Lecture 8: Bayesian Estimation: Direct Sampling

Big Data and Machine Learning for Applied Economics
Econ 4676

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September 2, 2021

Agenda

Bayesian Estimation

- The Bayesian approach to stats is fundamentally different from the classical approach we have been taking
- In the classical approach, the parameter β is thought to be an unknown, but fixed quantity, e.g., $X_i \sim f(\beta)$
- In the Bayesian approach β is considered to be a quantity whose variation can be described by a probability distribution (*prior distribution*)
- Then a sample is taken from a population indexed by β and the prior is updated with this sample
- The resulting updated prior is the *posterior distribution*

Bayes Approach

Bayes Theorem

$$\pi(\beta|X) = \frac{f(X|\beta)p(\beta)}{m(X)} \tag{1}$$

with $m(X)$ is the marginal distribution of X, i.e.

$$m(X) = \int f(X|\beta)p(\beta)d\beta \tag{2}$$

It is important to note that Bayes' theorem does not tell us what our beliefs should be, it tells us how they should change after seeing new information.

Frequentist Approach

- The interest is on β, frequentist estimation procedures give us that, for example

$$\hat{\beta}_{MLE} = (X'X)^{-1}X'y \tag{3}$$

- Now in Bayes world, I have the full distribution. Which β I use?
- I can use any moment, but usually the interest lies on

$$E(\beta) = \int \beta \pi(\beta|X) \tag{4}$$

- Why? note that if you use MSE as loss function, the Bayes estimate of the unknown parameter is the mean of the posterior distribution

Bayesian Estimation

- We are going to have an overview simulation-based methods for Bayesian analysis.
    1. Direct sampling algorithm
    2. Gibbs sampling algorithm
- As a running example, we use the linear regression framework

$$y_i = \beta x_i + u_i \ , \ \ u_i \sim N(0, \sigma^2 \ ) \tag{5}$$

- with σ^2 known, and
- with prior distribution $\beta \sim N(\beta_0, \tau^2)$

Direct Sampling

- Using the knowledge of conjugate priors + the trick for exponentials
- The posterior distribution β follows the normal distribution:

$$\beta | Y, X \sim N\left(\frac{\frac{1}{\sigma^2}\sum_{i=1}^{N} y_i x_i + \frac{1}{\tau^2}\beta_0}{\frac{1}{\sigma^2}\sum_{i=1}^{N} x_i^2 + \frac{1}{\tau^2}}, \frac{1}{\frac{1}{\sigma^2}\sum_{i=1}^{N} x_i^2 + \frac{1}{\tau^2}}\right) \tag{6}$$

- We where able to characterize the full posterior distribution for the unknown object.

Direct Sampling

- Suppose now, that our object of interest is not β per se, but some nonlinear function of unknown parameter β, e.g. $h(\beta)$.
- For example:
  - $h(\beta) = \beta$
  - $h(\beta) = |\beta|$
  - $h(\beta) = \alpha\%\ quantile\ of\ \beta$
  - $h(\beta) = \beta^3$
  - $h(\beta) = \beta_1\beta_2$
- The goal is to obtain posterior moments of $h(\beta)$.

Direct Sampling

Side note: Frequentist´s approach

- Frequentist obtain the sampling distribution of $h(\beta)$ using the delta method:
- If we have

$$\sqrt{N}\left(\widehat{\beta}-\beta_0\right) \rightarrow_d \mathrm{N}\left(0,\ V_{\text{asy}}\right) \tag{7}$$

- Then we

$$\sqrt{N}\left(h\left(\widehat{\beta}\right)-h(\beta_0)\right) \rightarrow_d \mathrm{N}\left(0,\ V_{\text{asy}}\left[h'(\beta_0)\right]^2\right] \tag{8}$$

- As $N \rightarrow \infty$ where n is the number of observations.

Direct Sampling

- Idea: Monte Carlo integration.
- Requirement
  - Know how to generate i.i.d. samples from the posterior distribution of β, $\pi(\beta|Y)$
- The requirement is satisfied for our linear regression example:
  - The posterior distribution of β follows the normal distribution.
  - Most modern statistical program languages provide random number generators for many parametric distributions including the normal distribution.

