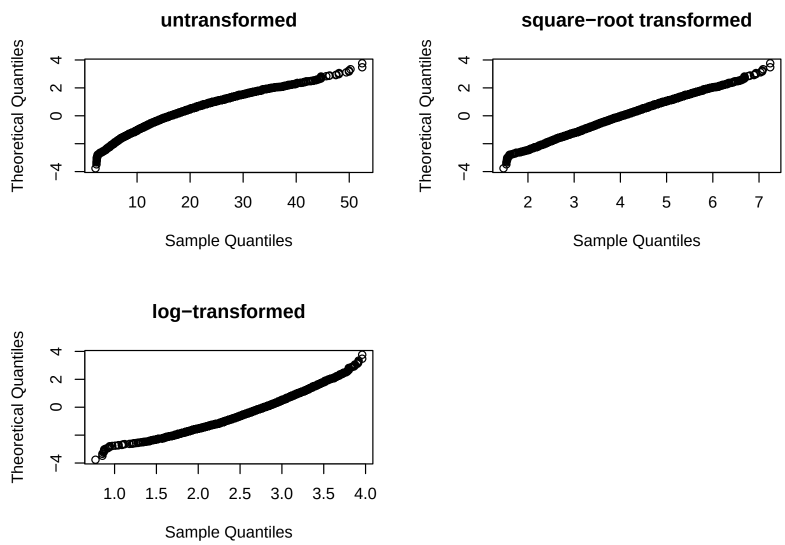
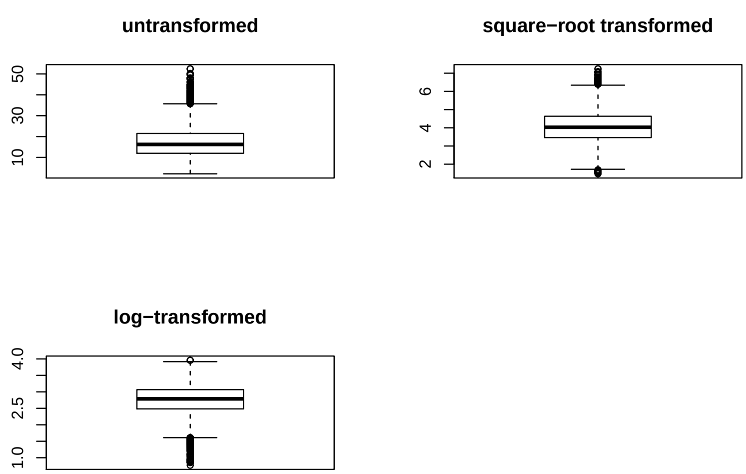
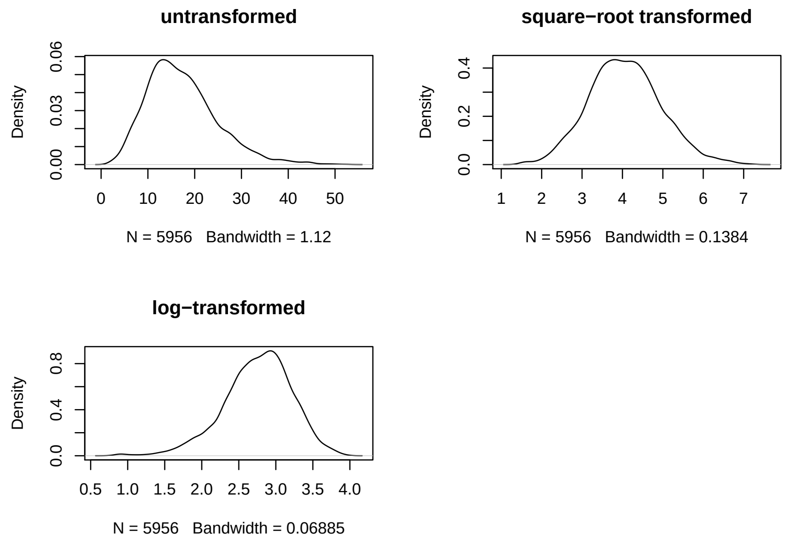
Solutions to Selected Computer Lab Problems and Exercises in Chapter 5 of *Statistics and Data Analysis for Financial Engineering, 2nd ed.* by David Ruppert and David S. Matteson

