Chapter 6
The Normal Distribution

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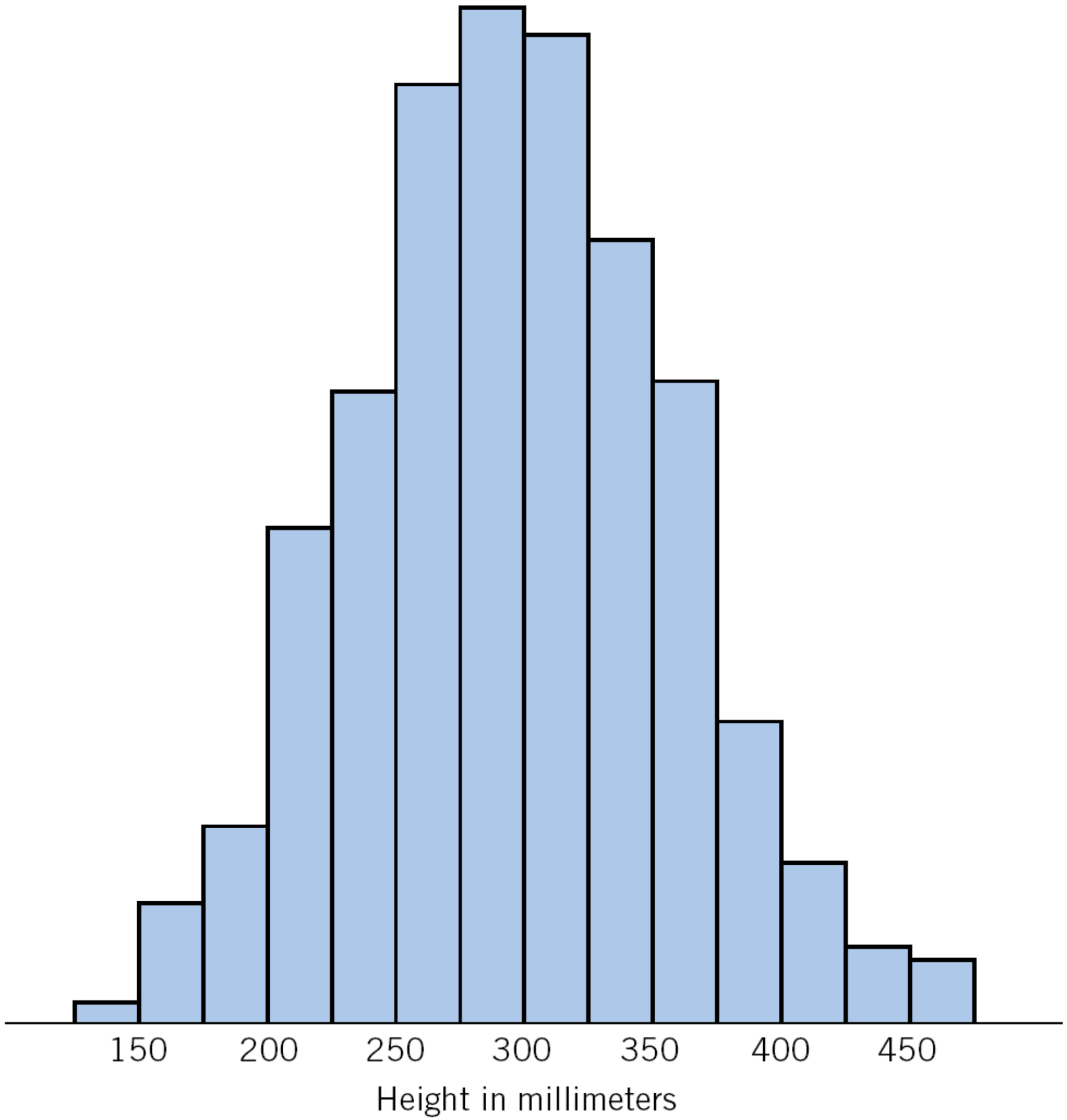
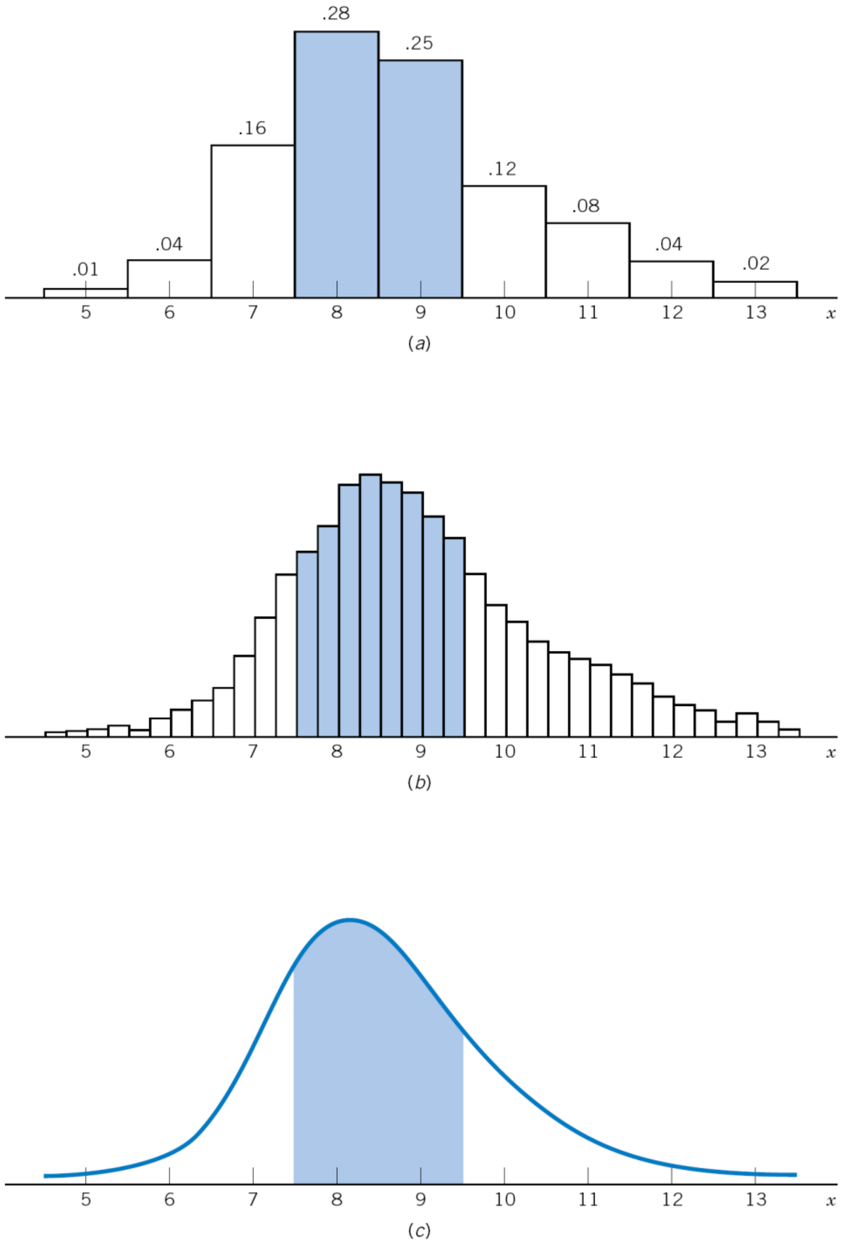
Probability Model for Continuous Random Variables



Probability density curve viewed as a limited form of relative frequency histograms

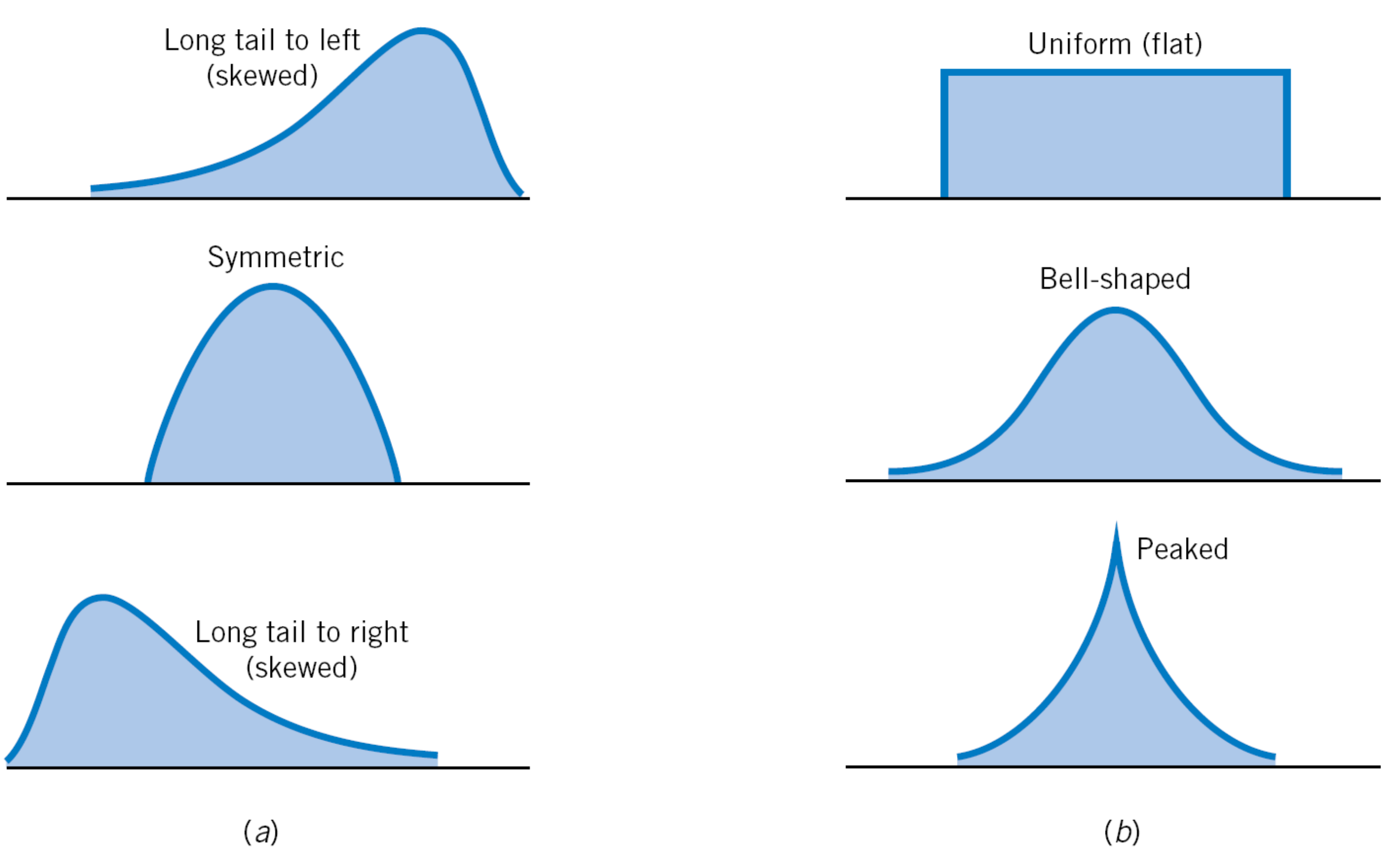


Probability Density function of a Continuous Random Variable

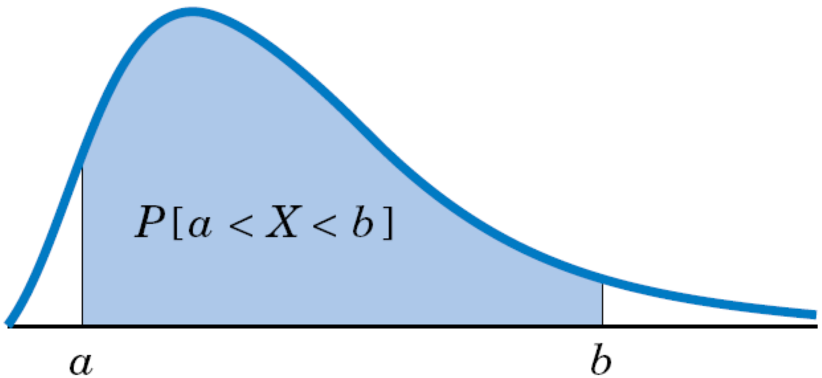
The **probability density function** $f(x)$ describes the distribution of probability for a continuous random variable. It has the properties:

