0.1 Previous lecture summary

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We arrived at an equation for the motion of light:

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2}; \quad W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right); \quad b^2 = \frac{l^2}{e^2}$$

If:

$$\frac{e^2}{l^2} = \frac{1}{b^2} < \frac{1}{27G^2M^2}$$

the photon has enough angular momentum ($\sqrt{27}eGM$) to "bounce back": it reaches a minimum distance from the black hole, and then escapes to infinity.

Note: the geodesic followed by a photon *does not depend* on its energy (nor on its wavelength).

0.2 Scattering

Consider a $\hat{x}\hat{y}$ plane, with a photon travelling from $x = +\infty$ and $y = d$ (**impact parameter**) directed as $-\hat{x}$. At the origin there is a point mass M. We study the photon's motion in polar coordinates (r, φ).
When it's very far away, the metric is that of Minkowski, and so:

$$d \approx r\varphi$$

Then:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -1 \cdot \frac{-d}{r^2} = \frac{d}{r^2}$$

because the photon is far away, $r \approx x$, and so the photon moves *radially*. Then we note that:

$$b = \frac{l}{e} =$$

And so b is the impact parameter d. After approaching M, the photon will be scattered at an angle $\Delta\varphi$, maintaining the same impact parameter b relative to a *tilted* axis. We define the angle of deflection as $\delta\varphi_{\text{defl}} = \Delta\varphi - \pi$, so that if the motion is *perfectly straight* we have $\varphi_{\text{in}} = 0$, $\varphi_{\text{out}} = \pi$ and $\delta\varphi = 0$. We want to know the trajectory $r(\varphi)$, and so we change variables:

$$l = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}$$

We assume that M is small, and search an explicit expression for $\delta\varphi_{\text{defl}}$.

Substituting in the geodesics equation for light:

$$\frac{1}{l^2}\frac{l^2}{r^5}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2+\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)=\frac{1}{b^2}$$

To simplify the problem we introduce $u \equiv 1/r$, so that:

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi}=-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}$$

Substituting back:

$$\cancel{u^4}\frac{1}{\cancel{u^4}}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2+u^2(1-2GMu)=\frac{1}{b^2}$$

As we did before, we differentiate wrt u, leading to:

$$2\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}+2u\frac{\mathrm{d}u}{\mathrm{d}\varphi}-6GMu^2\frac{\mathrm{d}u}{\mathrm{d}\varphi}=0$$

and then divide by $2\,\mathrm{d}u\,/\,\mathrm{d}\varphi$, arriving at:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}+u=3GMu^2$$

Note that $\mathrm{d}u\,/\,\mathrm{d}\varphi$ will be $=0$ at only *one point*, so we can work around it by solving the equation in the two sides and matching the two solutions.
We will solve this in a *perturbative* way, assuming that GM is small. Here, the solution will be almost a straight line, parametrized by $b = r\sin\varphi$ (ignoring the u^2 term leads to the *harmonic oscillator* differential equation, and the only solution that satisfies the boundary conditions is the one with the sin), so that:

$$b=\frac{\sin\varphi}{u}\Rightarrow u=\frac{1}{b}\sin\varphi$$

We add a small perturbation to it:

$$u(\varphi)=\frac{1}{b}\left[\sin\varphi+W(\varphi)\right]\qquad w\ll 1$$

Substituting in the equation:

$$-\cancel{\frac{1}{b}\sin}\varphi+\frac{1}{b}\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2}+\cancel{\frac{1}{b}\sin}\varphi+\frac{1}{b}w\approx 3GM\frac{\sin^2\varphi}{b^2}$$

As the u^2 term is already small, we can substitute in it only the *unperturbed* solution (straight line), as adding the perturbation will only lead to higher order terms. So, we arrive at:

$$\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2}+w\approx\frac{3GM}{b}\sin^2\varphi$$

To solve this, we make an *ansatz*:

$$w = A + B\sin^2\varphi$$

with A and B constants. This is because w'' will produce something of the form of what appears in the equation.
So, computing the two derivatives:

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}\varphi} &= 2B\sin\varphi\cos\varphi \\
\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2} &= 2B[\cos^2\varphi - \sin^2\varphi] \underset{(a)}{=} 2B - 4B\sin^2\varphi
\end{aligned}$$

where in (a) we used $\cos^2\varphi = 1 - \sin^2\varphi$. Substituting in the equation:

$$(2B - 4B\sin^2\varphi) + (A + B\sin^2\varphi) = \frac{3GM}{b}\sin^2\varphi$$

Equating the left and right sides leads to the following conditions:

$$\begin{cases} 2B + A = 0 \\ -3B = \frac{3GM}{b} \end{cases} \Rightarrow \begin{cases} A = \frac{2GM}{b} \\ B = -\frac{GM}{b} \end{cases}$$

and so the final solution is:

$$w = \frac{2GM}{b}\left[1 - \frac{\sin^2\varphi}{2}\right] \qquad \left(\frac{GM}{b}\right) \ll 1$$

