Control & Guidance of Multiple Air-Vehicle Systems (MAS)

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Part IV

Task Assignment Problem for UAVs.

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Chapter 1

Introduction

1.0.1 Specific problem formulation

General description : To assign air-vehicles to perform as many simultaneous service requests as possible.

Starting conditions :

Given :

1. The MAS is performing coverage
2. Miltipble requests for service with the following inormation:
   - Number of air-vehicles required
   - Location where air vehicles need to visit
   - Earliest time of 1-st visit
   - Latest time of 1-st visit
   - Minimum duration per visit
   - Maximum interval between visits
   - Time of last visit

Mission Objective :

Find :

1. Air-vehicle(s) to be assigned to request and the corresponding paths to take to the service request
2. Variations to requests with minimal change if the request cannot be met

That :

1. Maximises the number of service requests that can be serviced

Constraints :

Subject to :

1. Air-vehicle performance and dynamics
2. Sensor performance
3. Air to Air datalink performance
4. LOS occlusion in area of operations
5. At least one air-vehicle being directly connected to GCS 90 procents of the time (for MAS to be provided feedback to GCS)

From mathematical point of view this problem can be reformulated as extremal problem which is includes nonlinear systems, PDE and ODE, conflicting situation, vector cost functions, incomplete information, constraints. It is clear that effective algorithms for the solution of this problem must be effective if we use them to solve:

- optimization problems with one costs function;
- optimization problem for ODEs;
- optimization problem for linear ODEs;
- optimization problems for linear discrete processes;
  Thus the logic in consideration inevitable leads to
- linear programming.

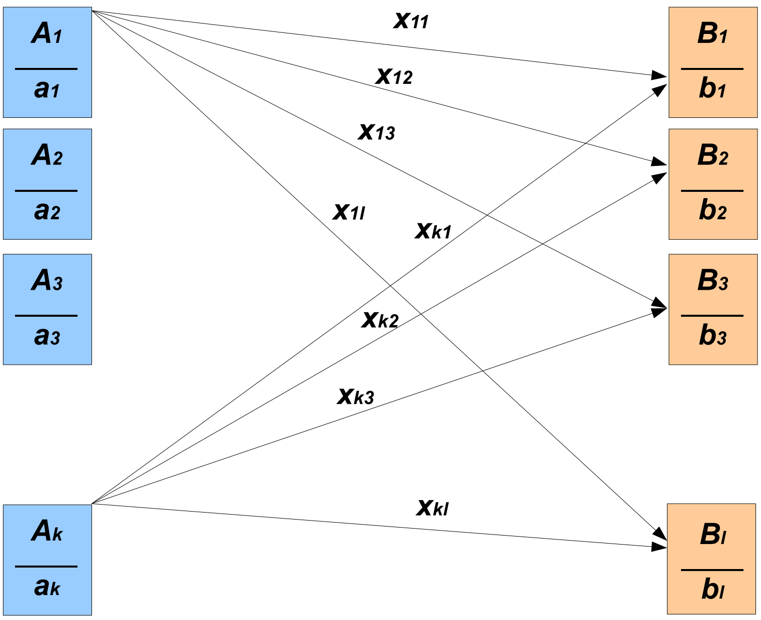
By this reason for the formulated problem above in a reduced form the special class of linear programming (LP) method will be considered in next chapter.

Chapter 2

LP assignment problem for group of UAVs

2.1 Formal problem statement

Assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$. Denote by $a_1, a_2, ..., a_i, ..., a_k$ their capacity, namely the maximal number of homogenous UAVs located in aerobase. Also we have l zones of area of operation $B_1, B_2, ..., B_j, ..., B_l$. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \ldots, l$.



Also assume that the sum of all requests are equal to the total number of available UAVs.

$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j \tag{2.1}$$

Next, for all i and j denote by c_{ij} the benefit of sending the UAV from i-th aerobase to j-th zone of area of operation. Note that the benefit of service can be given by different values, for example, by service time T_{ij}, fuel consumption, total number of UAVs involved etc. Then the problem is to define the plan of service for UAVs, in order to complete all incoming requests for service with maximal benefits. Denote by x_{ij} the number of UAVs from i-th aerobase which are send to service j-th zone. Then using our notation we can formulate the problem statement as the following integer programming problem: To find $x_{ij}, (i = 1, 2, ..., k; j = 1, 2, ..., l)$ such that, the total cost function for all services performed by all UAVs takes a maximal value

$$F = \sum_{i=1}^{k} \sum_{j=1}^{l} c_{ij} x_{ij} \to \min_{x_{ij}} \tag{2.2}$$

subject to

$$\begin{aligned} \sum_{i=1}^{k} x_{ij} = b_j, \quad j = 1, 2, ..., l \\ \sum_{j=1}^{l} x_{ij} = a_i, \quad i = 1, 2, ..., k \\ \sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j \\ x_{ij} \geq 0, \qquad x_{ij} \ \text{ are integer numbers.} \end{aligned} \tag{2.3}$$

Here the first two constraints means that we are need to satisfy all service request from each area of operation and to use for that purpose all available UAVs.

Remark 1.

Also it should be noted that in case of inequalities constraints instead of 2.3, the problem can be reduced to the equality case by introducing an additional variables. For example, for $\sum_{i=1}^{k} a_i \geq \sum_{j=1}^{l} b_j$ we will need to introduce some "artificial" aerobase or area of operation with correspondent capacity(for aerobase) and requirements (for area of operation):

$$b_{l+1} = \sum_{i=1}^{k} a_i - \sum_{j=1}^{l} b_j$$
$$a_{k+1} = \sum_{j=1}^{l} b_j - \sum_{i=1}^{k} a_i$$

According to the given Remark 1 it is assumed below that the constraints of (2.3) hold as equalities.

Remark 2. There are a lot of methods to solve such kind of problem. In this report the formulated simple integer optimization problem will be extended to search service schedular for dynamical case of service requests. For this purpose a new supporting optimization method will be developed on the base of the so-called constructive approach by Gabasov R. and Kirillova F.M. [2]. For linear programming problem this method was compared with classical simplex method in the paper [4], also was applied in gas industry [5] The proposed method allows to use both initial information and current one produced during the solution process. This together with developed for this case duality theory leads to high efficiency of the associated numerical algorithms. Here we will present a specific case of simplex method developed for assignment problem and transportation problems. This method include the following basic steps:
1. To find initial plan,
2. Check optimality condition for that plan,
3. Construct the improved plan in case of nonoptimality.
In order to demonstrate a key elements of this method, let us to consider the following example:

Example: Assume that three, five, and two homogeneous UAVs located on three aerobases correspondently. It is necessary to send them into 4 area of operations (zones), namely four UAVs into zone number 1, two UAVs into zone number 2, three UAVs into zone number 3 and one UAV into zone number 4. The distances between aerobases and zones are known, and given in the following table:

	B_1	B_2	B_3	B_4
A_1	400	600	800	200
A_2	400	1200	500	100
A_3	800	1000	600	400

Table 1: Distances between aerobases A_i and area of operations B_j

The problem is to define the plan of service of area of operation by UAVs, in order to complete all incoming requests for service with minimal fuel consumption. (Average fuel consumption for distance unit is 0,01 units of fuel.)

These problem can be formulated as follows:

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}} \tag{2.4}$$

subject to

$$\begin{aligned} \sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1, 2, 3, 4 \\ \sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1, 2, 3 \\ \sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j \\ x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}. \end{aligned} \tag{2.5}$$

The condition of that problem can be represented in table form:

	B_1	B_2	B_3	B_4	a_i
A_1	x_{11}	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	x_{21}	x_{22}	x_{23}	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

Below we give the detailed step-by-step procedure to determine the optimal solution.

2.2 Initial feasible solution

To construct the initial feasible solution we will use "North-West corner" method. The construction of the initial supporting feasible solution consist from the several steps on each of them are filled either a row or a table column. The procedure begins with the left top ("northwest") element x_{11} of the plan.

$$x_{11} = \min(a_1; b_1) \tag{2.6}$$

If $a_1 < b_1$, i.e. $x_{11} = a_1$, than from the further consideration we eliminate all elements from the first row. If $a_1 \geq b_1$, i.e. $x_{11} = b_1$, than all elements from the first column are eliminated. In the case $a_1 < b_1$ the next element of feasible solution will be chosen from the second row by the rule $x_{21} = \min(a_2; b_1 - a_1)$. Next, if $a_2 < b_1$, i.e. $x_{21} = a_2$, and in this case we eliminated from our further consideration all elements from the second row. If $a_2 \geq b_1 - a_1$, i.e. $x_{21} = b_1 - a_1$, and further we will not consider the elements from the first column. The next assignment will be made on the intersection of the second column and second row as follows: $x_{22} = \min(a_1 + a_2 - b_1; b_2)$. Then repeated this procedure we will find all elements of the initial supporting feasible solution.
In our case we have the following:

$$\begin{aligned}
x_{11} &= \min(a_1; b_1) = \min(3; 4) = 3 \\
\Downarrow & \quad \text{(next consider the element from the second row)} \\
x_{21} &= \min(a_2; b_1 - a_1) = \min(5; 1) = 1 \\
\Downarrow & \quad \text{(consider second column, element } A_2B_2 \text{)} \\
x_{22} &= \min(a_1 + a_2 - b_1; b_2) = \min(4; 2) = 2 \\
\Downarrow & \quad \text{(consider third column, element } A_2B_3 \text{)} \\
x_{23} &= \min(a_1 + a_2 - b_1 - b_2; b_3) = \min(2; 3) = 2 \\
\Downarrow & \quad \text{(consider third row, element } A_3B_3 \text{)} \\
x_{33} &= \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2)) = \min(2; 1) = 1 \\
\Downarrow & \quad \text{(consider fourth column, element } A_3B_4 \text{)} \\
x_{34} &= \min(a_1 + a_2 + a_3 - b_1 - b_2 - b_3; b_4) = \min(1; 1) = 1
\end{aligned} \tag{2.7}$$

	B_1	B_2	B_3	B_4	a_i
A_1	$x_{11} = 3$				$a_1 = 3$
A_2	$x_{21} = 1$	$x_{22} = 2$	$x_{23} = 2$		$a_2 = 5$
A_3			$x_{33} = 1$	$x_{34} = 1$	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum\limits_{i=1}^{3} a_i = \sum\limits_{j=1}^{4} b_j = 10$

It is easy to check that this solution is feasible and the fuel consumption is

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 60$$

Next go to the next step and check the optimality of feasible solution.

2.3 Optimality condition

We will use the so called method of potentials, also known as "$u - \nu$" method . Consider auxiliary numbers $u_1, u_2, ..., u_k$ and $\nu_1, \nu_2, ..., \nu_l$. For any admissible solution the value $\sum\limits_{i=1}^{k} \sum\limits_{j=l}^{l} (u_i + \nu_j) x_{ij}$ is the same and constant:

$$\sum_{i=1}^{k} \sum_{j=1}^{l} (u_i + \nu_j) x_{ij} = \sum_{i=1}^{k} u_i \sum_{j=1}^{l} x_{ij} + \sum_{j=1}^{l} \nu_j \sum_{i=1}^{k} x_{ij} = \sum_{i=1}^{k} u_i a_i + \sum_{j=1}^{l} \nu_j b_j = C$$

Next, assume that for some admissible solution we found the numbers u_i and ν_j such that the following conditions

$$\begin{aligned} u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0, \\ u_i + \nu_j \leq c_{ij},\ for\ x_{ij} = 0 \end{aligned} \tag{2.8}$$

hold.

