Project Proposal:
Controller design for USV path planning, tracking, and enemy-block processes

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Abstract

In this project some proposals of the controller design problem for USV are discussed. It is assumed that the USV maneuvering can be described in the framework of dynamical system theory. On the basis of methods for solving linear optimal control problems supplemented with optimization methods with respect to parameters, a dual method is developed for calculating open loop optimal controls. The method is also used in the synthesis of closed loop optimal controls such that the optimal controller designed on this base will be able to generate the values of an optimal feedback in real time. Also, the optimization problem for piecewise linear systems in the class of discrete controls is considered.

Keywords:

1 Introduction

The aim of this project proposal is to find an adequate problem statement and some collection of accompanying mathematical models for USV path planning, tracking, and enemy-block processes. As an starting position we can consider the following subproblems:

1. USV pursue, intercept and block processes with the proper mathematical models

2. Open Loop and Closed Loop optimization for piecewise linear systems

3. Optimal Control based on imperfect measurements of input and output profiles

4. Centralized and decentralized control problem

5. Control synthesis under uncertainty

2 USV pursue, intercept and block processes

2.1 USV,s Missions: single Evader & Pursuer case

Consider the game of two objects E (evader) and P (pursuer)on the plane $\mathbb{R}^2$. Let us the state of the object E be described by the variable $x \doteq (x_1(t), x_2(t))$ and the state of object P by $y \doteq (y_1(t), y_2(t))$. Also assume that the motion law of these two objects are described as follows (after the proper simplification due to the 2-nd Newton law when the acceleration is proportional to the acting force):

$$\begin{aligned} \ddot{x} &= u \\ \ddot{y} &= v \end{aligned} \tag{1}$$

where u and v are the applied controls (forces).

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Control system (1) can be rewritten in the form of the system of differential equations of 1-st order. Introduce the following auxiliary variables:

$$
\begin{aligned}
z_1 = x_1 &\quad\rightarrow\quad z_2 = \dot{x_1} = \dot{z_1}, \quad \dot{z_2} = \ddot{x_1} = u_1 \\
z_3 = x_2 &\quad\rightarrow\quad z_4 = \dot{x_2} = \dot{z_3}, \quad \dot{z_4} = \ddot{x_2} = u_2
\end{aligned}
\tag{2}
$$

Then the first equation of (1) $\ddot{x} = u$, where $u = (u_1(t), u_2(t)) \in R^2$
can be rewritten as

$$
\begin{cases}
\dot{z_1} = z_2 \\
\dot{z_2} = u_1 \\
\dot{z_3} = z_4 \\
\dot{z_4} = u_2
\end{cases}
\quad \textit{or as two subsystems}: \quad
\begin{cases}
\dot{z_1} = z_2 \\
\dot{z_2} = u_1
\end{cases}
\quad \textit{and} \quad
\begin{cases}
\dot{z_3} = z_4 \\
\dot{z_4} = u_2
\end{cases}
$$

By analogy we can rewrite the second equation of (1) $\ddot{y} = v$, where $v = (v_1(t), v_2(t)) \in R^2$
as

$$
\begin{cases}
\dot{w_1} = w_2 \\
\dot{w_2} = v_1
\end{cases}
\quad \textit{and} \quad
\begin{cases}
\dot{w_3} = w_4 \\
\dot{w_4} = v_2
\end{cases}
$$

where

$$
\begin{aligned}
w_1 = y_1 &\quad\rightarrow\quad w_2 = \dot{y_1} = \dot{w_1}, \quad \dot{w_2} = \ddot{y_1} = v_1 \\
w_3 = y_2 &\quad\rightarrow\quad w_4 = \dot{y_2} = \dot{w_3}, \quad \dot{w_4} = \ddot{y_2} = v_2
\end{aligned}
\tag{3}
$$

Also we should take into account the following assumption about control functions:

$$
\begin{aligned}
|u(t)| &\leq 1 \\
|v(t)| &\leq 1,
\end{aligned}
\tag{4}
$$

since the applied forces for E and P are bounded by some value in each time interval.

Also assume that E is trying to reach the line (or frontline)in R^2:

$$
Hz = g, \ z \in \mathbb{R}^2 \tag{5}
$$

or in coordinate form:

$$
h_1 z_1 + h_2 z_2 = g \tag{6}
$$

Note that the target set can be presented by more complicated geometric figure $M \in \mathbb{R}^2$.

Then the following problem statements can be considered:

- E is look forward to reach frontline at minimal time $T \rightarrow \min$ and P is trying not allows E to reach this goal.
- E is look forward to reach frontline at the given interval $[0, T]$ with minimal cost, and P is trying not allows E to reach this goal.

For the problem above the following quadratic cost function is often considered:

$$
J(u, v) = \rho(E, P), \tag{7}
$$

where $\rho(E, P) = \|x - y\|^2$ denote the euclidian distance between E and P.

In the most classic cases the unknown control functions for E and P is determined as "$minmax$" solution. In other words, it is proposed that the control function of E is a worst case for better choice of pursuer P or vice versa, i.e.

$$J(u^0, v^0) = \min_u \max_v J(u, v) \tag{8}$$

In context of the modern IT-technology achievements some perspective solutions can be achieved by new methods of ultrarapid getting and processing of the available information. This, in turn, demands the revision of the existing models and developing some new ones.

Question : What kind of information are available to the objects E and P?

For example:

i) we can assume that pursuer P knows the state $x(t_i)$ and control function (forces) of evader E on a small interval $t_i \leq t \leq t_{i+1}$. In addition, the number of such kind intervals can be arbitrary.

ii) we can know how the evader E will change the force (i.e. control function) on a small t_i to t_{i+1})

iii) we can assume that the information about the state $x(t)$ of E we will receive almost continuously. In that case the information about control function will be not so significant, since it will be automatically taken into account by state $x(t)$.

The being such information reduces the complexity of the problem which, in turn, gives an ability to design the simple and reliable computer codes.

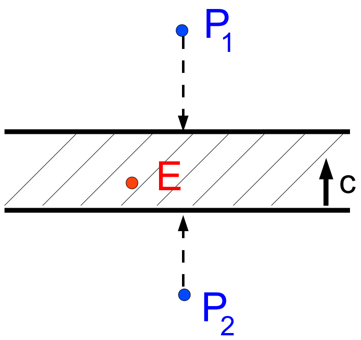
2.2 USV,s Missions: Evader & multiple Pursuers case

Consider the game of three objects E (evader) and P_1 (pursuer 1), P_2(pursuer 2). Let the state of object E is described by the variable $e \doteq (x_1(t), x_2(t))$ and the state of object $P_i, i = 1, 2$ by $p_i \doteq (y_{i1}(t), y_{i2}(t))$. Also assume that motion law of these objects are described as follows (see Wei Zhang paper, for example)

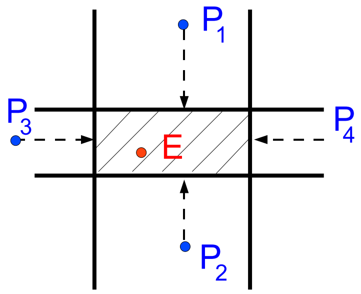
$$\begin{cases} \dot{e} = u_e \\ \dot{p_1} = u_1 \\ \dot{p_2} = u_2 \end{cases}$$

where u_e and $u_i, i = 1, 2$ are velocity control inputs of evader E and pursuers P_i. The problem here is to construct the control functions in order to capture the E in a strip (see figure (a)), i.e.

$$\begin{cases} c'p_1 - c'p_2 \longrightarrow \min \\ c'p_2 \leq c'e \leq c'p_1 \end{cases}$$



(a)



(b)

The proposed idea can be extended for the case when we have four pursuers $P_i, i = 1, 2, 3, 4$ and one evader E. And the problem is to capture E in a box (see figure (b)).

2.3 USV Missions: Docking USV&Evader case

Under some assumptions the pursuer P and evader E can be imaged an united single body on which two opposite forces u and v are applied. Then the dynamic of this object can be given in the form (for simplicity sake we consider one dimensional case):

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = u + v \end{cases}$$

The problem can be formulated as follows: P wishes to remove at final time T the integrated body by control input u to the pre-assigned position

$$x_1(T) = x^* \tag{9}$$

where x^* is the given data, and E counteracts this action with control force v. This controlled opposition should be realized with minimal velocity at the final moment

$$x_2(T) \to \min_u \max_v \tag{10}$$

Under various information interchange the proper optimal control problems are obtained. In particular, in the framework of the complete information the influence v of E can be represented as an disturbances of the state (trajectory) position of the united body $E\&P$. In this case the sensitivity analysis and the design of the optimal feedback control law to damp (reduce) the permanent opposite force v is a perspective way. This analysis can be effectively realized on the base of the so-called constructive methods of supporting control functions [1]. This methods can be used to design the optimal feedback control law in linear systems which allows to realize automatic blocking-out of evader activity.

Examples.

In order to demonstrate the advantages of the supporting control function approach, we now give the following examples.

Example 1. Consider the following optimal control problem

$$\max_{|u|\leq 1} J(u) \triangleq x^{(2)}(1) \tag{11}$$

for

$$\begin{aligned} \frac{dx^{(1)}(t)}{dt} &= x^{(2)}, \qquad x^{(1)}(t),\ x^{(2)}(t) \in \mathbb{R} \quad t \in [s, 1], \\ \frac{dx^{(2)}(t)}{dt} &= u(t), \quad x^{(1)}(s) = z_1, \quad x^{(2)}(s) = z_2 \end{aligned} \tag{12}$$

subject to the following constraints on control variables and a terminal state constraint

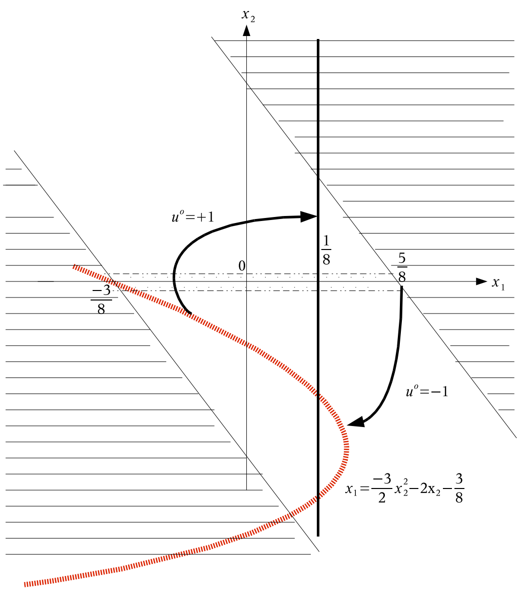
$$|u(t)| \leq 1, \qquad x^{(1)}(1) = 1/8, \tag{13}$$

respectively.

