Fundamentals of Vehicle Dynamics

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LIST OF SYMBOLS

a	Tire cornering stiffness parameter
b	Tire cornering stiffness parameter
A	Frontal area of a vehicle
A_f	Lateral force compliance steer coefficient on the front axle
A_r	Lateral force compliance steer coefficient on the rear axle
a_x	Acceleration in the x-direction
a_y	Acceleration in the lateral direction
b	Longitudinal distance from front axle to center of gravity
c	Longitudinal distance from center of gravity to rear axle
C_α	Cornering stiffness of the tires on an axle
C_α'	Cornering stiffness of one tire
CC_α	Tire cornering coefficient
C_γ	Tire camber stiffness
C_D	Aerodynamic drag coefficient
C_h	Road surface rolling resistance coefficient
C_L	Aerodynamic lift coefficient
C_{PM}	Aerodynamic pitching moment coefficient
C_{RM}	Aerodynamic rolling moment coefficient
C_{YM}	Aerodynamic yawing moment coefficient
C_s	Suspension damping coefficient
C_S	Aerodynamic side force coefficient
CP	Center of pressure location of aerodynamic side force
d	Lateral distance between steering axis and center of tire contact at the ground
d_h	Distance from axle to the hitch point
d_{ns}	Distance from center of mass to the neutral steer point
D	Tire diameter
DI	Dynamic index
D_x	Linear deceleration
D_A	Aerodynamic drag force
e	Height of the pivot for an "equivalent torque arm"
	Drum brake geometry factor
$E[y^2]$	Mean square vibration response
f	Longitudinal length for an "equivalent torque arm"
f_a	Wheel hop resonant frequency (vertical)

f_n	Undamped natural frequency of a suspension system (Hz)
f_r	Rolling resistance coefficient
F_b	Braking force
	Vertical disturbance force on the sprung mass
F_i	Imbalance force in a tire
F_x	Force in the x-direction (tractive force)
F_{xm}	Maximum brake force on an axle
F_{xt}	Total force in the x-direction
F_y	Force in the y-direction (lateral force)
	Lateral force on an axle
F_y'	Lateral force on one tire
F_z	Force in the z-direction (vertical force)
F_{zi}	Vertical force on inside tire in a turn
F_{zo}	Vertical force on outside tire in a turn
F_w	Tire/wheel nonuniformity force on the unsprung mass
g	Acceleration of gravity (32.2 ft/sec^2, 9.81 m/sec^2)
G	Brake gain
G_o	Road roughness magnitude parameter
G_z	Power spectral density amplitude of road roughness
G_{zs}	Power spectral density amplitude of sprung mass acceleration
h	Center of gravity height
h_a	Height of the aerodynamic drag force
h_h	Hitch height
h_1	Height of the sprung mass center of gravity above the roll axis
h_r	Height of suspension roll center
h_t	Tire section height
HP	Engine or brake horsepower
HP_A	Aerodynamic horsepower
HP_R	Rolling resistance horsepower
HP_{RL}	Road load horsepower
H_v	Response gain function
I_d	Moment of inertia of the driveshaft
I_e	Moment of inertia of the engine
I_t	Moment of inertia of the transmission
I_w	Moment of inertia of the wheels
I_{xx}	Moment of inertia about the x-axis