The solution is called potential solution if it satisfies to condition (2.8) and the sum $u_i + \nu_j = \bar{c_{ij}}$ called pseudocost. Then the condition for potential solution can be rewritten

(2.8) as

$$\bar{c_{ij}} - c_{ij} = 0,\ for\ x_{ij} > 0, \tag{2.9}$$
$$\bar{c_{ij}} - c_{ij} \leq 0,\ for\ x_{ij} = 0$$

The general cost of fuel consumption for potential solution is equal C. Indeed, we can replace c_{ij} in (2.2) by the sum $u_i + \nu_j$, since the component for which $u_i + \nu_j < c_{ij}$ are equal to zero. If change the potential solution in a such way that some positive components $x_{ij} > 0$ becomes zero and some zero components $x_{ij} = 0$ becomes positive (i.e. change basis), then new solution will be not a potential, while for some new $x'_{ij} > 0$ we will have $u_i + \nu_j \leq c_{ij}$. The cost function for this new solution is

$$F' = \sum_{i=1}^{k}\sum_{j=1}^{l} c_{ij}x'_{ij} \geq \sum_{i=1}^{k}\sum_{j=1}^{l}(u_i + \nu_j)x'_{ij} = C$$

or in other words

$$F' \geq F.$$

Thus, the general cost of fuel consumption for any solution can not be less then for potential solution. It means that potential solution is optimal. In another words if we have even one component $x_{ij} = 0$ with $\bar{c_{ij}} - c_{ij} > 0$ then the solution is not optimal.

Let us check the optimality condition for our problem. Consider the following table

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	$\bar{c_{11}} = 4$ $c_{11} = 4$ $x_{11} = 3$				3	
A_2	4 4 1	12 12 2	5 5 2		5	
A_3			6 6 1	4 4 1	2	
b_j	4	2	3	1	F=60	
ν_j						

Then we should find potential u_i and ν_j such that for $x_{ij} > 0$ the condition $c_{ij} = u_i + \nu_j$ hold. One of the potentials can be chosen arbitrary. Let $\nu_1 = 0$, since $u_1 + \nu_1 = 4$ then $u_1 = 4$. Next following this logic we found step by step:

$$u_2 = 4 \longrightarrow \nu_2 = 8 \longrightarrow \nu_3 = 1 \longrightarrow u_3 = 5 \longrightarrow \nu_4 = -1$$

and calculate $\bar{c_{ij}}$ for zero components $x_{ij} = 0$. Than we will have the following table:

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6	5 8	3 2	3	4
A_2	4 4 1	12 12 2	5 5 2	3 1	5	4
A_3	5 8	13 10	6 6 1	4 4 1	2	5
b_j	4	2	3	1		
ν_j	0	8	1	-1		

Now we are ready to check our initial supporting feasible solution for optimality. Namely to check the condition $\bar{c_{ij}} - c_{ij} \leq 0$ for $x_{ij} = 0$. In our case for several zero components of our feasible solution this conditions are not satisfied. Hence our solution is not optimal. In next section we will consider how to improve feasible solution.

2.4 Improvement of the feasible solution

In order to improve the feasible solution we should find the zero components for which the difference $\bar{c_{ij}} - c_{ij} > 0$ is maximal and to define the new value $\theta > 0$ for this component. Then we make a necessary corrections of the previous solution in order it remains feasible (i.e. the constraints of our problem should by satisfy).

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 $3-\theta$	12 6 θ	5 8	3 2	3	4
A_2	4 4 $1+\theta$	12 12 $2-\theta$	5 5 2	3 1	5	4
A_3	5 8	13 10	6 6 1	4 4 1	2	5
b_j	4	2	3	1		
ν_j	0	8	1	-1		

Now we can define the value of the θ by replacement one of the positive components from previous solution by zero component, and also we should not have the negative value of x_{ij}. The maximal admissible value of θ to provide the condition above can be find from $\min(3-\theta; 2-\theta) = 0 \Longrightarrow \theta = 2$. Then we have the new feasible solution:

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2			3	
A_2	4 4 3		5 5 2		5	
A_3			6 6 1	4 4 1	2	
b_j	4	2	3	1	F=48	
ν_j						

Now, we are need to repeat the described procedure again, namely we will need to calculate new potentials: Let $\nu_1 = 0$, since $u_1 + \nu_1 = 4$ then $u_1 = 4$. Next

$$u_2 = 4 \longrightarrow \nu_2 = 2 \longrightarrow \nu_3 = 1 \longrightarrow u_3 = 5 \longrightarrow \nu_4 = -1.$$

And then new value $\bar{c_{ij}}$ for zero components of the solution.

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 8	3 2	3	4
A_2	4 4 3	6 6	5 5 2	3 1	5	4
A_3	5 8	7 10	6 6 1	4 4 1	2	5
b_j	4	2	3	1		
ν_j	0	2	1	-1		

The optimality conditions are not holds for several zero components. Find the maximal value of $\bar{c_{ij}} - c_{ij}$. It is easy to check that maximal value located in A_2B_4.

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 8	3 2	3	4
A_2	4 4 3	6 6	5 5 $2-\theta$	3 1 θ	5	4
A_3	5 8	7 10	6 6 $1+\theta$	4 4 $1-\theta$	2	5
b_j	4	2	3	1		
ν_j	0	2	1	-1		

Find the value $\theta = \min(2-\theta, 1-\theta) = 0 \longrightarrow \theta = 1$. Then we will have the following new feasible solution:

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2			3	
A_2	4 4 3		5 5 1	1 1 1	5	
A_3			6 6 2		2	
b_j	4	2	3	1	F=46	
ν_j						

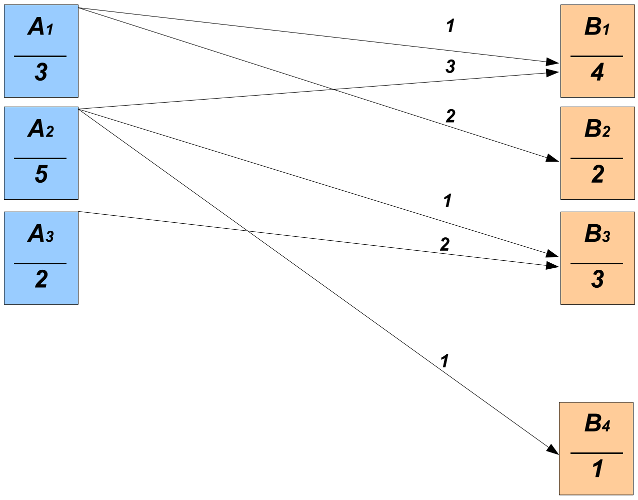
Let $\nu_1 = 0$, since $u_1 + \nu_1 = 4$ then $u_1 = 4$. Next

$$u_2 = 4 \longrightarrow \nu_2 = 2 \longrightarrow \nu_3 = 1 \longrightarrow u_3 = 5 \longrightarrow \nu_4 = -3.$$

And calculate the new value $\bar{c_{ij}}$ for zero components of the solution.

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 8	1 2	3	4
A_2	4 4 3	6 12	5 5 1	1 1 1	5	4
A_3	5 8	7 10	6 6 2	2 4	2	5
b_j	4	2	3	1		
ν_j	0	2	1	-3		

It is easy to see that optimality condition are holds for all zero components of the feasible solution, that means this solution is optimal. Thus we will need to send our UAVs as follows:



To area of operation number one B_1: 1 UAV from aerobase A_1 and 3 UAVs from A_2;
To area of operation B_2: 2 UAVs from A_1;
To area of operation B_3: 1 UAV from A_2 and 2 UAVs from A_3;
To area of operation B_4: 1 UAV from A_2.

Remark This method as well as simplex methods are iterative, finite, exact(satisfied all constraints) and relaxed(in a sense of the value of objective function). Thus in some sense this method is analog of simplex method, but the ideas of this method is more naturally can be applied to assignments LP problems.

Chapter 3

Dynamical assignment of UAVs for multiple areas of operation

3.1 Problem statement

Let $[0, H]$ is the given period for service of $B_1, B_2, ..., B_j, ..., B_l$ zones of area of operation. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \ldots, l$. Also, assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$ with $a_1, a_2, ..., a_i, ..., a_k$ number of homogenous UAVs, respectively. The problem is to assign UAVs between areas of operations $B_j, j = 1, \ldots, l$ in a such way that the total service time will be maximal.

3.2 Variables and constants

Divide the interval $[0, H]$ by the moments $t = i\Delta,\ i = 1, 2, ..., \nu$ where $\nu = \left[\dfrac{H}{\Delta}\right]$ denotes the integer part of the fraction $\dfrac{H}{\Delta}$, and Δ is a small number the concrete value of that depends on efficiency of numerical algorithms and will be determined later. Hence, we have the time interval partition

$$0 < \Delta < 2\Delta < ... < i\Delta < (i + 1)\Delta < .... < H.$$

For each discrete moment $t = i\Delta$, $i = 1, 2, ..., \nu$, introduce the following variables:

1. $x_{ij}(t)$ is the number of UAVs from i-th aerobase send to j-th zone at the moment t;

2. $a_i(t)$ is the number of UAVs at $i-th$ aerobase at the moment t;

3. $b_j(t)$ is the number of UAVs that are serving the $j-th$ zone at the moment t;

4. t_{ij} is the flight time from $i-th$ aerobase to $j-th$ zone;

5. k and l are the number of aerobases and zones for service, respectively;

6. h_i is the flight endurance for UAVs from $i-th$ aerobase.

Obviously, at the initial moment $t=0$ we have $b_j(0)=b_j,\ j=1,2,....,l;\quad a_i(0)=a_i,\ i=1,2,...,k.$

Now, we obtain the relation describing the dynamic of introduced variables.

3.3 Constraints

1) The number of UAVs at $i-th$ aerobase at the next moment $t+\Delta$ is composed of UAVs that are being at the previous moment t, plus UAVs that are returned during the period $[t,t+\Delta]$, and minus UAVs that were send to zones at the moment t. These facts give the following equalities

$$a_i(t+\Delta)=a_i(t)-\sum_{j=1}^{l}x_{ij}(t)+\sum_{j=1}^{l}x_{ij}(t+\Delta-h_i),\ i=1,...,k. \tag{3.1}$$

The term $\sum\limits_{j=1}^{l}x_{ij}(t+\Delta-h_i)$ denotes UAVs that were send early, and that should come back due to their flight endurance

Here we consider those objects where argument $t+\Delta-h_i>0$. Otherwise, the term $x_{ij}(t+\Delta-h_i)$ means that $i-th$ UAV has sufficient endurance to continue service of $j-th$ zone, and hence, it can not come back to aerobase.

The initial conditions are $a_i(0)=a_i,\ i=1,2,...,k.$

2) The number of UAVs that will serve the $j-th$ zone at the next moment $t+\Delta$ is composed of UAVs that are serving this zone at the previous moment t and having sufficient flight endurance, plus UAVs that reach this zone during the period $(t,t+\Delta]$, and

minus UAVs that are out-of-fuel to the moment t. These facts lead the following equalities

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}),\ j = 1, ..., l. \tag{3.2}$$

The term $\sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij})$ denotes UAVs that should leave the $j-th$ zone due to their out-of-fuel. The term $\sum_{i=1}^{k} x_{ij}(t - t_{ij})$ denotes UAVs that were send early and should reach the $j-th$ zone during the period $(t, t+\Delta]$.

Here we consider those objects where arguments $t - h_i + t_{ij} > 0$ and $t - t_{ij} > 0$.

The initial conditions are $b_j(0) = b_j,\ j = 1, 2, ..., l$.

3) The variables $x_{ij}(t)$ at each moment t satisfy the following conditions

$$\begin{aligned} a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k. \\ b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) = b_j,\ (t - t_{ij} > 0)\ j = 1, ..., l. \end{aligned} \tag{3.3}$$

The first equation images the fact that the being UAVs can be allocated among zones. The second equation means that at each moment the service request should be satisfied.

3.4 Types of objective function

4) The cost value function can be determined as follows:

a) the total service time for multiple zones

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}). \tag{3.4}$$

b) the total number of UAVs "circles"

$$J_2(x) = \sum_{t=0}^{\nu} x_{ij}(t) \tag{3.5}$$

c) the total unobservable time for multiple zones

$$J_3(x) = \sum_{t=0}^{\nu} x_{ij}(t)(H - h_i - 2t_{ij}) \tag{3.6}$$

Thus, the optimal schedule problem of UAVs for multiple zones can be formulated as, for example, the following special integer dynamical linear programming problem:

maximize the cost value function

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\dots,\nu} \tag{3.7}$$

subject to

$$\begin{aligned} a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t+\Delta-h_i),\ i=1,...,k. \\ b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}),\ j=1,...,l. \\ a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i=1,...,k. \\ b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j=1,...,l. \end{aligned} \tag{3.8}$$

where $\nu = \left[\frac{H}{\Delta}\right]$ denotes the integer part of the fraction $\frac{H}{\Delta}$.

