Lecture 10: Bayesian Estimation & Empirical Bayes

Big Data and Machine Learning for Applied Economics
Econ 4676

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Agenda

Bayes Theorem

$$\pi(\theta|X) = \frac{f(X|\theta)p(\theta)}{m(X)} \tag{1}$$

with $m(X)$ is the marginal distribution of X, i.e.

$$m(X) = \int f(X|\theta)p(\theta)d\theta \tag{2}$$

It is important to note that Bayes' theorem does not tell us what our beliefs should be, it tells us how they should change after seeing new information.

Motivation

- The constraints of slow mechanical computation molded classical statistics into a mathematically ingenious theory of sharply delimited scope.
- After WW2, computers allowed a more expansive and useful statistical methodology.
- However, Some revolutions start slowly. The journals of the 1950s continued to emphasize classical themes
- Change came gradually, but by the 1990s a new statistical technology, computer enabled, was firmly in place.
- Empirical Bayes methodology, has been a particularly slow developer despite an early start in the 1940s.
- The roadblock here was not so much the computational demands of the theory as a lack of appropriate data sets.

Motivation

- In Economics this revolution is starting to catch up, fueled by Big Data

4. Our methodology contributes to a recent literature that builds on empirical Bayes methods dating to Robbins (1956) by using shrinkage estimators to reduce MSE (risk) when estimating a large number of parameters. For instance, Angrist et al. (2017) combine experimental and observational estimates to improve forecasts of school value added. Our methodology differs from theirs because we have unbiased (quasi-experimental) estimates of causal effects for every area, whereas Angrist et al. have unbiased (experimental) estimates of causal effects for a subset of schools. Hull (2017) develops methods to forecast hospital quality, permitting nonlinear and heterogeneous causal effects. Abadie and Kasy (2017) show how machine learning methods can be used to reduce risk, using the fixed effect estimates constructed in this article as an application.

THE IMPACTS OF NEIGHBORHOODS ON INTERGENERATIONAL MOBILITY II: COUNTY-LEVEL ESTIMATES*

RAJ CHETTY AND NATHANIEL HENDREN

Chetty, R., & Hendren, N. QJE (2018).

Empirical Bayes

Consider the following standard Bayesian model:

$$X \sim N(\theta, 1) \tag{3}$$
$$\theta \sim N(0, \tau^2) \tag{4}$$

- Standard approach the experimenter would specify a prior value for τ^2

Empirical Bayes

- However, note that the marginal distribution of X is $N(0,\tau^2+1)$, Why?

Sketch

$$m(X) = \int f(X|\theta)p(\theta)d\theta \tag{5}$$

$$\propto \int exp\left(\frac{-1}{2}(x-\theta)^2\right) exp\left(\frac{-\theta^2}{2\tau^2}\right) d\theta$$

$$\propto \int exp\left(\frac{-1}{2}\left((x-\theta)^2 - \frac{\theta^2}{\tau^2}\right)\right) d\theta$$

Let's focus here $(x-\theta)^2 + \frac{\theta^2}{\tau^2}$

$$\approx \left(\frac{\tau^2}{(\tau^2+1)^2}(x)^2\right) + \left(\theta - \left(\frac{\tau^2}{\tau^2+1}\right)x\right)^2 \tag{6}$$

Empirical Bayes

then

$$X \sim N(0, \tau^2 + 1) \tag{7}$$

- Empirical Bayes uses this "shortcut".
- Use the data to obtain the "unknown parameters"

Robbins' Formula

Example: an insurance company is concerned about the claims each policy holder will make in the next year.

Table 1: Claims data for a European automobile insurance company

Claims	0	1	2	3	4	5	6	7
Counts	7840	1317	239	42	14	4	4	1

Robbins' Formula

- It seems that we can use Bayes formula to get next years expected number of accidents
- We suppose that x_k, the number of claims to be made in a single year by policy holder k,
- This follows a Poisson distribution with parameter θ_k
- Recall that the mean and variance are θ_k

$$Pr(x_k = x) = p_{\theta_k}(x) = \frac{e^{-\theta_k}\theta_k^x}{x!} \;\; for \;\; x = 0, 1, 2, 3, ... \tag{8}$$

Robbins' Formula

Suppose now, that we know the prior density $g(\theta)$. Then using Bayes rule we would have

$$E(\theta|x) = \int_0^\infty \theta\pi(\theta)d\theta \tag{9}$$

$$= \frac{\int_0^\infty \theta p_{\theta_k}(x)g(\theta)d\theta}{\int_0^\infty p_{\theta_k}(x)g(\theta)d\theta} \tag{10}$$

is the expected value of θ of a customer observed to make x claims in a single year. This would answer the insurance company's questions of what numbers of claims X to expect the next year from the same customer

