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Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming

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This note formulates a convex mathematical programming problem in which the usual definition of the feasible region is replaced by a significantly different strategy. Instead of specifying the feasible region by a set of convex inequalities, $f_i(x) \leqq b_i$, $i=1, 2, \cdots, m$, the feasible region is defined via set containment. Here n convex activity sets $\{K_j, j=1, 2, \cdots, n\}$ and a convex resource set K are specified and the feasible region is given by

$$X = \{x \epsilon R^n \mid x_1K_1 + x_2K_2 + \cdots + x_nK_n \subseteq K, x_j \geqq 0\},$$

where the binary operation $+$ refers to addition of sets. The problem is then to find $\bar{x} \epsilon X$ that maximizes the linear function $c \cdot x$. When the resource set has a special form, this problem is solved via an auxiliary linear-programming problem and application to inexact linear programming is possible.

THIS NOTE adopts a different strategy for defining the feasible region of a convex programming problem: The usual convex inequalities $f_i(x) \leqq b_i$ are replaced by the restriction that the sum of a finite number of convex sets is contained in another convex set. Let $\{K_j\}$ be n nonempty convex sets and K another nonempty convex set, all in Euclidean m space. We are concerned with the following optimization problem:

$$\sup c \cdot x, \quad \text{subject to } x_1K_1 + x_2K_2 + \cdots + x_nK_n \subseteq K \quad \text{and} \quad x_j \geqq 0, \qquad \text{(I)}$$

where the binary operation $+$ refers to addition of sets. The n sets $\{K_j\}$ can be identified as activity sets, the set K as the resource set. A set of n scalars $\{\bar{x}_j\}$ is feasible for (I) if and only if, for every possible set of one activity vector $a_j \epsilon K_j$ for each j, one has $\bar{x}_1a_1 + \bar{x}_2a_2 + \cdots + \bar{x}_na_n \epsilon K$ $(\bar{x}_j \geqq 0)$. Note that, if K contains the zero vector, then (I) has the feasible solution $\bar{x}_j = 0$ for all j.

When K has a certain natural structure, the essence of (I) can be enlightened by comparing it with the notion of a generalized linear program, as set forth by Dantzig and Wolfe.[1] Let $b \epsilon R^m$ and define $K(b) = \{y \epsilon R^m \mid y \leqq b\}$, where of course $K(b)$ is a convex set. Now define

$$\sup c \cdot x, \quad \text{subject to } x_1K_1 + x_2K_2 + \cdots + x_nK_n \subseteq K(b) \quad \text{and} \quad x_j \geqq 0. \qquad \text{(Ib)}$$

In generalized linear programming one is given the freedom to choose *any* vector $a_j \epsilon K_j$ for each j to maximize $c \cdot x$, i.e.,

$$\sup c\cdot x, \qquad \text{subject to } x_1a_1+x_2a_2+\cdots+x_na_n \leqq b \quad \text{and} \quad x_j \geqq 0, \quad a_j \epsilon K_j. \qquad \text{(GLP)}$$

In (GLP) the activity vectors a_j are decision quantities along with the scalar quantities x_j. But a set of scalars $\{\bar{x}_j\}$ is feasible for (Ib) if and only if $\bar{x}_1a_1+\bar{x}_2a_2+\cdots+\bar{x}_na_n \leqq b$ and $x_j \geqq 0$ for *all* possible sets of activity vectors.

Note that ordinary linear programming is a special case of (Ib) in which the convex sets contain a single vector. In fact, this observation is the motivation for considering (Ib) as an inexact linear programming problem. The term inexact (as suggested by K. O. Kortanek) applies to the situation in which the activity vectors for a linear program are not known with certainty; all that is known is that the jth activity vector will be a member of the convex set K_j. A different approach to inexact linear programming has been given by Gould.[2]

THE FEASIBLE REGION AND AN AUXILIARY LINEAR PROGRAM

This section shows that the set of feasible solutions to problem (I) determines a convex set, and furthermore that the optimal solution to problem (Ib) can be determined by solving an auxiliary linear programming problem.

Now define X as the set of feasible solutions to (I), i.e., $X=\{x \epsilon R^n | x \text{ is feasible for (I)}\}$.