Here we consider those objects where arguments $t - h_i + t_{ij} > 0$ and $t - t_{ij} > 0$.

Remark 2.

The proposed partition of the planing horizon $[0, H]$ with small step Δ yields an ability to produce optimal schedule for UAVs, in fact, in regime of real time. The realization of this idea demands the development of some fast numerical algorithms for solution of the special classes of linear programming problems. Some new approaches to accelerate the solution of general linear programming problem is discussed in the paper [**?**]

Remark 3. The proposed dynamical transportation problem (3.7)—(3.8) for allocation of MAS can be presented as a statistic problem given in the previous paragraph. But this way leads to the huge dimensions of the variables involved, and this together the specific structure of the considered problem are a serious obstacle for suitable solution for reasonable time. By this reason the development of special methods and design on this base of fast numerical methods for assignment problems of MAS with next their realization in the corresponding computer chips are actual and will be done at nearest period.

Chapter 4

Dynamical programming approach

It is known that dynamical programming method can be used for optimization problems. The method is attributed to Professor R.Bellman (USA) (see Bellman, 1957). Its continuous-time version is very broad generalization of classical Hamilton-Jacobi techniques to variational problems of control. The method is connected with embedding the construction of the optimal process into a family of identical problems with arbitrary initial conditions. This requires that the control would depend both on time and the state space variable, being presented in feedback (closed loop) form. The first indications on engineering solutions to specific problem of control synthesis were given by Flugge-Lotz in Germany, D.W. Bushaw in USA and A.A. Feldbaum in USSR. In fact, the work of Feldbaum on feedback control for automation served as an applied motivation for the development of Pontryagin Maximum Principle. In fact, a considerable amount of research was fullfilled by research groups at the Institute of Control Problem in Moscow. Applied problems of flight control were investigated by A.M. Letov, B.T. Polyak, Y.Z. Tsypkin.

4.1 Minimal path track for single UAV

Assume that we are need to provide the observation of $Z_2, ..., Z_n$ zones by single UAV Z_1. These object can be represented by the collection of n points from $\mathbb{R}^2$. It is supposed initially that the endurance T of the given UAV is sufficient to visit the given set of zones. The problem is to construct a shortest pass way from Z_1 to all zones $Z_2, ..., Z_n$ and come back to Z_1.

Solution by dynamical programming method

According to this method first we are needed to realize the so-called invariant embedding of the problem into parametric set of the similar problems. In our case instead of the fixed number n of zones we will consider the group composed by arbitrary s zones $1 \leq s \leq n-1$. That is the each group of s zones will be constitutes from arbitrary $Z_{i_1}, Z_{i_2}, ..., Z_{i_s}$ zones from the given collection. Moreover, instead of the pre-assigned initial point Z_1 consider the arbitrary zone Z_i from the given collection $Z_1, Z_2, Z_3, ..., Z_n$.

Next define the so-called Bellman function $B_s(i|i_1, i_2, ..., i_s)$ as a shortest pass way between the point Z_i and Z_1, and trough-passing the given zones $Z_{i_1}, Z_{i_2}, ..., Z_{i_s}$.

In order to construct the equation such that the Bellman function will satisfy this equation, we will make the trial moving from point Z_i to point Z_{i_k} where $k = 1, ..., s$. Then, in accordance with the definition of Bellman function given above, the shortest pass way from Z_{i_k} to Z_1 trough the zones $Z_{i_1}, ..., Z_{i_{k-1}}, Z_{i_{k+1}}, ..., Z_{i_s}$ (note that the zone Z_{i_k} is absent in this collection) is denotes as

$$B_{s-1}(i_k|i_1, ..., i_{k-1}, i_{k+1}, ..., i_s).$$

Hence the length of a route from Z_i to Z_1 will be not less then

$$d_{ii_k} + B_{s-1}(i_k|i_1, ..., i_{k-1}, i_{k+1}, ..., i_s)$$

where d_{ij} denotes here the distance between Z_i and Z_j.

Therefore, the minimal route from $Z_i \longrightarrow$ to Z_1 is

$$B_s(i|i_1, i_2, ..., i_s) = \min_{1 \leq k \leq s} \Big[d_{ii_k} + B_{s-1}(i_k|i_1, ..., i_{k-1}, i_{k+1}, ..., i_s) \Big] \tag{4.1}$$

and it is defined from the the expression $d_{ii_k} + B_{s-1}(i_k|i_1, ..., i_{k-1}, i_{k+1}, ..., i_s)$ by enumeration of possibilities of all trial steps from Z_i to Z_{i_k}, where $k,\ 1 \leq k \leq s$. The boundary condition for the Bellman equation (4.1) follows immediately from the definition of the Bellman function at $s = 1$:

$$B_1(i|i_1) = d_{ii_1} + d_{i_1 1} \tag{4.2}$$

Using the initial conditions (4.2) and the equation (4.1) we can calculate sequentially the functions

$$B_1(i|i_1), B_2(i|i_1, i_2), ..., B_{n-2}(i|i_1, i_2, ..., i_{n-2}), B_{n-1}(i|i_1, i_2, ..., i_{n-1}).$$

In final, the obtained the function value

$$B_{n-1}(1|2, ..., n) \tag{4.3}$$

presents the length of the shortest pass way for the considered optimization problem.

Next, the optimal pass way can be designed by the following procedure. From (4.1) we find for $s = n-1$

$$\begin{aligned} B_{n-1}(1|2,3,..,n) = \min_{2\le i\le n}\left[d_{1i} + B_{n-2}(i|2,3,...,i-1,i+1,...,n)\right] = \\ = d^0_{1i^0} + B_{n-2}(i^0|2,3,...,i^0-1,i^0+1,...,n) \end{aligned} \tag{4.4}$$

Note that the last equality denotes that the minimum in (4.4) is reached on the index i^0. This gives that the first zone of optimal pass way from Z_1 is the zone Z_{i^0}.

Further, again from (4.1) we find at $s = n-2$:

$$\begin{aligned} B_{n-2}(i^0|2,3,...,i^0-1,i^0+1,...,n) = \\ \min_{i\in[2,...,n]\setminus\{i^0\}}\left[d_{i^0 i} + B_{n-3}(i^0|2,3,...,i-1,i+1,...,i^0-1,i^0+1,...,n)\right] = \\ = d_{i^0 i^1} + B_{n-3}(i^0|2,3,...,i^1-1,i^1+1,...,i^0-1,i^0+1,...,n) \end{aligned} \tag{4.5}$$

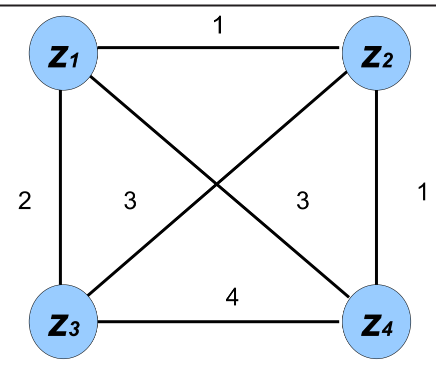
Thus the optimal pass way is continued by zones Z_{i^1} such that the optimal route is

$$Z_1 \longrightarrow Z_{i^o} \longrightarrow Z_{i^1} \longrightarrow (etc).$$

The described procedure is repeated next by the obvious manner.

Example

The problem is to find the shortest pass way for UAV of Z_1 to service the set of zones Z_2, Z_3, Z_4, the mutual distances between of which are given by the following Table A and imaged by the graph



	z_1	z_2	z_3	z_4
z_1	0	1	2	3
z_2	1	0	3	1
z_3	2	3	0	4
z_4	3	1	4	0

Table A

The solution of the given problem find by the dynamic programming method described above. The definition of Bellman function and (4.3) show that the minimal value of the desired pass way is equal to $B_3(1|2,3,4)$. By definition of the function $B_3(1|2,3,4)$ we have:

$$B_3(1|2,3,4) = \min\Big(d_{12} + B_2(2|3,4); d_{13} + B_2(3|2,4); d_{14} + B_2(4|2,3)\Big) \tag{4.6}$$

Hence we have to know the values $B_2(2|3,4), B_2(3|2,4), B_2(4|2,3)$ that are defined as follows:

$$B_2(2|3,4) = \min\left(d_{23} + B_1(3|4); d_{24} + B_1(4|3)\right) \tag{4.7}$$

$$B_2(3|2,4) = \min\left(d_{32} + B_1(2|4); d_{34} + B_1(4|2)\right) \tag{4.8}$$

$$B_2(4|2,3) = \min\left(d_{42} + B_1(2|3); d_{43} + B_1(3|2)\right) \tag{4.9}$$

Again we have to know the values $B_1(3|4); B_1(4|3); B_1(2|4); B_1(4|2); B_1(2|3); B_1(3|2)$. Find them in accordance with (4.2) and the distances given by the table (graph):

$$B_1(3|4) = 7; B_1(4|3) = 6;$$

$$B_1(2|4) = 4; B_1(4|2) = 2;$$

$$B_1(2|3) = 5; B_1(3|2) = 4$$

Now we are able to find the needed values of Bellman function:

$$B_2(2|3,4) = \min\left(3+7; 1+6\right) = 7$$

$$B_2(3|2,4) = \min\left(3+4; 4+2\right) = 6$$

$$B_2(4|2,3) = \min\left(2+5; 4+4\right) = 7$$

and finally

$$B_3(1|2,3,4) = \min\left(1+7, 2+6, 3+7\right) = \min\left(8,8,10\right) = 8$$

Thus, the minimal length of the pass way is equal 8.

Note, that the optimal value is reached on two variants

$$8 = B_3(1|2,3,4) = \begin{cases} d_{12} + B_2(2|3,4) \\ d_{13} + B_2(3|2,4) \end{cases} \tag{4.10}$$

For brevity sake, choose the second variant. That is the minimum in (4.6) (see, also, (4.4)) is reached on the index $i^0 = 3$ such that

$$B_3(1|2,3,4) = d_{13} + B_2(3|2,4) = 8 \tag{4.11}$$

This means that the first zone to be visit is $Z_1 \to Z_3$.

From (4.11) follows that further procedure is to consider the value of $B_2(3|2,4)$. As follows from (4.8)

$$B_2(3|2,4) = \min\left(d_{32} + B_1(2|4); d_{34} + B_1(4|2)\right) = \min\left(3+4; 4+2\right) = 6. \tag{4.12}$$

Since the minimum in the last is reached on the value $d_{34} + B_1(4|2)$ then the next index for the desired optimal pass way is $i^1 = 4$ (see, also (4.5)). Hence the next zone to visit is Z_4 such that the optimal pass is continued as $Z_1 \to Z_3 \to Z_4$. It is obviously that the last remained zone to be visited is Z_2. Hence the optimal route is

$$Z_1 \to Z_3 \to Z_4 \to Z_2 \to Z_1.$$

Obviously, the back pass

$$Z_1 \to Z_2 \to Z_4 \to Z_3 \to Z_1$$

is optimal, too. Note that this fact is detected by (4.10), if the first variant for minimization will be chosen in order to continue the given procedure.

The considered problem is simplest one and can be solved by usual examination of options. Since for the set of indexes $\{2,3,4\}$ (these indexes correspond to the numeration of zones Z_2, Z_3, Z_4)) there is $3! = 1 \cdot 2 \cdot 3 = 6$ variants. They and their total route length are as follows

$$\begin{aligned}
1 \to 2 \to 3 \to 4 \to 1 &= 1+3+4+3 = 11\\
1 \to 3 \to 2 \to 4 \to 1 &= 2+3+1+3 = 9\\
1 \to 4 \to 3 \to 2 \to 1 &= 3+4+3+1 = 11\\
1 \to 2 \to 4 \to 3 \to 1 &= 1+1+4+2 = 8\\
1 \to 3 \to 4 \to 2 \to 1 &= 2+4+1+1 = 8\\
1 \to 4 \to 2 \to 3 \to 1 &= 3+1+3+2 = 9.
\end{aligned} \tag{4.13}$$

Again, we have the optimal track as $Z_1 \to Z_3 \to Z_4 \to Z_2 \to Z_1$ or $Z_1 \to Z_2 \to Z_4 \to Z_3 \to Z_1$.