I_{yy}	Moment of inertia about the y-axis
I_{zz}	Moment of inertia about the z-axis
k	Radius of gyration
K	Understeer gradient
K_{at}	Understeer gradient due to aligning torque
K_{llt}	Understeer gradient due to lateral load transfer on the axles
K_{lfcs}	Understeer gradient due to lateral force compliance steer
K_s	Vertical stiffness of a suspension
K_{ss}	Steering system stiffness
K_{strg}	Understeer gradient due to the steering system
K_t	Vertical stiffness of a tire
K_ϕ	Suspension roll stiffness
L	Wheelbase
L_A	Aerodynamic lift force
m	Drum brake geometry parameter
M	Mass of the vehicle
M_{AT}	Moment around the steer axis due to tire aligning torques
M_L	Moment around the steer axis due to tire lateral forces
M_r	Equivalent mass of the rotating components
M_{SA}	Moment around the steer axis due to front-wheel-drive forces and torques
M_T	Moment around the steer axis due to tire tractive forces
M_V	Moment around the steer axis due to tire vertical forces
M_ϕ	Rolling moment
n	Drum brake geometry parameter
N	Normal force
N_t	Numerical ratio of the transmission
N_f	Numerical ratio of the final drive
N_{tf}	Numerical ratio of the combined transmission and final drive
NSP	Neutral steer point
p	Pneumatic trail
P_a	Brake application pressure/effort
P_{atm}	Atmospheric pressure
P_f	Front brake application pressure
P_r	Rear brake application pressure
P_s	Static pressure
P_t	Total pressure

PM	Aerodynamic pitching moment
p	Roll velocity about the x-axis of the vehicle
q	Pitch velocity about the y-axis of the vehicle
q	Dynamic pressure
r	Yaw velocity about the z-axis of the vehicle
r	Rolling radius of the tires
r_k	Ratio of tire to suspension stiffness
R	Radius of turn
R_h	Hitch force
R_g	Grade force
R_x	Rolling resistance force
R_{RL}	Road load
RM	Aerodynamic rolling moment
RR	Ride rate of a tire/suspension system
R_ϕ	Roll rate of the sprung mass
s	Lateral separation between suspension springs
S_A	Aerodynamic side force
S_o	Spectral density of white-noise
SD	Stopping distance
t	Tread
t_s	Length of time of a brake application
T_a	Torque in the axle
T_b	Brake torque
T_c	Torque at the clutch
T_d	Torque in the driveshaft
T_e	Torque of the engine
T_{sf}	Roll torque in a front suspension
T_{sr}	Roll torque in a rear suspension
T_{amb}	Ambient temperature
T_x	Torque about the x-axis
V	Forward velocity
V_w	Ambient wind velocity
V_f	Final velocity resulting from a brake application
V_o	Initial velocity in a brake application
w	Tire section width
W	Weight of the vehicle

W_a	Axle weight
W_d	Dynamic load transfer
W_f	Dynamic weight on the front axle
W_r	Dynamic weight on the rear axle
W_{rr}	Dynamic weight on the right rear wheel
W_{fs}	Static weight on the front axle
W_{rs}	Static weight on the rear axle
W_y	Lateral weight transfer on an axle
x	Forward direction on the longitudinal axis of the vehicle
y	Lateral direction out the right side of the vehicle
YM	Aerodynamic yawing moment
z	Vertical direction with respect to the plane of the vehicle
X	Forward direction of travel
Y	Lateral direction of travel
Z	Vertical direction of travel
	Vertical displacement of the sprung mass
Z_r	Road profile elevation
Z_u	Vertical displacement of the unsprung mass
α	Tire slip angle
	Coefficient in the pitch plane equations
α_{cw}	Aerodynamic wind angle
α_d	Rotational acceleration of the driveshaft
α_e	Rotational acceleration of the engine
α_w	Rotational acceleration of the wheels
α_x	Rotational acceleration about the x-axis
β	Sideslip angle Rotation angle of a U-joint Coefficient in the pitch plane equations
γ	Camber angle Coefficient in the pitch plane equations
γ_g	Wheel camber with respect to the ground
γ_b	Wheel camber with respect to the vehicle body
δ	Steer angle
δ_c	Compliance steer
δ_i	Steer angle of the inside wheel in a turn

