fast numerical methods for assignment problems of MAS with next their realization in the corresponding computer chips are actual and will be done at this work.

Introduce the following matrixes

$$A_{k\times\nu} = \begin{pmatrix} a_1(\Delta) & a_1(2\Delta) & ... & a_1(\nu\Delta) \\ a_2(\Delta) & a_2(2\Delta) & ... & a_2(\nu\Delta) \\ a_3(\Delta) & a_3(2\Delta) & ... & a_3(\nu\Delta) \\ ... & ... & ... & ... \\ a_k(\Delta) & a_k(2\Delta) & ... & a_k(\nu\Delta) \end{pmatrix}, \tag{3.31}$$

$$H^{-}_{\nu\times(\nu-1)} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}, \tag{3.32}$$

$$H^{+}_{\nu\times(\nu-1)} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \tag{3.33}$$

$$X_i = \begin{pmatrix} x_{i1}(\Delta) & x_{i1}(2\Delta) & ... & x_{i1}(\nu\Delta) \\ x_{i2}(\Delta) & x_{i2}(2\Delta) & ... & x_{i2}(\nu\Delta) \\ ... & ... & ... & ... \\ x_{il}(\Delta) & x_{il}(2\Delta) & ... & x_{il}(\nu\Delta) \end{pmatrix}_{l\times\nu}, i = 1, ..., k \tag{3.34}$$

Introduce the block matrixes of the form

$$X = \begin{pmatrix} X_1 \\ X_2 \\ ... \\ X_k \end{pmatrix}_{kl\times\nu}, \ \Pi = \begin{pmatrix} e_l & 0_l & ... & 0_l \\ 0_l & e_l & ... & 0_l \\ ... & ... & ... & ... \\ 0_l & 0_l & ... & e_l \end{pmatrix}_{k\times kl} \tag{3.35}$$

where

$$e_l = \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix}_{1\times l}, 0_l = \begin{pmatrix} 0 & 0 & ... & 0 \end{pmatrix}_{1\times l}, \tag{3.36}$$

Remark. Since the unknown variables of the optimization problem are $x_{ij}(\Delta), x_{ij}(2\Delta), ..., x_{ij}(\nu\Delta)$, then the other variables $x_{ij}(t)$ with argument t that is not coincide with arguments $\Delta, 2\Delta, ..., \nu\Delta$ will be approximated by the variables $x_{ij}(s\Delta)$ where $s = \left[\frac{t}{\Delta}\right]$ is the integer part of the number $s = \frac{t}{\Delta}$ such that the argument $s\Delta$ is the nearest to the argument

t. Such kind approximation is admissible due to the freedom in choice of sampling step Δ. We assume, in fact, that for the considered optimization problem the unknown continuous function $x_{ij}(\tau)$ of the real variable τ can be approximated by piecewise constant function $x_{ij}(s\Delta),\ s = 1, ..., \nu$.

Noting the given remark, introduce the following matrixes

$$
\begin{aligned}
h(X_i) &= \begin{pmatrix}
x_{i1}\left(\Delta\left[\frac{\Delta - h_i}{\Delta}\right]\right) & x_{i1}\left(\Delta\left[\frac{2\Delta - h_i}{\Delta}\right]\right) & ... & x_{i1}\left(\Delta\left[\frac{\nu\Delta - h_i}{\Delta}\right]\right) \\
x_{i2}\left(\Delta\left[\frac{\Delta - h_i}{\Delta}\right]\right) & x_{i2}\left(\Delta\left[\frac{2\Delta - h_i}{\Delta}\right]\right) & ... & x_{i2}\left(\Delta\left[\frac{\nu\Delta - h_i}{\Delta}\right]\right) \\
... & ... & ... & ... \\
x_{il}\left(\Delta\left[\frac{\Delta - h_i}{\Delta}\right]\right) & x_{il}\left(\Delta\left[\frac{2\Delta - h_i}{\Delta}\right]\right) & ... & x_{il}\left(\Delta\left[\frac{\nu\Delta - h_i}{\Delta}\right]\right)
\end{pmatrix}_{l\times\nu}, \\
i &= 1, ..., k
\end{aligned}
$$

Introduce the block matrixes of the form

$$
h(X) = \begin{pmatrix} h(X_1)H^- \\ h(X_2)H^- \\ ... \\ h(X_k)H^- \end{pmatrix}_{kl\times(\nu)}, \ a_{k\times 1} = \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_k \end{pmatrix}, e_{1\times\nu} \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix} \tag{3.37}
$$

