CS 333: Computer Organization II
Floating Point Arithmetic

IEEE Floating Point Arithmetic

Topics:

IEEE Floating-Point Representation
Floating-Point Addition
Other Floating-Point Arithmetic
Pipelined Floating-Point Arithmetic

An Overview of the IEEE Floating-Point Standard 754 (1985)

Basic Concepts for Floating-Point Numbers

Motivation:

- Need to represent very small fractions
  e.g., Planck's $6.6254 * 10^{-27}$
- Need to represent very large numbers
  e.g., Avo's $6.0247 * 10^{23}$

Format:

- sign
- exponent
- mantissa (significand)

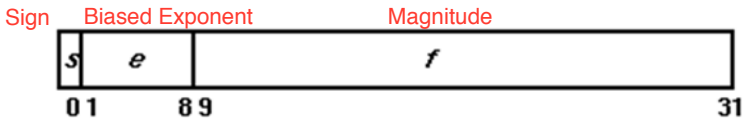
Normalization:

- eliminates leading zeros of mantissa
- adjusts exponent accordingly

Floating-Point Representation

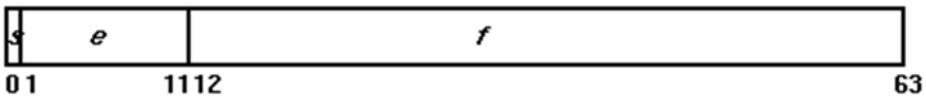
Single-Precision Numbers *(32 bits)*

- 1 bit for sign
  s (bit 0)
- 23 bits for magnitude
  f (bits 9-31)
- 8 bits for biased exponent
  e (bits 1-8)



Double-Precision Numbers *(64 bits)*

- 1 bit for sign
  s (bit 0)
- 52 bits for magnitude
  f (bits 12-63)
- 11 bits for biased exponent
  e (bits 1-11)



Single-Precision Numbers

Sign (*s*)

- FP number is + , if *s* is zero
- FP number is - , if *s* is one

Fraction (*f*):

- 23-bit binary representation is $0.f_2$
- True magnitude value = $0.f_2 + 1.0_2$
- Called a **normalized** representation
- In other words, the magnitude of the value 1.75_{10} would be represented
  as `11000000000000000000000`,
  not as `11100000000000000000000`,

because the first `1.0` is assumed to be there

Biased Exponent (*e*) *(Bias is 127)*

where $e = p + 127$

such that *p* is the true exponent

and *e* is the representation of the exponent to the base 2

Example: Single-Precision Numbers

Example

```
0   0000 1110   1010 0000 0000 0000 0000 000
s       e                   f
```

Magnitude = $1.1010\ 0000\ 0000\ 0000\ 0000\ 0000_2$
or 1.625

Exponent = $-0111\ 0001_2$
because p = e - 127

= ((0000 1110) - (0111 1111))$_2$
(since $(0111\ 1111)_2 = 127$)
So $p = -(0111\ 0001)_2$
$= -(64 + 32 + 16 + 1)$
$= -113$

Thus, the number represented in binary is

$(1.1010\ 0000\ 0000\ 0000\ 0000\ 000 \times 2^{-0111\ 0001})_2$

or 1.625×2^{-113}

Exercise: Single-Precision Numbers

Suppose

- $s = 0$ (=> positive)
- $f = 0110\ 0000\ 0000\ 0000\ 0000\ 0000$ (= 0)
- $e = 1000\ 1111$ (= 143)

What are the values of the magnitude and exponent?
What is the binary number represented?
What is the decimal number represented?
Answers: 1.375×2^{16}

Special Single-Precision Values

When $e = 0000\ 0000_2$
or $e = 1111\ 1111_2$,
the number is considered to be a special value
+0 as an IEEE single-precision floating-point number

```
= ( 0   0000 0000   0000 0000 0000 0000 0000 000 )2
```

-0 as an IEEE single-precision floating-point number

```
= ( 1   0000 0000   0000 0000 0000 0000 0000 000 )2
```

+infinity as an IEEE single-precision floating-point number

```
= ( 0   1111 1111   0000 0000 0000 0000 0000 000 )2
```

-infinity as an IEEE single-precision floating-point number

```
= ( 1   1111 1111   0000 0000 0000 0000 0000 000 )2
```

