EKF for Hexapod State Estimation
by Hassan Khalili
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1 Introduction

An important application of legged robots has been highlighted in extreme environments for exploring and mapping an unknown terrain. Corin, a hexapod robot, has been designed specifically to be able to perform advanced motions which allow it to navigate through these terrains [??]. These motions include wall walking, for passing through narrow pathways, and chimney walking, for traversing over large holes or obstacles on the ground. Implementing these motions requires accurate knowledge of the position and orientation of the robot's body. State-of-the-art state estimators for legged robots make use of an extended Kalman filter (EKF) to fuse measurements from an inertial measurement unit (IMU) with leg kinematic measurements.

2 Related Work

3 EKF Overview

In its original form, an extended Kalman filter (EKF) involves combining the state predictions from the non-linear system dynamics model with external measurements. This is a direct EKF, based on total state space formulation, where the filter states comprise of total states such as position and orientation [1]. However, in inertial navigation systems (INSs), indirect (error state) Kalman filters are the preferred method.

With an indirect Kalman filter formulation [2, 1], the high-frequency large-signal motions of the vehicle or robot are estimated outside the Kalman filter, based on the integration of the non-linear system dynamics. These estimations form the nominal system state $\hat{\mathbf{x}}$ which does not take into account system noise and modelling errors.

The indirect EKF is then used to estimate the corresponding system errors through a model of error propagation dynamics. The filter state is hence termed the error state $\delta\mathbf{x}$ which consists of small-signal variables which change at a slower rate and whose dynamics can be linearised with smaller errors than a direct EKF.

The error state is corrected by the EKF based on the external measurements, and this estimate is subsequently used to correct the nominal state. The true system state $\mathbf{x}$ can therefore be thought of as a composition of the nominal state and the error state.

The advantages of using an indirect EKF include ([2, 1]):

- minimal orientation error-state;
- operation near the origin and away from singularities;
- small error-state values and hence having negligible second-order products (more accurate linearisation);
- slow error-state dynamics requiring slower sampling rates;
- the possibility of operating the state-estimation when the EKF fail.

3.1 EKF States

The main states that are of benefit to the robot and hence have to be estimated are the position vector $\mathbf{r}$ and orientation quaternion q of the hexapod body. This is the midpoint of the main body of the robot. Additionally, for us to obtain position from acceleration measurements, we need to keep track of the robot velocity $\mathbf{v}$, as the acceleration has to be integrated twice.

In order to use the kinematic measurements to update the state estimates, there has to be states that are initially estimated using the dynamics model then corrected using the measurements. These are chosen to be the foot positions $\mathbf{p}_i$ where i is the foot index.

Finally, IMUs are often modelled with bias terms for both the accelerometer and the gyroscope measurements, given as $\mathbf{a}_b$ and $\boldsymbol{\omega}_b$, respectively.

Using the pre-superscript notation to denote the frame in which the vector is defined, the full-state vector can be written as below, where ${}^{I}<>$ corresponds to the inertial (world) frame, and ${}^{B}<>$ refers to the body (robot) frame:

$$\mathbf{x} = \begin{bmatrix} {}^{I}\mathbf{r} & {}^{I}\mathbf{v} & {}^{B}_{I}q & {}^{I}\mathbf{p}_0 & \dots & {}^{I}\mathbf{p}_5 & {}^{B}\mathbf{a}_b & {}^{B}\boldsymbol{\omega}_b \end{bmatrix}^T \tag{1}$$

The state vector $\mathbf{x}$ represents the true states of the system which, in the context of an indirect Kalman filter, are a composition of nominal states and error states [2]. The hat symbol $\hat{<>}$ denotes a nominal state, which is the best available estimate of the corresponding true state, and the prefix δ represents the error states. For most of the states the composition is a simple addition as shown below:

$$ {}^{I}\mathbf{r} = {}^{I}\hat{\mathbf{r}} + {}^{I}\delta\mathbf{r} \tag{2} $$
$$ {}^{I}\mathbf{v} = {}^{I}\hat{\mathbf{v}} + {}^{I}\delta\mathbf{v} \tag{3} $$
$$ {}^{I}\mathbf{p}_i = {}^{I}\hat{\mathbf{p}}_i + {}^{I}\delta\mathbf{p}_i \tag{4} $$
$$ {}^{B}\mathbf{a}_b = {}^{B}\hat{\mathbf{a}}_b + {}^{B}\delta\mathbf{a}_b \tag{5} $$
$$ {}^{B}\boldsymbol{\omega}_b = {}^{B}\hat{\boldsymbol{\omega}}_b + {}^{B}\delta\boldsymbol{\omega}_b \tag{6} $$

However, the rotation error is represented minimally using a three-dimensional angular error vector $\delta\boldsymbol{\theta}$. This avoids the over-parameterisation of having a four-dimensional quaternion error describing a three degree-of-freedom (DOF) orientation state. The error rotation quaternion and matrix, δq and $\delta\mathbf{C}$, are both a function of this angular error vector.

Using the Hamilton convention, where ${}^{I}_{B}\mathbf{C}$ is the true rotation matrix that maps a point from the body frame to the inertial frame [2]:

$$ {}^{I}_{B}\mathbf{C} = {}^{I}_{\hat{B}}\hat{\mathbf{C}}\ {}^{\hat{B}}_{B}\delta\mathbf{C} \tag{7} $$

where $\hat{\mathbf{C}}$ is the nominal rotation matrix and $\delta\mathbf{C}$ is the error rotation matrix. Similarly, the relationship between the quaternion variables is:

$$ {}^{I}_{B}q = {}^{I}_{\hat{B}}\hat{q} \otimes {}^{\hat{B}}_{B}\delta q \tag{8} $$

However, in a JPL convention, since the rotation operation relates the global frame to the local frame with the angular error still defined in the local frame, the error is multiplied as follows:

$$ {}^{B}_{I}\mathbf{C} = {}^{B}_{\hat{B}}\delta\mathbf{C}\ {}^{\hat{B}}_{I}\hat{\mathbf{C}} \tag{9} $$
$$ {}^{B}_{I}q = {}^{B}_{\hat{B}}\delta q \otimes {}^{\hat{B}}_{I}\hat{q} \tag{10} $$

3.2 IMU and Foot Position Models

The robot dynamic equations could be used to calculate the body position, velocity, and orientation, but that would be an arduous task. Alternatively, the IMU sensor is used as a simple method to measure the robot body kinematics. The two sensors of interest in an IMU are the accelerometer and the gyroscope. A common model for describing the angular velocity output from a gyroscope, $\boldsymbol{\omega}_m$, is as a combination of the true angular velocity ${}^B\boldsymbol{\omega}$, a bias vector inherent to the sensor ${}^B\boldsymbol{\omega}_b$, and additive white Gaussian noise (AWGN) ${}^B\mathbf{n}_\omega$ with a covariance $\mathbf{Q}_\omega$. The bias term is often modelled in IMUs as a random walk described by Brownian motion, where the derivative of the bias is an AWGN process ($\boldsymbol{w}_\omega$) with a covariance $\mathbf{Q}_{w\omega}$.

$$
{}^B\boldsymbol{\omega}_m = {}^B\boldsymbol{\omega} + {}^B\boldsymbol{\omega}_b + {}^B\mathbf{n}_\omega \tag{11}
$$

$$
{}^B\dot{\boldsymbol{\omega}}_b = {}^B\boldsymbol{w}_\omega \tag{12}
$$

Similarly, the accelerometer model is given by:

$$
{}^B\mathbf{f}_m = {}^B\mathbf{f} + {}^B\mathbf{a}_b + {}^B\mathbf{n}_a \tag{13}
$$

$$
{}^B\dot{\mathbf{a}}_b = {}^B\boldsymbol{w}_a \tag{14}
$$

where $\mathbf{f}_m$ is the proper linear acceleration (acceleration relative to free-fall) as measured by the accelerometer, $\mathbf{a}_n$ is the AWGN with a covariance $\mathbf{Q}_a$ affecting the accelerometer output, and $\boldsymbol{w}_a$ is the AWGN, with a covariance $\mathbf{Q}_{wa}$, used to model the bias random walk. The relationship between the true proper acceleration $\mathbf{f}$ and the absolute acceleration $\mathbf{a}$ is simply:

$$
{}^B\mathbf{f} = {}^B_I\mathbf{C}({}^I\mathbf{a} - \mathbf{g}) \tag{15}
$$

where the gravity vector $\mathbf{g} = \begin{bmatrix}0 & 0 & -9.8\end{bmatrix}^T$.

The position of a foot is important only when the foot is in contact with the ground, i.e. during the support phase. This is when the foot positions, given in the body frame, can be predicted based on the knowledge of the position and orientation of the robot body, as can be seen in Figure 1.

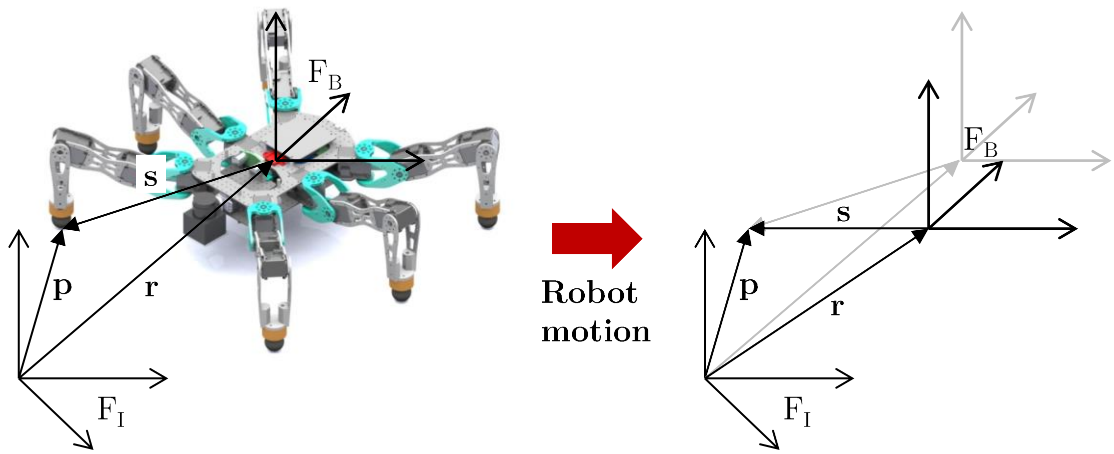

Figure 1: Corin foot position position model state vectors.

When the foot is on the ground, its position with respect to the inertial frame, $\mathbf{p}_i$, is assumed to be constant. As the robot body moves, $\mathbf{r}$ changes but can be predicted using the IMU. Knowing the orientation of the body, the foot positions with respect to the body frame, ${}^B\mathbf{s}_i$, can be predicted and compared against forward kinematic measurements which use joint encoder readings. This is expressed mathematically as:

$$ {}^B\mathbf{s}_i = {}^B_I\mathbf{C}({}^I\mathbf{p}_i - {}^I\mathbf{r}) \tag{16} $$

The robot feet are susceptible to external interference during the support phase, such as slippage, and hence their positions can be affected. This behaviour can also be modelled using a random walk as follows:

$$ {}^B\dot{\mathbf{p}}_i = {}^B\mathbf{n}_{p,i} \tag{17} $$
$$ {}^I\dot{\mathbf{p}}_i = {}^I_B\mathbf{C}\,{}^B\mathbf{n}_{p,i} \tag{18} $$
$$ = {}^B_I\mathbf{C}^T\,{}^B\mathbf{n}_{p,i} \tag{19} $$

where $\mathbf{n}_{p,i}$ is an AWGN process with covariance $\mathbf{Q}_{p,i}$. When a foot is not in contact with the ground, the covariance is set to a very large number and it will not be used in the estimation algorithm.

3.3 Leg Kinematics Model

Leg kinematic measurements use the three joint angles on each leg to obtain the position of each of the six hexapod feet with respect to the robot base frame, as defined in the same frame. For a vector of joint angles of leg i, the measurement joint angle vector $\boldsymbol{\alpha}_m$ is modelled as:

$$\boldsymbol{\alpha}_m = \boldsymbol{\alpha} + \mathbf{n}_\alpha \tag{20}$$

$\boldsymbol{\alpha}$ is a vector of the true joint angles, and $\mathbf{n}_\alpha$ is a vector of discrete AWGN terms with a covariance matrix $\mathbf{R}_\alpha$.

Applying the forward leg kinematic equations $\mathrm{FK}_i(<>)$ for the i^{th} leg to the true joint angles would yield:

$$\mathrm{FK}_i(\boldsymbol{\alpha}) = {}^B\mathbf{s}_i + {}^B\mathbf{n}_{\text{fk}} \tag{21}$$

where ${}^B\mathbf{s}_i$ is the true foot position with respect to the robot base frame, and ${}^B\mathbf{n}_{\text{fk}}$ is an AWGN vector with covariance $\mathbf{R}_{\text{fk}}$ which represents noise in the leg kinematic calculations. The measured foot positions ${}^B\mathbf{s}_{m,i}$, however, are performed with the measured joint angles:

$$^B\mathbf{s}_{m,i} = \mathrm{FK}_i(\boldsymbol{\alpha}_m) \tag{22}$$

which can be linearised using the Taylor series expansion which results in the approximation:

$$\begin{aligned}
{}^B\mathbf{s}_{m,i} &\approx \mathrm{FK}_i(\boldsymbol{\alpha}) + \mathbf{J}_{\text{fk},i}\mathbf{n}_\alpha && (23)\\
&\approx {}^B\mathbf{s}_i + {}^B\mathbf{n}_{\text{fk}} + \mathbf{J}_{\text{fk},i}\mathbf{n}_\alpha && (24)\\
&\approx {}^B\mathbf{s}_i + \mathbf{n}_s && (25)
\end{aligned}$$

where

$$\mathbf{J}_{\text{fk},i} = \frac{\partial \mathbf{fk_i}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \tag{26}$$

is the Jacobian of the leg kinematics for the i^{th} leg, and $\mathbf{n}_s$ is the combined measurement noise with covariance $\mathbf{R}_s$:

$$\mathbf{R}_s = \mathbf{R}_{\text{fk}} + \mathbf{J}_{\text{fk},i}\mathbf{R}_\alpha\mathbf{J}_{\text{fk},i}^T \tag{27}$$

4 EKF Description

4.1 EKF True and Nominal Kinematics

The linear kinematics are straight-forward, where the derivative of the position vector is the velocity vector, and derivative of the velocity vector is the acceleration vector. However, for the orientation, quaternion kinematics are used. In the Hamilton convention, the rate of change of the quaternion defining the orientation of a local frame with respect to the global frame is given by [3]:

$$ {}^{I}_{B}\dot{q} = \frac{1}{2}\, {}^{I}\omega \otimes {}^{I}_{B}q \tag{28} $$

where $\omega = [0 \quad \boldsymbol{\omega}^T]^T$ is the quaternion representing the angular velocity vector. This angular velocity is that of the body (local) frame with respect to the inertial (global or world) frame and it is defined in the inertial frame. However, since the angular velocity is often known directly in the local frame, for example in an IMU, the following conversion can be made:

$$ {}^{B}\omega = {}^{B}_{I}q \otimes {}^{I}\omega \otimes {}^{B}_{I}q^{*} \tag{29} $$

$$ = {}^{I}_{B}q^{*} \otimes {}^{I}\omega \otimes {}^{I}_{B}q \tag{30} $$

and the result substituted in equation (28) in order to obtain the quaternion derivative in terms of the locally-defined angular velocity:

$$ {}^{I}_{B}\dot{q} = \frac{1}{2}\, {}^{I}_{B}q \otimes {}^{B}\omega \tag{31} $$

In the JPL convention, the orientation of a local frame is given by a quaternion defined in the reverse direction, i.e. a vector is rotated from the global frame to the local frame. The equations can then be easily formulated as:

$$ {}^{B}_{I}\dot{q} = \frac{1}{2}\, {}^{B}\omega \otimes {}^{B}_{I}q \tag{32} $$

$$ = \frac{1}{2}\, {}^{B}_{I}q \otimes {}^{I}\omega \tag{33} $$

The derivations can be found in [4].

