ASTM Manual on

FITTING STRAIGHT LINES

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ASTM MANUAL ON FITTING STRAIGHT LINES

Prepared by

COMMITTEE E-11 ON QUALITY CONTROL OF MATERIALS

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PREFACE

The ASTM Manual on Quality Control of Materials includes the presentation and treatment of data pertaining to observations on a single variable. When more than one variable is involved, statistical methods other than those discussed are required for treating the data, for presenting in a concise form the essential information contained therein, and for drawing valid conclusions therefrom.

In ASTM work a frequent problem is to discover from an examination of the observations on two variables the nature and extent of the relationship, if any, between them. Examples of pairs of variables are: the hardness of a material and its tensile strength; aging time and strength of cement; concentration of a solute and the spectrophotometric reading; assays by a standard method and by a proposed new method.

The relationship between two variables may be linear or nonlinear. The treatment of data from two nonlinearly related variables is beyond the scope of this Manual. However, such data may often be made to assume a linear, or approximately linear, relationship by suitably transforming them, as by converting each observation to its logarithm or some other appropriate function. Such transformed data if linear may be treated by the procedures here presented.

This Manual discusses some of the ways for determining when a straight line may be judged to be inadequate. It is here that a plot of the data and the fitted straight line is often helpful. This is particularly useful in identifying wild observations that need to be rechecked. Also, if the relationship is curvilinear rather than straight line, bunching of points is very likely to occur, that is, an abnormal grouping of successive points on one side of the fitted line. It is apparent, then, that the straight line obtained by following the procedures outlined in this manual is reasonable from an engineering viewpoint only when the assumption of linearity is appropriate and is not contradicted by the data.

This manual makes no attempt at a complete survey of all the statistical methods available for treating experimental data from linearly related variables. It limits itself to a consideration of selected aspects of substantial importance in ASTM work and of frequent occurrence. Among the topics included are:

1. Presentation of the essential information in a set of linearly related pairs of observations by means of an equation (Sections 1, 2, and 3),
2. Formulas for establishing confidence limits for the constants of the line (Sections 4, 5 and 6),
3. Predictions about future observations based on the existing data (Section 7), and
4. Procedures for testing the hypotheses about the slope and intercept as well as the underlying linear relationship (Sections 8, 10, and 11).

Much of the material presented in this manual is based on Chapter 9 of *Engineering Statistics* by A. H. Bowker and G. J. Lieberman (6) © 1959 Prentice-

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This Manual was prepared by a Task Group of ASTM Committee E-11 on Quality Control of Materials. The personnel of the Task Group at the time of publication was as follows: L. Tanner, chairman, S. Collier, H. F. Dodge, R. J. Hader, G. J. Lieberman, and W. J. Youden. The Task Group gratefully acknowledges its indebtedness to its former chairman, J. H. Curtiss, to F. S. Acton who made available for the committee's use the manuscript of his book (7), and to the many who reviewed the manuscript of this Manual and have offered helpful suggestions for its improvement, and in particular to G. J. Lieberman.

CONTENTS

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ASTM MANUAL ON FITTING STRAIGHT LINES

1. Introduction

ASTM work often requires the presentation of observed data on two variables. The objective of the presentation generally is to show the nature and extent of the relationship between the variables and the uses to which the relationship may be used. Pairs of measurements may exhibit a linear or a nonlinear relationship. The treatment of nonlinear relationships is beyond the scope of this manual. However, it is often possible to transform nonlinearly related measurements by some suitable mathematical process so that a linear, or approximately linear, relationship is assumed by the data. Some of the commonly used transformations are the logarithms of the data, their roots, reciprocals, or other suitable functions. A discussion of transformations is given in Section 11.

It is usually advantageous to plot paired data so that the existence and nature of any relationship between them may be more readily observed. There are several ways of drawing a straight line through the plotted points. The simplest is by eye. When the scatter of the points is slight the line so drawn may be quite adequate for most purposes. For example, if experimental results showing the relationship between proportional limit and tensile strength of dental alloys are as shown in Fig. 1, a statistical analysis is probably unnecessary. Similarly, if experimental data calibrating a new method of determining calcium in the presence of large amounts of magnesium are as shown in Fig. 2, the analysis is very straight-forward. Different people observing these data would probably draw the same graph, and furthermore, such plots would be representative of the true relationship between the variables.

On the other hand, when the scatter of the points is considerable, visual plotting is too uncertain; different lines may be drawn by different individuals, thereby leading to nonobjectivity in the results. An example of such data is found in Table I and is plotted in Fig. 3. These data are taken from an actual experiment (1)[1] where a statistical analysis was carried out on certain mechanical properties of cast and wrought gold dental alloys. Several relationships were found between the mechanical properties, with the relationship between proportional limit and tensile strength shown in Fig. 3. If quantitative use is to be made of these data, visual plotting is unsatisfactory, and a more objective procedure is necessary.

As a further example where visual plotting is inadequate, suppose the data obtained for calibration of a new method of determining calcium in the presence of large amounts of magnesium are as shown in Table II and as plotted in Fig. 4. Ten different samples containing known amounts of calcium oxide were analyzed by the new method. Here again it is evident that a more objective method of determining the relationship between two variables is needed. The method used should be such that not only will all observers derive the same relationship from a given set of data, but it should also describe how the data can be used to predict results about future observations. The following sections will deal with these problems.

[1] The boldface numbers in parentheses refer to the list of references appended to this paper.

2. Types of Linear Relationships:

Interest in pairs of measurements often occurs when (1) there exists an underlying physical relationship between the variables or (2) there exists a degree of association between the variables.

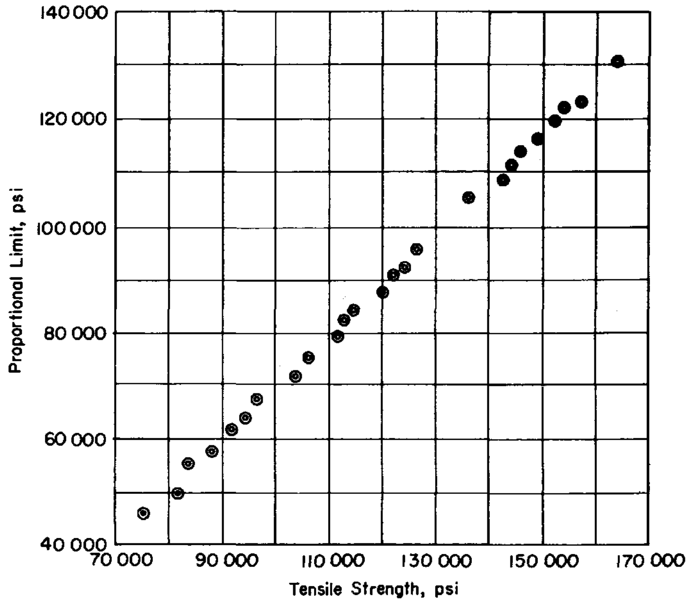

FIG. 1.—Relationship Between Proportional Limit and Tensile Strength of Dental Alloys.

In the first case, a functional relationship between Y and X is assumed. Observations are made on a response variable, Y, and a controlled variable, X. Since the response variable, Y, varies in a random fashion about its true value for any fixed value of X, the variable, Y, is called a random variable. The controlled variable, X, is assumed to be fixed. An example of such an experiment is the determination of the effect of time of aging on the strength of cement. The time corresponds to the X variate, the values of which are predetermined in the experiment. For a given value of time, the strength (corresponding to the Y variate) is the random variable. Many problems which involve time usually fall into the above framework. Time can frequently be measured to a sufficient degree of accuracy so that it can be assumed to be a known constant.

Other examples of a functional relationship are problems which involve calibration against a known standard. A

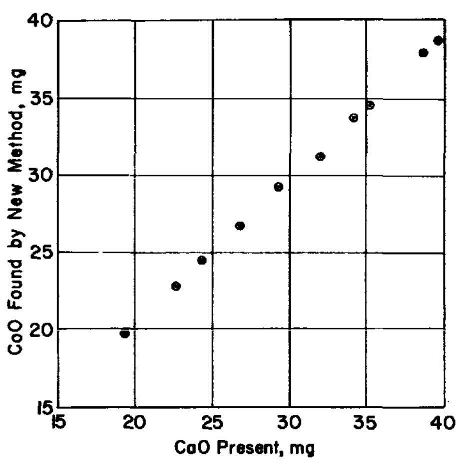

FIG. 2.—Gravimetric Determination of Calcium in the Presence of Magnesium.

TABLE I.—MECHANICAL PROPERTIES OF DENTAL GOLD ALLOYS.[a]

Tensile Strength, psi	Proportional Limit, psi
114 800	72 400
163 800	129 300
167 800	129 600
129 200	92 500
142 200	107 800
128 500	94 800
115 200	89 300
135 700	107 700
86 700	55 000
115 800	87 000
108 200	66 900
90 700	51 700
75 200	49 500
111 500	69 200
130 700	101 400
152 800	128 200
135 700	100 700
140 500	97 800
112 700	84 900
107 300	67 000
130 800	95 300
81 200	49 700
126 000	79 800
100 800	65 700
87 500	58 000

[a] Data from Reference (1).

series of observations may be taken by a laboratory on material whose contents are known accurately by design. The random variable, Y, can be considered as the laboratory measurement, whereas the true composition may be regarded as the X variate. The example of calibrating a new method of determining CaO falls into this category. In each of these examples, and in the general situation, interest is centered on determining the average value of the random variable, Y, as a function of the fixed value, X. Naturally, X is never known exactly, but it is sufficient to have the error in X small. More precisely, the ratio of the error of X to that of Y must be small as compared to the reciprocal of the slope.