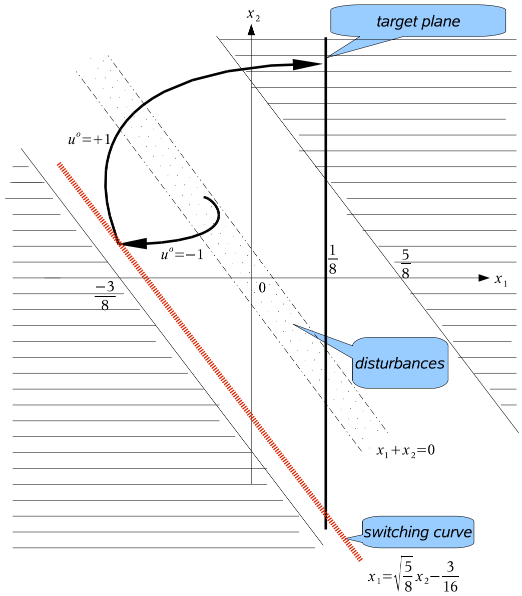
Consider the disturbances of the initial state (s, z_1, z_2) in some neighborhood of the point $(s = 0, z_1 = 0, z_2 = 0)$. It is easy to verify that for the case $s = 0$ and $x_1(0) = 0,\ x_2(0) = 0$ the optimal control for (11)–(13) is given by

In this case it is easy to verify that for $s = 0$ and $x^{(1)}(0) = 0,\ x^{(2)}(0) = 0$ the optimal control signal is given by

$$u^0(t) = -1 \quad \text{for} \quad 0 \leq t \leq 1 - \sqrt{5/8}; \qquad \text{and} \quad u^0(t) = +1 \quad \text{for} \quad 1 - \sqrt{5/8} < t \leq 1.$$



(c) Switching curve is parabolic curve



(d) Switching curve is line

Synthesis of the optimal control can be realized using switching instance function $\tau = \tau(z_1, z_2, s)$, which has to satisfy the following differential equations

$$
\begin{aligned}
\frac{\partial \tau}{\partial z_1} &= \frac{1}{2(1-\tau)} \\
\frac{\partial \tau}{\partial z_2} &= \frac{1-s}{2(1-\tau)} \\
\frac{\partial \tau}{\partial s} &= \frac{1-s-z_2}{2(1-\tau)}
\end{aligned}
\tag{14}
$$

with initial condition

$$\tau(0,0,0) = 1 - \sqrt{5/8}.$$

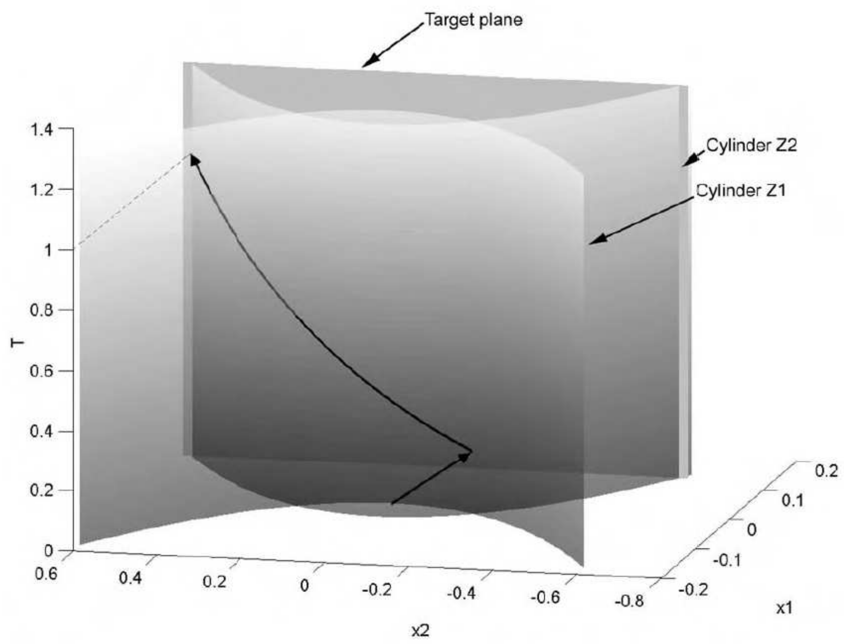
The solution of this Pffaf differential system is given by

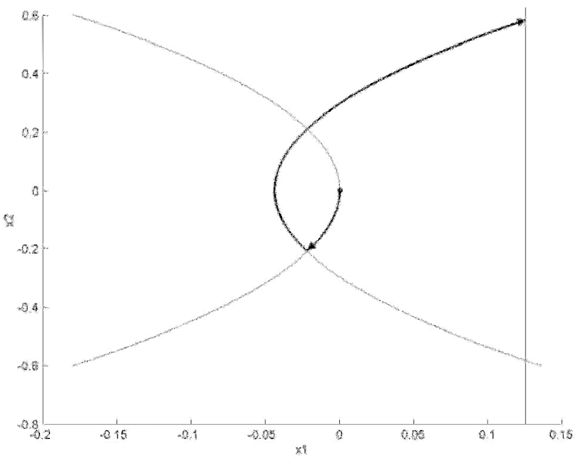
$$\tau(z_1, z_2, s) = 1 - \sqrt{5/8 + (s-1)z_2 - z_1 - s + s^2/2}$$

Without loss of generality, assume $s = 0$ and then the optimal switching function is

$$\tau(z_1, z_2, 0) = 1 - \sqrt{5/8 - z_1 - z_2}.$$

Figures 1 and 2 illustrate the form of this solution below. In Figure 1 is shown the state space variables together with additional variable t. The optimal trajectories (11)–(13) corresponding to the bang-bang control law lie on the parabolic cylinders $(Z_1): \ x^{(1)} = -\frac{1}{2}(x^{(2)})^2 + C_1 + C_2$ and $(Z_2): \ x^{(1)} = +\frac{1}{2}(x^{(2)})^2 + \tilde{C}_1 + \tilde{C}_2$ where the constants $C_i, \tilde{C}_i, \ i = 1,2$ are determined by the initial data $x^{(1)}(0) = z_1, \ x^{(2)}(0) = z_2$. These cylinders correspond to the solutions of differential equations (12) with $u \equiv -1$ or $u \equiv +1$, respectively. It can also be shown that the admissible initial domain for which the problem can be solved is determined by the inequalities: $-\frac{3}{8} \leq z_1 + z_2 \leq \frac{5}{8}$. The switching





manifold Z_h is described in parametric form by

$$\begin{cases} x^{(1)} = -\frac{\left(1-\sqrt{5/8-z_2-z_1}\right)^2}{2} + z_2\left(1-\sqrt{5/8-z_2-z_1}\right) + z_1, \\ x^{(2)} = -1 + \sqrt{5/8-z_2-z_1} + z_2, \\ T = 1 - \sqrt{5/8-z_2-z_1}, \\ -\frac{3}{8} \leq z_1 + z_2 \leq \frac{5}{8} \end{cases}$$

Finally, each optimal trajectory consists of two parts — first it evolves along the vertical parabolic cylinder Z_1 until $\tau = 1 - \sqrt{5/8 - z_2 - z_1}$ when it meets the switching manifold Z_h, and then immediately is switched to continue along the second vertical cylinder Z_2 to meet the target plane $x^{(1)} = 1/8$. Fig. 1 shows the optimal trajectory in the space $\mathbb{R}^3$ for zero initial data, and Fig. 2 shows the projection of this trajectory onto the $x^{(1)}, x^{(2)}$ plane.

Example 2. Consider the following optimization problem

$$\max_{u_1,u_2} J(u) = x_1^{(2)}(1) + x_2^{(2)}(1) \tag{15}$$

for the process

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= x_1^{(2)}(t), \qquad \frac{dx_2^{(1)}(t)}{dt} = x_2^{(2)}(t),\ t \in [s,1] \\ \frac{dx_1^{(2)}(t)}{dt} &= u_1(t), \qquad \frac{dx_2^{(2)}(t)}{dt} = x_1^{(1)}(t) + u_2(t), \end{aligned} \tag{16}$$

with boundary conditions of the form

$$x_1^{(1)}(s) = z_1^{(1)},\ x_1^{(2)}(s) = z_1^{(2)},\ x_2^{(1)}(s) = z_2^{(1)},\ x_2^{(2)}(s) = z_2^{(2)} \tag{17}$$

subject to

$$x_1^{(1)}(1) = 1/8, \quad x_2^{(1)}(1) = 1/384,\ |u_1(t)| \leq 1,\ |u_2(t)| \leq 1, \tag{18}$$

The dynamic here can be written as a stationary differential linear repetitive process of the form

$$\begin{bmatrix} \dot{x}_{k+1}^{(1)}(t) \\ \dot{x}_{k+1}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1}^{(1)}(t) \\ x_{k+1}^{(2)}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k^{(1)}(t) \\ x_k^{(2)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k+1}(t),\ k = 0, 1. \tag{19}$$

Without loss of generality we set $x_0(t) = 0,\ t \in [s,1]$.

To apply the results developed here to this example we first rewrite (16)–(18) in the following integral form:

$$\max_{u_1,u_2} \left\{ z_2^{(1)} + z_2^{(2)} + (1-s)z_1^{(1)} + \frac{(1-s)^2}{2} z_2^{(2)} + \int_s^1 \frac{(1-t)^2+2}{2} u_1(t)dt + \int_s^1 u_2(t)dt \right\} \tag{20}$$

subject to

$$\begin{aligned} &\int_s^1 (1-t)u_1(t)dt = \frac{1}{8} - z_1^{(1)} + (1-s)z_1^{(2)}, \\ &\int_s^1 \left[\frac{(1-t)^3}{6} u_1(t) + (1-t)u_2(t) \right] dt = \frac{1}{384} - z_2^{(1)} - (1-s)z_2^{(2)} - \frac{(1-s)^2}{2} z_1^{(1)} - \frac{(1-s)^3}{6} z_1^{(2)}. \end{aligned} \tag{21}$$

Hence

$$g_{11}(t) = 1-t, \quad g_{21}(t) = \frac{(1-t)^3}{6}, \quad g_{22}(t) = 1-t, \tag{22}$$

$$c_1(t) = \frac{(1-t)^2+2}{2}, \quad c_2(t) = 1 \tag{23}$$

and the multipliers required to design the co-control function $\Delta_i(t),\ i = 1, 2$, noting

$$\begin{aligned} \nu^{(2)} g_{22}(\tau_{2sup}) - c_2(\tau_{2sup}) &= 0, \\ \nu^{(1)} g_{11}(\tau_{1sup}) + \nu^{(2)} g_{21}(\tau_{1sup}) - c_1(\tau_{1sup}) &= 0 \end{aligned} \tag{24}$$