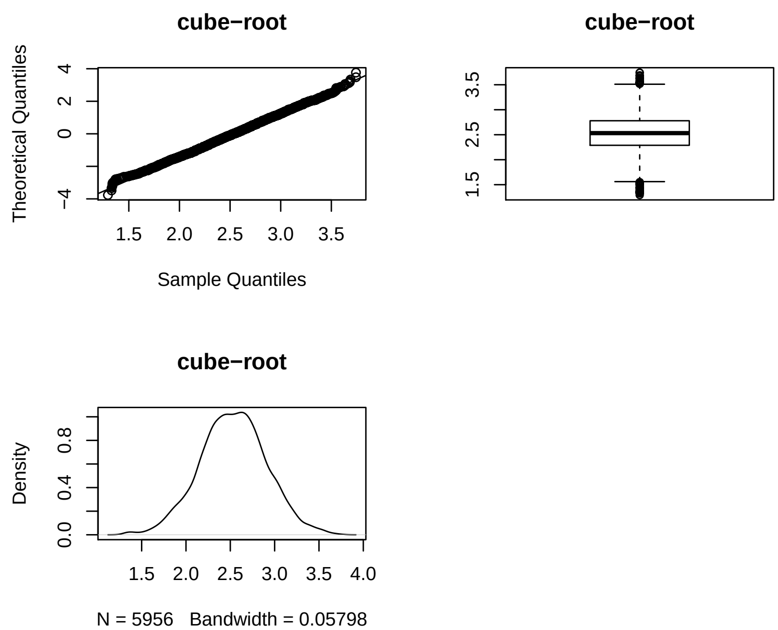
Problem 1 The plots are below. The square-root transformation does a good job of symmetrizing the data although there is a slight amount of right-skewness in the density plot. The log transformation over-transforms to left skewness.







Perhaps a transformation a little stronger than the square-root might be somewhat better. Below are plots using the cube-root transformation. That's about as symmetric as one can expect. I don't see much advantage to the cube-root over the square-root transfromation.



Problem 2 (a) `ind` indicates which value of λ is the MLE. `ind2` indicates which values of λ are in a 95% confidence interval.

(b) `interp = TRUE` indicates that spline interpolation should be used to produce a smoother plot of the log-likelihood.

```
> bc$x[ind]
```

```
[1] 0.36
> bc$x[ind2]
[1] 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.40
```

(c) The MLE of λ is 0.36, rounded to 2 digits.

(d) The 95% confidence interval of (0.32, 0.40), rounded to 2 digits.

```
> ind3 = (bc$y > max(bc$y) - qchisq(0.99, df = 1) / 2)
> bc$x[ind3]
 [1] 0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.40 0.41
```

The 99% confidence interval of (0.31, 0.41), rounded to 2 digits.

Problem 6. From the R output below, we see that the MLE's of the mean, standard deviation, and degrees-of-freedom parameter are 0.00078, 0.01058, and 4.03515, respectively, and AIC is -11960.47.

```
> data(Garch, package = "Ecdat")
> library("fGarch")
> data(EuStockMarkets)
> Y = diff(log(EuStockMarkets[ ,1])) # DAX
> ##### std #####
> loglik_std = function(x) {
+   f = -sum(dstd(Y, x[1], x[2], x[3], log = TRUE))
+   f}
> start = c(mean(Y), sd(Y), 4)
> fit_std = optim(start, loglik_std, method = "L-BFGS-B",
+                 lower = c(-0.1, 0.001, 2.1),
+                 upper = c(0.1, 1, 20), hessian = TRUE)
> cat("MLE =", round(fit_std$par, digits = 5))
MLE = 0.00078 0.01058 4.03515
> minus_logL_std = fit_std$value # minus the log-likelihood
> AIC_std = 2 * minus_logL_std + 2 * length(fit_std$par)
> AIC_std
[1] -11960.47
```

Problem 7. We see below that the MLE's are 0.00074, 0.0106, 4.03, and 0.987 for the mean, standard deviation, degrees-of-freedom, and ξ. AIC is -11958.6 which is larger than -11960.47, the AIC of the symmetric-t fit. Since smaller values of AIC are preferred, AIC would select the symmetric-t rather than the skewed-t.

