# Lecture 2:
# The classic and the predictive paradigms
# Decision Theory

## Big Data and Machine Learning for Applied Economics
## Econ 4676

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# Agenda

1 Review

2 Shifting Paradigms

3 How to Evaluate Estimators?

4 Statistical Decision Theory

5 Linear Regression

6 Recap

# Review

## Motivation

- We discussed the examples of Google Flu and Facebook face detection
  - Take away, the success was driven by an empiric approach
  - Given data estimate a function $f(x)$ that predicts $y$ from $x$
- This is basically what we do as economists everyday so:
  - Are these algorithms merely applying standard techniques to novel and large datasets?
  - If there are fundamentally new empirical tools, how do they fit with what we know?
  - As empirical economists, how can we use them?

# Big vs Small, Classic vs Predictive

- Classical Stats (small data?)
  - Get the most of few data (Gosset)
  - Lots of structure, e.g. $X_1, X_2, ..., X_n \sim t_v$
  - Carefully curated $\rightarrow$ approximates random sampling (expensive, slow) but very good and reliable

- Big Data (the 4 V's)
  - Data **V**olume
  - Data **V**ariety
  - Data **V**elocity
  - Data **V**alue

## The Classic Paradigm

$$Y = f(X) + u \tag{1}$$

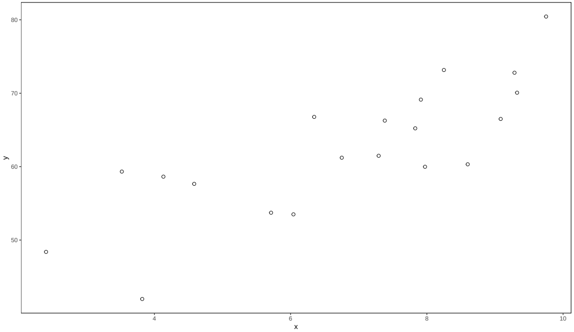
- Interest lies on inference
- "Correct" $f()$ to understand how Y is affected by X
- Model: Theory, experiment
- Hypothesis testing (std. err., tests)

# The Predictive Paradigm

$$Y = f(X) + u \tag{2}$$

- Interest on predicting $Y$
- "Correct" $f()$ to be able to predict (no inference!)
- Model?

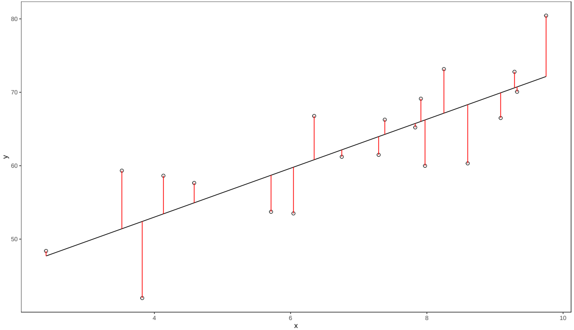
# How to choose $f(.)$




Source: simulated data, see `figures` folder for scripts

# How to choose $f(.)$

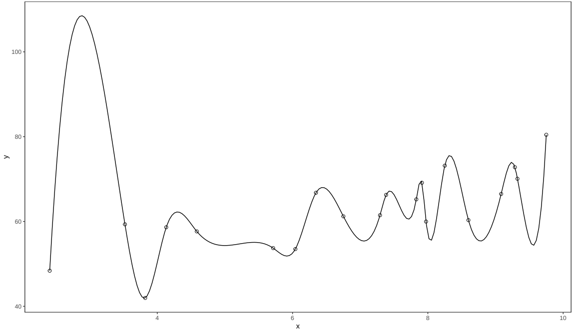
- Linear $f(X) = X\beta$




Source: simulated data, see `figures` folder for scripts

# How to choose $f(.)$

- Spline $f(X) = g(X)$, where g is a spline




Source: simulated data, see `figures` folder for scripts

# Statistical Decision Theory: A bit of theory

- We need a bit of theory to give us a framework for choosing $f$
- A decision theory approach involves an **action space** $\mathcal{A}$
- The **action space** $\mathcal{A}$ specify the possible "actions we might take"
- Some examples

Table 1: Action Spaces

| Inference | Action Space |
|---|---|
| Estimation $\theta, g(\theta)$ | $\mathcal{A} = \Theta$ |
| Prediction | $\mathcal{A} = \textit{space of } X_{n+1}$ |
| Model Selection | $\mathcal{A} = \{\textit{Model I}, \textit{Model II}, ...\}$ |
| Hyp. Testing | $\mathcal{A} = \{\textit{Reject} \mid \textit{Accept } H_0\}$ |

