Assignment problem for MAS

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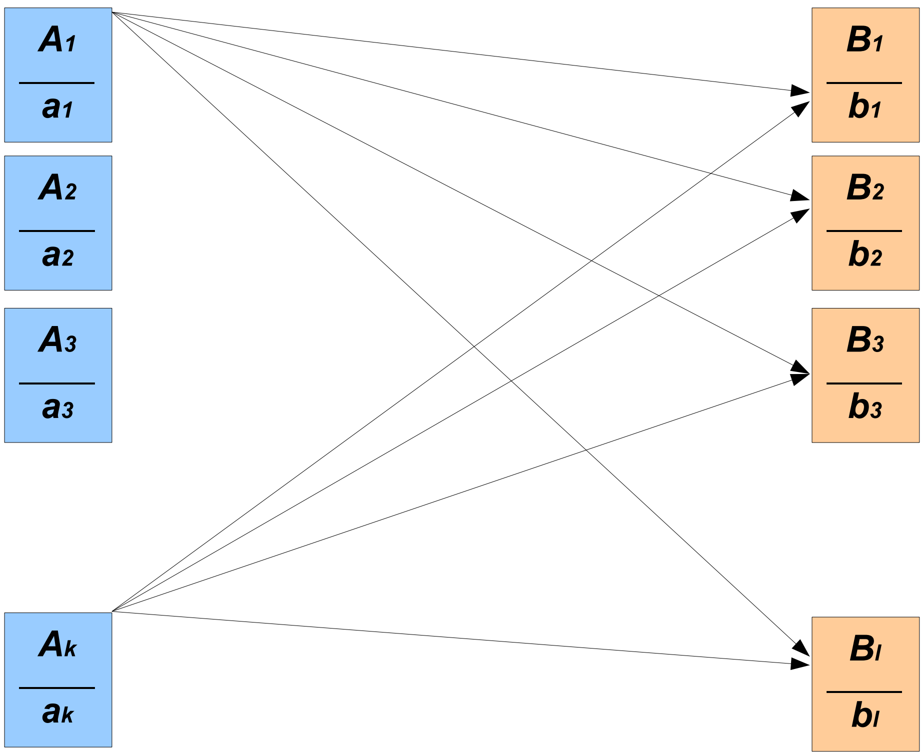
Assignment problem by cost criteria



$A_i, i = 1, ..., k$- number of aerobases,
$a_i, i = 1, ..., k$ - capacity (maximal number of homogenous UAVs located in aerobase),
$B_j, j = 1, ..., l$- areas of operations,
$b_j, j = 1, \ldots, l$- numbers of UAVs required for service of B_j zones,

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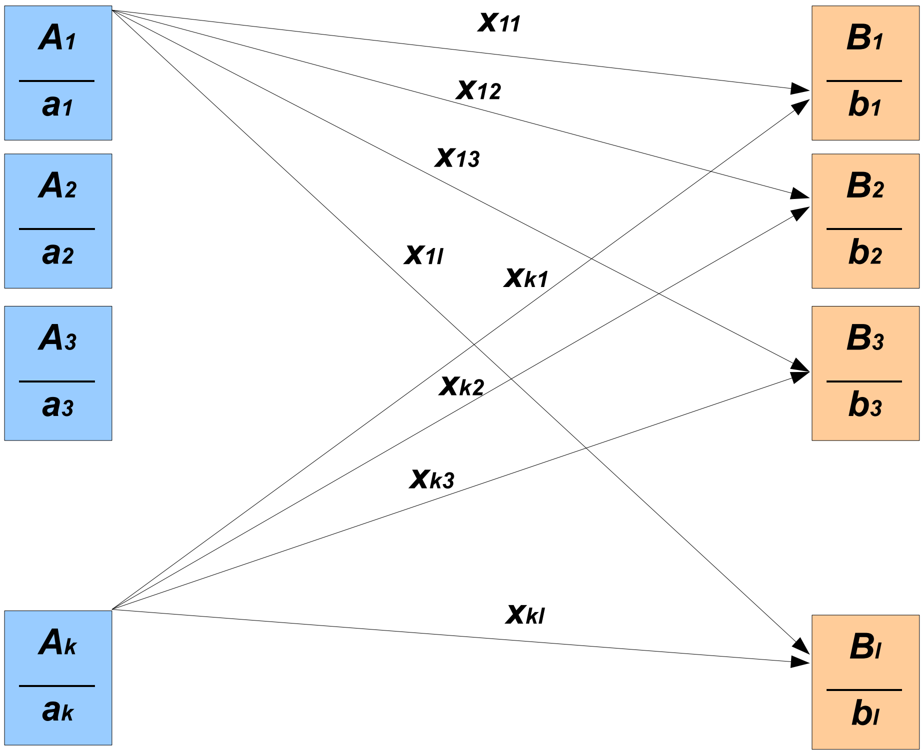


$$F=\sum_{i=1}^{k}\sum_{j=1}^{l}c_{ij}x_{ij}\to\min_{x_{ij}}$$

$$\sum_{i=1}^{k}x_{ij}=b_j,\quad j=1,2,\ldots,l$$

$$\sum_{j=1}^{l}x_{ij}=a_i,\quad i=1,2,\ldots,k$$

$$\sum_{i=1}^{k}a_i=\sum_{j=1}^{l}b_j$$

$$x_{ij}\geq 0, x_{ij}\in\mathbb{N}.$$

$A_i, i=1,\ldots,k$- number of aerobases,

$a_i, i=1,\ldots,k$ - capacity (maximal number of homogenous UAVs located in aerobase),

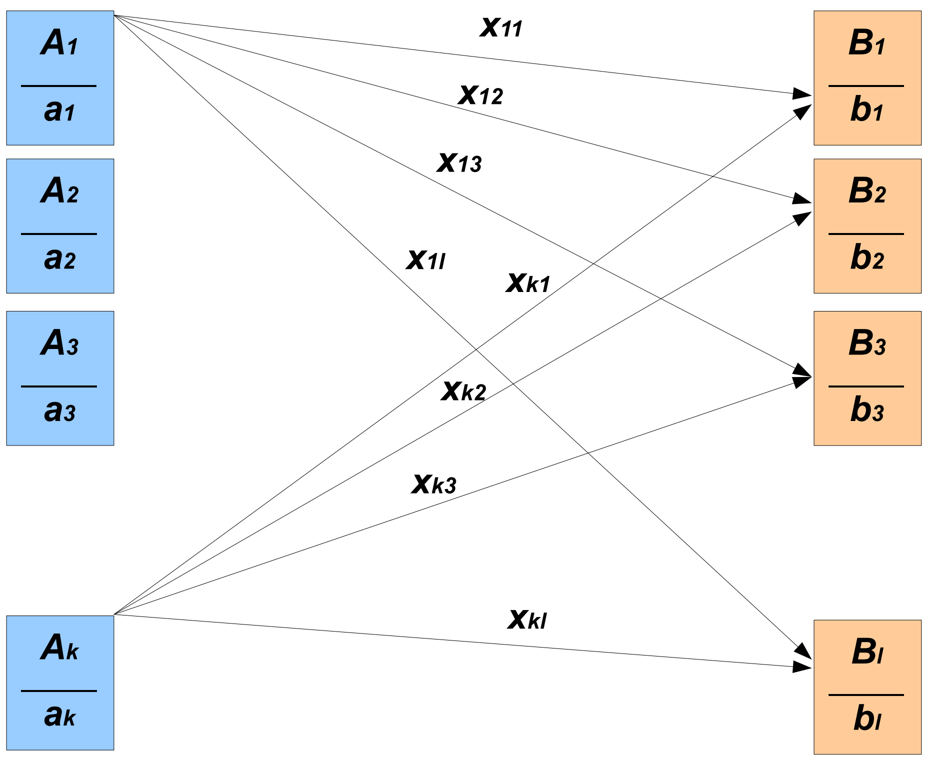
$B_j, j=1,\ldots,l$- areas of operations,

$b_j, j=1,\ldots,l$- numbers of UAVs required for service of B_j zones,

c_{ij}- benfits,

x_{ij}-number of UAVs from i-th aerobase to j-th zone of area of operation.