Then co-control function can be written as

$$\begin{aligned}\Delta_1(t) =&(1-t)\left[\frac{1}{1-\tau_{1sup}}+\frac{1-\tau_{1sup}}{2}-\frac{(1-\tau_{1sup})^2}{6(1-\tau_{2sup})}\right]+\frac{(1-t)^3}{6(1-\tau_{2sup})}-\frac{(1-t)^2}{2}-1,\\ \Delta_2(t) =&\frac{1-t}{1-\tau_{2sup}}-1\end{aligned} \tag{25}$$

Now the problem is how to find the basic optimal trajectory when all variables in (17) are zero, i.e.

$$s=0,\ x_1^{(1)}(0)=0,\ x_1^{(2)}(0)=0,\ x_2^{(1)}(0)=0,\ x_2^{(2)}(0)=0. \tag{26}$$

and take the supporting instances as

$$\tau_{1sup}=1-\sqrt{\frac{5}{8}},\quad \tau_{2sup}=1-\sqrt{\frac{131}{256}} \tag{27}$$

Then it follows immediately that the optimal control functions for (15)–(18) with the initial data (26) are given by

$$u_1^0(t)=\begin{cases}-1, & 0\le t<1-\sqrt{\frac{5}{8}},\\ +1, & 1-\sqrt{\frac{5}{8}}\le t\le 1\end{cases},\quad u_2^0(t)=\begin{cases}-1, & 0\le t<1-\sqrt{\frac{131}{256}},\\ +1, & 1-\sqrt{\frac{131}{256}}\le t\le 1\end{cases} \tag{28}$$

and the differential equations give the switching functions $\tau_1\equiv\tau_1(z_1^{(1)},z_1^{(2)},s),\ \tau_2\equiv\tau_2(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)$ as

$$\begin{aligned}&-2\frac{\partial\tau_2}{\partial s}(1-\tau_2)-\frac{2(1-\tau_1)^3}{6}\frac{\partial\tau_1}{\partial s}=\frac{(1-s)^2}{2}z_1^{(2)}+(1-s)z_1^{(1)}+z_2^{(2)}-\frac{(1-s)^3}{6}-(1-s),\\ &-2\frac{\partial\tau_2}{\partial z_1^{(1)}}(1-\tau_2)-\frac{(1-\tau_1)^3}{3}\frac{\partial\tau_1}{\partial z_1^{(1)}}=-\frac{(1-s)^2}{2},\\ &-2\frac{\partial\tau_2}{\partial z_1^{(2)}}(1-\tau_2)-\frac{(1-\tau_1)^3}{3}\frac{\partial\tau_1}{\partial z_1^{(2)}}=-\frac{(1-s)^3}{6},\\ &-2\frac{\partial\tau_2}{\partial z_2^{(1)}}(1-\tau_2)=-1,\quad -2\frac{\partial\tau_2}{\partial z_2^{(2)}}(1-\tau_2)=-(1-s),\end{aligned} \tag{29}$$

with initial conditions

$$\tau_1(0,0,0)=1-\sqrt{\frac{5}{8}},\qquad \tau_2(0,0,0,0,0)=1-\sqrt{\frac{131}{16^2}} \tag{30}$$

The solutions of this differential system are

$$\begin{aligned}\tau_1(z_1^{(1)},z_1^{(2)},s)&=1-\sqrt{SR_1(z_1^{(1)},z_1^{(2)},s)}\\ \tau^{(2)}(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)&=1-\sqrt{SR_2(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)}\end{aligned} \tag{31}$$

where

$$\begin{aligned}SR_1(z_1^{(1)},z_1^{(2)},s)&=\frac{5}{8}+(s-1)z_1^{(2)}-z_1^{(1)}-s+s^2/2,\\ SR_2(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)&=\frac{131}{256}+\frac{2s^4-8s^3+59s^2-102s}{96}+\\ \frac{-20s^2+40s-19}{48}z_1^{(1)}-\frac{1}{12}z_1^{(1)2}&+\frac{4s^3-12s^2+11s-3}{48}z_1^{(2)}+\frac{-s^2+2s-1}{12}z_1^{(2)2}+\\ &+\frac{sz_1^{(1)}z_1^{(2)}}{6}-\frac{z_1^{(1)}z_1^{(2)}}{6}-z_1^{(2)}+(s-1)z_2^{(2)}\end{aligned} \tag{32}$$

It easy to see that the solution of the differential equations describing the process dynamics with $u_1 = const,\ u_2 = const$ are

$$\begin{aligned} x_1^{(1)}(t) =& u_1\frac{t^2}{2} + tC_1 + C_2, \\ x_1^{(2)}(t) =& u_1 t + C_1 \\ x_2^{(1)}(t) =& u_1\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + u_2\frac{t^2}{2} + tC_3 + C_4, \\ x_2^{(2)}(t) =& u_1\frac{t^3}{6} + C_1\frac{t^2}{2} + tC_2 + tu_2 + C_3 \end{aligned} \tag{33}$$

and in this case that the optimal control for pass $k = 1$ coincides with that of Example 1.

Now consider disturbances Ω such that the optimal control is preserved for the case of zero initial conditions, i.e. $u_1 = -1$ for $t \le \tau_1(z_1^{(1)}, z_1^{(2)}, s)$; $(u_2^0 = -1$, for $t \le \tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)$ and the inequality $\tau_1(z_1^{(1)}, z_1^{(2)}, s) < \tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)$ holds. Using (31) we have that the domain Ω is described by

$$0 \le \tau_1(z_1^{(1)}, z_1^{(2)}, s) < \tau_2(z_1^{(1)}, z_1^{(2)}, z_2^{(1)}, z_2^{(2)}, s) \le 1$$

$$SR_1(z_1^{(1)}, z_1^{(2)}, s) \ge 0, \qquad SR_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s) \ge 0$$

To construct the solution for pass $k = 2$, it is necessary to construct the switching surface $\mathfrak{F}$ which is defined by the vectors

$$x_2^{(1)}(t)\mid_{t=\tau_2} = x_2^{(1)}\big(\tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s), \tau)\big),$$

$$x_2^{(2)}(t)\mid_{t=\tau_2} = x_2^{(2)}\big(\tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)\big)$$

when the parameters $z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s$ are members of the set Ω. The parametric description of the switching surface $\mathfrak{F}$ is given by

$$\begin{aligned} x_2^{(1)}(t) =& -\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} - \frac{t^2}{2} + tC_3 + C_4, \\ x_2^{(2)}(t) =& -\frac{t^3}{6} + C_1\frac{t^2}{2} + tC_2 - t + C_3 \end{aligned} \tag{34}$$

where the coefficients C_i are found from the parameters $z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s$.

2.4 USV Missions: Formation modeling case

In this section the various dynamic USV formations are proposed to model USV pursue processes. We suggest the distributed control logic for USV formation subject to the restricted mutual communications requirements for USVs. The challenge is for the team of USV to maintain the formation on the base of information concerning their neighbour on the left [right]. Such kind information presents a special case of feedback control in multiagent dynamical systems [Wei(2011)]. As an starting position we can consider the following subproblems:

- Formation control in discrete time – the task here is to adapt Van-Loan method to USV to make more appropriate assumptions, which are more suitable for USV;
- Rewrite formation control problem in continuous time - the dynamics of agents will described by ODE;
- Formulate adaptive decentralized control problem for USV.

2.5 Formation control in discrete time

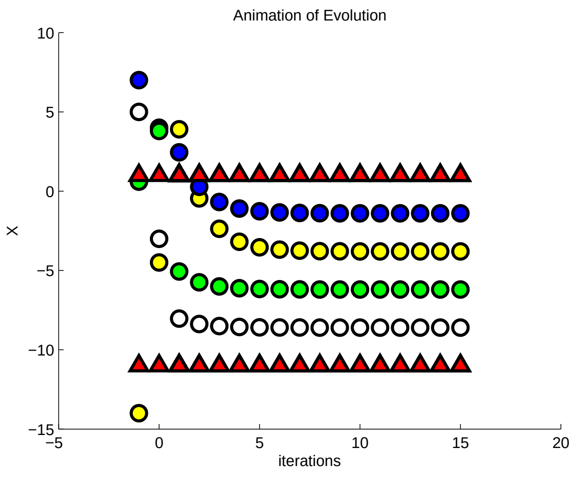
Let $p_i = (x_i, y_i),\ i = 1, ..., n$ are arbitrary n objects (agents) given on the plane $\mathbb{R}^2$. We suppose that for each object p_i the information on the position of the two nearest objects p_{i-1} and p_{i+1} is only available (the number n of objects can be unknown, also). In addition, we suppose that the last object p_n knows the position of the p_{n-1} object and the position of some given point $B = (b_1, b_2)$, and the first object p_1 knows the position of p_2 and some point $A = (a_1, a_2)$. Also, it is supposed that the all objects are moving according to the own trajectory.
The problem is: subject to the given imperfect information to design an algorithm for the object evolution such that the given objects will be placed in the preassigned order with the preassigned mutual distance and along the given straight line $[A, B]$. An essential feature of this problem is:
1) decentralized control by the group of objects
2) uncomplete information on the objects involved
3) adaptive control when each of objects chooses the own law according to existing information on the position of two nearest objects

Note, that the problem can be generalized as follows, for example:
i) among the given group of objects it can be chosen some "leader" that is moving according to the given law
ii) the target tracing can be realized along some given curve (circle, for example)

Algorithm of evolution

It is proposed to use the following algorithm: each inner object tends to occupy an assemble average between their nearest objects. In this case the new position of the objects are given as

$$\begin{aligned} x_i^{k+1} &= \frac{(x_{i-1}^k + x_{i+1}^k)}{2},\ i = 2, ..., n-1, \qquad x_1^{k+1} = \frac{(a_1 + x_2^k)}{2};\ \ x_n^{k+1} = \frac{(x_{n-1}^k + a_2)}{2}, \\ y_i^{k+1} &= \frac{(y_{i-1}^k + y_{i+1}^k)}{2},\ i = 2, ..., n-1, \qquad y_1^{k+1} = \frac{(b_1 + y_2^k)}{2};\ \ y_n^{k+1} = \frac{(y_{n-1}^k + b_2)}{2}, \end{aligned} \tag{35}$$



Results

The algorithm possesses global convergence at $k \to \infty$ to a unique limiting arrangement on straight line $[A, B]$ in the desired order $(A, p_1^*, p_2^*, \ldots, p_n^*, B)$ with equal mutual distance equal $\dfrac{\|B - A\|}{(n+1)}$ such that

$$\|A - p_1^*\| = \|p_1^* - p_2^*\| = \ldots = \|p_n^* - B\| = \frac{\|B - A\|}{(n+1)}$$

