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Scenario relaxation algorithm for finite scenario-based min–max regret and min–max relative regret robust optimization

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Abstract

Most practical decision-making problems are compounded in difficulty by the degree of uncertainty and ambiguity surrounding the key model parameters. Decision makers may be confronted with problems in which no sufficient historical information is available to make estimates of the probability distributions for uncertain parameter values. In these situations, decision makers are not able to search for the long-term decision setting with the best long-run average performance. Instead, decision makers are searching for the robust long-term decision setting that performs relatively well across all possible realizations of uncertainty without attempting to assign an assumed probability distribution to any ambiguous parameter. In this paper, we propose an iterative algorithm for solving min–max regret and min–max relative regret robust optimization problems for two-stage decision-making under uncertainty (ambiguity) where the structure of the first-stage problem is a mixed integer (binary) linear programming model and the structure of the second-stage problem is a linear programming model. The algorithm guarantees termination at an optimal robust solution, if one exists. A number of applications of the proposed algorithm are demonstrated. All results illustrate good performance of the proposed algorithm.
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1. Introduction

In this paper we address the two-stage decision-making problem under uncertainty (ambiguity), where the uncertainty appears in the values of key parameters of a mixed integer linear programming formulation. In the following model, let the vector $\vec{y}$ represent first-stage decisions made before the realization of ambiguity and let the vector $\vec{x}$ represent second-stage decisions made after the realization of ambiguity.

$$
\begin{aligned}
\max_{\vec{x},\vec{y}} \quad & \vec{p}_4^{\,\mathrm{T}} \vec{x} + \vec{p}_1^{\,\mathrm{T}} \vec{y}, \\
\text{s.t.} \quad & P_5 \vec{x} \leqslant \vec{p}_3 + P_2 \vec{y}, \\
& \vec{x} \geqslant \vec{0} \quad \text{and} \quad \vec{y} \in \{0,1\}^{|\vec{y}|}.
\end{aligned}
$$

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Let vector $\mathbf{p} = (\vec{p}_1, P_2, \vec{p}_3, \vec{p}_4, P_5)$ denote the parameters defining the objective function and the constraints of the optimization problem. In the problem considered, all or some elements of the vector $\mathbf{p}$ are uncertain with a finite number of possible realizations or scenarios and unknown joint probability distributions. Let scenario set $\bar{\Omega}$ denote the set of all possible realizations of the parameter vector $\mathbf{p}$.

Because there is a lack of complete knowledge about the probability distribution of uncertain parameters in the considered problem, decision makers are not able to search for the first-stage decision values (long-term decisions) that produce the best long-run average performance. Instead, decision makers are searching for a first-stage decision setting that performs well (provides a reasonable objective function value) across all possible input scenarios without attempting to assign an assumed probability distribution to any ambiguous parameter. This resulting first-stage decision setting is referred to as the robust decision setting. In this paper, we develop an iterative optimization algorithm for assisting decision makers who search for the robust first-stage decision setting under the deviation robustness definition (min–max regret robust solution) and the relative robustness definition (min–max relative regret solution) defined by Kouvelis and Yu [1].

Traditionally, a min–max regret or a min–max relative regret robust solution can be obtained by solving a scenario-based extensive form model of the problem, which is also a mixed integer (binary) linear programming model (explained in detail in Section 3). The size of this extensive form model grows substantially with the number of scenarios used to represent uncertainty. The proposed algorithm can be efficiently used to determine the robust first-stage decision setting when the only information available to decision makers at the time of making the first-stage decisions is a finite set of possible scenarios with unknown probability distribution. The proposed algorithm sequentially solves and updates a relaxation problem until both feasibility and optimality conditions of the overall problem are satisfied. At each iteration, the algorithm uses enumerative search to examine all possible scenarios and adds those scenarios violating optimality and/or feasibility conditions to the relaxation problem. The proposed algorithm is proven to terminate at an optimal min–max regret robust solution (if one exists) in a finite number of iterations.

In the following section, we summarize related literature in robust optimization and supporting topics. In Section 3, we detail the proposed iterative algorithm. In Section 4, we illustrate applications of the proposed algorithm for solving large-scale min–max regret and min–max relative regret robust optimization problems under ambiguity. All examples demonstrate excellent computational performance by the proposed algorithm.