Direct Sampling

- Direct sampling approach simply approximates the posterior expectations of a function $h(\beta)$ by

$$E_Y^{\beta}\left[h\left(\beta\right)\right] = \int h\left(\beta\right)\pi\left(\beta|Y\right)d\beta \tag{9}$$
$$\approx \frac{1}{S}\sum_{i=1}^{S} h(\beta^i)$$

- Where β^i is i.i.d. samples from $\pi(\beta|Y)$
- S is "number of random samples from the posterior" or "number of generated draws" NOT the number of observations.

Direct Sampling

- Provided that $E_Y^\beta\left[h(\beta)^2\right] < \infty$,
- we can use the Strong Law of Large Numbers (SLLN)

$$\frac{1}{S}\sum_{i=1}^{S} h(\beta^i) \rightarrow a.s. \int h(\beta)\, p(\beta|Y)\, d\beta$$

- and the Central Limit Theorem (CLT)

$$\sqrt{S}\left(\frac{1}{S}\sum_{i=1}^{N} h(\beta^i) - \int h(\beta)\, \pi(\beta|Y)\, d\beta\right) \rightarrow_d \; N(0, V_\pi)$$

- Where

$$V_\pi = \mathrm{Var}_Y^\beta(h(\beta)) = \int \left(h(\beta) - \; E_Y^\beta\left[h(\beta)\right]\right)^2 \pi(\beta|Y)\, d\beta$$

- S is the number of simulated draws from the posterior distribution

Direct Sampling

- Note that we turned a complicated integration into a simple average

$$\frac{1}{S}\sum_{i=1}^{S} h(\beta^i) \rightarrow a.s. \int h(\beta)\, p(\beta|Y)\, d\beta \tag{10}$$

- As the number of simulated draws increases, this simple average converges to the object of interest.
- Numerical accuracy?

Direct Sampling

- The CLT result provides a way to measure the numerical accuracy of this
- Monte Carlo approximation:

$$\sqrt{S}\left(\frac{1}{S}\sum_{i=1}^{N}h(\beta^i)-\int h(\beta)\,p(\beta|Y)\,d\beta\right)\to_d\; N(0,\;V_\pi) \tag{11}$$

- That is,

$$\frac{1}{S}\sum_{i=1}^{S}h(\beta^i)\approx_d N\left(E_Y^{\beta}\left[h(\beta)\right],\frac{V_\pi}{S}\right) \tag{12}$$

- Where $V_\pi = \; \mathrm{Var}_Y^{\beta}\left(h(\beta)\right)$. Posterior variance of $h(\beta)$ scaled by $1/S$ determines the numerical accuracy. As $S\to\infty$, numerical approximation goes to zero
- Trade-off
  - Large N: high computational cost (time) but more accurate approximation
  - Small N: low computation cost (time) but less accurate approximation

Direct Sampling

Example: Linear regression

- Consider the following linear regression model

$$y_i = \beta x_i + u_i, \quad \mathrm{u_i} \sim N\left(0, \sigma^2\right) \tag{13}$$

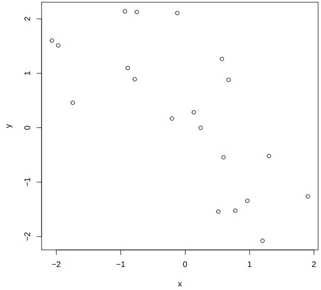
- with prior distribution$\beta \sim N\left(\beta_0, \tau^2\right)$, and suppose σ^2 is known.
- Then, we now all know that the posterior distribution β follows the normal distribution:

$$\beta|Y, X \sim N\left(\frac{\frac{1}{\sigma^2}\sum_{i=1}^N y_i x_i + \frac{1}{\tau^2}\beta_0}{\frac{1}{\sigma^2}\sum_{i=1}^N x_i^2 + \frac{1}{\tau^2}}, \frac{1}{\frac{1}{\sigma^2}\sum_{i=1}^N x_i^2 + \frac{1}{\tau^2}}\right) \tag{14}$$

Direct Sampling

Example: Linear regression

- Goal: posterior mean and equal-tail-probability credible set for $|\beta|$
- I generate data y_i x_i with
  - $y_i = \beta x_i + u_i, \quad \mathrm{u_i} \sim N\left(0, \sigma^2\right)$
  - $N = 20$
  - $\beta_{\text{true}} = -1$ *and* $\sigma^2 = 1$
  - $\beta_0 = 0$ *and* $i = 100$