Modification of the problem (for given relation of distances)

For the previous problem statement we will change the requirements for final placement of the objects. Namely to place objects so that the relation of distances between them will be the following: $\lambda_1 : \lambda_2 : ... : \lambda_n : \lambda_{n+1}$, Then instead of arithmetic mean as in algorithm (35) the new objects positions we will be calculated as weighed sum of its neighbor coordinates:

$$x_i^{k+1} = \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}} x_{i-1}^k + \frac{\lambda_i}{\lambda_i + \lambda_{i+1}} x_{i+1}^k, \; i = \overline{2, n-1} \tag{36}$$

$$x_1^{k+1} = \frac{\lambda_2}{\lambda_1 + \lambda_2} a_1 + \frac{\lambda_1}{\lambda_1 + \lambda_2} x_2^k; \tag{37}$$

$$x_n^{k+1} = \frac{\lambda_{n+1}}{\lambda_n + \lambda_{n-1}} x_{n-1}^k + \frac{\lambda_n}{\lambda_n + \lambda_{n+1}} a_2. \tag{38}$$

$$= \text{analogically for components } \; y_i \tag{39}$$

Modification of the problem (circle object arrangement)

The algorithm (35) is easily modified for the following situation. On a circle with the center in zero are given n numbered objects p_i, position of which is defined by their angles $\theta_i \in (0, 2\pi)$. The each object is know the angle of neighbors (under numbers) and position of the center of a circle. A problem the same: to place objects in uniform intervals on a circle. One of possible algorithms is obvious: to "cut" circle in some arbitrary point $\theta_c \in [0, 2\pi]$ and apply algorithm for a line.

Also it is possible to generalize the given models by adding some external influence $\omega(t) \in \mathbb{R}^2$ to the dynamics (evolution algorithm). For example, we can set the group speed $\omega(t)$. In final, the objects should be placed in the desired order that is moving with the preassigned speed. It is assumed to being the previous week information contacts.

Also, we can set some speed $\varphi(t)$ for the chosen "leader" among the given objects (for point A or B, for example.

Thus, here we propose to create linear algorithms for USV using a minimum of a priori information and extend these algorithms to the multidimensional cases. Also, it will be interesting to investigate the alternative schemes, where the external influences are take into account explicitly.

2.6 Formation control problem in continuous time

The results section above can be extended to continuous case when the discrete (difference) dynamic equations is replaced by differential equations.

The problem is: to design a simple and effective local algorithms for agents movement and allocation along a line (circle, other type of curves). They will be based on the following information:
1) the total number of agents in the system is unknown;
2) the future position of each agent is defined by its own coordinates and coordinates of its closest neighbors;
3) one or both of end agents may be fixed or movable.

Algorithm of evolution

For the two-dimensional case the dynamic of objects is determined by the following differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{x_{i-1} + x_{i+1}}{2}, \; i = 2, ..., n-1, \qquad x_1 = x_1(0), \quad x_n(0) = x_n, \\ \frac{dy_i}{dt} &= \frac{y_{i-1} + y_{i+1}}{2}, \; i = 2, ..., n-1, \qquad y_1(0) = y_1; \quad y_n(0) = y_n \end{aligned} \tag{40}$$

Here without loss of generality we assume that the A and B are the first and the last objects, respectively, of the given group.

Result

The objects of the group tend to limit position of the form

$$\Big(x_i(t), y_i(t)\Big) \to \Big(x_1 + \frac{i}{n+1}(x_n - x_1), y_1 + \frac{i}{n+1}(y_n - x_1)\Big) \qquad \text{at} \qquad t \to \infty \tag{41}$$

for $i = 1, 2, \ldots, n-2$.

2.6.1 Illustrative examples

We start with the simple one-dimensional cases when the finite collection of the objects $x_1, \dots, x_n$ are arbitrary given on the line. Let the position of the first and the last objects are fixed and their position is constant with the course of time.

The problem is to occupy the segment $[x_1, x_n]$ equally-spaced by the $x_2, \dots, x_{n-1}$ objects. The trajectory of this motion based on state information of their neighbours are described by the special linear differential equations.

Illustrative example (one dimensional case with $n = 4$ agents and two Leaders)

Consider the collection of objects $x_1, \dots, x_4$ where the Leaders x_1 and x_4 are moving along preassigned passway such that the motion of this group is described by the equations

$$
\begin{aligned}
&x_1(t) \equiv \varphi_1(t), \\
&\frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0, \\
&\frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0,, \\
&x_4(t) \equiv \varphi_2(t).
\end{aligned}
\tag{42}
$$

For example, put $\varphi_1(t) = e^{-t},\ \varphi_2(t) = e^{-t} + 10$. In this case the solution of (42) is

$$
\begin{aligned}
x_2(t) = &\ \frac{10}{3} + e^{-\frac{3}{2}t}\left[\frac{x_2^0 - x_3^0}{2} + \frac{5}{3}\right] + e^{-\frac{1}{2}t}\left[\frac{x_2^0 + x_3^0}{2} - 4\right] - e^{-t} \\
x_3(t) = &\ \frac{20}{3} + e^{-\frac{3}{2}t}\left[\frac{x_2^0 - x_3^0}{2} - \frac{5}{3}\right] + e^{-\frac{1}{2}t}\left[\frac{x_2^0 + x_3^0}{2} - 4\right] - e^{-t}
\end{aligned}
\tag{43}
$$

Illustrative example (one dimensional case with $n = 4$ agents and single Leader)

Consider the collection of objects $x_1, \dots, x_4$ where there is single Leaders x_1 is moving along preassigned passway. It is necessary to occupy the line $[x_1, x_1 + R]$ such that the last object x_4 will be in right end of this line and others objects x_2, x_3 are equally spaced along this line. Then the motion of this group is described by the equations

$$
\begin{aligned}
&x_1(t) \equiv \varphi_1(t), \\
&\frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0, \\
&\frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0, \\
&\frac{dx_4}{dt} = x_1 + R - x_4,\ x_4(0) = x_4^0,
\end{aligned}
\tag{44}
$$

For example, put $\varphi_1(t) = te^{-t} + 100,\ R = 60$. In this case the solution of (44) is

$$
\begin{aligned}
x_2(t) =& 120 + \frac{1}{2}e^{-\frac{3}{2}t}\Big[x_2^0 - x_3^0 + x_4^0 - 134\Big] + \frac{1}{2}e^{-\frac{1}{2}t}\Big[x_2^0 + x_3^0 + x_4^0 - 414\Big] + 12te^{-t} \\
x_2(t) =& 140 + \frac{1}{2}e^{-\frac{3}{2}t}\Big[-x_2^0 + x_3^0 - x_4^0 - 134\Big] + \frac{1}{2}e^{-\frac{1}{2}t}\Big[x_2^0 + x_3^0 + x_4^0 - 414\Big] + \frac{1}{2}e^{-t}\big[x_4^0 + t^2 - 308\big]
\end{aligned}
\tag{45}
$$

Illustrative example (one dimensional case with $n = 5$ agents)

Consider the collection of objects $x_1, \dots, x_5$ the motion of which is described by the equations

$$
\begin{aligned}
& x_1(t) \equiv x_1^0, \\
& \frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2, \;\; x_2(0) = x_2^0, \\
& \frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3, \;\; x_3(0) = x_3^0, , \\
& \frac{dx_4}{dt} = \frac{x_3 + x_5}{2} - x_4, \;\; x_4(0) = x_4^0, , \\
& x_5(t) \equiv x_5^0.
\end{aligned}
\tag{46}
$$

Denote $x = (x_2, x_3, x_4) \in \mathbb{R}^3$. Then the given linear differential can be presented in matrix form as follows

$$
\frac{dx}{dt} = Ax + b \tag{47}
$$

where

$$
A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}, \quad b = \left(\frac{x_1^0}{2}, 0, \frac{x_5^0}{2} \right)^T \in \mathbb{R}^3. \tag{48}
$$

The eigenvalues of the matrix A are distinct real numbers $\lambda_1 = -1,\ \lambda_2 = -1 + \frac{\sqrt{2}}{2},\ \lambda_3 = -1 - \frac{\sqrt{2}}{2}$. Note, that these eigenvalues can be written in compact form as $\lambda_k = -2\sin^2 \frac{k\pi}{8},\ k = 1, 2, 3.$

The corresponding eigenvectors α_k satisfy the linear algebraic equation

$$
(A - \lambda_k E)\alpha_k = 0, k = 1, 2, 3.
$$

It can be shown that they are

$$
\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}. \tag{49}
$$

Then the corresponding linear independent solutions of (47) are

$$
x_2(t) = \alpha_1 e^{-t},\ x_3(t) = \alpha_2 e^{(-1+\frac{\sqrt{2}}{2})t},\ x_4(t) = \alpha_3 e^{(-1-\frac{\sqrt{2}}{2})t}.
$$

Hence the fundamental matrix for solution of the differential equation (47) can be formed by these solutions $W(t) = \begin{pmatrix} x_2(t) & x_3(t) & x_4(t) \end{pmatrix}$.

Thus the fundamental matrix is

$$
W(t) = \begin{pmatrix} e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})t} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} \tag{50}
$$

It is known that the solution of nonhomogeneous differential equation of the form (47) can be presented as

$$
x(t) = K(t, 0)x(0) + \int_0^t K(t, \tau) b d\tau, \tag{51}
$$

where $K(t,\tau)$ is the so-called Cauchy matrix. In the considered case of linear differential with constant coefficients can be designed with the help of fundamental matrix $W(t)$ as follows

$$K(t,\tau) = K(t-\tau) = W(t-\tau)W^{-1}(0). \tag{52}$$

For the considered case we have

$$W(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad W^{-1}(0) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, \tag{53}$$

Then the solution of the differential equations (47) with the given initial conditions due to (51) is equal

$$\begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})t} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -e^{-t} & e^{(-1+\frac{\sqrt{2}}{2})t} & e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_2^0 \\ x_3^0 \\ x_4^0 \end{pmatrix} +$$

$$\int_0^t \begin{pmatrix} e^{-(t-\tau)} & e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \\ 0 & \sqrt{2}e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & -\sqrt{2}e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \\ -e^{-(t-\tau)} & e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} & e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{x_1^0}{2} \\ 0 \\ \frac{x_5^0}{2} \end{pmatrix} d\tau =$$

$$= \begin{pmatrix} \frac{x_2^0 - x_4^0}{2}e^{-t} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1+\frac{\sqrt{2}}{2})t} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1-\frac{\sqrt{2}}{2})t} \\ \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}}e^{(-1+\frac{\sqrt{2}}{2})t} - \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}}e^{(-1-\frac{\sqrt{2}}{2})t} \\ -\frac{x_2^0 - x_4^0}{2}e^{-t} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1+\frac{\sqrt{2}}{2})t} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}e^{(-1-\frac{\sqrt{2}}{2})t} \end{pmatrix} + \tag{54}$$

$$+\int_0^t \begin{pmatrix} \frac{x_1^0 - x_5^0}{4}e^{-(t-\tau)} + \frac{x_1^0 + x_5^0}{8}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} + e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \\ \frac{x_1^0 + x_5^0}{4\sqrt{2}}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} - e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \\ -\frac{x_1^0 - x_5^0}{4}e^{-(t-\tau)} + \frac{x_1^0 + x_5^0}{8}\left(e^{(-1+\frac{\sqrt{2}}{2})(t-\tau)} + e^{(-1-\frac{\sqrt{2}}{2})(t-\tau)}\right) \end{pmatrix} d\tau$$