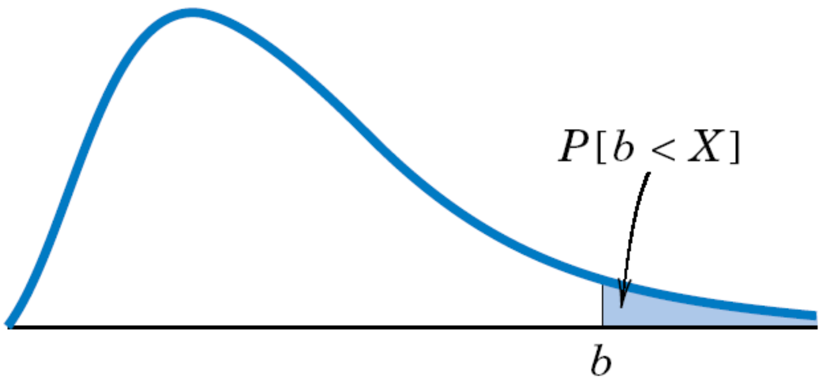
1. The total area under the probability density curve is 1.
2. $P[a \leq X \leq b]$ = area under the probability density curve between a and b.
3. $f(x) \geq 0$ for all x.

Different shapes of probability density curves

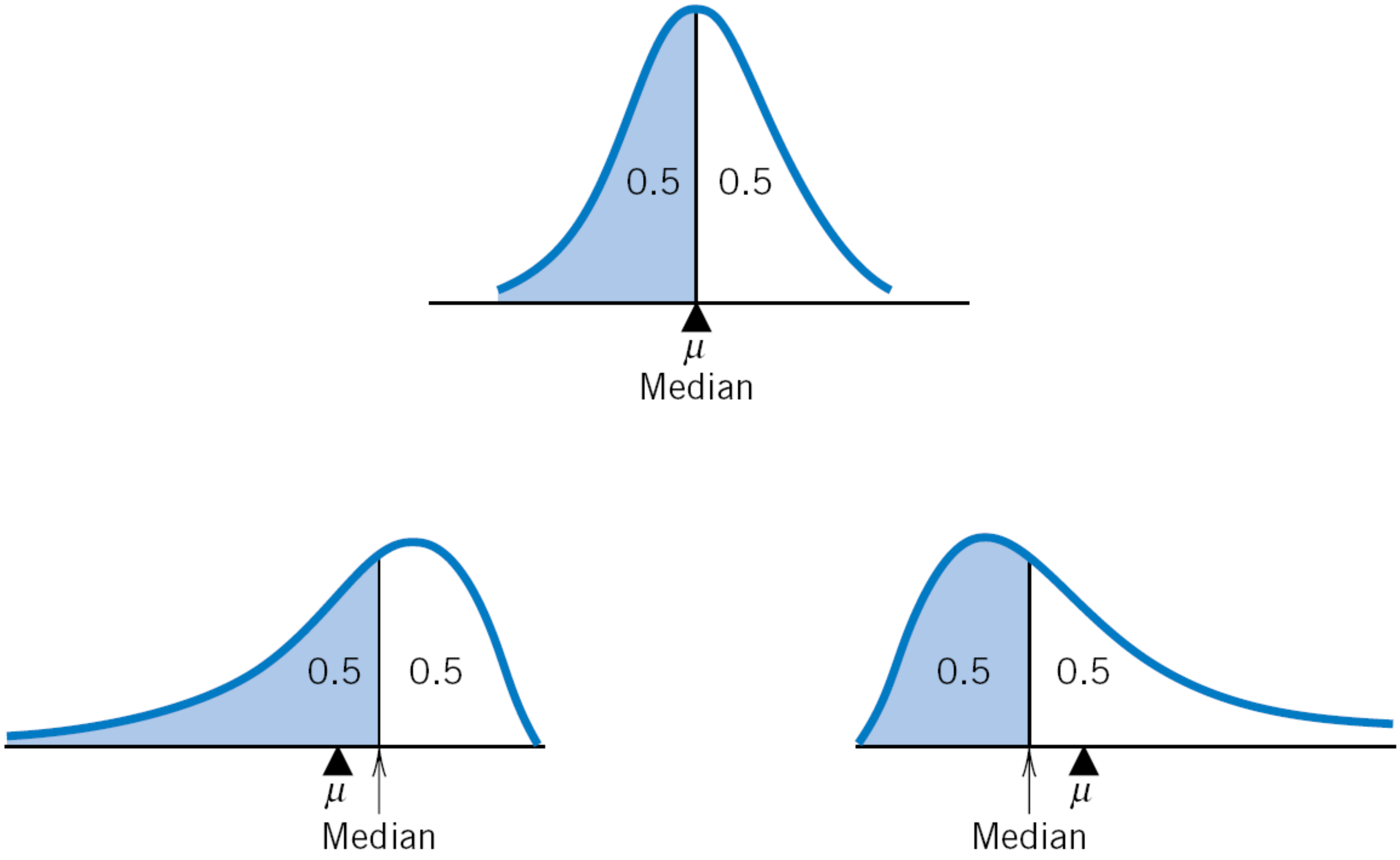


Probability as area under the curve.





$$P[b < X] = 1 - (\text{Area to left of } b)$$



Mean as the balance point and median as the point of equal division of the probability mass

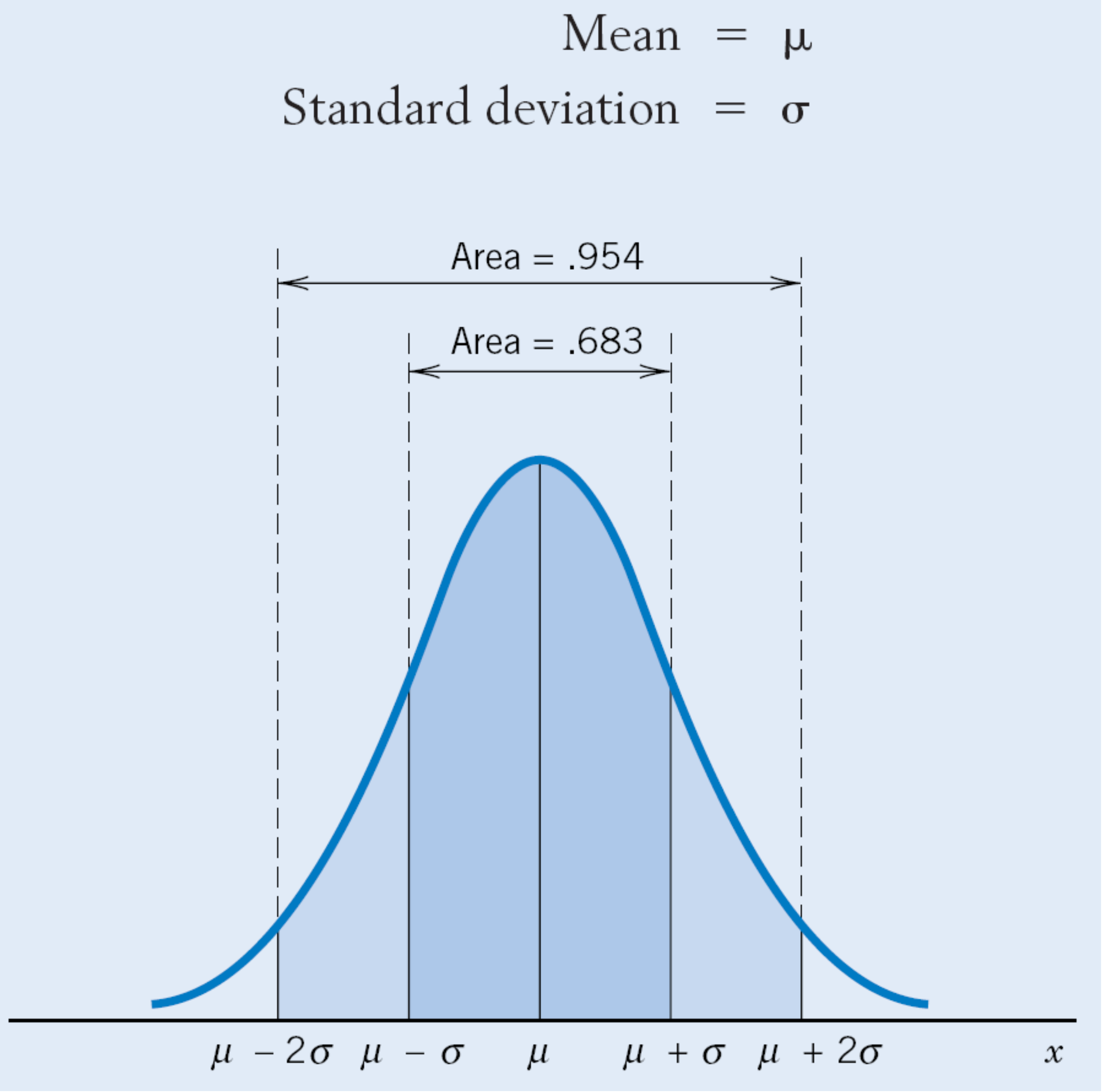
NORMAL PROBABILITY DISTRIBUTION

Normal Probability Distribution

A ***normal probability distribution*** , when plotted, gives a bell-shaped curve such that:

1. The total area under the curve is 1.0.
2. The curve is symmetric about the mean.
3. The two tails of the curve extend indefinitely.

The Normal Distribution-It's General Features



Properties:

- The total area under the normal curve is always equal to 1.

- The portion of the area under the curve within a given interval represents the probability that a measurement will lie in that interval.

