Assignment problem for MAS

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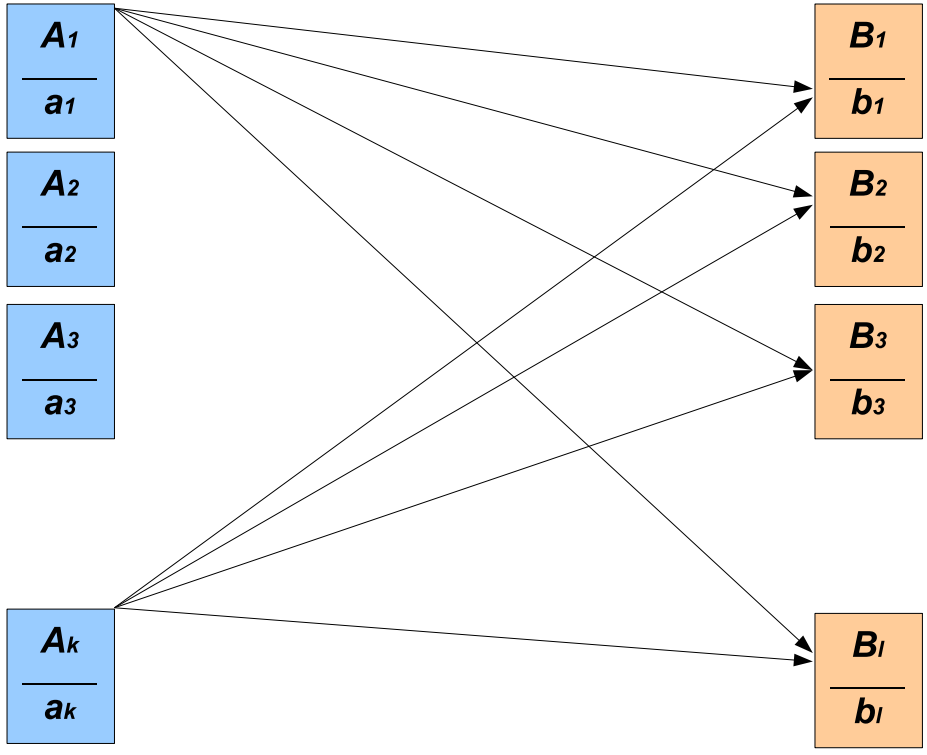
Assignment problem by cost criteria



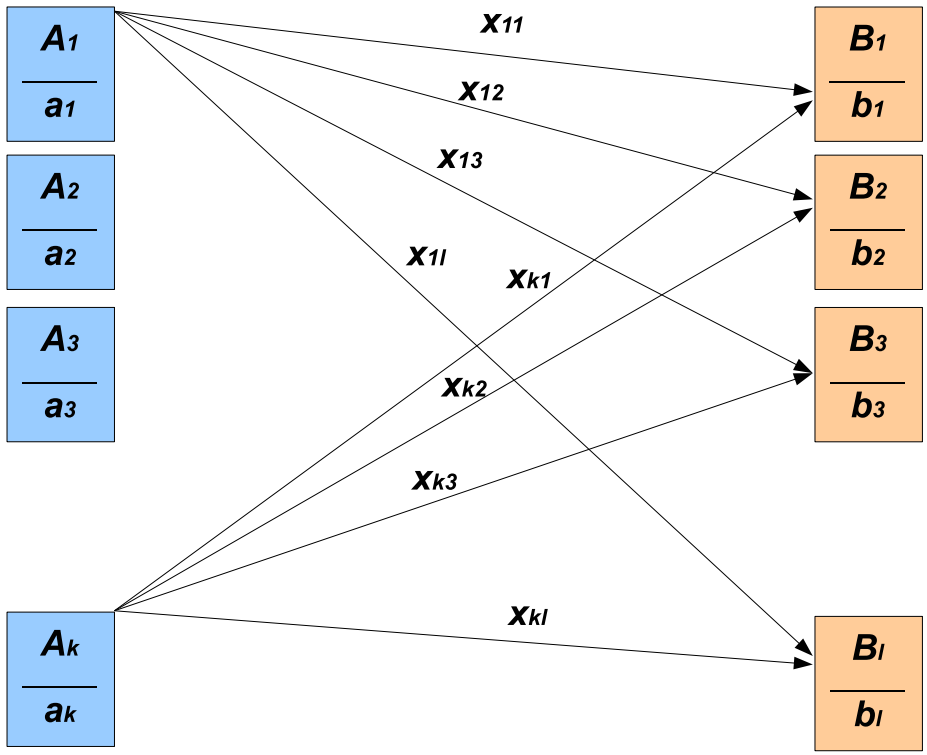
$A_i, i = 1, ..., k$- number of aerobases,