4.1.1 Minimal path track for the single UAV with restricted endurance

The method described above can be used for the case when the operational resource of UAV (endurance) is not sufficient to serve all the pre-assigned zones. Noting that by the definition of Bellman function the value $B_s(i|i_1, i_2, ..., i_s)$ is a shortest pass way between the point Z_i and Z_1, and trough-passing the given zones $Z_{i_1}, Z_{i_2}, ..., Z_{i_s}$ yields that the maximal number t^* of zones available for the visits from Z_1 and come back to Z_1 is the greatest number satisfying the following inequality

$$t^* : \quad B^* \doteq \max_t B_t(i|i_1, i_2, ..., i_t) \leq L \tag{4.14}$$

where L is maximal path length available for the given UAV.

Then the optimal collection zones to be visited is determined on the base of analysis of the term

$$B_{t^*}(1|i_1, i_2, ..., i_{t^*}) = B^*, \tag{4.15}$$

on which the last maximum is achieved. This analysis is realized by analogy with the Section above.

4.2 Optimal endurance allocation of UAVs for multiple zones

Consider the case when we have a set of UAV the total endurance of which is restricted by some value T. Let $Z_1, ..., Z_n$ be the zones that are required to serve. The endurance T (the total flying time of the given group of UAVs, for example) can be treated as a "resource" of the given UAVs. Then the problem is to distribute the given resource Tamong n zones.

Let $t_i, 0 \leq t_i \leq T$ is the portion of "resource" assigned for the zone Z_i. Then the "benefit" of this assignment we denote by $f_i(t_i)$. The benefit here can be treated as the probability of targets detection in i-th zone or the square D_i of observed area in i-th zone, for example.

1) We are assume that we have already some calculations of probability of targets detection depending on time.

2) Also it is possible to calculate the square of observed area in the zone in the given time.

3) For simplicity also assume that the time necessary to fly from zone to zone is not essential.

The problem statement are:

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \text{subject to} \quad \sum_{i=1}^{n} t_i \leq T,\ t_i \geq 0, i = 1, ..., n \tag{4.16}$$

Note that the specific feature of the problem (4.16) is separable form of the cost value function

In this paper the dynamic programming method is adopted for solution of the considered optimization problem.

First, this problem should be embed into the collection $P(k, y)$ of parametric problems of the form

$$P(k, y): \qquad \sum_{i=2}^{k} f_i(t_i) \to \max, \qquad \text{subject to} \quad \sum_{i=1}^{k} t_i \leq y,\ t_i \geq 0, i = 1, ..., k \tag{4.17}$$

where $0 \leq y \leq T,\ \ 1 \leq k \leq n$ are the parameters of the given collection. Obviously, that the initial problem can be obtained from this collection at $y = T$ and $k = n$.

Introduce now the Bellman function $B_k(y)$ as follows

$$B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i), \qquad \text{subject to} \quad \sum_{i=2}^{k} t_i \leq y,\ t_i \geq 0, i = 1, ..., k \tag{4.18}$$

Let $z,\ 0 \leq z \leq y$ be the portion of the endurance assigned for the zone Z_k. The corresponding cost value ("benefit") of such distribution is equal $f_k(z)$. Hence, the rest endurance $y - z$ should be distributed among the remained $(k-1)$ zones $Z_1, \ldots, Z_{k-1}$. In accordance with the definition of Bellman function the optimal distribution of the endurance $y - z$ among $(k-1)$ zones is determined as $B_{k-1}(y - z)$. Therefore, if the given resource is equal y, then after the assignment of z portion for zone Z_k the total profit of all k zones is

$$f_k(z) + B_{k-1}(y - z) \tag{4.19}$$

Hence, the optimal distribution z_k^0, $0 \leq z_k^0 \leq y$, for the given zone Z_k is determined by the following condition

$$f_k(z_k^0) + B_{k-1}(y - z_k^0) = \max_{0 \leq z \leq y} \Big[f_k(z) + B_{k-1}(y - z) \Big] \tag{4.20}$$

But in accordance with the definition (4.18) maximal profit from distribution of the initially given resource y among all k zones is equal $B_k(y)$. Thus, for the introduced Bellman function $B_k(y)$ we have the following Bellman equation

$$B_k(y) = \max_{0 \leq z \leq y} \Big[f_k(z) + B_{k-1}(y - z) \Big] \tag{4.21}$$

The initial condition for the obtained recurrent Bellman equation (4.21) follows from (4.18) at $k = 1$:

$$B_1(y) = \max f_1(t_1), \qquad \text{subject to} \quad t_1 = y,\ t_1 \geq 0$$

Hence the initial condition for the Bellman equation (4.21) is

$$B_1(y) = f_1(y). \tag{4.22}$$

Now we can design an optimal solution for the allocation problem. Put $k = 2$ in the equation (4.21):

$$B_2(y) = \max_{0\leq z\leq y} \Big[f_2(z) + B_1(y-z)\Big] = \max_{0\leq z\leq y} \Big[f_2(z) + f_1(y-z)\Big] \tag{4.23}$$

Since f_1, f_2 are known functions then using the initial conditions (4.22) we are able calculate the maximum in (4.24) and, hence, find the function $B_2(y)$. Put $k = 3$ in the equation (4.21):

$$B_3(y) = \max_{0\leq z\leq y} \Big[f_3(z) + B_2(y-z)\Big] = \max_{0\leq z\leq y} \Big[f_3(z) + f_2(y-z) + f_1(2y-z)\Big] \tag{4.24}$$

Since f_3 and B_2 are known functions, we can calculate $B_3(y)$. Proceeding sequentially this procedure we will determine the functions

$$B_4(y), B_5(y), ..., B_n(y).$$

In final, the function value

$$B_n(T) \tag{4.25}$$

presents the maximal profit for the initial endurance allocation problem (4.16).

In order to design the optimal distribution of the available endurance of UAV we realize the reverse motion in the procedure of solution of Bellman equation.

Put in (4.20) $k = n,\ y = T$ and find the value $t_n^0 \doteq z^0(T)$ on which the maximum in (4.20) is achieved for the given data. According to the definition (4.18)of Bellman function this value means the optimal portion of the endurance allocated for the zone Z_n.

Further, if for the zone Z_n the portion t_n^0 of the initial endurance T was allocated then the rest $T - t_n^0$ of endurance are available to allocate among the remainder zones $Z_1, \ldots, Z_{n-1}$. Put in (4.20) $k = n-1,\ y = T - t_n^0$ and find the value $t_{n-1}^0 \doteq z^0(T - t_n^0)$ on which the maximum in (4.20) is achieved for the given data. Again in accordance with the definition (4.18)of Bellman function the obtained value means that the optimal portion of the remainder endurance allocated for the zone Z_{n-1} is equal t_{n-1}^0. Ongoing analogously this procedure gives the desired optimal allocation of the endurance T among the given zones $Z_n, Z_{n-1}, \ldots, Z_1$ in the form

$$t_n^0, t_{n-1}^0, \ldots, t_2^0, t_1^0.$$

Illustrative example

Let Z_1, Z_2, Z_3 be the zones under service of five UAVs. Assume that the "benefits" of their service in each zone are given in Table 1. For example, the "benefit" of service in the given zone for each UAV can be interpreted as the number of the detected targets. Then $f_i(x)$ means the "benefit" of using x UAVs for service of zone Z_i,where x denotes the number of the used UAV.

x	0	1	2	3	4	5
$f_1(x)$	0	1	2	3	4	5
$f_2(x)$	0	0	1	2	4	7
$f_3(x)$	0	2	2	3	3	5

Table 1

The problem is to find an optimal allocation of the given UAVs to serve the given zone Z_1, Z_2, Z_3 such that to maximize the total "benefit".

For the considered example we have $n = 3$ and $T = 5$. According to the definition of Bellman function (4.18) the optimal solution of the considered allocation problem is determined by the function value

$$B_3(5) = \max_{0 \le z \le 5} \Big[f_3(z) + B_2(5-z) \Big] = \\ \max \Big\{ f_3(0) + B_2(5); f_3(1) + B_2(4); f_3(2) + B_2(3); f_3(3) + B_2(2); f_3(4) + B_2(1); f_3(5) + B_2(0) \Big\} \tag{4.26}$$

where the required here the functions $B_2(y)$ are determined by the following formula

$$B_2(y) = \max_{0 \le z \le y} \Big[f_2(z) + B_1(y-z) \Big] \tag{4.27}$$

Since the initial conditions gives $B_1(y) = f_1(y)$ then from Table 1 follows

$$B_1(0) = 0,\ B_1(1) = 1,\ B_1(2) = 2,\ B_1(3) = 3,\ B_1(4) = 4,\ B_1(5) = 5.$$

Using the equation (4.27) and the obtained $B_1(y)$ we can find the required function values

$$B_2(1), B_2(2), ..., B_2(5).$$

For example,

$$B_2(4) = \max_{0 \le z \le 4} \Big[f_2(z) + B_1(4-z) \Big] = \\ = \max \Big\{ f_2(0) + B_1(4); f_2(1) + B_1(3); f_2(2) + B_1(2); f_2(3) + B_1(1); f_2(4) + B_1(0) \Big\} \tag{4.28}$$

$$= \max \Big\{ 0+4; 0+3; 1+2; 2+1; 4+0 \Big\} = 4$$

It is convenient the obtained values of the functions $B_2(y)$ ($B_1(y)$ and $B_3(y)$, also) to collect in the Table 2. Also, in order to simplify the next calculation the obtained Bellman function values $B_k(y)$ is accompanied (in braces) by the arguments $z_k^0(y)$ at which this value is achieved (that is the right hand side of (4.27) reaches their maximum). Using the obtained values $B_2(1), B_2(2), ..., B_2(5)$ we are able to end (4.26) as follows

$$B_3(5) =$$
$$\max\Big\{f_3(0)+B_2(5); f_3(1)+B_2(4); f_3(2)+B_2(3); f_3(3)+B_2(2); f_3(4)+B_2(1); f_3(5)+B_2(0)\Big\} = \tag{4.29}$$
$$= \max\Big\{0+7; 2+4; 2+3; 3+2; 3+1; 5+0\Big\} = 7.$$

Note, that by analogy with this calculations we can find the other function values $B_3(y)$ that are presented in Table 2.

y	0	1	2	3	4	5
$B_1(y)$	0	1	2	3	4	5
$B_2(y)$	0	1(0)	2(0)	3(0)	4(0,4)	7(5)
$B_3(y)$	0	2(1)	3(1)	4(1)	5(1)	7(0)

Table 2

Thus, the the maximal profit available in the considered example is equal $B_3(5) = 7$. The corresponding optimal allocation is designed on the base of analysis of the obtained maximal Bellman function value $B_3(5) = 7$;

$$7 = B_3(5) = f_3(0) + B_2(5), \tag{4.30}$$

The last denotes that the maximal efficiency is achieved at $t_3^0 \doteq z^0(5) = 0$. This means that no UAVs are not directed for service of the zone Z_3. Hence, the rest $5 - 0 = 5$ of UAVs should be optimal allocated among the remainder zones Z_2 and Z_1.

Again, accordingly to the Definition of Bellman function (4.18) the maximal efficiency in this case is equal $B_2(5)$, and Table 2 gives that this value is $B_2(5) = 7$ that is achieved at $t_2^0 \doteq z^0(5) = 5$. This means that five UAVs should be directed to the zone Z_2 to get the maximal profit. Hence, for the zone Z_1 is no UAVs for their service.

In final, the optimal distribution of five UAVs for the considered allocation problem with the given initial data is

$$t_1^0 = 0,\ t_2^0 = 5,\ t_3^0 = 0.$$

The obtained solution can be easy find by the careful analysis of Table 1, where the maximal efficiency value 7 for the case of the total endurance $T = 5$ is given by the allocation of all 5 UAVs to zone Z_2.

But some changing initial data leads to a nontrivial optimization problems. For example, let the initial group formed by 5 UAVs is decreased to four UAVs such that $T = 4$. In this case the maximal efficiency is equal 5. This fact follows from Table 2 where we have $B_3(4) = 5$. The corresponding optimal allocation of 4 UAVs among 3 zones is determined by the following: since the maximal value function $B_3(4) = 5$ is achieved at $t_3^0 \doteq z_3^0(4) = 1$ (this value is written in the braces of the corresponding cell) then one UAV is directed to zone Z_3. Further, the rest $4 - 1 = 3$ of UAVs should be allocated among the remainder zones Z_1, Z_2. The Table 2 gives the optimal value of efficiency $B_2(3) = 3$ which is achieved at $t_2^0 \doteq z_2^0(3) = 0$. This means that UAVs are not allocated to zone Z_2. Therefore, for service of the zone Z_1 we have $3 - 0 = 3$ UAVs.