This concludes the derivations of the true system kinematics which are shown below [5]:

$$
\begin{aligned}
{}^{I}\dot{\mathbf{r}} &= {}^{I}\mathbf{v} && (34)\\
{}^{I}\dot{\mathbf{v}} &= {}^{I}\mathbf{a} = {}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\mathbf{f} + \mathbf{g} && (35)\\
&= {}^{B}_{I}\mathbf{C}^{T}({}^{B}\mathbf{f}_{m} - {}^{B}\mathbf{a}_{b} - \mathbf{a}_{n}) + \mathbf{g} && (36)\\
{}^{B}_{I}\dot{q} &= \frac{1}{2}\,{}^{B}\omega \otimes {}^{B}_{I}q && (37)\\
&= \frac{1}{2}({}^{B}\omega_{m} - {}^{B}\omega_{b} - \omega_{n}) \otimes {}^{B}_{I}q && (38)\\
{}^{B}\dot{\mathbf{a}}_{b} &= {}^{B}\boldsymbol{w}_{a} && (39)\\
{}^{B}\dot{\boldsymbol{\omega}}_{b} &= {}^{B}\boldsymbol{w}_{\omega} && (40)\\
{}^{I}\dot{\mathbf{p}}_{i} &= {}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\mathbf{n}_{p,i} && (41)
\end{aligned}
$$

It would follow that by removing the noise processes, we obtain the nominal-state kinematics [2]:

$$
\begin{aligned}
{}^{I}\dot{\hat{\mathbf{r}}} &= {}^{I}\hat{\mathbf{v}} && (42)\\
{}^{I}\dot{\hat{\mathbf{v}}} &= {}^{B}_{I}\hat{\mathbf{C}}^{T}({}^{B}\mathbf{f}_{m} - {}^{B}\hat{\mathbf{a}}_{b}) + \mathbf{g} && (43)\\
{}^{B}_{I}\dot{\hat{q}} &= \frac{1}{2}({}^{B}\omega_{m} - {}^{B}\hat{\omega}_{b}) \otimes {}^{B}_{I}\hat{q} && (44)\\
{}^{B}\dot{\hat{\mathbf{a}}}_{b} &= 0_{3\times1} && (45)\\
{}^{B}\dot{\hat{\boldsymbol{\omega}}}_{b} &= 0_{3\times1} && (46)\\
{}^{I}\dot{\hat{\mathbf{p}}}_{i} &= 0_{3\times1} && (47)
\end{aligned}
$$

4.2 EKF Error-state Kinematics

4.2.1 The Trivial Derivations

The position error kinematics are straight-forward to obtain as shown below:

$$
\begin{aligned}
{}^{I}\mathbf{r} &= {}^{I}\hat{\mathbf{r}} + {}^{I}\delta\mathbf{r} && (48)\\
{}^{I}\dot{\mathbf{r}} &= {}^{I}\dot{\hat{\mathbf{r}}} + {}^{I}\dot{\delta\mathbf{r}} && (49)\\
{}^{I}\dot{\delta\mathbf{r}} &= {}^{I}\dot{\mathbf{r}} - {}^{I}\dot{\hat{\mathbf{r}}} && (50)\\
&= {}^{I}\mathbf{v} - {}^{I}\hat{\mathbf{v}} = {}^{I}\delta\mathbf{v} && (51)
\end{aligned}
$$

The variables defined by a random walk, namely the IMU biases and the foot positions, will have similar error-state equations. Since their nominal states are defined to be constant (zero derivative), the error-state derivates are equal to the true-state derivatives:

$$^{B}\dot{\delta\mathbf{a}}_b = {}^{B}\dot{\mathbf{a}}_b = {}^{B}\boldsymbol{w}_a \tag{52}$$

$$^{B}\dot{\delta\boldsymbol{\omega}}_b = {}^{B}\dot{\boldsymbol{\omega}}_b = {}^{B}\boldsymbol{w}_\omega \tag{53}$$

$$^{I}\dot{\delta\mathbf{p}}_i = {}^{I}\dot{\mathbf{p}}_i = {}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\mathbf{n}_{p,i} \tag{54}$$

4.2.2 Rotation Notation Conversions

For the subsequent derivations, rotations have to be represented in terms of the error angles, as the error-state kinematics are defined with the error angle vector as a state. This subsection attempts to summarise the method.

The rotation vector (Euler vector) $\delta\boldsymbol{\theta}$, representing the angular errors, can be written in angle-axis format [6]:

$$\delta\boldsymbol{\theta} = \delta\theta\ \hat{\mathbf{e}} \tag{55}$$

$$\delta\theta = ||\delta\boldsymbol{\theta}|| \tag{56}$$

$$\hat{\mathbf{e}} = \frac{\delta\boldsymbol{\theta}}{\delta\theta} \tag{57}$$

where $\delta\theta$ is the magnitude of rotation about $\hat{\mathbf{e}}$, which is a unit vector pointing along the rotation vector.

For very small rotations, the relationship between δq and $\delta\boldsymbol{\theta}$ is simplified using the small angle approximation [4]:

$$\delta q = \begin{bmatrix} \delta q_r \\ \delta\mathbf{q} \end{bmatrix} \tag{58}$$

$$= \begin{bmatrix} \cos(\delta\theta/2) \\ \hat{\mathbf{e}}\sin(\delta\theta/2) \end{bmatrix} \tag{59}$$

$$\approx \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\theta} \end{bmatrix} \tag{60}$$

In the Hamilton convention, a rotation matrix can be defined in terms of its corresponding quaternion as follows [2]:

$$^{I}_{B}\mathbf{C} = (q_r^2 - \mathbf{q}^T\mathbf{q})\mathbf{I} + 2\mathbf{q}\mathbf{q}^T + 2q_r[\mathbf{q}]_\times \tag{61}$$

where q_r and $\mathbf{q}$ are the scalar and vector parts of the quaternion ${}^{I}_{B}q$, respectively. Using the approximation given in equation 60 and ignoring the higher-order terms, the small-angle rotation matrix can be written as:

$$ {}^{\hat{B}}_{B}\delta\mathbf{C} \approx \mathbf{I} + [\delta\boldsymbol{\theta}]_{\times} \tag{62} $$

resulting in the following error composition equation:

$$ {}^{I}_{B}\mathbf{C} = {}^{I}_{\hat{B}}\hat{\mathbf{C}}\ {}^{\hat{B}}_{B}\delta\mathbf{C} = {}^{I}_{\hat{B}}\hat{\mathbf{C}}(\mathbf{I} + [\delta\boldsymbol{\theta}]_{\times}) \tag{63} $$

In the JPL convention, a rotation matrix is defined as follows [4]:

$$ {}^{B}_{I}\mathbf{C} = (2q_r^2 - 1)\mathbf{I}_{3\times3} - 2q_r[\mathbf{q}]_{\times} + 2\mathbf{q}\mathbf{q}^T \tag{64} $$

where q_r and $\mathbf{q}$ are part of the quaternion ${}^{B}_{I}q$. This leads to a different expression for the small-angle rotation matrix and hence error composition:

$$ {}^{B}_{\hat{B}}\delta\mathbf{C} \approx \mathbf{I} - [\delta\boldsymbol{\theta}]_{\times} \tag{65} $$

$$ {}^{B}_{I}\mathbf{C} = {}^{B}_{\hat{B}}\delta\mathbf{C}\ {}^{\hat{B}}_{I}\hat{\mathbf{C}} \approx (\mathbf{I} - [\delta\boldsymbol{\theta}]_{\times})\, {}^{\hat{B}}_{I}\hat{\mathbf{C}} \tag{66} $$

By taking the transpose of this equation, we arrive at the error composition given in the Hamilton convention, as ${}^{B}_{I}\mathbf{C}^T = {}^{I}_{B}\mathbf{C}$:

$$ {}^{B}_{I}\mathbf{C}^T = {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T\ {}^{B}_{\hat{B}}\delta\mathbf{C}^T \approx {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T(\mathbf{I} + [\delta\boldsymbol{\theta}]_{\times}) \tag{67} $$

4.2.3 Error-velocity Kinematics

The derivation of the velocity error kinematics is presented in [2] and is summarised here based on the JPL convention:

$$ {}^{I}\dot{\hat{\mathbf{v}}} + {}^{I}\dot{\delta\mathbf{v}} = {}^{I}\dot{\mathbf{v}} = {}^{B}_{I}\mathbf{C}^T({}^{B}\mathbf{f}_m - {}^{B}\mathbf{a}_b - {}^{B}\mathbf{n}_a) + \mathbf{g} \tag{68} $$

$$ {}^{B}_{I}\hat{\mathbf{C}}^T({}^{B}\mathbf{f}_m - {}^{B}\hat{\mathbf{a}}_b) + \mathbf{g} + {}^{I}\dot{\delta\mathbf{v}} = {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T(\mathbf{I} + [\delta\boldsymbol{\theta}]_{\times})({}^{B}\mathbf{f}_m - {}^{B}\hat{\mathbf{a}}_b - {}^{B}\delta\mathbf{a}_b - {}^{B}\mathbf{n}_a) + \mathbf{g} \tag{69} $$

$$ {}^{I}\dot{\delta\mathbf{v}} = -\,{}^{\hat{B}}_{I}\hat{\mathbf{C}}^T({}^{B}\delta\mathbf{a}_b + {}^{B}\mathbf{n}_a) + {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T[\delta\boldsymbol{\theta}]_{\times}\, {}^{B}\mathbf{f} \tag{70} $$

$$ {}^{I}\dot{\delta\mathbf{v}} = -\,{}^{\hat{B}}_{I}\hat{\mathbf{C}}^T\, {}^{B}\delta\mathbf{a}_b - {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T\, {}^{B}\mathbf{n}_a - {}^{\hat{B}}_{I}\hat{\mathbf{C}}^T[{}^{B}\mathbf{f}]_{\times}\delta\boldsymbol{\theta} \tag{71} $$

The equation can also be expressed in terms of the estimated (nominal) proper acceleration ${}^{B}\hat{\mathbf{f}}$ which is evaluated by removing the best estimate of the bias from the accelerometer output value:

$$ {}^B\mathbf{f}_m = {}^B\mathbf{f} + {}^B\hat{\mathbf{a}}_b + {}^B\delta\mathbf{a}_b + {}^B\mathbf{n}_a \tag{72} $$

$$ {}^B\hat{\mathbf{f}} = {}^B\mathbf{f}_m - {}^B\hat{\mathbf{a}}_b = {}^B\mathbf{f} + {}^B\delta\mathbf{a}_b + {}^B\mathbf{n}_a \tag{73} $$

The corresponding velocity error kinematics are listed below:

$$ {}^I\dot{\delta\mathbf{v}} = -{}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\delta\mathbf{a}_b - {}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\mathbf{n}_a - {}^{\hat{B}}_I\hat{\mathbf{C}}^T[{}^B\mathbf{f}]_\times\delta\boldsymbol{\theta} \tag{74} $$

$$ {}^I\dot{\delta\mathbf{v}} = -{}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\delta\mathbf{a}_b - {}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\mathbf{n}_a - {}^{\hat{B}}_I\hat{\mathbf{C}}^T[{}^B\hat{\mathbf{f}} - {}^B\delta\mathbf{a}_b - {}^B\mathbf{n}_a]_\times\delta\boldsymbol{\theta} \tag{75} $$

$$ {}^I\dot{\delta\mathbf{v}} \approx -{}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\delta\mathbf{a}_b - {}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\mathbf{n}_a - {}^{\hat{B}}_I\hat{\mathbf{C}}^T[{}^B\hat{\mathbf{f}}]_\times\delta\boldsymbol{\theta} \tag{76} $$

The kinematics can also be written in terms of the true-state rotation matrix. The corresponding derivations are included in Appendix A.