$a_i, i = 1, ..., k$ - capacity (maximal number of homogenous UAVs located in aerobase),

$B_j, j = 1, ..., l$- areas of operations,

$b_j, j = 1, \ldots, l$- numbers of UAVs required for service of B_j zones,

Assignment problem by cost criteria



$$F = \sum_{i=1}^{k} \sum_{j=1}^{l} c_{ij} x_{ij} \to \min_{x_{ij}}$$

$$\sum_{i=1}^{k} x_{ij} = b_j, \quad j = 1, 2, ..., l$$

$$\sum_{j=1}^{l} x_{ij} = a_i, \quad i = 1, 2, ..., k$$

$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j$$

$$x_{ij} \geq 0, x_{ij} \in \mathbb{N}.$$

$A_i, i = 1, ..., k$- number of aerobases,

$a_i, i = 1, ..., k$ - capacity (maximal number of homogenous UAVs located in aerobase),

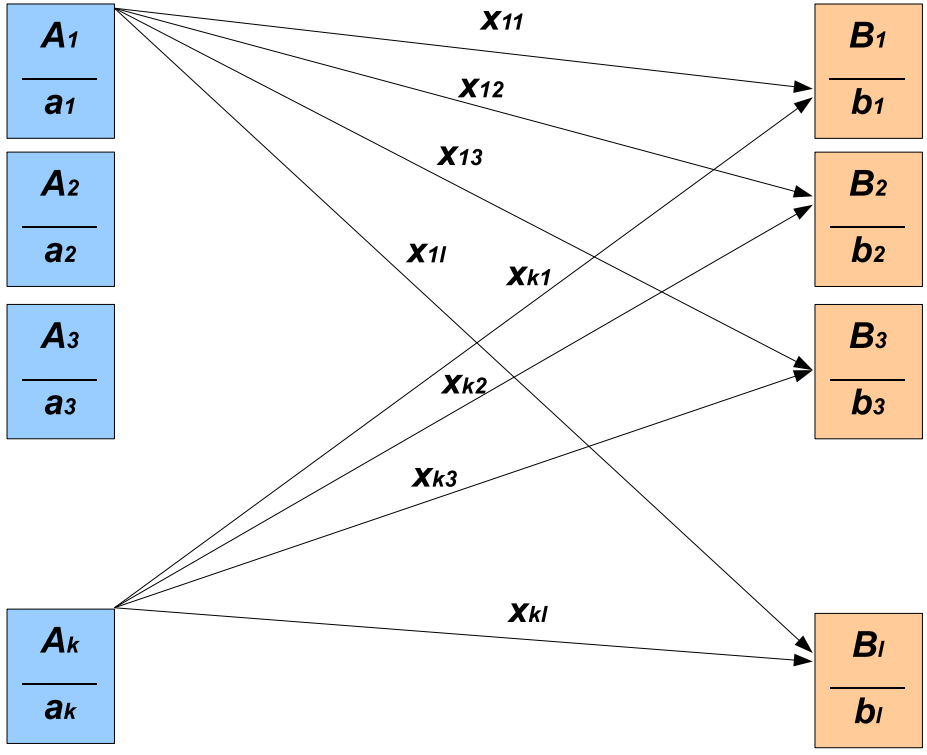
$B_j, j = 1, ..., l$- areas of operations,

$b_j, j = 1, \ldots, l$- numbers of UAVs required for service of B_j zones,

c_{ij}- benfits,

x_{ij}-number of UAVs from i-th aerobase to j-th zone of area of operation.

