0.1 Frame Dragging

(Lesson ? of 06/12/19) Compiled: December 6, 2019

Recall the line element of a slowly rotating geometry:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \frac{4GJ}{r}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi$$

We consider a gyroscope falling in the direction of the axis of rotation. We then use a cartesian coordinate system (as $\hat{z}$ is singular in spherical coordinates). So, we need to change coordinates in the line element. Starting from the polar coordinates definition:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{cases} \Rightarrow \varphi = \arctan\frac{y}{x}$$

Differentiating:

$$\mathrm{d}\varphi = \frac{1}{1+y^2/x^2}\,\mathrm{d}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\frac{\mathrm{d}y\,x - y\,\mathrm{d}x}{x^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{r^2\sin^2\theta} = \mathrm{d}\varphi$$

Substituting in $\mathrm{d}s^2$ leads to:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \frac{4GJ}{r^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{r}\,\mathrm{d}t$$

We then expand in first order of J:

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{O(J^0)} + \underbrace{\delta g_{\mu\nu}}_{O(J^1)}$$

with:

$$\delta g_{01} = \delta g_{10} = \frac{2GJy}{(x^2+y^2+z^2)^{3/2}}$$
$$\delta g_{02}\delta g_{20} = -\frac{2GJx}{(x^2+y^2+z^2)^{3/2}}$$

We can then compute the Christoffel's symbols in a perturbative manner:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left[\delta g_{\lambda\beta,\alpha} + \delta g_{\alpha\lambda,\beta} - \delta g_{\alpha\beta,\lambda}\right] + O(J^2)$$

The 4-velocity for the falling gyroscope will be only on $\hat{z}$:

$$u^\alpha = (u^t, 0, 0, u^z)$$

And we choose the initial spin aligned with $\hat{x}$:

$$S^\alpha_{\text{in}} = (0, S^x_{\text{in}}, 0, 0)$$

We now show that $S^z \equiv 0$ at all times. We start with the spin-equation:

$$\frac{\mathrm{d}S^z}{\mathrm{d}\tau} + \Gamma^3_{\alpha\beta} u^\alpha S^\beta = 0$$

Looking at u^α, we will have non-zero results only for $\alpha = 0, 3$:

$$\frac{\mathrm{d}S^z}{\mathrm{d}\tau} + \Gamma^3_{0\beta} u^t S^\beta + \Gamma^3_{3\beta} u^z S^\beta = 0$$

Then we compute the required Christoffel symbols:

$$\Gamma^3_{0\beta} = \frac{1}{2}\eta^{33}\left[\cancel{\delta g_{3\beta,0}} + \cancel{\delta g_{03,\beta}} - \delta g_{0\beta,3}\right]$$

For the last term we have two options:

$$\delta g_{01,3}\Big|_{x=y=0} = \frac{\partial}{\partial z}\frac{2GJy}{(x^2+y^2+z^2)}\Big|_{x=y=0} = 0$$

and the same happens for $\delta g_{02,3}$. Note that they are not *always* null, but they vanish *at the rotation axis.*

Then:

$$\Gamma^3_{3\beta} = \frac{1}{2}\eta^{33}(\cancel{\delta g_{3\beta,3}} + \cancel{\delta g_{33,\beta}} - \cancel{\delta g_{3\beta,3}}) = 0$$

and so:

$$\frac{\mathrm{d}S^z}{\mathrm{d}\tau} = 0 \Rightarrow S^z = 0 \text{ at all times}$$

The same happens with S^t, as we now show:

$$\frac{\mathrm{d}S^t}{\mathrm{d}\tau} + \Gamma^0_{\alpha\beta} u^\alpha S^\beta = 0$$

As $\alpha = 0, 3$:

$$\frac{\mathrm{d}S^t}{\mathrm{d}\tau} + \Gamma^0_{0\beta} u^t S^\beta + \Gamma^0_{3\beta} u^z S^\beta = 0$$