Robbins' Formula

What happens if we don't know the prior?
Note the following:

$$E(\theta|x) = \frac{\int_0^\infty \theta[e^{-\theta}\theta^x/x!]g(\theta)d\theta}{\int_0^\infty [e^{-\theta}\theta^x/x!]g(\theta)d\theta} \tag{11}$$

$$E(\theta|x) = \frac{(x+1)\int_0^\infty [e^{-\theta}\theta^{x+1}/(x+1)!]g(\theta)d\theta}{\int_0^\infty [e^{-\theta}\theta^x/x!]g(\theta)d\theta} \tag{12}$$

$$E(\theta|x) = \frac{(x+1)m(x+1)}{m(x)} \tag{13}$$

Robbins' Formula

The obvious estimate of the marginal density $f(x)$ is the proportion of total counts in category x,

$$\hat{m}(x) = \frac{y_x}{N} \tag{14}$$

where $N = \sum_x y_x$

Table 2: Claims data for a European automobile insurance company

Claims	0	1	2	3	4	5	6	7
Counts	7840	1317	239	42	14	4	4	1
Mean	.168	.363	.527	1.33	1.43	46	1.75	.

Sabermetrics: Batting Averages

Sabermetrics: Batting Averages

- One of the most commonly used statistics in baseball is the batting average

$$\text{Batting Average} = \frac{\text{number of hits (H)}}{\text{number of at-bats (AB)}} \tag{15}$$

Today we are going to explore two additional problems and use EB:

1. You want to recruit two players: One has achieved 4 hits in 10 chances, the other 300 hits in 1000 chances.
2. Based on first few performances, can we predict what is going to be the season-long batting averages

Sabermetrics: Recruiting

- Let's see some data (Here I'm using a "clean" version of `Batting` data from the `Lahman` package)

```
require("dplyr")
require("tidyr")
require("ggplot2")
```

```
career<-readRDS("baseball.rds")
head(career)
```

```
## # A tibble: 6 x 4
##   name                H     AB average
##   <chr>           <int> <int>   <dbl>
## 1 Hank Aaron       3771 12364  0.305
## 2 Tommie Aaron      216   944  0.229
## 3 Andy Abad           2    21  0.0952
## 4 John Abadie        11    49  0.224
## 5 Ed Abbaticchio    772  3044  0.254
## 6 Fred Abbott       107   513  0.209
```

Batting Averages

- Best Batting Averages?

```
## # A tibble: 6 x 4
##   name              H    AB average
##   <chr>         <int> <int>   <dbl>
## 1 Roe Skidmore      1     1       1
## 2 Charlie Snow      1     1       1
## 3 Matt Tupman       1     1       1
## 4 Allie Watt        1     1       1
## 5 Al Wright         1     1       1
## 6 George Yantz      1     1       1
```

Batting Averages

- Worst Batting Averages?

```
## # A tibble: 6 x 4
##   name                  H    AB average
##   <chr>             <int> <int>   <dbl>
## 1 Frank Abercrombie     0     4       0
## 2 Horace Allen          0     7       0
## 3 Pete Allen            0     4       0
## 4 Walter Alston         0     1       0
## 5 Bill Andrus           0     9       0
## 6 Wyman Andrus          0     4       0
```

Sabermetrics: Recruiting

- You want a “true” measure of batting performance
- We know by history that most batting averages are between .210 and .360
- We can model

$$\text{Batting Average} \sim \textit{Binomial}(N, \theta) \tag{16}$$

- where N is the times at bat and θ is the proportion of successes

Sabermetrics: Recruiting

- We can incorporate historical data with a prior
- We use a conjugate prior for simplicity.

$$p(\theta) \sim Beta(\alpha_0, \beta_0) \tag{17}$$

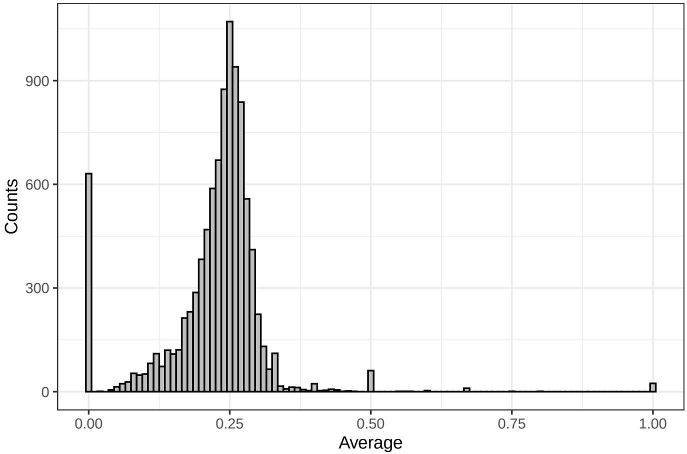
The posterior is:

$$\pi(\theta) \sim Beta(\alpha_0 + hits, \beta_0 + N - hits) \tag{18}$$

- We don't know α_0 and β_0. We could use the fact that most batting averages are between .210 and .360. Select α_0 and β_0 accordingly.
- Or we can use Empirical Bayes: estimate these parameters from the data.