Assignment problem by cost criteria



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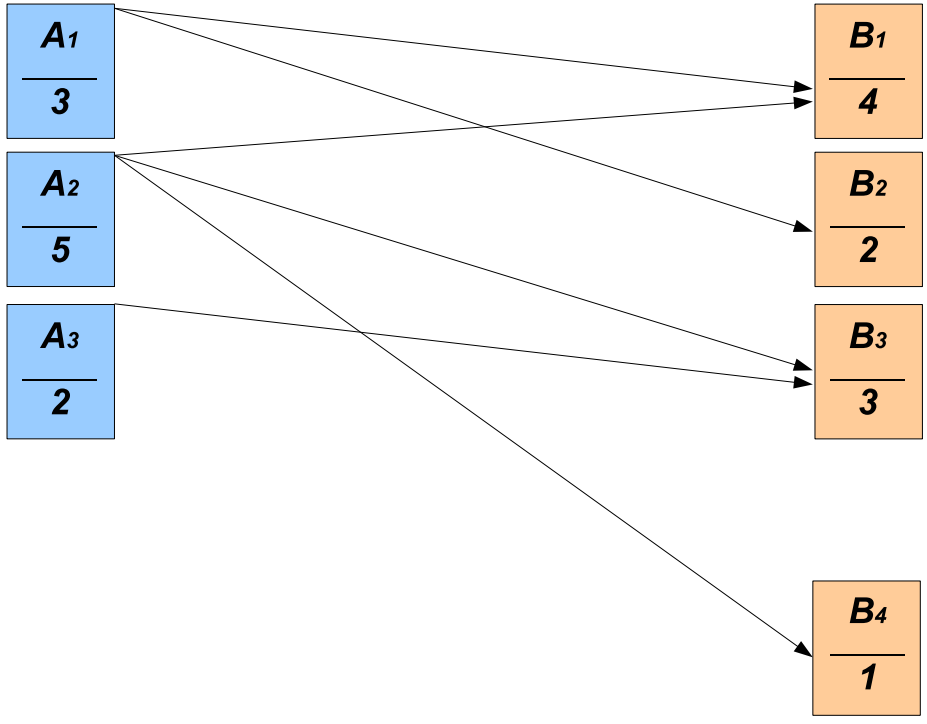
$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j$$

$$x_{ij} \geq 0, x_{ij} \in \mathbb{N}.$$

The most of methods include the following basic steps:

- To find initial plan x_{ij};
- Check optimality condition for that plan;
- Construct the improved plan in case of nonoptimality.

Example (Initial plan)



$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

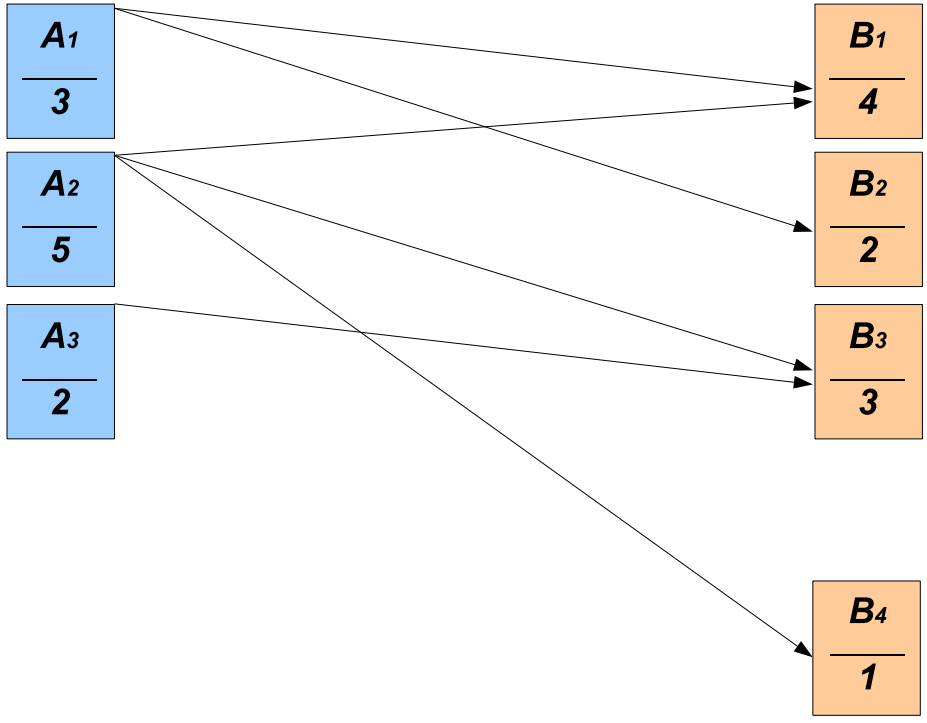
$$\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1, 2, 3, 4$$