δ_o	Steer angle of the outside wheel in a turn
Δ	Off-tracking distance in a turn
ε	Roll steer coefficient
	Inclination of the roll axis
ζ	Moment arm related to tire force yaw damping
	Half-shaft angle on a front-wheel drive
ζ_s	Damping ratio of the suspension
η_b	Braking efficiency
η_t	Efficiency of the transmission
η_f	Efficiency of the final drive
η_{tf}	Combined efficiency of the transmission and final drive
θ	Pitch angle
	Angle of a U-joint
θ_p	Body pitch due to acceleration squat or brake dive
Θ	Grade angle
λ	Lateral inclination angle of the steer axis (kingpin inclination angle)
μ	Coefficient of friction
μ_p	Peak coefficient of friction
μ_s	Sliding coefficient of friction
ν	Wavenumber of road roughness spectrum
ξ	Fraction of the drive force developed on the front axle of a 4WD
	Fraction of the brake force developed on the front axle
	Rear steer proportioning factor on a 4WS vehicle
ρ	Density of air
υ	Caster angle of the steer axis
ϕ	Roll angle
φ	Road cross-slope angle
χ	Ratio of unsprung to sprung mass
ψ	Heading angle
	Yaw angle
ω	Rotational speed
ω_d	Damped natural frequency of a suspension system (radians/second)
	Rotational speed of the driveshaft
ω_e	Rotational speed of the engine
ω_i	Rotational speed at the input of a U-joint
ω_n	Undamped natural frequency of a suspension system (radians/second)

ω_o Rotational speed at the output of a U-joint
ω_u Natural frequency of the unsprung mass
ω_w Rotational speed of the wheels

Although tire cornering stiffness was used as the basis for developing the equations for understeer/oversteer, there are multiple factors in vehicle design that may influence the cornering forces developed in the presence of a lateral acceleration. Any design factor that influences the cornering force developed at a wheel will have a direct effect on directional response. The suspensions and steering system are the primary sources of these influences. In this section the suspension factors affecting handling will be discussed.

Roll Moment Distribution

For virtually all pneumatic tires the cornering forces are dependent on, and nonlinear with, load. This is important because load is transferred in the lateral direction in cornering due to the elevation of the vehicle CG above the ground plane. Figure 6.11 shows a typical example of how lateral force varies with vertical load.

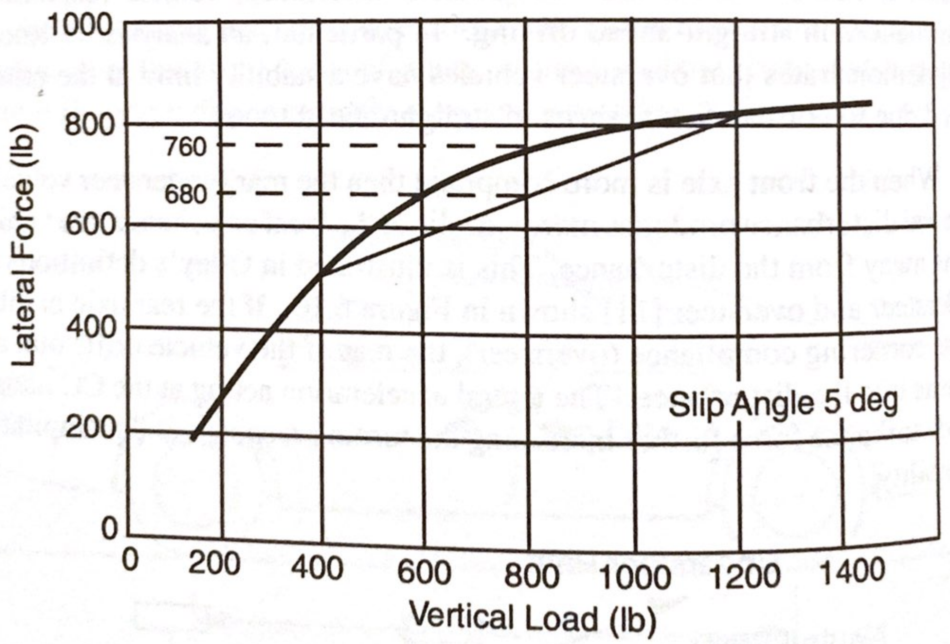

Fig. 6.11 Lateral force-vertical load characteristics of tires.