4.2.4 Rotation Error Kinematics

The derivation of the rotation error kinematics can be found in [2] for the Hamilton convention, and in [4] for the JPL convention. This is reproduced below, where, for simplicity, the pre-superscripts defining the frames of reference are dropped after initial use:

$$ \frac{d}{dt}({}^B_{\hat{B}}\delta q \otimes {}^{\hat{B}}_I\hat{q}) = {}^B_I\dot{q} = \frac{1}{2}\,{}^B\omega \otimes {}^B_I q \tag{77} $$

$$ (\delta q \dot{\otimes} \hat{q}) = \dot{q} = \frac{1}{2}\omega \otimes q \tag{78} $$

$$ \dot{\delta q} \otimes \hat{q} + \delta q \otimes \dot{\hat{q}} = \frac{1}{2}\omega \otimes \delta q \otimes \hat{q} \tag{79} $$

$$ \dot{\delta q} \otimes \hat{q} + \delta q \otimes \frac{1}{2}\hat{\omega} \otimes \hat{q} = \frac{1}{2}\omega \otimes \delta q \otimes \hat{q} \tag{80} $$

$$ \dot{\delta q} = \frac{1}{2}(\omega \otimes \delta q - \delta q \otimes \hat{\omega}) \tag{81} $$

$$ 2\dot{\delta q} = \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}]_\times \end{bmatrix} \delta q - \begin{bmatrix} 0 & -\hat{\boldsymbol{\omega}}^T \\ \hat{\boldsymbol{\omega}} & [\hat{\boldsymbol{\omega}}]_\times \end{bmatrix} \delta q \tag{82} $$

The equation is then simplified by writing the true angular velocity in terms of the estimated (nominal) angular velocity ${}^B\hat{\boldsymbol{\omega}}$:

$$ {}^B\boldsymbol{\omega}_m = {}^B\boldsymbol{\omega} + {}^B\hat{\boldsymbol{\omega}}_b + {}^B\delta\boldsymbol{\omega}_b + {}^B\mathbf{n}_\omega \tag{83} $$

$$ {}^B\hat{\boldsymbol{\omega}} = {}^B\boldsymbol{\omega}_m - {}^B\hat{\boldsymbol{\omega}}_b = {}^B\boldsymbol{\omega} + {}^B\delta\boldsymbol{\omega}_b + {}^B\mathbf{n}_\omega \tag{84} $$

which leads to the expression below, following the elimination of higher-order terms:

$$2\dot{\delta q} = \begin{bmatrix} 0 & -[\hat{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}_b - \mathbf{n}_\omega - \hat{\boldsymbol{\omega}}]^T \\ \hat{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}_b - \mathbf{n}_\omega - \hat{\boldsymbol{\omega}} & -[\hat{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}_b - \mathbf{n}_\omega + \hat{\boldsymbol{\omega}}]_\times \end{bmatrix} \delta q \tag{85}$$

$$\begin{bmatrix} 1 \\ \dot{\delta\boldsymbol{\theta}} \end{bmatrix} \approx \begin{bmatrix} 0 & [\delta\boldsymbol{\omega}_b + \mathbf{n}_\omega]^T \\ -[\delta\boldsymbol{\omega}_b + \mathbf{n}_\omega] & -[2\hat{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}_b - \mathbf{n}_\omega]_\times \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\theta} \end{bmatrix} \tag{86}$$

$$\dot{\delta\boldsymbol{\theta}} = -\delta\boldsymbol{\omega}_b - \mathbf{n}_\omega - \frac{1}{2}[2\hat{\boldsymbol{\omega}} - \delta\boldsymbol{\omega}_b - \mathbf{n}_\omega]_\times \delta\boldsymbol{\theta} \tag{87}$$

$$\dot{\delta\boldsymbol{\theta}} \approx -\delta\boldsymbol{\omega}_b - \mathbf{n}_\omega - [\hat{\boldsymbol{\omega}}]_\times \delta\boldsymbol{\theta} \tag{88}$$

$${}^B\dot{\delta\boldsymbol{\theta}} \approx -{}^B\delta\boldsymbol{\omega}_b - {}^B\mathbf{n}_\omega - [{}^B\hat{\boldsymbol{\omega}}]_\times {}^B\delta\boldsymbol{\theta} \tag{89}$$

4.2.5 Summary of Equations

This concludes the derivations of the system error kinematics which are shown below [5]:

$${}^I\dot{\delta\mathbf{r}} = {}^I\delta\mathbf{v} \tag{90}$$

$${}^I\dot{\delta\mathbf{v}} = -{}^{\hat{B}}_I\hat{\mathbf{C}}^T[{}^B\hat{\mathbf{f}}]_\times \delta\boldsymbol{\theta} - {}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\delta\mathbf{a}_b - {}^{\hat{B}}_I\hat{\mathbf{C}}^T\,{}^B\mathbf{n}_a \tag{91}$$

$${}^B\dot{\delta\boldsymbol{\theta}} = -[{}^B\hat{\boldsymbol{\omega}}]_\times {}^B\delta\boldsymbol{\theta} - {}^B\delta\boldsymbol{\omega}_b - {}^B\mathbf{n}_\omega \tag{92}$$

$${}^B\dot{\delta\mathbf{a}}_b = {}^B\boldsymbol{w}_a \tag{93}$$

$${}^B\dot{\delta\boldsymbol{\omega}}_b = {}^B\boldsymbol{w}_\omega \tag{94}$$

$${}^I\dot{\delta\mathbf{p}}_i = {}^B_I\mathbf{C}^T\,{}^B\mathbf{n}_{p,i} \tag{95}$$

4.3 Output Matrix

The measurements performed to correct the EKF are forward leg kinematic measurements. These result in foot positions ${}^B\mathbf{s}_{m,i}$, described in Subsection 3.3, with respect to the robot base frame. Hence, the true-state system output is a vector of the true foot positions of all six legs:

$$\mathbf{y} = \begin{bmatrix} {}^B\mathbf{s}_0 & {}^B\mathbf{s}_1 & \dots & {}^B\mathbf{s}_5 \end{bmatrix}^T \tag{96}$$

where $\mathbf{s}_i$ denotes the position of foot i and is expressed as:

$${}^B\mathbf{s}_i = {}^B_I\mathbf{C}({}^I\mathbf{p}_i - {}^I\mathbf{r}) \tag{97}$$

Similarly, the nominal output vector is given in terms of the nominal foot positions which are defined as:

$$ {}^{B}\hat{\mathbf{s}}_i = {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}}) \tag{98} $$

The error-state output is the error vector of the estimated foot positions:

$$ \delta\mathbf{y} = \begin{bmatrix} {}^{B}\delta\mathbf{s}_0 & {}^{B}\delta\mathbf{s}_1 & \dots & {}^{B}\delta\mathbf{s}_5 \end{bmatrix}^T \tag{99} $$

${}^{B}\delta\mathbf{s}_i$ is the error of the i^{th} foot position. This is shown below where the derivation uses the state composition equations (2), (4), and (66), and the relationship $[a]_\times b = -[b]_\times a$, and ignores all second-order terms:

$$ \begin{aligned} {}^{B}\delta\mathbf{s}_i &= {}^{B}\mathbf{s}_i - {}^{B}\hat{\mathbf{s}}_i && (100) \\ &= {}^{B}_{I}\mathbf{C}({}^{I}\mathbf{p}_i - {}^{I}\mathbf{r}) - {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}}) && (101) \\ &\approx (\mathbf{I} - {}^{B}[\delta\boldsymbol{\theta}]_\times)\, {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i + {}^{I}\delta\mathbf{p} - {}^{I}\hat{\mathbf{r}} - {}^{I}\delta\mathbf{r}) - {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}}) && (102) \\ &= {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{p} - {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{r} - {}^{B}[\delta\boldsymbol{\theta}]_\times\, {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i + {}^{I}\delta\mathbf{p} - {}^{I}\hat{\mathbf{r}} - {}^{I}\delta\mathbf{r}) && (103) \\ &\approx {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{p} - {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{r} - {}^{B}[\delta\boldsymbol{\theta}]_\times\, {}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}}) && (104) \\ &= {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{p} - {}^{B}_{I}\hat{\mathbf{C}}\,{}^{I}\delta\mathbf{r} + \left[{}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}})\right]_\times {}^{B}\delta\boldsymbol{\theta} && (105) \end{aligned} $$

The system output matrix $\mathbf{H}$ can then be easily obtained using the Jacobian:

$$ \begin{aligned} \mathbf{H} &= \frac{\partial \mathbf{y}}{\partial \delta\mathbf{x}} && (106) \\ &= \begin{bmatrix} -{}^{B}_{I}\hat{\mathbf{C}} & \mathbf{0} & \left[{}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}})\right]_\times & \mathbf{0} & \mathbf{0} & {}^{B}_{I}\hat{\mathbf{C}} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -{}^{B}_{I}\hat{\mathbf{C}} & \mathbf{0} & \left[{}^{B}_{I}\hat{\mathbf{C}}({}^{I}\hat{\mathbf{p}}_i - {}^{I}\hat{\mathbf{r}})\right]_\times & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & {}^{B}_{I}\hat{\mathbf{C}} \end{bmatrix} && (107) \end{aligned} $$

5 System Discretisation

All the system kinematics derived in the previous section are in continuous time. To implement the EKF on a digital system, these equations have to be transformed to discrete time. For the nominal state $\hat{\mathbf{x}}$, the resulting equations are used for discrete integration. However, the error state $\delta\mathbf{x}$ is always zero during the state propagation (prediction) stage of the Kalman filter, since when it is non-zero, it is used to correct the nominal state and reset back to zero. However, the error state covariance propagates through to the next iteration. Given that the error covariance propagation is dependent on the state propagation matrix, the error-state kinematics must be also discretised, as presented later in this section.

5.1 Nominal-state Kinematics

By assuming that the system states are constant during a sampling period of length T (zero-order hold), linear equations of motions can be used to describe the translational system kinematics. As for the constant states, the kinematics is trivial. That leaves the quaternion expression.

The zeroth order integration of a changing quaternion can be derived analytically, as shown in [2] and [4] for the Hamilton and JPL conventions, respectively. Intuitively, the integration can be thought of as a combination of two consecutive rotations: the rotation describing the previous orientation, and the rotation corresponding to the orientation increment. In the Hamilton convention, this is expressed as:

$$ {}^{I}_{B_{k+1}}q = {}^{I}_{B_k}q \otimes {}^{B_k}_{B_{k+1}}\Delta q \tag{108} $$

where ${}^{B_k}_{B_{k+1}}\Delta q$ corresponds to the rotation vector $\Delta\boldsymbol{\theta} = {}^{B}\boldsymbol{\omega}T$. Using equations (55) and (59), the incremental quaternion can be expressed as:

$$ {}^{B_k}_{B_{k+1}}\Delta q(\Delta\boldsymbol{\theta}) = \begin{bmatrix} \cos\left(\dfrac{||\Delta\boldsymbol{\theta}||}{2}\right) \\ \dfrac{\Delta\boldsymbol{\theta}}{||\Delta\boldsymbol{\theta}||}\sin\left(\dfrac{||\Delta\boldsymbol{\theta}||}{2}\right) \end{bmatrix} \tag{109} $$

The reversed definition of the quaternion in the JPL convention yields a reversed integration expression:

$$ {}^{B_{k+1}}_{I}q = {}^{B_{k+1}}_{B_k}\Delta q \otimes {}^{B_k}_{I}q \tag{110} $$

where ${}^{B_{k+1}}_{B_k}\Delta q$ is still defined as given in equation (109).

The discrete-time nominal-state kinematics are summarised below:

$$
\begin{aligned}
{}^{I}\hat{\mathbf{r}}_{k+1} &= {}^{I}\dot{\hat{\mathbf{r}}}_{k} + T\,{}^{I}\hat{\mathbf{v}}_{k} + \frac{T^2}{2}\left({}^{B}_{I}\hat{\mathbf{C}}_{k}^{T}({}^{B}\mathbf{f}_{m,k} - {}^{B}\hat{\mathbf{a}}_{b,k}) + \mathbf{g}\right) && (111)\\
{}^{I}\hat{\mathbf{v}}_{k+1} &= T\left({}^{B}_{I}\hat{\mathbf{C}}_{k}^{T}({}^{B}\mathbf{f}_{m,k} - {}^{B}\hat{\mathbf{a}}_{b,k}) + \mathbf{g}\right) && (112)\\
{}^{B}_{I}\hat{q}_{k+1} &= \frac{1}{2}\Delta q([{}^{B}\boldsymbol{\omega}_{m,k} - {}^{B}\hat{\boldsymbol{\omega}}_{b,k}]T) \otimes {}^{B}_{I}\hat{q}_{k} && (113)\\
{}^{B}\hat{\mathbf{a}}_{b,k+1} &= {}^{B}\hat{\mathbf{a}}_{b,k} && (114)\\
{}^{B}\hat{\boldsymbol{\omega}}_{b,k+1} &= {}^{B}\hat{\boldsymbol{\omega}}_{b,k} && (115)\\
{}^{I}\hat{\mathbf{p}}_{i,k+1} &= {}^{I}\hat{\mathbf{p}}_{i,k} && (116)
\end{aligned}
$$

5.2 Error-state Kinematics

5.2.1 Background Theory

A state-space equation describing the dynamics of a system in continuous time is given as [1] (page 42 and 169):

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{n}(t) \tag{117}$$

where $\mathbf{x}$ is a vector of the system states, $\mathbf{u}$ is the system input vector, and $\mathbf{n}$ is the noise (disturbance) vector with a covariance $\mathbf{Q}$. The solution for this equation can be shown to be:

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t}\boldsymbol{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau + \int_{t_0}^{t}\boldsymbol{\Phi}(t,\tau)\mathbf{G}(\tau)\mathbf{n}(\tau)d\tau \tag{118}$$

where the state-transition matrix $\boldsymbol{\Phi}$ is evaluated as:

$$\boldsymbol{\Phi}(t,t_0) = e^{\mathbf{F}(t-t_0)} = L^{-1}\{[s\mathbf{I} - \mathbf{F}]^{-1}\} \tag{119}$$

where L^{-1} is the inverse Laplace transform. Given that the state-transition matrix is only a function of the time difference between consecutive samples, in a discrete system with a fixed sampling rate:

$$\boldsymbol{\Phi}(t_{k+1},t_k) = \boldsymbol{\Phi}(T) = e^{\mathbf{F}T} \tag{120}$$

If we assume that the control input is constant for the duration of the sample, the equivalent discrete-time system model can be written as:

$$\mathbf{x}(t_{i+1}) = \mathbf{F}_d(t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{n}_d(t_i) \tag{121}$$

where the discretised system matrices are evaluated as:

$$\mathbf{F}_d(t_i) = e^{\mathbf{F}T} \tag{122}$$

$$\mathbf{B}_d(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)d\tau \tag{123}$$

$$\mathbf{n}_d(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{G}(\tau)\mathbf{n}(\tau)d\tau \tag{124}$$

The covariance of the discrete noise vector is calculated using the expectation operator based on the assumption that all the noise variables are centred around zero:

$$\mathbf{Q}_d(t_i) = E\{\mathbf{n}_d(t_i)\mathbf{n}_d^T(t_i)\} \tag{125}$$

$$= E\left[\left(\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_1)\mathbf{G}(\tau_1)\mathbf{n}(\tau_1)d\tau_1\right)\left(\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_2)\mathbf{G}(\tau_2)\mathbf{n}(\tau_2)d\tau_2\right)^T\right] \tag{126}$$

$$= E\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_1)\mathbf{G}(\tau_1)\mathbf{n}(\tau_1)\mathbf{n}^T(\tau_2)\mathbf{G}^T(\tau_2)\mathbf{\Phi}^T(t_{i+1}, \tau_2)d\tau_2 d\tau_1\right] \tag{127}$$

$$= \int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_1)\mathbf{G}(\tau_1)E\left[\mathbf{n}(\tau_1)\mathbf{n}^T(\tau_2)\right]\mathbf{G}^T(\tau_2)\mathbf{\Phi}^T(t_{i+1}, \tau_2)d\tau_2 d\tau_1 \tag{128}$$

$$= \int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_1)\mathbf{G}(\tau_1)\mathbf{Q}(\tau_1)\delta(\tau_1 - \tau_2)\mathbf{G}^T(\tau_2)\mathbf{\Phi}^T(t_{i+1}, \tau_2)d\tau_2 d\tau_1 \tag{129}$$

$$= \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau_1)\mathbf{G}(\tau_1)\mathbf{Q}(\tau_1)\mathbf{G}^T(\tau_1)\mathbf{\Phi}^T(t_{i+1}, \tau_1)d\tau_1 \tag{130}$$

5.2.2 IMU Discrete Kinematics

In order to slightly simplify the calculation, the discretisation of the error-state kinematics is split into two parts. Initially, the IMU kinematics are considered and then expanded to include the foot error state.