$$\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1, 2, 3$$

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

$$x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}.$$

Example (Initial plan)



$$F = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \to \min_{x_{ij}}$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3,4$$

$$\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1,2,3$$

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

$$x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}.$$

The condition of that problem can be represented in table form.

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	x_{11}	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	x_{21}	x_{22}	x_{23}	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

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A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

$$x_{11} = \min(a_1; b_1);$$

Example (Initial plan)

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A_1	3	x_{12}	x_{13}	x_{14}	$a_1 = 3$
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$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	3	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	1	x_{22}	x_{23}	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$

Example (Initial plan)

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$
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Example (Initial plan)

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$
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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$
$$x_{23} = \min(a_1 + a_2 - b_1 - b_2; b_3);\ x_{33} = \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2));$$

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$$\begin{aligned} x_{11} &= \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2); \\ x_{23} &= \min(a_1 + a_2 - b_1 - b_2; b_3);\ x_{33} = \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2)); \\ x_{34} &= \min(a_1 + a_2 + a_3 - b_1 - b_2 - b_3; b_4). \end{aligned}$$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
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A_3	x_{31}	x_{32}	1	1	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$
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$$x_{34} = \min(a_1 + a_2 + a_3 - b_1 - b_2 - b_3; b_4).$$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	3	0	0	0	3
A_2	1	2	2	0	5
A_3	0	0	1	1	2
b_j	4	2	3	1	

$$\begin{aligned} x_{11} &= 3;\ x_{21} = 1;\ x_{22} = 2; \\ x_{23} &= 2;\ x_{33} = 1; \\ x_{34} &= 1. \end{aligned}$$

Example (Initial plan)

Check the optimality

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	3	0	0	0	3	
A_2	1	2	2	0	5	
A_3	0	0	1	1	2	
b_j	4	2	3	1	F=60	
ν_j						

Optimality condition

Find a number u_i and ν_j such that

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	3	0	0	0	3	
A_2	1	2	2	0	5	
A_3	0	0	1	1	2	
b_j	4	2	3	1	F=60	
ν_j						

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

$$u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0$$

Optimality condition

Find a number u_i and ν_j such that

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	
A_2	4 1	12 2	5 2	0	5	
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
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A_2	4 1	12 2	5 2	0	5	
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0					

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Optimality condition

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	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0					

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b_j	4	2	3	1		F=60
ν_j	0					

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0					

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0					

$$u_i + \nu_j = c_{ij}, \ for\ x_{ij} > 0,$$

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Optimality condition

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	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0	8				

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

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Optimality condition

Find a number u_i and ν_j such that

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0	8				

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

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Optimality condition

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	B_1	B_2	B_3	B_4	a_i	u_i
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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1		F=60
ν_j	0	8	1			

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Optimality condition

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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j	0	8	1			

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A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1			

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A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1			

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Optimality condition

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	B_1	B_2	B_3	B_4	a_i	u_i
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A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

$$u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0$$

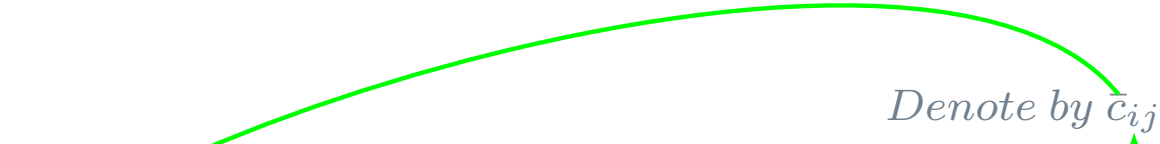
Optimality condition

$Denote\ by\ \bar{c}_{ij}$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$
$$u_i + \nu_j \leq c_{ij},\ for\ x_{ij} = 0$$