For a vehicle at 800 lb load on each wheel, about 760 lb of lateral force will be developed by each wheel at the 5-degree slip angle. In hard cornering, the loads might typically change to 400 lb on the inside wheel and 1200 lb on the outside. Then the average lateral force from both tires will be reduced to about

680 lb. Consequently, the tires will have to assume a greater slip angle to maintain the lateral force necessary for the turn. If these are front tires, the front will plough out and the vehicle will understeer. If on the rear, the rear will slip out and the vehicle will oversteer.

Actually, this mechanism is at work on both axles of all vehicles. Whether it contributes to understeer or oversteer depends on the balance of roll moments distributed on the front and rear axles. More roll moment on the front axle contributes to understeer, whereas more roll moment on the rear axle contributes to oversteer. Auxiliary roll stiffeners (stabilizer bars) alter handling performance primarily through this mechanism—applied to the front axle for understeer, and to the rear for oversteer.

The mechanics governing the roll moment applied to an axle are shown in the model of Figure 6.12. All suspensions are functionally equivalent to the two springs. The lateral separation of the springs causes them to develop a roll resisting moment proportional to the difference in roll angle between the body and axle. The stiffness is given by:

$$K_\phi = 0.5\, K_s\, s^2 \tag{6-26}$$

where:

K_ϕ = Roll stiffness of the suspension
K_s = Vertical rate of each of the left and right springs
s = Lateral separation between the springs

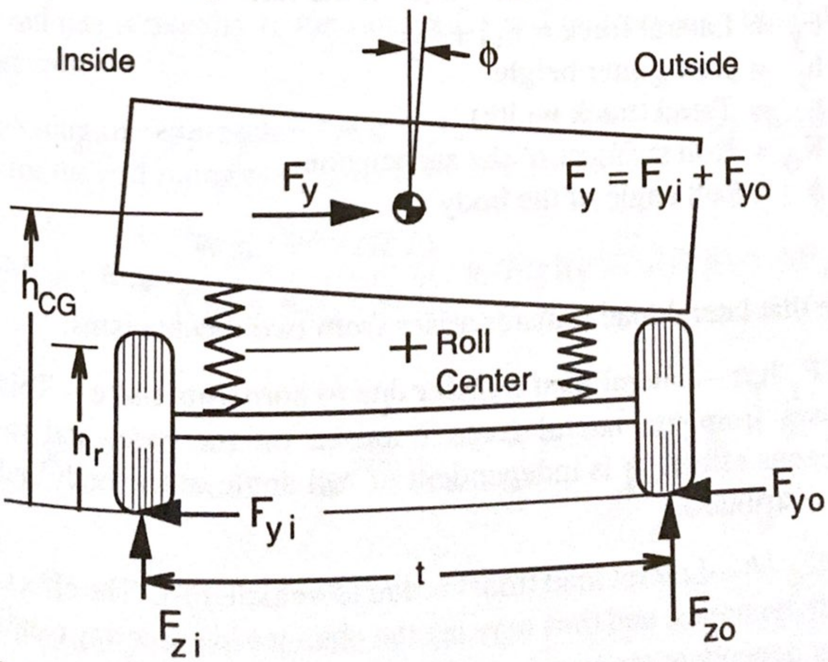

Fig. 6.12 Force analysis of a simple vehicle in cornering.

(In the case of an independent suspension, the above expression can be used by substituting the rate at the wheel for K_s and using the tread as the separation distance. When a stabilizer bar is present, the roll stiffness of the bar must be added to the stiffness calculated above.)