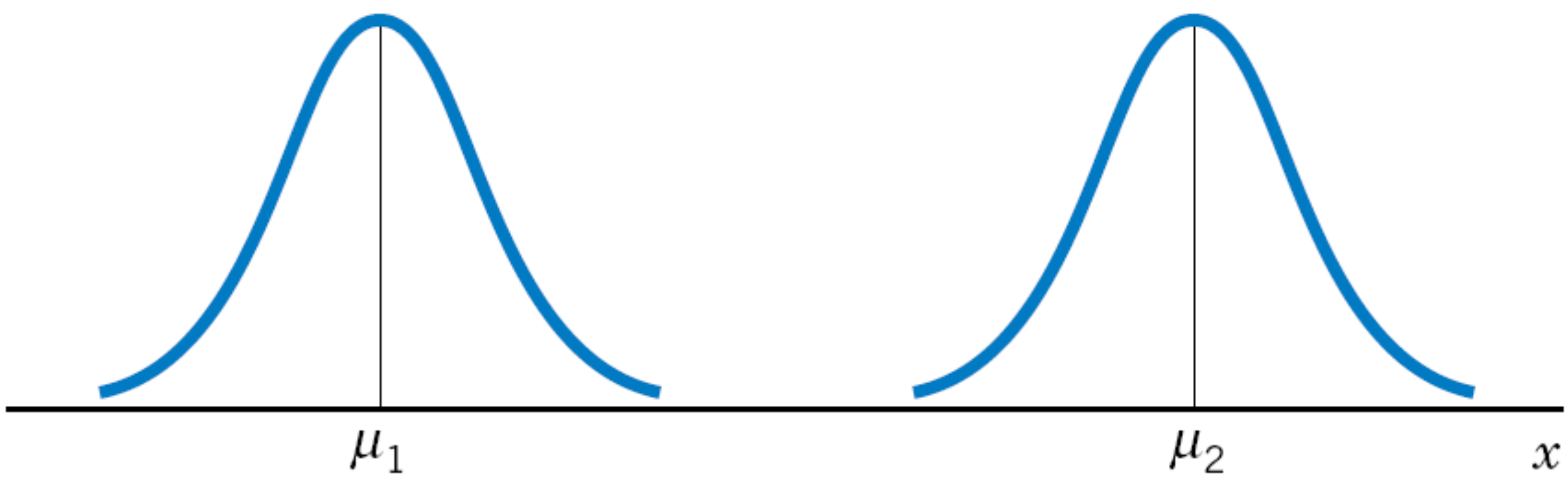
- The probability that z equals a certain number is always 0 i.e. $P(z = a) = 0$

- Therefore, $<$ and $\leq$ can be used interchangeably.

  Similarly, $>$ and $\geq$ can be used interchangeably.
  - $P(z < b) = P(z \leq b)$
  - $P(z > c) = P(z \geq c)$

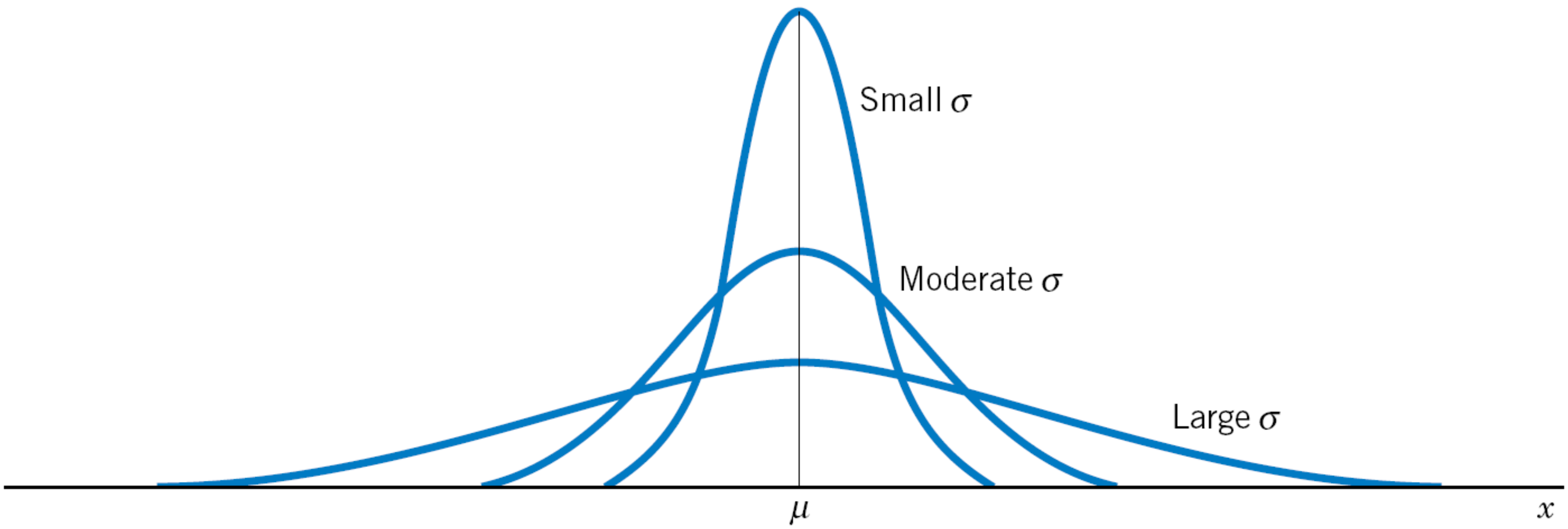
Notation

The normal distribution with a mean of μ and a standard deviation of σ is denoted by $N(\mu,\sigma)$.



Figure, Two normal distributions with different means but the same standard deviation.

Normal Distribution Curves(Same mean different standard deviation



Decreasing σ increases the maximum height and the concentration of probability about μ,

The Standard Normal Distribution

The **standardized variable**

$$Z = \frac{X - \mu}{\sigma} = \frac{\text{Variable} - \text{Mean}}{\text{Standard deviation}}$$

has mean 0 and sd 1.

The **standard normal distribution** has a bell-shaped density with

$$\text{Mean } \mu = 0$$
$$\text{Standard deviation } \sigma = 1$$

The standard normal distribution is denoted by $N(0, 1)$.

6.9 Find the standardized variable Z if X has

(a) Mean 15 and standard deviation 4.

(b) Mean 61 and standard deviation 9.

(c) Mean 161 and variance 25.

Example

Comparing Scores: A students take three tests graded as follows:

Test 1: a score of 18 of a test with a mean of 16 and a standard deviation of 6

Test 2: a score of 44 on a test with a mean of 38 and a standard deviation of 14

Test 3: a score of 95 on a test with mean of 87 and a standard deviation of 36

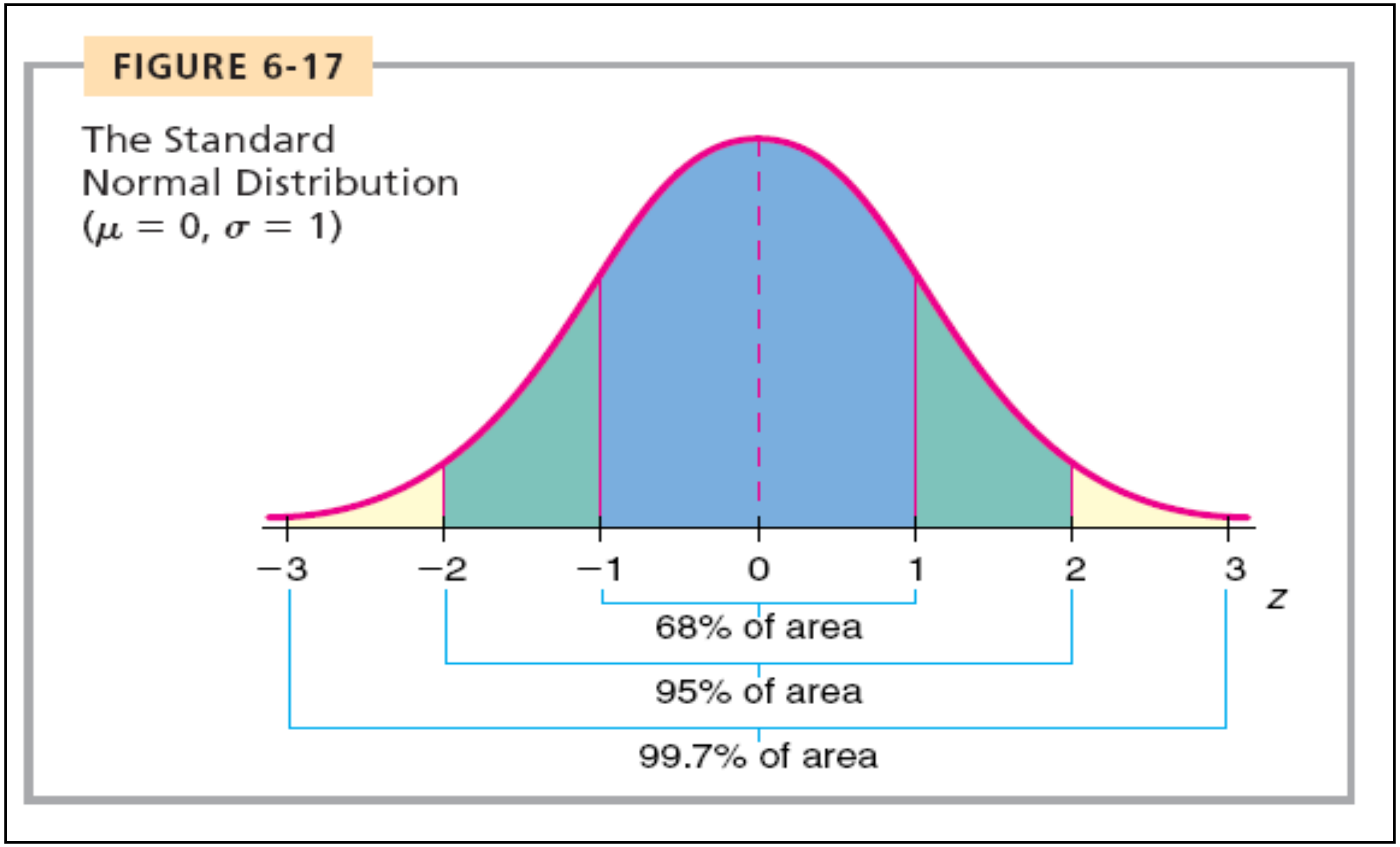
a) Give the transformed **z-scores**, z_1, z_2 and z_3; and b) determine which is the **higher relative score**?

$$z_1 = \frac{18-16}{6} = 0.33 \quad z_2 = \frac{44-38}{14} = 0.43 \quad z_3 = \frac{95-87}{36} = 0.22$$