Integrating (54) yields then

$$x_2(t) =\frac{3}{2}x_1^0 + \frac{1}{2}x_5^0 + e^{-t}\Big[\frac{x_2^0 - x_1^0 - x_4^0 + x_5^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_1^0 + x_5^0}{2(\sqrt{2}-2)} + \frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4}\Big]+$$

$$+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{x_5^0 + x_1^0}{2\sqrt{2}} + \frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4}\Big],$$

$$x_3(t) = - x_1^0 - x_5^0 + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}} + \frac{x_1^0 + x_5^0}{2-2\sqrt{2}}\Big]+$$

$$+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{2\sqrt{2}} + \frac{x_1^0 + x_5^0}{2+2\sqrt{2}}\Big], \tag{55}$$

$$x_4(t) =\frac{x_1^0}{2} + \frac{x_3^0}{2\sqrt{2}} + \frac{3\sqrt{2}-1}{2\sqrt{2}}x_5^0 + e^{-t}\Big[\frac{x_1^0 - 2x_2^0 + x_4^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 + \sqrt{2}x_3^0 + x_4^0}{4} + \frac{x_1^0 + x_5^0}{2\sqrt{2}}\Big]$$

$$+ e^{(-1-\frac{\sqrt{2}}{2})t}\Big[\frac{x_2^0 - \sqrt{2}x_3^0 + x_4^0}{4} - \frac{x_1^0 + x_5^0}{2(\sqrt{2}+2)}\Big].$$

Note, that the objects x_1 and x_5 are fixed as the group leaders such that their motion is given by simple formulas, for example, as

$$x_1(t) = x_1^0 + v_x^1 \cdot t,\ x_5(t) = x_5^0 + v_x^5 \cdot t$$

where $v_x^1,\ v_x^5$ are given velocities.

Illustrative example (two dimensional case)

Consider the collection of two dimensional objects $M_1(x_1, y_1), \dots, M_5(x_5, y_5)$ the motion of which is described by the couple of equations

$$\begin{aligned}
&x_1(t) \equiv x_1^0,\ x_5(t) \equiv x_5^0,\\
&\frac{dx_2}{dt} = \frac{x_1 + x_3}{2} - x_2,\ x_2(0) = x_2^0,\\
&\frac{dx_3}{dt} = \frac{x_2 + x_4}{2} - x_3,\ x_3(0) = x_3^0,\\
&\frac{dx_4}{dt} = \frac{x_3 + x_5}{2} - x_4,\ x_4(0) = x_4^0,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&y_1(t) \equiv y_1^0,\ y_5(t) \equiv y_5^0,\\
&\frac{dy_2}{dt} = \frac{y_1 + y_3}{2} - y_2,\ y_2(0) = y_2^0,\\
&\frac{dy_3}{dt} = \frac{y_2 + y_4}{2} - y_3,\ y_3(0) = y_3^0,\\
&\frac{dy_4}{dt} = \frac{y_3 + y_5}{2} - y_4,\ y_4(0) = y_4^0,
\end{aligned} \tag{57}$$

Denote $z = (x, y) \in \mathbb{R}^6$ where $x = (x_2, x_3, x_4) \in \mathbb{R}^3$ and $y = (y_2, y_3, y_4) \in \mathbb{R}^3$. Then the given linear differential can be presented in matrix form as follows

$$\frac{dz}{dt} = Az + B \tag{58}$$

where

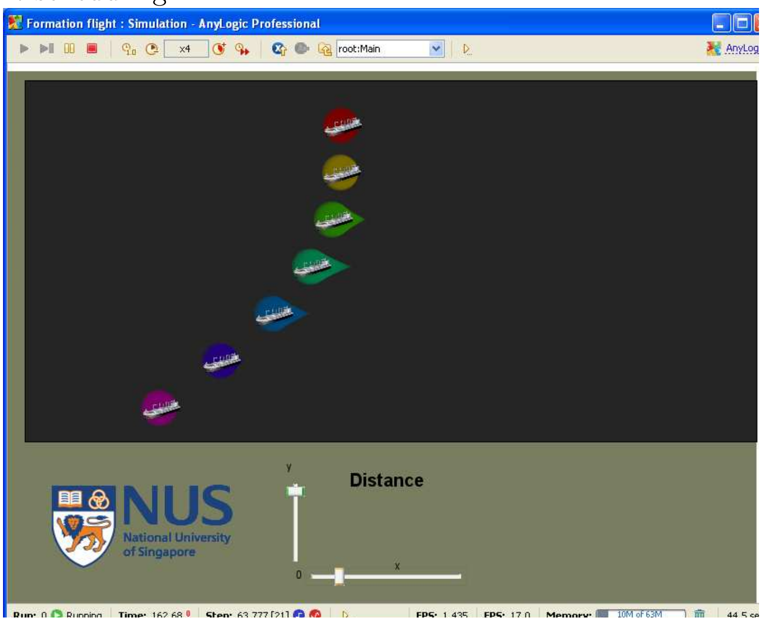
$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \left(\frac{x_1^0}{2}, 0, \frac{x_5^0}{2}, \frac{y_1^0}{2}, 0, \frac{y_5^0}{2}\right)^T \in \mathbb{R}^6. \tag{59}$$

and the matrices A_1, A_2 are of the form (48).

Since the equation (56) and (57) are independent then their solutions are given by analogy with formulas (55) with the proper changes in notations. Thus, in addition to the solutions $x(t)$ of (56) given by the formulas (55) we have the similar formulas for solutions $y(t)$ of equations (57) written as follows

$$\begin{aligned} y_2(t) =& \frac{3}{2}y_1^0 + \frac{1}{2}y_5^0 + e^{-t}\Big[\frac{y_2^0 - y_1^0 - y_4^0 + y_5^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_1^0 + y_5^0}{2(\sqrt{2}-2)} + \frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{4}\Big] + \\ &+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{y_5^0 + y_1^0}{2\sqrt{2}} + \frac{y_2^0 - \sqrt{2}y_3^0 + y_4^0}{4}\Big], \\ y_3(t) =& -y_1^0 - y_5^0 + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{2\sqrt{2}} + \frac{y_1^0 + y_5^0}{2 - 2\sqrt{2}}\Big] + \\ &+ e^{-(1+\frac{\sqrt{2}}{2})t}\Big[-\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{2\sqrt{2}} + \frac{y_1^0 + y_5^0}{2 + 2\sqrt{2}}\Big], \\ y_4(t) =& \frac{y_1^0}{2} + \frac{y_3^0}{2\sqrt{2}} + \frac{3\sqrt{2}-1}{2\sqrt{2}}y_5^0 + e^{-t}\Big[\frac{y_1^0 - 2y_2^0 + y_4^0}{2}\Big] + e^{(-1+\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 + \sqrt{2}y_3^0 + y_4^0}{4} + \frac{y_1^0 + y_5^0}{2\sqrt{2}}\Big] \\ &+ e^{(-1-\frac{\sqrt{2}}{2})t}\Big[\frac{y_2^0 - \sqrt{2}y_3^0 + y_4^0}{4} - \frac{y_1^0 + y_5^0}{2(\sqrt{2}+2)}\Big]. \end{aligned} \tag{60}$$

Remark. The simulation model for two dimensional case can be created on "AnyLogic" simulation software using the agent-based modeling approach, since it is the only simulation tool, which allows creating flexible models with agents, interacting with each other and their environment. "AnyLogic" supports all known ways of specifying the agent behavior statecharts, synchronous and asynchronous event scheduling.



In [Wei(2011)] the multi-agent system is considered in the form

$$\frac{dx_i}{dt} = u_i, \ i = 1, \ldots, n$$

where the control input u_i is given in feedback form as follows

$$u_i = \sum_{j=1}^{n} a_{ij}(x_i - x_j),\ i = 1, \dots, n.$$

Here the coefficients a_{ij} characterize the mutual interaction among the n agents. Note, that our cases can be treated as a special case of such interaction when the information is only available from nearest neighbors.

Remark. By analogy with Section above we can add some external input influence $u(t)$ into described dynamics. Also, we can extend the proposed dynamic model by some general linear control system.

Some generalization of this idea we will present in the next Sections.

3 Adaptive decentralized control problem for USV

Let $T = [t_*, t^*]$ be the control interval; $T_h = \{t_*, t_* + h, ..., t^* - h\}; h = \frac{t^*-t_*}{N}$; N be a natural number; $I = \{1, 2, ..., q\},\ I_i = I \setminus i; A_{ij}(t) \in \mathbb{R}^{n_i \times n_j}; B_{ij}(t) \in \mathbb{R}^{n_i \times r_j}(t \in T, i, j \in I)$ be piecewise continuous matrix functions; $A_i(t) = A_{ii}(t)$ and $B_i = B_{ii}(t), t \in T, i \in I; H_i \in \mathbb{R}^{m \times n_i}, g_0 \in \mathbb{R}^m, c_i \in \mathbb{R}^{n_i}, u_{i*}, u_i^* \in \mathbb{R}^{r_i}$ for $i \in I$ be given matrices and vectors; and $n = \sum\limits_{i \in I} n_i$.

On interval T, consider the group of q objects to be controlled assuming that the i-th object $(i \in I)$ is governed by the equation

$$\dot{x}_i \quad = \quad A_i(t)x_i + \sum_{j \in I_i} A_{ij}(t)x_j + B_i(t)u_i + \sum_{j \in I_i} B_{ij}(t)u_j, \quad x_i(t_*) = x_{i0} \tag{61}$$

Here, $x_i = x_i(t) \in \mathbb{R}^{n_i}$ is the state of the ith object at the time t, $u_i = u_i(t) \in U_i$ is the value of the discrete control at the time t, and $U_i = \{u \in \mathbb{R}^{r_i} : u_{i*} \le u \le u_i^*\}$ is a bounded set of available values of the i-th control. If $u(t) \equiv u(s)$ for $t \in [s, s+h[, (s \in T_h)$, then the function $u(t), (t \in T)$ is said to be discrete with sampling period h. In (77), the function $A_i(t), t \in T$ characterizes the self-dynamics of ith object; the function $(A_{ij}(t), t \in T, j \in I_i)$ describes the influence of other objects on object i; the function $B_i(t), t \in T$ characterizes the input properties of the object i; and the functions $B_{ij}(t), (t \in T, j \in I_i)$ describes the influence of the controls of other objects on object i.