Lemma. *X is a convex set.*

Proof. Let $x^i=(x_1^i, x_2^i, \cdots, x_n^i)$, $i=1, 2$, be members of X, let $a_j \epsilon K_j$, $j=1, 2, \cdots, n$, be arbitrary, and suppose that $\lambda \epsilon (0, 1)$. Then $x_1^i a_1+\cdots+x_n^i a_n=k^i \epsilon K$, $i=1, 2$, so

$$[\lambda x_1^1+(1-\lambda)x_1^2]a_1+\cdots+[\lambda x_n^1+(1-\lambda)x_n^2]a_n=\lambda k_1+(1-\lambda)k_2 \epsilon K,$$

and thus $\lambda x^1+(1-\lambda)x^2 \epsilon X$.

On the basis of this lemma, problem (I) is appropriately classified as a convex programming problem, for, in addition to the convex feasible region, the objective function is linear. These same remarks apply as well to problem (Ib), for it is a special case of problem (I). The specialization in (Ib) is that the resource set is a special type of polyhedral convex set. This simplification permits the optimal solution of problem (Ib) to be obtained with an auxiliary linear programming problem.

The activity vectors $\{\bar{a}_j | j=1, 2, \cdots, n\}$ for this auxiliary linear programming problem are obtained through the support functionals $\delta^*(\cdot|K_j)$ of the n convex sets $\{K_j\}$, where $\delta^*(y|K_j)$ is by definition equal to $\sup_{a_j \epsilon K_j} y\cdot a_j$. For each j define the column $\bar{a}_j$ whose ith component is equal to $\delta^*(e_i|K_j)=\sup_{a_j \epsilon K_j} a_{ij}$. Now note that, if, for some $i \epsilon \{1, 2, \cdots, m\}$, $\delta^*(e_i|K_j)=\infty$, then $\sum_{j=1}^{j=n} \bar{x}_j K_j \subseteq K(b)$ necessarily implies that $\bar{x}_j=0$. Since the decision variable $\bar{x}_j$ must necessarily be zero, there is no loss of generality in simply omitting the activity set K_j from problem (Ib). If all the activity sets $\{K_j\}$ must be omitted in this fashion, then the only possibility for a solution to (Ib) is $\bar{x}=(0, 0, 0, \cdots, 0)$, and this vector is feasible if and only if $b \geqq 0$. Hence, we shall assume that, whenever necessary, certain activity sets $\{K_j\}$ are deleted from (Ib), so that $\delta^*(e_i|K_j)<\infty$ for all i and j. Note that, when all the activity sets $\{K_j\}$ are compact, this latter condition is automatically ensured.

Now define the $m\times n$ matrix $\bar{A}$ as $\bar{A}=(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n)$.
The auxiliary linear program to (Ib) will be denoted LP$(\bar{A})$ as follows:

$$\text{LP}(\bar{A}): \max c\cdot x, \qquad \text{subject to } \bar{A}\cdot x\leqq b \quad \text{and} \quad x\geqq 0.$$

In the following theorem it will be shown that the optimal solution to LP$(\bar{A})$ is also the optimal solution to (Ib). For this theorem define a set M of $m\times n$ matrices as follows:

$$M=\{(a_1, a_2, \cdots, a_n)|a_1\epsilon K_1, a_2\epsilon K_2, \cdots, a_n\epsilon K_n\}.$$

With this definition, $\bar{x}$ is a feasible solution to problem (Ib) if and only if $A\cdot x\leqq b\forall A\epsilon M$ and $x\geqq 0$.

Theorem. *If $\bar{x}$ is a feasible solution to* LP$(\bar{A})$, *then $\bar{x}$ is feasible for* (Ib) *and conversely. Hence, the sets of feasible solutions to the two problems are identical, which means that the optimal solution to* (Ib) *can be obtained by finding the optimal solution to the ordinary linear programming problem* LP$(\bar{A})$.

Proof. First let $\bar{x}$ be a feasible solution to LP$(\bar{A})$, so that $\bar{A}\bar{x}\leqq b$ and $\bar{x}\geqq 0$. Now, if $A\epsilon M$, then by construction $A\leqq \bar{A}$, which means that $A\bar{x}\leqq\bar{A}\bar{x}\leqq b\forall A\epsilon M$. Hence, $\bar{x}$ is feasible for problem (Ib).