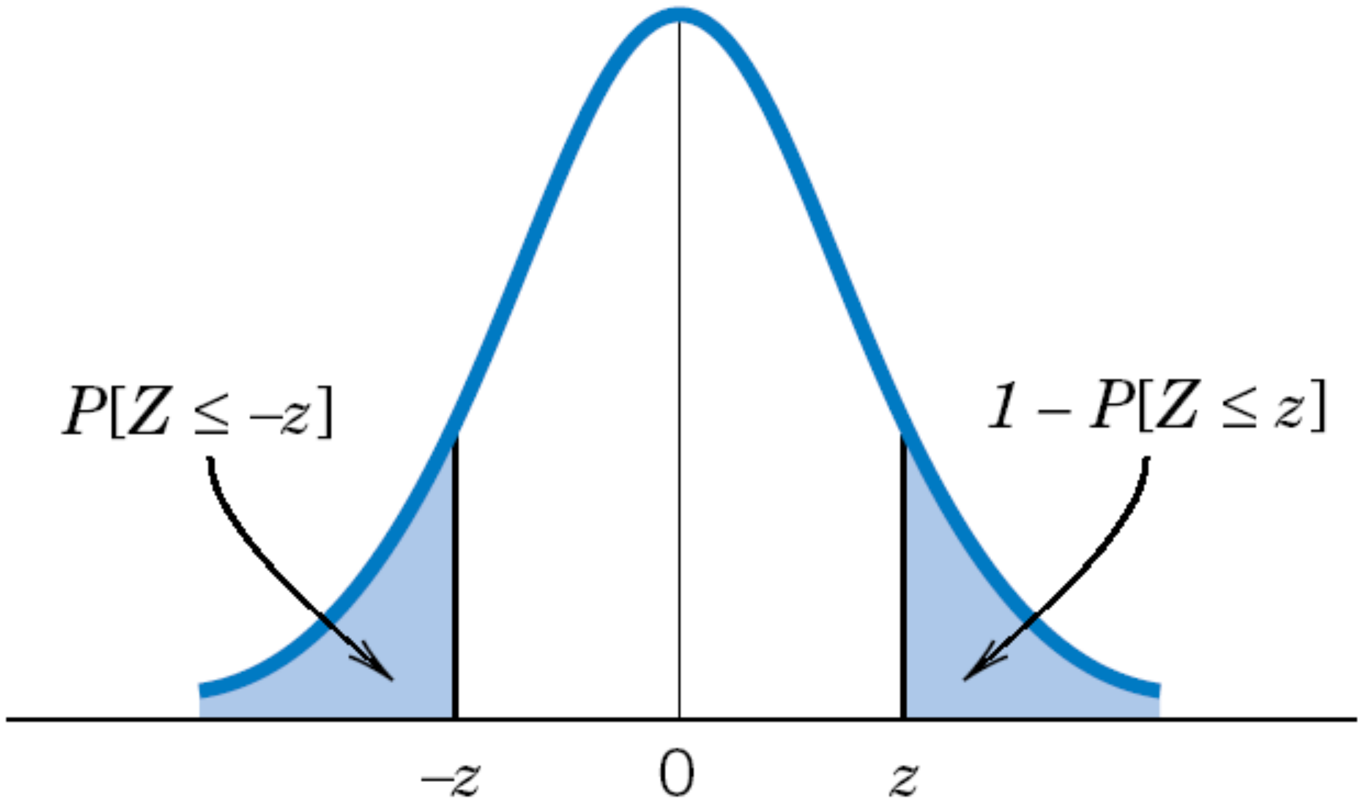
Hence, the second score (test 2) has the relatively higher score

(i.e. Relatively better score)

The Standard Normal Curve



Finding the Probability :



Equal normal tail probabilities

V

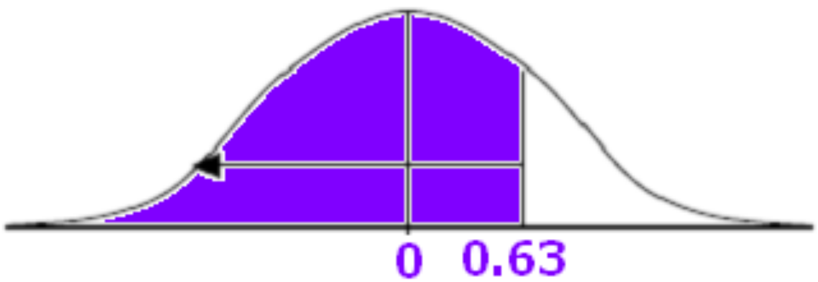
How to Read from Appendix B, Table 4 for $z = 1.37 = 1.3 + .07$

TABLE 1 How to Read from Appendix B, Table ~~3~~ 4 for $z = 1.37 = 1.3 + .07$

z	.00	···	.07	···
.0				
·				
·				
·				
1.3			.9147	
·				
·				
·				

Estimating Normal Probability (Table 4 text page 656)

Example 1: $P(z \leq 0.63) = 0.7357$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

Example

z	.00	···	.06	···
.0				
.				
.				
.				
1.9			.9750	
.				
.				
.				

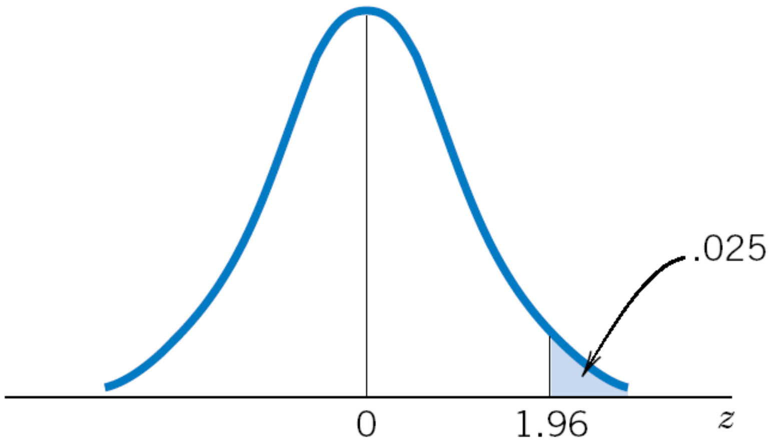
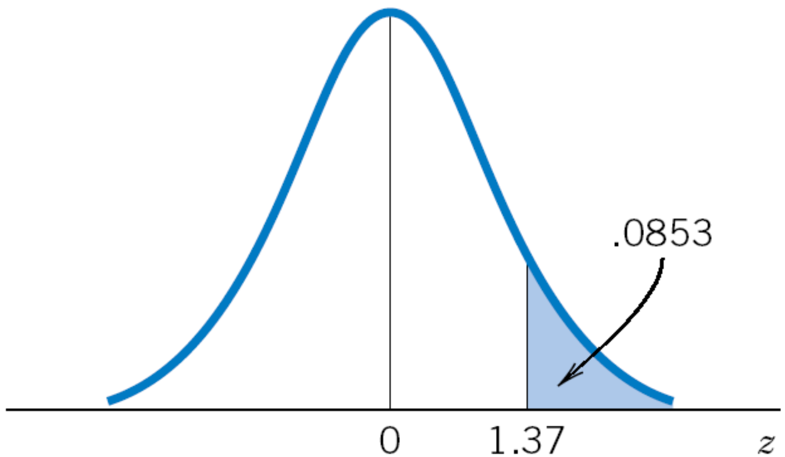


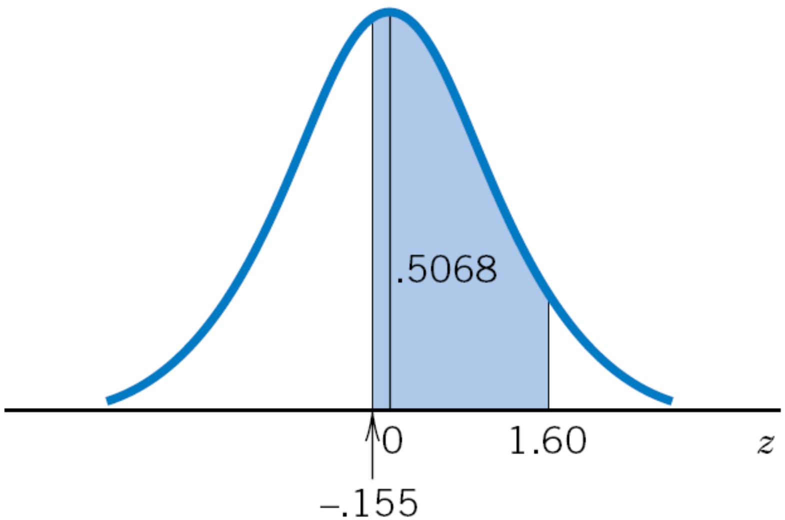
Figure 13 $P[Z > 1.96] = .025.$

$P[Z > 1.96] = 1\text{-}P(Z<1.96) = 1\text{-}0.975 = .025,$

Calculating Probability using Table



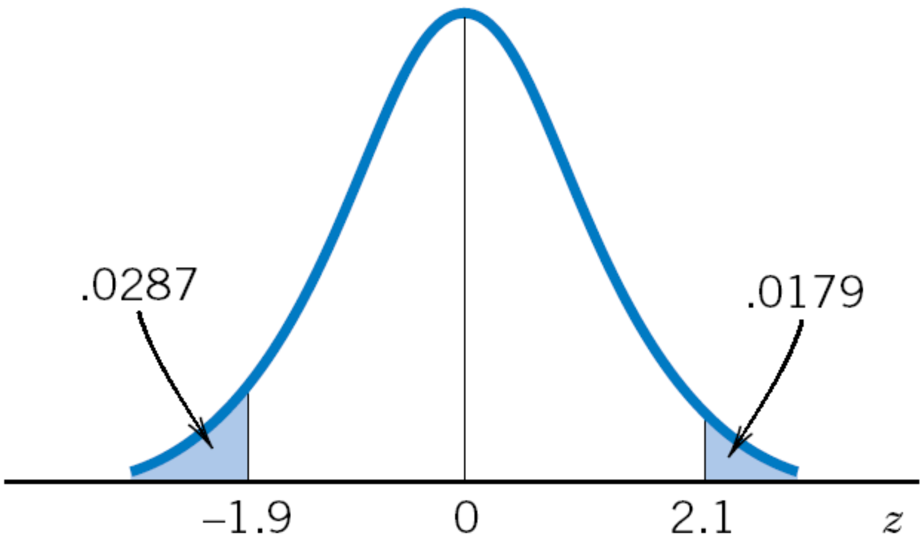
P(Z> 1.37)
=1- P(Z<1.37)
=1-9147
= 0.0853