The degree of association case deals with observations X and Y, each of which represents measurements on random variables associated with different characteristics of the same item. Interest is centered on determining the relationship between the two variables for any of a number of reasons. Measurements

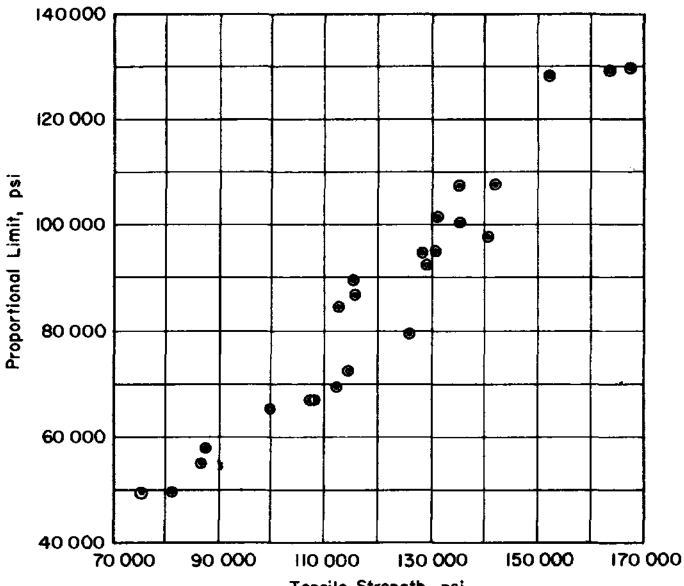

FIG. 3.—Relationship Between Proportional Limit and Tensile Strength.

on one variable, say X, may be relatively inexpensive compared with measurements of Y, thereby resulting in a monetary saving if Y can be predicted from a knowledge of X. For example, determining abrasion loss is difficult, whereas measuring hardness by means of a Rockwell hardness machine is relatively simple. There exists a degree of association between abrasion loss and hardness. The example of proportional limit and tensile strength also falls into this category. In these examples, and in this general degree of association model, interest is centered on representing the average value of the random variable, Y, as a function of a given value that the random variable, X, takes on.

TABLE II.—GRAVIMETRIC DETERMINATION OF CALCIUM IN THE PRESENCE OF MAGNESIUM.

CaO Present, mg	CaO Found by New Method, mg
20.0	19.8
22.5	22.8
25.0	24.5
28.5	27.3
31.0	31.0
33.5	35.0
35.5	35.1
37.0	37.1
38.0	38.5
40.0	39.0

Thus, both the underlying physical relationship model and the degree of association model, each arising from different physical situations, are concerned with the average value of Y for a given value of X. Hence, it is not surprising to find that the two methods of analysis are essentially equivalent. Thus, once the experimenter recognizes the distinction in models, the formal mechanics of analysis can be carried out without regard to which of the physical situations exist.

3. Least Squares Estimates of the Slope and Intercept:

Both of the models presented in the previous section have the property that the average value of the random variable, Y, can be expressed as a linear function of a known variate, X, that is,

$$\text{average value of } Y = \beta_0 + \beta_1 X.$$

The values of β_0 and β_1 are usually unknown and are to be estimated from experimental data. The estimate of β_0 will be denoted by b_0 and of β_1 by b_1. Thus, the estimated relationship will be of the form

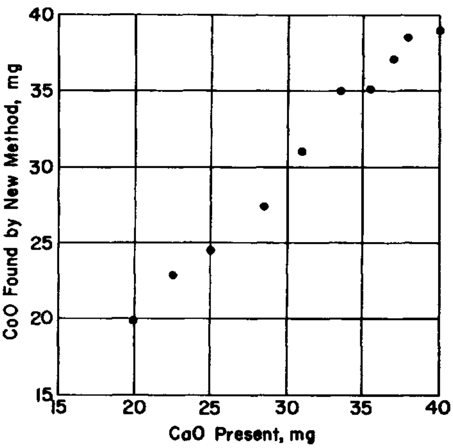

FIG. 4.—Gravimetric Determination of Calcium in the Presence of Magnesium.

$$\bar{Y} = b_0 + b_1 X.$$

Returning to the properties of gold dental alloys, the underlying (unknown) relationship between proportional limit and tensile strength may be expressed as:

$$\text{average value of proportional limit} = \beta_0 + \beta_1 \times \text{tensile strength}.$$

β_0 and β_1 are estimated from the experimental data, and the estimated relationship is:

estimated proportional limit

$$= b_0 + b_1 \times \text{tensile strength.}$$

In the calibration of CaO, the underlying (unknown) relationship between the amount of CaO present and the amount determined by the new method may be expressed as:

average value of CaO determined by the

$$\text{new method} = \beta_0 + \beta_1 \times \text{CaO present.}$$

The data will yield an estimated relation of the form:

estimated value of CaO determined by

$$\text{the new method} = b_0 + b_1 \times \text{CaO present.}$$

The proportional limit versus tensile strength example illustrates the degree of association model, whereas the new CaO method versus the known quantity illustrates the underlying physical relationship model.

The usual technique for estimating the intercept β_0 and slope β_1 is by the method of least squares. The values of b_0 and b_1 are determined such that the sum of the squares of the deviation of Y about the fitted line $\tilde{Y} = b_0 + b_1X$ is a minimum; that is, such that $(Y_1 - \tilde{Y}_1)^2 + (Y_2 - \tilde{Y}_2)^2 + \cdots + (Y_n - \tilde{Y}_n)^2$ is a minimum. If (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n) are the n pairs of observations obtained experimentally,

$$b_1 = \frac{(X_1 - \bar{X})(Y_1 - \bar{Y}) + (X_2 - \bar{X})(Y_2 - \bar{Y}) + \cdots + (X_n - \bar{X})(Y_n - \bar{Y})}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

and $b_0 = \bar{Y} - b_1\bar{X}$ are the least squares estimates of β_1 and β_0, respectively, where:

$$\bar{Y} = \frac{Y_1 + Y_2 + \cdots + Y_n}{n} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

and

$$\bar{X} = \frac{X_1 + X_2 + \cdots X_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

A computational procedure for calculating b_0 and b_1 is given in Section 12.

If the Y's are independent (in a probability sense) and the variability of Y is constant for all X's (for a discussion of this point, see Section 11) the values of b_0 and b_1 given above are unbiased estimates of β_0 and β_1, respectively; that is, b_0, on the average, will equal β_0, and b_1, on the average, will equal β_1. Further, the estimates, b_0 and b_1, are the *best* in the sense of having the smallest variance of all unbiased estimates that are linear functions of the observed Y's. Thus, the variance of the estimated line $\tilde{Y} = b_0 + b_1X$ will be less than the variance of any other linear estimates. The least squares estimate is therefore also called the best linear unbiased estimate.

4. Confidence Interval Estimates of the Slope and Intercept:

The method of least squares for obtaining estimates of the slope and intercept does not depend upon any assumptions about the distribution of Y. Furthermore, it was pointed out in the last section that under limited assumptions, these estimates are the "best linear unbiased estimates" of the slope and intercept. The least squares estimates have other desirable properties[2] provided the following conditions are satisfied:

[2] The least squares estimates are under these conditions the maximum likelihood estimates.

TABLE III.—VALUES OF t.[a]

n	P, per cent						
	50	75	90	95	97.5	99	99.5
3	1.00000	2.4142	6.3138	12.706	25.452	63.657	127.32
4	0.81650	1.6036	2.9200	4.3027	6.2053	9.9248	14.089
5	0.76489	1.4226	2.3534	3.1825	4.1765	5.8409	7.4533
6	0.74070	1.3444	2.1318	2.7764	3.4954	4.6041	5.5976
7	0.72669	1.3009	2.0150	2.5706	3.1634	4.0321	4.7733
8	0.71756	1.2733	1.9432	2.4469	2.9687	3.7074	4.3168
9	0.71114	1.2543	1.8946	2.3646	2.8412	3.4995	4.0293
10	0.70639	1.2403	1.8595	2.3060	2.7515	3.3554	3.8325
11	0.70272	1.2297	1.8331	2.2622	2.6850	3.2498	3.6897
12	0.69981	1.2213	1.8125	2.2281	2.6338	3.1693	3.5814
13	0.69745	1.2145	1.7959	2.2010	2.5931	3.1058	3.4966
14	0.69548	1.2089	1.7823	2.1788	2.5600	3.0545	3.4284
15	0.69384	1.2041	1.7709	2.1604	2.5326	3.0123	3.3725
16	0.69242	1.2001	1.7613	2.1448	2.5096	2.9768	3.3257
17	0.69120	1.1967	1.7530	2.1315	2.4899	2.9467	3.2860
18	0.69013	1.1937	1.7459	2.1199	2.4729	2.9208	3.2520
19	0.68919	1.1910	1.7396	2.1098	2.4581	2.8982	3.2225
20	0.68837	1.1887	1.7341	2.1009	2.4450	2.8784	3.1966
21	0.68763	1.1866	1.7291	2.0930	2.4334	2.8609	3.1737
22	0.68696	1.1848	1.7247	2.0860	2.4231	2.8453	3.1534
23	0.68635	1.1831	1.7207	2.0796	2.4138	2.8314	3.1352
24	0.68580	1.1816	1.7171	2.0739	2.4055	2.8188	3.1188
25	0.68531	1.1802	1.7139	2.0687	2.3979	2.8073	3.1040
26	0.68485	1.1789	1.7109	2.0639	2.3910	2.7969	3.0905
27	0.68443	1.1777	1.7081	2.0595	2.3846	2.7874	3.0782
28	0.68405	1.1766	1.7056	2.0555	2.3788	2.7787	3.0669
29	0.68370	1.1757	1.7033	2.0518	2.3734	2.7707	3.0565
30	0.68335	1.1748	1.7011	2.0484	2.3685	2.7633	3.0469
40	0.68099	1.1682	1.6860	2.0244	2.3337	2.7116	2.9803
60	0.67876	1.1620	1.6716	2.0017	2.3011	2.6633	2.9184
120	0.67660	1.1560	1.6579	1.9802	2.2704	2.6181	2.8608
∞	0.67449	1.1503	1.6449	1.9600	2.2414	2.5758	2.8070

[a] Computed by Maxine Merrington from Reference **(2)**.

(*a*) the average value of Y for a given X is equal to $\beta_0 + \beta_1 X$,

(*b*) the variability of Y is constant for all X's,

(*c*) the distribution of Y for the universe sampled is approximately normal, and

(*d*) the sample is a random sample.

It can be shown that if the above conditions are satisfied,[3] the estimates of the slope and intercept are normally distributed with means β_1 and β_0, respectively. Therefore, in addition to having b_1 and b_0 as point estimates of β_1 and β_0, respectively, confidence interval estimates with fixed confidence limits can also be given.

The confidence interval for β_1 is given by:

$$b_1 \pm \frac{ts}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2}}.$$

[3] In the following sections these conditions will be assumed to be satisfied.

and for β_0 is given by:

$$b_0 \pm ts \sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2}}$$

where the value of t is given in Table III for seven values of P, the P-per cent confidence limit, and for various values of n, the number of pairs of observations. s is an estimate of the variability about the line and is given by:

$$s = \sqrt{\frac{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}{n - 2}}$$

$$= \sqrt{\frac{(Y_1 - \bar{Y})^2 + (Y_2 - \bar{Y})^2 + \cdots + (Y_n - \bar{Y})^2}{n - 2}}.$$

A computational form for s is given in Section 12. It is evident that $\sum_{i=1}^{n} (Y_i - \bar{Y})^2$ is just the sum of squares of the deviations about the fitted line. Finally, $\sum_{i=1}^{n} (X_i - \bar{X})^2 = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2$ is just the sum of squares of the X value about $\bar{X}$ where $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$. A computational form for this value is also given in Section 12.