A group of dynamical objects can be controlled in two different ways- in a centralized or decentralized fashion. In the first case, there is a common control center that, given perfect (complete and accurate) information about the current state $x^*(\tau) = (x_i^*(\tau), i \in I)$ of the group, produces for each time interval $[\tau, \tau + h[, \tau \in T_h$, a control $u^*(t) = (u_i^*(t), i \in I), t \in [\tau, \tau + h[$ for all the objects. In the second case, each ith object of the group has a particular (local) control center that produces at each time interval $[\tau, \tau + h[, (\tau \in T_h)$ a control action $u_i^*(t)(t \in [\tau, \tau + h[, \tau \in T_h)$ based on the perfect information about its own current state $x_i^*(\tau)$ and about the states $x_k^*(\tau - h)(k \in I_i)$ of other objects. Also assume that delay of the information about the state of the other objects coincides with the sampling period h. The aim of the control is follows:

(1) Steer the group to a given(common) terminal set at the time t^*:

$$x(t^*) \in X^* = \{x = (x_i, i \in I) : \sum_{i \in I} H_i x_i = g_0, \} \tag{62}$$

(2) Achieve the maximum value of the terminal objective function

$$J(u) = \sum_{i \in I} c_i' x_i(t^*) \to \max \tag{63}$$

Depending on the way used to control the group, we have two optimal control problems a centralized and a decentralized problem. Assume that centralized real-time control of the group of objects is impossible for some reason.

3.1 Decentralized close-loop control for USV in classical form

Before starting the control, the set of functions

$$\begin{aligned}&u_i(t_*,x), x=(x_i, i\in I)\in\mathbb{R}^n,\\&u_i(\tau,x_i;x_k,k\in I_i), \qquad \tau\in T_h\setminus t_*,\ x_i\in\mathbb{R}^{n_i},\ x_k\in\mathbb{R}^{n_k},\ k\in I_i,\ i\in I\end{aligned} \tag{64}$$

is chosen that is called the (discrete) decentralized feedback.
Mathematical models (77)($i\in I$) are closed by feedback (80):

$$\begin{aligned}\dot{x}_i(t)=A_i(t)x_i(t)+\sum_{j\in I_i}A_{ij}(t)x_j(t)+B_i(t)u_i(t,x(t))+\sum_{j\in I_i}B_{ij}(t)u_j(t,x(t)),\\ x_i(t_*)=x_{i0},\ t\in[t_*,t_*+h[,\\ \dot{x}_i(t)=A_i(t)x_i(t)+\sum_{j\in I_i}A_{ij}(t)x_j(t)+B_i(t)u_i(t,x_i(t);x_k(t-h),k\in I_i)+\\ +\sum_{j\in I_i}B_{ij}(t)u_j(t,x_j(t);x_k(t-h),k\in I_j),t\in[\tau,\tau+h[,\tau\in T_h\setminus t_*,\ i\in I.\end{aligned} \tag{65}$$

Here $x_0=(x_{i0},i\in I);\ u_i(t,x(t))\equiv u_i(t_*,x_0)$ for $t\in[t_*,t_*+h[$, and $u_i(t,x_i(t);x_k(t-h),k\in I_i)\equiv u_i(\tau,x_i(\tau);x_k(\tau-h),k\in I_i)$ for $t\in[\tau,\tau+h[,\tau\in T_h\setminus t_*,i\in I$. The trajectory of nonlinear system (81) is defined as the unique function $x(t\mid x_0;u_i,i\in I),(t\in T)$ composed of the continuously connected solutions to the linear differential equations

$$\begin{aligned}\dot{x}_i=A_i(t)x_i+\sum_{j\in I_i}A_{ij}(t)x_j+B_i(t)u_i+\sum_{j\in I_i}B_{ij}(t)u_j,\quad x_i(t_*)=x_{i0}\\ u_i(t)=u_i(t_*,x_0),\quad t\in[t_*,t_*+h[,\\ u_i(t)=u_i(\tau,x_i(\tau);x_k(\tau-h),k\in I_i),t\in[\tau,tau+h[,\tau\in T_h\setminus t_*,\\ i\in I.\end{aligned}$$

Feedback (80) is said to be admissible for the state x_0 if
1) $u_i(t)\in U_i$ for $t\in T$ and $i\in I$ and
2) $x(t^*\mid x_0;u_i,i\in I)\in X^*$.

Let $X(t_*)$ be the set of initial states x_0 for which there exists an admissible feedback, and let $X_i(\tau)$ be the set of states $x_i;x_k,k\in I_i$ for which function (80) is define at the time $\tau\in T_h\setminus t^*$. The quality of an admissible feedback for the state $x_0\in X(t_*)$ is evaluated using the functional

$$J(u,x_0)=\sum_{i\in I}c_i'x_i(t^*\mid x_0;u_i,i\in I).$$

The admissible feedback

$$u_i(t_*,x),x\in X(t_*);\quad u_i^0(\tau,x_i;x_k,k\in I_i),(x_i;x_k,k\in I)\in X_i(\tau),\quad \tau\in T_h\setminus t_*,i\in I, \tag{66}$$

is said to be optimal for $x_0\in X_{t_*}$ if $J(u^0,x_0)=\max\limits_u J(u,x_0),x_0\in X(t_*)$.

Similarly to the classical centralized optimal feedback, the decentralized optimal feedback is determined on the basis of the mathematical model but is designed for controlling its physical prototype. The decentralized optimal control of a group of dynamical objects designed using the classical closed-loop principle assumes that feedback (82) is preliminary synthesized, and group (77) $i\in I$ is closed by this feedback, which yields the optimal automatic control system governed by the equations

$$\dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x(t)) + \sum_{j \in I_i} B_{ij}(t)u_j(t, x(t)) + w_i,$$
$$x_i(t_*) = x_{i0},\ t \in [t_*, t_* + h[,$$

$$\dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x_i(t); x_k(t-h), k \in I_i) + \tag{67}$$
$$+ \sum_{j \in I_i} B_{ij}(t)u_j(t, x_j(t); x_k(t-h), k \in I_j) + w_i,$$
$$t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*,\ i \in I.$$

where $w = (w_i \in \mathbb{R}^{n_i}, i \in I)$ is a collection of terms describing inaccuracies of the mathematical modeling and of the implementation of the optimal feedback and the perturbations that affect the objects in the course of control. For simplicity, we will call w the perturbation, and we will assume that it is realized as unknown bounded piecewise continuous functions $w_i(t), (t \in T, i \in I)$. The problem of synthesizing system (83) is a very difficult one and has not yet been solved even for centralized control.

The purpose of our investigation is to propose algorithms for synthesis of decentralized automatic control systems using decentralized real-time optimal control of a group of objects; in this case, the control function among individual systems of which each solves an individual autonomous problem (performs self-control) taking into account the actions of the other members of the group. In other words, in this case, the control functions are distributed among q controllers that compute the current values of the feedback components for their own object in the group. Our approach will be based on a fast implementation of the dual method and the learning algorithm from information obtained by each controller from the other controllers working at the previous iterations.

4 Open Loop and Closed Loop optimization

In the mathematical theory of optimal processes, to analyze nonlinear systems of the form

$$\dot{x}_i = f(x) + bu,\ (x \in \mathbb{R}^n,\ u \in \mathbb{R}) \tag{68}$$

it is often sufficient to use linear approximations

$$\dot{x}_i = A(t)x + bu,\ A(t) = \frac{\partial f(x(t))}{\partial x} \tag{69}$$

constructed along certain trajectories $x(t)(t \geq 0)$ of the nonlinear system (68). Linear approximations (69) are frequently used to develop approximate methods for solving optimal control problems. It is clear that they can only provide satisfactory descriptions of local behavior of non- linear systems in the neighborhoods of reference trajectories. Therefore, these approximations have a limited scope. One natural way to expand the scope of linear optimization methods is to use piecewise linear approximations (first-order splines) of the nonlinear elements of a problem. Even though the model remains non- linear in this approximation, effective optimization methods can be developed by taking into account specific properties of the piecewise linear model.

Let $T = [0, t*]$ be the control time interval, $h = t*/N$ be the quantization interval, Nbe a positive integer, and $Th = \{0, h, ..., t * -h\}$. The function $u(t)(t \in T)$ is called the discrete control (with the quantization interval h) if $u(t) = u(kh)$ for $t \in [kh, (k+1)h[(k = 0, ..., N-1)$.. We consider the optimal control problem for the piecewise linear system

$$c^{'}x(t^*) \to \max_u,\ \dot{x} = A(t)x + bu,\ x(0) = x_0 \tag{70}$$

$$Hx(t^*) = g,\ |u(t)| \leq 1.\ t \in T = [0, t^*]$$

in the class of discrete controls. Here, $x = x(t)$ is the state vector of the dynamical system at an instant t, and $u = u(t)$ is the value of a scalar control, $b \in \mathbb{R}^n$, $g \in \mathbb{R}^m$. The discrete control $u(\cdot) = (u(t), t \in T)$ is called feasible for the problem if it satisfies the condition $|u(t)| \leq 1$ for $t \in T$ and the corresponding trajectory $x(t)(t \in T)$ satisfies the terminal constraint $Hx(t*) = g$. A feasible control $u(\cdot)$ is called the optimal open loop control for problem if the corresponding trajectory $x(t)(t \in T)$ maximizes the objective functional of problem.

To introduce the concept of the closed loop optimal control for problem (70), we embed problem (70) in the family of problems

$$c^{'}x(t^*) \rightarrow \max_{u},\ \dot{x} = A(t)x + bu,\ x(\tau) = z \tag{71}$$

$$Hx(t^*) = g\ |u(t)| \leq 1,\ t \in T(\tau) = [\tau, t^*]$$

which depends on a scalar $\tau \in T_h$ and an $n-$dimensional vector z. Let $u^0(t|\tau, z)(t \in T(\tau))$ be the optimal open loop control for problem (71) for (τ, z) and X^τ be the set of states z for which problem (71) has a solution. The function

$$u^0(\tau, z) = u^0(t|\tau, z),\ z \in X^\tau,\ \tau \in T_h \tag{72}$$

is called the closed loop (discrete) optimal control for problem (70).