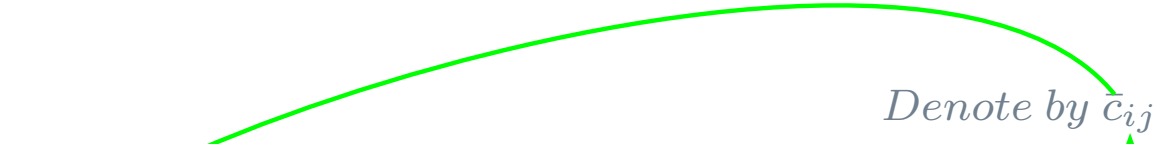
Optimality condition



	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	0	0	0	3	4
A_2	4 4 1	12 12 2	5 5 2	0	5	4
A_3	0	0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$
$$u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0$$

Optimality condition



	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	6 0	8 0	2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	1 0	5	4
A_3	8 0	10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$
$$u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0$$

Optimality condition

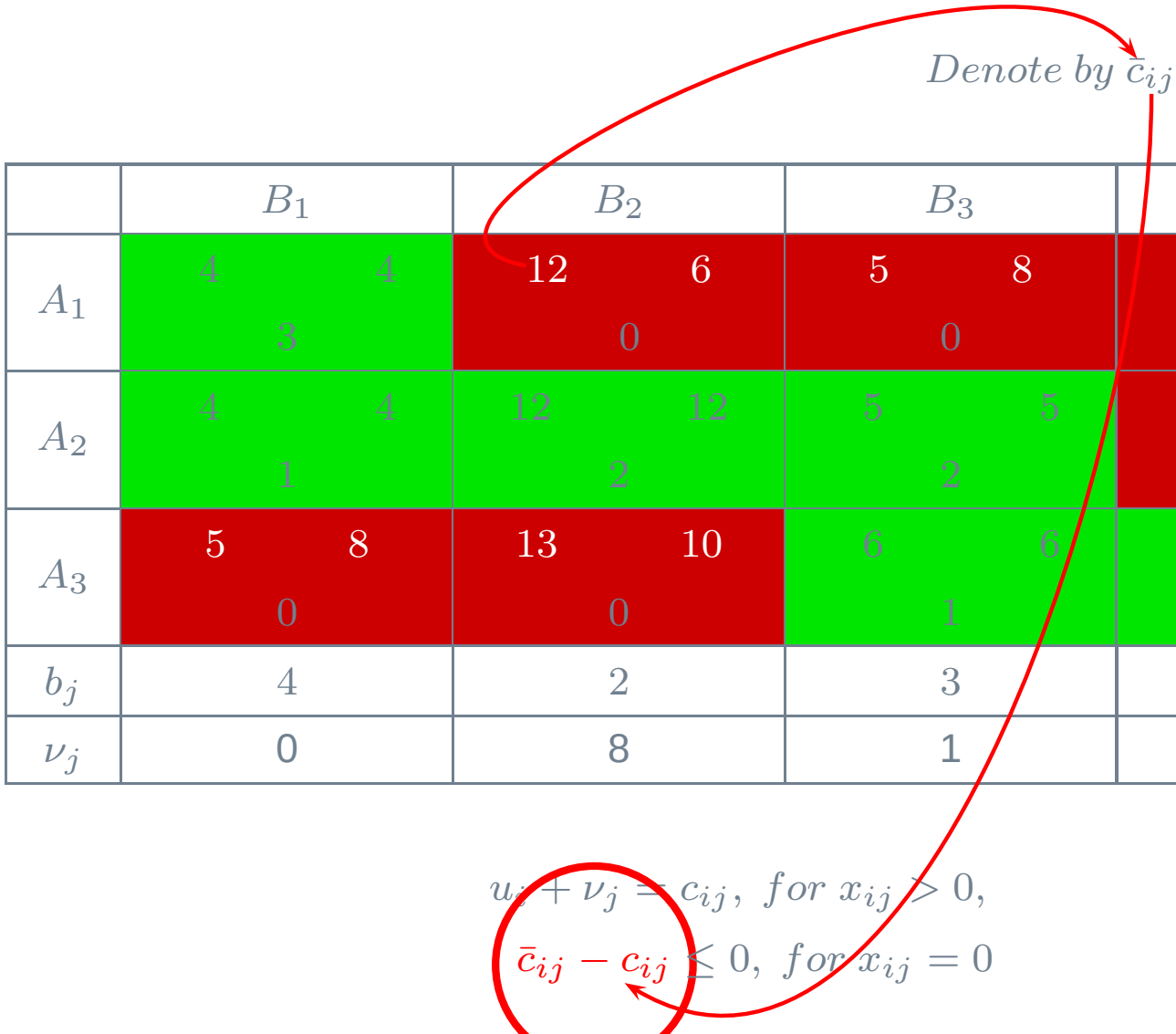
Denote by c_{ij}

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 0	5 8 0	3 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 1 0	5	4
A_3	5 8 0	13 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

$$u_i + \nu_j \leq c_{ij}, \ for \ x_{ij} = 0$$

Optimality condition



	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 0	5 8 0	3 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 1 0	5	4
A_3	5 8 0	13 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1		F=60
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \ for\ x_{ij} > 0,$$

$$\bar{c}_{ij} - c_{ij} \leq 0, \ for\ x_{ij} = 0$$

Optimality condition

Check our conditions

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 0	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij}, \quad for\ x_{ij} > 0,$$

$$\bar{c}_{ij} - c_{ij} \leq 0, \quad for\ x_{ij} = 0$$

Improvement procedure

Check our conditions

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 0	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\bar{c_{ij}} - c_{ij} > 0 \rightarrow \max, \ for\ x_{ij} = 0$$

Improvement procedure

Find the maximal admissible value of θ