Range of Single-Precision Numbers

Because of the special cases above,
0 < e < 255
Thus, *-127 < p < +128*
or *-126 <= p <= +127*
The smallest possible single precision positive number:

```
  1.0000 0000 0000 0000 0000 000 x 2^0000 0001   (biased exponent)
= 1.0000 0000 0000 0000 0000 000 x 2^-0111 1110
```

or 1.0×2^{-126}
or approximately 1.2×10^{-38}
The largest possible single precision positive number:

```
  1.1111 1111 1111 1111 1111 111 x 2^1111 1110   (biased exponent)
= 1.1111 1111 1111 1111 1111 111 x 2^0111 1111
```

or $(2 - 2^{-23}) \times 2^{+127}$
or approximately $3.4 \times 10^{+38}$

Denormalized Numbers

Suppose $N < 2^{-126}$

- May be set to zero
- May be considered **denormalized**

For denormalized single-precision values,
the magnitude is in the form *0.f*

The smallest possible single precision denormalized number:
$2^{-23} \times 2^{-126}$ which is represented as

```
0  0000 0001  0000 0000 0000 0000 0000 001
```

This is called **graceful underflow**
or **gradual underflow**

Numerical Precision

Consider

$x_1 = 1.0000\ 0000\ 0000\ 0000\ 0000\ 000 \times 2^{-3}$

and

$x_2 = 1.0000\ 0000\ 0000\ 0000\ 0000\ 001 \times 2^{-3}$

No value between these two numbers
can be represented as a single-precision FP number **exactly**

Any value between these two numbers
must be rounded up to x_1 or down to x_2

x_1 and x_2 are called **machine numbers**,
because they can be represented **exactly** in the machine

Values between x_1 and x_2 are **not** machine numbers

Machine epsilon: distance between x_1 and x_2

Double-Precision Numbers

These are basically the same as single-precision numbers,
but with more bits representing the exponent and the magnitude

Sign (*s*)

- Value of number is **+** , if *s* is zero
- Value of number is **-** , if *s* is one

Fraction (*f*):

- 52-bit binary representation is $0.f_2$
- True magnitude value = $0.f_2 + 1.0_2$
- Called a **normalized** representation

Biased Exponent (*e*) *(Bias is 1023)*
where $e = p + 1023$
such that *p* is the true exponent

Range of Double-Precision Numbers

Exercise: Provide this range of values

Infinity, NaN, and zero

Since the IEEE standard permits the representation of values like infinity, it must also be able to perform operations on such values

Infinity

treated like a large number, usually

Examples: Given a valid normalized floating-point number Z:

- *+infinity + Z --> +infinity*
- *+infinity x Z --> +infinity*
- *Z / +infinity --> +0*
- *+infinity / Z --> +infinity*

+infinity and negative Z ==> similar results

NaN (Not A Number)

used when arithmetic operations produce an indeterminate value

Examples:

- *0 / 0 --> NaN*
- *NaN + Z --> NaN*
- *+infinity - (+infinity) --> NaN*

-0: a mechanism for recognizing that a number is zero to machine precision

Addition of Floating-Point Numbers

The steps (or stages) of a floating-point addition:

1. The exponents of the two floating-point numbers to be added are compared to find the number with the smallest magnitude
2. The significand of the number with the smaller magnitude is shifted so that the exponents of the two numbers agree
3. The significands are added
4. The result of the addition is normalized
5. Checks are made to see if any floating-point exceptions occurred during the addition, such as overflow
6. Rounding occurs

Floating-Point Addition Example

Example: $s = x + y$

- numbers to be added are $x = 1234.00$ and $y = -567.8$
- these are represented in decimal notation with a mantissa (significand) of four digits
- six stages (A - F) are required to complete the addition

Step	A	B	C	D	E	F
x	0.1234E4	0.12340E4				
y	-0.05678E3	-0.05678E4				
s			0.066620E4	0.6662E3	0.6662E3	0.6662E3

(For this example, we are throwing out biased exponents and the assumed 1.0 before the magnitude. Also all numbers are in the decimal number system and no complements are used.)

Time for Floating-Point Addition

Consider a set of floating-point additions
sequentially following one another
(as in adding the elements of two arrays)
Assume that each stage of the addition takes *t* time units

Time:	***t***	***2t***	***3t***	***4t***	***5t***	***6t***	***7t***	***8t***
Step								
A	$x_1 + y_1$						$x_2 + y_2$	
B		$x_1 + y_1$						$x_2 + y_2$
C			$x_1 + y_1$					
D				$x_1 + y_1$				
E					$x_1 + y_1$			
F						$x_1 + y_1$		