# Statistical Decision Theory: A bit of theory

- After the data $X = x$ is observed, where $X \sim f(X|\theta)$, $\theta \in \Theta$
- A decision is made
- The set of allowable decisions is the action space ($\mathcal{A}$)
- The loss function in an estimation problem reflects the fact that if an action $a$ is close to $\theta$,
  - then the decision $a$ is reasonable and little loss is incurred.
  - if it is far then a large loss is incurred

$$L : \mathcal{A} \rightarrow [0, \infty] \tag{3}$$

# Statistical Decision Theory: A bit of theory

## Loss Function

- If $\theta$ is real valued, two of the most common loss functions are
  - Squared Error Loss:

$$L(a,\theta) = (a-\theta)^2 \tag{4}$$

  - Absolute Error Loss:

$$L(a,\theta) = |a-\theta| \tag{5}$$

- These two are symmetric functions. However, there's no restriction. For example in hypothesis testing a "0-1" Loss is common.
- Loss is minimum if the action is correct

# Statistical Decision Theory: A bit of theory

## Risk Function

In a decision theoretic analysis, the quality of an estimator is quantified by its risk function, that is, for an estimator $\delta(x)$ of $\theta$, the risk function is

$$R(\theta,\delta) = E_\theta(L(\theta,\delta(X))) \tag{6}$$

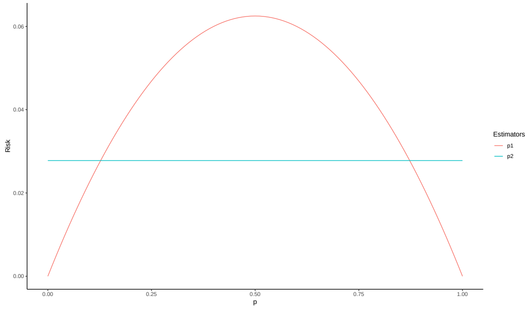
at a given $\theta$, the risk function is the average loss that will be incurred if the estimator $\delta(X)$ is used

- Since $\theta$ is unknown we would like to use an estimator that has a small value of $R(\theta,\delta)$ for all values $\theta$
- Loss is minimum if the action is correct
- If we need to compare two estimators ($\delta_1$ and $\delta_2$) then we will compare their risk functions
- If $R(\delta_1,\theta) < R(\delta_2,\theta)$ for all $\theta \in \Theta$, then $\delta_1$ is preferred because it performs better for all $\theta$

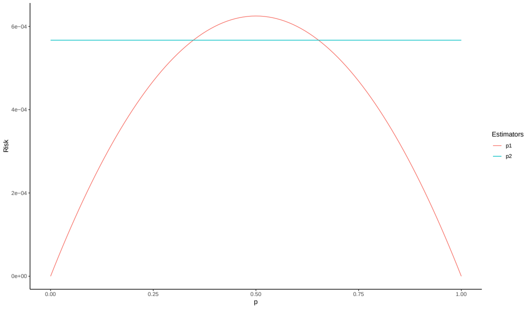
# Statistical Decision Theory: A bit of theory

## Example: Binomial Risk Function

- Let $X_1, X_2, ...X_n \sim Bernoulli(p)$
- Consider 2 estimators for $p$: $\hat{p}^1 = \frac{1}{n}\sum X_i$ and $\hat{p}^2 = \frac{\sum X_i + \sqrt{n/4}}{n+\sqrt{n}}$
- Their risks are: $R(\hat{p}^1, p) = \frac{p(1-p)}{n}$ and $R(\hat{p}^2, p) = \frac{n}{4(n+\sqrt{n})^2}$




(a) n=4




(b) n=400

# Decision Theory for prediction

## How to choose f?

- In a prediction problem we want to predict $Y$ from $f(X)$ in such a way that the loss is minimum
- Assume also that $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}$ with joint distribution $Pr(X,Y)$

$$R(Y,f(X)) = E[(Y - f(X))^2] \tag{7}$$

$$= \int (y - f(x))^2 Pr(dx, dy) \tag{8}$$

conditioning on X we have that

$$R(Y,f(X)|X) = E_X E_{Y|X}[(Y - f(X))^2 | X] \tag{9}$$

this risk is also know as the **mean squared (prediction) error** $MSE(f)$

# Decision Theory for prediction

It suffices to minimize the $MSE(f)$ point wise so

$$f(x) = argmin_m E_{Y|X}[(Y-m)^2|X=x) \tag{10}$$

Y a random variable and m a constant (predictor)

$$min_m E(Y-m)^2 = \int (y-m)^2 f(y)dy \tag{11}$$

**Result**: The best prediction of $Y$ at any point $X = x$ is the conditional mean, when best is measured using a square error loss

# Decision Theory for prediction

**Proof**

FOC

$$\int -2(y - m)f(y)dy = 0 \tag{12}$$

Dividing by -2 and reorganizing

$$m \int (y)dy = \int yf(y)dy \tag{13}$$

# Decision Theory for prediction

$$m \int (y)dy = \int yf(y)dy \tag{14}$$

$$m = E(Y|X = x) \tag{15}$$

The best prediction of $Y$ at any point $X = x$ is the conditional expectation function (CEF), when best is measured using a square error loss

- What shape does the CEF take?
- Linear
  - $(y, X)$ are jointly normal
  - When models are saturated.