What is the physical meaning of this solution? First we substitute back to compute $u(\varphi)$:

$$u(\varphi) = \frac{1}{b}\left[\sin\varphi + \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right)\right]$$

We are interested in the behaviour in the infinite past/future, i.e. its asymptotic behaviour, where $r = \infty \Rightarrow u = 0$. We already now that $\varphi = 0$ will be approximately a solution (as we are working perturbatively). So we know that:

$$\varphi_{\text{in}} = \epsilon_{\text{in}} \qquad \varphi_{\text{out}} = \pi + \epsilon_{\text{out}}$$

with $\epsilon_{\text{in}}, \epsilon_{\text{out}} \ll 1$. Ignoring higher order terms:

$$0 = \left[\sin\epsilon_{\text{in}} + \frac{2GM}{b}\right] \approx \epsilon_{\text{in}} + \frac{2GM}{b} \Rightarrow \epsilon_{\text{in}} \approx -\frac{2GM}{b}$$

where we used $\sin(x) \approx x$ for $x \approx 0$. The other solution will be:

$$0 = \sin\left(\pi + \epsilon_{\text{out}} + \frac{2GM}{b}\right) \approx -\epsilon_{\text{out}} + \frac{2GM}{b} \Rightarrow \epsilon_{\text{out}} \approx \frac{2GM}{b}$$

and so:

$$\varphi_{\text{in}} \approx -\frac{2GM}{b}; \qquad \varphi_{\text{out}} \approx \pi + \frac{2GM}{b}$$

So the path of light is slightly *bent* by the presence of the central mass M, with a deflection:

$$\delta\varphi_{\text{defl}} \approx \frac{4GM}{b}$$

This result was used in the first proof of GR. In 1919, sir Arthur Eddington observed a deviation in the position of a star when the Sun passed close to its line of sight (the observation was made during a total solar eclipse, otherwise it would've been impossible to see). However, this is a really tiny effect:

$$\delta\varphi_{\text{defl}} \approx 1.7''$$

0.3 Schwarzschild Horizon

Recall the Schwarzschild line element:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2$$

To study the *structure* of a geometry is very useful to plot *light cones*. Let's ignore angular motion and consider only the radial one. So $\mathrm{d}s^2 = 0$ when:

$$0 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

When $r = \infty$ (huge distance from the central mass) $\mathrm{d}t = \mathrm{d}r$, and so the light cones are the same as the ones from Minkowski's spacetime. Approaching $r = 2GM$, the light cones (plotted on a t over r plane) become "thinner", meaning that photons appear to cover less and less $\mathrm{d}s$ for a given $\mathrm{d}t$ as their source approaches the horizon (and massimve particles move even less). Explicitly, we can integrate the $\mathrm{d}t$ and $\mathrm{d}r$ relation:

$$\int_0^t \mathrm{d}t = \int_{r_*}^{r(t)} \frac{\mathrm{d}r}{1 - 2GM/r} \qquad r_* > 2GM$$

This integral evaluates to:

$$t = \int_{r(t)}^{r_*} \mathrm{d}r\,\frac{r}{r - 2GM} = \int_{r(t)}^{r_*} \mathrm{d}r + 2GM\int_{r(t)}^{r_*} \mathrm{d}r\,\frac{1}{r - 2GM}$$

and so:

$$t = r_* - r(t) + 2GM\ln(r_* - 2GM)\ -2GM\ln(r(t) - 2GM)$$

the yellow term is always finite, but the blue ones diverges as $r(t) \to 2GM$, meaning that an object can never reach the horizon.

