



# 1 abs\_integrate

## 1.1 Introduction to abs\_integrate

The package `abs_integrate` extends Maxima's integration code to some integrands that involve the absolute value, max, min, signum, or unit step functions. For integrands of the form  $p(x)|q(x)|$ , where  $p$  is a polynomial and  $q$  is a polynomial that `factor` is able to factor into a product of linear or constant terms, the `abs_integrate` package determines an antiderivative that is continuous on the entire real line. Additionally, for an integrand that involves one or more parameters, the function `conditional_integrate` tries to determine an antiderivative that is valid for all parameter values.

**Examples** To use the `abs_integrate` package, you'll first need to load it:

```
(%i1) load("abs_integrate.mac")$
```

```
(%i2) integrate(abs(x),x);
```

```
(%o2) (x*abs(x))/2
```

To convert (%o2) into an expression involving the signum function, apply `convert_to_signum`; thus

```
(%i3) convert_to_signum(%);
```

```
(%o3) (x^2*signum(x))/2
```

When the integrand has the form  $p(x)|x-c1||x-c2|\dots|x-cn|$ , where  $p(x)$  is a polynomial and  $c1, c2, \dots, cn$  are constants, the `abs_integrate` package returns an antiderivative that is valid on the entire real line; thus *without* making assumptions on  $a$  and  $b$ ; for example

```
(%i4) factor(convert_to_signum(integrate(abs((x-a)*(x-b)),x,a,b)));
```

```
(%o4) ((b-a)^3*signum(b-a)^2)/6
```

Additionally, `abs_integrate` is able to find antiderivatives of some integrands involving max, min, signum, and unit\_step; examples:

```
(%i5) integrate(max(x,x^2),x);
```

```
(%o5) (signum(x-1)*((2*x^3-3*x^2)/12+1/12)+1/12)*signum(x)+  
(x^3/3+x^2/2)/2
```

```
(%i6) integrate(signum(x) - signum(1-x),x);
```

```
(%o6) abs(x)+abs(x-1)
```

A plot indicates that indeed (%o5) and (%o6) are continuous at zero and at one.

For definite integrals with numerical integration limits (including both minus and plus infinity), the `abs_integrate` package converts the integrand to signum form and then it tries to subdivide the integration region so that the integrand simplifies to a non-signum expression on each subinterval; for example

```
(%i1) integrate(1 / (1 + abs(x-5)),x,-5,6);
```

```
(%o1) log(11)+log(2)
```

Finally, `abs_integrate` is able to determine antiderivatives of *some* functions of the form  $F(x, |x-a|)$ ; examples

```
(%i2) integrate(1/(1 + abs(x)),x);
(%o2) ((signum(x)+1)*log(x+1))/2-(log(1-x)*(1-signum(x)))/2

(%i3) integrate(cos(x + abs(x)),x);
(%o3) ((signum(x)+1)*sin(2*x)-2*x*signum(x)+2*x)/4
```

Barton Willis (Professor of Mathematics, University of Nebraska at Kearney) wrote the `abs_integrate` package and its English language user documentation. This documentation also describes the `partition` package for integration. Richard Fateman wrote `partition`. Additional documentation for `partition` is located at <https://people.eecs.berkeley.edu/~fateman/papers/partition.pdf>.

To use `load(abs_integrate)`

## 1.2 Definitions for `abs_integrate`

`extra_integration_methods` [Option]

Default value: `['signum_int', 'abs_integrate_use_if']`

The list `extra_integration_methods` is a list of functions for integration. When `integrate` is unable to find an antiderivative, Maxima uses the methods in `extra_integration_methods` to attempt to determine an antiderivative.

Each function `f` in `extra_integration_methods` should have the form `f(integrand, variable)`. The function `f` may either return `false` to indicate failure, or it may return an expression involving an integration noun form. The integration methods are tried from the first to the last member of `extra_integration_methods`; when no method returns an expression that does not involve an integration noun form, the value of the integral is the last value that does not fail (or a pure noun form if all methods fail).