Assignment problem by cost criteria



$$F = \sum_{i=1}^{k}\sum_{j=1}^{l} c_{ij}x_{ij} \to \min_{x_{ij}}$$

$$\sum_{i=1}^{k} x_{ij} = b_j, \quad j = 1, 2, ..., l$$

$$\sum_{j=1}^{l} x_{ij} = a_i, \quad i = 1, 2, ..., k$$

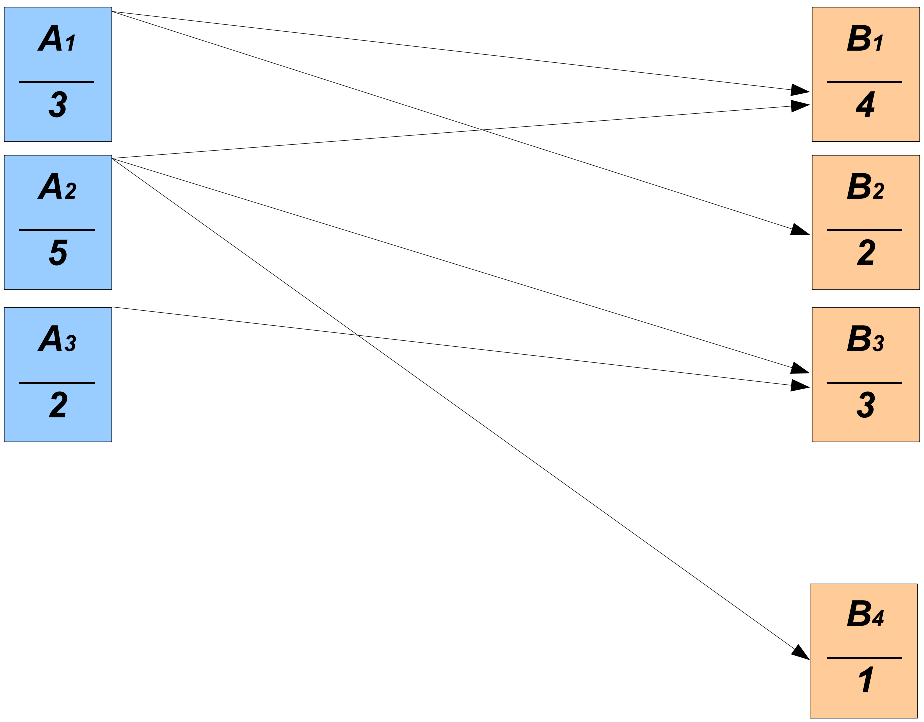
$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j$$

$$x_{ij} \geq 0, x_{ij} \in \mathbb{N}.$$

The most of methods include the following basic steps:

- To find initial plan x_{ij};
- Check optimality condition for that plan;
- Construct the improved plan in case of nonoptimality.

Example (Initial plan)



$$F = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \to \min_{x_{ij}}$$

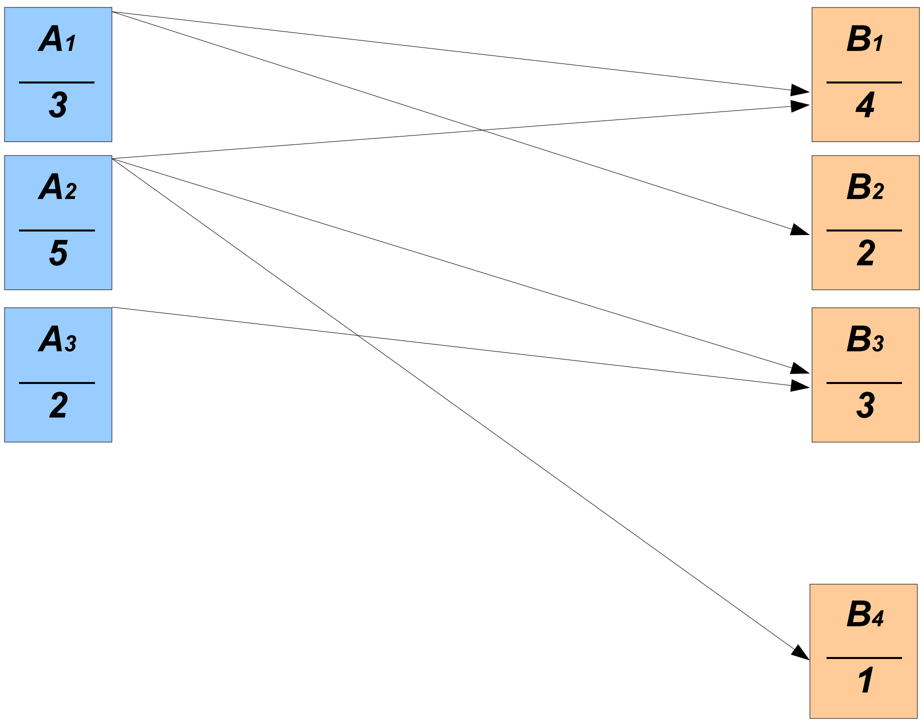
$$\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1, 2, 3, 4$$

$$\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1, 2, 3$$

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

$$x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}.$$

Example (Initial plan)



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$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

$$x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}.$$

The condition of that problem can be represented in table form.

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	x_{11}	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	x_{21}	x_{22}	x_{23}	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

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A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

$$x_{11} \quad = \min(a_1; b_1);$$

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$$x_{11} \quad = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);$$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

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A_1	3	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	1	x_{22}	x_{23}	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	x_{33}	x_{34}	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$

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$$x_{11} = \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2);$$

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$$\begin{aligned} x_{11} &= \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2); \\ x_{23} &= \min(a_1 + a_2 - b_1 - b_2; b_3); \end{aligned}$$

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$$\begin{aligned} x_{11} &= \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2); \\ x_{23} &= \min(a_1 + a_2 - b_1 - b_2; b_3); \end{aligned}$$

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$$\begin{aligned} x_{11} &= \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2); \\ x_{23} &= \min(\color{red}{a_1 + a_2 - b_1 - b_2}; b_3);\ x_{33} = \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2)); \end{aligned}$$

Example (Initial plan)

$$F=\sum_{i=1}^{3}\sum_{j=1}^{4}c_{ij}x_{ij}\to\min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
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A_3	x_{31}	x_{32}	1	x_{34}	$a_3=2$
b_j	$b_1=4$	$b_2=2$	$b_3=3$	$b_4=1$	$\sum_{i=1}^{3}a_i=\sum_{j=1}^{4}b_j=10$

$$\begin{aligned}
x_{11}&=\min(a_1;b_1);\ x_{21}=\min(a_2;b_1-a_1);\ x_{22}=\min(a_1+a_2-b_1;b_2);\\
x_{23}&=\min(a_1+a_2-b_1-b_2;b_3);\ x_{33}=\min(a_3;b_3-(a_1+a_2-b_1-b_2));\\
x_{34}&=\min(a_1+a_2+a_3-b_1-b_2-b_3;b_4).
\end{aligned}$$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	3	x_{12}	x_{13}	x_{14}	$a_1 = 3$
A_2	1	2	2	x_{24}	$a_2 = 5$
A_3	x_{31}	x_{32}	1	1	$a_3 = 2$
b_j	$b_1 = 4$	$b_2 = 2$	$b_3 = 3$	$b_4 = 1$	$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 10$

$$\begin{aligned} x_{11} &= \min(a_1; b_1);\ x_{21} = \min(a_2; b_1 - a_1);\ x_{22} = \min(a_1 + a_2 - b_1; b_2); \\ x_{23} &= \min(a_1 + a_2 - b_1 - b_2; b_3);\ x_{33} = \min(a_3; b_3 - (a_1 + a_2 - b_1 - b_2)); \\ x_{34} &= \min(a_1 + a_2 + a_3 - b_1 - b_2 - b_3; b_4). \end{aligned}$$

Example (Initial plan)

$$F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \to \min_{x_{ij}}$$

	B_1	B_2	B_3	B_4	a_i
A_1	3	0	0	0	3
A_2	1	2	2	0	5
A_3	0	0	1	1	2
b_j	4	2	3	1	

$$\begin{aligned} x_{11} &= 3;\ x_{21} = 1;\ x_{22} = 2; \\ x_{23} &= 2;\ x_{33} = 1; \\ x_{34} &= 1. \end{aligned}$$

Example (Initial plan)

Check the optimality

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	3	0	0	0	3	
A_2	1	2	2	0	5	
A_3	0	0	1	1	2	
b_j	4	2	3	1	F=60	
ν_j						

Optimality condition

Find a number u_i and ν_j such that

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	3	0	0	0	3	
A_2	1	2	2	0	5	
A_3	0	0	1	1	2	
b_j	4	2	3	1	F=60	
ν_j						

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

$$u_i + \nu_j \le c_{ij},\ for\ x_{ij} = 0$$

Optimality condition

$$Find\ a\ number\ u_i\ and\ \nu_j\ such\ that$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	
A_2	4 1	12 2	5 2	0	5	
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j						

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Optimality condition

$$Find\ a\ number\ u_i\ and\ \nu_j\ such\ that$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 / 3	0	0	0	3	
A_2	4 / 1	12 / 2	5 / 2	0	5	
A_3	0	0	6 / 1	4 / 1	2	
b_j	4	2	3	1	F=60	
ν_j	0					

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

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Optimality condition

$$Find\ a\ number\ u_i\ and\ \nu_j\ such\ that$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	
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b_j	4	2	3	1	F=60	
ν_j	0					

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A_2	4 1	12 2	5 2	0	5	
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j	0					

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j	0					

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

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Optimality condition

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	B_1	B_2	B_3	B_4	a_i	u_i
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$$u_i + \nu_j = c_{ij}, \ for \ x_{ij} > 0,$$

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	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j	0	8				

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

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Optimality condition

Find a number u_i and ν_j such that

	B_1	B_2	B_3	B_4	a_i	u_i
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b_j	4	2	3	1	F=60	
ν_j	0	8				

$$u_i + \nu_j = c_{ij}, \; for \; x_{ij} > 0,$$

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A_3	0	0	6 1	4 1	2	
b_j	4	2	3	1	F=60	
ν_j	0	8	1			