The meaning of a confidence statement is as follows. If the values of t given in Table III for $P = 95$ per cent are used in a series of problems involving β_1, then in the long run we may expect 95 per cent of the interval bounded by the limits so computed to include the true slope β_1. If in each instance we were to assert that β_1 lies within the limits computed, we should expect to be correct 95 times in 100 and in error 5 times in 100; that is, the statement "β_1 lies within the interval so computed" has a 95 per cent probability of being correct. But there would be no operational meaning in the following statement made in any one instance: "The probability is 95 per cent that β_1 falls within the limits computed in this case" since β_1 either does or does not fall within the limits. It should also be emphasized that even in repeated sampling from the same universe the interval defined by the limits

$$b_1 \pm \frac{ts}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2}}$$

will vary in width and position from sample to sample, particularly with small samples. It is this series of intervals fluctuating in size and position which will include, ideally, the true slope β_1 95 times out of 100 for $P = 95$ per cent.

These limits are commonly referred to as confidence limits; to construct P-per cent confidence limits, the appropriate column of Table III is entered.

It is evident that the confidence interval for β_1 is a minimum whenever $\sum_{i=1}^{n} (X_i - \bar{X})^2$ is a maximum. Since the X's are assumed to be fixed, the choice of values of X's is often available. The value of $\sum_{i=1}^{n} (X_i - \bar{X})^2$ is maximized when the X observations are divided equally at the two extreme points of the interested range of the X's. However, this should be done only when there is strong *a priori* knowledge that the relationship is linear since such a division of the points precludes picking up any nonlinear effects. A method of testing for the lack of fit of the fitted straight line is given in Section 10.

5. Point Estimates and Confidence Interval Estimates of the Average Value of Y for a Given X:

Very often estimates of straight-line relationships are desired in order to get point estimates or interval estimates of the average value of Y corresponding to a given X. For example, in the calcium oxide calibration, the experimenters in presenting their results may be interested in estimating by a confidence interval, or a point estimate, the average value of CaO present as determined by their new

TABLE IV —SUMMARY OF POINT ESTIMATES, CONFIDENCE INTERVAL ESTIMATES, AND PREDICTION INTERVAL ESTIMATES.

Parameter	Symbol for Estimate	Computation Formula for Estimate	P, per cent Confidence Interval
β_1	b_1	$\frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2}$	$b_1 \pm \frac{ts}{\sqrt{\Sigma(X_i - \bar{X}^2}}$
β_0	b_0	$\bar{Y} - b_1\bar{X}$	$b_0 \pm ts \sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\Sigma(X_i - \bar{X})^2}}$
Average value of Y for a given value of $X = X^*$	$\tilde{Y}^*$	$b_0 + b_1\bar{X}^*$	$b_0 + b_1X^* \pm ts \sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}$
Value of X corresponding to an observed value of Y' for the case where there is an underlying physical relationship	X'	$\frac{Y' - b_0}{b_1}$	$\frac{Y' - b_0}{b_1} \pm \frac{ts}{b_1}\sqrt{1 + \frac{1}{n} + \frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\Sigma(X_i - \bar{X})^2}}$ [a]
Parameter	**Symbol for Estimate**	**Computation Formula for Estimate**	**P, per cent Prediction Interval**
Future observation on Y, Y^0, corresponding to $X = X^0$	Y^0	$b_0 + b_1X^0$	$b_0 + b_1X^0 \pm ts \sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}$

[a] Approximate form.

method for a fixed amount of CaO present. Another example deals with the relationship between average tensile strength of cement as a function of curing time, t; that is, tensile strength is related to time by the function, $C\ e^{-\beta_1/t}$, or, in straight-line form, the average value of ln (strength) $=$ ln (C) $-$ β_1/t, where ln means the natural logarithm, ln (strength) corresponds to Y, ln (C) corresponds to β_0, $1/t$ corresponds to X, and β_1 and C are constants. A cement manufacturer is interested in either a point estimate or an interval estimate of the average tensile strength of his cement after a particular period of time. A point estimate for the average value of Y for a given value of X, say X^*, is given by

$$\bar{Y}^* = b_0 + b_1X^*.$$

A confidence interval for the average value of Y for a given value of X, say X^*, is given by:

$$b_0 + b_1X^* \pm ts\sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2}}$$

where the value of t is given in Table III for seven values of P, the P-per cent confidence coefficient, and for various values of n. The quantities s, n, and $\bar{X}$ are as defined in the previous section.

This interval becomes increasingly large as X^* moves away from $\bar{X}$. This is intuitively sound since estimates are expected to become poorer when one extrapolates away from the range used in the original experiment. The interval is narrowest whenever $X^* = \bar{X}$.

6. Point Estimates and Interval Estimates of the Independent Variable X Associated with an Observation on the Dependent Variable Y:

A point estimate or an interval estimate of the independent variable associated with an observation on the dependent variable Y is often appropriate when there is an underlying physical relation. For example, in the determination of calcium oxide, the experimenter may wish to estimate the true amount of CaO present, given a determination of this amount using the new method. A point estimate of X for a given observation on Y, say Y', is given by

$$X' = \frac{Y' - b_0}{b_1}.$$

A confidence interval estimate of X for a given observation in Y, say Y', is given approximately by

$$\frac{Y' - b_0}{b_1} \pm \frac{ts}{b_1}\sqrt{1 + \frac{1}{n} + \frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\sum (X_i - \bar{X})^2}}$$

where the value of t is given in Table III for seven values of P, the P-per cent confidence coefficient, and for various values of n.

A summary of point estimates and confidence interval estimates is given in Table IV.

7. Prediction Interval for a Future Observation on the Dependent Variable:

Section 5 dealt with determining a confidence interval for the mean value of Y corresponding to a given X, say X^*. The average value of Y is a fixed, but unknown, constant and a confidence interval is appropriate. It is often the case, however, that a confidence statement about the mean value is not of major importance, whereas a probability statement about a future observation is relevant. For example, in the discussion about the relationship between tensile strength of cement and curing time, it was pointed out that the cement manufacturer is interested in the average tensile strength of his cement after a particular period of time, that is, a confidence statement, whereas a builder is interested in the tensile strength of his particular batch of cement to determine whether it will carry the required load. The builder will find a confidence statement inadequate. He requires assurance that after a specified period of time, say 28 days, the probability is P that the tensile strength of his particular single batch of cement will lie in a specified interval. In the CaO determination, the experimenter may be interested in making a probability statement about an interval containing a future determination by the new method for a fixed concentration of CaO rather than making a confidence statement about the average value of CaO determined by the new method for this given concentration.

Finally, in the determination of the mechanical properties of gold dental alloys, the particular dentist using the alloy may be interested in the average value of proportional limit for an alloy having a fixed tensile strength, whereas the patient is interested in the proportional limit of the gold in his mouth for a given tensile strength; that is, the patient is interested in a prediction interval in which the proportional limit of the gold alloy in his mouth will lie.

TABLE V.—VALUES OF G.

n	P, per cent					
	90	95	97.5	99	99.5	99.9
3	49.5	200	800	5 000	20 000	500 000
4	9.00	19.0	39.0	99.0	199	999
5	5.46	9.55	16.0	30.8	49.8	148
6	4.32	6.94	10.6	18.0	26.3	61.2
7	3.78	5.79	8.43	13.3	18.3	36.6
8	3.46	5.14	7.26	10.9	14.5	27.0
9	3.26	4.74	6.54	9.55	12.4	21.7
10	3.11	4.46	6.06	8.65	11.0	18.5
11	3.01	4.26	5.71	8.02	10.1	16.4
12	2.92	4.10	5.46	7.56	9.43	14.9
13	2.86	3.98	5.26	7.21	8.91	13.8
14	2.81	3.89	5.10	6.93	8.51	13.0
15	2.76	3.81	4.97	6.70	8.19	12.3
16	2.73	3.74	4.86	6.51	7.92	11.8
17	2.70	3.68	4.76	6.36	7.70	11.3
18	2.67	3.63	4.69	6.23	7.51	11.0
19	2.64	3.59	4.62	6.11	7.35	10.7
20	2.62	3.55	4.56	6.01	7.21	10.4
22	2.59	3.49	4.46	5.85	6.99	9.95
24	2.56	3.44	4.38	5.72	6.81	9.61
26	2.54	3.40	4.32	5.61	6.66	9.34
28	2.52	3.37	4.27	5.53	6.54	9.12
30	2.50	3.34	4.22	5.45	6.44	8.93
∞	2.30	3.00	3.69	4.61	5.30	6.91

The following statement can be made: The probability is P that a future observation, Y^0, corresponding to X^0 will lie in the interval

$$b_0 + b_1X^0 \pm ts\sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2}}$$

Values of P and t can be found in Table III.

Again, this interval is narrowest when $X^0 = \bar{X}$, and increases as X^0 becomes more distant from $\bar{X}$.

8. Tests of Hypothesis about the Slope and Intercept:

It is sometimes useful to determine whether the true value of the slope or intercept is equal to some hypothesized values. For example, in the CaO determination, the experimenters may wish to determine whether their new method of determination was perfect, that is, $\beta_0 = 0$, or $\beta_1 = 1$, or both $\beta_0 = 0$ and $\beta_1 = 1$ simultaneously. The fact that the experimental results may not lead to $b_0 = 0$ and $b_1 = 1$ may just be attributed to experimental error. Hence, some objective procedures are in order to answer the above questions.

The hypothesis that $\beta_0 = \beta_0'$ is rejected whenever the absolute value of

$$\frac{b_0 - \beta_0'}{s\sqrt{\dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum (X_i - \bar{X})^2}}}$$

exceeds t. The hypothesis that $\beta_1 = \beta_1'$ is rejected whenever the absolute value of

$$\frac{b_1 - \beta_1'}{s\sqrt{\dfrac{1}{\sum (X_i - \bar{X})^2}}}$$

exceeds t. The hypothesis that $\beta_0 = \beta_0$ and $\beta_1 = \beta_1'$ simultaneously is rejected whenever

$$g = [n(b_0 - \beta_0')^2 + 2n\bar{X}(b_0 - \beta_0')(b_1 - \beta_1') + (\textstyle\sum X_i^2)(b_1 - \beta_1')^2]/(2)s^2$$

exceeds G. Whenever one of the hypotheses above is true, the probability is P-per cent that the rule will not lead to rejection; that is, the probability (in per cent) of rejection when the hypothesis is true (the probability of an error being made) is $1 - P$. The value of t in the above equations is given in Table III for seven values of P and for various values of n. The value of G is given in Table V for six values of P and various values of n.[4] It should be emphasized that rejection of the hypothesis when using the "g test" implies that either β_0 does not equal β_0', β_1 does not equal β_1', or both.