The Christoffel symbols:

$$\Gamma^0_{0\beta} = \frac{1}{2}\eta^{00}\left(\cancel{\delta g_{0\beta,0}} + \cancel{\delta g_{00,\beta}} - \cancel{\delta g_{0\beta,0}}\right) = 0$$
$$\Gamma^0_{3\beta} = \frac{1}{2}\eta^{00}\left(\delta g_{0\beta,3} + \cancel{\delta g_{30,\beta}} - \cancel{\delta g_{3\beta,0}}\right) = 0$$

and the highlighted term, as seen before, vanishes on $\hat{z}$. So:

$$\frac{\mathrm{d}S^t}{\mathrm{d}\tau} = 0 \Rightarrow S^t \equiv 0 \text{ at all times}$$

Summarizing:

$$u^\alpha = (u^t, 0, 0, u^z) \qquad S^\alpha = (0, S^x(\tau), S^y(\tau), 0)$$

and $\boldsymbol{u} \cdot \boldsymbol{S} = 0$ immediately holds.
All that's left is to write the system of differential equations for S^x and S^y:

$$\begin{cases} \frac{\mathrm{d}S^1}{\mathrm{d}\tau} + \Gamma^1_{\alpha\beta} u^\alpha S^\beta = 0 \\ \frac{\mathrm{d}S^2}{\mathrm{d}\tau} + \Gamma^2_{\alpha\beta} u^\alpha S^\beta = 0 \end{cases}$$

Since only $\delta g_{01,10,02,20} \neq 0$, one lower index in $\Gamma^{1,2}_{\alpha\beta}$ must be 0. However, $\beta = 0$ multiplies $S^t = 0$, and so the 0 index must be α. This leads to:

$$\begin{cases} \frac{\mathrm{d}S^x}{\mathrm{d}\tau} + \Gamma^1_{0\beta} \frac{\mathrm{d}t}{\mathrm{d}\tau} S^\beta = 0 \\ \frac{\mathrm{d}S^y}{\mathrm{d}\tau} + \Gamma^2_{0\beta} \frac{\mathrm{d}t}{\mathrm{d}\tau} S^\beta = 0 \end{cases}$$

Changing variables and expanding:

$$\begin{cases} \frac{\mathrm{d}S^x}{\mathrm{d}t} + \Gamma^1_{01} S^x + \Gamma^1_{02} S^y = 0 \\ \frac{\mathrm{d}S^y}{\mathrm{d}t} + \Gamma^2_{01} S^x + \Gamma^2_{02} S^y = 0 \end{cases}$$

The Christoffel symbols:

$$\begin{aligned} \Gamma^1_{01} &= \frac{1}{2}\eta^{11}(\delta g_{11,0} + \delta g_{01,1} - \delta g_{0\vec{1},1}) = 0 \\ \Gamma^2_{02} &= 0 \\ \Gamma^1_{02} &= \frac{1}{2}\eta^{11}\left[\cancel{\delta g_{12,0}} + \delta g_{01,2} - \delta g_{02,1}\right] \\ \Gamma^2_{01} &= \frac{1}{2}\eta^{22}\left(\cancel{\delta g_{21,0}} + \delta g_{0\vec{2},1} - \delta g_{01,2}\right) \end{aligned}$$

and so $\Gamma^2_{01} = -\Gamma^1_{02}$ are the only non-vanishing symbols.

$$\begin{aligned} \Gamma^1_{02} &= \frac{1}{2}\left[\frac{\partial}{\partial y}\frac{2GJy}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial x}\frac{-2GJx}{(x^2+y^2+z^2)^{3/2}}\right]\Bigg|_{x=y=0} = \\ &\underset{(a)}{=} \frac{2GJ}{z^3} \end{aligned}$$

where in (a) we note that, to have non-zero results, the derivatives must *kill* the variables at the numerators. This leads to *two equal terms*.
We can finally substitute back in the equations:

$$\begin{cases} \frac{\mathrm{d}S^x}{\mathrm{d}t} + \frac{2GJ}{z^3} S^y = 0 \\ \frac{\mathrm{d}S^y}{\mathrm{d}t} - \frac{2GJ}{z^3} S^x = 0 \end{cases}$$

If z were constant, this would lead to an *harmonic oscillator* with an *instantaneous angular velocity* of the spin-vector $\Omega_{\mathrm{LT}} = 2GJ/z^3$ (Lens-Thirring). However, z is a function of time, so this will be valid only for a *very slowly moving object*.
In the case of a gyroscope *not on the rotation axis*, but at a direction $\boldsymbol{e}^{\hat{r}}$ we would have (calculations omitted):

$$\boldsymbol{\Omega}_{\mathrm{LT}} = \frac{GJ}{c^2 r^3}\left[3(\boldsymbol{J}\cdot\boldsymbol{e}^{\hat{r}})\boldsymbol{e}^{\hat{r}} - \boldsymbol{J}\right]$$

Note that this reduces to the formula we found if $\boldsymbol{e}^{\hat{r}} \parallel \boldsymbol{J}$, and also has the same pattern as an electric field of an electric dipole.