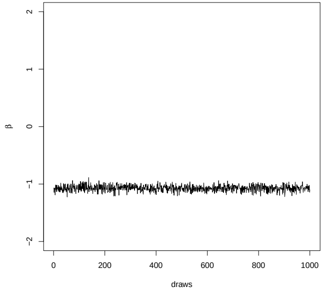


Direct Sampling

Example: Linear regression

- Step 1: we generate N draws from the $N\left(m, V\right)$, $\left\{\beta^i\right\} 1, \ldots, S$
- $m = \ -1.07$
- $V = \ 0.0510$

Figure 1: Example of draws $\left(\left\{\beta^i\right\}_{1,\ldots,N}\right)$, $S = 1,000$

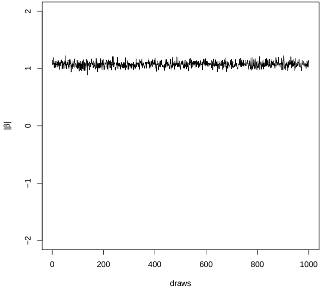


Direct Sampling

Example: Linear regression

- Step 2: we are interested in posterior moments of $|\beta|$.
- Turn draws into $\{|\beta|\}_{1,\ldots,S}$

Figure 2: Example of draws $\left(\{|\beta^i|\}_{1,\ldots,S}\right)$

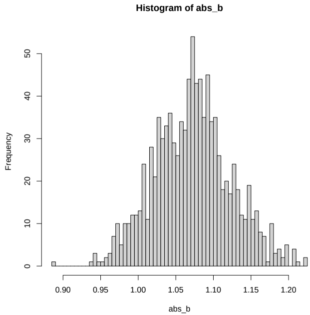


Direct Sampling

Example: Linear regression

- Histogram approximation to $\pi\left(|\beta| \;|Y\right)$ using $\left\{\left|\beta^i\right|\right\}_{1,\dots,S}$

Figure 3: Example of draws $\left(\left\{|\beta^i|\right\}_{1,\dots,S}\right)$



Direct Sampling

Example: Linear regression

- The posterior mean of $|\beta|$ is approximated by

$$E_Y^{\beta}\left[|\beta|\right] \approx \frac{1}{S}\sum_{i=1}^{S}\left|\beta^i\right| = 1.0719 \tag{15}$$

- The 90% equal-tail-probability interval is approximated by

$$C_Y = [q_1,\ q_u] = [0.719,\ 1.441] \tag{16}$$

- Where q_1 and q_u such that

$$5\% = \frac{1}{S}\sum_{i=1}^{S} 1\{|\beta| < q_u\} \tag{17}$$

Direct Sampling

Example: Linear regression

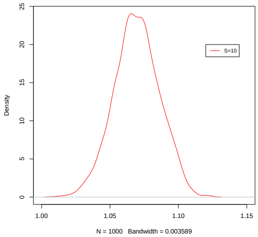
- Numerical accuracy of $\frac{1}{N} \sum_{i=1}^{N} 1 \left|\beta^i\right|$
- We know that if we generate enough number of β^i, we get an accurate approximation to the posterior moments
- How many draws are enough?
- In other words, "Will I get different answer if I construct the same quantity using different set of draws $\{\beta^i\}$?
- Is $S = 10$ enough? Or, is $S = 10,000$ enough?

Direct Sampling

Example: Linear regression

- To see the numerical error I generate 1,000 sets of $\{\beta^i\}_{i=1,\ldots,S}$
- Compute 1,000 of $\frac{1}{S}\sum_{i=1}^{S}|\beta^i|$ to see how variable this Monte Carlo approximation with different S

Figure 4: Distribution of $\frac{1}{S}\sum_{i=1}^{S}|\beta^i|$ over $\{\beta^i\}_{i=1,\ldots,S}$



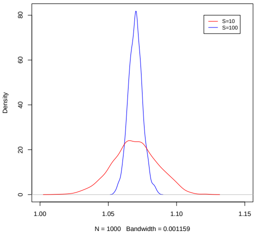
SD $\frac{1}{N}\sum_{i=1}^{N}|\beta^i| = 0.688$