2. Background

Min–max regret and min–max relative regret robust models address optimization problems where some of the model parameters are ambiguous when the first-stage decisions are made. In many strategic decision problems, mixed integer (binary) linear programming models are often applied to find optimal solutions (e.g., network design problems and facility location problems in supply chains). In most cases, decision makers are facing long-term decisions (e.g., capacity decisions and/or location decisions) that have to be made before the realization occurs for uncertain parameter values. Many of these long-term decisions are typically represented by binary variables. After the first-stage decisions are made and the decision maker obtains the realization of model parameters, the second-stage decisions are then made given fixed settings for the first-stage decisions.

In our problem setting, all uncertain parameters in the mixed integer linear programming model can take theirs values from a finite set of possible scenarios with unknown joint probability distribution. Because of the lack of complete knowledge about the joint probability distribution of uncertain parameters in the considered problem, decision makers are not able to search for the first-stage decision setting with the best long-run average performance (stochastic programming approach). In this situation, criteria for the first-stage decisions can be to minimize the maximum regret or the relative regret between the optimal objective function value under perfect information and the resulting objective function value under the robust decisions over all possible realizations of the uncertain parameters (scenarios) in the model.

The book by Kouvelis and Yu [1] summarizes the state-of-the-art in min–max regret and min–max relative regret robust optimization up to 1997 and provides a comprehensive discussion of the motivation for the min–max regret and min–max relative regret approaches and various aspects of applying it in practice. Ben-Tal and Nemirovski [2–5] address robust solutions (min–max/max–min objectives) by allowing the uncertainty sets for the data to be ellipsoids, and propose efficient algorithms to solve convex optimization problems under data uncertainty. However, as the resulting robust formulations involve conic quadratic problems, such methods cannot be directly applied to discrete optimization. Averbakh [6,7] shows that polynomial solvability is preserved for a specific discrete optimization problem (selecting p elements of minimum total weight out of a set of m elements with uncertainty in weights of the elements) when each weight can vary within an interval under the min–max regret robustness definition.

Bertsimas and Sim [8,9] propose an approach to address data uncertainty for discrete optimization and network flow problems that allows the degree of conservatism of the solution (min–max/max–min objective) to be controlled. They show that the robust counterpart of an *NP*-hard α-approximable 0–1 discrete optimization problem remains α-approximable. They also propose an algorithm for robust network flows that solves the robust counterpart by solving a polynomial number of nominal minimum cost flow problems in a modified network.

The procedure of the algorithm presented in this paper has similarities with the solution methodology for the semi-infinite programming problem. The algorithm iteratively solves a series of relaxation problems as the approximation to the full problem formulation until both optimality and feasibility conditions are satisfied. Zakovic and Rustem [10] present the application of semi-infinite programming in solving the minimax optimization problems. Breton and El Hachem [11] develop a bundle methods-based scheme for solving convex multi-stage minimax stochastic programming problems by solving a sequence of linear and quadratic programs. Additional algorithms for worst-case design and some applications to risk management can be found in the work by Rustem and Howe [12].

In the following section, we present and summarize the theory and a new iterative algorithm for min–max regret and min–max relative regret robust optimization problems when the structure of parametric uncertainty allows any finite number of realizations of the parameter vector **p**. Further, in Section 4 we illustrate a number of applications for the new iterative algorithm to solve robust optimization problems with a significant number of scenarios.

3. Related theoretical methodology and the proposed iterative algorithm

In this section we begin by addressing traditional robust approaches for two-stage mixed integer linear programming problems under parametric uncertainty. The key approach here for capturing parametric uncertainty is the definition of a finite set of all possible realizations of parameter vector **p**. Let scenario set $\bar{\Omega}$ denote the set of all possible realizations of the parameter vector **p**. The problem contains two types of decision variables. The first-stage variables model binary choice decisions, which have to be made before the realization of uncertainty. The second-stage decisions are continuous recourse decisions, which can be made after the realization of uncertainty. Let vector $\vec{y}$ denote binary choice decision variables and let vector $\vec{x}_\omega$ denote continuous recourse decision variables for each scenario $\omega \in \bar{\Omega}$. If the realization of model parameters is known to be scenario ω a priori, the optimal choice for the decision variables $(\vec{y}, \vec{x}_\omega)$ can be obtained by solving the following model:

$$O^*_\omega = \left\{ \begin{array}{ll} \max\limits_{\vec{x}_\omega, \vec{y}} & Z(\vec{x}_\omega, \vec{y}) = \vec{p}^{\,\mathrm{T}}_{4\omega} \vec{x}_\omega + \vec{p}^{\,\mathrm{T}}_{1\omega} \vec{y} \\ \text{s.t.} & P_{5\omega} \vec{x}_\omega - P_{2\omega} \vec{y} \leqslant \vec{p}_{3\omega} \\ & \vec{x}_\omega \geqslant \vec{0} \quad \text{and} \quad \vec{y} \in \{0,1\}^{|\vec{y}|} \end{array} \right\}. \tag{1}$$

When parameters' uncertainty (ambiguity) exists, the search for the min–max regret or the min–max relative regret robust solution comprises finding binary choice decisions, $\vec{y}$, such that the function $\max_{\omega \in \bar{\Omega}} (O^*_\omega - Z^*_\omega(\vec{y}))$ or $\max_{\omega \in \Omega} ((O^*_\omega - Z^*_\omega(\vec{y}))/|O^*_\omega|)$ are minimized, respectively, where

$$Z^*_\omega(\vec{y}) = \left\{ \begin{array}{ll} \max\limits_{\vec{x}_\omega} & \vec{p}^{\,\mathrm{T}}_{4\omega} \vec{x}_\omega \\ \text{s.t.} & P_{5\omega} \vec{x}_\omega \leqslant \vec{p}_{3\omega} + P_{2\omega} \vec{y} \\ & \vec{x}_\omega \geqslant \vec{0} \end{array} \right\} + \vec{p}^{\,\mathrm{T}}_{1\omega} \vec{y} \quad \forall \omega \in \bar{\Omega}.$$

In the case when the scenario set $\bar{\Omega}$ is a finite set, the optimal choice of decision variables $\vec{y}$ (the min–max regret and min–max relative regret robust solutions) can be obtained by solving the following models:

$$\begin{aligned} \min_{\delta,\vec{y},\vec{x}_\omega} \quad & \delta \\ \text{s.t.} \quad & \left.\begin{aligned} & \delta \geqslant O^*_\omega - \vec{p}_{4\omega}^{\,\mathrm{T}}\vec{x}_\omega - \vec{p}_{1\omega}^{\,\mathrm{T}}\vec{y} \\ & P_{5\omega}\vec{x}_\omega - P_{2\omega}\vec{y} \leqslant \vec{p}_{3\omega} \\ & \vec{x}_\omega \geqslant \vec{0} \ \text{ and } \ \vec{y} \in \{0,1\}^{|\vec{y}|} \end{aligned}\right\} \ \forall \omega \in \bar{\Omega}, \end{aligned} \tag{2}$$

$$\begin{aligned} \min_{\delta,\vec{y},\vec{x}_\omega} \quad & \delta \\ \text{s.t.} \quad & \left.\begin{aligned} & \delta|O^*_\omega| \geqslant O^*_\omega - \vec{p}_{4\omega}^{\,\mathrm{T}}\vec{x}_\omega - \vec{p}_{1\omega}^{\,\mathrm{T}}\vec{y} \\ & P_{5\omega}\vec{x}_\omega - P_{2\omega}\vec{y} \leqslant \vec{p}_{3\omega} \\ & \vec{x}_\omega \geqslant \vec{0} \ \text{ and } \ \vec{y} \in \{0,1\}^{|\vec{y}|} \end{aligned}\right\} \ \forall \omega \in \bar{\Omega}. \end{aligned} \tag{3}$$

Models (2) and (3) are referred to as the extensive form models of the problem under deviation and relative robust definitions, respectively. If an optimal solution for model (2) (model (3)) exists, the deviation (relative) robust decision setting for the problem is the optimal setting of decision variables $\vec{y}$. Unfortunately, the size of the extensive form model can become unmanageably large as does the required computation time to find the optimal setting of $\vec{y}$.

The implementation of this extensive form model obviously becomes computationally prohibitive for the robust design problem with a large number of scenarios. For this reason, we propose a new iterative optimization algorithm referred to as the scenario relaxation algorithm that can generate optimal robust first-stage decisions for the deviation and relative robust objectives.