Formatting the continuous-time error-state kinematics described in equations

(90) to (94) as a state-space model results in the following matrices:

$$\begin{bmatrix} \delta\dot{\mathbf{r}} \\ \delta\dot{\mathbf{v}} \\ \delta\dot{\boldsymbol{\theta}} \\ \delta\dot{\mathbf{a}}_b \\ \delta\dot{\boldsymbol{\omega}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{C}^T\mathbf{f}_\times & -\mathbf{C}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\omega}_\times & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \\ \delta\boldsymbol{\theta} \\ \delta\mathbf{a}_b \\ \delta\boldsymbol{\omega}_b \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{C}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{n}_a \\ \mathbf{n}_\omega \\ \boldsymbol{w}_a \\ \boldsymbol{w}_\omega \end{bmatrix} \tag{131}$$

where, for simplicity, the variables have been presented without the frame in which they are defined, and also $\mathbf{0} = 0_{3\times 3}$. As can be seen, the model does not include any inputs as expected.

The first step to evaluating the state-transition matrix is to obtain the inverse of $[s\mathbf{I} - \mathbf{F}]$:

$$\boldsymbol{\Phi}(t) = L^{-1}\{[s\mathbf{I} - \mathbf{F}]^{-1}\} \tag{132}$$

$$= L^{-1}\left\{ \left[\begin{array}{cc:ccc} s\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I} & \mathbf{C}^T\mathbf{f}_\times & \mathbf{C}^T & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & s\mathbf{I} + \omega_\times & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & s\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & s\mathbf{I} \end{array} \right]^{-1} \right\} \tag{133}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{1}{s}\mathbf{I} & \frac{1}{s^2}\mathbf{I} & -\frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^3}\mathbf{C}^T & \frac{1}{s^3}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & -\frac{1}{s}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^2}\mathbf{C}^T & \frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & [s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I} + \boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \right\} \tag{134}$$

$$= \begin{bmatrix} \mathbf{I} & t\mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\boldsymbol{\Gamma}_2^T(t) & -\frac{t^2}{2}\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\boldsymbol{\Gamma}_3^T(t) \\ \mathbf{0} & \mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\boldsymbol{\Gamma}_1^T(t) & -T\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\boldsymbol{\Gamma}_2^T(t) \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_0^T(t) & \mathbf{0} & -\boldsymbol{\Gamma}_1^T(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{135}$$

where

$$\boldsymbol{\Gamma}_n(t) = \sum_{i=0}^{\infty} \frac{t^{i+n}}{(i+n)!}\omega^{\times i} \tag{136}$$

which can been approximated to the first term in the series. The details of the matrix inversion, performed blockwise, is included in Appendix B, and the evaluation of the inverse Laplace transform is presented in Appendix C. Using the linear approximation, the discrete-time state transition matrix $\mathbf{F}_d$ is given as:

$$\mathbf{F}_d = \mathbf{\Phi}(T) \approx \begin{bmatrix} \mathbf{I} & T\mathbf{I} & -\frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times & -\frac{T^2}{2}\mathbf{C}^T & \frac{T^3}{3!}\mathbf{C}^T\mathbf{f}_\times \\ \mathbf{0} & \mathbf{I} & -T\mathbf{C}^T\mathbf{f}_\times & -T\mathbf{C}^T & \frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{137}$$

The next step involves evaluating the discrete covariance matrix:

$$\mathbf{Q}_d(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1},\tau_1)\mathbf{G}(\tau_1)\mathbf{Q}(\tau_1)\mathbf{G}^T(\tau_1)\mathbf{\Phi}^T(t_{i+1},\tau_1)d\tau_1 \tag{138}$$

$$\mathbf{Q}_d(T) = \int_0^T \mathbf{\Phi}(T-\tau)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(T-\tau)d\tau \tag{139}$$

$$= \int_0^T \mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t)dt \tag{140}$$

which has been simplified based on the following:

- The matrices $\mathbf{G}$ and $\mathbf{Q}$ are considered to be constant during our integration interval.
- The state transition matrix is a function of the constant time difference between the consecutive samples.
- Use the substitution $t = T - \tau$.

The evaluation of the integral is presented in detail in Appendix D. The resulting discretised Covariance matrix is:

$$\mathbf{Q}_d(T) = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} & -\frac{T^3}{3!}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^4}{4!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{k}_{12}^T & \mathbf{k}_{22} & \mathbf{k}_{23} & -\frac{T^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^3}{3!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{k}_{13}^T & \mathbf{k}_{23}^T & \mathbf{k}_{33} & \mathbf{0} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} \\ -\frac{T^3}{3!}\mathbf{Q}_{wa}\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{wa}\mathbf{C} & \mathbf{0} & T\mathbf{Q}_{wa} & \mathbf{0} \\ \frac{T^4}{4!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & \frac{T^3}{3!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} & \mathbf{0} & T\mathbf{Q}_{w\omega} \end{bmatrix} \tag{141}$$

where

$$\mathbf{k}_{11} = \frac{T^3}{3}\mathbf{Q}_a + \frac{T^5}{20}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^5}{20}\mathbf{Q}_{wa} + \frac{T^7}{252}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{142}$$

$$\mathbf{k}_{12} = \frac{T^2}{2}\mathbf{Q}_a + \frac{T^4}{8}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^4}{8}\mathbf{Q}_{wa} + \frac{T^6}{72}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{143}$$

$$\mathbf{k}_{13} = -\frac{T^3}{6}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{T^5}{30}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{144}$$

$$\mathbf{k}_{22} = T\mathbf{Q}_a + \frac{T^3}{3}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^3}{3}\mathbf{Q}_{wa} + \frac{T^5}{20}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{145}$$

$$\mathbf{k}_{23} = -\frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{T^4}{8}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{146}$$

$$\mathbf{k}_{33} = T\mathbf{Q}_\omega + \frac{T^3}{3}\mathbf{Q}_{w\omega} \tag{147}$$

5.2.3 Corin Discrete Kinematics

The continuous-time kinematics for the i^{th} foot of the hexapod, described in equation (95), can be reformatted as:

$$^{I}\delta\dot{\mathbf{p}}_i = 0(^{I}\delta\mathbf{p}_i) + \mathbf{C}^{T}\,{}^{B}\mathbf{n}_{p,i} \tag{148}$$

where the corresponding discrete-time state transition matrix $\mathbf{\Phi}_{p,i}(T)$ simply becomes the identity matrix:

$$\mathbf{\Phi}_{p,i}(T) = e^{\mathbf{0}T} = \mathbf{I} \tag{149}$$

and the discretised error covariance matrix is evaluated as:

$$\mathbf{Q}_{pd,i}(T) = \int_0^T \mathbf{C}^T\mathbf{Q}_{p,i}\mathbf{C}\;dt \tag{150}$$

$$= T\mathbf{C}^T\mathbf{Q}_{p,i}\mathbf{C} \tag{151}$$

Subsequently, adding the foot position kinematics to the IMU motion kinematics, results in the full error state:

$$\delta\mathbf{x} = \begin{bmatrix} ^{I}\delta\mathbf{r} & ^{I}\delta\mathbf{v} & ^{B}\delta\boldsymbol{\theta} & ^{B}\delta\mathbf{a}_b & ^{B}\delta\boldsymbol{\omega}_b & ^{I}\delta\mathbf{p}_0 & \dots & ^{I}\delta\mathbf{p}_5 \end{bmatrix}^T \tag{152}$$

with a discrete-time state transition matrix:

$$
\mathbf{F}_d = \mathbf{\Phi}(T) \approx \begin{bmatrix}
\mathbf{I} & T\mathbf{I} & -\frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times & -\frac{T^2}{2}\mathbf{C}^T & \frac{T^3}{3!}\mathbf{C}^T\mathbf{f}_\times & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & -T\mathbf{C}^T\mathbf{f}_\times & -T\mathbf{C}^T & \frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -T & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I}
\end{bmatrix} \tag{153}
$$

and an error covariance matrix:

$$
\mathbf{Q}_d(T) =
$$

$$
\begin{bmatrix}
\mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} & -\frac{T^3}{3!}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^4}{4!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{k}_{12}^T & \mathbf{k}_{22} & \mathbf{k}_{23} & -\frac{T^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^3}{3!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{k}_{13}^T & \mathbf{k}_{23}^T & \mathbf{k}_{33} & \mathbf{0} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} & \mathbf{0} & \dots & \mathbf{0} \\
-\frac{T^3}{3!}\mathbf{Q}_{wa}\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{wa}\mathbf{C} & \mathbf{0} & T\mathbf{Q}_{wa} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\frac{T^4}{4!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & \frac{T^3}{3!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} & \mathbf{0} & T\mathbf{Q}_{w\omega} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{pd,0} & \dots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Q}_{pd,5}
\end{bmatrix} \tag{154}
$$

6 EKF Implementation

6.1 Time Update

The Time Update step, also known as the prediction step, updates the states of the Kalman filter using the system model kinematics. Since, in our case, these are based on the IMU measurements, the Time Update loop will run at the same frequency as the IMU data output.

In an indirect EKF, the states are updated using the nominal-state kinematics described in equations (111) to (116), which have been reproduced below:

$$\hat{\mathbf{r}}_{k+1}^{-} = \dot{\hat{\mathbf{r}}}_k + T\hat{\mathbf{v}}_k + \frac{T^2}{2}\left(\hat{\mathbf{C}}_k^{+T}(\mathbf{f}_{m,k} - \hat{\mathbf{a}}_{b,k}^{+}) + \mathbf{g}\right) \tag{155}$$

$$\hat{\mathbf{v}}_{k+1}^{-} = T\left(\hat{\mathbf{C}}_k^{+T}(\mathbf{f}_{m,k} - \hat{\mathbf{a}}_{b,k}^{+}) + \mathbf{g}\right) \tag{156}$$

$${}_{I}^{B}\hat{q}_{k+1}^{-} = \frac{1}{2}\Delta q([\boldsymbol{\omega}_{m,k} - \hat{\boldsymbol{\omega}}_{b,k}]T) \otimes {}_{I}^{B}\hat{q}_k^{+} \tag{157}$$

$$\hat{\mathbf{a}}_{b,k+1}^{-} = \hat{\mathbf{a}}_{b,k}^{+} \tag{158}$$

$$\hat{\boldsymbol{\omega}}_{b,k+1}^{-} = \hat{\boldsymbol{\omega}}_{b,k}^{+} \tag{159}$$

$$\hat{\mathbf{p}}_{i,k+1}^{-} = \hat{\mathbf{p}}_{i,k}^{+} \tag{160}$$

where, for simplicity, the frame notation has been removed, and where the $<>^{-}$ refers to an *a priori* state estimate and $<>^{+}$ denotes an *a posteriori* estimate.

As far as this stage is concerned, the nominal states are the best estimates of the true states. Hence, the estimate of the error state vector, $\hat{\delta\mathbf{x}}$, is always zero and the following expression is eliminated from the EKF processing:

$$\hat{\delta\mathbf{x}}_{k+1}^{-} = \mathbf{F}_{d,k}\hat{\delta\mathbf{x}}_k^{+} \tag{161}$$

but with the knowledge that there is an uncertainly associated with the current states and with process noise. Hence, the covariance matrix $\mathbf{P}_k$ corresponding to the error states is propagated according to:

$$\mathbf{P}_{k+1}^{-} = \mathbf{F}_{d,k}\mathbf{P}_k^{+}\mathbf{F}_{d,k} + \mathbf{Q}_{d,k} \tag{162}$$

where $\mathbf{F}_{d,k}$ and $\mathbf{Q}_{d,k}$ are evaluated using equations (153) and (154), respectively, at time index k.

6.2 Measurement Update

The Measurement Update step, updates the state estimates using external measurements. For the hexapod, the measurements are foot positions obtained using forward leg kinematic measurements evaluated based on joint encoder output values. Hence, this loop runs as fast as the sampling frequency of the encoders.

The first step involves calculating the Kalman gain which is performed as with any Kalman filter:

$$\mathbf{K}_k = \mathbf{P}_k^{-}\mathbf{H}_k^{T}(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{T} + \mathbf{R}_k)^{-1} \tag{163}$$

where $\mathbf{H}_k$ is the observation matrix defined in equation (107) and evaluated at time index k.