P(-0.155<Z<1.60)= P(Z≤1.60)-P(Z≤-0.155)
=0.9452-0.4384
=0.5068

Example: Determine the Standard Normal Probability of

$P\ (Z<-1.9 \text{ or } Z>2.1)$



Hence, $P\ (Z<-1.9 \text{ or } Z>2.1) = P\ (Z<-1.9) + P(Z>2.1)$

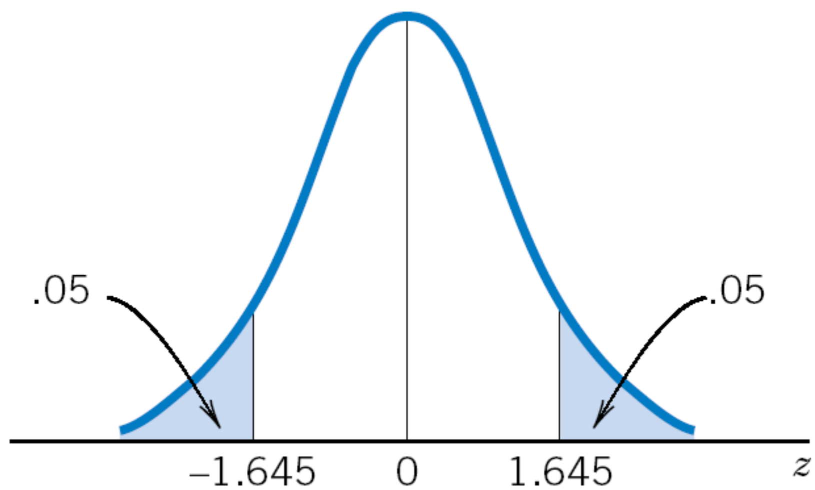
$= P\ (Z<-1.9) + 1 - P(Z<2.1)$

$= 0.0287 + 1 - 0.9821$

$= 0.0287 + 0.0179$

$= 0.0466$

Example: $P[Z > -1.645 \text{ or } Z > 1.645] = .10$



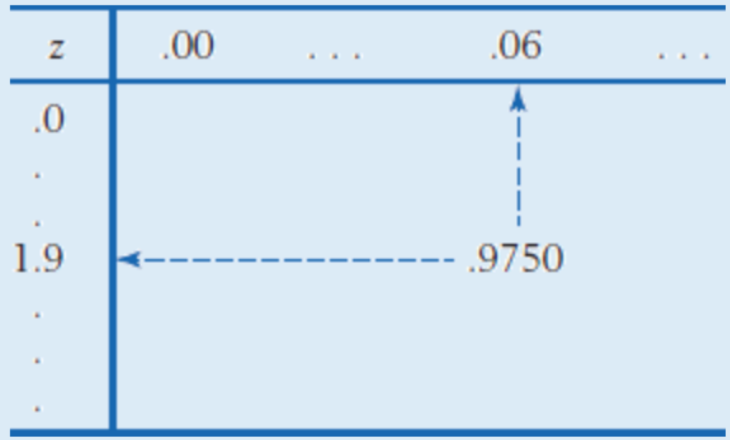
$$P[Z < -1.645 \text{ or } Z > 1.645] = P[Z < -1.645] + P[Z > 1.645]$$
$$= 0.05 + 1 - P[Z<1.645]$$
$$= 0.05+1-0.95$$
$$= 0.10$$

Determining an Upper Percentile of the Standard Normal Distribution

Locate the value of z that satisfies $P[Z > z] = .025$

SOLUTION

If we use the property that the total area is 1, the area to the left of z must be $1 - .0250 = .9750$. The marginal value with the tabular entry .9750 is $z = 1.96$ (diagrammed in Figure 13).



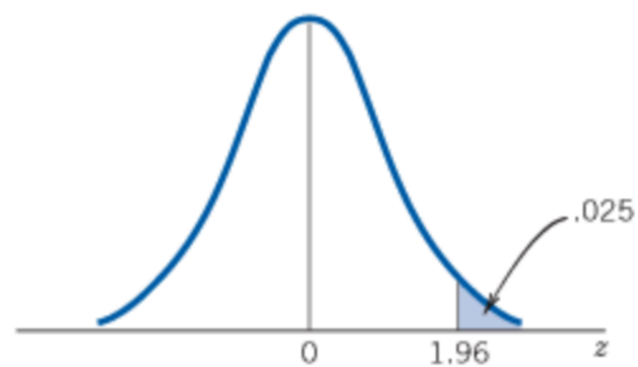


Figure 13 $P[Z > 1.96] = .025$.

Find the probability

a. $P(z \le -2.43)$

b. $P(z \ge -1.78)$

c. $P(z \ge 3.09)$

d. $P(z \le 0.227)$

e. $P(-2.18 \le z \le 1.34)$

Find the z value in each of the following cases

1. $P(Z<z)= .1762$

2. $P(Z>z)= 0.10$

3. $P(-z<Z<z) = 0.954$

4. $P(-0.6<Z<z)= 0.50$

Recall: Probability Calculations With Normal Distribution

If X is distributed as $N(\mu, \sigma)$, then the standardized variable

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution.

The **standard normal distribution** has a bell-shaped density with

$$\text{Mean } \mu = 0$$

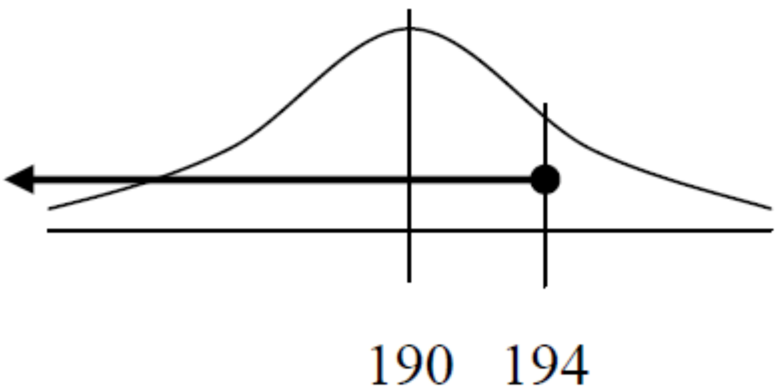
$$\text{Standard deviation } \sigma = 1$$

The standard normal distribution is denoted by $N(0, 1)$.

Example

Given that the **POPULATION MEAN** is 190 and the **STANDARD DEVIATION** is 36 on a continuous **NORMALLY DISTRIBUTED** scale. Find the following.

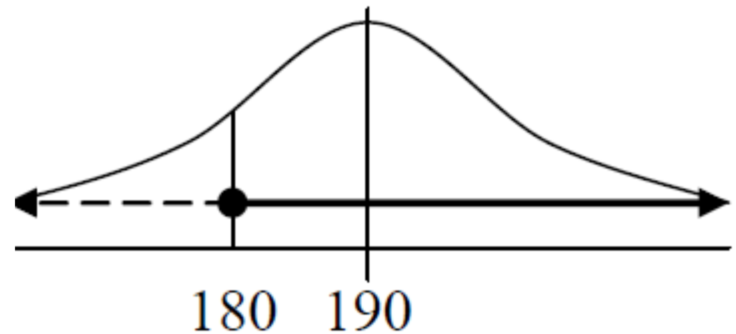
a. $P(x \leq 194)$



$$z = \frac{194 - 190}{36} = 0.11$$

$$P(z \leq 0.11) = 0.5438$$

b. $P(x \geq 180)$



$$z = \frac{180 - 190}{36} = -0.28$$

$$P(z \geq -0.28) = 0.6103$$

Recall: *P*[Z > a]= 1-P[Z<a]

Probability Calculations with Normal Distributions

If X is distributed as $N(\mu, \sigma)$, then

$$P[a \le X \le b] = P\left[\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right]$$

where Z has the standard normal distribution.

$$Example\_6\_Textbook$$

$$X \sim N(60, 4)$$

$$P(55 \leq X \leq 63)$$

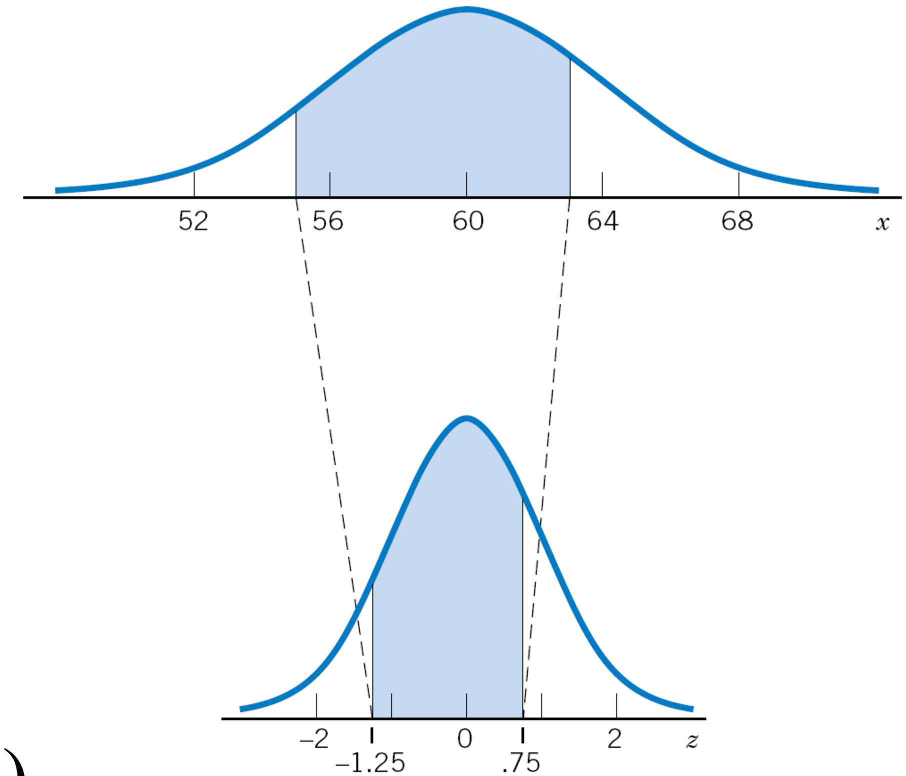
$$z_1 = \frac{55-60}{4} = -1.25$$

$$z_2 = \frac{63-60}{4} = 0.75$$

$$P(-1.25 \leq Z \leq 0.75)$$

$$= P(Z \leq 0.75) - P(Z \leq -1.25)$$

$$= 0.7734 - 0.1056 = 0.6678$$



Converting to the z scale

Practice: If X is distributed as N(190,36), find the Probability that

$$P(140 \le x \le 194)$$

$$= 0.5438 - 0.0823$$

$$= 0.4615$$

Example:

The number of calories in a salad on the lunch menu is normally distributed with mean = 200 and sd=5. Find the probability that the salad you select will contain:

(a) More than 208 calories

(b) Between 190 and 200 calories

(c) Find the 95^{th} percentile of the distribution for the number of calories.

Solution:

$X \sim N(200,5)$

(a) $P(X>208) = ?$

Example: Contd.

(b) $P(190<X<200)$ =?

(c) First we find the 95th percentile in the Z scale then convert it to the X scale. Using the Standard Normal Table,

we can fill the dots: $P(Z< \ldots\ldots) = 0.95$, where…..=1.645

And using this in $X= \mu+\sigma^* 1.645$

$= 200+5 * 1.645$

$= 208.225$

This means 95th percentile of lunch salad is about 208.225 calories.

Example

- Suppose the duration of trouble-free operation of a new robotic vacuum cleaner is normally distributed with mean 750 days and standard deviation 100 days.

- What is the probability that the vacuum cleaner will work for at least two years without trouble?

- What is the probability that the vacuum cleaner will work no more than 300 days?

- What is the probability that the vacuum cleaner will work in between 400 to 600 days?

- The company wishes to set the warranty period so that no more than 100% of the vacuum cleaners would need repair services while under warranty. How long a warranty period must be set?