The key insight of the scenario relaxation algorithm is that in a problem with a large number of possible scenarios only a small subset of scenarios actually has to be explicitly examined when searching for the optimal deviation (or relative) robust solution. This subset will be comprised of two types of scenarios. The first type consists of scenarios required to ensure that the resulting solution is feasible for all possible scenarios. The second type consists of scenarios required to establish the optimal robust solution.

The scenario relaxation algorithm starts by establishing a subset containing the first type of scenarios, starting with the scenario set $\Omega \subseteq \bar{\Omega}$ (a best guess scenario set informed by knowledge of the problem). A relaxed model (2), model (3), is solved by considering only the scenario set Ω instead of $\bar{\Omega}$. The algorithm continues to enhance the relaxed model by iteratively adding additional infeasible scenarios to the scenario set Ω.

The second subset of explicitly considered scenarios is constructed by a very simple procedure. At each iteration, a candidate robust solution is used to calculate the regrets (or relative regrets) from optimality for all unconsidered scenarios in $\bar{\Omega}\backslash\Omega$. If the optimal objective function value of the relaxed model (2) (model (3)), δ^*, is greater than or equal to all of these regrets (relative regrets), the optimal condition is confirmed and the algorithm terminates at the optimal deviation (relative) robust solution. Otherwise, a subset of these unconsidered scenarios with the regret (relative regret) values greater than δ^* will be explicitly included to the scenario set Ω in the next iteration. As described below, we propose an iterative optimization algorithm for solving the min–max regret robust optimization problem under scenario set $\bar{\Omega}$ that proceeds as follows.

Scenario relaxation algorithm

Step 0: (Initialization) Choose a subset $\Omega \subseteq \bar{\Omega}$ and set UB $=\infty$ and LB $=0$. Determine the value of ε (predetermined small nonnegative real value) and proceed to Step 1.

Step 1: Solve model (1) to obtain $O^*_\omega\ \forall\omega \in \bar{\Omega}$. If model (1) is infeasible for any scenario in the scenario set $\bar{\Omega}$, the algorithm is terminated; the problem is ill posed. Otherwise the optimal objective function value to model (1) for scenario ω is designated as O^*_ω. Proceed to Step 2.

Step 2: (Solving the relaxation problem and optimality check) Solve the relaxation of model (2) by considering only the scenario set Ω instead of $\bar{\Omega}$. If the relaxed model (2) is infeasible, the algorithm is terminated with the confirmation that no robust solution exists for the problem. Otherwise, set $Y_\Omega = \vec{y}^*$ (optimal solution from the relaxed model (2)) and set $\mathrm{LB} = \delta^*$ (optimal objective function value from the relaxed model (2)). If $\{\mathrm{UB} - \mathrm{LB}\} \leqslant \varepsilon$, the robust solution associated with UB is the globally ε-optimal robust solution and the algorithm is terminated. Otherwise the algorithm proceeds to Step 3.

Step 3: Solve the linear programming version of model (1) for all scenarios $\omega \in \bar{\Omega}\backslash\Omega$ by fixing the binary decision vector $\vec{y}$ at Y_Ω. Let $W_1 \subseteq \bar{\Omega}\backslash\Omega$ be the scenario subset that includes all scenarios such that the linear programming version of the model (1) is infeasible and let $W \subseteq \bar{\Omega}\backslash(\Omega \cup W_1)$ be the scenario subset that includes all scenarios $\omega \in \bar{\Omega}\backslash(\Omega \cup W_1)$ such that $O^*_\omega - Z^*_\omega(Y_\Omega) > \delta^*$.

Step 4: If $W_1 \neq \phi$, proceed to Step 5. Otherwise, update $\mathrm{UB} \leftarrow \min(\mathrm{UB}, \max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(Y_\Omega)))$. If $\mathrm{UB} - \mathrm{LB} \leqslant \varepsilon$, the algorithm is terminated and the resulting ε-optimal robust solution is $\vec{y}^*$ such that $\mathrm{UB} = \max_{\omega\in\bar{\Omega}}(O^*_\omega - Z^*_\omega(\vec{y}^*))$. Otherwise, proceed to Step 6.

Step 5: Select a nonempty subset $W'_1 \subseteq W_1$and update subset $\Omega \leftarrow \Omega \cup W'_1$ and proceed to Step 2.