Thus, for the case $T = 4$ the solution of the form

$$t_1^0 = 3,\ t_2^0 = 0,\ t_3^0 = 1$$

is the optimal allocation for the given 4 UAVs among 3 zones.

Note, that the search of optimal solution for $T = 4$ is a nontrivial in contrast the case $T = 5$. As it follows the remark mentioned above the being a "schedule" like to the Table 2 gives a good tool for the fast solution of the allocation problems with disturbed data.

4.3 Schedule representation for hierarchical missions of UAVs

In some cases, the hierarchical missions can be presented by the collection I of heterogeneous tasks that should be served by some group of UAVs. It is supposed that the given tasks should be served by UAVs in the pre-assigned order such that every next task cannot be served before the previous task is completed.

Let $S \triangleq \{I, U\}$ denotes a network, where I means the set of the nodes(tasks) and U is the set of arcs (possible services). The service of task $j \in I$ which is available only after the ending task $i \in I$ we will denote by the arc $(i, j) \in U$. Hence any service $(i, j) \in U$ is not available if the services $(k, i) \in U,\ k \in I_i^- \triangleq \{k \in I : \exists\ (k, i) \in U\}$ are not completed. The moment of the service end for the task i is determined by the moments of the **completion of all services** of $(k, i),\ k \in I_i^-$.

In the network $S = \{I, U\}$ choose two nodes s and t, where s denotes the starting node (that is the start of the mission) and t is the final node where the mission is completed.

Denote $I_i^+ = \{j \in I : \exists\ (i,j) \in U\}$. It is obvious that $I_s^- = \varnothing,\ I_t^+ = \varnothing$. Denote by c_{ij} the service time for the arc $(i,j) \in U$. Also, let $x_i, i \in I$ denotes the moment of the service completion for the task i.

The given request of the pre-assigned order in service leads to the following inequalities

$$x_i + c_{ij} \le x_j,\ i \in I_j^-,\ j \in I. \tag{4.31}$$

These inequalities image the fact that the service of j cannot be realized before completion of all services of $(i,j),\ i \in I_j^-$ of the previous task i.

Then the minimal service time of the mission is determined as the smallest number x_t^0 that together the numbers $x_i^0 \ge 0,\ i \in I \setminus \{t\};\ x_s^0 = 0$ satisfy the inequalities (4.31).

Since for ending mission it is necessary to finish all of services then the length $\sum c_{ij}$ of each route $s \to t$ formed by the set of available arcs is not less x_t^0. Therefore, the search of the optimal solution x_t^0 is equivalently to the problem: find the route from the nodes s to the node t such that the length $\sum c_{ij}$ of this rout is **maximal**. The route with such kind of property is called **extremal route**.

The formalization of this problem can be given by the following

$$x_t - x_s \to \min \tag{4.32}$$

subject to

$$\begin{aligned} x_i + c_{ij} \le x_j,\ i \in I_j^-,\ j \in I \\ x_i^0 \ge 0,\ i \in I \setminus \{t\} \end{aligned} \tag{4.33}$$

For solution of this optimization problem we use the dynamic programming method.

For this purpose embed this problem into the parametric collection of optimization problems when instead the concrete terminal node t we will consider an arbitrary node j. For the extended optimization problem introduce the Bellman function B_j as a largest pass way from the node s to the node j.

In order to get the corresponding Bellman equation to which the introduced function satisfies we investigate the pass way from s to j. Suppose that the last arc of this pass way is the arc (i,j) where $i \in I_j^-$ and, moreover, the previous pass way from s to i is optimal such that the length of this pass $s \to i$ is equal B_i. Then the total length of the pass way

$s \to i \to j$ is equal $B_i + c_{ij}$. Therefore, the optimal (maximal) length of the pass $s \to j$ is given as $\max\limits_{i \in I_j^-}(B_i + c_{ij})$. Then from the definition of Bellman function follows immediately that the function B_j satisfies the following equation

$$B_j = \max_{i \in I_j^-}(B_i + c_{ij}) \tag{4.34}$$

The initial condition for this Bellman equation is given

$$B_s = 0 \tag{4.35}$$

To solve equation (4.34)–(4.35) we use the following approach. Denote by I_* the set of nodes from $i \in I$, for which the values of Bellman function B_i are known yet. The set I_* is not empty since $s \in I_*$, at least. If $t \in I_*$ then the considered optimization problem is solved since in this case B_t is optimal length of the pass $s \to t$, and the design of the corresponding pass way is realized by the analysis of the optimal value B_t by "backward motion" in Bellman equation. To simplify this "backward motion" for the nodes $i \in I_*$ introduce the additional function $f(i)$ $i \in I_*$. At the first stage we have

$$I_* = s, \; B_s = 0, \; f(s) = 0. \tag{4.36}$$

Let us now $t \notin I_*$. In the network $S = (I, U)$ we find the set of nodes $w(I_*)$ neighboring with the nodes of I_* :

$$w(I_*) = \{j \in I : (i, j) \in U, \; j \notin I_*, \; i \in I_*\}. \tag{4.37}$$

It can be shown that in the set $w(I_*)$ there exists the node $j_* \in w(I_*)$ such that $I_{j^*}^- \subset I_*$. Since for all nodes $i \in I_*$ the values of the Bellman function B_i are known then the Bellman equation (4.34) gives

$$B_{j^*} = \max_{i \in I_{j^*}^-}(B_i + c_{ij^*}) = c_{i^*j^*} + B_{i^*}. \tag{4.38}$$

Then the second stage of the procedure is to extend the set I_* and to calculate the associated function value $f(i)$ as

$$I_* \triangleq I_* \cup \{j_*\}, \; f(j_*) = i_*. \tag{4.39}$$

Note that the nodes of j_* satisfying the property $I_{j^*}^- \subset I_*$ can be nonunique. In this case the corresponding values of the functions $B_j, f(j)$ are determined by the same procedure of (4.38). Further the next iterations are continued by analogy with given above. It is obviously that the number of the described iterations does not exceed the number $|I|$ of elements in the set I.

In final, these iterations will lead to the case when $t \in I_*$. This means that the considered optimization problem is solved, and B_t is the optimal value of the length $s \to t$. The corresponding optimal " backward pass way"

$$\{t \leftarrow i_1 \leftarrow i_2 \leftarrow \ldots \leftarrow i_k \leftarrow s\}$$

is defined by the following formula

$$i_1 = f(t),\ i_2 = f(i_1), \ldots, i_k = f(i_{k-1}),\ s = f(i_k). \tag{4.40}$$

4.3.1 Illustrative example

Consider the complex mission given by the network of Figure 1 where $s = 1,\ t = 4$. Here the node 1 is the beginning of the mission. The nodes 2 and 6 image the tasks of the first submission, where there are two tasks in the node 2 and one task in the node 6. In addition, one of the tasks of the node 2 can be served only after the completion of the task in the node 6. Moreover, the service time of the first and the second task from the node 6 are equal 4 and 2, respectively, and the service time for the task of the node 6 is equal 1. The second submission is presented by two tasks in the node 3 and one task in the node 5, where the first task in the node 3 can be served after the completion of all tasks in the node 2, and the second task of 3 can be done after the completion of the task in 6. The corresponding service time are 2 and 4, respectively. The third submission is presented by two tasks in the node 5, where the first task can be done after the completion of the task in 6, and the second task is available after the ending tasks in 3. The corresponding service time are 1 for the both tasks. The node 4 presented by two tasks in the node 4 is the final submission. The service time of these tasks are 8 and 1.

The numbers marked under the arcs of Figure 1 denote the service time of the corresponding tasks of submissions.

Additionally, on the Figure 1 the Bellman function values $B_j,\ j = 1, \ldots, 6$ and function $f(j),\ j = 1, \ldots, 6$ calculated in accordance with (4.36)–(4.40) are marked. These calculations are realized by the following step-by-step procedure.

1) For $s = 1$ we have $I_*^{(1)} = \{s\} = \{1\}$ and $B_s = B_1 = 0,\ f(s) = f(1) = 0$. Then the set

$$w(I_*^{(1)}) = \{j \in I : (i,j) \in U,\ j \notin I_*,\ i \in I_*\}$$

in this case is

$$w(I_*^{(1)}) = \{j \in I : (1,j) \in U\} = \{(1,2) \in U, (1,6) \in U\} = \{2; 6\}. \tag{4.41}$$

Find now the node $j_* \in w(I_*^{(1)})$ such that $I_{j^*}^- \subset I_*^{(1)}$ where the set I_i^- is defined as $I_i^- \triangleq \{k \in I : \exists \text{ arc } (k,i) \in U\}$. In the considered case the needed node is unique $j_* = \{6\}$ because the corresponding set $I_6^- = \{k \in I : \exists \text{ arc } (k,6) \in U\} = \{k = 1 : \exists \text{ arc } (1,6) \in U\} = \{1\}$ satisfies the condition $\{1\} = I_6^- \subset I_*^{(1)} = \{1\}$. Note, that the set

$$I_2^- = \{k \in I : \exists \,\text{arc}\,(k,2) \in U\} = \{k = 1, k = 6 : \exists \,\text{arc}\,(1,2) \in U, (6,2) \in U\} = \{1;6\} \not\subset I_*^{(1)} = \{1\},$$

and hence, the node $j_* = \{2\}$ is not suitable for the choice.

Thus, the associated Bellman function value of

$$B_{j^*} = \max_{i \in I_6^-}(B_i + c_{ij^*}) = c_{i^*j^*} + B_{i^*}.$$

in the considered case is equal

$$B_6 = \max_{i \in I_6^-}(B_i + c_{i6}) = c_{16} + B_1 = 1 + 0 = 1 \tag{4.42}$$

where $i_* = \{1\}$ and the corresponding function $f(j_*) = i_*$ is equal $f(6) = 1$.

2) Extend the set $I_*^{(2)} \triangleq I_*^{(1)} \cup \{j_*\} = \{1;6\}$ and find

$$w(I_*^{(2)}) = w(\{1;6\}) = \{j \in I : (i,j) \in U,\ j \notin I_*^{(2)},\ i \in I_*^{(2)}\} =$$

$$= \{(1,2) \in U, (6,2) \in U, (6,3) \in U, (6,3) \in U\} = \{2;3;5\}.$$

It can be checked that $I_2^- = \{k \in I : \exists \text{ arc } (k,2) \in U\} = \{k = 1, k = 6 : \exists \text{ arc } (1,2) \in U, (6,2) \in U\} = \{1;6\} \subset I_*^{(2)} = \{1;6\}$ such that the required node from the set $w(I_*^{(2)})$ is $j_* = \{2\}$. Note that the nodes $j_* = \{3\}$ and $j_* = \{5\}$ from the considered set $w(I_*^{(2)})$ are not suitable for the choice since the corresponding sets

$$I_3^- = \{k \in I : \exists \,\text{arc}\,(k,3) \in U\} = \{k = 2, k = 6 : \exists \,\text{arc}\,(2,3) \in U, (6,3) \in U\} = \{2;6\} \not\subset I_*^{(2)} = \{1;6\}$$

and

$$I_5^- = \{k \in I : \exists \,\text{arc}\,(k,5) \in U\} = \{k = 3, k = 6 : \exists \,\text{arc}\,(3,5) \in U, (6,5) \in U\} = \{3;6\} \not\subset I_*^{(2)} = \{1;6\}$$

do not satisfy the condition $I_{j^*}^- \subset I_*^{(2)}$.

Hence

$$B_2 = \max_{i \in I_2^-}(B_i + c_{i2}) = \max\{c_{12} + B_1; c_{62} + B_6\} = \max\{2 + 0; 4 + 1\} = 5 \tag{4.43}$$

Moreover, the corresponding elements are $i_* = \{6\},\ f(j_*) = f(2) = i_* = 6$.