The standard errors are 0.000238, 0.000367, 0.400424, 0.029603 for these parameters. The 95% confidence intervals are in the matrix `ci`. The confidence interval for ξ includes 1, which suggests that a t-distribution fits adequately, at least compared to a skewed-t. This result agrees with the conclusion from the AIC values.

```
> loglik_sstd = function(x){
+   f = -sum(dsstd(Y, x[1], x[2], x[3], x[4], log = TRUE))
+   f}
> start = c(mean(Y), sd(Y), 4, 1)
> fit_sstd = optim(start, loglik_sstd, method = "L-BFGS-B",
+                  lower = c(-0.1, 0.001, 2.1,0.25),
+                  upper = c(0.1, 1, 20,4), hessian = TRUE,
+                  control=list(maxit=1000, tmax=100))
> cat("MLE =", round(fit_sstd$par, digits = 5))
MLE = 0.00074 0.0106 4.03 0.987
>
> minus_logL_sstd = fit_sstd$value # minus the log-likelihood
> AIC_sstd = 2 * minus_logL_sstd + 2 * length(fit_sstd$par)
> AIC_sstd
[1] -11958.6
> se = sqrt(diag(solve(fit_sstd$hessian)))
> se
[1] 0.000238 0.000367 0.400424 0.029603
> ci = matrix(nrow=4,ncol=2)
> for (i in 1:4){
+   ci[i,] = fit_sstd$par[i] + c(-1,1)*se[i]*qnorm(0.975)
+ }
> options(digits=3)
> ci
         [,1]   [,2]
[1,] 0.000272 0.0012
[2,] 0.009846 0.0113
[3,] 3.247316 4.8170
[4,] 0.929380 1.0454
```

Exercise 4. (a)

$$\text{Var}\left\{\left(\frac{X-\mu}{\sigma}\right)^2\right\} = E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} - \left\{E\left(\frac{X-\mu}{\sigma}\right)^2\right\}^2 = \text{Kur}(X) - 1,$$

so $\text{Kur}(X) = \text{Var}\left\{\left(\frac{X-\mu}{\sigma}\right)^2\right\} + 1$.

(b) Since a variance is non-negative and is zero only if the random variable is degenerate, $\text{Kur}(X)$ must be at least 1 and is equal 1 only if $\left(\frac{X-\mu}{\sigma}\right)^2$ is constant.

(c) If $P(X = a) = P(X = b) = 1/2$, then $\left(\frac{X-\mu}{\sigma}\right)^2$ is equal to the constant $\{(a-b)/2\}^2$, so $\text{Kur}(X)$ must be 1 by part (b).

Exercise 8.

$$
\begin{aligned}
\mathrm{MSE}(\widehat{\theta}) &= E\{(\widehat{\theta}-\theta)^2\} \\
&= E\Big[\{\widehat{\theta}-E(\widehat{\theta})\} - \{E(\widehat{\theta})-\theta\}\Big]^2 \\
&= E\{\widehat{\theta}-E(\widehat{\theta})\}^2 + \{E(\widehat{\theta})-\theta\}^2 - 2E\Big[\{\widehat{\theta}-E(\widehat{\theta})\}\{E(\widehat{\theta})-\theta\}\Big]
\end{aligned}
$$

and

$$E\Big[\{\widehat{\theta}-E(\widehat{\theta})\}\{E(\widehat{\theta})-\theta\}\Big] = \{E(\widehat{\theta})-\theta\}E\{\widehat{\theta}-E(\widehat{\theta})\} = 0,$$

since

$$E\{\widehat{\theta}-E(\widehat{\theta})\} = E(\widehat{\theta}) - E(\widehat{\theta}) = 0.$$

Exercise 10. Since the kurtosis of a t-distribution is $3+6/(\nu-4)$, given only that the sample kurtosis is 9, we would estimate ν by solving

$$9 = 3 + \frac{6}{\widehat{\nu}-4},$$

which has solution $\widehat{\nu} = 5$.