The suspension is further characterized by a "roll center," the point at which the lateral forces are transferred from the axle to the sprung mass. The roll center can also be thought of as the point on the body at which a lateral force application will produce no roll angle, and it is the point around which the axle rolls when subjected to a pure roll moment.

By writing Newton's Second Law for moments on the axle, we can determine the relationship between wheel loads and the lateral force and roll angle. In addition to the vertical forces imposed at the tires there is a net lateral force, F_y (the sum of the lateral forces on the inside and outside wheels), acting to the right on the axle at its roll center. The body roll acting through the springs imposes a torque on the axle proportional to the roll stiffness, K_ϕ, times the roll angle, ϕ. This results in an equation for the load difference from side to side of the form:

$$F_{zo} - F_{zi} = 2\, F_y\, h_r/t + 2\, K_\phi\, \phi/t = 2\, \Delta F_z \qquad (6\text{-}27)$$

where:

F_{zo} = Load on the outside wheel in the turn
F_{zi} = Load on the inside wheel in the turn
F_y = Lateral force = $F_{yi} + F_{yo}$
h_r = Roll center height
t = Tread (track width)
K_ϕ = Roll stiffness of the suspension
ϕ = Roll angle of the body

Note that lateral load transfer arises from two mechanisms:

1) $2\, F_y\, h_r/t$—Lateral load transfer due to cornering forces. This mechanism arises from the lateral force imposed on the axle, and is thus an instantaneous effect. It is independent of roll angle of the body and the roll moment distribution.

2) $2\, K_\phi\, \phi/t$—Lateral load transfer due to vehicle roll. The effect depends on the roll dynamics, and thus may lag the changes in cornering conditions. It is directly dependent on front/rear roll moment distribution.

The total vehicle must be considered to obtain the expression for the roll moment distribution on the front and rear axles. In this case, we define a roll axis as the line connecting the roll centers of the front and rear suspensions, as shown in Figure 6.13. Now the moment about the roll axis in this case is:

$$M_\phi = [W\, h_1 \sin\phi + W\, V^2/(R\, g)\, h_1 \cos\phi] \cos\varepsilon \tag{6-28}$$

For small angles, cos ϕ and cos ε may be assumed as unity, and sin $\phi = \phi$. Then:

$$M_\phi = W\, h_1\, [V^2/(R\, g) + \phi] \tag{6-29}$$

But:

$$M_\phi = M_{\phi f} + M_{\phi r} = (K_{\phi f} + K_{\phi r})\, \phi \tag{6-30}$$

Equations (6-28) and (6-29) can be solved for the roll angle, ϕ:

$$\phi = \frac{W\, h_1\, V^2/(R\, g)}{K_{\phi f} + K_{\phi r} - W\, h_1} \tag{6-31}$$

The derivative of this expression with respect to the lateral acceleration produces an expression for the roll rate of the vehicle:

$$R_\phi = d\phi/da_y = W\, h_1/[K_{\phi f} + K_{\phi r} - W\, h_1] \tag{6-32}$$

The roll rate is usually in the range of 3 to 7 degrees/g on typical passenger cars.

Combining the expression for ϕ from Eq. (6-31) with Eq. (6-29) allows solution for the roll moments on the front and rear axles:

$$M'_{\phi f} = K_{\phi f} \frac{W\, h_1\, V^2/(R\, g)}{K_{\phi f} + K_{\phi r} - W\, h_1} + W_f\, h_f\, V^2/(R\, g) = \Delta F_{zf}\, t_f \tag{6-33}$$

$$M'_{\phi r} = K_{\phi r} \frac{W\, h_1\, V^2/(R\, g)}{K_{\phi f} + K_{\phi r} - W\, h_1} + W_r\, h_r\, V^2/(R\, g) = \Delta F_{zr}\, t_r \tag{6-34}$$

where:

$$\Delta F_{zf} = F_{zfo} - W_f/2 = -(F_{zfi} - W_f/2)$$
$$\Delta F_{zr} = F_{zro} - W_r/2 = -(F_{zri} - W_r/2)$$