When the function `abs_integrate_use_if` is successful, it returns a conditional expression; for example

```
(%i2) integrate(1/(1 + abs(x+1) + abs(x-1)),x);
(%o2) %if(-(x+1)>0,-log(1-2*x)/2+log(3)-2/3,
      %if(-(x-1)>0,x/3+log(3)/2-1/3,log(2*x+1)/2))

(%i3) integrate(exp(-abs(x-1) - abs(x)),x);
(%o3) %if(-x>0,%e^(2*x-1)/2-2*e^(-1),
      %if(-(x-1)>0,%e^(-1)*x-(3*e^(-1))/2,-%e^(1-2*x)/2))
```

For definite integration, these conditional expressions can cause trouble:

```
(%i4) integrate(exp(-abs(x-1) - abs(x)),x, minf,inf);
(%o4) 'limit(%if(-x > 0, (%e^-1*(%e^(2*x)-4))/2,
      %if(-(x-1) > 0, (%e^-1*(2*x-3))/2, -%e^(1-2*x)/2)),
      x,inf,minf)
- 'limit(%if(-x > 0, (%e^-1*(%e^(2*x)-4))/2,
      %if(-(x-1) > 0, (%e^-1*(2*x-3))/2, -%e^(1-2*x)/2)),x,
      minf,plus)
```

For such definite integrals, try disallowing the method `abs_integrate_use_if`:

```
(%i9) integrate(exp(-abs(x-1) - abs(x)),x, minf,inf),
      extra_integration_methods : ['signum_int'];
(%o9) 2*%e^(-1)
```

**Related options** *extra\_definite\_integration\_methods*

**To use** load(abs\_integrate)

**extra\_definite\_integration\_methods** [Option]

Default value: ['abs\_defint']

The list `extra_definite_integration_methods` is a list of extra functions for *definite* integration. When `integrate` is unable to find a definite integral, Maxima uses the methods in `extra_definite_integration_methods` to attempt to determine an antiderivative.

Each function `f` in `extra_definite_integration_methods` should have the form `f(integrand, variable, lo, hi)`, where `lo` and `hi` are the lower and upper limits of integration, respectively. The function `f` may either return `false` to indicate failure, or it may return an expression involving an integration noun form. The integration methods are tried from the first to the last member of `extra_definite_integration_methods`; when no method returns an expression that does not involve an integration noun form, the value of the integral is the last value that does not fail (or a pure noun form if all methods fail).

**Related options** *extra\_integration\_methods*

**To use** load(abs\_integrate)

**intfudu (e, x)** [Function]

This function uses the derivative divides rule for integrands of the form  $f(w(x)) * diff(w(x), x)$ . When `intfudu` is unable to find an antiderivative, it returns `false`.

```
(%i1) intfudu(cos(x^2) * x,x);
(%o1) sin(x^2)/2

(%i3) intfudu(x * sqrt(1+x^2),x);
(%o3) (x^2+1)^(3/2)/3

(%i4) intfudu(x * sqrt(1 + x^4),x);
(%o4) false
```

For the last example, the derivative divides rule fails, so `intfudu` returns `false`.

A hashed array `intable` contains the antiderivative data. To append a fact to the hash table, say  $integrate(f) = g$ , do this:

```
(%i1) intable[f] : lambda([u], [g(u),diff(u,%voi)]);
(%o1) lambda([u], [g(u),diff(u,%voi)])

(%i2) intfudu(f(z),z);
(%o2) g(z)

(%i3) intfudu(f(w(x)) * diff(w(x),x),x);
(%o3) g(w(x))
```