9. Estimation of the Slope β_1 when the Intercept β_0 is Known to be Zero:

Situations frequently arise in which it is possible to assume beforehand that the intercept β_0 is zero. The average value of Y at different levels of X is given by

$$\text{average } (Y) = \beta_1 X.$$

The least squares estimate for β_1 is given by

$$b_1 = \frac{\sum X_i Y_i}{\sum X_i^2}$$

This is similar to the previous result except that the actual values of X and Y are used rather than the deviations about their mean. The estimated line is given by

$$\tilde{Y} = b_1 X.$$

A confidence interval for β_1 is given by

$$b_1 \pm \frac{t's}{\sqrt{\sum X_i^2}}$$

where the value of t' can be found in Table III for seven values of P, the P-

[4] To get a joint simultaneous confidence interval estimate of β_0 and β_1 with confidence coefficient, P, the sample values of b_0, b_1, and s^2 are substituted into the expression for g. The confidence region is defined by $g \leq G$. To determine whether a given pair, β_0, β_1, is included in the confidence region, one substitutes these given values of β_0 and β_1 into the expression for g; then, if $g \leq G$, the pair, β_0, β_1, is included in the confidence region. The values of G appearing in Table V correspond to the percentage points of the F distribution with 2 and n-2 deg of freedom and are excerpted from this distribution.

per cent confidence coefficient. For n pairs of observations, the value of t' is read from the row corresponding to $n + 1$. s is an estimate of the variability about the line and is given by

$$s = \sqrt{\frac{\sum (Y_i - \hat{Y})^2}{n - 1}}$$

$$= \sqrt{\sum Y_i^2 - \frac{(\sum X_i Y_i)^2}{\sum X_i^2}} \Big/ \sqrt{n - 1}.$$

A point estimate of the average value of Y for a given value of X, say X^*, is given by

$$\hat{Y}^* = b_1 X^*.$$

A confidence interval estimate for the average value of Y for a given value of X, say X^*, is given by

$$b_1 X^* \pm t's \sqrt{\frac{X^{*2}}{\sum X_i^2}}$$

where the values of t' and s can be found as described above.

A point estimate of the value of X corresponding to a given observation on Y, say Y', is given by

$$X' = \frac{Y'}{b_1}.$$

A confidence interval estimate for the value of X corresponding to a given observation on Y, say Y', is given approximately by

$$\frac{Y'}{b_1} \pm \frac{t's}{b_1} \sqrt{1 + \frac{(Y'/b_1)^2}{\sum X_i^2}}$$

where t' and s are as described above.

A prediction interval having the property that the probability is P that a future observation Y^0 corresponding to X^0 will lie in the interval is given by

$$b_1 X^0 \pm t's \sqrt{1 + \frac{(X^0)^2}{\sum X_i^2}}$$

Values of t' can be found in Table III as described above.

Finally, the hypothesis that $\beta_1 = \beta_1$ can be rejected whenever the absolute value of

$$\frac{b_1 - \beta_1'}{\sqrt{s^2/\sum X_i^2}} \quad \text{exceeds } t'.$$

The probability of rejection when the hypothesis is true is $(100 - P)$ per cent, and values of P and t' can be found in Table III where the value of t' is read from the row corresponding to $n + 1$.

10. Determining the Adequacy of the Straight Line Fit:

In all of the previous sections, it was assumed that the curve to be estimated was linear in X. If the experiment is performed so that there are k values of Y for each of the X's, a test for linearity can be made.[5] Let $\bar{Y}_i$ be the mean of the k observations on the Y_i and Y_{ij} be the jth of the k observations of Y corresponding to X_i. Thus, if there are four Y's corresponding to X_3, the values of Y_i are Y_{31}, Y_{32}, Y_{33}, Y_{34}, and $\bar{Y}_3$ is the average of these four observations.

The least squares estimate of the line can be obtained in the usual fashion using the nk individual observations, or the equivalent result may be obtained from the relations

$$b_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(\bar{Y}_i - \bar{\bar{Y}})}{\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

and

$$b_0 = \bar{\bar{Y}} - b_1 \bar{X}$$

where:

$$\bar{\bar{Y}} = \frac{\sum_{i=1}^{n} \bar{Y}_i}{n} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k} Y_{ij}}{nk}.$$

[5] Actually, all that is necessary is that there be more than one value of Y for some X.

The hypothesis of linearity is rejected if

$$\frac{\sum_{i=1}^{n} k(\bar{Y}_i - \tilde{Y}_i)^2/(n-2)}{\sum_{i=1}^{n}\sum_{j=1}^{k} (Y_{ij} - \bar{Y}_i)^2/n(k-1)}$$

exceeds F, where the values of F are given in Table VI for two values of P, the probability (in per cent) of the rule not rejecting the hypothesis when there is a true linear relationship. The table is a double entry table with the column corresponding to $n - 2$ and the row to $n(k - 1)$.

This test is reasonable in that it compares the variability about the fitted line (numerator) with the inherent variability of the Y's which is independent of the form of the relationship (denominator). If the functional relationship between Y and X is a straight line, these variabilities should compare favorably.

The test depends upon having more than one Y for the settings on X. If there is just one Y for each X, a supplementary investigation should be carried out by studying the quantities $(Y_i - \tilde{Y}_i)/s$. These values are approximately normally distributed with mean 0 and variance 1 for large n provided the relationship is linear with constant variability. A control chart procedure may be instituted by plotting $(Y_i - \tilde{Y}_i)/s$ against the original X_i. Three sigma control limits can be placed at -3 and $+3$. Points falling outside of these limits are an indication of the inadequacy of the straight line fit with constant variability.

TABLE VI.—DATA ON ANALYSIS OF CALCIUM OXIDE.

X_i	Y_{i1}	Y_{i2}	$\bar{Y}_i$	$\tilde{Y}_i$	$(\bar{Y}_i - \tilde{Y}_i)^2$	$(Y_{i1} - \bar{Y}_i)^2$	$(Y_{i2} - \bar{Y}_i)^2$	$X_i\bar{Y}_i$
20.0	19.8	19.6	19.70	19.61	0.0081	0.0100	0.0100	394.000
22.5	22.8	22.1	22.45	22.15	0.0900	0.1225	0.1225	505.125
25.0	24.5	24.3	24.40	24.69	0.0841	0.0100	0.0100	610.000
28.5	27.3	28.4	27.85	28.24	0.1521	0.3025	0.3025	793.725
31.0	31.0	30.0	30.50	30.77	0.0729	0.2500	0.2500	945.500
33.5	35.0	33.0	34.00	33.31	0.4761	1.0000	1.0000	1 139.000
35.5	35.1	35.0	35.05	35.34	0.0841	0.0025	0.0025	1 244.275
37.0	37.1	36.8	36.95	36.86	0.0081	0.0225	0.0225	1 367.150
38.0	38.5	38.0	38.25	37.88	0.1369	0.0625	0.0625	1 453.500
40.0	39.0	40.2	39.60	39.90	0.0900	0.3600	0.3600	1 584.000
					1.2024	2.1425	2.1425	10 036.275

11. Transforming to a Straight Line:

If there is prior knowledge that the relationship is nonlinear, or if this has been determined by a test such as is described in the previous section, it is often possible to transform the data to a linear form. An example of such a transformation was given in Section 5 where the relationship between tensile strength of cement and curing time was expressed as

$$Y = C\, e^{-\beta_1/t}.$$

The transformation consisted of taking logarithms of both sides, whereby we obtain a linear relation of the form

$$\text{average value of } \ln(Y) = \ln(C) - \beta_1/t,$$

where $\ln(C)$ corresponds to β_0 and $\frac{1}{t}$ to X.

Generally, then, it is often possible by simple transformations of the variables to represent the relationship as a straight line in the transformed variates. The transformations are usually chosen on the

General Data:

X represents ____________.

$\sum X_i =$ ________; $\bar{X} = \sum X_i / n =$ ________; $(\sum X_i)\bar{X} =$ ________; $\sum X_i^2 =$ ________;

$\sum X_i^2 - (\sum X_i)\bar{X} = \sum (X_i - \bar{X})^2 =$ ________.

Y represents ____________.

$\sum Y_i =$ ________; $\bar{Y} = \sum Y_i / n =$ ________; $(\sum Y_i)\bar{Y} =$ ________; $\sum Y_i^2 =$ ________;

$\sum Y_i^2 - (\sum Y_i)\bar{Y} = \sum (Y_i - \bar{Y})^2 =$ ________.

$n =$ ________; $\frac{1}{n} =$ ________; $(\sum X_i)(\sum Y_i)/n = \bar{X} \sum Y_i =$ ________;

$\sum X_i Y_i =$ ________; $\sum X_i Y_i - (\sum X_i)(\sum Y_i)/n = \sum (X_i - \bar{X})(Y_i - \bar{Y}) =$ ________.

$\left[\sum (X_i - \bar{X})(Y_i - \bar{Y})\right]^2 =$ ________; $\dfrac{\left[\sum (X_i - \bar{X})(Y_i - \bar{Y})\right]^2}{\sum (X_i - \bar{X})^2} =$ ________.

Data for Equation of Line:

$b = \dfrac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} =$ ________; $b_0 = \bar{Y} - b_1 \bar{X} =$ ________;

Equation of Line $= \tilde{Y} = b_0 + b_1 X =$ ________ + ________ X.

Estimation of Standard Deviation:

$(n-2)s^2 = \sum (Y_i - \bar{Y})^2 - \dfrac{\left[\sum (X_i - \bar{X})(Y_i - \bar{Y})\right]^2}{\sum (X_i - \bar{X})^2} =$ ________;

$s^2 =$ ________; $s =$ ________.

Data for Significance Tests and for Confidence Intervals for β_0 and β_1:

$\dfrac{1}{\sum (X_i - \bar{X})^2} =$ ________; $\sqrt{\dfrac{1}{\sum (X_i - \bar{X})^2}} =$ ________; $s\sqrt{\dfrac{1}{\sum (X_i - \bar{X})^2}} =$ ________.

$\bar{X}^2 =$ ________; $\dfrac{\bar{X}^2}{\sum (X_i - \bar{X})^2} =$ ________; $\dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum (X_i - \bar{X})^2} =$ ________;

$\sqrt{\dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum (X_i - \bar{X})^2}} =$ ________; $s\sqrt{\dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum (X_i - \bar{X})^2}} =$ ________.

