Hypothesis Testing

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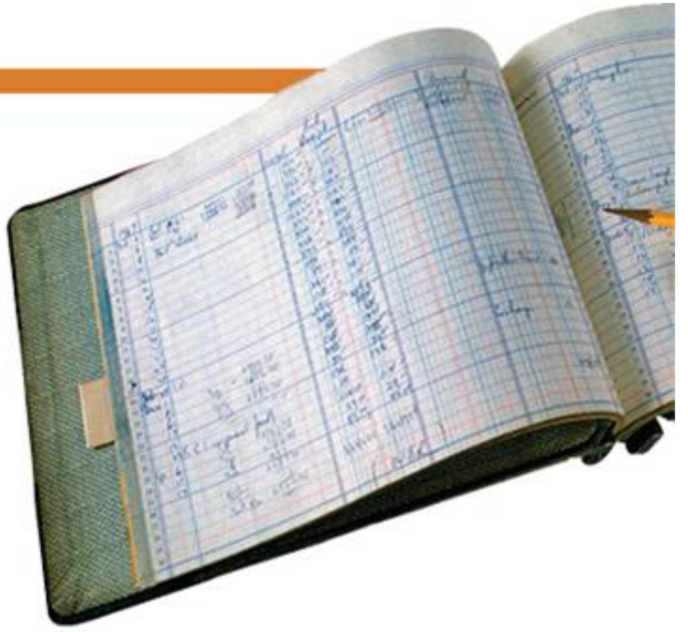
Section 8.3

Testing a Proportion *p*



Focus Points

- Identify the components needed for testing a proportion.

- Compute the sample test statistic.

- Find the *P*-value and conclude the test.

Testing a Proportion p

Throughout this section, we will assume that the situations we are dealing with satisfy the conditions underlying the binomial distribution.

In particular, we will let r be a binomial random variable. This means that r is the number of successes out of n independent binomial trials.

We will use $\hat{p} = r/n$ as our estimate for p, the population probability of success on each trial.

Testing a Proportion p

The letter q again represents the population probability of failure on each trial, and so $q = 1 – p$. We also assume that the samples are large (i.e., $np > 5$ and $nq > 5$).

For large samples, $np > 5$, and $nq > 5$, the distribution of $\hat{p} = r/n$ values is well approximated by a *normal curve* with mean μ and standard deviation σ as follows:

$$\mu = p$$

$$\sigma = \sqrt{\frac{pq}{n}}$$

Testing a Proportion p

The null and alternate hypotheses for tests of proportions are

Left-Tailed Test	**Right-Tailed Test**	**Two-Tailed Test**
H_0: $p = k$	H_o: $p = k$	H_0: $p = k$
H_1: $p < k$	H_1: $p > k$	H_1: $p \neq k$

depending on what is asked for in the problem. Notice that since p is a probability, the value k must be between 0 and 1.

Testing a Proportion *p*

For tests of proportions, we need to convert the sample test statistic $\hat{p}$ to a z value. Then we can find a P-value appropriate for the test. The $\hat{p}$ distribution is approximately normal, with mean p and standard deviation $\sqrt{pq/n}$. Therefore, the conversion of $\hat{p}$ to z follows the formula

$$z = \frac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$$

where $\hat{p} = r/n$ is the sample test statistic

n = number of trials

p = proportion specified in H_0

$q = 1 - p$

Testing a Proportion *p*

Using this mathematical information about the sampling distribution for $\hat{p}$, the basic procedure is similar to tests you have conducted before.

Testing a Proportion p

Procedure:

How to Test a Proportion p

Requirements

Consider a binomial experiment with n trials, where p represents the population probability of success and $q = 1 - p$ represents the population probability of failure. Let r be a random variable that represents the number of successes out of the n binomial trials. The number of trials n should be sufficiently large so that both $np > 5$ and $nq > 5$ (use p from the null hypothesis). In this case, $\hat{p} = r/n$ can be approximated by the normal distribution.

Testing a Proportion p

cont'd

Procedure:

HOW TO TEST A PROPORTION p

Procedure

1. In the context of the application, state the *null and alternate hypotheses* and set the *level of significance* α.
2. Compute the standardized *sample test statistic*

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

   where p is the value specified in H_0 and $q = 1 - p$.
3. Use the standard normal distribution and the type of test, one-tailed or two-tailed, to find the *P-value* corresponding to the test statistic.
4. *Conclude* the test. If P-value $\leq \alpha$, then reject H_0. If P-value $> \alpha$, then do not reject H_0.
5. *Interpret your conclusion* in the context of the application.