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

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Optimality condition

$$Find\ a\ number\ u_i\ and\ \nu_j\ such\ that$$

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A_1	4 3	0	0	0	3	4
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b_j	4	2	3	1	F=60	
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b_j	4	2	3	1	F=60	
ν_j	0	8	1			

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b_j	4	2	3	1	F=60	
ν_j	0	8	1			

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A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

$$u_i + \nu_j \le c_{ij},\ for\ x_{ij} = 0$$

Optimality condition

$Denote\ by\ \bar{c}_{ij}$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 3	0	0	0	3	4
A_2	4 1	12 2	5 2	0	5	4
A_3	0	0	6 1	4 1	2	5
b_j	4	2	3	1	F=60	
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A_1	4 4 3	0	0	0	3	4
A_2	4 4 1	12 12 2	5 5 2	0	5	4
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A_1	4 4 3	6 0	8 0	2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	1 0	5	4
A_3	8 0	10 0	6 6 1	4 4 1	2	5
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	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 0	5 8 0	3 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 1 0	5	4
A_3	5 8 0	13 10 0	6 6 1	4 4 1	2	5
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A_2	4 4 1	12 12 2	5 5 2	3 1 0	5	4
A_3	5 8 0	13 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$u_i + \nu_j = c_{ij},\ for\ x_{ij} > 0,$$

$$\bar{c}_{ij} - c_{ij} \leq 0,\ for\ x_{ij} = 0$$

Optimality condition

Check our conditions

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 0	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
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$$u_i + \nu_j = c_{ij}, \; for \; x_{ij} > 0,$$

$$\bar{c}_{ij} - c_{ij} \leq 0, \; for \; x_{ij} = 0$$

Improvement procedure

Check our conditions

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 0	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\bar{c_{ij}} - c_{ij} > 0 \to \max,\ for\ x_{ij} = 0$$

Improvement procedure

Find the maximal admissible value of θ

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3	12 6 6 θ	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1	12 12 2	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\bar{c_{ij}} - c_{ij} > 0 \rightarrow \max, \ for\ x_{ij} = 0$$

Improvement procedure

The maximal admissible value of θ

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 3 $-$ θ	12 6 6 θ	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 1 $+$ θ	12 12 2 $-$ θ	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=60	
ν_j	0	8	1	-1		

$$\min(3-\theta; 2-\theta) = 0 \Longrightarrow \theta{=}2$$

Improvement procedure

$$New\ feasible\ solution:$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	12 6 6 2	5 ✓ 8 0	3 1 2 0	3	4
A_2	4 4 3	12 12 0	5 5 2	3 2 1 0	5	4
A_3	5 ✓ 8 0	13 3 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=48	
ν_j	0	8	1	-1		

Iterations

$$Find\ new\ potentials:$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 2	✓ 8 0	2 0	3	
A_2	4 4 3	12 0	5 5 2	1 0	5	
A_3	✓ 8 0	10 0	6 6 1	4 4 1	2	
b_j	4	2	3	1	F=48	
ν_j						

Iterations

$$Find\ new\ potentials:$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 ✓ 8 0	3 2 0	3	4
A_2	4 4 3	6 ✓ 12 0	5 5 2	3 1 0	5	4
A_3	5 ✓ 8 0	7 ✓ 10 0	6 6 1	4 4 1	2	5
b_j	4	2	3	1	F=48	
ν_j	0	2	1	-1		

Iterations

Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 $\checkmark$ 8 0	3 2 0	3	4
A_2	4 4 3	6 $\checkmark$ 12 0	5 5 2 $-$ θ	3 1 θ	5	4
A_3	5 $\checkmark$ 8 0	7 $\checkmark$ 10 0	6 6 1 $+$ θ	4 4 1 $-$ θ	2	5
b_j	4	2	3	1	F=48	
ν_j	0	2	1	-1		

Iterations

$$Find\ new\ potentials:$$

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	✓ 8 0	2 0	3	
A_2	4 4 3	✓ 12 0	5 5 1	1 1	5	
A_3	✓ 8 0	✓ 10 0	6 2	4 0	2	
b_j	4	2	3	1	F=46	
ν_j						

Iterations

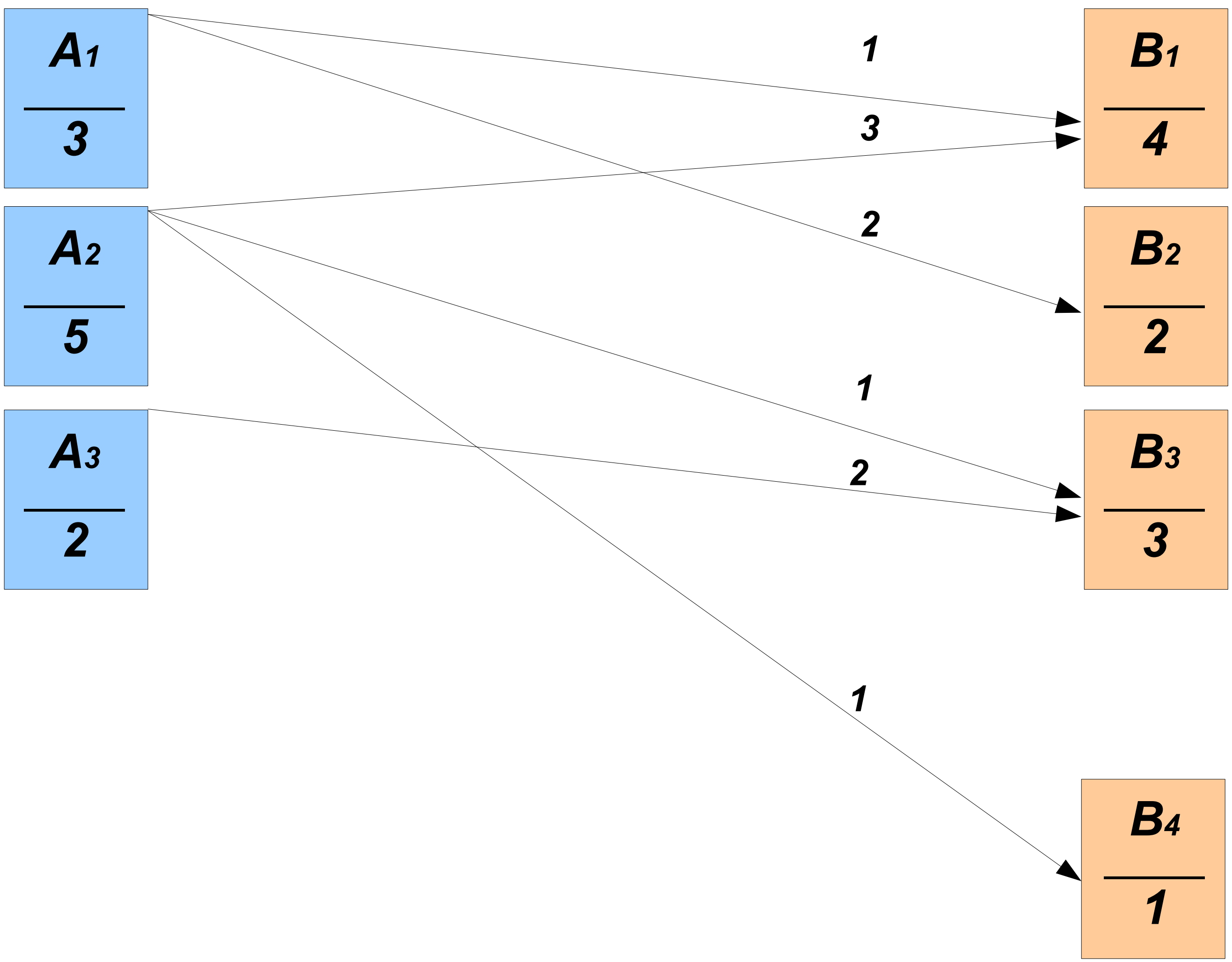
Find new potentials :

	B_1	B_2	B_3	B_4	a_i	u_i
A_1	4 4 1	6 6 2	5 ✓ 8 0	1 ✓ 2 0	3	4
A_2	4 4 3	6 ✓ 12 0	5 5 1	1 1 1	5	4
A_3	5 ✓ 8 0	7 ✓ 10 0	6 6 2	2 ✓ 4 0	2	5
b_j	4	2	3	1		
ν_j	0	2	1	-3		

F=46

$$\boxed{\min F = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij}}$$

Example (Optimal solution)



Dynamical assignment of UAVs

Optimal schedule problem on time interval $[0, H]$ can be formulated as:

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i), \ i = 1, ..., k;$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}), \ j = 1, ..., l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i, \ i = 1, ..., k; \quad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j, \ j = 1, ..., l.$$

$[0, H]$ is the given period for service of $B_1, B_2, ..., B_j, ..., B_l$ zones of area of operation. It is assumed that each onetime service of each zone B_j requests includes at least b_j numbers of UAVs, $j = 1, \ldots, l$. Also, assume that we have k aerobases $A_1, A_2, ..., A_i, ..., A_k$ with $a_1, a_2, ..., a_i, ..., a_k$ number of UAVs.