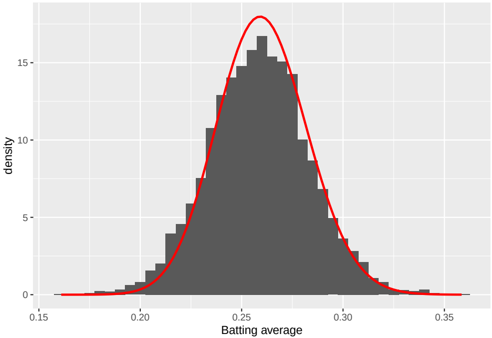
Batting Averages

Histogram of batting averages



Batting Averages

Restrict our sample to those data points that are informative (individuals that have gone at bat at least 500 times)



Batting Averages

How we find the parameters that find the red line $\rightarrow$ MLE! We know that

$$f(x_i|\alpha_0,\beta_0) = \frac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} x_i^{\alpha_0-1}(1-x_i)^{\beta_0-1} \tag{19}$$

The log likelihood

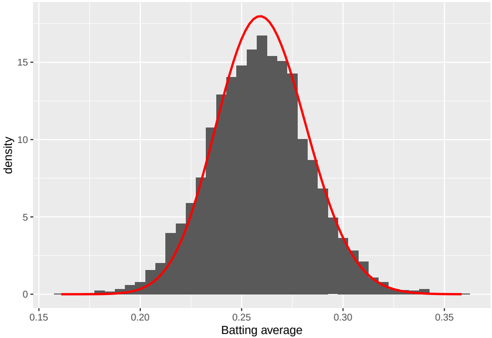
$$l(\alpha_0,\beta_0|X) = n.log(\frac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)}) + \sum_{i=1}^{n}((\alpha_0-1)log(x_i) + (\beta_0-1)log(1-x_i)) \tag{20}$$

In R

```
# log-likelihood function
ll <- function(alpha, beta) {
-sum(VGAM::dbetabinom.ab(x, total, alpha, beta, log = TRUE))
}
# maximum likelihood estimation
m <- mle(ll, start = list(alpha = 1, beta = 10),
method = "L-BFGS-B", lower = c(0.0001, .1))
ab <- coef(m)
```

Batting Averages

```
alpha0 <- ab[1]
101.7319
beta0 <- ab[2]
289.046
```



Batting Averages

We can use the estimated average based on the posterior mean

$$E(\theta|X) = \frac{\alpha_0 + hits}{\alpha_0 + \beta_0 + N} \tag{21}$$

- Now we can ask again: who are the best batters by this improved estimate?

Sabermetrics: Recruiting

- Now we can ask again: who are the best batters by this improved estimate?

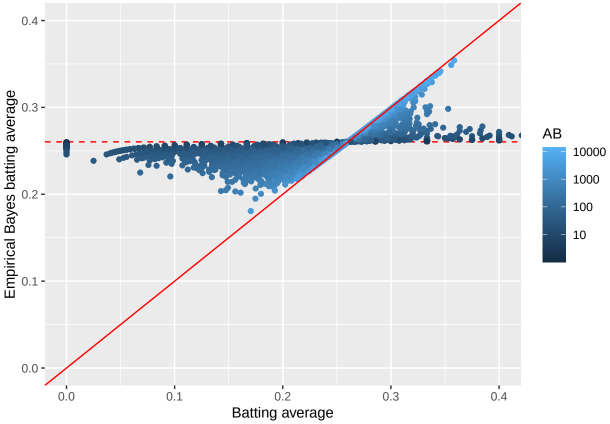
```
## # A tibble: 5 x 5
##   name                    H    AB average eb_estimate
##   <chr>               <int> <int>   <dbl>       <dbl>
## 1 Rogers Hornsby       2930  8173   0.358       0.354
## 2 Shoeless Joe Jackson 1772  4981   0.356       0.349
## 3 Ed Delahanty         2597  7510   0.346       0.342
## 4 Billy Hamilton       2164  6283   0.344       0.339
## 5 Willie Keeler        2932  8591   0.341       0.338
```

- Who are the *worst* batters?

```
## # A tibble: 5 x 5
##   name               H    AB average eb_estimate
##   <chr>          <int> <int>   <dbl>       <dbl>
## 1 Bill Bergen      516  3028   0.170       0.181
## 2 Ray Oyler        221  1265   0.175       0.195
## 3 Henry Easterday  203  1129   0.180       0.201
## 4 John Vukovich     90   559   0.161       0.202
## 5 George Baker      74   474   0.156       0.203
```

Sabermetrics: Recruiting

We can see how EB changed all of the batting average estimates:



Sabermetrics: Predicting Batting Averages

- Now supposed you want to know the end of season final batting average of players, after observing them their 45 first times at bat.