0.2 Kerr geometry (1963)

The Kerr metric is a *vacuum solution* ($R_{\mu\nu} = 0$) of a *rotating spherical mass*.

$$\begin{aligned}
\mathrm{d}s^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4GMar\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2 + \\
&+\left(r^2 + a^2 + \frac{2GMra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2 \\
a \equiv \frac{J}{M} \qquad &\rho^2 \equiv r^2 + a^2\cos^2\theta \qquad \Delta \equiv r^2 - 2GMr + a^2
\end{aligned}$$

Some properties

- Note that in $c = 1$ units, velocities are dimensionless. So: $[J] = [\text{Mass}][\text{Length}]$.
  Note that $a \equiv J/M$, and $[a] = \text{Length}$.
- $O(a^0)$ term is Schwarzschild
- $O(a^0) + O(a^1)$ leads to the slowly rotating geometry previously seen.
- $r \gg GM$ we have,at first order in $1/r$:

  $$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{4GMa}{r}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\theta + \left(1 + \frac{2GM}{r}\right)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta)\,\mathrm{d}\varphi^2$$

  So it is possible to orbit with a gyroscope far away from M and measure M an J using this line element.
- As the metric is stationary and axis-symmetric, there are still two Killing vectors:

  $$\xi^\alpha = (1,0,0,0) \qquad \xi^\alpha = (0,0,0,1)$$

- Symmetry $\theta \to \pi - \theta$

- Real singularity at $\rho = 0$, meaning that $r = 0$ **and** $\cos\theta = 0 \Rightarrow \theta = \pi/2$. Note that $r = 0$ *is not a single point*: the metric *changes* depending on ρ, and $r = 0$ does not identify a single value of ρ (there are different properties by approaching $r = 0$ from different directions, and in fact the singularities appears only if $\theta = \pi/2$). In a certain sense, the rotation *smears* the single point $r = 0$ into *a disk*. Another way to see it is by looking at $\mathrm{d}s^2$ and noting that two points at $r = 0$ and different values of θ are *separated by a non-zero distance*, as $\rho \neq 0$.

- There is a **coordinate singularity** (horizon) when $\Delta = 0$ (generalization of what happens in Schwarzschild).

  $$\Delta = 0 = r^2 - 2GMr + a^2 = 0 \Rightarrow r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

  So there are *two horizons*, and we will limit our discussion at the outer one (r_+). Note that if $a = 0$ we get back $r_+ = 2GM$ (Schwarzschild horizon).
  Note that if $a > GM$ *there is no horizon*, producing a **naked singularity**. We postulate (**cosmic censorship**) that *naked singularities do not exist in nature*. This principle *is not proven*, but it's suggested by the mechanism of black-hole formation.
  The case where $a = GM$ is called *extreme Kerr solution*.

We show now that $r = r_+$ defines a *null surface*, that is a surface separating a region where light can go to $r \to \infty$ from a region where light goes to $r \to 0$. So light entering it is "trapped inside" this surface.

0.2.1 Null surfaces

Consider a light cone in Minkowski spacetime, which is defined by $r = t$ (3-surface). Any vector on the light cone is of the form (in $\{t, r, \theta, \varphi\}$ coordinates):

$$x^\mu = (\alpha, \alpha, \beta, \gamma) = \alpha \underbrace{(1,1,0,0)}_{l^\alpha} + \beta \underbrace{(0,0,1,0)}_{m^\alpha} + \gamma \underbrace{(0,0,0,1)}_{n^\alpha}$$