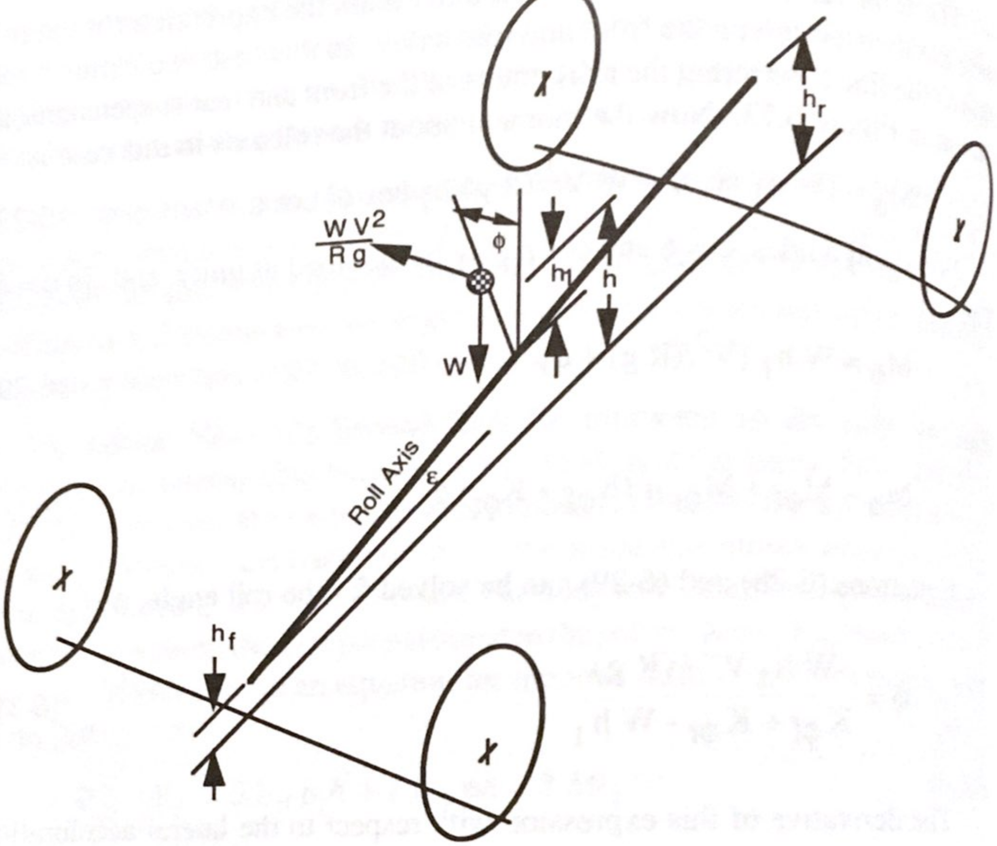

Fig. 6.13 Force analysis for roll of a vehicle.

In general, the roll moment distribution on vehicles tends to be biased toward the front wheels due to a number of factors:

1) Relative to load, the front spring rate is usually slightly lower than that at the rear (for flat ride), which produces a bias toward higher roll stiffness at the rear. However, independent front suspensions used on virtually all cars enhance front roll stiffness because of the effectively greater spread on the front suspension springs.

2) Designers usually strive for higher front roll stiffness to ensure understeer in the limit of cornering.

3) Stabilizer bars are often used on the front axle to obtain higher front roll stiffness.

4) If stabilizer bars are needed to reduce body lean, they may be installed on the front or the front and rear. Caution should be used when adding a stabilizer bar only to the rear because of the potential to induce unwanted oversteer.