Player	Observed	Final
1	0.395	0.346
2	0.355	0.279
3	0.313	0.276
4	0.291	0.266
5	0.247	0.271
6	0.224	0.266
7	0.175	0.318

Sabermetrics: Predicting Batting Averages

- Recall that we can think each time at bat can be thought as a binomial trial, with θ the probability of success equal to the player's true batting average.
- With 45 trials, we can "reasonably" use a Normal Approximation.

$$X_i \sim N(\theta_i, \sigma^2) \tag{22}$$

where

- θ_i is the true batting average for player i
- σ^2 is the known variance that equals $(0.0659)^2$

We are going to use also a normal prior

$$\theta_i \sim N(\mu, \tau^2) \tag{23}$$

Sabermetrics: Predicting Batting Averages

With this model the posterior mean for θ_i is $E(\theta_i|X_i)$

$$E(\theta_i|X_i) = \frac{\sigma^2}{\sigma^2+\tau^2}\mu + \frac{\tau^2}{\sigma^2+\tau^2}X_i \tag{24}$$

Note that the marginal of X_i

$$m(X_i) \sim N(\mu, \sigma^2+\tau^2) \;\; i = 1, \dots, n \tag{25}$$

with these we can construct estimates of $E(\theta_i|X_i)$, note that

$$E(\bar{X}) = \mu \tag{26}$$

$$E\left[\frac{(n-3)\sigma^2}{\sum(X_i-\bar{X})^2}\right] = \frac{\sigma^2}{\sigma^2+\tau^2} \tag{27}$$

Sabermetrics: Predicting Batting Averages

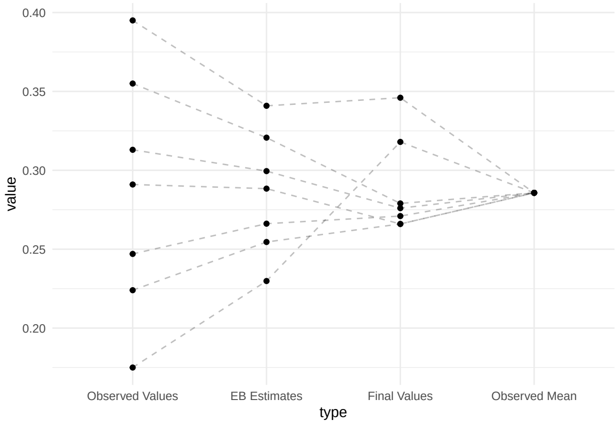
The empirical Bayes estimator of θ_i is then

$$\delta(X_i) = \left[\frac{(n-3)\sigma^2}{\sum(X_i-\bar{X})^2}\right]\bar{X} + \left[1 - \frac{(n-3)\sigma^2}{\sum(X_i-\bar{X})^2}\right]X_i \tag{28}$$

Player	Observed	Final	Empirical Bayes
1	0.395	0.346	0.341
2	0.355	0.279	0.321
3	0.313	0.276	0.299
4	0.291	0.266	0.288
5	0.247	0.271	0.266
6	0.224	0.266	0.255
7	0.175	0.318	0.230

- RMSE Observed 6.861903
- RMSE EB 3.918203

Sabermetrics: Predicting Batting Averages



Review & Next Steps

- Recap Bayesian
- Empirical Bayes Examples
- **Next Week: Spatial Econometrics**
- **Next Week:** Problem set 2.

Further Readings

- Casella, G., & Berger, R. L. (2002). Statistical inference (Vol. 2, pp. 337-472). Pacific Grove, CA: Duxbury. Chapter 7
- Casella, G. (1985). An introduction to empirical Bayes data analysis. The American Statistician, 39(2), 83-87.
- Chetty, R., & Hendren, N. (2018). The impacts of neighborhoods on intergenerational mobility II: County-level estimates. The Quarterly Journal of Economics, 133(3), 1163-1228.
- Efron, B., & Hastie, T. (2016). Computer age statistical inference (Vol. 5). Cambridge University Press. Chapter 6
- Robinson, D. (2017). Introduction to Empirical Bayes: Examples from Baseball Statistics. 2017.
- Gu, J., & Koenker, R. (2017). Empirical Bayesball remixed: Empirical Bayes methods for longitudinal data. Journal of Applied Econometrics, 32(3), 575-599.