FIG. 5.—Worksheet for the

Data for Confidence Intervals for Average Value of Y Corresponding to $X = X^*$:

$X^* =$ ________; $(X^* - \bar{X})^2 =$ ________; $\dfrac{(X^* - \bar{X})^2}{\Sigma(X_i - \bar{X})^2} =$ ________;

$\dfrac{1}{n} + \dfrac{(X^* - \bar{X})^2}{\Sigma(X_i - \bar{X})^2} =$ ________; $\sqrt{\dfrac{1}{n} + \dfrac{(X^* - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}} =$ ________;

$s\sqrt{\dfrac{1}{n} + \dfrac{(X^* - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}} =$ ________.

Data for Prediction Intervals for a Future Value of Y Corresponding to $X = X^0$:

$X^0 =$ ________; $(X^0 - \bar{X})^2 =$ ________; $\dfrac{(X^0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2} =$ ________;

$1 + \dfrac{1}{n} + \dfrac{(X^0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2} =$ ________; $\sqrt{1 + \dfrac{1}{n} + \dfrac{(X^0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}} =$ ________;

$s\sqrt{1 + \dfrac{1}{n} + \dfrac{(X^0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}} =$ ________.

Data for Confidence Intervals for X Corresponding to an Observed Value of $Y = Y'$ for the Case when there is an Underlying Physical Relationship:

$Y' =$ ________; $X' = \dfrac{Y' - b_0}{b_1} =$ ________; $\left(\dfrac{Y' - b_0}{b_1} - \bar{X}\right)^2 =$ ________;

$\dfrac{\left(\dfrac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\Sigma(X_i - \bar{X})^2} =$ ________; $1 + \dfrac{1}{n} + \dfrac{\left(\dfrac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\Sigma(X_i - \bar{X})^2} =$ ________;

$\sqrt{1 + \dfrac{1}{n} + \dfrac{\left(\dfrac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\Sigma(X_i - \bar{X})^2}} =$ ________; $s\sqrt{1 + \dfrac{1}{n} + \dfrac{\left(\dfrac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\Sigma(X_i - \bar{X})^2}} =$ ________.

basis of a graphical analysis of the observations, using all the prior knowledge available about the theoretical underlying relationship. Caution must be exercised to verify that the assumptions underlying the use of the previously described techniques are satisfied for the transformed variates.

In some instances, Y may have a linear relationship with some function of X, in which case no transformation of Y is required. For example, if the average value of Y is given by $\beta_1 \cos Z$ the relationship is linear with respect to the variate $\cos Z$, though not with respect to Z. In any case, the assumptions underlying the use of the previously described techniques must be satisfied for Y or the function of Y, whichever is the appropriate variate.

It sometimes occurs that although the underlying relationship is linear, the variability is not constant for all X's. Variance stabilizing transformations are often applied in this case. If the variability in Y tends to increase linearly with X, using the square roots (or higher roots) of the data often stabilizes the variability. If the variability increases at a higher rate, a logarithm transformation is sometimes appropriate. If the variability tends to increase and then decrease with increasing values of X, an inverse sine transformation often is fruitful. It must be emphasized that a constant variance is not the only condition sought, and precautions are still necessary when using the techniques described previously with the transformed variables. Fortunately, however, the transformation of scale to meet the condition of constant variability often has the effect of improving the closeness of the distribution to normality; a correlation of variability of Y with X in the original scale often supplies excessive skewness which tends to be eliminated after the transformation. But the validity of any assumption of normality should be watched, for while moderate departures from normality are known not to be serious, any large departures in the region of more outlying observations are likely to affect the validity of the probability statements made.

12. Work Sheets for Fitting Straight Lines:

A worksheet for fitting straight lines is given in Fig. 5. This worksheet has been arranged for ease of computation when using a hand calculator. Completion of this table will result in obtaining all the quantities described in the previous sections. It is important to carry more decimal places in the calculations than are necessary for the final results, because rounding intermediate calculations can adversely effect the final answer.

13. Illustrative Examples:

Example 1: The data on proportional limit and tensile strength of gold dental alloys given in Table I, in Section 1, will be analyzed. The following quantities will be obtained:

1. Least squares estimates of β_0 and β_1,

2. 95 per cent confidence interval estimates of β_0 and β_1,

3. A point estimate of the average value of the proportional limit, $\bar{Y}^*$, corresponding to a tensile strength of 129,000 psi, X^*,

4. A 90 per cent confidence interval estimate of the average value of the proportional limit corresponding to a tensile strength of 129,000 psi, X^*,

5. A 95 per cent prediction interval for the proportional limit corresponding to a tensile strength of 129,000 psi, X^0, and

6. A test of the hypothesis that $\beta_1 = 1.5$ such that the probability of accepting the hypothesis when it is true is 95 per cent.

From the data given on the worksheet for Example 1, the following are obtained for each of the catagories listed:

1. The least squares estimates of β_0 and β_1 are computed directly on the worksheet; the rounded off values of $b_0 = -30800$ and $b_1 = 0.970$ are obtained. The least squares line is shown in Fig. 5.

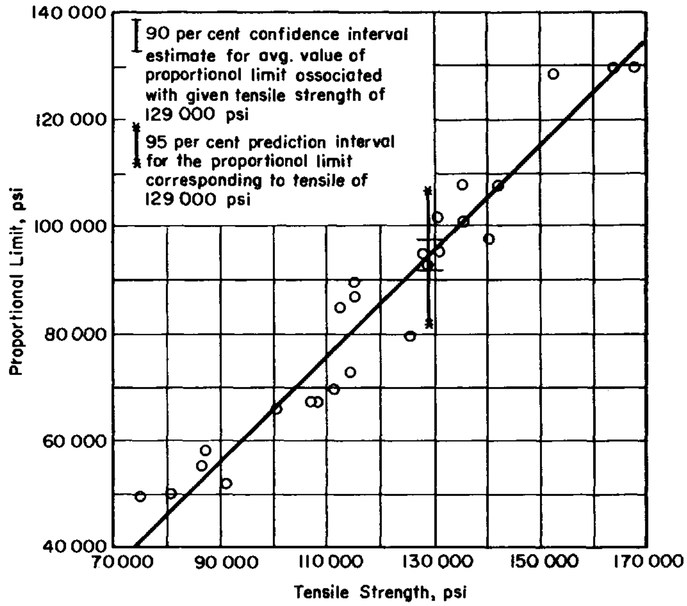

FIG. 6.—Fitted Relationship Between Proportional Limit and Tensile Strength of Dental Alloys.

2. To compute the confidence interval estimates of β_0 and β_1, the value of $t = 2.0687$ corresponding to $P = 95$ per cent and $n = 25$ is obtained from Table III. Substitute into the formulas given in Section 4; the confidence interval for β_0 is

$$- 30793.79165 \pm (2.0687)(6242.826839)$$

$$= - 30793.79165 \pm 12914.53588$$

$$= [-43708.32753, -17879.25577)$$

$$= [-43700, -17900].$$

Similarly, the confidence interval estimate for β_1 is

$$0.96982743 \pm (2.0687)(0.0511487401)$$

$$= 0.96982743 \pm 0.10581140$$

$$= [0.86401603, 1.07563883] = [0.864, 1.076].$$

3. A point estimate of the average value of the proportional limit corresponding to a tensile strength of $X^* =$ 129,000 psi is given by

$$\bar{Y}^* = b_0 + b_1X^* = - 30793.79165 + (0.96982743)(129000)$$

$$= 94313.946 = 94300$$

4. To compute the confidence interval estimate of the average value of the proportional limit associated with a given tensile strength of $X^* = 129{,}000$ psi, the value of $t = 1.7139$ corresponding to $P = 90$ per cent and $n = 25$ is obtained from Table III. One substitutes into the formula given in Section 5; the required

confidence interval is given by

$$- 30793.79165 + (0.96982743)(129000) \pm (1.7139)(1321.553192)$$
$$= 94313.94682 \pm 2265.01002$$
$$= [92048.93680, 96578.95684]$$
$$= [92000, 96600].$$

This interval is shown in Fig. 6.

for the prediction interval

$$- 30793.79165 + (0.96982743)(129000) \pm (2.0687)(6300.290867)$$
$$= 94313.94682 \pm 13033.41172$$
$$= [81280.53510, 107347.35854]$$
$$= [81300, 107300].$$

This interval is shown in Fig. 6.

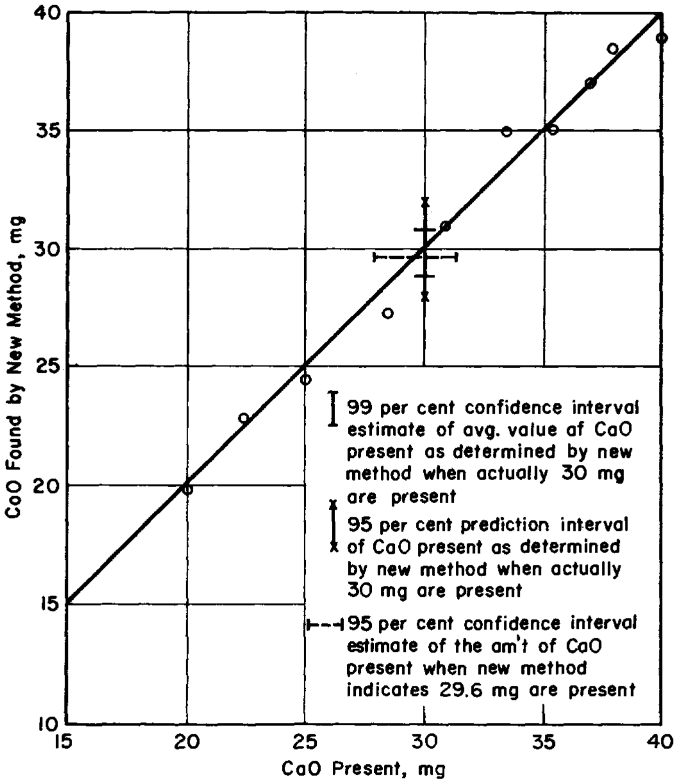

FIG. 7.—Gravimetric Determination of Calcium in the Presence of Magnesium Fitted Relationship

5. To compute the prediction interval for the proportional limit corresponding to a tensile strength of $X^0 = 129{,}000$ psi, a value of $t = 2.0687$ associated with $P = 95$ per cent and $n = 25$ is obtained from Table III. Then substituting into the formula given in Section 7, one obtains

6. The hypothesis that $\beta_1 = 1.5$ is rejected at the $P = 95$ per cent level if the absolute value of

$$\frac{b_1 - 1.5}{s\sqrt{\frac{1}{\sum (X_i - \bar{X})^2}}} = \frac{0.96982743 - 1.5}{0.0511487401} = -10.3653$$

exceeds t corresponding to $P = 95$ per cent and $n = 25$. This value of t is 2.0687. Thus:

$$|-10.3653| = 10.3653 > 2.0687$$

and, therefore, the hypothesis that $\beta_1 = 1.5$ is rejected.