Example

According to the Kaiser Family Foundation, U.S. workers who had employer-provided health insurance paid an average premium of $4129 for family coverage during 2011 (*USA TODAY*, October 10, 2011). Suppose that the premiums for family coverage paid this year by all such workers are normally distributed with a mean of $4129 and a standard deviation of $600. Find the probability that such premium paid this year by a randomly selected such worker is between $3331 and $4453.

Solution

For x = \$3331:

$$z = 3331 - 4129/600 = -1.33$$

For x = \$4453:

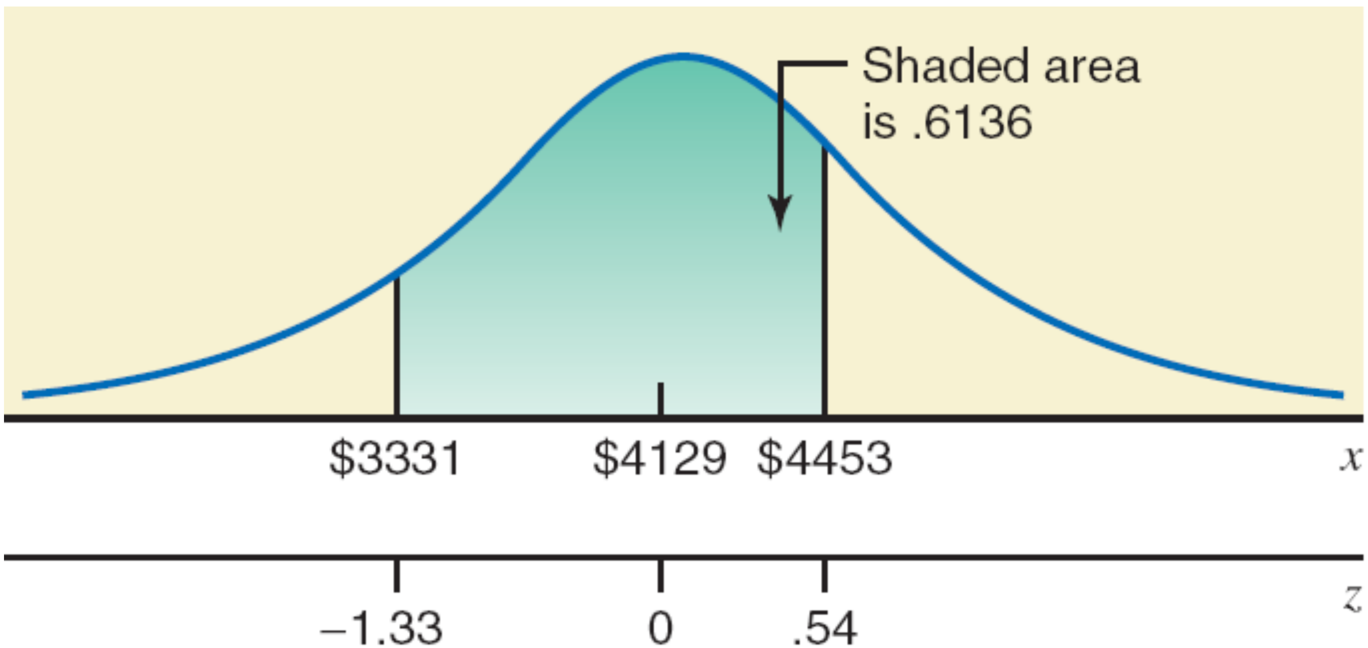
$$z = 4453 - 4129/600 = .54$$

P (\$3331 < x < \$4453) = P (-1.33 < z < .54)

= .7054 - .0918

= .6136 = 61.36%

Area between x = \$3331 and x = \$4453.



Example

A racing car is one of the many toys manufactured by Mack Corporation. The assembly times for this toy follow a normal distribution with a mean of 55 minutes and a standard deviation of 4 minutes. The company closes at 5 p.m. every day. If one worker starts to assemble a racing car at 4 p.m., what is the probability that she will finish this job before the company closes for the day?

Example 6-12: Solution

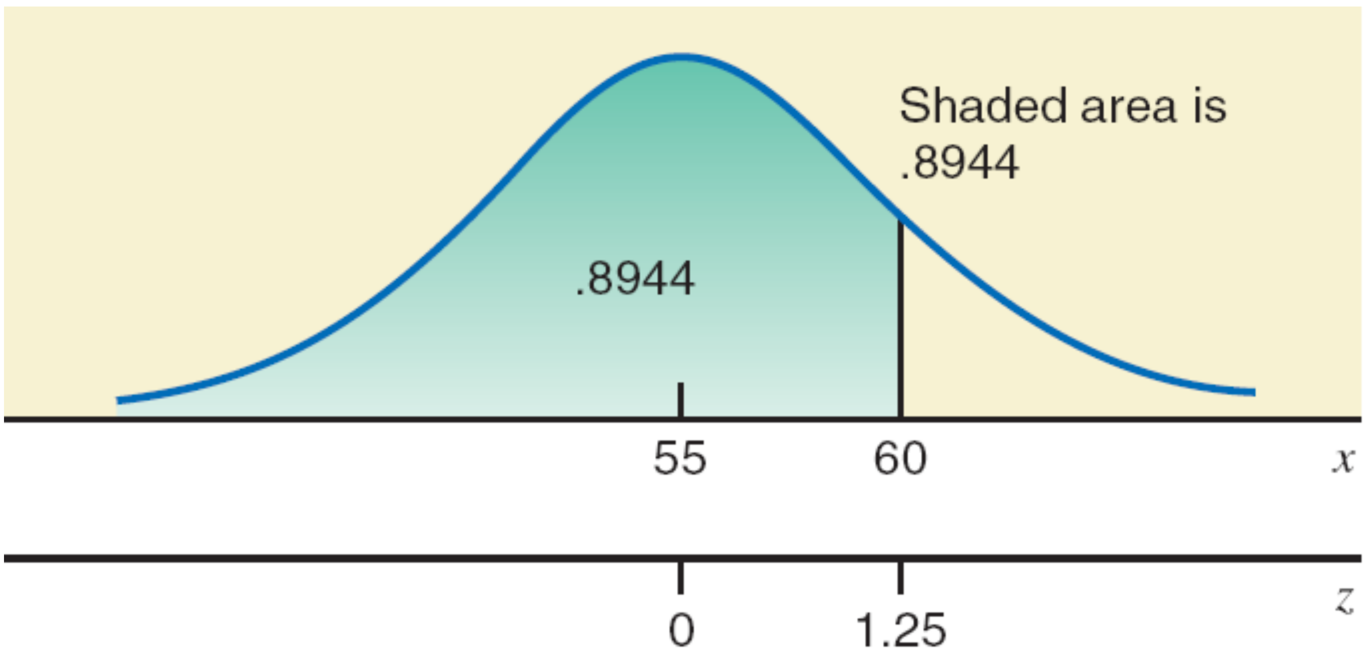
For x = 60:

$$z = 60 - 55/4 = 1.25$$

$P(x \le 60) = P(z \le 1.25) = .8944$

Thus, the probability is .8944 that this worker will finish assembling this racing car before the company closes for the day.

Area to the left of x = 60.



Example

Hupper Corporation produces many types of soft drinks, including Orange Cola. The filling machines are adjusted to pour 12 ounces of soda into each 12-ounce can of Orange Cola. However, the actual amount of soda poured into each can is not exactly 12 ounces; it varies from can to can. It has been observed that the net amount of soda in such a can has a normal distribution with a mean of 12 ounces and a standard deviation of .015 ounce.

(a) What is the probability that a randomly selected can of Orange Cola contains 11.97 to 11.99 ounces of soda?

(b) What percentage of the Orange Cola cans contain 12.02 to 12.07 ounces of soda?

Solution

(a) For x = 11.97:

$$z = \frac{11.97 - 12}{.015} = -2.00$$

For x = 11.99:

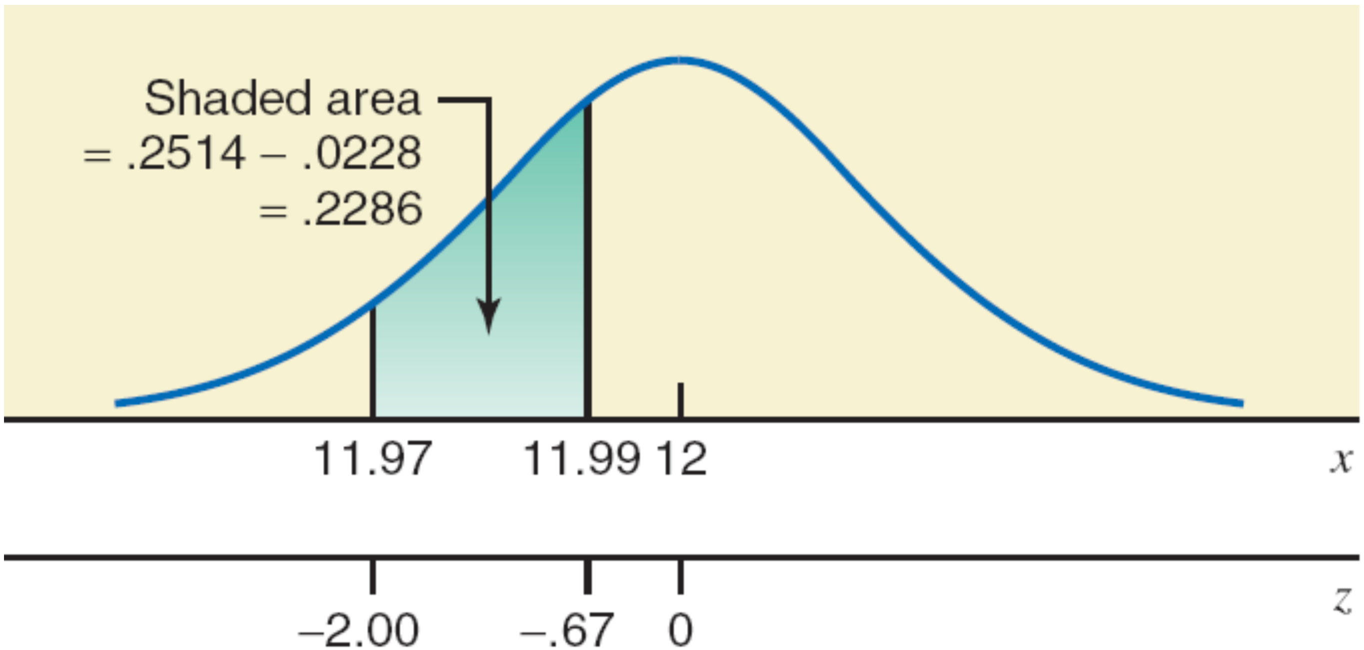
$$z = \frac{11.99 - 12}{.015} = -.67$$

P (11.97 ≤ x ≤ 11.99) = P (-2.00 ≤ z ≤ -.67)

= .2514 - .0228

= .2286

Area between x = 11.97 and x = 11.99.