Conversely, if $\bar{x}$ is feasible for (Ib), then, for $i=1, 2, \cdots, m$, $\bar{x}_1a_{i1}+\cdots+\bar{x}_na_{in}\leqq b_i$, $a_j\epsilon K_j$. Consequently, for $i=1, 2, \cdots, m$,

$$\bar{x}_1\sup_{a_1\epsilon K_1}(a_{i1})+\cdots+\bar{x}_n\sup_{a_n\epsilon K_n}(a_{in})\leqq b_i,$$

which means that $\bar{x}$ is feasible for LP$(\bar{A})$.

The relation between problems LP$(\bar{A})$ and (Ib) is somewhat more general than is indicated by the theorem. The sets of the form

$$K(b)=\{y\epsilon R^m|y_i\leqq b_i, i=1, 2, \cdots, m\}$$

can be extended by replacing the ith constraint by $y_i\geqq b_i$ or $y_i=b_i$. In the former case, replace $\bar{a}_{ij}$ for $j=1, 2, \cdots, n$ by $\inf_{a_j\epsilon K_j} e_i\cdot a_j$ and use $\geqq b_i$ in LP$(\bar{A})$. In the latter case, if the ith component of K_j is not fixed, then necessarily $\bar{x}_j=0$ and K_j can be omitted from (Ib) as above. Otherwise, the ith component of K_j is fixed at $\bar{a}_{ij}$ for $j=1, \cdots, n$ and use $=b_i$ in $LP(\bar{A})$. This extension was communicated to the author by T. L. Magnanti.

INEXACT LINEAR PROGRAMMING

Problem (Ib) can be viewed as a linear programming problem in which the activity vectors are not known with certainty and before the exact activity vectors are known one must make an assignment of the decision variables $\{x_j\}$. Hence (Ib) is a type of stochastic linear program for which the solution of LP$(\bar{A})$ provides an ultraconservative strategy. The only allowable decision variables are the ones that permit a feasible solution no matter what $a_j\epsilon K_j$ is generated. Of course, the sets $\{K_j\}$ could represent, for example, hyperspheres that contain a certain probability that a_j lies in a given set.

To illustrate this inexact extension of linear programming, consider the following problem:

$$\max c \cdot x, \qquad \text{subject to } Ax \leqq b \quad \text{and} \quad x \geqq 0,$$

where $A = (a_1, a_2, \cdots, a_n)$ and each $a_j \epsilon R^m$. Now suppose that the activity vectors a_j are only estimates of the true activities; all that is known with certainty is that the true jth activity vector lies in a hypersphere with center at a_j and radius whose magnitude is ρ_j.

Now define

$$K_j = \{a \epsilon R^m | \, \|a - a_j\| \leqq \rho_j\},$$

so that the deterministic linear program is replaced by a problem of the form (Ib). Note that ρ_j measures the amount of inexactness associated with the vectors a_j, e.g., if $\rho_j = 0$, then the jth activity vector is known exactly.

According to the previous theorem, problem (Ib) can be solved via the associated linear program LP$(\bar{A})$, where the vectors $\bar{a}_j$ comprising the matrix $\bar{A}$ must be obtained via the support functionals of the sets $\{K_j\}$. But when the sets $\{K_j\}$ are hyperspheres, these vectors $\bar{a}_j$ are easily determined. Since the convex set K_j is a hypersphere, $\sup_{a_j \epsilon K_j} e_i \cdot a_j = a_{ij} + \rho_j \; \forall i$; that is, $\bar{a}_j = a_j + \rho_j \cdot e$, where e is the vector of all ones. Hence, the optimal solution to (Ib) can be determined by solving

$$\max c \cdot x, \quad \text{subject to } x_1(a_1 + \rho_1 \cdot e) + x_2(a_2 + \rho_2 \cdot e) + \cdots + x_n (a_n + \rho_n \cdot e) \leqq b \quad \text{and} \quad x_j \geqq 0.$$

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Decomposition Techniques for the Chebyshev Problem

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This note investigates a decomposition procedure for the problem of maximizing the minimum of a finite number of functions over a common domain. Results, in the form of nonlinear programming subproblems with linear contraints, are obtained for two different classes of functions with domains defined by convex polyhedra. The relation between the subproblems for both classes of functions is shown.