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$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, \ldots, k; \quad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1, \ldots, l.$$

Divide the interval $[0, H]$ by the moments $t = i\Delta,\ i = 1, 2, \ldots\ldots, \nu$, where $\nu = \left[\frac{H}{\Delta}\right]$ denotes the integer part of the fraction $\frac{H}{\Delta}$.

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Δ – is a small number

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Δ – is a small number

Divide the interval $[0, H]$ by the moments $t = i\Delta,\ i = 1, 2, \ldots.., \nu$, where $\nu = \left[\frac{H}{\Delta}\right]$ denotes the integer part of the fraction $\frac{H}{\Delta}$. Hence, we have the time interval partition $0 < \Delta < 2\Delta < \ldots < i\Delta < (i+1)\Delta < \ldots. < H.$

Variables

For each discrete moment $t = i\Delta$, $i = 1, 2, ..., \nu$, introduce the following variables:

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,...,\nu}$$

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- $x_{ij}(t)$- number of UAVs from $A_i \to B_j$ at t;
- $a_i(t)$ - number of UAVs at A_i at t;
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- k and l are the number of aerobases and zones respectively;
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Constraints

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$The\ number\ of\ UAVs\ at\ A_i\ aerobase\ at\ the\ next\ moment\ t+\Delta\ is\ composed\ by:$

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$\boxed{a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1,\ldots,k;}$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ j = 1,\ldots,l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1,\ldots,k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1,\ldots,l.$$

$The\ number\ of\ UAVs\ at\ A_i\ aerobase\ at\ the\ next\ moment\ t+\Delta\ is\ composed\ by:$

- by the UAVs that are being at the previous moment t

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \rightarrow \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$\boxed{a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1, ..., k;}$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}),\ j = 1, ..., l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) = b_j,\ j = 1, ..., l.$$

$The\ number\ of\ UAVs\ at\ A_i\ aerobase\ at\ the\ next\ moment\ t+\Delta\ is\ composed\ by:$

- by the UAVs that are being at the previous moment t
- UAVs that are returned during the period $[t, t+\Delta]$

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t),t=0,\Delta,2\Delta,....,\nu}$$

$$\boxed{a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1,...,k;}$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ j = 1,...,l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1,...,k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1,...,l.$$

$The\ number\ of\ UAVs\ at\ A_i\ aerobase\ at\ the\ next\ moment\ t+\Delta\ is\ composed\ by:$

- by the UAVs that are being at the previous moment t
- UAVs that are returned during the period $[t, t+\Delta]$
- UAVs that were send to zones at the moment t

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$\boxed{a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1, \ldots, k;}$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ j = 1, \ldots, l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, \ldots, k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1, \ldots, l.$$

$$\textit{The initial conditions are } a_i(0) = a_i,\ i = 1, 2, \ldots, k.$$

- by the UAVs that are being at the previous moment t
- UAVs that are returned during the period $[t, t+\Delta]$
- UAVs that were send to zones at the moment t

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1,\ldots,k;$$

$$\boxed{b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ j = 1,\ldots,l;}$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1,\ldots,k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1,\ldots,l.$$

The number of UAVs that will serve the B_j zone at the next moment $t+\Delta$

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1, ..., k;$$

$$\boxed{b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}),\ j = 1, ..., l;}$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) = b_j,\ j = 1, ..., l.$$

$$The\ number\ of\ UAVs\ that\ will\ serve\ the\ B_j\ zone\ at\ the\ next\ moment\ t+\Delta$$

- UAVs that are serving this zone at before time moment t and having sufficient flight endurance;
- plus UAVs that reach this zone during the period $[t, t+\Delta]$;
- minus UAVs that are out-of-fuel to the moment t

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1, ..., k;$$

$$\boxed{b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ j = 1, ..., l;}$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ j = 1, ..., l.$$

The initial conditions are $b_j(0) = b_j,\ j = 1, 2, ..., l.$

- UAVs that are serving this zone at before time moment t and having sufficient flight endurance;
- plus UAVs that reach this zone during the period $[t, t+\Delta]$;
- minus UAVs that are out-of-fuel to the moment t

Constraints

$$J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t), t=0,\Delta,2\Delta,\ldots,\nu}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ \ i = 1, ..., k;$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ \ j = 1, ..., l;$$

$$\boxed{a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ \ i = 1, ..., k;} \qquad \boxed{b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ \ j = 1, ..., l.}$$

$The\ variables\ x_{ij}(t)\ at\ each\ moment\ t\ satisfy\ the\ following\ conditions:$

The first one images the fact that the being UAVs can be allocated among zones.
The second one means that at each moment the service request should be satisfied.

Objective functions

$$\boxed{J_1(x) = \sum_{t=0}^{\nu} x_{ij}(t)(h_i - 2t_{ij}) \to \max_{x_{ij}(t),t=0,\Delta,2\Delta,\ldots,\nu}}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t - h_i),\ i = 1, ..., k;$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}),\ j = 1, ..., l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) = b_j,\ j = 1, ..., l.$$

1. *The total service time for multiple zones*

2. *The total number of UAVs "circles"*

3. *The total unobservable time for multiple zones*

Objective functions

$$\boxed{J_2(x) = \sum_{t=0}^{\nu} x_{ij}(t)}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ \ i = 1, ..., k;$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}) + \sum_{i=1}^{k} x_{ij}(t-h_i+t_{ij}),\ \ j = 1, ..., l;$$

$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ \ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t-t_{ij}) = b_j,\ \ j = 1, ..., l.$$

1. *The total service time for multiple zones*

2. *The total number of UAVs "circles"*

3. *The total unobservable time for multiple zones*

Objective functions

$$\boxed{J_3(x) = \sum_{t=0}^{\nu} x_{ij}(t)(H - h_i - 2t_{ij})}$$

$$a_i(t+\Delta) = a_i(t) - \sum_{j=1}^{l} x_{ij}(t) + \sum_{j=1}^{l} x_{ij}(t-h_i),\ i = 1, ..., k;$$

$$b_j(t+\Delta) = b_j(t) - \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}) + \sum_{i=1}^{k} x_{ij}(t - h_i + t_{ij}),\ j = 1, ..., l;$$

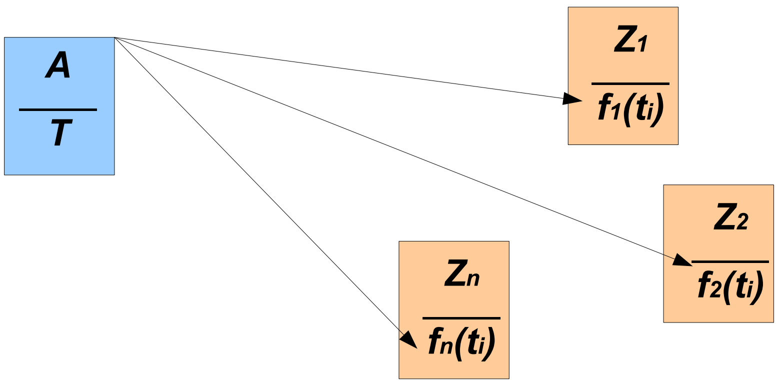
$$a_i(t) + \sum_{j=1}^{l} x_{ij}(t) = a_i,\ i = 1, ..., k; \qquad b_j(t) + \sum_{i=1}^{k} x_{ij}(t - t_{ij}) = b_j,\ j = 1, ..., l.$$

1. *The total service time for multiple zones*

2. *The total number of UAVs "circles"*

3. *The total unobservable time for multiple zones*

DP

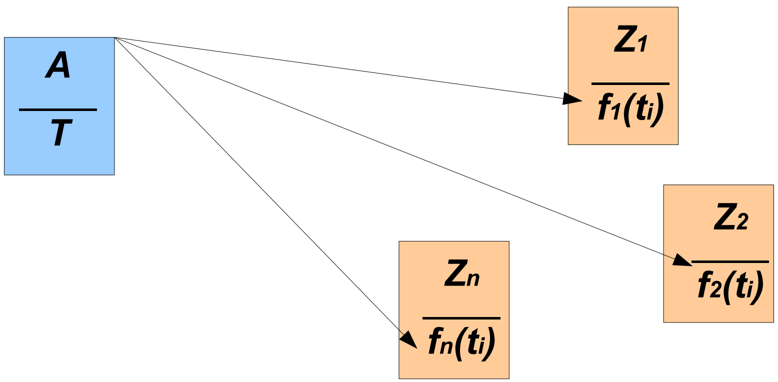


$$\sum_{i=1}^{n} f_i(t_i) \to \max$$

$$\sum_{i=1}^{n} t_i \le T,\; t_i \ge 0, i = 1, ..., n$$

- $0 \le y \le T$, - the portion of "resource" assigned for the zone $0 \le k \le n$
- $f_i(t_i)$ - the "benefit" of this assignment (*probability of targets detection in* Z_i *or the square of observed area in* Z_i)