Example 1 – *Testing p*

A team of eye surgeons has developed a new technique for a risky eye operation to restore the sight of people blinded from a certain disease.

Under the old method, it is known that only 30% of the patients who undergo this operation recover their eyesight.

Suppose that surgeons in various hospitals have performed a total of 225 operations using the new method and that 88 have been successful (i.e., the patients fully recovered their sight). Can we justify the claim that the new method is better than the old one? (Use a 1% level of significance.)

Example 1– *Solution*

(a) Establish H_0 and H_1 and note the level of significance.

The level of significance is $\alpha = 0.01$. Let p be the probability that a patient fully recovers his or her eyesight. The null hypothesis is that p is still 0.30, even for the new method.

The alternate hypothesis is that the new method has improved the chances of a patient recovering his or her eyesight. Therefore,

$$H_0: p = 0.30 \qquad \text{and} \qquad H_1: p > 0.30$$

Example 1 – *Solution*

cont'd

(b) *Check Requirements* Is the sample sufficiently large to justify use of the normal distribution for $\hat{p}$? Find the sample test statistic $\hat{p}$ and convert it to a z value, if appropriate.

Using p from H_0 we note that $np = 225(0.3) = 67.5$ is greater than 5 and that $nq = 225(0.7) = 157.5$ is also greater than 5, so we can use the normal distribution for the sample statistic $\hat{p}$.

$$\hat{p} = \frac{r}{n} = \frac{88}{225} \approx 0.39$$

Example 1 – *Solution*

cont'd

The z value corresponding to $\hat{p}$ is

$$z = \frac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}} \approx \frac{0.39 - 0.30}{\sqrt{\dfrac{0.30(0.70)}{225}}} \approx 2.95$$

In the formula, the value for p is from the null hypothesis. H_0 specifies that

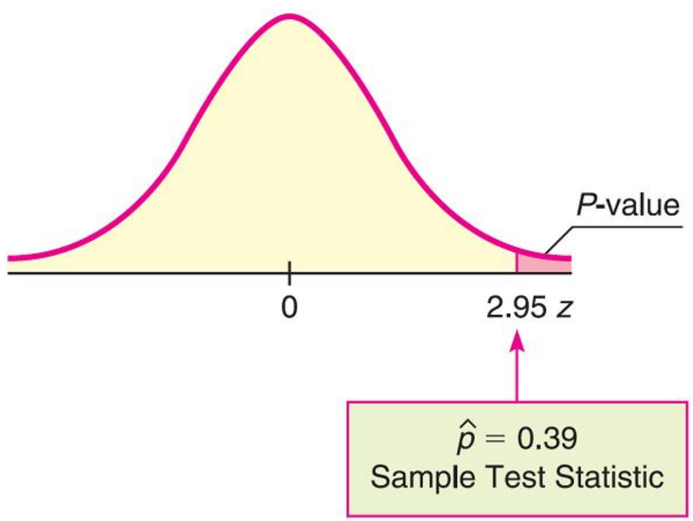
$$p = 0.30, \text{ so } q = 1 - 0.30$$
$$= 0.70.$$

Example 6 – *Solution*

cont'd

(c) Find the *P*-value of the test statistic.

Figure 8-10 shows the *P*-value. Since we have a right tailed test, the *P*-value is the area to the right of z = 2.95



P-value Area

Figure 8.10

Example 6 – *Solution*

cont'd

Using the normal distribution (Table 5 of Appendix II), we find that *P*-value = $P(z > 2.95) \approx 0.0016$.

(d) Conclude the test.

Since the *P*-value of $0.0016 \leq 0.01$ for α, we reject H_0.

(e) *Interpretation* Interpret the results in the context of the problem.

At the 1% level of significance, the evidence shows that the population probability of success for the new surgery technique is higher than that of the old technique.