In an indirect EKF, the state update equation would take the following form:

$$\hat{\delta\mathbf{x}}_k^+ = \hat{\delta\mathbf{x}}_k^- + \mathbf{K}_k(\delta\mathbf{y}_m - \hat{\delta\mathbf{y}}) \tag{164}$$

where

$$\delta\mathbf{y}_m = \begin{bmatrix} {}^B\delta\mathbf{s}_{m,0} & {}^B\delta\mathbf{s}_{m,1} & \dots & {}^B\delta\mathbf{s}_{m,5} \end{bmatrix}^T \tag{165}$$

which is the measurement residual, comprising of the measured foot position errors:

$${}^B\delta\mathbf{s}_{m,i} = {}^B\mathbf{s}_{m,i} - {}^B_I\hat{\mathbf{C}}({}^I\hat{\mathbf{p}}_i - {}^I\hat{\mathbf{r}}) \tag{166}$$

Given that the error state estimate from the previous iteration is zero, equation (164) reduces to the following expression:

$$\hat{\delta\mathbf{x}}_k^+ = \begin{bmatrix} \hat{\delta\mathbf{r}}_k^+ \\ \hat{\delta\mathbf{v}}_k^+ \\ \hat{\delta\boldsymbol{\theta}}_k^+ \\ \hat{\delta\mathbf{a}}_{b,k}^+ \\ \hat{\delta\boldsymbol{\omega}}_{b,k}^+ \\ \hat{\delta\mathbf{p}}_{i,k}^+ \end{bmatrix} = \mathbf{K}_k(\delta\mathbf{y}_m) \tag{167}$$

and the updated error estimates can now used to correct the nominal states, resulting in the *a posteriori* state estimates. Following the measurement update:

$$\hat{\mathbf{r}}_k^+ = \hat{\mathbf{r}}_k^- + \hat{\delta\mathbf{r}}_k^+ \tag{168}$$

$$\hat{\mathbf{v}}_k^+ = \hat{\mathbf{v}}_k^- + \hat{\delta\mathbf{v}}_k^+ \tag{169}$$

$$\hat{q}_k^+ = \Delta q(\hat{\delta\boldsymbol{\theta}}_k^+) \otimes \hat{q}_k^- \tag{170}$$

$$\hat{\mathbf{a}}_{b,k}^+ = \hat{\mathbf{a}}_{b,k}^- + \hat{\delta\mathbf{a}}_{b,k}^+ \tag{171}$$

$$\hat{\boldsymbol{\omega}}_{b,k}^+ = \hat{\boldsymbol{\omega}}_{b,k}^- + \hat{\delta\boldsymbol{\omega}}_{b,k}^+ \tag{172}$$

$$\hat{\mathbf{p}}_{i,k}^+ = \hat{\mathbf{p}}_{i,k}^- + \hat{\delta\mathbf{p}}_{i,k}^+ \tag{173}$$

the error state vector is immediately reset to zero:

$$\hat{\delta\mathbf{x}}_k^+ = 0_{33\times 1} \tag{174}$$

Finally, the corresponding error covariance matrix is updated:

$$\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^- \tag{175}$$

7 Transforming IMU Frames

All the equations considered so far in the EKF, assume that the IMU measurements are at the robot base. However, in reality, the IMU is placed at an arbitrary location on the robot body. This section discussed how the IMU measurements are transformed from the IMU frame to the robot body frame.

The acceleration of a point $\mathbf{p}$ with respect to the inertial frame, ${}^{i}\ddot{\mathbf{r}}_{i}$, can be written in terms of its acceleration with respect to a moving frame, F_m, and the kinematics of the moving frame. This is described in Appendix F and the following expression is derived:

$$
\begin{aligned}
{}^{i}\ddot{\mathbf{r}}_{i} &= {}^{i}\ddot{\mathbf{o}} + {}^{i}\ddot{\mathbf{r}}_{m} && (176)\\
&= {}^{i}\ddot{\mathbf{o}} + {}^{i}_{m}\mathbf{R}\left({}^{m}\ddot{\mathbf{r}}_{m} + 2\,{}^{m}\mathbf{\Omega}_{im}\,{}^{m}\dot{\mathbf{r}}_{m} + {}^{m}\dot{\mathbf{\Omega}}_{im}\,{}^{m}\mathbf{r}_{m} + {}^{m}\mathbf{\Omega}_{im}\,{}^{m}\mathbf{\Omega}_{im}\,{}^{m}\mathbf{r}_{m}\right) && (177)
\end{aligned}
$$

where:

- ${}^{i}\mathbf{r}_{i}$ is the position of the point $\mathbf{p}$ with respect to the inertial frame;
- ${}^{i}\mathbf{o} = {}^{i}\mathbf{o}_{i}$ is the position of the origin of the moving frame with respect to the inertial frame;
- ${}^{i}\mathbf{r}_{m}$ is the position of the point $\mathbf{p}$ with respect to the moving frame;
- ${}^{m}\mathbf{\Omega}_{im}$ is the skew-symmetric matrix of the angular velocity $\boldsymbol{\omega}_{im}$ of the moving frame with relative to the inertial frame.

For the hexapod state estimator, the IMU frame is the moving frame, and the point $\mathbf{p}$ defined above is set to be at the origin of the robot body frame F_b. This is shown graphically in Figure 2, where ${}^{I}\mathbf{r}$ replaces the vector ${}^{i}\mathbf{r}_{i}$ and ${}^{M}\mathbf{d}$ replaces the vector ${}^{i}\mathbf{r}_{m}$.

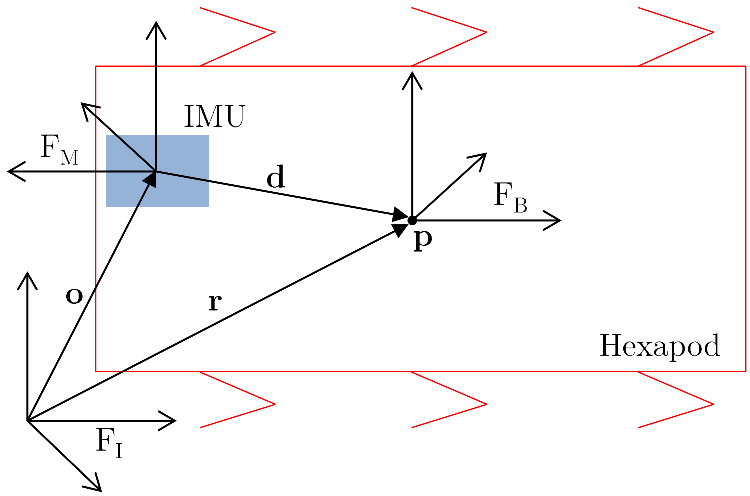

Figure 2: IMU frame transformation.

Since the position of this point $\mathbf{p}$ with respect to the IMU frame F_M (vector $\mathbf{d}$) is fixed, the acceleration, now written in terms of the new frames, can be simplified by eliminating the position derivative terms:

$$ {}^{I}\mathbf{a} = {}^{I}\ddot{\mathbf{r}} = {}^{I}\ddot{\mathbf{o}} + {}^{I}_{M}\mathbf{C}\left({}^{M}\dot{\mathbf{\Omega}}_{IM}\,{}^{M}\mathbf{d} + {}^{M}\mathbf{\Omega}_{IM}\,{}^{M}\mathbf{\Omega}_{IM}\,{}^{M}\mathbf{d}\right) \tag{178} $$

Based on equation (15), the true acceleration at the origin of the body frame can be written in terms of the proper acceleration:

$$ {}^{I}\mathbf{a} = {}^{I}_{B}\mathbf{C}\,{}^{B}\mathbf{f} + \mathbf{g} \tag{179} $$

and similarly for the acceleration of the moving frame:

$$ {}^{I}\ddot{\mathbf{o}} = {}^{I}_{M}\mathbf{C}\,{}^{M}\mathbf{f}' + \mathbf{g} \tag{180} $$

where ${}^{M}\mathbf{f}'$ is the true proper acceleration of moving frame. Substituting these two expressions and ${}^{M}\mathbf{\Omega}' = {}^{M}\mathbf{\Omega}_{IM}$ in equation 178 yields:

$$ {}^{I}_{B}\mathbf{C}\,{}^{B}\mathbf{f} + \mathbf{g} = {}^{I}_{M}\mathbf{C}\,{}^{M}\mathbf{f}' + \mathbf{g} + {}^{I}_{M}\mathbf{C}\left({}^{M}\dot{\mathbf{\Omega}}'\,{}^{M}\mathbf{d} + {}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{d}\right) \tag{181} $$

$$ {}^{B}\mathbf{f} = {}^{B}_{I}\mathbf{C}\,{}^{I}_{M}\mathbf{C}\left({}^{M}\mathbf{f}' + {}^{M}\dot{\mathbf{\Omega}}'\,{}^{M}\mathbf{d} + {}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{d}\right) \tag{182} $$

$$ {}^{B}\mathbf{f} = {}^{B}_{M}\mathbf{C}\left({}^{M}\mathbf{f}' + {}^{M}\dot{\mathbf{\Omega}}'\,{}^{M}\mathbf{d} + {}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{d}\right) \tag{183} $$

which transforms the acceleration of the moving frame to the acceleration of the robot base. By substituting the constituents of the true acceleration:

$$ {}^{B}\mathbf{f} = {}^{B}\mathbf{f}_m - {}^{B}\mathbf{a}_b - {}^{B}\mathbf{n}_a \tag{184} $$

$$ {}^{M}\mathbf{f}' = {}^{M}\mathbf{f}'_m - {}^{M}\mathbf{a}'_b - {}^{M}\mathbf{n}'_a \tag{185} $$

the relationship between the measured acceleration, bias, and noise terms is deduced:

$$ {}^{B}\mathbf{f}_m = {}^{B}_{M}\mathbf{C}\left({}^{M}\mathbf{f}'_m + {}^{M}\dot{\mathbf{\Omega}}'\,{}^{M}\mathbf{d} + {}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{\Omega}'\,{}^{M}\mathbf{d}\right) \tag{186} $$

$$ {}^{B}\mathbf{a}_b = {}^{B}_{M}\mathbf{C}\,{}^{M}\mathbf{a}'_b \tag{187} $$

$$ {}^{B}\mathbf{n}_a = {}^{B}_{M}\mathbf{C}\,{}^{M}\mathbf{n}'_a \tag{188} $$

By using equation 186, the measured IMU acceleration can be transformed from the IMU frame to the robot base frame. The transformed bias term will be automatically estimated by the EKF.

As for the angular velocity, the transformation is straight-forward. Since both the IMU and the robot body frames are fixed on a rigid body, they have the same angular velocity when resolved in the world frame:

$$ {}^{I}\boldsymbol{\omega} = {}^{I}\boldsymbol{\omega}' \tag{189} $$

$$ {}^{I}_{B}\mathbf{C}\,{}^{B}\boldsymbol{\omega} = {}^{I}_{M}\mathbf{C}\,{}^{M}\boldsymbol{\omega}' \tag{190} $$

$$ {}^{B}\boldsymbol{\omega} = {}^{B}_{M}\mathbf{C}\,{}^{M}\boldsymbol{\omega}' \tag{191} $$

8 Initialisation

To minimise the error in the EKF estimation, the states should be initialised to the their best estimates when the EKF is run. This process requires the robot to be stationary for more accurate initialisation. In such a case, both the position and velocity vectors are initialised to zero ($[0\ 0\ 0]^T$).

The orientation quaternion is initialised to the same value as that of the IMU output ${}^{M}_{I}q_{imu}$. The processor on-board the IMU uses a Kalman filter to estimate the orientation, which is accurate in terms or roll and pitch, but not the heading. This is especially important as knowledge of the direction of the gravity vector reduces the errors associated with the subtraction of this vector during integration. The heading angle does not affect this process. In the JPL convention, the initial orientation is evaluated as follows:

$$q_{av} = \frac{1}{N} \sum_{k=0}^{N} {}^{M}_{I}q_{imu,k} \tag{192}$$

$${}^{M}_{I}\hat{q}_0 = \frac{q_{av}}{\|q_{av}\|} \tag{193}$$

$${}^{B}_{I}\hat{q}_0^{+} = {}^{B}_{M}q \, {}^{M}_{I}\hat{q}_0 \tag{194}$$

where ${}^{B}_{M}q$ is a constant quaternion that defined the orientation of the robot body frame with respect to the IMU frame.

Since the robot is stationary during the initialisation phase, the acceleration term in equation (15) is set to zero, and the equation reduces to:

$${}^{B}\mathbf{f}_0 = -\,{}^{B}_{I}\mathbf{C}_0\mathbf{g} \tag{195}$$

and using the state estimates and the acceleration estimate expression (equation 73), the accelerometer bias, defined in the robot body frame, can be derived as follows:

$${}^{B}\hat{\mathbf{f}}_0^{+} = -\,{}^{B}_{I}\hat{\mathbf{C}}_0^{+}\mathbf{g} \tag{196}$$

$${}^{B}\mathbf{f}_{m,0} - {}^{B}\hat{\mathbf{a}}_{b,0}^{+} = -\,{}^{B}_{I}\hat{\mathbf{C}}_0^{+}\mathbf{g} \tag{197}$$

$${}^{B}\hat{\mathbf{a}}_{b,0}^{+} = {}^{B}\mathbf{f}_{m,0} + {}^{B}_{I}\hat{\mathbf{C}}_0^{+}\mathbf{g} \tag{198}$$

$${}^{B}\hat{\mathbf{a}}_{b,0}^{+} = {}^{B}_{M}\mathbf{C}\, {}^{M}\mathbf{f}'_{m,0} + {}^{B}_{I}\hat{\mathbf{C}}_0^{+}\mathbf{g} \tag{199}$$

Similarly, setting the angular velocity to zero, the bias of the gyroscope can be estimated as follows, based on equation (84):

$${}^{B}\hat{\boldsymbol{\omega}}_{b,0}^{+} = {}^{B}\boldsymbol{\omega}_{m,0} \tag{200}$$

$${}^{B}\hat{\boldsymbol{\omega}}_{b,0}^{+} = {}^{B}_{M}\mathbf{C}\, {}^{M}\boldsymbol{\omega}'_{m,0} \tag{201}$$

The foot positions are initialised by setting the measurement residual, given in equation (166), to zero which reduces the expression to:

$$[0\ 0\ 0]^T = {}^{B}\mathbf{s}_{m,i,0} - {}^{B}_{I}\hat{\mathbf{C}}_0^{+}({}^{I}\hat{\mathbf{p}}_{i,0}^{+} - {}^{I}\hat{\mathbf{r}}_0^{+}) \tag{202}$$

and since the body position vector is initialised to zero, the initial foot position states become trivial to evaluate:

$${}^{I}\hat{\mathbf{p}}_{i,0}^{+} = {}^{B}_{I}\hat{\mathbf{C}}_0^{+T}\, {}^{B}\mathbf{s}_{m,i,0} \tag{203}$$

$${}^{I}\hat{\mathbf{p}}_{i,0}^{+} = {}^{B}_{I}\hat{\mathbf{C}}_0^{+T}\mathrm{FK}_i(\boldsymbol{\alpha}_{m,0}) \tag{204}$$

9 Setting Covariance

Now that the EKF algorithm is ready to run, the last step involves setting the error covariance matrices $\mathbf{Q}$ and $\mathbf{R}$. The process noise covariance matrix is a function of the following covariance matrices:

- $\mathbf{Q}_a$: covariance of the accelerometer output of the IMU;
- $\mathbf{Q}_\omega$: covariance of the gyroscope output of the IMU;
- $\mathbf{Q}_{wa}$: covariance of the accelerometer offset derivative;
- $\mathbf{Q}_{w\omega}$ covariance of the gyroscope offset derivative;
- $\mathbf{Q}_{p,i}$: covariance of the i^{th} foot position derivative, caused by foot slippage and moving contact point.