DP

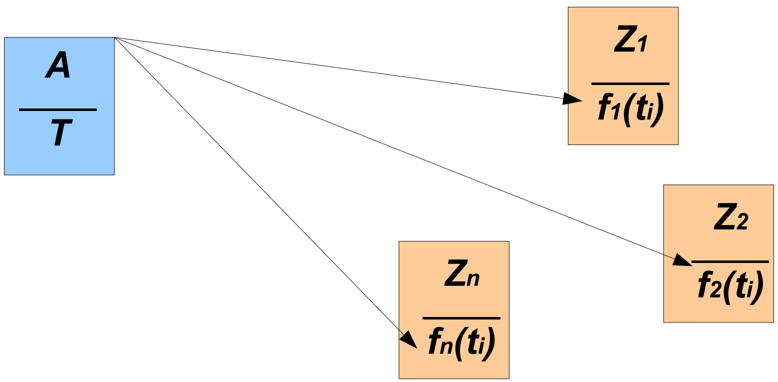


- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$,

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;

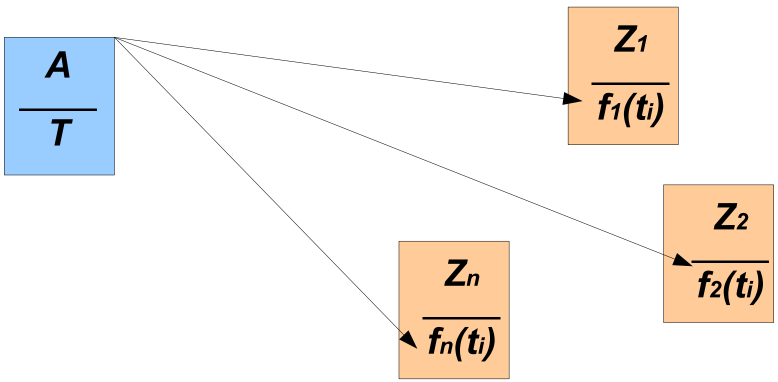
$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

$$B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i),$$

$$\sum_{i=2}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

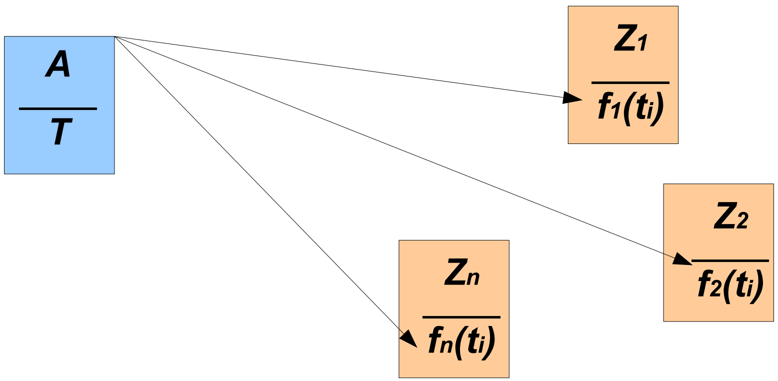
$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

Let $z,\ 0 \le z \le y$ be the number of UAVs assigned for the zone Z_k. Then benefit is equal $f_k(z)$. The rest UAVs $y - z$ should be distributed among the remained $(k-1)$ zones. And optimal distribution among $(k-1)$ zones is determined by $B_{k-1}(y-z)$.

$$\Longleftarrow \qquad B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i),$$

$$\sum_{i=2}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y): \sum_{i=1}^{k} f_i(t_i) \to \max,$$

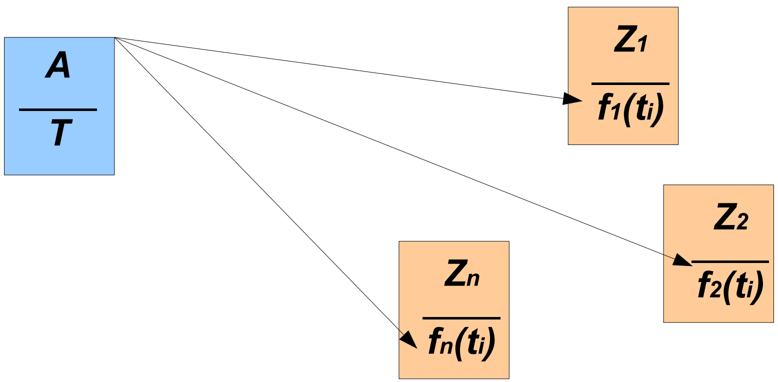
$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

Therefore, if the given resource is equal y, then after the assignment of z portion for zone Z_k the total profit of all k zones is $f_k(z) + B_{k-1}(y-z)$ $\Longleftarrow$

$$B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i),$$

$$\sum_{i=2}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
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$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y): \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

Hence, the optimal distribution z_k^0, $0 \le z_k^0 \le y$, for the given zone Z_k is determined by

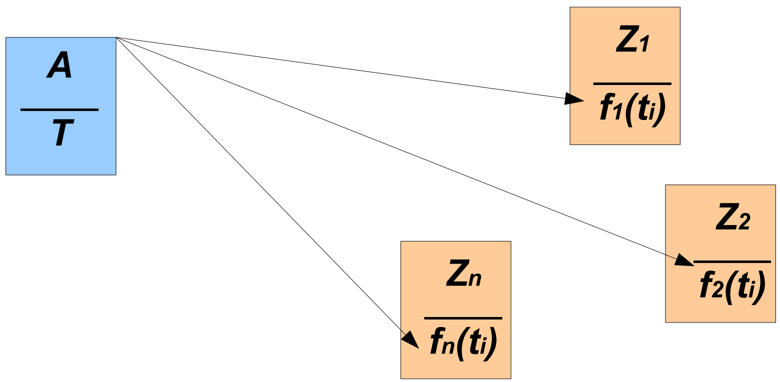
$$B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i),$$

$$\Longleftarrow$$

$$f_k(z_k^0) + B_{k-1}(y - z_k^0) = \max_{0 \le z \le y} \left[f_k(z) + B_{k-1}(y - z) \right]$$

$$\sum_{i=2}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \le T,\ t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

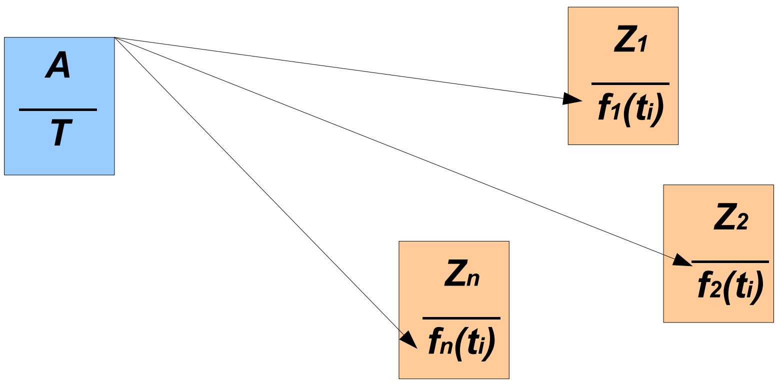
Optimal distribution of the initially given resource y among all k zones is equal $B_k(y)$

$$B_k(y) = \max_{t_i} \sum_{i=2}^{k} f_i(t_i),$$

$$\Longleftarrow$$

$$B_k(y) = \max_{0 \le z \le y} \left[f_k(z) + B_{k-1}(y-z) \right] \qquad \sum_{i=2}^{k} t_i \le y,\ t_i \ge 0, i = 1, ..., k$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
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$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

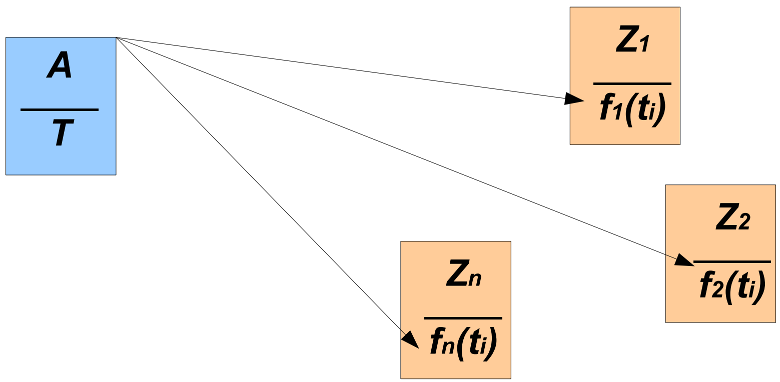
$$\sum_{i=1}^{n} t_i \le T,\; t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\; t_i \ge 0, i = 1, ..., k$$

Bellman equation

$$B_k(y) = \max_{0 \le z \le y} \left[f_k(z) + B_{k-1}(y-z) \right] \qquad \Longrightarrow \qquad B_1(y) = \max_{t_1} \sum_{i=1}^{1} f_1(t_1),$$

$$\sum_{i=1}^{1} t_1 \le y,\; t_1 \ge 0,$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;

$$\sum_{i=1}^{n} f_i(t_i) \to \max \qquad \Longrightarrow \qquad P(k,y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \le T,\; t_i \ge 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \le y,\; t_i \ge 0, i = 1, ..., k$$