Example 2 : National Crime Rates

Is the national crime rate really going down? They say that the reason for the decline in crime rates in the 1980s and 1990s is demographics. It seems that the population is aging and older people commit fewer crimes. According to the FBI and the Justice Department, 70% of all arrests are males aged 15 to 34 years. A random sample found that out of 32 arrest, 24 were males aged 15 to 34 years. Use a 1% level of significance to test the claim that the population of such arrests is different from 70%.

$$np = 32 \times 0.7 = 22.4 > 5 \qquad H_0 : p = 0.70 \qquad \alpha = 0.01 = 1\%$$

$$n(1-p) = 9.6 > 5 \qquad H_a : p \neq 0.70 \qquad c = 0.99 = 99\%$$

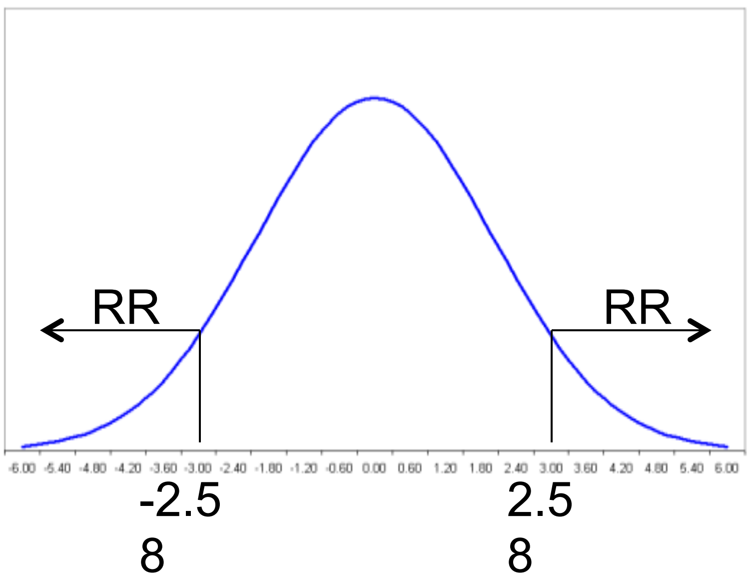
National Crime Rates (cont)

$$np = 32 \times 0.7 = 22.4 > 5 \qquad H_0 : p = 0.70 \qquad \alpha = 0.01 = 1\%$$

$$n(1-p) = 9.6 > 5 \qquad H_a : p \neq 0.70 \qquad c = 0.99 = 99\%$$

$$\hat{p} = \frac{24}{32} = 0.75$$

$$z = \frac{0.75 - 0.70}{\sqrt{\dfrac{0.70(1-0.70)}{32}}} \approx 0.62$$



$$p = 2 * P(z > 0.62)$$

$$= 2(0.2676)$$

$$= 0.5352$$

$$z_{\alpha/2} = \pm 2.58$$

National Crime Rates (Cont)

Decision:
Fail to reject the null hypothesis.

Hence, the significance statement is:
At the 1% level of significance, there is insufficient evidence to reject the null hypothesis.

Conclusion:
That is, the proportion of males aged 15 to 34 arrested is
not different from the claim by the FBI and Justice Department at the 1% level of significance.

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Example 3: Proposed Taxation

To test the proportion within a population that agree with a newly proposed tax a sample of 200 individuals were surveyed and it was found that 6 stated that they in fact did agree. At the 1% level of significance, test the claim that 5% of the population agree with the new taxation.

$$np = 200 \times 0.05 = 10 > 5 \qquad H_0 : p = 0.05 \qquad \alpha = 0.01 = 1\%$$

$$n(1 - p) = 190 > 5 \qquad H_a : p \neq 0.05 \qquad c = 0.99 = 99\%$$

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Proposed Taxations (cont)

$$np = 200 \times 0.05 = 10 > 5 \qquad H_0 : p = 0.05 \qquad \alpha = 0.01 = 1\%$$