3) Extend the set $I_*^{(3)} \triangleq I_*^{(2)} \cup \{j_*\} = \{1;6;2\}$ and find

$$w(I_*^{(3)}) = w(\{1;6;2\}) = \{j \in I : (i,j) \in U,\ j \notin I_*^{(3)},\ i \in I_*^{(3)}\} =$$

$$= \{(2,3) \in U, (6,3) \in U, (6,5) \in U\} = \{3; 5\}.$$

It can be checked that $I_3^- = \{k \in I : \exists \text{ arc } (k,3) \in U\} = \{k = 2, k = 6 : \exists \text{ arc } (2,3) \in U, (6,3) \in U\} = \{2; 6\} \subset I_*^{(3)} = \{1; 6; 2\}$ such that the required node from the set $w(I_*^{(3)})$ is $j_* = \{3\}$. Note that the node $j_* = \{5\}$ from the considered set $w(I_*^{(3)})$ is not suitable for the choice since the corresponding set

$$I_5^- = \{k \in I : \exists \, \text{arc} \, (k,5) \in U\} = \{k = 3, k = 6 : \exists \, \text{arc} \, (3,5) \in U, (6,5) \in U\} = \{3; 6\} \not\subset I_*^{(3)} = \{1; 6; 2\}$$

does not satisfy the condition $I_{j^*}^- \subset I_*^{(3)}$. Hence

$$B_3 = \max_{i \in I_3^- = \{2;6\}} (B_i + c_{i3}) = \max\{c_{23} + B_2; c_{63} + B_6\} = \max\{2 + 5; 4 + 1\} = 7. \quad (4.44)$$

Moreover, the corresponding elements are $i_* = \{2\}, \ f(j_*) = f(3) = i_* = 2$.

The remainder elements for nodes 5 and 4 are calculated by analogy with given above, and it is omitted here.

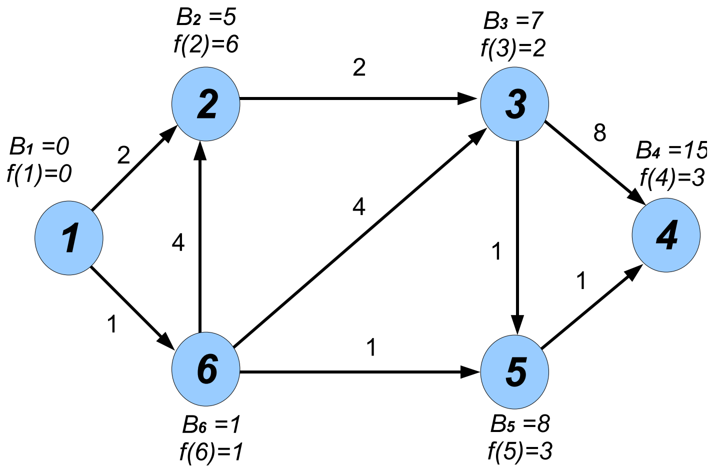


Figure 1

From this Figure it follows that the minimal time to serve mission is equal $B_4 = 15$. The obtained values of the function $f(j)$ make it possible to re-construct the optimal pass way $1 \to 4$ by the formula (4.40):

$$i_1 = f(t) = 3, \ i_2 = f(i_1) = f(3) = 2, \ i_3 = f(i_2) = f(2) = 6, \ i_4 = f(i_3) = f(6) = 1 (4.45)$$

Hence, the optimal order to serve the tasks of the given mission is

$$s = 1 \to 6 \to 2 \to 3 \to 4 = t.$$

Note that the tasks of each node from the extremal route $(i, j,) \in U_{extrem}$ should be started exactly at the calculated moments $x_i^0, i \in I_{extrem}$. But the tasks of the nodes which are out the extremal route can be served early. Let $(i, j) \notin U_{extrem}$. Then the tasks in nonextremal node j can be served starting at any moment of the form

$$x_i^0 + \Delta t_{ij}, \text{ where } \Delta t_{ij} \in [0, x_j^0 - c_{ij} - x_i^0]$$

such that the optimal service time of whole mission is not changed. It should be emphasized that the given ability "to shift" the start for service of some tasks can be used to minimize the number of the UAVs involved in missions. In particular, since the moments $x_i^0, i \in I$ is determined then the optimization problem can be formulated as follows: find the moments $\Delta t_{ij},\ (i, j) \in U$ to minimize the number of UAVs needed to complete the all tasks of the given mission with minimal service time.

Chapter 5

Multiagent Formation with incomplete data

5.1 Formation flight

In this chapter we will study the distributed control logic for formation flight subject to the restricted mutual communications requirements for UAVs. The challenge is for the air vehicles to maintain the formation on the base of information concerning their neighbour on the left [right]. Such kind information presents a special case of feedback control in multiagent dynamical systems [9]. As an starting position we can consider the following subproblems:

- Formation control in discrete time – the task here is to adapt Van-Loan method to MAS to make more appropriate assumptions, which are more suitable for MAS;
- Rewrite formation control problem in continuous time - the dynamics of agents will described by ODE;
- Formulate adaptive decentralized control problem for MAS.

5.2 Formation control in discrete time

Let $p_i = (x_i, y_i),\ i = 1, ..., n$ are arbitrary n objects (agents) given on the plane $\mathbb{R}^2$. We suppose that for each object p_i the information on the position of the two nearest objects p_{i-1} and p_{i+1} is only available (the number n of objects can be unknown, also). In addition, we suppose that the last object p_n knows the position of the p_{n-1} object and the position of some given point $B = (b_1, b_2)$, and the first object p_1 knows the position of p_2 and some point $A = (a_1, a_2)$. Also, it is supposed that the all objects are moving according to the own trajectory.

The problem is: subject to the given imperfect information to design an algorithm for the object evolution such that the given objects will be placed in the preassigned order with the preassigned mutual distance and along the given straight line $[A, B]$. An essential feature of this problem is:

1) decentralized control by the group of objects
2) uncomplete information on the objects involved
3) adaptive control when each of objects chooses the own law according to existing information on the position of two nearest objects

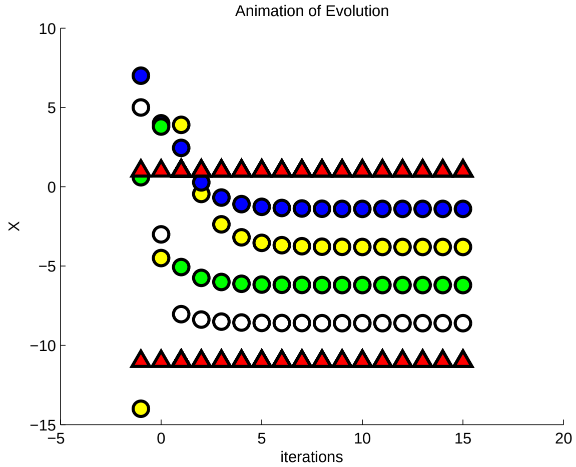
Note, that the problem can be generalized as follows, for example:

i) among the given group of objects it can be chosen some "leader" that is moving according to the given law
ii) the target tracing can be realized along some given curve (circle, for example)

Algorithm of evolution

It is proposed to use the following algorithm: each inner object tends to occupy an assemble average between their nearest objects. In this case the new position of the objects are given as

$$x_i^{k+1} = \frac{(x_{i-1}^k + x_{i+1}^k)}{2},\ i = 2, ..., n-1, \qquad x_1^{k+1} = \frac{(a_1 + x_2^k)}{2};\ \ x_n^{k+1} = \frac{(x_{n-1}^k + a_2)}{2},$$
$$y_i^{k+1} = \frac{(y_{i-1}^k + y_{i+1}^k)}{2},\ i = 2, ..., n-1, \qquad y_1^{k+1} = \frac{(b_1 + y_2^k)}{2};\ \ y_n^{k+1} = \frac{(y_{n-1}^k + b_2)}{2}, \tag{5.1}$$



Results

The algorithm possesses global convergence at $k \to \infty$ to a unique limiting arrangement on straight line $[A, B]$ in the desired order $(A, p_1^*, p_2^*, \ldots, p_n^*, B)$ with equal mutual distance equal $\dfrac{\|B-A\|}{(n+1)}$ such that

$$\|A - p_1^*\| = \|p_1^* - p_2^*\| = \ldots = \|p_n^* - B\| = \frac{\|B-A\|}{(n+1)}$$

Modification of the problem (for given relation of distances)

For the previous problem statement we will change the requirements for final placement of the objects. Namely to place objects so that the relation of distances between them will be the following: $\lambda_1 : \lambda_2 : \ldots : \lambda_n : \lambda_{n+1}$, Then instead of arithmetic mean as in algorithm (5.1) the new objects positions we will be calculated as weighed sum of its neighbor coordinates:

$$x_i^{k+1} = \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}} x_{i-1}^k + \frac{\lambda_i}{\lambda_i + \lambda_{i+1}} x_{i+1}^k, \ i = \overline{2, n-1} \tag{5.2}$$

$$x_1^{k+1} = \frac{\lambda_2}{\lambda_1 + \lambda_2} a_1 + \frac{\lambda_1}{\lambda_1 + \lambda_2} x_2^k; \tag{5.3}$$

$$x_n^{k+1} = \frac{\lambda_{n+1}}{\lambda_n + \lambda_{n-1}} x_{n-1}^k + \frac{\lambda_n}{\lambda_n + \lambda_{n+1}} a_2. \tag{5.4}$$

$$= \text{ analogically for components } \ y_i \tag{5.5}$$

Modification of the problem (circle object arrangement)

The algorithm (5.1) is easily modified for the following situation. On a circle with the center in zero are given n numbered objects p_i, position of which is defined by their angles $\theta_i \in (0, 2\pi)$. The each object is know the angle of neighbors (under numbers) and position of the center of a circle. A problem the same: to place objects in uniform intervals on a circle. One of possible algorithms is obvious: to "cut" circle in some arbitrary point $\theta_c \in [0, 2\pi]$ and apply algorithm for a line.

Also it is possible to generalize the given models by adding some external influence $\omega(t) \in \mathbb{R}^2$ to the dynamics (evolution algorithm). For example, we can set the group speed $\omega(t)$. In final, the objects should be placed in the desired order that is moving with the preassigned speed. It is assumed to being the previous week information contacts.

Also, we can set some speed $\varphi(t)$ for the chosen "leader" among the given objects (for point A or B, for example.

Thus, here we propose to create linear algorithms for MAS using a minimum

of a priori information and extend these algorithms to the multidimensional cases. Also, it will be interesting to investigate the alternative schemes, where the external influences are take into account explicitly.

5.3 Formation control problem in continuous time

The results section above can be extended to continuous case when the discrete (difference) dynamic equations is replaced by differential equations.

The problem is: to design a simple and effective local algorithms for agents movement and allocation along a line (circle, other type of curves). They will be based on the following information:
1) the total number of agents in the system is unknown;
2) the future position of each agent is defined by its own coordinates and coordinates of its closest
neighbors;
3) one or both of end agents may be fixed or movable.

Algorithm of evolution

For the two-dimensional case the dynamic of objects is determined by the following differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{x_{i-1}+x_{i+1}}{2}, \; i=2,...,n-1, \qquad x_1 = x_1(0), \;\; x_n(0)=x_n, \\ \frac{dy_i}{dt} &= \frac{y_{i-1}+y_{i+1}}{2}, \; i=2,...,n-1, \qquad y_1(0) = y_1; \;\; y_n(0)=y_n \end{aligned} \tag{5.6}$$

Here without loss of generality we assume that the A and B are the first and the last objects, respectively, of the given group.

Result

The objects of the group tend to limit position of the form

$$\Big(x_i(t), y_i(t)\Big) \rightarrow \Big(x_1 + \frac{i}{n+1}(x_n - x_1), y_1 + \frac{i}{n+1}(y_n - x_1)\Big) \qquad \text{at} \qquad t \rightarrow \infty \tag{5.7}$$

for $i = 1, 2, \dots, n-2$.

5.4 Illustrative examples

We start with the simple one-dimensional cases when the finite collection of the objects $x_1, \ldots, x_n$ are arbitrary given on the line. Let the position of the first and the last objects are fixed and their position is constant with the course of time.