Step 6: Select a nonempty subset $W' \subseteq W$ and update subset $\Omega \leftarrow \Omega \cup W'$ and proceed to Step 2.

In the case of relative robust optimization problems, the proposed algorithm can be utilized in completely the same fashion with four minor modifications. These four modifications include: (a) replacing model (2) with model (3) in the algorithm; (b) replacing the term $O^*_\omega - Z^*_\omega(Y_\Omega) > \delta^*$ in Step 3 with the term $O^*_\omega - Z^*_\omega(Y_\Omega) > \delta^*|O^*_\omega|$; (c) replacing the term $\min(\mathrm{UB}, \max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(Y_\Omega)))$ in Step 4 with the term $\min(\mathrm{UB}, \max_{\omega\in\bar{\Omega}}((O^*_\omega - Z^*_\omega(Y_\Omega))/|O^*_\omega|))$; and finally (d) replacing the term $\max_{\omega\in\Omega}(O^*_\omega - Z^*_\omega(\vec{y}^*))$ in Step 4 with the term $\max_{\omega\in\bar{\Omega}}((O^*_\omega - Z^*_\omega(\vec{y}^*))/|O^*_\omega|)$.

The following Lemma 1 and Corollary 1 provide the important result that the algorithm always terminates at a globally ε-optimal robust solution, if one exists, in finite number of algorithm steps.

Lemma 1. *The scenario relaxation algorithm terminates in finite number of steps. After the algorithm terminates with* $\varepsilon = 0$, *it has either detected infeasibility or has found an optimal robust solution to the original problem.*

Proof. Notice that the relaxed model (2) is a relaxation of the original min–max regret problem and the feasible region of model (1) contains the feasible region of the original problem. This has four important implications: (a) if model (1) is infeasible then the original min–max regret problem is also infeasible; (b) if the relaxed model (2) is infeasible then the original min–max regret problem is also infeasible; (c) $\mathrm{LB} \leqslant \min_{\vec{y}}(\max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(\vec{y})))$ under deviation robust definition for all iterations; and (d) if $\vec{y}^*$ is an optimal solution to the original problem, $\vec{y}^*$ is a feasible solution to the relaxed model (2). From the first and second implications, it is clear that if the algorithm terminates because either model (1) or the relaxed model (2) is infeasible then the original min–max regret problem is infeasible. Now suppose that the algorithm terminates in Step 2 or Step 4 with $\mathrm{UB} = \mathrm{LB}$ and the solution Y_Ω. Notice that the algorithm can terminate in these steps only if Y_Ω is feasible to the overall problem, or $W_1 = \phi$, and $\mathrm{UB} = \max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(Y_\Omega)) \geqslant \min_{\vec{y}}(\max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(\vec{y})))$ under the deviation robust definitions. Therefore if $\mathrm{UB} = \mathrm{LB}$, then $\max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(Y_\Omega)) = \min_{\vec{y}} (\max_{\omega\in\bar{\Omega}} (O^*_\omega - Z^*_\omega(\vec{y})))$ under the deviation robust definitions. Obviously, Y_Ω is an optimal solution to the original min–max regret problem. Because there is a finite number of possible combinations of Y_Ω and the parameters settings, the proposed three-stage algorithm always terminates in finite number of steps. The proof of Lemma 1 under relative robust definition can be obtained in the similar fashion. □

Corollary 1. *The proposed three-stage algorithm terminates in a finite number of steps. After the algorithm terminates with* $\varepsilon > 0$, *it has either detected infeasibility or has found an* ε*-optimal robust solution to the original problem.*

This result follows directly from the result of Lemma 1. On the other hand, there is no known theoretical result that determines the best methodologies for selecting set W'_1 and set W' in Steps 5 and 6, respectively, that will guarantee the best computation time required to solve the overall optimization problem. In the following subsections, we present some rules for determining these sets.