Direct Sampling

Example: Linear regression

- To see the numerical error I generate 1,000 sets of $\{\beta^i\}_{i=1,\dots,S}$
- Compute 1,000 of $\frac{1}{S}\sum_{i=1}^{S}|\beta^i|$ to see how variable this Monte Carlo approximation with different S

Figure 5: Distribution of $\frac{1}{S}\sum_{i=1}^{S}|\beta^i|$ over $\{\beta^i\}_{i=1,\dots,S}$



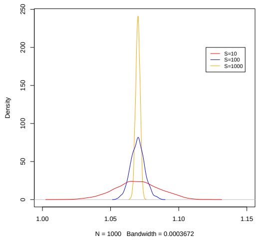
$$\text{SD}\ \frac{1}{N}\sum_{i=1}^{N}\left|\beta^i\right| = 0.022$$

Direct Sampling

Example: Linear regression

- To see the numerical error I generate 1,000 sets of $\{\beta^i\}_{i=1,\ldots,S}$
- Compute 1,000 of $\frac{1}{S}\sum_{i=1}^{S}\left|\beta^i\right|$ to see how variable this Monte Carlo approximation with different S

Figure 6: Distribution of $\frac{1}{S}\sum_{i=1}^{S}\left|\beta^i\right|$ over $\{\beta^i\}_{i=1,\ldots,S}$



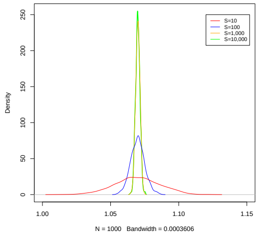
$$\text{SD}\ \frac{1}{N}\sum_{i=1}^{N}\left|\beta^i\right| = 0.0074$$

Direct Sampling

Example: Linear regression

- To see the numerical error I generate 1,000 sets of $\{\beta^i\}_{i=1,\ldots,S}$
- Compute 1,000 of $\frac{1}{S}\sum_{i=1}^{S}\left|\beta^i\right|$ to see how variable this Monte Carlo approximation with different S

Figure 7: Distribution of $\frac{1}{S}\sum_{i=1}^{S}\left|\beta^i\right|$ over $\{\beta^i\}_{i=1,\ldots,S}$



$$SD\ \frac{1}{N}\sum_{i=1}^{N}\left|\beta^i\right| = 0.0023$$

Direct Sampling
Example: Linear regression

- What do we try to capture in this exercise?
- We try to mimic the distribution of Monte Carlo approximation offered by

$$\sqrt{S}\left(\frac{1}{S}\sum_{i=1}^{N} h\left(\beta^{i}\right)-\int h(\beta)\,\pi(\beta|Y)\,d\beta\right)\rightarrow_{d} N(0,V_{\pi}) \tag{18}$$

- All variation in this Monte Carlo approximation is due to numerical ”simulation”.
- Throughout this example, we fix $Y_{1:N},\ X_{1:N}$ (not a sampling variation).

Recap

- If you know how to generate i.i.d draws from the posterior distribution of β,
- You also can posterior moments of $h(\beta)$ by simple average:

$$\int \mathrm{h}\ (\beta)\, p\,(\beta|Y)\, d\beta \;\approx\; \frac{1}{N}\sum_{i=1}^{N} h(\beta^i) \tag{19}$$

- SLLN guarantees this Monte Carlo average to the right limit:

$$\frac{1}{N}\sum_{i=1}^{N} h\left(\beta^i\right) \rightarrow_{\text{a.s}} \int h\,(\beta)\,\pi\,(\beta|Y)\,d\beta \tag{20}$$

- CLT tells you that the Monte Carlo average always has a numerical error:

$$\sqrt{S}\left(\frac{1}{S}\sum_{i=1}^{N} h\left(\beta^i\right) - \int h\,(\beta)\,\pi\,(\beta|Y)\,d\beta\right) \rightarrow_d \; N\,(0, V_\pi) \tag{21}$$

- It is important to check how good is your numerical approximation

Review & Next Steps

- Direct Sampler
- **Next Class:** Gibbs Sampler

Further Readings

- Casella, G., & Berger, R. L. (2002). Statistical inference (Vol. 2, pp. 337-472). Pacific Grove, CA: Duxbury. Chapter 7
- Hoff, P. D. (2009). A first course in Bayesian statistical methods (Vol. 580). New York: Springer.