Then the real trajectory of the control object can be presented as the following system

$$\dot{x} = A(t)x + bu^0(t, x) + w(t),\ x(0) = x_0 \tag{73}$$

closed by the optimal feedback (72) and subjected to a piecewise continuous perturbation $w(t)(t \in T)$. The purpose of this study is to develop effective algorithms for constructing the optimal open loop control and synthesize the optimal closed loop control for problem (70)

4.1 Optimal Controller

The main result of this study is the synthesis of closed loop optimal controls. Optimal synthesis is the key problem in control theory since open loop solutions are not used in actual controls; they are required only to reveal the potential capabilities of control systems. The approach described in this project essentially relies on the dynamic nature of the problem under consideration. The optimal controller generates the values of an optimal feedback in real time. This eliminates the main disadvantage of dynamic programming, i.e., the need to calculate the optimal feedback for all possible states of the system in advance (before the control process begins), which results in the so-called curse of dimensionality .

This approach will be based on the use of an optimal feedback (72) in a control process for system (70). Assume that the optimal feedback (72) has already been determined and the behavior of the closed loop system is described by the equation (73). The function $w(t)(t \in T)$ represents the influence of the perturbations neglected in the mathematical model. When piecewise linear approximations are used to optimize the nonlinear system (68), the function $w(t)(t \in T)$ includes, in addition to external perturbations, the deviation of the piecewise linear system (70) from the original nonlinear system (68). Assume that a perturbation $w^*(t)(t \in T)$ occurs in a certain control process of system (73). Driven by this perturbation and function (72), system (73) moves along a trajectory $x^*(t)(t \in T)$, while the control $u^*(t) = u^0(t, x^*(t))(t \in T)$ is applied to system (73). The function $u^*(t)(t \in T)$ is called the realization of the optimal feedback in a particular control process, and the device that calculates the values of this function in real time is called the optimal controller.

The testing examples demonstrated that the developed algorithm of the optimal controller is efficient and can therefore be implemented on modern computers for relatively complex control systems. This is illustrated by the numerical results obtained by solving two optimal control problems for a piecewise linear oscillatory system with one degree of freedom. In particular, the frictionless motion of a one-USVs oscillatory system along a horizontal line was considered. On different parts of the line, the system is driven by forces exerted by different elastic elements (springs). We seek a control that maximizes the velocity gained by the USVs in a given time. The mathematical model of the system is formulated as follows:

$$\dot{x}(t^*) \to \max_u, \tag{74}$$

$$\begin{cases} \ddot{x} + k_1 x = u, & \text{if } \; x \geq \alpha, \\ \\ \ddot{x} + k_1 x + k_2(x + \alpha) = u, & \text{if } \; x < \alpha, \end{cases} \tag{75}$$

$$x(0) = 0, \;\; \dot{x}(0) = 0, \;\; |u(t)| \leq 1, \;\; t \in T = [0, t^*] \tag{76}$$

where $x = x(t)$ is the deviation of the USVs from the equilibrium point $x = 0$ at the instant t,$u = u(t)$ is the control (force), and α is the distance from the equilibrium point to the right end of the second spring.

The calculated results expose the high efficiency of the method in constructing optimal open loop controls and the possibility of implementing closed loop controls on modern computers.

5 Optimal Control based on imperfect measurements of input and output

In this part of the project can be considered the optimal guaranteed control problems for linear nonstationary dynamical systems under set-membership uncertainties. It is supposed that in the course of control process states of control object are unknown and signals of two measurement devices are only available for use. The first of them implements incomplete and inexact measurements of input signals, the second one makes imperfect measurements of control object states (output signals). By preposterior analysis an optimal output (combined) closable loop is defined. Realization of this loop (forming current values of control actions) is carried out by optimal estimators and optimal regulator. According to the separation principle of control and observation processes, optimal estimators generate in real time estimates of uncertainty using signals of measurement devices. By obtained estimates the optimal regulator produces current values of optimal loop in the same mode.

6 Adaptive decentralized control problem

Let $T = [t_*, t^*]$ be the control interval; $T_h = \{t_*, t_* + h, ..., t^* - h\}; h = \frac{t^* - t_*}{N}$; N be a natural number; $I = \{1, 2, ..., q\}$, $I_i = I \setminus i; A_{ij}(t) \in \mathbb{R}^{n_i \times n_j}; B_{ij}(t) \in \mathbb{R}^{n_i \times r_j}(t \in T, i, j \in I)$ be piecewise continuous matrix functions; $A_i(t) = A_{ii}(t)$ and $B_i = B_{ii}(t), t \in T, i \in I; H_i \in \mathbb{R}^{m \times n_i}, g_0 \in \mathbb{R}^m, c_i \in \mathbb{R}^{n_i}, u_{i*}, u_i^* \in \mathbb{R}^{r_i}$ for $i \in I$ be given matrices and vectors; and $n = \sum\limits_{i \in I} n_i$.

On interval T, consider the group of q objects to be controlled assuming that the i-th object ($i \in I$) is governed by the equation

$$\dot{x}_i \;\; = \;\; A_i(t)x_i + \sum_{j \in I_i} A_{ij}(t)x_j + B_i(t)u_i + \sum_{j \in I_i} B_{ij}(t)u_j, \;\;\; x_i(t_*) = x_{i0} \tag{77}$$

Here, $x_i = x_i(t) \in \mathbb{R}^{n_i}$ is the state of the ith object at the time t, $u_i = u_i(t) \in U_i$ is the value of the discrete control at the time t, and $U_i = \{u \in \mathbb{R}^{r_i} : u_{i*} \leq u \leq u_i^*\}$ is a bounded set of available values of the i-th control. If $u(t) \equiv u(s)$ for $t \in [s, s+h[, (s \in T_h)$, then the function $u(t), (t \in T)$ is said to be discrete with sampling period h. In (77), the function $A_i(t), t \in T$ characterizes the self-dynamics of ith object; the function $(A_{ij}(t), t \in T, j \in I_i)$ describes the influence of other objects on object i; the function $B_i(t), t \in T$ characterizes the input properties of the object i; and the functions $B_{ij}(t), (t \in T, j \in I_i)$ describes the influence of the controls of other objects on object i.

A group of dynamical objects can be controlled in two different ways- in a centralized or decentralized fashion. In the first case, there is a common control center that, given perfect (complete and

accurate) information about the current state $x^*(\tau) = (x_i^*(\tau), i \in I)$ of the group, produces for each time interval $[\tau, \tau + h[, \tau \in T_h$, a control $u^*(t) = (u_i^*(t), i \in I), t \in [\tau, \tau + h[$ for all the objects. In the second case, each ith object of the group has a particular (local) control center that produces at each time interval $[\tau, \tau + h[, (\tau \in T_h)$ a control action $u_i^*(t)(t \in [\tau, \tau + h[, \tau \in T_h)$ based on the perfect information about its own current state $x_i^*(\tau)$ and about the states $x_k^*(\tau - h)(k \in I_i)$ of other objects. Also assume that delay of the information about the state of the other objects coincides with the sampling period h. The aim of the control is follows:
(1) Steer the group to a given(common) terminal set at the time t^*:

$$x(t^*) \in X^* = \{x = (x_i, i \in I) : \sum_{i \in I} H_i x_i = g_0, \} \tag{78}$$

(2) Achieve the maximum value of the terminal objective function

$$J(u) = \sum_{i \in I} c_i' x_i(t^*) \to \max \tag{79}$$

Depending on the way used to control the group, we have two optimal control problems a centralized and a decentralized problem. Assume that centralized real-time control of the group of objects is impossible for some reason.

6.1 Decentralized close-loop control in classical form

Before starting the control, the set of functions

$$\begin{aligned} &u_i(t_*, x), x = (x_i, i \in I) \in \mathbb{R}^n, \\ &u_i(\tau, x_i; x_k, k \in I_i), \qquad \tau \in T_h \setminus t_*,\ x_i \in \mathbb{R}^{n_i},\ x_k \in \mathbb{R}^{n_k},\ k \in I_i,\ i \in I \end{aligned} \tag{80}$$

is chosen that is called the (discrete) decentralized feedback. Mathematical models (77)($i \in I$) are closed by feedback (80):

$$\begin{aligned} \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x(t)) + \sum_{j \in I_i} B_{ij}(t)u_j(t, x(t)), \\ x_i(t_*) = x_{i0},\ t \in [t_*, t_* + h[, \\ \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x_i(t); x_k(t-h), k \in I_i) + \\ + \sum_{j \in I_i} B_{ij}(t)u_j(t, x_j(t); x_k(t-h), k \in I_j), t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*,\ i \in I. \end{aligned} \tag{81}$$

Here $x_0 = (x_{i0}, i \in I)$; $u_i(t, x(t)) \equiv u_i(t_*, x_0)$ for $t \in [t_*, t_* + h[$, and $u_i(t, x_i(t); x_k(t-h), k \in I_i) \equiv u_i(\tau, x_i(\tau); x_k(\tau - h), k \in I_i)$ for $t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*, i \in I$. The trajectory of nonlinear system (81) is defined as the unique function $x(t \mid x_0; u_i, i \in I), (t \in T)$ composed of the continuously connected solutions to the linear differential equations

$$\begin{aligned} \dot{x}_i = A_i(t)x_i + \sum_{j \in I_i} A_{ij}(t)x_j + B_i(t)u_i + \sum_{j \in I_i} B_{ij}(t)u_j, \quad x_i(t_*) = x_{i0} \\ u_i(t) = u_i(t_*, x_0), \quad t \in [t_*, t_* + h[, \\ u_i(t) = u_i(\tau, x_i(\tau); x_k(\tau - h), k \in I_i), t \in [\tau, tau + h[, \tau \in T_h \setminus t_*, \\ i \in I. \end{aligned}$$

Feedback (80) is said to be admissible for the state x_0 if
1) $u_i(t) \in U_i$ for $t \in T$ and $i \in I$ and

2) $x(t^* \mid x_0; u_i, i \in I) \in X^*$.

Let $X(t_*)$ be the set of initial states x_0 for which there exists an admissible feedback, and let $X_i(\tau)$ be the set of states $x_i; x_k, k \in I_i$ for which function (80) is define at the time $\tau \in T_h \setminus t^*$. The quality of an admissible feedback for the state $x_0 \in X(t_*)$ is evaluated using the functional

$$J(u, x_0) = \sum_{i \in I} c_i' x_i(t^* \mid x_0; u_i, i \in I).$$

The admissible feedback

$$u_i(t_*, x), x \in X(t_*); \quad u_i^0(\tau, x_i; x_k, k \in I_i), (x_i; x_k, k \in I) \in X_i(\tau), \quad \tau \in T_h \setminus t_*, i \in I, \tag{82}$$

is said to be optimal for $x_0 \in X_{t_*}$ if $J(u^0, x_0) = \max\limits_u J(u, x_0), x_0 \in X(t_*)$.