The IMU sensor output noise is of the form:

$$\begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix} \tag{205}$$

where the continuous-time noise power density σ_c is assumed to be the same across the different axes:

$$\sigma_c = \sigma_x = \sigma_y = \sigma_z \tag{206}$$

and is given in the IMU datasheet in $(m/s^2)/\sqrt{Hz}$ for the accelerometer and in rad$/\sqrt{Hz}$ for the gyroscope. The equivalent noise process can be simulated in discrete-time with an AWGN model of variance σ_d:

$$\sigma_d^2 = f_s \sigma_c^2 \tag{207}$$

where f_s is the sampling frequency.

At this stage $\mathbf{Q}_{wa}$, $\mathbf{Q}_{w\omega}$, $\mathbf{Q}_{p,i}$ have been evaluated through a non-exhaustive search. The foot error covariance is set to a very large number when not in contact with the ground so as to remove it from the estimation process. This is because the foot position is no longer constant when not on the ground and cannot be used to correct the EKF error state. On the hexapod, Corin, foot contact is determined using force sensors at the end of each foot.

As for the measurement noise covariance matrix $\mathbf{R}$, the constituent covariance matrices are:

- $\mathbf{R}_{\alpha}$: covariance of the joint encoders output;
- $\mathbf{R}_{\text{fk}}$: covariance of the kinematics error.

The joint encoder error can be estimated from the accuracy figure given in the datasheet. Whereas the error in the kinematics has been determined through a search operation.

10 EKF Validation

The orientation state estimate output by the EKF is in the world (inertial) frame set initially by the robot, F_I. This frame selection results in accurate robot roll and pitch angle values with respect to the gravity vector, but with an arbitrary yaw angle.

A VICON motion capture system has been used to validate the EKF state estimation against ground truth values output by the VICON system. The corresponding world frame of the VICON system, F_V, is set in the calibration phase by manually placing a calibration object in the motion capture space. Therefore, even though the user could make every effort to accurately align the calibration object, there is bound to be errors in the orientation of the VICON world frame, both with respect to gravity and heading directions.

Hence, the position and pose information output by the VICON system should be mapped into the inertial frame, as known by the robot, by removing the effect of the constant rotation between the two frames. This is shown more clearly in Figure 3.

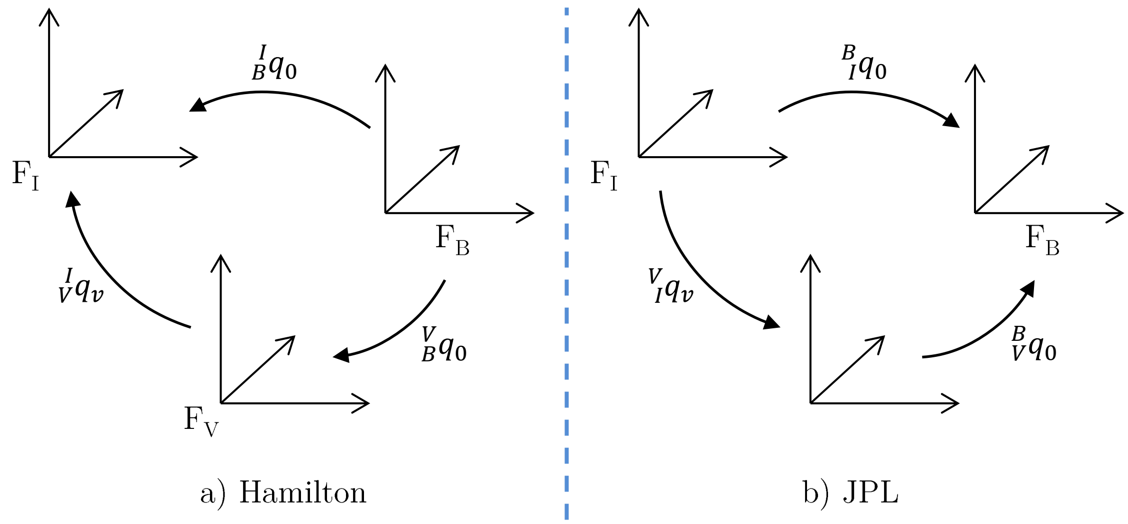

Figure 3: IMU frame transformation.

In the Hamilton convention, the rotation offset quaternion ${}^{I}_{V}q_v$ is evaluated as below:

$$ {}^{I}_{V}q_v = {}^{I}_{B}\hat{q}_0 \otimes {}^{V}_{B}q_0^* \tag{208} $$

and subsequent VICON position and pose information are mapped into the inertial frame as follows:

$$ {}^{I}_{B}q(t) = {}^{I}_{V}q_v \otimes {}^{V}_{B}q(t) \tag{209} $$

$$ {}^{I}r(t) = {}^{I}_{V}q_v \otimes \left({}^{V}r(t) - {}^{V}r_0\right) \otimes {}^{I}_{V}q_v^* \tag{210} $$

However, in the JPL convention, the multiplication order is reversed. Figure 3 can be used to derive the corresponding equations:

$$ {}^{V}_{I}q_v = {}^{B}_{V}q_0^* \otimes {}^{B}_{I}\hat{q}_0 \tag{211} $$

$$ {}^{B}_{I}q(t) = {}^{B}_{V}q(t) \otimes {}^{V}_{I}q_v \tag{212} $$

$$ {}^{I}r(t) = {}^{V}_{I}q_v^* \otimes \left({}^{V}r(t) - {}^{V}r_0\right) \otimes {}^{V}_{I}q_v \tag{213} $$

A Alternative Velocity Error Kinematics

The velocity error kinematics derived in Subsection 4.2.3 can be written in terms of the true rotation matrix instead of the estimated one, as shown below:

$$ {}^{I}\dot{\hat{\mathbf{v}}} + {}^{I}\delta\dot{\mathbf{v}} = {}^{B}_{I}\mathbf{C}^{T}({}^{B}\mathbf{f}_{m} - {}^{B}\mathbf{a}_{b} - {}^{B}\mathbf{n}_{a}) + \mathbf{g} \tag{214} $$

$$ {}^{B}_{I}\mathbf{C}^{T}(\mathbf{I} - [\delta\boldsymbol{\theta}]_{\times})({}^{B}\mathbf{f}_{m} - {}^{B}\hat{\mathbf{a}}_{b}) + {}^{I}\delta\dot{\mathbf{v}} = {}^{B}_{I}\mathbf{C}^{T}({}^{B}\mathbf{f}_{m} - {}^{B}\hat{\mathbf{a}}_{b} - {}^{B}\delta\mathbf{a}_{b} - {}^{B}\mathbf{n}_{a}) \tag{215} $$

$$ {}^{I}\delta\dot{\mathbf{v}} = -{}^{B}_{I}\mathbf{C}^{T}({}^{B}\delta\mathbf{a}_{b} + {}^{B}\mathbf{n}_{a}) + {}^{B}_{I}\mathbf{C}^{T}[\delta\boldsymbol{\theta}]_{\times}\,{}^{B}\hat{\mathbf{f}} \tag{216} $$

$$ {}^{I}\delta\dot{\mathbf{v}} = -{}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\delta\mathbf{a}_{b} - {}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\mathbf{n}_{a} - {}^{B}_{I}\mathbf{C}^{T}[{}^{B}\hat{\mathbf{f}}]_{\times}\delta\boldsymbol{\theta} \tag{217} $$

Using equation (73), and neglecting higher order terms, the equations can also be expressed in terms of the true proper acceleration:

$$ {}^{I}\delta\dot{\mathbf{v}} = -{}^{B}_{I}\mathbf{C}^{T}({}^{B}\delta\mathbf{a}_{b} + {}^{B}\mathbf{n}_{a}) - {}^{B}_{I}\mathbf{C}^{T}[{}^{B}\mathbf{f}_{m} - {}^{B}\hat{\mathbf{a}}_{b}]_{\times}\delta\boldsymbol{\theta} \tag{218} $$

$$ {}^{I}\delta\dot{\mathbf{v}} = -{}^{B}_{I}\mathbf{C}^{T}({}^{B}\delta\mathbf{a}_{b} + {}^{B}\mathbf{n}_{a}) - {}^{B}_{I}\mathbf{C}^{T}[{}^{B}\mathbf{f} + {}^{B}\delta\mathbf{a}_{b} + {}^{B}\mathbf{n}_{a}]_{\times}\delta\boldsymbol{\theta} \tag{219} $$

$$ {}^{I}\delta\dot{\mathbf{v}} \approx -{}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\delta\mathbf{a}_{b} - {}^{B}_{I}\mathbf{C}^{T}\,{}^{B}\mathbf{n}_{a} - {}^{B}_{I}\mathbf{C}^{T}[{}^{B}\mathbf{f}]_{\times}\delta\boldsymbol{\theta} \tag{220} $$

In summary, in the velocity error kinematics, the rotation matrix and acceleration vector can be expressed equivalently using their estimated or true values, when neglecting higher-order terms.

B Matrix Inversion of $[s\mathbf{I} - \mathbf{F}]$

The inversion of a large matrix can be made easier by using the expression below for a block matrix:

$$ \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix} \tag{221} $$

$$ \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0_{n\times m} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ 0_{n\times m} & \mathbf{D}^{-1} \end{bmatrix} \tag{222} $$

Hence, the matrix inverse

$$ [s\mathbf{I} - \mathbf{F}]^{-1} = \left[\begin{array}{cc:ccc} s\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I} & \mathbf{C}^{T}\mathbf{f}_{\times} & \mathbf{C}^{T} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & s\mathbf{I} + \boldsymbol{\omega}_{\times} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & s\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & s\mathbf{I} \end{array}\right]^{-1} \tag{223} $$

can be evaluated blockwise using equation (222). The formatting of the matrix as a block matrix has been achieved using the dashed lines shown in equation (223). The resulting top-left matrix “$\mathbf{A}^{-1}$” of the matrix inverse is:

$$\begin{bmatrix} s\mathbf{I} & -\mathbf{I} \\ \mathbf{0} & s\mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} (s\mathbf{I})^{-1} & -(s\mathbf{I})^{-1}(-\mathbf{I})(s\mathbf{I})^{-1} \\ \mathbf{0} & (s\mathbf{I})^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\mathbf{I} & \frac{1}{s^2}\mathbf{I} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{224}$$

and the bottom-right matrix “$\mathbf{D}^{-1}$ is:

$$\left[\begin{array}{c|cc} s\mathbf{I}+\boldsymbol{\omega}_\times & \mathbf{0} & \mathbf{I} \\ \hline \mathbf{0} & s\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & s\mathbf{I} \end{array}\right]^{-1} = \left[\begin{array}{c|cc} [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \multicolumn{2}{c}{-[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1}\begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix} \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix}} \\ \hline \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{array}\right]^{-1} \tag{225}$$

$$= \begin{bmatrix} [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{226}$$

and, finally, the top-right matrix “$-\mathbf{A}^{-1}\mathbf{B}\mathbf{D}^{-1}$” is:

$$-\begin{bmatrix} \frac{1}{s}\mathbf{I} & \frac{1}{s^2}\mathbf{I} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^T\mathbf{f}_\times & \mathbf{C}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{227}$$

$$= -\begin{bmatrix} \frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times & \frac{1}{s^2}\mathbf{C}^T & \mathbf{0} \\ \frac{1}{s}\mathbf{C}^T\mathbf{f}_\times & \frac{1}{s}\mathbf{C}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{228}$$

$$= -\begin{bmatrix} \frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \frac{1}{s^3}\mathbf{C}^T & -\frac{1}{s^3}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \frac{1}{s}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \frac{1}{s^2}\mathbf{C}^T & -\frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \end{bmatrix} \tag{229}$$

resulting in the inverse matrix:

$$[s\mathbf{I}-\mathbf{F}]^{-1} = \begin{bmatrix} \frac{1}{s}\mathbf{I} & \frac{1}{s^2}\mathbf{I} & -\frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^3}\mathbf{C}^T & \frac{1}{s^3}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & -\frac{1}{s}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^2}\mathbf{C}^T & \frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{230}$$