$$n(1-p) = 195 > 5 \qquad H_a : p \neq 0.05 \qquad c = 0.99 = 99\%$$

$$\hat{p} = \frac{6}{200} = 0.03$$

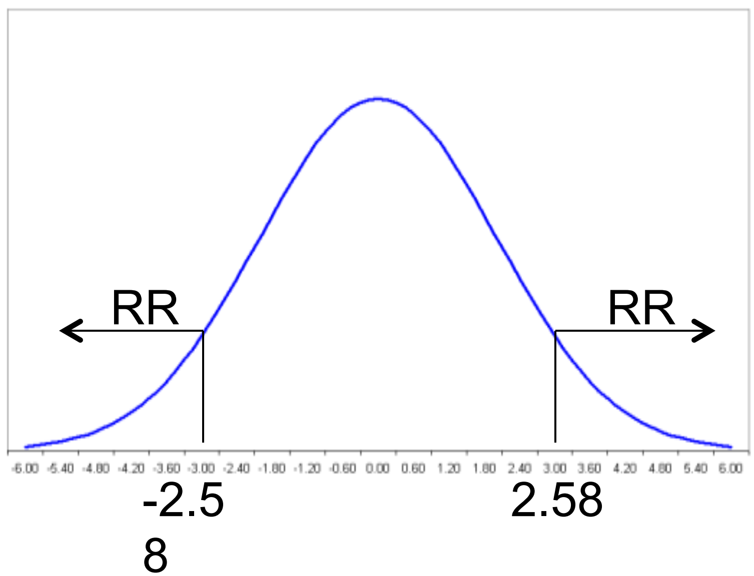
$$z = \frac{0.03 - 0.05}{\sqrt{\frac{0.05(1-0.05)}{200}}} \approx -1.298$$

$$p = 2 * P\left(z < -1.2978\right)$$

$$= 2(0.0972)$$

$$= 0.1944$$

$$z_{\alpha/2} = \pm 2.58$$



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Proposed Taxation (cont.)

Decision:
Fail to reject the null hypothesis.

Hence, the significance statement is:
At the 1% level of significance, there is insufficient evidence to reject the null hypothesis.

Conclusion:
That is, the proportion of individual that agree with the newly proposed tax is not different from 5% at the 1% level of significance.

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Example 16

A One-Sided Test Showing Smokers Have Fewer Males Babies

A study of 8960 births[4]The total sample size and percentages match G. Koshy et al., "*Parental smoking and increased likelihood of female births*," *Annals of Human Biology* 37(6), (2010), pp. 789-800. revealed that, among the $n = 1103$ births to parents who both smoked, there were 480 males.

Construct a test of hypotheses, with $\alpha = .01$, with the intent of establishing that, when both parents smoke, the population proportion of male births is less than .5.

SOLUTION

The sample size is $n = 1103$ and the parameter p is the population proportion of males born. We are trying to show that this proportion is less than .5 so we test $H_0 : p = .5$ against a one-sided alternative. That is, we test

$$H_0 : p = .5 \quad \text{versus} \quad H_1 : p < .5$$

Because the sample size $n = 1103$ is very large, the Z test is appropriate. The test statistic is

$$Z = \frac{\hat{p} - .5}{\sqrt{.5 \times .5 / 1103}}$$

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With $\alpha = .01$ the one-sided rejection region is $R: Z \le -2.33$. See Figure 16.

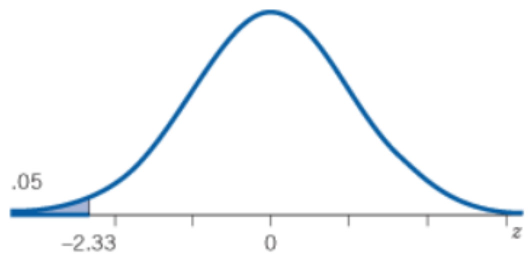

Figure 16 Left-sided rejection region $Z \le -2.33$.

From the sample data, $\hat{p} = .435$ and the computed value of the test statistic Z is

$$z = \frac{.435 - .5}{\sqrt{.5 \times .5 / 1103}} = \frac{-.065}{.01506} = -4.32$$

Because the observed value $z = -4.32$ is less than -2.33, we reject the null hypothesis in favor of the the alternative that p is less than .5, at level of significance $\alpha = .01$.

The observed value -4.32 of Z is to the left of the interval where the sampling distribution is exhibited in Figure 16. A computed calculation gives $P\text{–value} = P\ [Z < -4.32\ ] = .0000$ indicating that the positive P-value rounds to zero to four decimal places. This extremely small P-value greatly strengthens the evidence against the null hypothesis that males and females are equally likely in the subpopulation that was sampled. When both parents smoke, males constitute a smaller proportion of all babies.

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