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 θ	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\bar{c_{ij}} - c_{ij} > 0 \to \max, \ for\ x_{ij} = 0$$

Improvement procedure

The maximal admissible value of θ

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3 $-$ θ	12 6 6 θ	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1 $+$ θ	12 12 2 $-$ θ	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\min(3-\theta; 2-\theta) = 0 \Longrightarrow \theta{=}2$$

Improvement procedure

New feasible solution :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 (1)	12 6 6 (2)	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 (3)	12 12 0	5 5 (2)	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 (1)	4 4 (1)	2	5
b_j	4	2	3	1	F=48	
ν_j	0	8	1	-1		

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 2	✓ 8 0	2 0	3	
A_2	4 4 3	12 0	5 5 2	1 0	5	
A_3	✓ 8 0	10 0	6 6 1	4 4 1	2	
b_j	4	2	3	1	F=48	
ν_j						

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 ✓ 8 0	3 2 0	3	4
A_2	4 4 3	6 ✓ 12 0	5 5 2	3 1 0	5	4
A_3	5 ✓ 8 0	7 ✓ 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=48	
ν_j	0	2	1	-1		

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 ✓ 8 0	3 2 0	3	4
A_2	4 4 3	6 ✓ 12 0	5 5 2 − θ	3 1 θ	5	4
A_3	5 ✓ 8 0	7 ✓ 10 0	6 6 1 + θ	4 4 1 − θ	2	5
b_j	4	2	3	1	F=48	
ν_j	0	2	1	-1		

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	✓ 8 0	2 0	3	
A_2	4 4 3	✓ 12 0	5 5 1	1 1	5	
A_3	✓ 8 0	✓ 10 0	6 2	4 0	2	
b_j	4	2	3	1	F=46	
ν_j						

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 ✓ 8 0	1 ✓ 2 0	3	4
A_2	4 4 3	6 ✓ 12 0	5 5 1	1 1 1	5	4
A_3	5 ✓ 8 0	7 ✓ 10 0	6 6 2	2 ✓ 4 0	2	5
b_j	4	2	3	1		F=46
ν_j	0	2	1	-3		

$$\min F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij}$$

Example (Optimal solution)