We now have the solution for roll moments front and rear, and can calculate the difference in load between the left and right wheels on the axle. To translate the lateral load transfer into an effect on understeer gradient, it is necessary to have data which relates the tire cornering force to slip angle and load. At the given conditions, the slip angle on each axle will change when the load transfer is taken into account. The difference between the change on the front and rear (normalized by the lateral acceleration) represents the understeer effect. The effect can be modeled by expressing the tire load sensitivity as a polynomial. In the first analysis the cornering characteristics of the tires on an axle were described simply by a constant called the cornering stiffness, C_α. The cornering force developed on the axle was given by:

$$F_y = C_\alpha \, \alpha \tag{6-35}$$

where:

F_y = Lateral force developed on the axle
C_α = Cornering stiffness of two tires, each at one-half the axle load
α = Slip angle

To represent load sensitivity effect, the two tires (inside and outside) must be treated separately. The cornering stiffness of each tire can be represented by a second- or higher-order polynomial, and the lateral force developed by either will be given by:

$$F_y' = C_\alpha' \, \alpha = (a\,F_z - b\,F_z^2)\,\alpha \tag{6-36}$$

where:

F_y' = Lateral force of one tire
C_α' = Cornering stiffness of one tire
a = First coefficient in the cornering stiffness polynomial (lb_y/lb_z/deg)
b = Second coefficient in the cornering stiffness polynomial (lb_y/lb_z^2/deg)
F_z = Load on one tire (assumed equal on both tires in previous analysis)

For a vehicle cornering as shown in Figure 6.12, the lateral force of both tires, F_y, is given by:

$$F_y = (a\,F_{zo} - b\,F_{zo}^2 + a\,F_{zi} - b\,F_{zi}^2\;)\,\alpha \tag{6-37}$$

Now, let the load change on each wheel be given by ΔF_z.

$$F_{zo} = F_z + \Delta F_z \qquad F_{zi} = F_z - \Delta F_z \tag{6-38}$$

Then:

$$F_y = [a\,(F_z + \Delta F_z) - b\,(F_z + \Delta F_z)^2 + a\,(F_z - \Delta F_z) - b\,(F_z - \Delta F_z)^2\,]\,\alpha \tag{6-39}$$

This equation reduces to:

$$F_y = [2\,a\,F_z - 2\,b\,F_z{}^2 - 2\,b\;\;\Delta F_z{}^2\,]\,\alpha \tag{6-40}$$

The equation can be simplified if we recognize that the first two terms in the brackets are equivalent to the cornering stiffness of the tires at their static load conditions (as it has been defined in the previous analysis). Namely:

$$C_\alpha = 2\,a\,F_z - 2\,b\,F_z{}^2 \tag{6-41}$$

or:

$$F_y = [C_\alpha - 2\,b\;\;\Delta F_z{}^2\,]\,\alpha \tag{6-42}$$

Recall that the steer angle necessary to maintain a turn is given by:

$$\delta = 57.3\ L/R + \alpha_f - \alpha_r \tag{6-43}$$

For the two tires on the front we can write:

$$F_{yf} = [C_{\alpha f} - 2\,b\;\;\Delta F_{zf}{}^2\,]\,\alpha_f = W_f\,V^2/(R\ g) \tag{6-44}$$

and on the rear:

$$F_{yr} = [C_{\alpha r} - 2\,b\;\;\Delta F_{zr}{}^2\,]\,\alpha_r = W_r\,V^2/(R\ g) \tag{6-45}$$

Substituting to eliminate the slip angles in Eq. (6-43):

$$\delta = 57.3\,\frac{L}{R} + \frac{W_f\,V^2/(R\ g)}{(C_{\alpha f} - 2\,b\,\Delta F_{zf}^2)} - \frac{W_r\,V^2/(R\ g)}{(C_{\alpha r} - 2\,b\,\Delta F_{zr}^2)} \tag{6-46}$$