Example 2: The data on calibrating the new method of determining CaO given in Table II will be analyzed. The following quantities will be obtained:

1. Least squares estimates of β_0 and β_1,
2. 90 per cent confidence interval estimate of β_0 and β_1,
3. A point estimate of the average value of the amount of CaO present obtained by the new method, Y^*, when there is actually 30 mg present, X^*,
4. A 99 per cent confidence interval estimate of the average value of the amount of CaO present obtained by the new method when there is actually 30 mg present, X^*,
5. A 95 per cent confidence interval estimate of the true amount of CaO present when the new method gives a quantity equal to 29.6 mg, Y',
6. A 95 per cent prediction interval for the amount of CaO present determined by the new method when there is actually 30 mg present, X^0,
7. A test of the hypothesis that $\beta_1 = 1$ such that the probability of accepting the hypothesis when it is true is 90 per cent, and
8. A joint test of the hypothesis that $\beta_0 = 0$ and $\beta_1 = 1$ such that the probability of accepting the hypothesis when it is true is 90 per cent.

From the data given on the worksheet for Example 2, we obtain:

1. The least squares estimates of β_0 and β_1 are computed directly on the worksheet; the rounded off values of $b_0 = -0.293$ and $b_1 = 1.007$ are obtained. The least squares line is shown in Fig. 7.

2. To compute the confidence interval estimates of β_0 and β_1, the value of $t = 1.8595$ corresponding to $P = 90$ per cent and $n = 10$ is obtained from Table III. Substitute into the formulas given in Section 4; the confidence interval for β_0 is

$$\begin{aligned}&-0.2927786865 \pm 1.8595\,(1.261163829)\\ &= -0.2927786865 \pm 2.3451341400\\ &= [-2.637912827, 2.052355454]\\ &= [-2.6, 2.1].\end{aligned}$$

Similarly, the confidence interval estimate for β_1 is

$$\begin{aligned}&1.006520215 \pm (1.8595)(0.03968357772)\\ &= 1.006520215 \pm 0.073791613\\ &= [0.932728602, 1.080311828]\\ &= [0.933, 1.080].\end{aligned}$$

3. A point estimate of the average value of the amount of CaO present obtained by the new method when there is actually $X^* = 30$ mg present is given by

$$\tilde{Y}^* = -0.2927786865 + (1.006520215)(30) = 29.90282776 = 29.9.$$

4. To compute the confidence interval estimate of the average value of the amount of CaO present obtained by the new method when there is actually $X^* = 30$ mg present, the value of $t = 3.3554$ corresponding to $P = 99$ per cent and $n = 10$ is obtained from Table III. One substitutes into the formula given in Section 5; the required confidence interval is given by

$$\begin{aligned}&-0.2927786865 + (1.006520215)(30) \pm (3.3554)(0.2632310752)\\ &= 29.90282776 \pm 0.88324555)\\ &= [29.01958221, 30.78607331] = [29.0, 30.8]\end{aligned}$$

This interval is shown in Fig. 7.

5. To compute the confidence interval estimate of the true amount of CaO present when the new method gives a

TABLE VII.—VALUES OF F.[a]

$n\ (k-1)$	$n-2$								
	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54[a]
	4052.2	4999.5	5403.3	5624.6	5763.7	5859.0	5928.3	5981.6	6022.5[a]
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385
	98.503	99.000	99.166	99.249	99.299	99.332	99.356	99.374	99.388
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8868	8.8452	8.8123
	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345
4	7.7086	6.9443	6.5914	6.3883	6.2560	6.1631	6.0942	6.0410	5.9988
	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8759	4.8183	4.7725
	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158
6	5.9874	5.1433	4.7571	4.5337	4.3874	4.2839	4.2066	4.1468	4.0990
	13.745	10.925	9.7795	9.1483	8.7459	8.4661	8.2600	8.1016	7.9761
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7870	3.7257	3.6767
	12.246	9.5466	8.4513	7.8467	7.4604	7.1914	6.9928	6.8401	6.7188
8	5.3177	4.4590	4.0662	3.8378	3.6875	3.5806	3.5005	3.4381	3.3881
	11.259	8.6491	7.5910	7.0060	6.6318	6.3707	6.1776	6.0289	5.9106
9	5.1174	4.2565	3.8626	3.6331	3.4817	3.3738	3.2927	3.2296	3.1789
	10.561	8.0215	6.9919	6.4221	6.0569	5.8018	5.6129	5.4671	5.3511
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.1355	3.0717	3.0204
	10.044	7.5594	6.5523	5.9943	5.6363	5.3858	5.2001	5.0567	4.9424
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	3.0123	2.9480	2.8962
	9.6460	7.2057	6.2167	5.6683	5.3160	5.0692	4.8861	4.7445	4.6315
12	4.7472	3.8853	3.4903	3.2592	3.1059	2.9961	2.9134	2.8486	2.7964
	9.3302	6.9266	5.9526	5.4119	5.0643	4.8206	4.6395	4.4994	4.3875
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.8321	2.7669	2.7144
	9.0738	6.7010	5.7394	5.2053	4.8616	4.6204	4.4410	4.3021	4.1911
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.7642	2.6987	2.6458
	8.8616	6.5149	5.5639	5.0354	4.6950	4.4558	4.2779	4.1399	4.0297
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.7066	2.6408	2.5876
	8.6831	6.3589	5.4170	4.8932	4.5556	4.3183	4.1415	4.0045	3.8948
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.6572	2.5911	2.5377
	8.5310	6.2262	5.2922	4.7726	4.4374	4.2016	4.0259	3.8896	3.7804
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.6143	2.5480	2.4943
	8.3997	6.1121	5.1850	4.6690	4.3359	4.1015	3.9267	3.7910	3.6822

[a] In each row, the top figures are values of F corresponding to P = 95 per cent, and the bottom figures correspond to P = 99 per cent.

TABLE VII.—(*Continued*).

$n(k-1)$	$n-2$								
	1	2	3	4	5	6	7	8	9
18	4.4139 8.2854	3.5546 6.0129	3.1599 5.0919	2.9277 4.5790	2.7729 4.2479	2.6613 4.0146	2.5767 3.8406	2.5102 3.7054	2.4563 3.5971
19	4.3808 8.1850	3.5219 5.9259	3.1274 5.0103	2.8951 4.5003	2.7401 4.1708	2.6283 3.9386	2.5435 3.7653	2.4768 3.6305	2.4227 3.5225
20	4.3513 8.0960	3.4928 5.8489	3.0984 4.9382	2.8661 4.4307	2.7109 4.1027	2.5990 3.8714	2.5140 3.6987	2.4471 3.5644	2.3928 3.4567
21	4.3248 8.0166	3.4668 5.7804	3.0725 4.8740	2.8401 4.3688	2.6848 4.0421	2.5727 3.8117	2.4876 3.6396	2.4205 3.5056	2.3661 3.3981
22	4.3009 7.9454	3.4434 5.7190	3.0491 4.8166	2.8167 4.3134	2.6613 3.9880	2.5491 3.7583	2.4638 3.5867	2.3965 3.4530	2.3419 3.3458
23	4.2793 7.8811	3.4221 5.6637	3.0280 4.7649	2.7955 4.2635	2.6400 3.9392	2.5277 3.7102	2.4422 3.5390	2.3748 3.4057	2.3201 3.2986
24	4.2597 7.8229	3.4028 5.6136	3.0088 4.7181	2.7763 4.2184	2.6207 3.8951	2.5082 3.6667	2.4226 3.4959	2.3551 3.3629	2.3002 3.2560
25	4.2417 7.7698	3.3852 5.5680	2.9912 4.6755	2.7587 4.1774	2.6030 3.8550	2.4904 3.6272	2.4047 3.4568	2.3371 3.3239	2.2821 3.2172
26	4.2252 7.7213	3.3690 5.5263	2.9751 4.6366	2.7426 4.1400	2.5868 3.8183	2.4741 3.5911	2.3883 3.4210	2.3205 3.2884	2.2655 3.1818
27	4.2100 7.6767	3.3541 5.4881	2.9604 4.6009	2.7278 4.1056	2.5719 3.7848	2.4591 3.5580	2.3732 3.3882	2.3053 3.2558	2.2501 3.1494
28	4.1960 7.6356	3.3404 5.4529	2.9467 4.5681	2.7141 4.0740	2.5581 3.7539	2.4453 3.5276	2.3593 3.3581	2.2913 3.2259	2.2360 3.1195
29	4.1830 7.5976	3.3277 5.4205	2.9340 4.5378	2.7014 4.0449	2.5454 3.7254	2.4324 3.4995	2.3463 3.3302	2.2782 3.1982	2.2229 3.0920
30	4.1709 7.5625	3.3158 5.3904	2.9223 4.5097	2.6896 4.0179	2.5336 3.6990	2.4205 3.4735	2.3343 3.3045	2.2662 3.1726	2.2107 3.0665
40	4.0848 7.3141	3.2317 5.1785	2.8387 4.3126	2.6060 3.8283	2.4495 3.5138	2.3359 3.2910	2.2490 3.1238	2.1802 2.9930	2.1240 2.8876
60	4.0012 7.0771	3.1504 4.9774	2.7581 4.1259	2.5252 3.6491	2.3683 3.3389	2.2540 3.1187	2.1665 2.9530	2.0970 2.8233	2.0401 2.7185
120	3.9201 6.8510	3.0718 4.7865	2.6802 3.9493	2.4472 3.4796	2.2900 3.1735	2.1750 2.9559	2.0867 2.7918	2.0164 2.6629	1.9588 2.5586
∞	3.8415 6.6349	2.9957 4.6052	2.6049 3.7816	2.3719 3.3192	2.2141 3.0173	2.0986 2.8020	2.0096 2.6393	1.9384 2.5113	1.8799 2.4073

TABLE VII.—(*Continued*).