An alternative to calling `intfudu` directly is to use the `extra_integration_methods` mechanism; an example:

```
(%i1) load("abs_integrate.mac")$
(%i2) load(basic)$
(%i3) load("partition.mac")$

(%i4) integrate(bessel_j(1,x^2) * x,x);
(%o4) integrate(bessel_j(1,x^2)*x,x)

(%i5) push('intfudu, extra_integration_methods)$

(%i6) integrate(bessel_j(1,x^2) * x,x);
(%o6) -bessel_j(0,x^2)/2
```

To use `load(partition)`

#### Additional documentation

<https://people.eecs.berkeley.edu/~fateman/papers/partition.pdf>

**Related functions** *intfugudu*

`intfugudu (e, x)` [Function]

This function uses the derivative divides rule for integrands of the form  $f(w(x)) * g(w(x)) * diff(w(x), x)$ . When `infudu` is unable to find an antiderivative, it returns false.

```
(%i1) diff(jacobi_sn(x,2/3),x);
(%o1) jacobi_cn(x,2/3)*jacobi_dn(x,2/3)

(%i2) intfugudu(%,x);
(%o2) jacobi_sn(x,2/3)

(%i3) diff(jacobi_dn(x^2,a),x);
(%o3) -2*a*x*jacobi_cn(x^2,a)*jacobi_sn(x^2,a)

(%i4) intfugudu(%,x);
(%o4) jacobi_dn(x^2,a)
```

For a method for automatically calling `infugudu` from `integrate`, see the documentation for `intfudu`.

To use `load(partition)`

#### Additional documentation

<https://people.eecs.berkeley.edu/~fateman/papers/partition.pdf>

**Related functions** *intfudu*

`signum_to_abs (e)` [Function]

This function replaces subexpressions of the form  $qsignum(q)$  by  $abs(q)$ . Before it does these substitutions, it replaces subexpressions of the form  $signum(p)*signum(q)$  by  $signum(p*q)$ ; examples:

```
(%i1) map('signum_to_abs, [x * signum(x),
```

```

                                x * y * signum(x) * signum(y)/2]);
(%o1) [abs(x),(abs(x)*abs(y))/2]

```

To use load(abs\_integrate)

**conditional\_integrate (e, x)** [Function]

For an integrand with one or more parameters, this function tries to determine an antiderivative that is valid for all parameter values. When successful, this function returns a conditional expression for the antiderivative.

```

(%i1) conditional_integrate(cos(m*x),x);
(%o1) %if(m#0,sin(m*x)/m,x)

(%i2) conditional_integrate(cos(m*x)*cos(x),x);
(%o2) %if((m-1 # 0) %and (m+1 # 0),
        ((m-1)*sin((m+1)*x)+((-m)-1)*sin((1-m)*x))/(2*m^2-2),
        (sin(2*x)+2*x)/4)

(%i3) sublis([m=6],%);
(%o3) -(5*cos(7*x)+7*cos(5*x))/70

(%i4) conditional_integrate(exp(a*x^2+b*x),x);
(%o4) %if(a # 0,
        -(sqrt(%pi)*%e^(-(b^2/(4*a)))*erf((2*a*x+b)/(2*sqrt(-a))))
        /(2*sqrt(-a)),%if(b # 0,%e^(b*x)/b,x))

```

**convert\_to\_signum (e)** [Function]

This function replaces subexpressions of the form *abs(q)*, *unit\_step(q)*, *min(q1,q2,...,qn)* and *max(q1,q2,...,qn)* by equivalent *signum* terms.

```

(%i1) map('convert_to_signum, [abs(x), unit_step(x),
                                max(a,2), min(a,2)]);
(%o1) [x*signum(x),(signum(x)*(signum(x)+1))/2,
        (a+signum(a-2)*(a-2)+2)/2,
        (a-signum(a-2)*(a-2)+2)/2]

```

To convert *unit\_step* to *signum* form, the function *convert\_to\_signum* uses  $unit\_step(x) = (1 + signum(x))/2$ .

To use load(abs\_integrate)

**Related functions** *signum\_to\_abs*

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