Similarly to the classical centralized optimal feedback, the decentralized optimal feedback is determined on the basis of the mathematical model but is designed for controlling its physical prototype. The decentralized optimal control of a group of dynamical objects designed using the classical closed-loop principle assumes that feedback (82) is preliminary synthesized, and group (77) $i \in I$ is closed by this feedback, which yields the optimal automatic control system governed by the equations

$$\begin{aligned} \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x(t)) + \sum_{j \in I_i} B_{ij}(t)u_j(t, x(t)) + w_i, \\ x_i(t_*) = x_{i0}, \ t \in [t_*, t_* + h[, \end{aligned}$$

$$\begin{aligned} \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j \in I_i} A_{ij}(t)x_j(t) + B_i(t)u_i(t, x_i(t); x_k(t-h), k \in I_i) + \\ + \sum_{j \in I_i} B_{ij}(t)u_j(t, x_j(t); x_k(t-h), k \in I_j) + w_i, \\ t \in [\tau, \tau + h[, \tau \in T_h \setminus t_*, \ i \in I. \end{aligned} \tag{83}$$

where $w = (w_i \in \mathbb{R}^{n_i}, i \in I)$ is a collection of terms describing inaccuracies of the mathematical modeling and of the implementation of the optimal feedback and the perturbations that affect the objects in the course of control. For simplicity, we will call w the perturbation, and we will assume that it is realized as unknown bounded piecewise continuous functions $w_i(t), (t \in T, i \in I)$. The problem of synthesizing system (83) is a very difficult one and has not yet been solved even for centralized control.

The purpose of our investigation is to propose algorithms for synthesis of decentralized automatic control systems using decentralized real-time optimal control of a group of objects; in this case, the control function among individual systems of which each solves an individual autonomous problem (performs self-control) taking into account the actions of the other members of the group. Our approach will be based on a fast implementation of the dual method and the learning algorithm from information obtained by each controller from the other controllers.

6.2 Control problem with moving targets

Two optimal control problems of approaching and aiming with moving targets are under consideration.

Let $T = [t_*, t^*]$ be the control interval.

Object of control:

$$\dot{x} = A(t)x + b(t)u + M(t)w, t \in T; x(t_*) = x_0 \tag{84}$$

and moving target:

$$\dot{\tilde{x}} = \tilde{A}(t)\tilde{x} + \tilde{b}(t)v + \tilde{M}(t)w, t \in T; \tilde{x}(t_*) = \tilde{x}_0 \tag{85}$$

Here
$x = x(t) \in \mathbb{R}^n, \tilde{x} = \tilde{x}(t) \in \mathbb{R}^n$ are the state of control object and state of the target.
$u = u(t) \in \mathbb{R}$ - control input; $v = v(t) \in \mathbb{R}$ - maneuvering effort of target; $w = w(t) \in \mathbb{R}^{n_w}$ - disturbances.
If $\tilde{b}(t) = 0, t \in T$ then target is *not maneuvering*, otherwise it is *maneuvering target*.

Admissible discrete control function:

$$u(\cdot) = (u(t) \in U, t \in T)$$

Unknown maneuvering efforts $v(t), t \in T$ and disturbances $w(t), t \in T$ represented in the following form:

$$v(t) = v_1(t) + v_2(t),\ w(t) = w_1(t) + w_2(t),\ t \in T$$

where

$$v_1(t) = K_v(t)v, v \in V_1;\ w_1(t) = K_w(t)w, w \in W_1, t \in T$$

- regular components,

$$v_2(t) \in V_2;\ w_2(t) \in W_2, t \in T$$

not regular components $(V_1, W_1, V_2, W_2; K_v(t), K_w(t), t \in T)$ - are known; $v, w; v_2(t), w_2(t), t \in T$ - are arbitrary.

In case,when $M(t) = 0, \tilde{M}(t) = 0, \tilde{b}(t) = 0, t \in T; x_0, \tilde{x}_0$ are fixed vectors, the models (84)-(85) *deterministic*,otherwise *indeterministic*.

Also let $X_\rho(\tilde{x}) = \{x \in \mathbb{R}^n : \rho g_* \leq H(x - \tilde{x}) \leq \rho g^*\}$ - ρ-neighborhood of a point $\tilde{x}$, where

$$H \in \mathbb{R}^{m \times n}; g_*, g^* \in \mathbb{R}^m, -\infty < g_* < 0 < g^* < \infty; \rho \geq 0.$$

The problem of optimal approaching consist in the way of choosing $u^0(\cdot)$,such that (84) at final moment t^* will be at the set $X_{\rho^0(\tilde{x}(t^*))}$ with minimal ρ^0.

The problem of aiming- to be on $X^* = X_1(\tilde{x}(t^*))$ with minimal $J(u) = \int\limits_{t_*}^{t^*} |u(t)|dt$.

For the problems above can be investigated the following situations:

- Deterministic problem statement;
- Maneuvering target;
- Not maneuvering target with noncompletely defined motion, imperfect measuring;
- Maneuvering target,imperfect measuring;
- Indeterministic models, imperfect measuring.

7 Control synthesis under uncertainty

A considerable amount of work was done by the control community in coping with uncertainty and incomplete information in the field of control. The adopted scheme is based on constructing superpositions of value functions for open-loop control problems. In the limit these relations reflect the Principle of Optimality under set-membership uncertainty. This principle then allows one to describe the closed loop reach set as a level set for the solution to the forward HJBI (Hamilton JacobiBellmanIsaacs) equation. The final results are then presented either in terms of value functions for this equation or in terms of setvalued relations. Schemes of such type have been used in synthesizing solution strategies for guaranteed control,dynamic games and related problems of feedback control under realistic data, including those of control under incomplete information and measurement feedback control. The control schemes were constructed in backward time in more or less equivalent forms of solvability sets, stable bridges of N. N. Krasovski (1968; 1998), the alternated integrals of L. S. Pontryagin (1980), the scheme of B. N. Pschenichniy in Kiev (Pschenichniy and Ostapenko, 1992). Effective and original dynamic programming-type constructions were developed in USA by R. Isaacs, P. Varaiya, G.Leitmann, R. J. Elliot and N. J. Kalton, T.Basar and others, as well as in France, by A. Blaquiere and later, through the notion of H^∞-control by P.Bernhard and the idea of capture basins by J. P. Aubin and his associates P.Saint-Pierre, M.Quincampoix and others.

7.1 Uncertain dynamics. The Standard Model.

This is given by differential equation

$$\dot{x} = f(t, x, u, v), \tag{86}$$

with properties of continuous function f defined as in control theory, with inputs representing controls u to be specified and unknown disturbances (v). Here $x \in \mathbb{R}^n$ as always is the state and $u \in \mathbb{R}^p$is the control that may be selected either as an open loop control - a measurable function of time t, restricted by the inclusion

$$u(t) \in \mathbb{P}(t), a.e.,$$

or as a closed-loop control a feedback strategy which is either sought for either as a multivalued map

$$u = \mathbb{U}(t, x) \subseteq \mathbb{P}(t).$$

or as a single-valued function $u(t, x) \in U_c$ which ensure existence in some appropriate sense of solutions to differential inclusion

$$\dot{x} \in f(t, x, U(t, x), v), \tag{87}$$

or to differential equation

$$\dot{x} = f(t, x, u(t, x), v), \tag{88}$$

respectively.

Here $v \in \mathbb{R}^q$ is the unknown input disturbance with values

$$v(t) \in (\mathbb{Q}(t), a.e., \tag{89}$$

$\mathbb{P}(t), \mathbb{Q}(t)$ are set-valued continuous functions with compact values, $(\mathbb{P} \in comp\mathbb{R}^p, \mathbb{Q} \in comp\mathbb{R}^q)$. Given also is a closed target set $\mathbb{M}$.

The Problem of Control Synthesis under Uncertainty is to find an admissible feedback control strategy $U = u(t, x)$ or $U = U(t, x)$ which steers system (86) to reach the target set $\mathbb{M}$ despite the unknown disturbances v. Such problems are usually treated within the notions of game-type dynamics introduced by R. Isaacs.

Typical admissible classes of feedback controls and trajectory solutions involved for the given problem are due to N. N. Krasovski in the singlevalued case (Krasovski, 1968; Krasovski and Subbotin, 1998) and A. F. Filippov in the multivalued case (Filippov, 1959). It is N. N. Krasovski who introduced the most developed formalized and integrated solution theory for problems in "gametype" controlled dynamics, which was developed further by him and his associates for a broad class of control problems under uncertainty or conflict.

Continuing with the last problem, we are to find the value function

$$V(t,x) = \min_{\mathbb{U}} \max_{x(\cdot)} \{d^2(x[t_1], \mathbb{M}) \mid \mathbb{U} \in U_c, x(\cdot) \in \mathbb{X}_{\mathbb{U}}\}.$$

Here $\mathbb{X}_{\mathbb{U}}$ is the variety of all trajectories of equation (87) or (88).

The formal solution equation for the problem is the HamiltonJacobiBellmanIsaacs (HJBI) equation

$$V_t = \min_u \max_v (V_x, f(t,x,u,v)) = 0, \tag{90}$$

with minmax often assumed interchangeable and with control $u(t,x)$ to be found from the solution to the minmax problem in (90). However in reality this is just a symbolic relation as the function $V(t,x)$ in general turns out to be nondifferentiable.

The solutions to the control synthesis problem under uncertainty are then found, for example, through procedures of constructing solvability tubes in the form of stable bridges

$$W[t] = \{x : V(t,x) \leq 0\},$$

as introduced by N. N. Krasovski with control strategy further found from conditions of "extremal aiming":

$$u(t,x) = arg \min_u \{\max_v d(x, W[t])\},$$

and trajectories interpreted as "constructive motions" of (Krasovski and Subbotin, 1998). Here $V(t,x)$ is a generalized solution to equation (90).

Along with the theory, the numerical methods of constructing solutions were developed. Several global successive approximation numerical schemes were proposed as well schemes to approximate the deterministic control problem or game by a stochastic discrete-time process. The game-theoretic approaches in conjunction with set-valued techniques and new results in nonlinear analysis allowed to formalize basic problems of control under measurement feedback and unknown but bounded disturbances. Various formalizations and applications of the theory of control under incomplete information may be found in books and papers by N. N. and A. N. Krasovski (1994), A. B. Kurzhanski (1977), Yu. S. Osipov and A. V. Kryazhimski (1995), F. L. Chernousko and A. A. Melikyan (1978), V. M. and A. V. Kuntsevich (2002), B. N. Pschenichniy and V. V. Ostapenko (1992).

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