Solution

(b) For x = 12.02:

$$z = \frac{12.02 - 12}{.015} = 1.33$$

For x = 12.07:

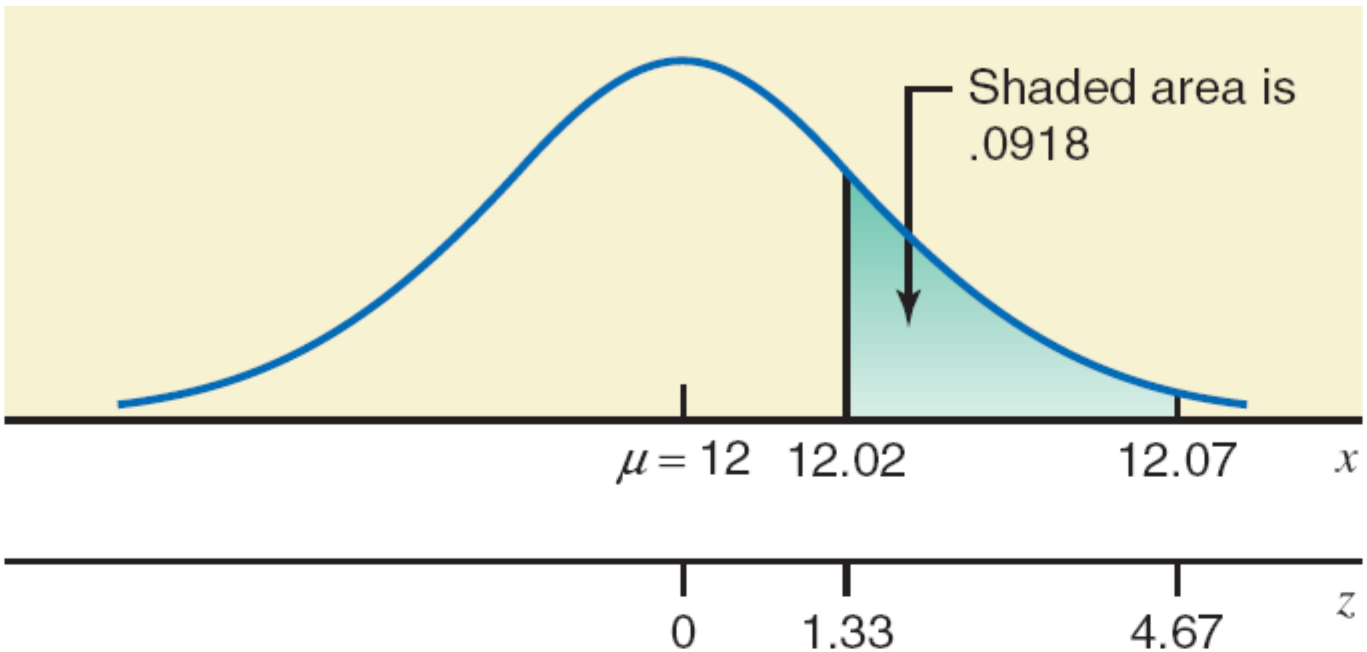
$$z = \frac{12.07 - 12}{.015} = 4.67$$

$P\,(12.02 \le x \le 12.07) = P\,(1.33 \le z \le 4.67)$

$= 1 - .9082$

$= .0918$

Area from x = 12.02 to x = 12.07.



Example

Suppose the life span of a calculator manufactured by Calculators Corporation has a normal distribution with a mean of 54 months and a standard deviation of 8 months. The company guarantees that any calculator that starts malfunctioning within 36 months of the purchase will be replaced by a new one. About what percentage of calculators made by this company are expected to be replaced?

Solution

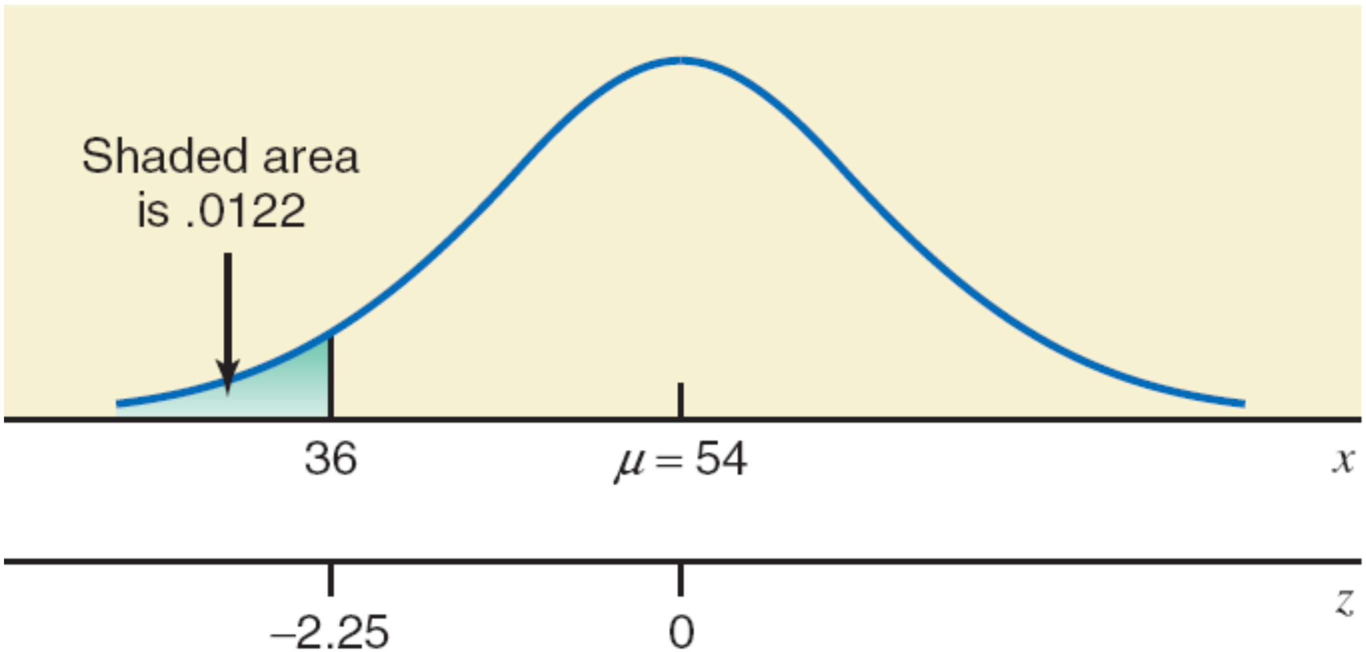
For x = 36:

$$z = \frac{36 - 54}{8} = -2.25$$

$P(x < 36) = P\ (z < -2.25) = .0122$

Hence, 1.22% of the calculators are expected to be replaced.

Area to the left of x = 36.



DETERMINING THE z AND x VALUES WHEN AN AREA UNDER THE NORMAL DISTRIBUTION CURVE IS KNOWN

Now we learn how to find the corresponding value of z or x when an area under a normal distribution curve is known.

Example

Find a point z such that the area under the standard normal curve to the left of z is .9251.

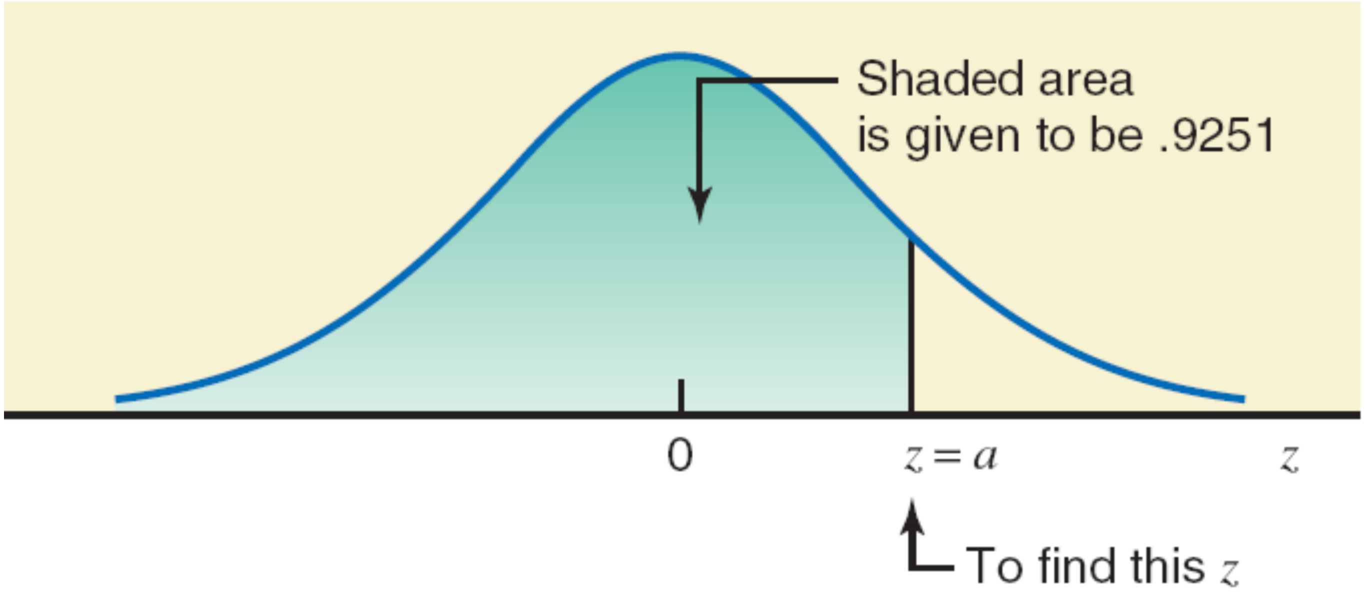


Table 6.4 Finding the z Value When Area Is Known.

z	**.00**	**.01**	⋯	**.04**	⋯	**.09**
−3.4	.0003	.0003	⋯		⋯	.0002
−3.3	.0005	.0005	⋯		⋯	.0003
−3.2	.0007	.0007	⋯		⋯	.0005
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
1.4				.9251	⋯	⋯
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
3.4	.9997	.9997	⋯	.9997	⋯	.9998

We locate this value in Table IV of Appendix C

Example

Find the value of z such that the area under the standard normal curve in the right tail is .0050.

