Solutions to Selected Computer Lab Problems and Exercises in Chapter 7 of *Statistics and Data Analysis for Financial Engineering, 2nd ed.* by David Ruppert and David S. Matteson

Problem 7. (a) `A` is an upper triangular matrix, and, as can be seen below, the sample covariance matrix of Y is equal to $A^{\mathsf{T}}A$.

```
> A
        ibm     crsp
ibm  0.0175 0.003773
crsp 0.0000 0.006779
> cov(Y)
           ibm      crsp
ibm  3.061e-04 6.602e-05
crsp 6.602e-05 6.019e-05
> t(A)%*%A
           ibm      crsp
ibm  3.061e-04 6.602e-05
crsp 6.602e-05 6.019e-05
```

(b) The MLE of θ is given in the `R` output below.

```
> fit_mvt$par
[1] 0.0003789 0.0008317 0.0126907 0.0026859 0.0051011 4.2618395
```

We see that the estimated mean vector is (0.0003789, 0.0008317), the estimated Cholesky factor of the covariance matrix is

$$\begin{pmatrix} 0.01269 & 0.00268 \\ 0 & 0.00510 \end{pmatrix},$$

and the estimated degrees of freedom parameter is 4.26.

(c) The Fisher information matrix is printed below.

```
> fisher = fit_mvt$hessian
> fisher
           [,1]       [,2]      [,3]       [,4]      [,5]      [,6]
[1,]  1.533e+07 -1.572e+07   -337423  1.804e+05     87121   -474.73
[2,] -1.572e+07  7.444e+07    145737  1.014e+06    838199    938.50
[3,] -3.374e+05  1.457e+05  23313101 -1.365e+07  -8785834  -7420.14
[4,]  1.804e+05  1.014e+06 -13648160  7.138e+07  -5213629   -608.94
[5,]  8.712e+04  8.382e+05  -8785834 -5.214e+06 147902020 -19875.27
[6,] -4.747e+02  9.385e+02     -7420 -6.089e+02    -19875     21.77
```

(d) The standard error are below:

```
> se = sqrt(diag(solve(fisher)))
> se
[1] 2.887e-04 1.310e-04 2.506e-04 1.292e-04 9.373e-05 2.570e-01
```

(e) The MLE of the covariance matrix is printed as COV_Y in the output below. For comparison, the sample covariance is printed in the last line.

```
> ML = fit_mvt$par
> Ahat = matrix(c(ML[3:4],0,ML[5]),nrow=2,byrow=TRUE)
> Ahat
        [,1]     [,2]
[1,] 0.01269 0.002686
[2,] 0.00000 0.005101
> COV_Y = t(Ahat)%*%Ahat * ML[6]/(ML[6]-2)
> COV_Y
          [,1]      [,2]
[1,] 3.035e-04 6.423e-05
[2,] 6.423e-05 6.262e-05
> cov(Y)
           ibm      crsp
ibm  3.061e-04 6.602e-05
crsp 6.602e-05 6.019e-05
```

(f) The MLE of ρ is 0.4659. For comparison, the sample correlation is 0.4864.

```
> rho = COV_Y[1,2]/sqrt(COV_Y[1,1]*COV_Y[2,2])
> rho
[1] 0.4659
> cor(Y)
        ibm   crsp
ibm  1.0000 0.4864
crsp 0.4864 1.0000
```

Exercise 5. $E(X) = 0$ by symmetry of X about 0. Then by (A.27), $\text{cov}(X, Y) = E(XY) = E(X^3) = (2a)^{-1} \int_{-a}^{a} x^3 dx = 0$. Since the correlation coefficient is the covariance divided by the two standard deviation, the correlation coefficient must also be 0.

To show that X and Y are not independent, we need only find sets A and B such that $P(X \in A \text{ and } Y \in B) \neq P(X \in A)P(Y \in B)$. This is easy to do. Let $A = (0.9a, a)$, that is, the interval from $0.9a$ to a, and $B = ((0.9a)^2, a^2)$. Then

$$0.05 = P(X \in A \text{ and } Y \in B) \neq P(X \in A)P(Y \in B) = (0.05)(0.1).$$

The point here is that $X \in A \Rightarrow Y \in B$ so $P(X \in A) = P(X \in A \text{ and } Y \in B)$.

Exercise 7. If $\boldsymbol{x}$ is an eigenvector of a matrix $\boldsymbol{A}$ with negative eigenvalue λ, then

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\mathsf{T}(\lambda\boldsymbol{x}) = \lambda\|\boldsymbol{x}\|^2$$

is negative, so $\boldsymbol{A}$ cannot be a covariance matrix, since, if $\boldsymbol{A}$ were the covariance matrix of a random vector $\boldsymbol{Y}$, then by (7.7) the variance of $\boldsymbol{x}^\mathsf{T}\boldsymbol{Y}$ would be negative.

We see from the output below that $\mathrm{COV}(\boldsymbol{Y})$ would have a negative eigenvalue, -0.2728, if $a = 0$.

```
> A = matrix(c(1, 0.9, 0, 0.9, 1, 0.9, 0, 0.9, 1),nrow=3)
> A
     [,1] [,2] [,3]
[1,]  1.0  0.9  0.0
[2,]  0.9  1.0  0.9
[3,]  0.0  0.9  1.0
> eigen(A)
$values
[1]  2.2728  1.0000 -0.2728

$vectors
       [,1]       [,2]    [,3]
[1,] 0.5000 -7.071e-01  0.5000
[2,] 0.7071 -8.723e-16 -0.7071
[3,] 0.5000  7.071e-01  0.5000
```