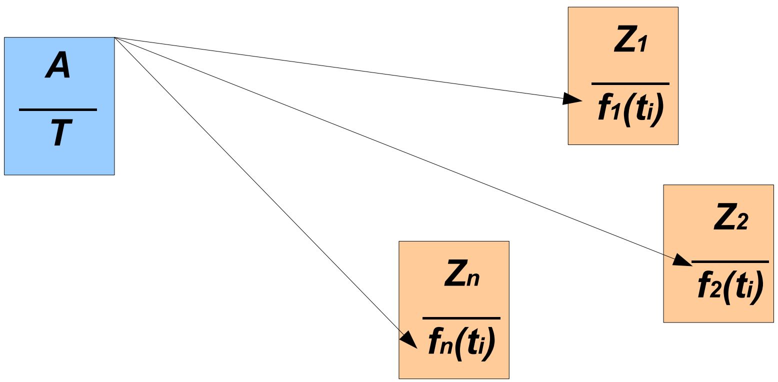
Bellman equation

$$B_k(y) = \max_{0 \le z \le y} \left[f_k(z) + B_{k-1}(y-z) \right] \qquad \Longrightarrow$$

with initial condition

$$B_1(y) = f_1(y)$$

DP



- Invariant embedding of the problem into $P(k, y), k \in [1; n]; y \in [0; T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

$$\sum_{i=1}^{n} f_i(t_i) \to \max \quad \Longrightarrow \quad P(k, y) : \sum_{i=1}^{k} f_i(t_i) \to \max,$$

$$\sum_{i=1}^{n} t_i \leq T, \; t_i \geq 0, i = 1, ..., n \qquad \sum_{i=1}^{k} t_i \leq y, \; t_i \geq 0, i = 1, ..., k$$

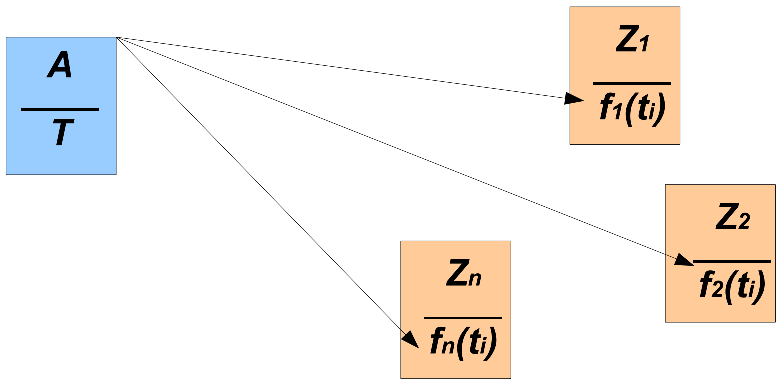
Bellman equation

$$B_k(y) = \max_{0 \leq z \leq y} \left[f_k(z) + B_{k-1}(y - z) \right] \quad \Longrightarrow$$

with initial condition

$$B_1(y) = f_1(y)$$

DP



- Invariant embedding of the problem into $P(k, y), k \in [1; n]; y \in [0; T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

$$k = 2: \; B_2(y) = \max_{0 \le z \le y} \Big[f_2(z) + B_1(y - z) \Big] = \max_{0 \le z \le y} \Big[f_2(z) + f_1(y - z) \Big]$$

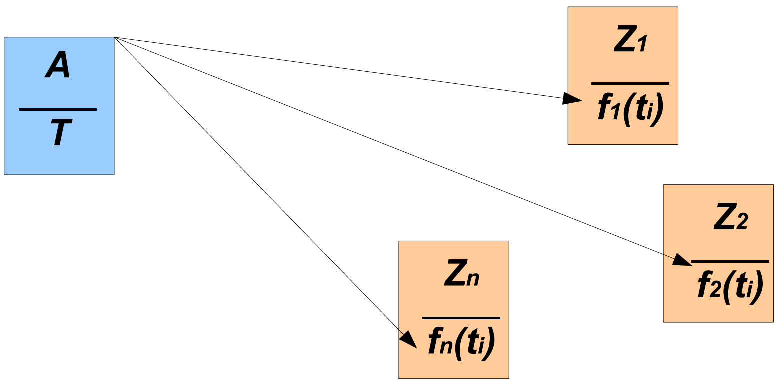
Bellman equation

$$B_k(y) = \max_{0 \le z \le y} \Big[f_k(z) + B_{k-1}(y - z) \Big]$$

$\Longrightarrow$ *with initial condition*

$$B_1(y) = f_1(y)$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

$$k = 2: \; B_2(y) = \max_{0\le z\le y}\left[f_2(z) + B_1(y-z)\right] = \max_{0\le z\le y}\left[f_2(z) + f_1(y-z)\right]$$

$$k = 3: \quad B_3(y) = \max_{0\le z\le y}\left[f_3(z) + B_2(y-z)\right]$$

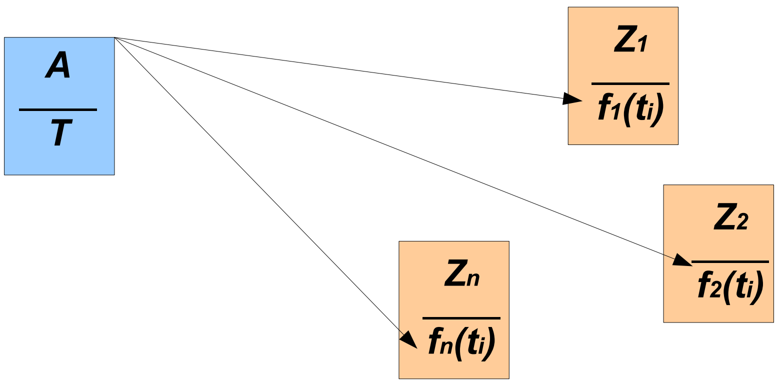
Bellman equation

$$B_k(y) = \max_{0\le z\le y}\left[f_k(z) + B_{k-1}(y-z)\right] \Longrightarrow$$

with initial condition

$$B_1(y) = f_1(y)$$

DP



- Invariant embedding of the problem into $P(k,y), k \in [1;n]; y \in [0;T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

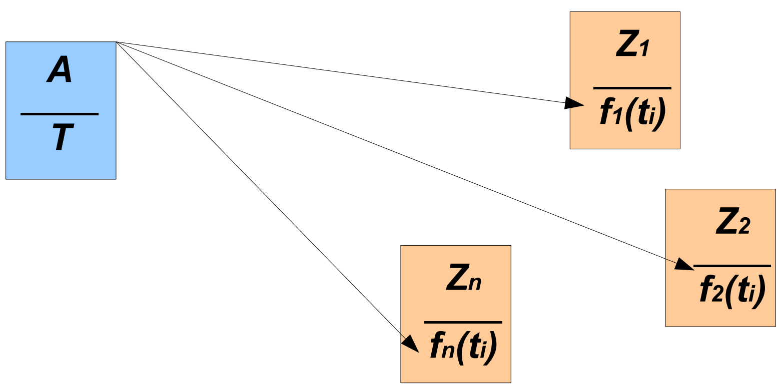
$$k=2: \; B_2(y) = \max_{0\le z\le y}\Big[f_2(z)+B_1(y-z)\Big] = \max_{0\le z\le y}\Big[f_2(z)+f_1(y-z)\Big]$$

$$k=3: \quad B_3(y) = \max_{0\le z\le y}\Big[f_3(z)+B_2(y-z)\Big]$$

Proceeding sequentially this procedure we will determine the functions

$$B_4(y), B_5(y), ..., B_n(y).$$

DP



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$$k = 2: \; B_2(y) = \max_{0 \le z \le y} \left[f_2(z) + B_1(y - z) \right]$$
$$= \max_{0 \le z \le y} \left[f_2(z) + f_1(y - z) \right]$$

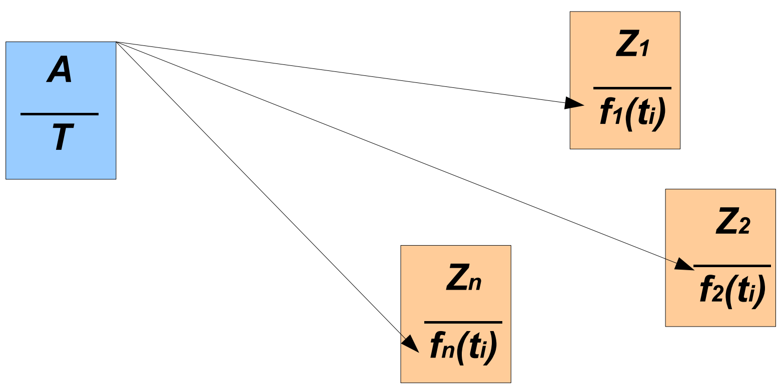
$$k = 3:$$
$$B_3(y) = \max_{0 \le z \le y} \left[f_3(z) + B_2(y - z) \right]$$

Proceeding sequentially this procedure we will determine the functions

$$B_4(y), B_5(y), ..., B_n(y).$$

In final, the function value $B_n(T)$ presents the maximal profit for the initial allocation problem.