The problem is to occupy the segment $[x_1, x_n]$ equally-spaced by the $x_2, \ldots, x_{n-1}$ objects. The trajectory of this motion based on state information of their neighbours are described by the special linear differential equations.

Illustrative example (one dimensional case with $n = 4$ agents and two Leaders)

Consider the collection of objects $x_1, \ldots, x_4$ where the Leaders x_1 and x_4 are moving along preassigned passway such that the motion of this group is described by the equations

$$\begin{aligned} & x_1(t) \equiv \varphi_1(t), \\ & \frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2, \ x_2(0) = x_2^0, \\ & \frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3, \ x_3(0) = x_3^0, , \\ & x_4(t) \equiv \varphi_2(t). \end{aligned} \tag{5.8}$$

For example, put $\varphi_1(t) = e^{-t}, \ \varphi_2(t) = e^{-t} + 10$. In this case the solution of (5.8) is

$$\begin{aligned} x_2(t) = & \ \frac{10}{3} + e^{-\frac{3}{2}t}\left[\frac{x_2^0 - x_3^0}{2} + \frac{5}{3}\right] + e^{-\frac{1}{2}t}\left[\frac{x_2^0 + x_3^0}{2} - 4\right] - e^{-t} \\ x_3(t) = & \ \frac{20}{3} + e^{-\frac{3}{2}t}\left[\frac{x_2^0 - x_3^0}{2} - \frac{5}{3}\right] + e^{-\frac{1}{2}t}\left[\frac{x_2^0 + x_3^0}{2} - 4\right] - e^{-t} \end{aligned} \tag{5.9}$$

Illustrative example (one dimensional case with $n = 4$ agents and single Leader)

Consider the collection of objects $x_1, \ldots, x_4$ where there is single Leaders x_1 is moving along preassigned passway. It is necessary to occupy the line $[x_1, x_1 + R]$ such that the last object x_4 will be in right end of this line and others objects x_2, x_3 are equally spaced along this line. Then the motion of this group is described by the equations

$$
\begin{aligned}
&x_1(t) \equiv \varphi_1(t), \\
&\frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0, \\
&\frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0, \\
&\frac{dx_4}{dt} = x_1 + R - x_4,\ x_4(0) = x_4^0,
\end{aligned} \tag{5.10}
$$

For example, put $\varphi_1(t) = te^{-t} + 100,\ R = 60$. In this case the solution of (5.10) is

$$
\begin{aligned}
x_2(t) =&120 + \frac{1}{2}e^{-\frac{3}{2}t}\Big[x_2^0 - x_3^0 + x_4^0 - 134\Big] + \frac{1}{2}e^{-\frac{1}{2}t}\Big[x_2^0 + x_3^0 + x_4^0 - 414\Big] + 12te^{-t} \\
x_2(t) =&140 + \frac{1}{2}e^{-\frac{3}{2}t}\Big[- x_2^0 + x_3^0 - x_4^0 - 134\Big] + \frac{1}{2}e^{-\frac{1}{2}t}\Big[x_2^0 + x_3^0 + x_4^0 - 414\Big] + \frac{1}{2}e^{-t}\big[x_4^0 + t^2 - 308\big]
\end{aligned} \tag{5.11}
$$

Illustrative example (one dimensional case with $n = 5$ agents)

Consider the collection of objects $x_1, \dots, x_5$ the motion of which is described by the equations

$$
\begin{aligned}
&x_1(t) \equiv x_1^0, \\
&\frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0, \\
&\frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0,, \\
&\frac{dx_4}{dt} = \frac{x_3 + x_5}{2} - x_4,\ x_4(0) = x_4^0,, \\
&x_5(t) \equiv x_5^0.
\end{aligned} \tag{5.12}
$$

Denote $x = (x_2, x_3, x_4) \in \mathbb{R}^3$. Then the given linear differential can be presented in matrix form as follows

$$
\frac{dx}{dt} = Ax + b \tag{5.13}
$$

where

$$
A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}, \quad b = \left(\frac{x_1^0}{2}, 0, \frac{x_5^0}{2}\right)^T \in \mathbb{R}^3. \tag{5.14}
$$

The eigenvalues of the matrix A are distinct real numbers $\lambda_1 = -1,\ \lambda_2 = -1 + \frac{\sqrt{2}}{2},\ \lambda_3 = -1 - \frac{\sqrt{2}}{2}$. Note, that these eigenvalues can be written in compact form as $\lambda_k = -2\sin^2\frac{k\pi}{8},\ k = 1, 2, 3.$

The corresponding eigenvectors α_k satisfy the linear algebraic equation

$$(A - \lambda_k E)\alpha_k = 0, k = 1, 2, 3.$$

It can be shown that they are

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},\ \alpha_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}. \tag{5.15}$$

Then the corresponding linear independent solutions of (5.13) are

$$x_2(t) = \alpha_1 e^{-t},\ x_3(t) = \alpha_2 e^{(-1+\frac{\sqrt{2}}{2})t},\ x_4(t) = \alpha_3 e^{(-1-\frac{\sqrt{2}}{2})t}.$$

Hence the fundamental matrix for solution of the differential equation (5.13) can be formed by these solutions $W(t) = \begin{pmatrix} x_2(t) & x_3(t) & x_4(t) \end{pmatrix}$.

Thus the fundamental matrix is

$$W(t) = \begin{pmatrix} e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})t} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} \tag{5.16}$$

It is known that the solution of nonhomogeneous differential equation of the form (5.13) can be presented as

$$x(t) = K(t, 0)x(0) + \int_0^t K(t, \tau)b\,d\tau, \tag{5.17}$$

where $K(t, \tau)$ is the so-called Cauchy matrix. In the considered case of linear differential with constant coefficients can be designed with the help of fundamental matrix $W(t)$ as follows

$$K(t, \tau) = K(t - \tau) = W(t - \tau)W^{-1}(0). \tag{5.18}$$

For the considered case we have

$$W(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad W^{-1}(0) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, \tag{5.19}$$

Then the solution of the differential equations (5.13) with the given initial conditions due to (5.17) is equal

$$\begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})t} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_2^0 \\ x_3^0 \\ x_4^0 \end{pmatrix} +$$

$$\int\limits_0^t \begin{pmatrix} e^{-(t-\tau)} & e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \\ -e^{-(t-\tau)} & e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{x_1^0}{2} \\ 0 \\ \frac{x_5^0}{2} \end{pmatrix} d\tau =$$

$$= \begin{pmatrix} \frac{x_2^0 - x_4^0}{2}e^{-t} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1+\frac{\sqrt{2}}{2})t} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1-\frac{\sqrt{2}}{2})t} \\ \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}}e^{(-1+\frac{\sqrt{2}}{2})t} - \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -\frac{x_2^0 - x_4^0}{2}e^{-t} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1+\frac{\sqrt{2}}{2})t} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} + \tag{5.20}$$

$$+ \int\limits_0^t \begin{pmatrix} \frac{x_1^0 - x_5^0}{4}e^{-(t-\tau)} + \frac{x_1^0 + x_5^0}{8}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} + e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \\ \frac{x_1^0 + x_5^0}{4\sqrt{2}}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} - e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \\ -\frac{x_1^0 - x_5^0}{4}e^{-(t-\tau)} + \frac{x_1^0 + x_5^0}{8}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} + e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \end{pmatrix} d\tau$$

Integrating (5.20) yields then

$$
\begin{aligned}
x_2(t) =& \frac{3}{2}x_1^0 + \frac{1}{2}x_5^0 + e^{-t}\Big[\frac{x_2^0 - x_1^0 - x_4^0 + x_5^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_1^0 + x_5^0}{2(\sqrt{2}-2)} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}\Big] + \\
&+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{x_5^0 + x_1^0}{2\sqrt{2}} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}\Big], \\
x_3(t) =& -x_1^0 - x_5^0 + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}} + \frac{x_1^0 + x_5^0}{2 - 2\sqrt{2}}\Big] + \\
&+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}} + \frac{x_1^0 + x_5^0}{2 + 2\sqrt{2}}\Big], \\
x_4(t) =& \frac{x_1^0}{2} + \frac{x_3^0}{2\sqrt{2}} + \frac{3\sqrt{2}-1}{2\sqrt{2}}x_5^0 + e^{-t}\Big[\frac{x_1^0 - 2x_2^0 + x_4^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4} + \frac{x_1^0 + x_5^0}{2\sqrt{2}}\Big] \\
&+ e^{(-1-\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4} - \frac{x_1^0 + x_5^0}{2(\sqrt{2}+2)}\Big].
\end{aligned}
\tag{5.21}
$$

Note, that the objects x_1 and x_5 are fixed as the group leaders such that their motion is given by simple formulas, for example, as

$$x_1(t) = x_1^0 + v_x^1 \cdot t,\ x_5(t) = x_5^0 + v_x^5 \cdot t$$

where $v_x^1,\ v_x^5$ are given velocities.

Illustrative example (two dimensional case)

Consider the collection of two dimensional objects $M_1(x_1, y_1), \ldots, M_5(x_5, y_5)$ the motion of which is described by the couple of equations

$$
\begin{aligned}
& x_1(t) \equiv x_1^0,\ x_5(t) \equiv x_5^0, \\
& \frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0, \\
& \frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0, \\
& \frac{dx_4}{dt} = \frac{x_3 + x_5}{2} - x_4,\ x_4(0) = x_4^0,
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
&y_1(t) \equiv y_1^0, \;\; y_5(t) \equiv y_5^0, \\
&\frac{dy_2}{dt} = \frac{y_1 + y_3}{2} - y_2, \;\; y_2(0) = y_2^0, \\
&\frac{dy_3}{dt} = \frac{y_2 + y_4}{2} - y_3, \;\; y_3(0) = y_3^0, \\
&\frac{dy_4}{dt} = \frac{y_3 + y_5}{2} - y_4, \;\; y_4(0) = y_4^0,
\end{aligned}
\tag{5.23}
$$

Denote $z = (x, y) \in \mathbb{R}^6$ where $x = (x_2, x_3, x_4) \in \mathbb{R}^3$ and $y = (y_2, y_3, y_4) \in \mathbb{R}^3$. Then the given linear differential can be presented in matrix form as follows

$$\frac{dz}{dt} = Az + B \tag{5.24}$$

where

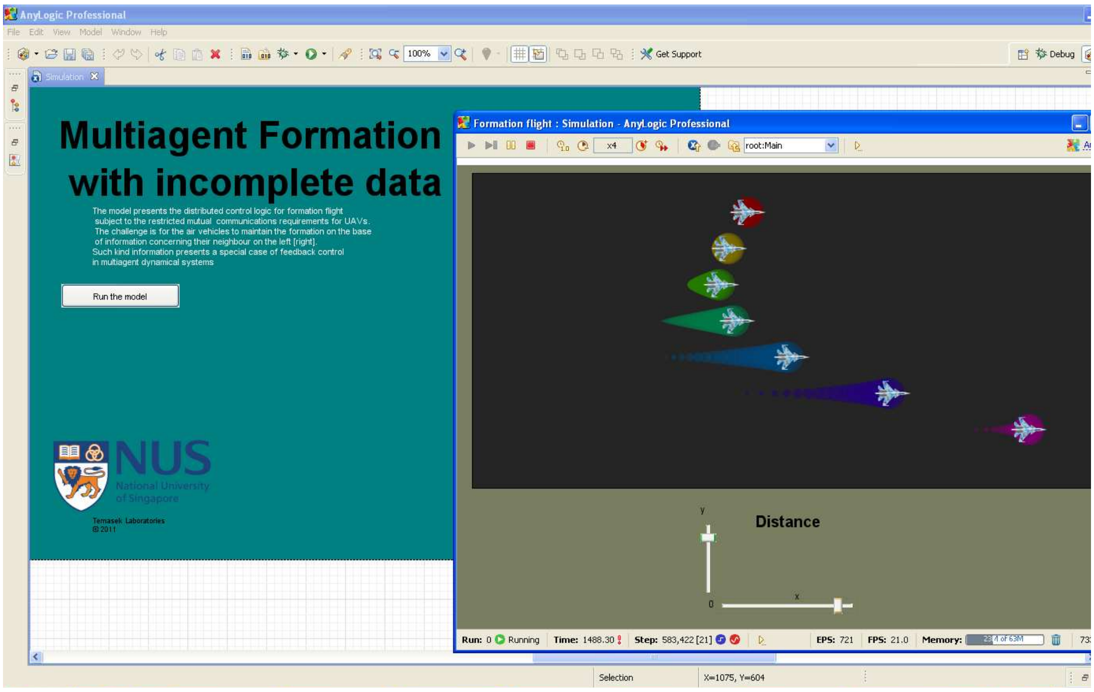
$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \left(\frac{x_1^0}{2}, 0, \frac{x_5^0}{2}, \frac{y_1^0}{2}, 0, \frac{y_5^0}{2}\right)^T \in \mathbb{R}^6. \tag{5.25}$$

and the matrices A_1, A_2 are of the form (5.14).