C Inverse Laplace Transformation

The state-transition matrix can be evaluated using an inverse Laplace transform operation as follows:

$$\mathbf{\Phi}(t) = L^{-1}\{[s\mathbf{I} - \mathbf{F}]^{-1}\} \tag{231}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{1}{s}\mathbf{I} & \frac{1}{s^2}\mathbf{I} & -\frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^3}\mathbf{C}^T & \frac{1}{s^3}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \frac{1}{s}\mathbf{I} & -\frac{1}{s}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & -\frac{1}{s^2}\mathbf{C}^T & \frac{1}{s^2}\mathbf{C}^T\mathbf{f}_\times[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & [s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} & \mathbf{0} & -\frac{1}{s}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{s}\mathbf{I} \end{bmatrix}\right\} \tag{232}$$

which involves taking the inverse Laplace transform of the individual sub-matrices. The transforms of the matrices of the form $\frac{1}{s^n}$ are straight-forward to evaluate and are listed below:

$$L^{-1}\left\{\frac{1}{s}\mathbf{I}\right\} = \mathbf{I} \tag{233}$$

$$L^{-1}\left\{\frac{1}{s^2}\mathbf{I}\right\} = t\mathbf{I} \tag{234}$$

$$L^{-1}\left\{\frac{1}{s^3}\mathbf{I}\right\} = \frac{t^2}{2}\mathbf{I} \tag{235}$$

The remaining transforms are of the form $\frac{1}{s^n}[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1}$ which can be calculated using:

$$L^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(t-\tau)g(\tau)d\tau \tag{236}$$

which, in the case of $F(s) = \frac{1}{s}$, simplifies to an integral:

$$L^{-1}\left\{\frac{1}{s}G(s)\}\right\} = \int_0^t g(\tau)d\tau \tag{237}$$

For easier readability, the following matrix variable is defined:

$$\mathbf{\Gamma}_n(t) = \sum_{i=0}^{\infty} \frac{t^{i+n}}{(i+n)!}\boldsymbol{\omega}^{\times i} \tag{238}$$

and the inverse Laplace transforms are evaluated as follows:

$$L^{-1}\left\{[s\mathbf{I}+\boldsymbol{\omega}_\times]^{-1}\right\} = e^{-t\boldsymbol{\omega}_\times} = e^{t\boldsymbol{\omega}_\times^T} \tag{239}$$

$$= \mathbf{I} + \boldsymbol{\omega}_\times^T\tau + (\boldsymbol{\omega}_\times^T)^2\frac{\tau^2}{2!} + (\boldsymbol{\omega}_\times^T)^3\frac{\tau^3}{3!} + ... \tag{240}$$

$$= \boldsymbol{\Gamma}_0^T(t) \tag{241}$$

$$L^{-1}\left\{\tfrac{1}{s}\left[s\mathbf{I}+\boldsymbol{\omega}_\times\right]^{-1}\right\} = \int_0^t \boldsymbol{\Gamma}_0^T(\tau)d\tau = \int_0^t e^{-\tau\boldsymbol{\omega}_\times}d\tau = \int_0^t e^{\tau\boldsymbol{\omega}_\times^T}d\tau \tag{242}$$

$$= \int_0^t \left(\mathbf{I} + \boldsymbol{\omega}_\times^T\tau + (\boldsymbol{\omega}_\times^T)^2\frac{\tau^2}{2!} + (\boldsymbol{\omega}_\times^T)^3\frac{\tau^3}{3!} + ...\right)d\tau \tag{243}$$

$$= \mathbf{I}t + \boldsymbol{\omega}_\times^T\frac{t^2}{2!} + (\boldsymbol{\omega}_\times^T)^2\frac{t^3}{3!} + (\boldsymbol{\omega}_\times^T)^3\frac{t^4}{4!} + ... \tag{244}$$

$$= \boldsymbol{\Gamma}_1^T(t) \tag{245}$$

$$L^{-1}\left\{\tfrac{1}{s^2}\left[s\mathbf{I}+\boldsymbol{\omega}_\times\right]^{-1}\right\} = \int_0^t \boldsymbol{\Gamma}_1^T(\tau)d\tau \tag{246}$$

$$= \int_0^t \left(\mathbf{I}\tau + \boldsymbol{\omega}_\times^T\frac{\tau^2}{2!} + (\boldsymbol{\omega}_\times^T)^2\frac{\tau^3}{3!} + (\boldsymbol{\omega}_\times^T)^3\frac{\tau^4}{4!} + ...\right)d\tau \tag{247}$$

$$= \mathbf{I}\frac{t^2}{2!} + \boldsymbol{\omega}_\times^T\frac{t^3}{3!} + (\boldsymbol{\omega}_\times^T)^2\frac{t^4}{4!} + (\boldsymbol{\omega}_\times^T)^3\frac{t^5}{5!} + ... \tag{248}$$

$$= \boldsymbol{\Gamma}_2^T(t) \tag{249}$$

$$L^{-1}\left\{\tfrac{1}{s^3}\left[s\mathbf{I}+\boldsymbol{\omega}_\times\right]^{-1}\right\} = \int_0^t \boldsymbol{\Gamma}_2^T(\tau)d\tau \tag{250}$$

$$= \int_0^t \left(\mathbf{I}\frac{\tau^2}{2!} + \boldsymbol{\omega}_\times^T\frac{\tau^3}{3!} + (\boldsymbol{\omega}_\times^T)^2\frac{\tau^4}{4!} + (\boldsymbol{\omega}_\times^T)^3\frac{\tau^5}{5!} + ...\right)d\tau \tag{251}$$

$$= \mathbf{I}\frac{t^3}{3!} + \boldsymbol{\omega}_\times^T\frac{t^4}{4!} + (\boldsymbol{\omega}_\times^T)^2\frac{t^5}{5!} + (\boldsymbol{\omega}_\times^T)^3\frac{t^6}{6!} + ... \tag{252}$$

$$= \boldsymbol{\Gamma}_3^T(t) \tag{253}$$

All these evaluations are now put together to form the state-transition matrix:

$$\mathbf{\Phi}(t) = \begin{bmatrix} \mathbf{I} & t\mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t) & -\frac{t^2}{2}\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t) \\ \mathbf{0} & \mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t) & -T\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_0^T(t) & \mathbf{0} & -\mathbf{\Gamma}_1^T(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{254}$$

The inverse Laplace transforms described in equations (246) and (250) can also be computed directly without the knowledge of the previous transforms, as shown below:

$$L^{-1}\left\{\tfrac{1}{s^2}\left[s\mathbf{I}+\boldsymbol{\omega}_\times\right]^{-1}\right\} = \int_0^t (t-\tau)e^{\tau\boldsymbol{\omega}_\times^T}d\tau \tag{255}$$

$$= t\int_0^t \left(\mathbf{I}+\boldsymbol{\omega}_\times^T\tau+(\boldsymbol{\omega}_\times^T)^2\frac{\tau^2}{2!}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^3}{3!}+...\right)d\tau \tag{256}$$

$$-\int_0^t \left(\mathbf{I}\tau+\boldsymbol{\omega}_\times^T\tau^2+(\boldsymbol{\omega}_\times^T)^2\frac{\tau^3}{2!}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^4}{3!}+...\right)d\tau \tag{257}$$

$$= \mathbf{I}t^2+\boldsymbol{\omega}_\times^T\frac{t^3}{2!}+(\boldsymbol{\omega}_\times^T)^2\frac{t^4}{3!}+(\boldsymbol{\omega}_\times^T)^3\frac{t^5}{4!}+... \tag{258}$$

$$-\left(\mathbf{I}\frac{t^2}{2}+\boldsymbol{\omega}_\times^T\frac{t^3}{3}+(\boldsymbol{\omega}_\times^T)^2\frac{t^4}{2!(4)}+(\boldsymbol{\omega}_\times^T)^3\frac{t^5}{3!(5)}+...\right) \tag{259}$$

$$= \mathbf{I}\frac{t^2}{2!}+\boldsymbol{\omega}_\times^T\frac{t^3}{3!}+(\boldsymbol{\omega}_\times^T)^2\frac{t^4}{4!}+(\boldsymbol{\omega}_\times^T)^3\frac{t^5}{5!}+... \tag{260}$$

$$= \mathbf{\Gamma}_2^T(t) \tag{261}$$

$$
\begin{aligned}
L^{-1}\left\{\tfrac{1}{s^3}\left[s\mathbf{I}+\boldsymbol{\omega}_\times\right]^{-1}\right\} &= \int_0^t \frac{(t-\tau)^2}{2} e^{\tau\boldsymbol{\omega}_\times^T} d\tau && (262)\\
&= \frac{t^2}{2}\int_0^t \left(\mathbf{I}+\boldsymbol{\omega}_\times^T\tau+(\boldsymbol{\omega}_\times^T)^2\frac{\tau^2}{2!}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^3}{3!}+...\right)d\tau && (263)\\
&\quad - t\int_0^t \left(\mathbf{I}\tau+\boldsymbol{\omega}_\times^T\tau^2+(\boldsymbol{\omega}_\times^T)^2\frac{\tau^3}{2!}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^4}{3!}+...\right)d\tau && (264)\\
&\quad + \frac{1}{2}\int_0^t \left(\mathbf{I}\tau^2+\boldsymbol{\omega}_\times^T\tau^3+(\boldsymbol{\omega}_\times^T)^2\frac{\tau^4}{2!}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^5}{3!}+...\right)d\tau && (265)\\
&= \frac{1}{2}\left(\mathbf{I}t^3+\boldsymbol{\omega}_\times^T\frac{t^4}{2!}+(\boldsymbol{\omega}_\times^T)^2\frac{t^5}{3!}+(\boldsymbol{\omega}_\times^T)^3\frac{t^6}{4!}+...\right) && (266)\\
&\quad - \left(\mathbf{I}\frac{t^3}{2}+\boldsymbol{\omega}_\times^T\frac{t^4}{3}+(\boldsymbol{\omega}_\times^T)^2\frac{t^5}{2!(4)}+(\boldsymbol{\omega}_\times^T)^3\frac{t^6}{3!(5)}+...\right) && (267)\\
&\quad + \frac{1}{2}\left(\mathbf{I}\frac{t^3}{3}+\boldsymbol{\omega}_\times^T\frac{t^4}{4}+(\boldsymbol{\omega}_\times^T)^2\frac{t^5}{2!(5)}+(\boldsymbol{\omega}_\times^T)^3\frac{\tau^6}{3!(6)}+...\right) && (268)\\
&= \mathbf{I}\frac{t^3}{3!}+\boldsymbol{\omega}_\times^T\frac{t^4}{4!}+(\boldsymbol{\omega}_\times^T)^2\frac{t^5}{5!}+(\boldsymbol{\omega}_\times^T)^3\frac{t^6}{6!}+... && (269)\\
&= \boldsymbol{\Gamma}_3^T(t) && (270)
\end{aligned}
$$

D Discretisation of Covariance

The discrete covariance matrix of the EKF process noise is given by:

$$\mathbf{Q}_d(t_i) = \int_0^T \mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t)dt \tag{271}$$

The matrix multiplication $\mathbf{GQG}^T$ is firstly evaluated as below:

$$\mathbf{GQG}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{C}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{272}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^T\mathbf{Q}_a\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \tag{273}$$

and simplified to:

$$\mathbf{GQG}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \tag{274}$$

since

$$\mathbf{C}^T\mathbf{Q}_a\mathbf{C} = \mathbf{C}^T(q_a\mathbf{I})\mathbf{C} = (q_a\mathbf{I})\mathbf{C}^T\mathbf{C} = \mathbf{Q}_a \tag{275}$$

Secondly, the state-transition matrix and its transpose are multiplied by the previous resulting matrix as shown below:

$$\boldsymbol{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T = \tag{276}$$

$$\begin{bmatrix} \mathbf{I} & t\mathbf{I} & -\frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times & -\frac{t^2}{2}\mathbf{C}^T & -\frac{t^3}{3!}\mathbf{C}^T\mathbf{f}_\times \\ \mathbf{0} & \mathbf{I} & -t\mathbf{C}^T\mathbf{f}_\times & -t\mathbf{C}^T & -\frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & t\mathbf{Q}_a & -\frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega & -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^3}{3!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{Q}_a & -t\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega & -t\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & -t\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \tag{277}$$

$$\boldsymbol{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\boldsymbol{\Phi}^T(t) = \tag{278}$$

$$\begin{bmatrix} \mathbf{0} & t\mathbf{Q}_a & -\frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega & -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^3}{3!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{Q}_a & -t\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega & -t\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & -t\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ t\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\frac{t^2}{2}\mathbf{f}_\times^T\mathbf{C} & -t\mathbf{f}_\times^T\mathbf{C} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\frac{t^2}{2}\mathbf{C} & -t\mathbf{C} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \frac{t^3}{3!}\mathbf{f}_\times^T\mathbf{C} & \frac{t^2}{2}\mathbf{f}_\times^T\mathbf{C} & -t & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^3}{6}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{h}_{12}^T & \mathbf{h}_{22} & \mathbf{h}_{23} & -t\mathbf{C}^T\mathbf{Q}_{wa} & \frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{h}_{13}^T & \mathbf{h}_{23}^T & \mathbf{h}_{33} & \mathbf{0} & -t\mathbf{Q}_{w\omega} \\ -\frac{t^2}{2}\mathbf{Q}_{wa}\mathbf{C} & -t\mathbf{Q}_{wa}\mathbf{C} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \frac{t^3}{6}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & \frac{t^2}{2}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & -t\mathbf{Q}_{w\omega} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \tag{279}$$

where

$$\mathbf{h}_{11} = t^2\mathbf{Q}_a + \frac{t^4}{4}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{t^4}{4}\mathbf{Q}_{wa} + \frac{t^6}{36}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{280}$$

$$\mathbf{h}_{12} = t\mathbf{Q}_a + \frac{t^3}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{t^3}{2}\mathbf{Q}_{wa} + \frac{t^5}{12}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{281}$$

$$\mathbf{h}_{13} = -\frac{t^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{t^4}{6}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{282}$$

$$\mathbf{h}_{22} = \mathbf{Q}_a + t^2\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + t^2\mathbf{Q}_{wa} + \frac{t^4}{4}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{283}$$

$$\mathbf{h}_{23} = -t\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{t^3}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{284}$$

$$\mathbf{h}_{33} = \mathbf{Q}_\omega + t^2\mathbf{Q}_{w\omega} \tag{285}$$

The last step to evaluating the discrete covariance matrix is integrating the previous matrix:

$$\mathbf{Q}_d(t_i) = \int_0^T \mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t)dt \tag{286}$$

$$= \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} & -\frac{T^3}{3!}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^4}{4!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{k}_{21}^T & \mathbf{k}_{22} & \mathbf{k}_{23} & -\frac{T^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \frac{T^3}{3!}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \\ \mathbf{k}_{13}^T & \mathbf{k}_{23}^T & \mathbf{k}_{33} & \mathbf{0} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} \\ -\frac{T^3}{3!}\mathbf{Q}_{wa}\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{wa}\mathbf{C} & \mathbf{0} & T\mathbf{Q}_{wa} & \mathbf{0} \\ \frac{T^4}{4!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & \frac{T^3}{3!}\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} & -\frac{T^2}{2}\mathbf{Q}_{w\omega} & \mathbf{0} & T\mathbf{Q}_{w\omega} \end{bmatrix} \tag{287}$$

where

$$\mathbf{k}_{11} = \frac{T^3}{3}\mathbf{Q}_a + \frac{T^5}{20}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^5}{20}\mathbf{Q}_{wa} + \frac{T^7}{252}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{288}$$

$$\mathbf{k}_{12} = \frac{T^2}{2}\mathbf{Q}_a + \frac{T^4}{8}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^4}{8}\mathbf{Q}_{wa} + \frac{T^6}{72}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{289}$$

$$\mathbf{k}_{13} = -\frac{T^3}{6}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{T^5}{30}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{290}$$

$$\mathbf{k}_{22} = T\mathbf{Q}_a + \frac{T^3}{3}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega\mathbf{f}_\times^T\mathbf{C} + \frac{T^3}{3}\mathbf{Q}_{wa} + \frac{T^5}{20}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega}\mathbf{f}_\times^T\mathbf{C} \tag{291}$$

$$\mathbf{k}_{23} = -\frac{T^2}{2}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_\omega - \frac{T^4}{8}\mathbf{C}^T\mathbf{f}_\times\mathbf{Q}_{w\omega} \tag{292}$$

$$\mathbf{k}_{33} = T\mathbf{Q}_\omega + \frac{T^3}{3}\mathbf{Q}_{w\omega} \tag{293}$$