However, t for now is just a coordinate: what does it physically mean?
Consider a far away observer ($g_{\mu\nu} = \eta_{\mu\nu}$) that it's looking towards a particle falling toward M, and which is at rest wrt M (so that $\tau = t$). So t is the *proper time* of such an observer, meaning that from the point of view of this person no object can reach the horizon in a finite time. However, as we computed during last lecture, the *falling observer* will reach and traverse the horizon in a finite time.
We also note that the *far away* observer will see the falling object as *increasingly red*. In fact, if we computed the gravitational redshift effect (neglecting the one due to motion) we find that:

$$f_{\text{obs}} = f_{\text{emit}} \sqrt{\frac{-g_{00}(\text{emit})}{-g_{00}(\text{obs})}}$$

as the observer is at rest in a Minkowski's space, we have $g_{00}(\text{obs}) = -1$ and so:

$$f_{\text{obs}} = f_{\text{emit}} \sqrt{1 - \frac{2GM}{r}}$$

and as $r \to 2GM$ we have $f_{\text{obs}} \to 0$, meaning that, at some points, the received information *stops*. This is to be expected: the falling observer can send only a finite amount of information, as he reaches the horizon in a finite time, while the observer sees this phenomenon as *stretched* to an infinite time - so he cannot receive an *infinite* information.
This phenomenon suggests that the Schwarzschild horizon is just a by-product of coordinates.

0.4 Rindler Spacetime & Rindler Horizon

What is a Minkowski Spacetime seen by an accelerated observer?
Recall the homework from week 2, where we considered a trajectory:

$$x(t) = \frac{c}{k}\left[\sqrt{1 + k^2 t^2} - 1\right]$$

which is the trajectory of an observer experiencing constant acceleration $a = ck$.
Now let's consider $c = 1$, and ignore the constant -1 in the square parentheses, leading to:

$$x(t) = \frac{\sqrt{1 + k^2 t^2}}{k}$$

For $t = -\infty$, $x = +\infty$; $t = 0 \Rightarrow x = 1/k$ and $t = +\infty \Rightarrow x = +\infty$. The proper time of such observer is given by:

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 \Big|_{\text{trajectory}} = -\,\mathrm{d}t^2 \left[1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right]$$

Rearranging:

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2} = \mathrm{d}t\left[1 - \frac{k^2t^2}{1+k^2t^2}\right]^{1/2} = \frac{\mathrm{d}t}{\sqrt{1+k^2t^2}}$$

Integrating:

$$\int_0^\tau \mathrm{d}\tau = \int_0^t \frac{\mathrm{d}t}{\sqrt{1+k^2t^2}} \Rightarrow \tau = \frac{1}{k}\operatorname{arcsinh}(kt)$$

So we can parametrize the trajectory using the proper time:

$$t = \frac{1}{k}\sinh(k\tau)$$
$$x = \frac{1}{k}\cosh(k\tau)$$

We want now to construct a coordinate system where the observer is at a fixed spatial position and where time is equal to the proper time measured by the observer (up to a constant). We choose:

$$\begin{cases} t = \rho\sinh\eta \\ x = \rho\cosh\eta \end{cases}$$

Now the observer is at a fixed space coordinate $\rho_* = 1/k$ and measures a proper time $\tau = \eta/k = \eta\rho_*$.
We now consider a *family of observers* at different spatial locations ρ, each with his own constant accelerations. The lines of *constant* ρ are just their trajectories on the xt plane, while the lines of constant η are the ones that satisfy:

$$\frac{t}{x} = \tanh\eta \Rightarrow \eta = \tanh^{-1}\frac{t}{x}$$

So constant η means constant t/x, thus a line of *constant slope*. So the x axis corresponds to $\eta = 0$, and the two 45° degree boundaries of the light correspond to $\eta = \pm\infty$. Plotting these lines is useful to *see* the action of the change of variables. The coordinates so defined, named **Rindler Coordinates**, cover *one quadrant* of the Minkowski spacetime.
We can now compute the *line element* on this coordinate set:

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 = -(\mathrm{d}\rho\sinh\eta + \rho\cosh\eta\,\mathrm{d}\eta)^2 + (\mathrm{d}\rho\cosh\eta + \rho\sinh\eta\,\mathrm{d}\eta)^2 = \\ &= -\rho^2\,\mathrm{d}\eta^2 + \mathrm{d}\rho^2 \end{aligned}$$

This is the **Rindler Spacetime**, describing one quarter of Minkowski spacetime and adapted to an *accelerating* observer.
Consider now the worldline of an observer at rest (a vertical line in the xt plane) lying at $x = x_0$, and two events A and B with $\Delta t = t_B - t_A = x_0$. For a Rindler observer, however, the first event A is seen at $\eta_A = 0$ (more precisely, it can be *reconstructed* from a light signal arriving some time later), and the second at $\eta_B = \infty$. Note that, due to acceleration, at some point the emitted light from the resting observer cannot be seen by the accelerated one. So, an *accelerating observer* generates an horizon - similar to the one of a blackhole.