3.1. Selection methodology for set W'

In this subsection, we present the selection rules for set W' in Step 6 of the scenario relaxation algorithm for both the deviation robust and relative robust optimization problems. For the deviation robust optimization problem, we select subset $W' \subseteq W$ in Step 6 of the scenario relaxation algorithm such that W' contains the m scenarios with the highest $O^*_\omega - Z^*_\omega(Y_\Omega)$ values in W where $m = \min\{n_i, |W|\}$ and n_i is a pre-specified constant for iteration i of the algorithm. For the relative robust optimization problem, we select subset $W' \subseteq W$ in Step 6 of the scenario relaxation algorithm such that W' contains the m scenarios with the highest $(O^*_\omega - Z^*_\omega(Y_\Omega))/|O^*_\omega|$ values in W where $m = \min\{n_i, |W|\}$. These selection methods require fewer algorithm iterations when a large n_i value is used with the tradeoff of longer computation time per iteration, and vice versa when a small n_i value is used.

3.2. Selection methodology for set W'_1

In this subsection, we present two alternative selection methods for set W'_1 in Step 5 of the scenario relaxation algorithm for both the deviation robust and relative robust optimization problems. For both robust definitions, we select subset $W'_1 \subseteq W_1$ in Step 5 of the scenario relaxation algorithm such that W'_1 contains the m scenarios in W_1 with the highest objective function values for the phase I problem where $m = \min\{n_i, |W_1|\}$ and n_i is a pre-specified constant for iteration i of the algorithm. An alternative method is to select W'_1 such that W'_1 contains the m scenarios in W_1 with the lowest objective function values from the following linear programming problem:

$$\begin{aligned} \max \quad & \delta \\ \text{s.t.} \quad & \vec{1}\,\delta \leqslant \vec{s}, \\ & P_{5\omega}\vec{x}_\omega + \vec{s} = \vec{p}_{3\omega} + P_{2\omega}Y_\Omega, \\ & \vec{x}_\omega \geqslant \vec{0} \end{aligned}$$

Again these selection methods require fewer algorithm iterations when a large n_i value is used with the tradeoff of longer computation times per iteration, and vice versa when a small n_i value is used.

The following section illustrates computational results for the scenario relaxation algorithm on a number of example problems. These example problems illustrate the application of the scenario relaxation algorithm on facility location problems for a number of test problems under both the deviation and relative robust objectives with variation in the number of possible scenarios. All results demonstrate a significant improvement in the computation time required by the scenario relaxation algorithm over that required by the extensive form model.

4. Computational results

In this section, we apply the proposed algorithm to a hypothetical robust supply chain facility location problem under various numbers of possible scenarios. We consider the supply chain in which suppliers send material to factories that supply warehouses that supply markets as shown in Fig. 1 [13]. Location and capacity allocation decisions have to be made for both factories and warehouses. Multiple warehouses may be used to satisfy demand at a market and multiple factories may be used to replenish warehouses. It is also assumed that units have been appropriately adjusted such that one unit of input from a supply source produces one unit of the finished product. In addition, each factory and each

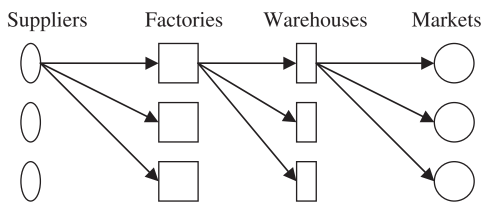

Fig. 1. Stages in the considered supply chain network (Chopra and Meindl [13]).

Table 1
Approximated parameters' information of the example problem

Supplier (h)	S_h	Factory (i)	f_{1i}	K_i	Warehouse (e)	f_{2e}	W_e	Market (j)	D_j	p_j
San Diego	2500	Seattle	75,000	3360	Sacramento	37,500	3000	Portland	800	150
Denver	3000	San Franc	75,000	2880	Oklahoma City	37,500	3120	LA	1050	168
Kansas City	2000	Salt Lake	75,000	3000	Lincoln	37,500	3000	Phoenix	600	144
El Paso	1000	Wilmington	60,000	2580	Nashville	40,500	3600	Houston	1800	180
Cincinnati	800	Dallas	75,000	3240	Cleveland	37,500	3120	Miami	1500	150
Boise	500	Minneapolis	60,000	2520	Fort Worth	36,000	2520	New York	1250	156
Austin	1100	Detroit	81,000	3600	Eugene	37,500	2760	St. Louis	1050	144
Charlotte	2000	Tampa	39,000	1680	Santa Fe	34,500	2040	Chicago	1750	168

warehouse cannot operate at more than its capacity and a linear penalty cost is incurred for each unit of unsatisfied demands. The model requires the following inputs:

m	number of markets
n	number of potential factory locations
l	number of suppliers
t	number of potential warehouse locations
D_j	annual demand from customer j
K_i	potential capacity of factory site i
S_h	supply capacity at supplier h
W_e	potential warehouse capacity at site e
f_{1i}	fixed cost of locating a plant at site i
f_{2e}	fixed cost of locating a warehouse at site e
c_{1hi}	cost of shipping one unit from supplier h to factory i
c_{2ie}	cost of shipping one unit from factory i to warehouse e
c_{3ej}	cost of shipping one unit from warehouse e to market j
p_j	penalty cost per unit of unsatisfied demand at market j