$n(k-1)$	$n-2$								
	10	12	15	20	30	40	60	120	∞
1	241.88	243.91	245.95	248.01	250.09	251.14	252.20	253.25	254.32
	6055.8	6106.3	6157.3	6208.7	6260.7	6286.8	6313.0	6339.4	6366.0
2	19.396	19.413	19.429	19.446	19.462	19.471	19.479	19.487	19.496
	99.399	99.416	99.432	99.449	99.466	99.474	99.483	99.491	99.501
3	8.7855	8.7446	8.7029	8.6602	8.6166	8.5944	8.5720	8.5494	8.5265
	27.229	27.052	26.872	26.690	26.505	26.411	26.316	26.221	26.125
4	5.9644	5.9117	5.8578	5.8025	5.7459	5.7170	5.6878	5.6581	5.6281
	14.546	14.374	14.198	14.020	13.838	13.745	13.652	13.558	13.463
5	4.7351	4.6777	4.6188	4.5581	4.4957	4.4658	4.4314	4.3984	4.3650
	10.051	9.8883	9.7222	9.5527	9.3793	9.2912	9.2020	9.1118	9.0204
6	4.0600	3.9999	3.9381	3.8742	3.8082	3.7743	3.7398	3.7047	3.6688
	7.8741	7.7183	7.5590	7.3958	7.2285	7.1432	7.0568	6.9690	6.8801
7	3.6365	3.5747	3.5108	3.4445	3.3758	3.3404	3.3043	3.2674	3.2298
	6.6201	6.4691	6.3143	6.1554	5.9921	5.9084	5.8236	5.7372	5.6495
8	3.3472	3.2840	3.2184	3.1503	3.0794	3.0428	3.0053	2.9669	2.9276
	5.8143	5.6668	5.5151	5.3591	5.1981	5.1156	5.0316	4.9460	4.8588
9	3.1373	3.0729	3.0061	2.9365	2.8637	2.8259	2.7872	2.7475	2.7067
	5.2565	5.1114	4.9621	4.8080	4.6486	4.5667	4.4831	4.3978	4.3105
10	2.9782	2.9130	2.8450	2.7740	2.6996	2.6609	2.6211	2.5801	2.5379
	4.8492	4.7059	4.5582	4.4054	4.2469	4.1653	4.0819	3.9965	3.9090
11	2.8536	2.7876	2.7186	2.6464	2.5705	2.5309	2.4901	2.4480	2.4045
	4.5393	4.3974	4.2509	4.0990	3.9411	3.8596	3.7761	3.6904	3.6025
12	2.7534	2.6866	2.6169	2.5436	2.4663	2.4259	2.3842	2.3410	2.2962
	4.2961	4.1553	4.0096	3.8584	3.7008	3.6192	3.5355	3.4494	3.3608
13	2.6710	2.6037	2.5331	2.4589	2.3803	2.3392	2.2966	2.2524	2.2064
	4.1003	3.9603	3.8154	3.6646	3.5070	3.4253	3.3413	3.2548	3.1654
14	2.6021	2.5342	2.4630	2.3879	2.3082	2.2664	2.2230	2.1778	2.1307
	3.9394	3.8001	3.6557	3.5052	3.3476	3.2656	3.1813	3.0942	3.0040
15	2.5437	2.4753	2.4035	2.3275	2.2468	2.2043	2.1601	2.1141	2.0658
	3.8049	3.6662	3.5222	3.3719	3.2141	3.1319	3.0471	2.9595	2.8684
16	2.4935	2.4247	2.3522	2.2756	2.1938	2.1507	2.1058	2.0589	2.0096
	3.6909	3.5527	3.4089	3.2588	3.1007	3.0182	2.9330	2.8447	2.7528
17	2.4499	2.3807	2.3077	2.2304	2.1477	2.1040	2.0584	2.0107	1.9604
	3.5931	3.4552	3.3117	3.1615	3.0032	2.9205	2.8348	2.7459	2.6530

TABLE VII.—(*Concluded*).

$n(k-1)$	$n-2$								
	10	12	15	20	30	40	60	120	∞
18	2.4117	2.3421	2.2686	2.1906	2.1071	2.0629	2.0166	1.9681	1.9168
	3.5082	3.3706	3.2273	3.0771	2.9185	2.8354	2.7493	2.6597	2.5660
19	2.3779	2.3080	2.2341	2.1555	2.0712	2.0264	1.9796	1.9302	1.8780
	3.4338	3.2965	3.1533	3.0031	2.8442	2.7608	2.6742	2.5839	2.4893
20	2.3479	2.2776	2.2033	2.1242	2.0391	1.9938	1.9464	1.8963	1.8432
	3.3682	3.2311	3.0880	2.9377	2.7785	2.6947	2.6077	2.5168	2.4212
21	2.3210	2.2504	2.1757	2.0960	2.0102	1.9645	1.9165	1.8657	1.8117
	3.3098	3.1729	3.0299	2.8796	2.7200	2.6359	2.5484	2.4568	2.3603
22	2.2967	2.2258	2.1508	2.0707	1.9842	1.9380	1.8895	1.8380	1.7831
	3.2576	3.1209	2.9780	2.8274	2.6675	2.5831	2.4951	2.4029	2.3055
23	2.2747	2.2036	2.1282	2.0476	1.9605	1.9139	1.8649	1.8128	1.7570
	3.2106	3.0740	2.9311	2.7805	2.6202	2.5355	2.4471	2.3542	2.2559
24	2.2547	2.1834	2.1077	2.0267	1.9390	1.8920	1.8424	1.7897	1.7331
	3.1681	3.0316	2.8887	2.7380	2.5773	2.4923	2.4035	2.3099	2.2107
25	2.2365	2.1649	2.0889	2.0075	1.9192	1.8718	1.8217	1.7684	1.7110
	3.1294	2.9931	2.8502	2.6993	2.5383	2.4530	2.3637	2.2695	2.1694
26	2.2197	2.1479	2.0716	1.9898	1.9010	1.8533	1.8027	1.7488	1.6906
	3.0941	2.9579	2.8150	2.6640	2.5026	2.4170	2.3273	2.2325	2.1315
27	2.2043	2.1323	2.0558	1.9736	1.8842	1.8361	1.7851	1.7307	1.6717
	3.0618	2.9256	2.7827	2.6316	2.4699	2.3840	2.2938	2.1984	2.0965
28	2.1900	2.1179	2.0411	1.9586	1.8687	1.8203	1.7689	1.7138	1.6541
	3.0320	2.8959	2.7530	2.6017	2.4397	2.3535	2.2629	2.1670	2.0642
29	2.1768	2.1045	2.0275	1.9446	1.8543	1.8055	1.7537	1.6981	1.6377
	3.0045	2.8685	2.7256	2.5742	2.4118	2.3253	2.2344	2.1378	2.0342
30	2.1646	2.0921	2.0148	1.9317	1.8409	1.7918	1.7396	1.6835	1.6223
	2.9791	2.8431	2.7002	2.5487	2.3860	2.2992	2.2079	2.1107	2.0062
40	2.0772	2.0035	1.9245	1.8389	1.7444	1.6928	1.6373	1.5766	1.5089
	2.8005	2.6648	2.5216	2.3689	2.2034	2.1142	2.0194	1.9172	1.8047
60	1.9926	1.9174	1.8364	1.7480	1.6491	1.5943	1.5343	1.4673	1.3893
	2.6318	2.4961	2.3523	2.1978	2.0285	1.9360	1.8363	1.7263	1.6006
120	1.9105	1.8337	1.7505	1.6587	1.5543	1.4952	1.4290	1.3519	1.2539
	2.4721	2.3363	2.1915	2.0346	1.8600	1.7628	1.6557	1.5330	1.3805
∞	1.8307	1.7522	1.6664	1.5705	1.4591	1.3940	1.3180	1.2214	1.0000
	2.3209	2.1848	2.0385	1.8783	1.6964	1.5923	1.4730	1.3246	1.0000

quantity $Y' = 29.6$ mg, the value of $t = 2.3060$ corresponding to $P = 95$ per cent and $n = 10$ is obtained from Table III. One substitutes into the formula given in Section 6; the required confidence interval is

$$\frac{29.6 - (-0.2927786865)}{1.006520215} \pm \frac{2.3060}{1.006520215}(0.8627436375)$$

$$= 29.69913395 \pm 1.97659898$$

$$= [27.72253497, \quad 31.67573293] = [27.7, \quad 31.7]$$

This interval is shown in Fig. 7.

6. To compute the prediction interval for the amount of CaO present determined by the new method corresponding to an actual value $X^0 = 30$ mg present, a value of $t = 2.3060$ associated with $P = 95$ per cent and $n = 10$ is obtained from Table III. Then substituting into the formula given in Section 7; one obtains for the prediction interval:

$$-0.2927786865 + (1.006520215)(30) \pm (2.3060)(0.8620566534)$$

$$= 29.90282776 \pm 1.98790264$$

$$= [27.91492512, 31.89073040] = [27.9, 31.9].$$

This interval is shown in Fig. 7.

7. To test the hypothesis that $\beta_1 = 1$, substitute into the expression given in Section 8;

$$\frac{b_1 - 1}{s\sqrt{\dfrac{1}{\sum (X_i - \bar{X})^2}}} = \frac{0.006520215}{0.03968357772} = 0.1643.$$

The absolute value of the number is compared with the value of t corresponding to $P = 90$ per cent and $n = 10$ which is obtained from Table III and is found to be 1.8595. Thus

$$|0.1643| = 0.1643 < 1.8595$$

and the hypothesis that $\beta_1 = 1$ is not rejected at the 90 per cent level.

8. To test the joint hypothesis, $\beta_0 = 0$ and $\beta_1 = 1$, substitute into the expression for g given in Section 8. For this example, one obtains

$$g = [10(-0.2927786865)^2 + 20(31.1)(-0.2927786865)(0.006520215) + 10100(0.006520215)^2]/1.34770215$$

$$= 0.0991914002/1.34770215$$

$$= 0.073600368 = 0.07.$$

This value of g is compared with a value of G obtained from Table V for $P = 90$ per cent and $n = 10$. In this case, $G = 3.11$.

$$g = 0.07 < 3.11$$

and the joint hypothesis, $\beta_0 = 0$ and $\beta_1 = 1$, is not rejected.

Example 3: As a final example, assume that the data in Table II are augmented so that corresponding to a given quantity of CaO present, X_i, there are two values of CaO determined by the new method, Y_{i1} and Y_{i2}. The data are given in Table VI. Using the expressions for b_0 and b_1 given in Section 10, that is,

$$b_0 = \bar{\bar{Y}} - b_1\bar{X}$$

and

$$b_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(\bar{Y}_i - \bar{\bar{Y}})}{\sum_{i=1}^{n} (X_i - \bar{X})^2}.$$

The least squares line is given by

$$\hat{Y} = -0.67925285 + 1.0146062X.$$

A test for linearity is called for.