E Accurate Covariance Discretisation

In this work, the discretised state-transition matrix was approximated using the first term of the Γ matrices. If this approximation wasn't performed, the discretisation of the covariance matrix would be as follows:

$$\mathbf{Q}_d(t_i) = \int_0^T \mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t)dt \tag{294}$$

where

$$\mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T = \tag{295}$$

$$\begin{bmatrix} \mathbf{I} & t\mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t) & -\frac{t^2}{2}\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t) \\ \mathbf{0} & \mathbf{I} & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t) & -T\mathbf{C}^T & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_0^T(t) & \mathbf{0} & -\mathbf{\Gamma}_1^T(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & T\mathbf{Q}_a & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_\omega & -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{Q}_a & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t)\mathbf{Q}_\omega & -T\mathbf{C}^T\mathbf{Q}_{wa} & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_0^T(t)\mathbf{Q}_\omega & \mathbf{0} & -\mathbf{\Gamma}_1^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix} \tag{296}$$

and subsequently

$$\mathbf{\Phi}(t)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t) = \tag{297}$$

$$= \begin{bmatrix} \mathbf{0} & T\mathbf{Q}_a & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_\omega & -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{Q}_a & -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t)\mathbf{Q}_\omega & -T\mathbf{C}^T\mathbf{Q}_{wa} & \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_0^T(t)\mathbf{Q}_\omega & \mathbf{0} & -\mathbf{\Gamma}_1^T(t)\mathbf{Q}_{w\omega} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{wa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{w\omega} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ T\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{\Gamma}_2(t)\mathbf{f}_\times^T C & -\mathbf{\Gamma}_1(t)\mathbf{f}_\times^T\mathbf{C} & \mathbf{\Gamma}_0(t) & \mathbf{0} & \mathbf{0} \\ -\frac{t^2}{2}\mathbf{C} & -T\mathbf{C} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{\Gamma}_3(t)\mathbf{f}_\times^T\mathbf{C} & \mathbf{\Gamma}_2(t)\mathbf{f}_\times^T\mathbf{C} & -\mathbf{\Gamma}_1(t) & \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & \mathbf{h}_{14} & \mathbf{h}_{15} \\ \mathbf{h}_{21}^T & \mathbf{h}_{22} & \mathbf{h}_{23} & \mathbf{h}_{24} & \mathbf{h}_{25} \\ \mathbf{h}_{13}^T & \mathbf{h}_{23}^T & \mathbf{h}_{33} & \mathbf{0} & \mathbf{h}_{35} \\ \mathbf{h}_{14}^T & \mathbf{h}_{24}^T & \mathbf{0} & \mathbf{h}_{44} & \mathbf{0} \\ \mathbf{h}_{15}^T & \mathbf{h}_{25}^T & \mathbf{h}_{35}^T & \mathbf{0} & k_{55} \end{bmatrix}$$

where

$$\mathbf{h}_{11} = t^2\mathbf{Q}_a + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_\omega\mathbf{\Gamma}_2(t)\mathbf{f}_\times^T\mathbf{C} + \frac{t^4}{4}\mathbf{Q}_{wa} + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega}\mathbf{\Gamma}_3(t)\mathbf{f}_\times^T\mathbf{C} \tag{298}$$

$$\mathbf{h}_{12} = T\mathbf{Q}_a + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_\omega\mathbf{\Gamma}_1(t)\mathbf{f}_\times^T\mathbf{C} + \frac{t^3}{2}\mathbf{Q}_{wa} + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega}\mathbf{\Gamma}_2(t)\mathbf{f}_\times^T\mathbf{C} \tag{299}$$

$$\mathbf{h}_{13} = -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_\omega\mathbf{\Gamma}_0(t) - \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega}\mathbf{\Gamma}_1(t) \tag{300}$$

$$\mathbf{h}_{14} = -\frac{t^2}{2}\mathbf{C}^T\mathbf{Q}_{wa} \tag{301}$$

$$\mathbf{h}_{15} = \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_3^T(t)\mathbf{Q}_{w\omega} \tag{302}$$

$$\mathbf{h}_{22} = \mathbf{Q}_a + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t)\mathbf{Q}_\omega\mathbf{\Gamma}_1(t)\mathbf{f}_\times^T\mathbf{C} + t^2\mathbf{Q}_{wa} + \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_{w\omega}\mathbf{\Gamma}_2(t)\mathbf{f}_\times^T\mathbf{C} \tag{303}$$

$$\mathbf{h}_{23} = -\mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_1^T(t)\mathbf{Q}_\omega\mathbf{\Gamma}_0(t) - \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_{w\omega}\mathbf{\Gamma}_1(t) \tag{304}$$

$$\mathbf{h}_{24} = -T\mathbf{C}^T\mathbf{Q}_{wa} \tag{305}$$

$$\mathbf{h}_{25} = \mathbf{C}^T\mathbf{f}_\times\mathbf{\Gamma}_2^T(t)\mathbf{Q}_{w\omega} \tag{306}$$

$$\mathbf{h}_{33} = \mathbf{\Gamma}_0^T\mathbf{Q}_\omega\mathbf{\Gamma}_0(t) + \mathbf{\Gamma}_1^T\mathbf{Q}_{w\omega}\mathbf{\Gamma}_1(t) \tag{307}$$

$$\mathbf{h}_{35} = -\mathbf{\Gamma}_1^T(t)\mathbf{Q}_{w\omega} \tag{308}$$

$$\mathbf{h}_{44} = \mathbf{Q}_{wa} \tag{309}$$

$$\mathbf{h}_{55} = \mathbf{Q}_{w\omega} \tag{310}$$

However, the integration step would be slightly more cumbersome and it is not shown here.

F Kinematics of Moving Frames

Consider an inertial (world) frame, F_i, that is fixed in space, and a frame, F_m, that is moving with respect to the world frame. A vector v can be mapped from one frame to another using the transformation matrix:

$$ {}^{i}\mathbf{v} = {}^{i}_{m}\mathbf{R}\,{}^{m}\mathbf{v} \tag{311} $$

It can be shown that the time derivative of the transformation matrix is given by (Chapter 2 of [7]):

$$ {}^{i}_{m}\dot{\mathbf{R}} = {}^{i}\mathbf{\Omega}_{im}\,{}^{i}_{m}\mathbf{R} \tag{312} $$

$$ = {}^{i}_{m}\mathbf{R}\,{}^{m}\mathbf{\Omega}_{im} \tag{313} $$

where ${}^{i}\mathbf{\Omega}_{im}$ is the skew-symmetric matrix of the angular velocity of the moving frame with respect to the inertial frame, as defined in the inertial frame.

We can define the position of a point in space with respect to either the inertial frame or the moving frame. In other terms, a position vector defined with respect to the moving frame extends from the origin of the moving frame to the described point. Additionally, each of these two vectors can be defined in either of the frames. For example, ${}^{i}\mathbf{r}_m$ represents the position vector relative to the moving frame, as resolved in the inertial frame.

Hence the relationship between two different vectors, each defining the position of a point $\mathbf{p}$ with respect to a different frame, is given as [8]:

$$ {}^{i}\mathbf{r}_i = {}^{i}\mathbf{o} + {}^{i}\mathbf{r}_m \tag{314} $$

$$ = {}^{i}\mathbf{o} + {}^{i}_{m}\mathbf{R}\,{}^{m}\mathbf{r}_m \tag{315} $$

where

- ${}^{i}\mathbf{r}_i$ is the position of the point $\mathbf{p}$ with respect to the inertial frame;
- ${}^{i}\mathbf{o} = {}^{i}\mathbf{o}_i$ is the position of the origin of the moving frame with respect to the fixed frame ($\mathbf{o}_m$ is obviously zero);
- ${}^{i}\mathbf{r}_m$ is the position of the point $\mathbf{p}$ with respect to the moving frame;

and all of the vectors are defined (resolved) in the inertial frame, and ${}^{m}\mathbf{r}_m$ is the position of the point $\mathbf{p}$ with respect to the moving frame as defined in the same frame.

The velocity of the point $\mathbf{p}$ with respect to the moving frame as described in the inertial frame is given by:

$$
\begin{aligned}
{}^{i}\dot{\mathbf{r}}_m &= \frac{d}{dt}\left({}^{i}_{m}\mathbf{R}\,{}^{m}\mathbf{r}_m\right) && (316)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\dot{\mathbf{r}}_m + {}^{i}_{m}\dot{\mathbf{R}}\,{}^{m}\mathbf{r}_m && (317)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\dot{\mathbf{r}}_m + {}^{i}\boldsymbol{\Omega}_{im}\,{}^{i}_{m}\mathbf{R}\,{}^{m}\mathbf{r}_m && (318)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\dot{\mathbf{r}}_m + {}^{i}\boldsymbol{\Omega}_{im}\,{}^{i}\mathbf{r}_m && (319)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\dot{\mathbf{r}}_m + {}^{i}\boldsymbol{\omega}_{im} \times {}^{i}\mathbf{r}_m \qquad \text{OR} && (320)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\dot{\mathbf{r}}_m + {}^{i}_{m}\mathbf{R}\,{}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m && (321)\\
&= {}^{i}_{m}\mathbf{R}\left({}^{m}\dot{\mathbf{r}}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right) && (322)\\
&= {}^{i}_{m}\mathbf{R}\left({}^{m}\dot{\mathbf{r}}_m + {}^{m}\boldsymbol{\omega}_{im} \times {}^{m}\mathbf{r}_m\right) && (323)\\
& && (324)
\end{aligned}
$$

The acceleration of this velocity vector is therefore given by [7]:

$$
\begin{aligned}
{}^{i}\ddot{\mathbf{r}}_m &= \frac{d}{dt}\left({}^{i}_{m}\mathbf{R}\left({}^{m}\dot{\mathbf{r}}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right)\right) && (325)\\
&= {}^{i}_{m}\dot{\mathbf{R}}\left({}^{m}\dot{\mathbf{r}}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right) + {}^{i}_{m}\mathbf{R}\left({}^{m}\ddot{\mathbf{r}}_m + {}^{m}\dot{\boldsymbol{\Omega}}_{im}\,{}^{m}\mathbf{r}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\dot{\mathbf{r}}_m\right) && (326)\\
&= {}^{i}_{m}\mathbf{R}\,{}^{m}\boldsymbol{\Omega}_{im}\left({}^{m}\dot{\mathbf{r}}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right) + {}^{i}_{m}\mathbf{R}\left({}^{m}\ddot{\mathbf{r}}_m + {}^{m}\dot{\boldsymbol{\Omega}}_{im}\,{}^{m}\mathbf{r}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\dot{\mathbf{r}}_m\right) && (327)\\
&= {}^{i}_{m}\mathbf{R}\left({}^{m}\ddot{\mathbf{r}}_m + 2\,{}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\dot{\mathbf{r}}_m + {}^{m}\dot{\boldsymbol{\Omega}}_{im}\,{}^{m}\mathbf{r}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right) && (328)
\end{aligned}
$$

which leads to the acceleration of the point $\mathbf{p}$ with respect to the inertial frame, as defined in this frame, in terms of the acceleration and rotation of the moving frame, and the motion of the point $\mathbf{p}$ with respect to the moving frame:

$$
\begin{aligned}
{}^{i}\ddot{\mathbf{r}}_i &= {}^{i}\ddot{\mathbf{o}} + {}^{i}\ddot{\mathbf{r}}_m && (329)\\
&= {}^{i}\ddot{\mathbf{o}} + {}^{i}_{m}\mathbf{R}\left({}^{m}\ddot{\mathbf{r}}_m + 2\,{}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\dot{\mathbf{r}}_m + {}^{m}\dot{\boldsymbol{\Omega}}_{im}\,{}^{m}\mathbf{r}_m + {}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\boldsymbol{\Omega}_{im}\,{}^{m}\mathbf{r}_m\right) && (330)
\end{aligned}
$$

G Noise Variance

For a continuous-time noise process $w(t)$, the autocorrelation function is given as:

$$R_w(t_1, t_2) = N\delta(t_1 - t_2) \tag{331}$$

where N is the power spectral density in $W/\sqrt{Hz}$. Hence, the following expectation holds true:

$$E[w(t_1)w(t_2)] = \infty, \quad t_1 = t_2 \tag{332}$$

For the equivalent discrete-time noise $w[n]$, the autocorrelation function is given by:

$$R_w[n] = \sigma_d^2\delta[n] \tag{333}$$

where

$$\sigma_d^2 = \frac{N}{T} \tag{334}$$

and T is the sampling period.

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