In the deterministic case, the goal is to identify factory and warehouse locations as well as quantities shipped between various points in the supply chain that minimize the total fixed and variable costs. In this case, the overall problem can be modeled as the mixed integer linear programming problem presented in the following model:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} f_{1i} y_i + \sum_{e=1}^{t} f_{2e} z_e + \sum_{h=1}^{l}\sum_{i=1}^{n} c_{1hi} x_{1hi} + \sum_{i=1}^{n}\sum_{e=1}^{t} c_{2ie} x_{2ie} + \sum_{e=1}^{t}\sum_{j=1}^{m} c_{3ej} x_{3ej} + \sum_{j=1}^{m} p_j s_j, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{1hi} \leqslant S_h \ \ \forall h \in \{1,\ldots,l\}, \quad \sum_{h=1}^{l} x_{1hi} - \sum_{e=1}^{t} x_{2ie} = 0 \ \ \forall i \in \{1,\ldots,n\}, \\
& \sum_{e=1}^{t} x_{2ie} \leqslant K_i y_i \ \ \forall i \in \{1,\ldots,n\}, \quad \sum_{i=1}^{n} x_{2ie} - \sum_{j=1}^{m} x_{3ej} = 0 \ \ \forall e \in \{1,\ldots,t\}, \\
& \sum_{j=1}^{m} x_{3ej} \leqslant W_e z_e \ \ \forall e \in \{1,\ldots,t\}, \quad \sum_{e=1}^{t} x_{3ej} + s_j = D_j \ \ \forall j \in \{1,\ldots,m\}, \\
& x_{1hi} \geqslant 0, \quad x_{2ie} \geqslant 0, \quad x_{3ej} \geqslant 0, \quad s_j \geqslant 0, \quad y_i \in \{0,1\}, \quad \text{and} \quad z_e \in \{0,1\}.
\end{aligned}$$

When some parameters in the model are ambiguous, the goal becomes to identify robust factory and warehouse locations (long-term decisions) with an objective that uses the deviation or relative robust formulation. Transportation decisions (short-term decisions) are now recourse decisions which can be made after all model parameters' values are realized. Table 1 summarizes the information on approximated parameters' values associated with suppliers, factories, warehouses, and markets. A variable transportation cost of $0.01 per unit per mile is assumed in this example.

Table 2
Alternative 23 settings in the case study

Setting number	Number of factories with ambiguous potential capacity	Number of warehouses with ambiguous potential capacity	Number of markets with ambiguous unit penalty cost	Number of possible scenarios
1	2	2	2	64
2	2	2	4	256
3	2	2	6	1024
4	2	4	2	256
5	2	4	4	1024
6	2	4	6	4096
7	2	6	2	1024
8	2	6	4	4096
9	2	6	6	16,384
10	4	2	2	256
11	4	2	4	1024
12	4	2	6	4096
13	4	4	2	1024
14	4	4	4	4096
15	4	4	6	16,384
16	4	6	2	4096
17	4	6	4	16,384
18	6	2	2	1024
19	6	2	4	4096
20	6	2	6	16,384
21	6	4	2	4096
22	6	4	4	16,384
23	6	6	2	16,384

The key uncertain (ambiguous) parameters that we consider in this example are the potential capacity at the factory, potential capacity at the warehouse, and the unit penalty cost for not meeting demand at the market. We assume that each uncertain (ambiguous) parameter can take its values from 67% to 100% of the approximated values reported in Table 1.

We apply the scenario relaxation algorithm and extensive form model to 23 different settings of the robust facility and warehouse location problem for each robustness objective. Each setting in this case study contains different sets of ambiguous parameters which results in different numbers of possible scenarios. The number of possible scenarios varies from 64 to 16,384 scenarios. Table 2 describes 23 settings of the case study.