DP

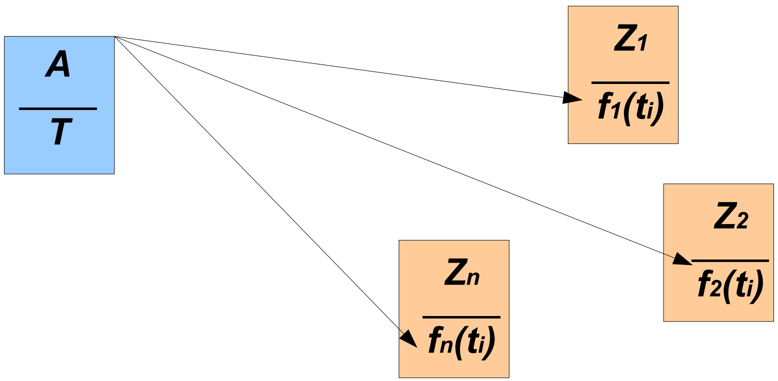


- Invariant embedding of the problem into $P(k, y), k \in [1; n]; y \in [0; T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

Optimal distribution z_k^0, $0 \leq z_k^0 \leq y$, for the given zone Z_k is determined by

$$f_k(z_k^0) + B_{k-1}(y - z_k^0) = \max_{0 \leq z \leq y} \left[f_k(z) + B_{k-1}(y - z) \right]$$

DP

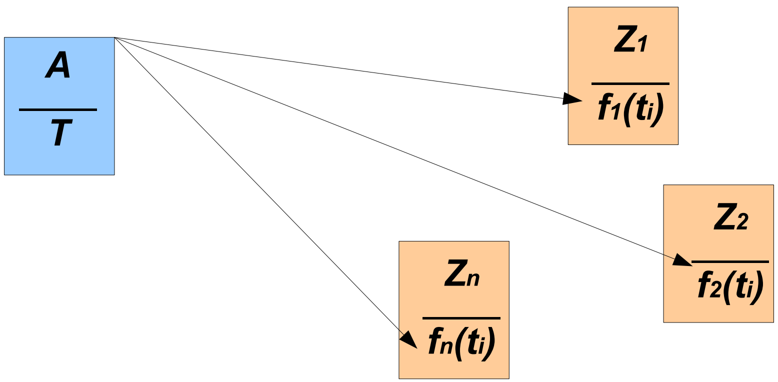


- Invariant embedding of the problem into $P(k, y), k \in [1; n]; y \in [0; T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

Put $k = n,\ y = T$ and
find the value $t_n^0 \doteq z^0(T)$ for the zone Z_n

$$f_n(z_n^0) + B_{n-1}(T - z_n^0) = \max_{0 \le z \le T} \left[f_n(z) + B_{n-1}(T - z) \right]$$

DP



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Put $k = n,\ y = T$ and
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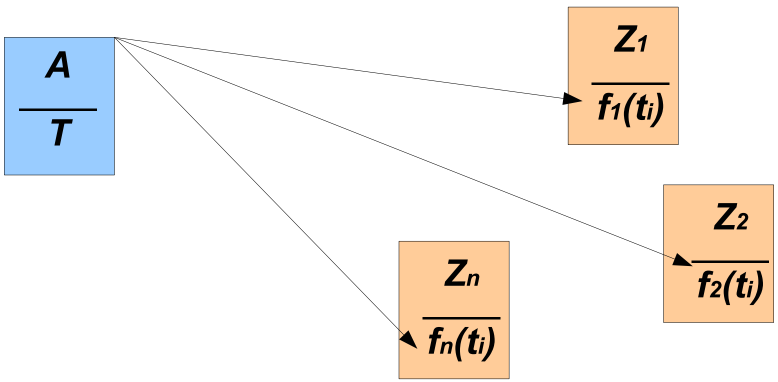
$$f_n(z_n^0) + B_{n-1}(T - z_n^0) = \max_{0 \le z \le T} \left[f_n(z) + B_{n-1}(T - z) \right]$$

$\Longrightarrow$

$k = n-1,\ y = T - t_n^0$ and
find the value $t_{n-1}^0 \doteq z^0(T - t_n^0)$

$$f_{n-1}(z_{n-1}^0) + B_{n-2}(T - z_{n-1}^0) = \max_{0 \le z \le T - t_n^0} \left[f_{n-1}(z) + B_{n-2}(T - t_n^0 - z) \right]$$

DP



- Invariant embedding of the problem into $P(k, y), k \in [1; n]; y \in [0; T]$, then define the Bellman function;
- Construct the Bellman equation;
- Solve the Bellman equation, and apply the solution of it to initial problem

$$t_n^0,\ t_{n-1}^0,\ \ldots,\ t_2^0,\ t_1^0 \quad \Longleftarrow \quad \begin{array}{c}\textit{Continue this procedure we will find}\\ \textit{the optimal solution of our problem}\end{array}$$

Put $k = n,\ y = T$ and
find the value $t_n^0 \doteq z^0(T)$ for the zone Z_n

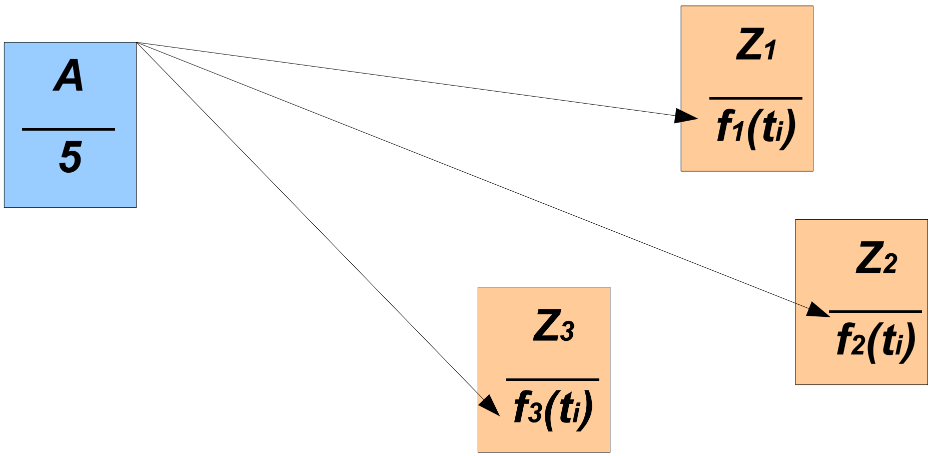
$$f_n(z_n^0) + B_{n-1}(T - z_n^0) = \max_{0 \le z \le T} \Big[f_n(z) + B_{n-1}(T - z) \Big]$$

$\Longrightarrow$

$k = n - 1,\ y = T - t_n^0$ and
find the value $t_{n-1}^0 \doteq z^0(T - t_n^0)$

$$f_{n-1}(z_{n-1}^0) + B_{n-2}(T - z_{n-1}^0) = \max_{0 \le z \le T - t_n^0} \Big[f_{n-1}(z) + B_{n-2}(T - t_n^0 - z) \Big]$$

DP example

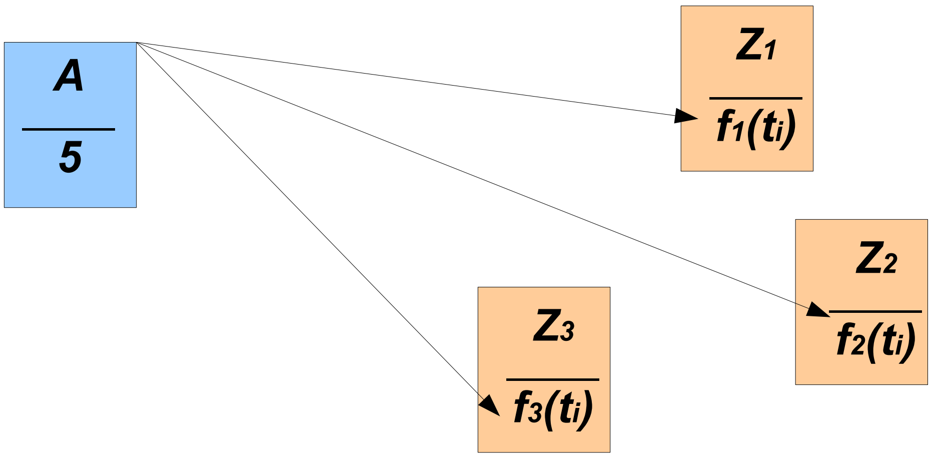


$n = 3, T = 5$

t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

The problem is to find an optimal allocation of the given UAVs to serve the given zone Z_1, Z_2, Z_3 such that to maximize the total "benefit" (for example, the "benefit" of service in the given zone for each UAV can be interpreted as the number of the detected targets).