# Linear Regression

- Note the following from the *Regression-CEF Theorem*
  The function $X'\beta$ provides the minimum risk linear approximation to $E(Y|X)$, that is

$$\beta = \underset{b}{argmin}\, E\left\{(E(Y|X) - X'b)^2\right\} \tag{16}$$

- Proof

$$(Y - X'b)^2 = (Y - E(Y|X)) + (E(Y|X) - X'b)^2 \tag{17}$$

$$= (Y - E(Y|X))^2 + (E(Y|X) - X'b)^2 + 2(Y - E(Y|X))(E(Y|X) - X'b) \tag{18}$$

- The CEF approximation problem then has the same solution as the population least square problems

# Linear Regression

- Regression provides the best linear predictor for the dependent variable in the same way that the CEF is the best unrestricted predictor of the dependent variable.
- The fact that Regression approximates the CEF is useful because it helps describe the essential features of statistical relationships, without necessarily trying to pin them down exactly.
- Linear regression is the "work horse" of econometrics and (supervised) machine learning.
- Very powerful in many contexts.
- Big 'payday' to study this model in detail.

# Linear Regression Model

$f(X) = X\beta$, estimating $f(.)$ boils down to estimating $\beta$

$$y = X\beta + u \tag{19}$$

where

- y is a vector $n \times 1$ with typical element $y_i$
- X is a matrix $n \times k$
  - Note that we can represent it as a column vector $\underset{n\times k}{X} = [\underset{n\times 1}{X_1} \; \underset{n\times 1}{X_2} \dots \underset{n\times 1}{X_k}]$
- $\beta$ is a vector $k \times 1$ with typical element $\beta_j$

Thus

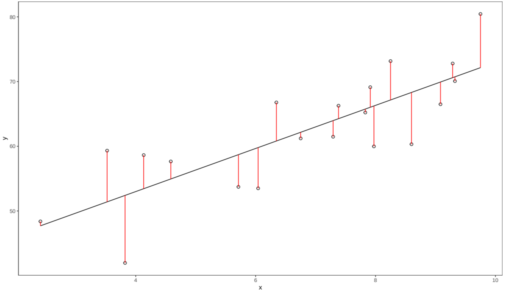
$$\begin{aligned} y_i &= X_i'\beta + u_i \\ &= \sum_{j=1}^{k} \beta_j X_{ji} + u_i \end{aligned} \tag{20}$$

# Linear Regression Model

How do we estimate $\beta$?

- Method of Moments (for HW)
- MLE (more on this later)
- OLS: minimize risk squared error loss $\rightarrow$ minimizes SSR ($e'e$)
  - where $e = Y - \hat{Y} = Y - X\hat{\beta}$
  - In the HW, you will show that min SSR same as max $R^2$

OLS solution: $\hat{\beta} = (X'X)^{-1}X'y$



# Gauss Markov Theorem

Gauss-Markov Theorem says that

$$\hat{\beta} = (X'X)^{-1}X'y \tag{21}$$

- The OLS estimator ($\hat{\beta}$) is BLUE, the more efficient than any other linear unbiased estimator,
- Efficiency in the sense that $Var(\tilde{\beta}) - Var(\hat{\beta})$ is positive semidefinite matrix.

Proof: HW. Tip: a matrix $M_{p\times p}$ is positive semi-definite iff $c'Mc \geq 0 \; \forall x \in \mathbb{R}^p$

# Gauss Markov Theorem

- Gauss Markov Theorem that says OLS is BLUE is perhaps one of the most famous results in statistics.
  - $E(\hat{\beta}) = \beta$
  - $V(\hat{\beta}) = \sigma^2(X'X)^{-1}$
- However, it is essential to note the limitations of the theorem.
  - Correctly specified with exogenous Xs,
  - The term error is homoscedastic
  - No serial correlation.
  - Nothing about the OLS estimator being the more efficient than any other estimator one can imagine.

# Prediction vs Estimation

- **Predicting well in this context $\rightarrow$ estimating well**
  - Note that the prediction of $y$ will be given by $\hat{y} = X\hat{\beta}$
  - Under Gauss-Markov framework
    - $E(\hat{y}) = X\beta$
    - $V(\hat{y}) = \sigma^2 X'(X'X)^{-1}X$
- Then if $\hat{\beta}$ is unbiased and of minimum variance,
- then $\hat{y}$ is an unbiased predictor and minimum variance, from the class of unbiased linear estimators/predictors
  - Proof: for HW similar to $\hat{\beta}$ proof

# Recap

- We start shifting paradigms
- Tools are not that different (so far)
- Decision Theory: Risk with square error loss $\rightarrow$ MSE
- OLS is a "work horse" approximates the $E[Y|X]$ quite well
- Next Class:
  - Next Class: OLS, Geometry, Properties

# Further Readings

- Angrist, J. D., & Pischke, J. S. (2008). Mostly harmless econometrics. Princeton university press.
- Casella, G., & Berger, R. L. (2002). Statistical inference (Vol. 2, pp. 337-472). Pacific Grove, CA: Duxbury.
- Tom Shaffer The 42 V's of Big Data and Data Science. https://www.kdnuggets.com/2017/04/42-vs-big-data-data-science.html