Since the equation (5.22) and (5.23) are independent then their solutions are given by analogy with formulas (5.21) with the proper changes in notations. Thus, in addition to the solutions $x(t)$ of (5.22) given by the formulas (5.21) we have the similar formulas for solutions $y(t)$ of equations (5.23) written as follows

$$
\begin{aligned}
y_2(t) =& \frac{3}{2}y_1^0 + \frac{1}{2}y_5^0 + e^{-t}\Big[\frac{y_2^0 - y_1^0 - y_4^0 + y_5^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_1^0 + y_5^0}{2(\sqrt{2}-2)} + \frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{4}\Big] + \\
&+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{y_5^0 + y_1^0}{2\sqrt{2}} + \frac{y_2^0 - \sqrt{2}y_3^0 + y_4^0}{4}\Big], \\
y_3(t) =& -y_1^0 - y_5^0 + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{2\sqrt{2}} + \frac{y_1^0 + y_5^0}{2 - 2\sqrt{2}}\Big] + \\
&+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{2\sqrt{2}} + \frac{y_1^0 + y_5^0}{2 + 2\sqrt{2}}\Big], \\
y_4(t) =& \frac{y_1^0}{2} + \frac{y_3^0}{2\sqrt{2}} + \frac{3\sqrt{2}-1}{2\sqrt{2}}y_5^0 + e^{-t}\Big[\frac{y_1^0 - 2y_2^0 + y_4^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{4} + \frac{y_1^0 + y_5^0}{2\sqrt{2}}\Big] \\
&+ e^{(-1-\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 - \sqrt{2}y_3^0 + y_4^0}{4} - \frac{y_1^0 + y_5^0}{2(\sqrt{2}+2)}\Big].
\end{aligned}
\tag{5.26}
$$

Remark. The simulation model for two dimensional case was created on "AnyLogic" simulation software using the agent-based modeling approach, since it is the only simulation tool, which allows creating flexible models with agents, interacting with each other and their environment. "AnyLogic" supports all known ways of specifying the agent behavior statecharts, synchronous and asynchronous event scheduling.



In [9] the multi-agent system is considered in the form

$$\frac{dx_i}{dt} = u_i, \ i = 1, \dots, n$$

where the control input u_i is given in feedback form as follows

$$u_i = \sum_{j=1}^{n} a_{ij}(x_i - x_j), \ i = 1, \dots, n.$$

Here the coefficients a_{ij} characterize the mutual interaction among the n agents. Note, that our cases can be treated as a special case of such interaction when the information is only available from nearest neighbors.

Remark. By analogy with Section above we can add some external input influence $u(t)$ into described dynamics. Also, we can extend the proposed dynamic model by some general linear control system.

Some generalization of this idea we will present in the next Sections.

5.5 Adaptive decentralized control problem for MAS

Let $T = [t_*, t^*]$ be the control interval; $T_h = \{t_*, t_* + h, ..., t^* - h\}; h = \frac{t^*-t_*}{N}$; N be a natural number; $I = \{1, 2, ..., q\}$, $I_i = I \setminus i; A_{ij}(t) \in \mathbb{R}^{n_i \times n_j}; B_{ij}(t) \in \mathbb{R}^{n_i \times r_j}(t \in T, i, j \in I)$ be piecewise continuous matrix functions; $A_i(t) = A_{ii}(t)$ and $B_i = B_{ii}(t), t \in T, i \in I; H_i \in \mathbb{R}^{m \times n_i}, g_0 \in \mathbb{R}^m, c_i \in \mathbb{R}^{n_i}, u_{i*}, u_i^* \in \mathbb{R}^{r_i}$ for $i \in I$ be given matrices and vectors; and $n = \sum_{i \in I} n_i$.

On interval T, consider the group of q objects to be controlled assuming that the i-th object $(i \in I)$ is governed by the equation

$$\dot{x}_i = A_i(t)x_i + \sum_{j \in I_i} A_{ij}(t)x_j + B_i(t)u_i + \sum_{j \in I_i} B_{ij}(t)u_j, \quad x_i(t_*) = x_{i0} \tag{5.27}$$

Here, $x_i = x_i(t) \in \mathbb{R}^{n_i}$ is the state of the ith object at the time t, $u_i = u_i(t) \in U_i$ is the value of the discrete control at the time t, and $U_i = \{u \in \mathbb{R}^{r_i} : u_{i*} \leq u \leq u_i^*\}$ is a bounded set of available values of the i-th control. If $u(t) \equiv u(s)$ for $t \in [s, s+h[, (s \in T_h)$, then the function $u(t), (t \in T)$ is said to be discrete with sampling period h. In (5.27), the function $A_i(t), t \in T$ characterizes the self-dynamics of ith object; the function $(A_{ij}(t), t \in T, j \in I_i)$ describes the influence of other objects on object i; the function $B_i(t), t \in T$ characterizes the input properties of the object i; and the functions $B_{ij}(t), (t \in T, j \in I_i)$ describes the influence of the controls of other objects on object i.

A group of dynamical objects can be controlled in two different ways- in a centralized or decentralized fashion. In the first case, there is a common control center that, given perfect (complete and accurate) information about the current state $x^*(\tau) = (x_i^*(\tau), i \in I)$ of the group, produces for each time interval $[\tau, \tau + h[, \tau \in T_h$, a control $u^*(t) = (u_i^*(t), i \in I), t \in [\tau, \tau + h[$ for all the objects. In the second case, each ith object of the group has a particular (local) control center that produces at each time interval $[\tau, \tau + h[, (\tau \in T_h)$ a control action $u_i^*(t)(t \in [\tau, \tau + h[, \tau \in T_h)$ based on the perfect information about its own current state $x_i^*(\tau)$ and about the states $x_k^*(\tau - h)(k \in I_i)$ of other objects. Also assume that delay of the information about the state of the other objects coincides with the sampling period h. The aim of the control is follows:

(1) Steer the group to a given(common) terminal set at the time t^*:

$$x(t^*) \in X^* = \{x = (x_i, i \in I) : \sum_{i \in I} H_i x_i = g_0, \} \tag{5.28}$$

(2) Achieve the maximum value of the terminal objective function

$$J(u) = \sum_{i \in I} c_i' x_i(t^*) \to \max \tag{5.29}$$

Depending on the way used to control the group, we have two optimal control problems a centralized and a decentralized problem. Assume that centralized real-time control of the group of objects is impossible for some reason.

5.5.1 Decentralized close-loop control for MAS in classical form

Before starting the control, the set of functions

$$\begin{aligned} &u_i(t_*, x), x = (x_i, i \in I) \in \mathbb{R}^n, \\ &u_i(\tau, x_i; x_k, k \in I_i), \qquad \tau \in T_h \setminus t_*,\ x_i \in \mathbb{R}^{n_i},\ x_k \in \mathbb{R}^{n_k},\ k \in I_i,\ i \in I \end{aligned} \tag{5.30}$$

is chosen that is called the (discrete) decentralized feedback.
Mathematical models (5.27)($i \in I$) are closed by feedback (5.30):

$$\begin{aligned} \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x(t)) + \sum_{j \in I_i} B_{ij}(t)u_j(t, x(t)), \\ x_i(t_*) = x_{i0},\ t \in [t_*, t_* + h[, \\ \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x_i(t); x_k(t-h), k \in I_i) + \\ + \sum_{j \in I_i} B_{ij}(t)u_j(t, x_j(t); x_k(t-h), k \in I_j), t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*,\ i \in I. \end{aligned} \tag{5.31}$$

Here $x_0 = (x_{i0}, i \in I);\ u_i(t, x(t)) \equiv u_i(t_*, x_0)$ for $t \in [t_*, t_* + h[$, and $u_i(t, x_i(t); x_k(t-h), k \in I_i) \equiv u_i(\tau, x_i(\tau); x_k(\tau - h), k \in I_i)$ for $t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*, i \in I$. The trajectory of nonlinear system (5.31) is defined as the unique function $x(t \mid x_0; u_i, i \in I), (t \in T)$ composed of the continuously connected solutions to the linear differential equations

$$\begin{aligned} \dot{x}_i = A_i(t)x_i + \sum_{j \in I_i} A_{ij}(t)x_j + B_i(t)u_i + \sum_{j \in I_i} B_{ij}(t)u_j, \quad x_i(t_*) = x_{i0} \\ u_i(t) = u_i(t_*, x_0), \quad t \in [t_*, t_* + h[, \\ u_i(t) = u_i(\tau, x_i(\tau); x_k(\tau - h), k \in I_i), t \in [\tau, tau + h[, \tau \in T_h \setminus t_*, \\ i \in I. \end{aligned}$$

Feedback (5.30) is said to be admissible for the state x_0 if
1) $u_i(t) \in U_i$ for $t \in T$ and $i \in I$ and
2) $x(t^* \mid x_0; u_i, i \in I) \in X^*$.

Let $X(t_*)$ be the set of initial states x_0 for which there exists an admissible feedback, and let $X_i(\tau)$ be the set of states $x_i; x_k, k \in I_i$ for which function (5.30) is define at the time $\tau \in T_h \backslash t^*$. The quality of an admissible feedback for the state $x_0 \in X(t_*)$ is evaluated using the functional

$$J(u, x_0) = \sum_{i \in I} c_i' x_i(t^* \mid x_0; u_i, i \in I).$$

The admissible feedback

$$u_i(t_*, x), x \in X(t_*); \quad u_i^0(\tau, x_i; x_k, k \in I_i), (x_i; x_k, k \in I) \in X_i(\tau), \quad \tau \in T_h \setminus t_*, i \in I, \tag{5.32}$$

is said to be optimal for $x_0 \in X_{t_*}$ if $J(u^0, x_0) = \max_u J(u, x_0), x_0 \in X(t_*)$.

Similarly to the classical centralized optimal feedback, the decentralized optimal feedback is determined on the basis of the mathematical model but is designed for controlling its physical prototype. The decentralized optimal control of a group of dynamical objects designed using the classical closed-loop principle assumes that feedback (5.32) is preliminary synthesized, and group (5.27) $i \in I$ is closed by this feedback, which yields the optimal automatic control system governed by the equations

$$\begin{gathered}
\dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x(t)) + \sum_{j \in I_i} B_{ij}(t)u_j(t, x(t)) + w_i, \\
x_i(t_*) = x_{i0}, \ t \in [t_*, t_* + h[, \\
\dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x_i(t); x_k(t-h), k \in I_i) + \\
+ \sum_{j \in I_i} B_{ij}(t)u_j(t, x_j(t); x_k(t-h), k \in I_j) + w_i, \\
t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*, \ i \in I.
\end{gathered} \tag{5.33}$$

where $w = (w_i \in \mathbb{R}^{n_i}, i \in I)$ is a collection of terms describing inaccuracies of the mathematical modeling and of the implementation of the optimal feedback and the perturbations that affect the objects in the course of control. For simplicity, we will call w the perturbation, and we will assume that it is realized as unknown bounded piecewise continuous functions $w_i(t), (t \in T, i \in I)$. The problem of synthesizing system (5.33) is a very difficult one and has not yet been solved even for centralized control.

The purpose of our investigation is to propose algorithms for synthesis of decentralized automatic control systems using decentralized real-time optimal control of a group of objects; in this case, the control function among individual systems of which each solves an individual autonomous problem (performs self-control) taking into account the actions of the other members of the group. In other words, in this case, the control functions are distributed among q controllers that compute the current values of the feedback components for their own object in the group. Our approach will be based on a fast implementation of the dual method and the learning algorithm from information obtained by each controller from the other controllers working at the previous iterations.

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