DP example



$n = 3, T = 5$

t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

By the definition of Bellman function the optimal solution of the problem is determined by the function value

$$B_3(5) = \max_{0 \le z \le 5} \Big[f_3(z) + B_2(5 - z) \Big] =$$

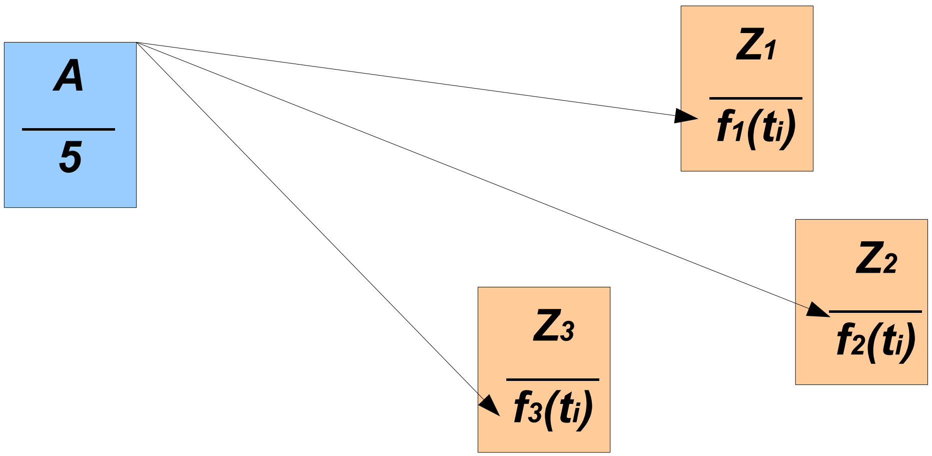
$$\max \Big\{ f_3(0) + B_2(5); f_3(1) + B_2(4); f_3(2) + B_2(3); f_3(3) + B_2(2); f_3(4) + B_2(1); f_3(5) + B_2(0) \Big\}$$

where $B_2(y)$ are determined by $B_2(y) = \max\limits_{0 \le z \le y} \Big[f_2(z) + B_1(y - z) \Big]$

$B_1(y)$ from initial conditions

$B_1(y) = f_1(y) : B_1(0) = 0,\ B_1(1) = 1,\ B_1(2) = 2,\ B_1(3) = 3,\ B_1(4) = 4,\ B_1(5) = 5.$

DP example



$n = 3, T = 5$

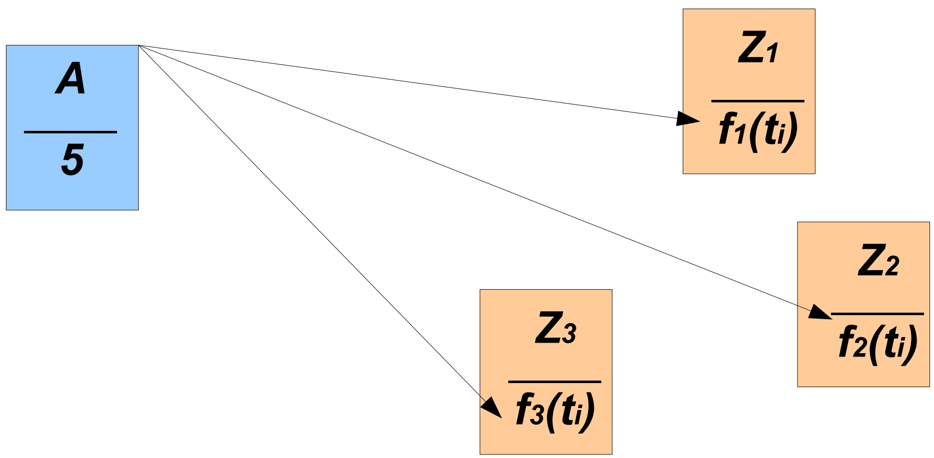
t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

y	0	1	2	3	4	5
$B_1(y)$	0	1	2	3	4	5
$B_2(y)$	0	1(0)	2(0)	3(0)	4(0,4)	7(5)
$B_3(y)$	0	2(1)	3(1)	4(1)	5(1)	7(0)

Bellman function values $B_k(y)$ is accompanied (in braces) by the arguments $z_k^0(y)$ at which this value is achieved.

$$B_3(5) =$$
$$\max\Big\{f_3(0) + B_2(5); f_3(1) + B_2(4); f_3(2) + B_2(3); f_3(3) + B_2(2); f_3(4) + B_2(1);$$
$$f_3(5) + B_2(0)\Big\} = \max\Big\{0 + 7; 2 + 4; 2 + 3; 3 + 2; 3 + 1; 5 + 0\Big\} = 7.$$

DP example



$n = 3, T = 5$

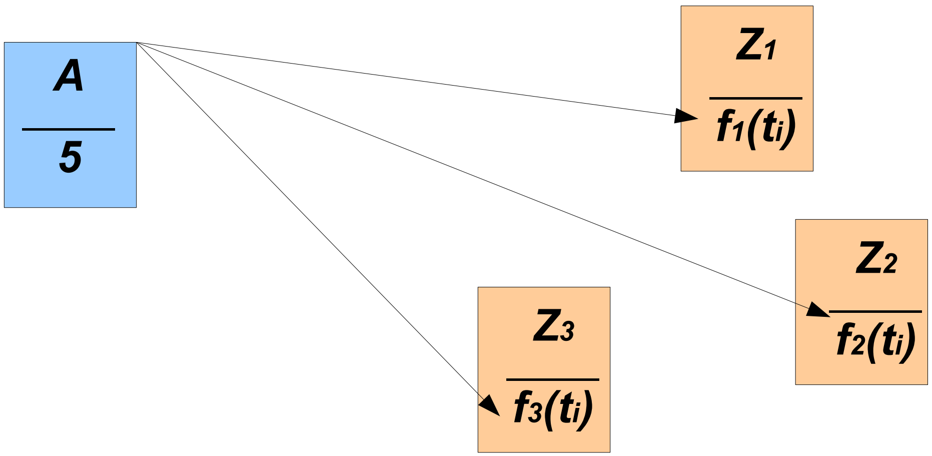
t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

y	0	1	2	3	4	5
$B_1(y)$	0	1	2	3	4	5
$B_2(y)$	0	1(0)	2(0)	3(0)	4(0,4)	7(5)
$B_3(y)$	0	2(1)	3(1)	4(1)	5(1)	7(0)

The maximal efficiency is achieved at $t_3^0 \doteq z^0(5) = 0$. Hence, the rest $5 - 0 = 5$ of UAVs should be distributed between zones Z_2 and Z_1.

$$B_3(5) = f_3(0) + B_2(5) = 7$$

DP example



$n = 3, T = 5$

t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

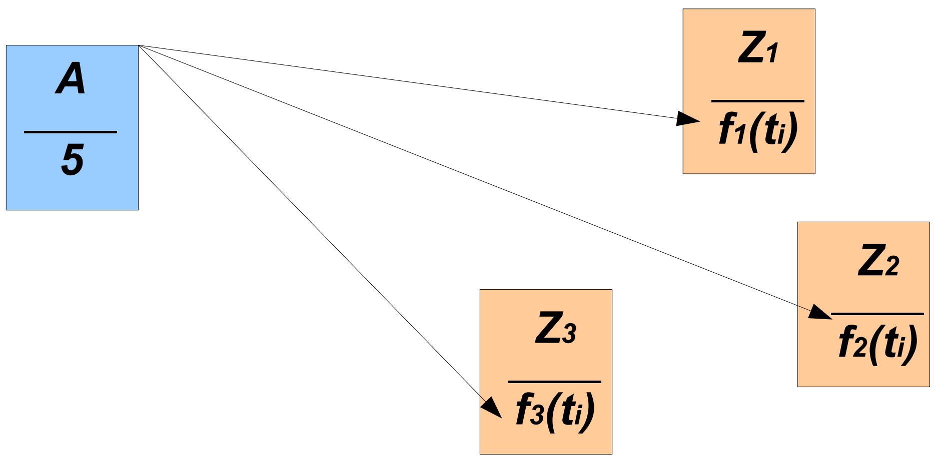
y	0	1	2	3	4	5
$B_1(y)$	0	1	2	3	4	5
$B_2(y)$	0	1(0)	2(0)	3(0)	4(0,4)	7(5)
$B_3(y)$	0	2(1)	3(1)	4(1)	5(1)	7(0)

The maximal efficiency is achieved at $t_2^0 \doteq z^0(5) = 5$. Hence, for the zone Z_1 is no UAVs for their service.

$$Z_3 : B_3(5) = f_3(0) + B_2(5) = 7$$

$$Z_2 : B_2(5) = 7$$

DP example



$n = 3, T = 5$

t	0	1	2	3	4	5
$f_1(t)$	0	1	2	3	4	5
$f_2(t)$	0	0	1	2	4	7
$f_3(t)$	0	2	2	3	3	5

y	0	1	2	3	4	5
$B_1(y)$	0	1	2	3	4	5
$B_2(y)$	0	1(0)	2(0)	3(0)	4(0,4)	7(5)
$B_3(y)$	0	2(1)	3(1)	4(1)	5(1)	7(0)

The maximal efficiency is achieved at $t_2^0 \doteq z^0(5) = 5$. Hence, for the zone Z_1 is no UAVs for their service.

$$Z_3 : B_3(5) = f_3(0) + B_2(5) = 7$$

$$Z_2 : B_2(5) = 7$$

Optimal distribution of five UAVs: $t_1^0 = 0,\ t_2^0 = 5,\ t_3^0 = 0.$

DP advantages/disatvantages

- Objectives with very general functional forms may be handled and a global optimal solution is always obtained
- "Curse of dimensionality"- the number of states grows exponentially with the number of dimensions of the problem