This equation can be simplified by utilizing the fact that $C_\alpha >> 2\,b\,\Delta F_z{}^2$. Then:

$$\frac{1}{(C_\alpha - 2\,b\,\Delta F_z^2)} = \frac{1}{C_\alpha\left(1 - \dfrac{2\,b\,\Delta F_z^2}{C_\alpha}\right)} \approx \frac{1}{C_\alpha}\left(1 + \frac{2\,b\,\Delta F_z^2}{C_\alpha}\right) \tag{6-47}$$

Equation (6-45) can be rewritten in the form:

$$\delta = 57.3\frac{L}{R} + [\,(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}) + (\frac{W_f}{C_{\alpha f}}\frac{2\,b\,\Delta F_{zf}^2}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\frac{2\,b\,\Delta F_{zr}^2}{C_{\alpha r}})]\frac{V^2}{R\,g}$$

$$\longleftarrow 1 \longrightarrow \quad \longleftarrow 2 \longrightarrow \tag{6-48}$$

Term number 1 inside the brackets is simply the understeer gradient arising from the nominal cornering stiffness of the tires, K_{tires}, as was developed earlier. The second term represents the understeer gradient arising from lateral load transfer on the tires; i.e.:

$$K_{llt} = \frac{W_f}{C_{\alpha f}}\frac{2\,b\,\Delta F_{zf}^2}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\frac{2\,b\,\Delta F_{zr}^2}{C_{\alpha r}} \tag{6-49}$$

The values for ΔF_{zf} and ΔF_{zr} can be obtained from Eqs. (6-33) and (6-34) as a function of lateral acceleration. Since all the variables in the above equation are positive, the contribution from the front axle is always understeer; that from the rear axle is always negative, meaning it is an oversteer effect.

Camber Change

The inclination of a wheel outward from the body is known as the camber angle [2]. Camber on a wheel will produce a lateral force known as "camber thrust." Figure 6.14 shows a typical camber thrust curve.

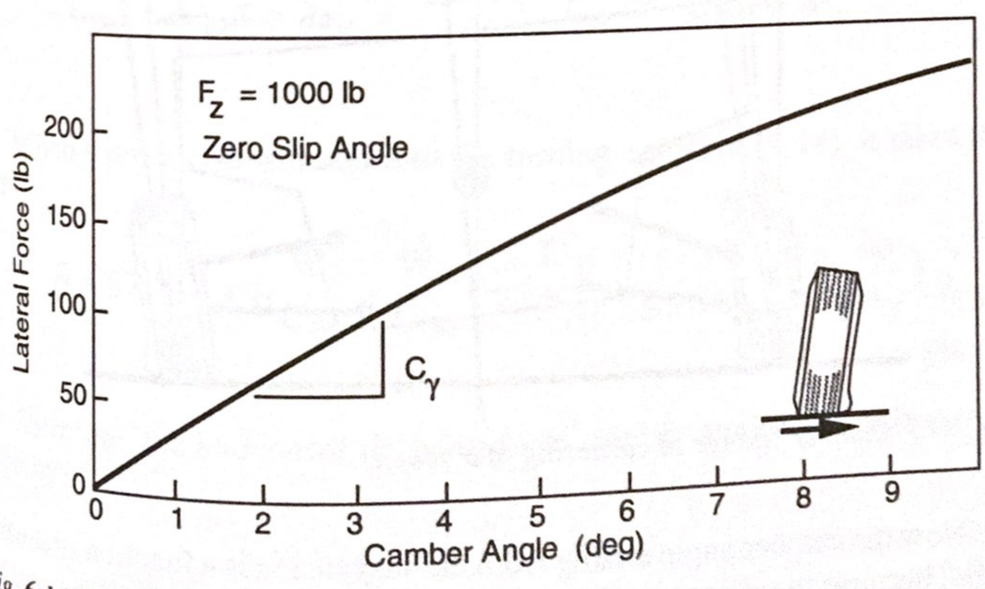

Fig. 6.14 Lateral force caused by camber of a tire.