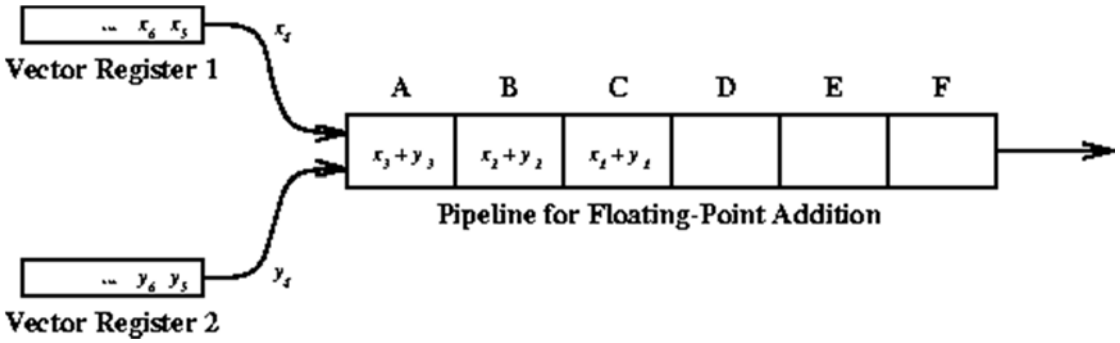
Each floating-point addition takes *6t* time units

Pipelined Floating-Point Addition

With the proper architectural design,
the floating-point addition stages can be overlapped

Time:	***t***	***2t***	***3t***	***4t***	***5t***	***6t***	***7t***	***8t***
Step								
A	$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$	$x_6 + y_6$	$x_7 + y_7$	$x_8 + y_8$
B		$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$	$x_6 + y_6$	$x_7 + y_7$
C			$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$	$x_6 + y_6$
D				$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$
E					$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$
F						$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$

This is called **pipelined** floating-point addition
Once the pipeline is full
and has produced the first result in *6t* time units,
it only takes *1t* time units to produce each succeeding sum
This is the idea behind vector computers
(Cray-1, Cray-2, Cyber 205)
where a vector can be viewed as a one-dimensional array of floating-point numbers



For an animated version of the figure above,
click here
For more information on vector computers,
click here

Main Features of the IEEE Floating-Point Standard 754

NaN: Not a Number

- allows computation to continue past exceptions
- if the argument of an expression is **NaN**,
  then so is the result
- ==> checking for exceptions and related measures
  can be done within the computation
  without having to involve the OS
- Example: suppose *x* = SQRT(*-1*)
  - SQRT(*-1*) = NaN
  - So *1* + *x* is NaN
  - and *1* - *x* is NaN

Infinity Arithmetic:

- Division by zero:
  - *1 / 0* is not NaN
  - *1 / 0* is infinity
- Division by infinity:
  - *1 / infinity* is not NaN
  - *1 / infinity* is not infinity
  - *1 / infinity* is zero
- All the standard rules for handling infinity apply
- Example: consider *arccos x = 2 arctan (SQRT((1-x) / (1+x)))*
  - *arctan x (as x --> infinity) = pi/2*
  - So *arctan (-1)*
    = 2 arctan (SQRT(2 / 0))
    = 2 arctan (SQRT(infinity))
    = 2 (pi/2)
    = pi

Rounding Rules: 4 rounding modes

1. **(Default)** Round to nearest value

-- round to even LSD on ties

2. Round toward zero
3. Round toward + infinity
4. Round toward - infinity

Representation Issues:

- Mantissas must be unpacked before used
  ie., the implicit bit must be made explicit
- Exponent field range (single-precision): $1 < e < 254$
- Representation as $a = s * 1.f * 2^{e-127}$
  where
    - *1.f*: significand (mantissa or integer part)
    - *f*: fraction
    - *e*: exponent (field or characteristic)
    - *s*: sign
- Special cases:

e = 0	***f = 0***	0 (zero), by convention
e = 0	***f != 0***	denormalized (subnormal) value
e = 255	***f = 0***	+ or = infinity
e = 255	***f != 0***	NaN

- Example: assume decimal representation with 4 digits
  and let $x = 1.234 * 10^{E_{min}}$
  (where E_{min} is the smallest possible exponent)
  Then
    - $x / 10 = 0.123$
    - $x / 100 = 0.012$
    - $x / 1000 = 0.001$
    - $x / 10000 = 0.000$

This is called **gradual underflow**
Alternative: **flush to zero**

- Note: Denormal values are **not** normalized

Advantage: $x != y \Rightarrow x - y != 0$
This might not be the case with **flush to zero**
Extra digits (or bits): used for rounding

- **guard digit**: first to the right of mantissa
  (needed for rounding addition results)
- **round digit**: second to the right of mantissa
  (needed for rounding multiplication results)
- **sticky bit**: third to the right of mantissa
  (needed in case of ties, 0.50..00 vs. 0.50..01)

Rounding Mode	Sum >= 0	Sum < 0
Toward - Infinity		+1 if (r or s)

Toward + Infinity	+1 if (r or s)	
Toward Zero (truncate)		
Toward Nearest Even	+1 if ((r and LSB) or (r and s))	+1 if ((r and LSB) or (r and s))

Based on notes provided by Carolyn J. C. Schauble, Colorado State University