Area to the left of z = 1.0 - .0050 = .9950

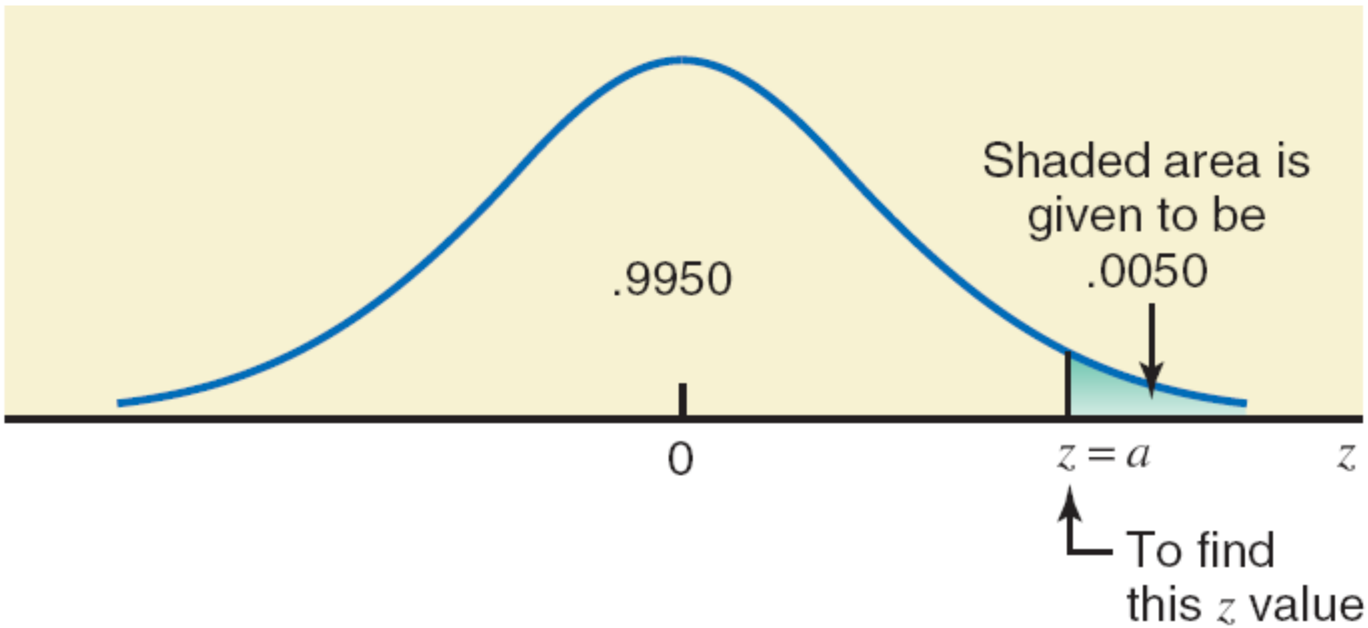
Look for .9950 in the body of the normal distribution table. Table VII does not contain .9950.

Find the value closest to .9950, which is either .9949 or .9951.

If we choose .9951, the z = **2.58**.

If we choose .9949, the z = **2.57**.

Finding the z value.



Example

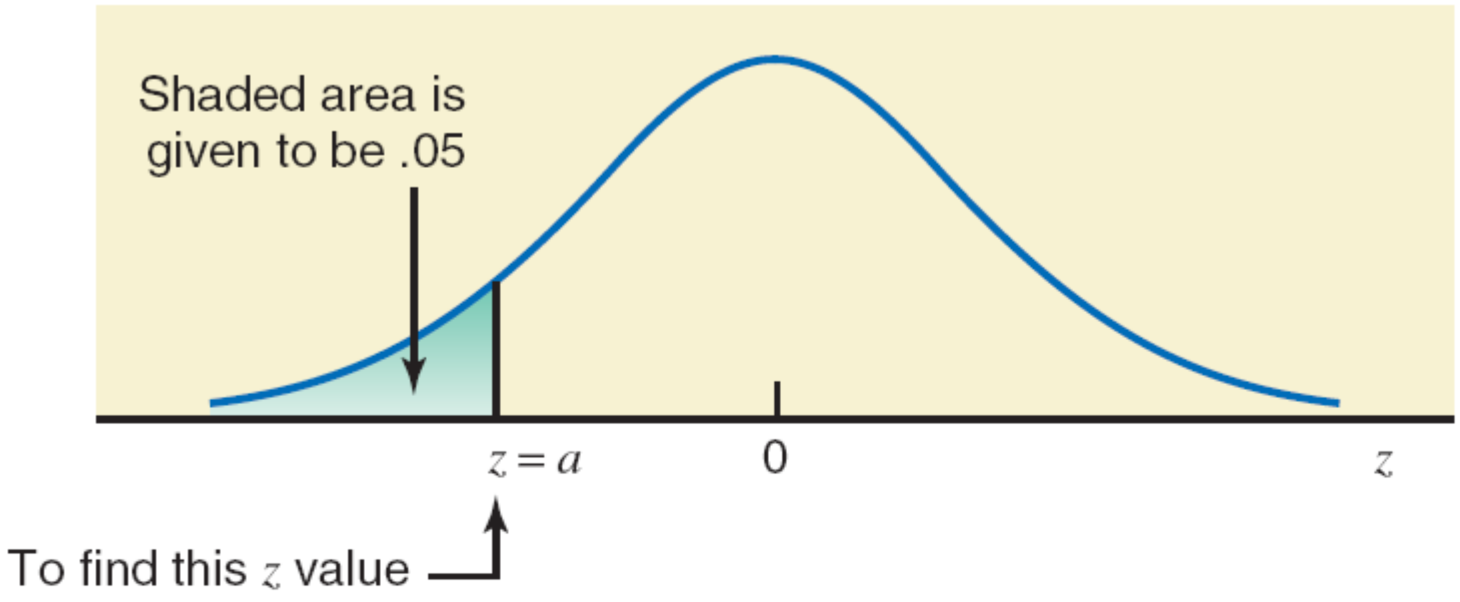
Find the value of z such that the area under the standard normal curve in the left tail is .05.

Because .05 is less than .5 and it is the area in the left tail, the value of z is negative.

Look for .0500 in the body of the normal distribution table. The value closest to .0500 in Table IV is either .0505 or .0495.

If we choose .0495, the $z = -1.65$.

Finding the z value.



Finding an x Value for a Normal Distribution

For a normal curve, with known values of μ and σ and for a given area under the curve to the left of x, the x value is calculated as

$$x = \mu + z\sigma$$

Example

It is known that the life of a calculator manufactured by Calculators Corporation has a normal distribution with a mean of 54 months and a standard deviation of 8 months. What should the warranty period be to replace a malfunctioning calculator if the company does not want to replace more than 1% of all the calculators sold?

Solution

Area to the left of x = .01 or 1%

Find the z value from the normal distribution table for .0100. Table IV does not contain a value that is exactly .0100.

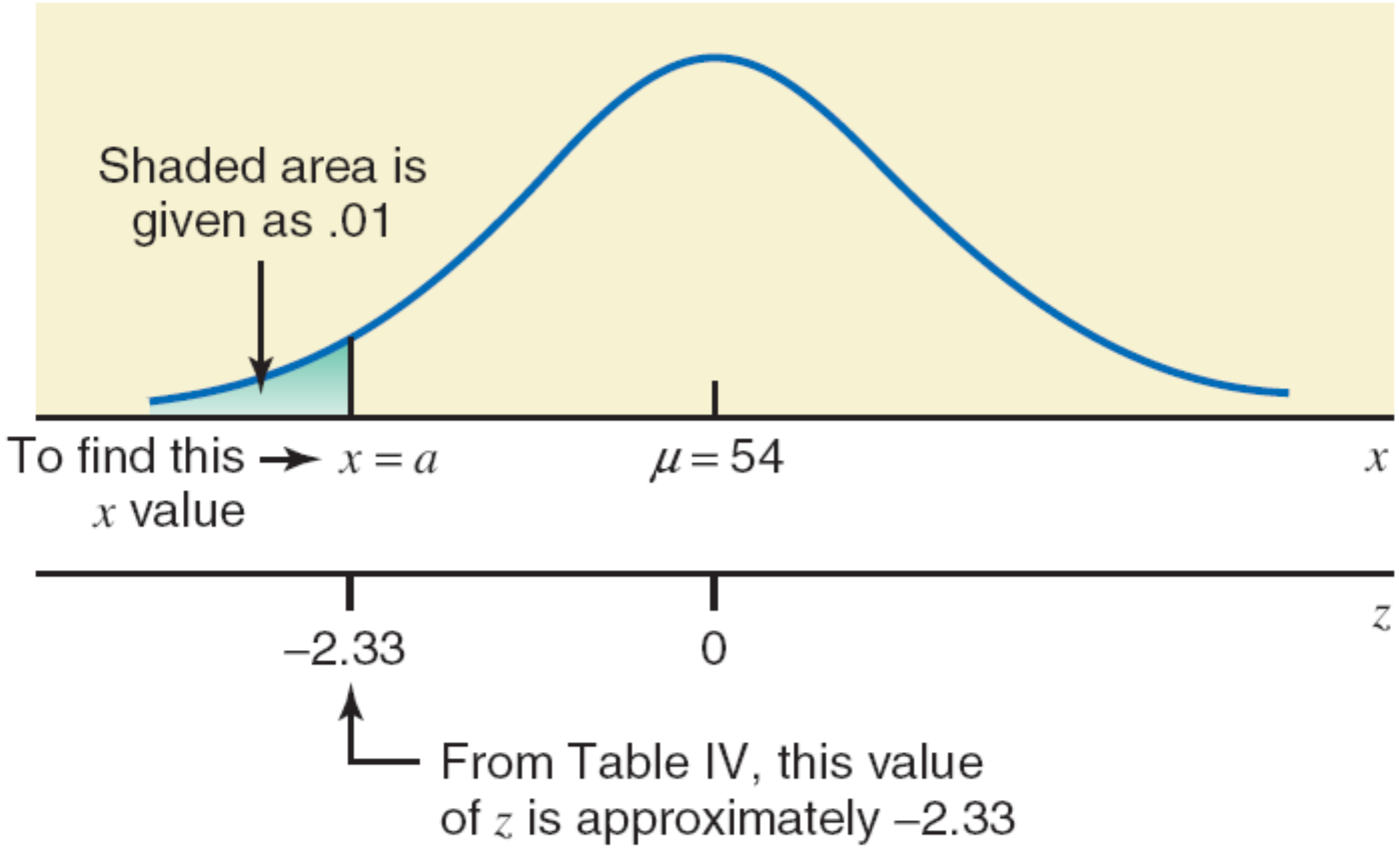
The value closest to .0100 in the table is .0099. The z = -2.33.

$$x = \mu + z\sigma = 54 + (-2.33)(8)$$
$$= 54 - 18.64 = 35.36$$

Solution

Thus, the company should replace all calculators that start to malfunction within 35.36 months (which can be rounded to 35 months) of the date of purchase so that they will not have to replace more than 1% of the calculators.

Finding an x value.



Example

According to the College Board, the mean combined (mathematics and critical reading) SAT score for all college-bound seniors was 1012 with a standard deviation of 213 in 2011. Suppose that the current distribution of combined SAT scores for all college-bound seniors is approximately normal with a mean of 1012 and a standard deviation of 213. Jennifer is one of the college-bound seniors who took this test. It is found that 10% of all current college-bound seniors have SAT scores higher than Jennifer. What is Jennifer's SAT score?

Solution

Area to the left of the x value = 1.0 - .10 = .9000

Look for .9000 in the body of the normal distribution table. The value closest to .9000 in Table IV is .8997, and the z value is 1.28.

$$x = \mu + z\sigma = 1012 + 1.28(213)$$
$$= 1012 + 272.64 = 1284.64 \approx 1285$$

Thus, Jennifer's combined SAT score is 1285.

Finding an x value.