All case study settings are solved by the scenario relaxation algorithm and extensive form model with $\varepsilon = 0\%$ on a Windows XP-based Pentium(R) 4 CPU 3.60 GHz personal computer with 2.00 GB RAM using a C + + program and CPLEX 9 for the optimization process. MS-Access is used for the case study input and output database. Tables 3 and 4 illustrate the performance comparison between the scenario relaxation algorithm and the extensive form model over all 23 settings under deviation and relative robust objective, respectively. If the algorithm fails to obtain an optimal robust solution within 12 h, the computation time of "–" is reported in these tables.

All results from our example problems illustrate a significant improvement in computation time of the scenario relaxation algorithm over the extensive form model. These results also illustrate the applicability and effectiveness of the scenario relaxation algorithm for solving min–max regret and min–max relative regret robust optimization problems which have a mixed integer (binary) linear programming base model under parametric uncertainty and ambiguity with a finite number of possible scenarios.

It is also worth noting that, in principle the proposed algorithm can terminate with $|\Omega|$ equal to $|\bar{\Omega}|$, which means that the proposed algorithm can take more computation time than solving the problem directly. However, based on our illustrated examples, there is evidence that the proposed algorithm will terminate with $|\Omega|$ much smaller than $|\bar{\Omega}|$ and with computation time much shorter than the one required by solving the problem directly.

Table 3
Performance comparison under deviation robust objective

Setting number	Total number of scenarios	Number of scenarios generated by the algorithm	Computation time of the algorithm (s)	Computation time of the extensive form model (s)
1	64	3	32	80
2	256	3	78	1495
3	1024	3	357	27,540
4	256	7	103	1923
5	1024	6	444	38,630
6	4096	6	1822	–
7	1024	7	441	39,757
8	4096	7	1936	–
9	16,384	7	7705	–
10	256	9	109	3043
11	1024	9	526	39,089
12	4096	9	1959	–
13	1024	11	611	40,801
14	4096	10	2053	–
15	16,384	11	9443	–
16	4096	15	2515	–
17	16,384	15	11,181	–
18	1024	10	581	41,539
19	4096	10	2060	–
20	16,384	10	9009	–
21	4096	16	2727	–
22	16,384	15	11,804	–
23	16,384	21	16,088	–

Table 4
Performance comparison under relative robust objective

Setting number	Total number of scenarios	Number of scenarios generated by the algorithm	Computation time of the algorithm (s)	Computation time of the extensive form model (s)
1	64	4	35	460
2	256	4	83	6378
3	1024	4	379	–
4	256	6	104	10,790
5	1024	6	450	–
6	4096	6	1827	–
7	1024	9	511	–
8	4096	9	2123	–
9	16,384	9	8895	–
10	256	8	107	15,154
11	1024	8	523	–
12	4096	8	1894	–
13	1024	11	634	–
14	4096	11	2146	–
15	16,384	11	9503	–
16	4096	13	2311	–
17	16,384	13	10,237	–
18	1024	12	666	–
19	4096	12	2271	–
20	16,384	12	9892	–
21	4096	14	2386	–
22	16,384	14	11,016	–
23	16,384	19	13,956	–

5. Summary and conclusion

This paper presents a new solution methodology for solving large scale min–max regret and min–max relative regret robust optimization problems with a finitely large number of possible scenarios when the direct solution methodology (extensive form model) fails to solve the problem in reasonable amount of time. Our new algorithm is referred to as the scenario relaxation algorithm, which capitalizes on the observation that a robust solution can be achieved by solving the problem considering only a small subset of all possible scenarios. This subset consists of two types of scenarios. The first type of scenarios prescribes the feasibility of the solution over all possible scenarios. The second type of scenarios controls the minimum maximum regret (or minimum maximum relative regret) of the problem. This paper also provides proof that the scenario relaxation algorithm converges to the optimal robust solution if one exists in finite number of iterations. Set selection rules in the scenario relaxation algorithm are presented as alternative procedures for decision makers. Even though there is no theoretical guarantee that the scenario relaxation algorithm always reduces computational effort, all results from our example problems illustrate the significant improvement in computation time required by the scenario relaxation algorithm over the extensive form model.

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