Then the first and third equations of (3.22) can be written in the matrix form as follows

$$
AH^- = AH^+ - \Pi X + \Pi h(X) \tag{3.38}
$$

$$
A + \Pi X = a_{k\times 1} e_{1\times\nu} \tag{3.39}
$$

In order to rewrite the remained equations of (3.22) introduce the matrixes

$$B = \begin{pmatrix} b_1(\Delta) & b_1(2\Delta) & ... & b_1(\nu\Delta) \\ b_2(\Delta) & b_2(2\Delta) & ... & b_2(\nu\Delta) \\ b_3(\Delta) & b_3(2\Delta) & ... & b_3(\nu\Delta) \\ ... & ... & ... & ... \\ b_l(\Delta) & b_l(2\Delta) & ... & b_l(\nu\Delta) \end{pmatrix}_{l\times\nu}, \quad (3.40)$$

$$T(X_i) = \begin{pmatrix} x_{i1}\Big(\Delta\Big[\frac{\Delta-t_{i1}}{\Delta}\Big]\Big) & x_{i1}\Big(\Delta\Big[\frac{2\Delta-t_{i1}}{\Delta}\Big]\Big) & ... & x_{i1}\Big(\Delta\Big[\frac{\nu\Delta-t_{i1}}{\Delta}\Big]\Big) \\ x_{i2}\Big(\Delta\Big[\frac{\Delta-t_{i1}}{\Delta}\Big]\Big) & x_{i2}\Big(\Delta\Big[\frac{2\Delta-t_{i1}}{\Delta}\Big]\Big) & ... & x_{i2}\Big(\Delta\Big[\frac{\nu\Delta-t_{i1}}{\Delta}\Big]\Big) \\ ... & ... & ... & ... \\ x_{il}\Big(\Delta\Big[\frac{\Delta-t_{i1}}{\Delta}\Big]\Big) & x_{il}\Big(\Delta\Big[\frac{2\Delta-t_{i1}}{\Delta}\Big]\Big) & ... & x_{il}\Big(\Delta\Big[\frac{\nu\Delta-t_{i1}}{\Delta}\Big]\Big) \end{pmatrix}_{l\times\nu}, \quad (3.41)$$

$$TH(X_i) = \begin{pmatrix} x_{i1}\Big(\Delta\Big[\frac{\Delta-h_i+t_{i1}}{\Delta}\Big]\Big) & x_{i1}\Big(\Delta\Big[\frac{2\Delta-h_i+t_{i1}}{\Delta}\Big]\Big) & ... & x_{i1}\Big(\Delta\Big[\frac{\nu\Delta-h_i+t_{i1}}{\Delta}\Big]\Big) \\ x_{i2}\Big(\Delta\Big[\frac{\Delta-h_i+t_{i2}}{\Delta}\Big]\Big) & x_{i2}\Big(\Delta\Big[\frac{2\Delta-h_i+t_{i2}}{\Delta}\Big]\Big) & ... & x_{i2}\Big(\Delta\Big[\frac{\nu\Delta-h_i+t_{i2}}{\Delta}\Big]\Big) \\ ... & ... & ... & ... \\ x_{il}\Big(\Delta\Big[\frac{\Delta-h_i+t_{il}}{\Delta}\Big]\Big) & x_{il}\Big(\Delta\Big[\frac{2\Delta-h_i+t_{il}}{\Delta}\Big]\Big) & ... & x_{il}\Big(\Delta\Big[\frac{\nu\Delta-h_i+t_{il}}{\Delta}\Big]\Big) \end{pmatrix}_{l\times\nu} \quad (3.42)$$

$$i = 1, ..., k \quad (3.43)$$

and

$$T(X) = \begin{pmatrix} T(X_1) \\ T(X_2) \\ ... \\ T(X_k) \end{pmatrix}_{lk\times\nu}, \; TH(X) = \begin{pmatrix} TH(X_1) \\ TH(X_2) \\ ... \\ TH(X_k) \end{pmatrix}_{lk\times\nu} \quad (3.44)$$

Then the second an forth equations of (3.22) can be written as

$$BH^- = BH^+ - \Pi TH(X) + \Pi T(X), \quad (3.45)$$

$$B + \Pi T(X) = b_{l\times1}e_{1\times\nu} \quad (3.46)$$

where

$$b_{l\times1} = \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_l \end{pmatrix}, e_{1\times\nu}\begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix}$$

Finally, for example, the cost function $J_2(x) = \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{s=1}^{\nu} x_{ij}(t_s)$ can be written as

$$J_2(X) = e_{kl}^T X e_{kl} \tag{3.47}$$

where $e_{kl}^T = \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix}_{1\times kl}$ is the unit vector.

Thus, the matrix optimization problem is to find the integer valued matrix X maximizing the cost function

$$J_2(X) = e_{kl}^T X e_{kl} \to \max_X \tag{3.48}$$

subject to

$$\begin{aligned} A(H^- - H^+) &= \Pi\left(h(X) - X\right) && (3.49)\\ A - a_{k\times 1}e_{1\times\nu} &= -\Pi X && (3.50)\\ B(H^- - H^+) &= \Pi\left(T(X) + TH(X)\right), && (3.51)\\ B - b_{l\times 1}e_{1\times\nu} &= -\Pi T(X) && (3.52) \end{aligned}$$

This problem can be rewritten in the coordinate form, also.