It is desired to test at the 95 per cent level. Table VII is entered under column value $n - 2 = 8$ and row $n(k-1) = 10$. The value of F obtained is 3.0717. Then substituting into the formula given in Section 10, one obtains

$$\frac{2(1.2024)/8}{4.2850/10} = 0.7015 < 3.0717.$$

Thus the hypothesis of linearity is not rejected.

Calculations of Example 1:

General Data:

X represents tensile strength

$\sum X_i = 2991300; \qquad \bar{X} = 119652;$

$(\sum X_i)\,\bar{X} = 357915027600;$

$\sum X_i^2 = 372419750000;$

$\sum X_i^2 - (\sum X_i)\bar{X} = \sum (X_i - \bar{X})^2$

$= 14504722400.$

Y represents proportional limit

$\sum Y_i = 2131210; \qquad \bar{Y} = 85248;$

$(\sum Y_i)\,\bar{Y} = 181680537600;$

$\sum Y_i^2 = 196195960000;$

$\sum Y_i^2 - (\sum Y_i)\bar{Y} = \sum\ Y_i - \bar{Y})^2$

$= 14515422400.$

$$n = 25; \qquad \frac{1}{n} = .04;$$

$(\sum X_i)(\sum Y_i)/n = \bar{X} \sum Y_i = 255002342400;$

$\sum X_iY_i = 269069420000;$

$\sum X_iY_i - (\sum X_i)(\sum Y_i)/n$

$= \sum (X_i - \bar{X})(Y_i - \bar{Y}) = 14067077600.$

$[\sum (X_i - \bar{X})(Y_i - \bar{Y})]^2 = 1978826722 \cdot 10^{11};$

$$\frac{[\sum (X_i - \bar{X})(Y_i - \bar{Y})]^2}{\sum (X_i - \bar{X})^2} = 13642637670.$$

Data for Equation of Line:

$$b_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = 0.96982743;$$

$$b_0 = \bar{Y} - b_1\bar{X} = -30793.79165;$$

equation of line $= \bar{Y} = b_0 + b_1X$

$= -30793.79165 + 0.96982743X.$

Estimation of Standard Deviation:

$$(n - 2)s^2 = \sum (Y_i - \bar{Y})^2 - \frac{[\sum (X_i - \bar{X})(Y_i - \bar{Y})]^2}{\sum (X_i - \bar{X})^2} = 872784730;$$

$s^2 = 37947162.17; \qquad s = 6160.126798.$

Data for Significance Tests and for Confidence Intervals for β_0 and β_1:

$$\frac{1}{\sum (X_i - \bar{X})^2} = 0.6894306367 \cdot 10^{-10};$$

$$\sqrt{\frac{1}{\sum (X_i - \bar{X})^2}} = 0.8303195991 \cdot 10^{-5};$$

$$s\sqrt{\frac{1}{\sum (X_i - \bar{X})^2}} = 0.0511487401.$$

$\bar{X}^2 = 14316601104;$

$$\frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} = 0.9870303415;$$

$$\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} = 1.0270303415;$$

$$\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2}} = 1.013425055;$$

$$s\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2}} = 6242.826839.$$

Data for Confidence Intervals for Average Value of Y Corresponding to $X = X^*$:

$X^* = 129000; \quad (X^* - \bar{X})^2 = 87385104;$

$$\frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 0.006024596789;$$

$$\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 0.046024596789;$$

$$\sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 0.2145334398;$$

$$s\sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 1321.553192.$$

Data for Prediction Intervals for a Future Value of Y Corresponding to $X = X^0$:

$X^0 = 129000, \quad (X^0 - \bar{X})^2 = 87385104;$

$$\frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2}\ 0.006024596789;$$

$$1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 1.0460245968;$$

$$\sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 1.022753439;$$

$$s\sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 6300.290867.$$

Calculations for Example 2:

General Data:

X represents CaO present

$\sum X_i = 311; \quad \bar{X} = \sum X_{i/n} = 31.1;$

$(\sum X_i)\bar{X} = 9672.1; \quad \sum X_i^2 = 10100;$

$\sum X_i^2 - (\sum X_i)\bar{X} = \sum (X_i - \bar{X})^2 = 427.9.$

Y represents CaO found by new method

$\sum Y_i = 310.1; \quad \bar{Y} = \sum Y_i/n = 31.01;$

$(\sum Y_i)\bar{Y} = 9616.201;$

$\sum Y_i^2 = 10055.09;$

$\sum Y_i^2 - (\sum Y_i)\bar{Y} = \sum (Y_i - \bar{Y})^2$
$= 438.889.$

$n = 10; \quad \frac{1}{n} = 0.1;$

$(\sum X_i)(\sum Y_i)/n = \bar{X} \sum Y_i = 9644.11;$

$\sum X_iY_i = 10074.8;$

$\sum X_iY_i - (\sum X_i)(\sum Y_i)/n$
$= \sum (X_i - \bar{X})(Y_i - \bar{Y}) = 430.69.$

$[\sum (X_i - \bar{X})(Y_i - \bar{Y})]^2 = 185493.8761;$

$$\frac{[\sum X_i - \bar{X})(Y_i - \bar{Y})]^2}{\sum (X_i - \bar{X})^2} = 433.4981914.$$

Data for Equation of Line:

$$b_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = 1.006520215;$$

$b_0 = \bar{Y} - b_1\bar{X} = -0.2927786865;$

equation of line $= \hat{Y} = b_0 + b_1X$

$= -0.2927786865 + 1.006520215\,X$

Estimation of Standard Deviation:

$$(n - 2)s^2 = \sum (Y_i - \bar{Y})^2 - \frac{[\sum (X_i - \bar{X})(Y_i - \bar{Y})]^2}{\sum (X_i - \bar{X})^2} = 5.3908086$$

$s^2 = 0.673851075; \quad s = 0.8208843250.$

Data for Significance Tests and for Confidence Intervals for β_0 and β_1:

$$\frac{1}{\sum (X_i - \bar{X})^2} = 0.002336994625;$$

$$\sqrt{\frac{1}{\sum (X_i - \bar{X})^2}}\ 0.04834247227;$$

$$s\sqrt{\frac{1}{\sum (X_i - \bar{X})^2}} = 0.03968357772.$$

$\bar{X}^2 = 967.21;$

$$\frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} = 2.260364571;$$

$$\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} = 2.360364571;$$

$$\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2}} = 1.536347802;$$

$$s\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2}}\ 1.261163829.$$

Data for Confidence Intervals for Average Value of Y Corresponding to $X = X^*$:

$X^* = 30; \quad (X^* - \bar{X})^2 = 1.21;$

$$\frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 0.0028277635;$$

$$\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 0.1028277635;$$

$$\sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 0.3206676839;$$

$$s\sqrt{\frac{1}{n} + \frac{(X^* - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 0.2632310752.$$

Data for Prediction Intervals for a Future Value of Y Corresponding to $X = X^0$:

$X^0 = 30; \quad (X^0 - \bar{X})^2 = 1.21;$

$$\frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 0.0028277635;$$

$$1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2} = 1.1028277635;$$

$$\sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 1.050156066;$$

$$s\sqrt{1 + \frac{1}{n} + \frac{(X^0 - \bar{X})^2}{\sum (X_i - \bar{X})^2}} = 0.8620566534.$$

Data for Confidence Intervals for X Corresponding to an Observed Value of $Y = Y'$ for the Case When There Is an Underlying Physical Relationship:

$$Y' = 29.6; \qquad X' = \frac{Y' - b_0}{b_1} = 29.69913395;$$

$$\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2 = 1.962425690;$$

$$\frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\sum (X_i - \bar{X})^2} = 0.004586178289;$$

$$1 + \frac{1}{n} + \frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\sum (X_i - \bar{X})^2} = 1.104586178;$$

$$\sqrt{1 + \frac{1}{n} + \frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\sum (X_i - \bar{X})^2}} = 1.050992948;$$

$$s\sqrt{1 + \frac{1}{n} + \frac{\left(\frac{Y' - b_0}{b_1} - \bar{X}\right)^2}{\sum (X_i - \bar{X})^2}} = 0.8627436375.$$

14. Glossary of Terms and Symbols.

β_0	= Intercept of the true underlying linear relationship (usually unknown).
β_0'	= Hypothesized value of β_0.
b_0	= Least squares estimate of the intercept as determined from the data.
$\lvert z \rvert$, the absolute value of z	= Absolute value of z equals z if z is positive; the absolute value of z equals $-z$ if z is negative.
β_1	= Slope of the true underlying linear relationship (usually unknown).
β_1	= Hypothesized value of β_1.
b_1	= Least squares estimate of the slope as determined from the data.
Confidence interval estimate	= An interval estimate of an unknown constant having the property that there is a known probability that this interval will include the constant.
F	= Criterion used in judging whether linearity exists.
g	= Function used to test the joint hypothesis that $\beta_0 = \beta_0'$ and $\beta_1 = \beta_1'$.
n	= Number of pairs of observations.
P	= When used in the context of a confidence interval estimate of a constant, P denotes the probability (in per cent) that the interval includes the constant. When used in the context of testing a hypothesis, P denotes the probability (in per cent) of not rejecting the hypothesis when the hypothesis is true. When used in the context of a prediction interval, P denotes the probability (in per cent) of the variable Y falling within the prediction interval.
Prediction interval estimate	= An interval within which a future value of Y will lie with known probability for a given value of X.
s^2	= An estimate of the variability of Y about the straight line.
$\sum_{i=1}^{n} z_i$ or $\sum z_i$	= Notation indicating the sum $z_1 + z_2 + \cdots + z_n$.
t	= A factor used in obtaining confidence interval estimates and in testing hypotheses.
t'	= A factor used in obtaining confidence interval estimates and in testing hypotheses when there is a priori knowledge that $\beta_0 = 0$.
$\bar{X}$	= The sample mean of the X's.
X^*	= A fixed value of X for which a point estimate or a confidence interval estimate of the average value of Y is desired.
X'	= The point estimate of X corresponding to the observed value Y'.
X^0	= Value of X used in obtaining a prediction interval within which Y^0 will lie with given probability.
$\bar{Y}$	= The sample mean of the Y's.
$\tilde{Y}$	= $\tilde{Y} = b_0 + b_1X$ is the line fitted to sample data.

Y^* = A point estimate of the average value of Y corresponding to a given X^*.

Y' = An observed value of Y for which a confidence interval on the value of X, which leads to this Y, is desired.

Y^0 = Value of a future observation of Y (unknown) corresponding to a given value X^0.

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