Note that:

$$\boldsymbol{l} \cdot \boldsymbol{l} = l^\mu \eta_{\mu\nu} l^\nu = (l^0)^2 + (l^1)^2 = 0$$

and so l^α is a null vector. Also $\boldsymbol{m} \cdot \boldsymbol{m} > 0$ and $\boldsymbol{n} \cdot \boldsymbol{n} > 0$ are *space-like* vectors, and $\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot \boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{n} = 0$ (they are orthogonal to each other). So $\{l^\alpha, m^\alpha, n^\alpha\}$ is a basis for all vectors on a light cone (and they are all elements of the tangent space).
Then consider, in the Schwarzschild metric:

$$l^\alpha = (1,0,0,0) \qquad m^\alpha = (0,0,1,0) \qquad n^\alpha = (0,0,0,1)$$

and $m^\alpha g_{\alpha\beta} m^\beta > 0$, $\boldsymbol{m} \cdot \boldsymbol{m}, \boldsymbol{n} \cdot \boldsymbol{n} > 0$, $\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot \boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{n} = 0$. Also:

$$\boldsymbol{l} \cdot \boldsymbol{l} = l^\alpha g_{\alpha\beta} l^\beta = g_{00}\Big|_{r=2GM} = 0$$

And so $\{l^\alpha, m^\alpha, n^\alpha\}$ are a basis for the null surface at the Schwarzschild horizon. Similarly, in the Kerr metric at the $r = r_+ = GM + \sqrt{G^2M^2 - a^2}$ horizon, the following vectors define a null surface:

$$\begin{aligned} l^\alpha &= (1, 0, 0, \Omega_H) \qquad \Omega_H \equiv \frac{a}{2GMr_+} \\ m^\alpha &= (0, 0, 1, 0) \\ n^\alpha &= (0, 0, 0, 1) \end{aligned}$$

Note that, while in Schwarzschild light *trapped* in the horizon *just moves in time*, for the Kerr geometry light is *orbiting* the blackhole with angular velocity Ω_H.
Let's now look at the Kerr horizon $r = r_+$ at fixed t (the value is not important, as the metric is stationary), so to have a 2D surface (θ, φ). It can be shown that, on the horizon:

$$\mathrm{d}s^2\Big|_{r=r_+} = \rho_+^2\,\mathrm{d}\theta^2 + \left(\frac{2GMr_+}{\rho_+}\right)^2 \sin^2\theta\,\mathrm{d}\varphi^2$$

Note that it is not spherical $\rho_+^2(\theta) = r^2 + a^2\cos^2\theta$, as the equator is *greater* than the corresponding equator for a sphere. To show this, we compute the *length* of the equator (motion at $\theta = \pi/2$ and $\varphi \in [0, 2\pi)$) and that of a full meridian (fixed φ, $\theta\colon 0 \to \pi \to 0$). For a sphere, the equator and a circle going through the poles *have the same length*. However, for the Kerr horizon, we will see that:

$$L_{\text{equator}} > L_{\text{N}-\text{S}-\text{N}}$$

(N-S-N stands for the path starting at the North pole, going to the South pole and returning to the North).

$$\begin{aligned} L_{\text{equator}} &= \int_0^{2\pi} \mathrm{d}\varphi\,\sqrt{g_{\varphi\varphi}}\Big|_{\theta=\pi/2} = 2GM\int_0^{2\pi}\mathrm{d}\varphi = 4\pi GM \\ L_{\text{NSN}} &= 2\int_0^{\pi}\mathrm{d}\theta\,\sqrt{g_{\theta\theta}} = 2\int_0^{\pi}\mathrm{d}\theta\,\sqrt{r_+^2 + a^2\cos^2\theta} \end{aligned}$$

For the second one we would need an elliptic integral. To simplify things, we expand it in the limit of small a:

$$L_{\text{NSN}} \approx 2\int_0^{\pi}\mathrm{d}\theta\left[2GM + \frac{a^2}{4GM}(-2 + \cos^2\theta) + O(a^4)\right]$$

As we are integrating $\cos^2\theta$ over a period, we can substitute it with its average $(1/2)$, leading to:

$$= \left[4GM - \frac{3a^2}{4GM}\right]\pi$$

And so $L_{\text{equator}} > L_{NSN}$ for $a \neq 0$ (they are the same for $a = 0$).