0.1 Einstein's equations

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We arrived at the Einstein's equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

These:

1. Reproduces Netown's Law at small curvature/non-relativistic source
2. Are covariant
3. Mathematically consistent:

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu G^{\mu\nu} = 0$$

Recall that a free particle in flat spacetime satisfies:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0 \Leftrightarrow \int_A^B \mathrm{d}\tau = \int_A^B \sqrt{-\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta} \text{ is minimum}$$

We supposed that a free particle in *curved spacetime* also minimizes the proper time, thus satisfying the following, more complex, equation of motion:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \Leftrightarrow \int_A^B \mathrm{d}\tau = \int \sqrt{-g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta} \text{ is minimunm}$$

We can rewrite this expression in terms of the 4-velocity. Recall that:

$$u^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$$

Thus:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

Also:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}x^\alpha} \underbrace{\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}}_{u^\alpha} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 \Rightarrow u^\alpha \left[\frac{\mathrm{d}u^\mu}{\mathrm{d}x^\alpha} + \Gamma^\mu_{\alpha\beta} u^\beta\right] = 0 \Rightarrow A^\mu \equiv u^\alpha \nabla_\alpha u^\mu$$

where A^μ represents the *acceleration felt by the moving particle*. So, a free particle, moving along a geodesic, feels *no acceleration* $A^\mu = 0$.
So, while a circular motion *caused by a centripetal force* (e.g. a rope) leads to a "feeling of acceleration", the same motion, only caused by gravity, does not lead to any feeling of acceleration, because it is a geodesic motion (if the speed is the right one for that trajectory).

0.1.1 Timelike geodesics

We call *timelike geodesics* the free motion of massive particles, following trajectories that *minimize the proper time*:

$$\mathrm{d}\tau = \sqrt{-\,\mathrm{d}s^2} \qquad (\mathrm{d}s^2 < 0 \Rightarrow \text{time-like})$$

As we demonstrated:

$$\int_A^B \mathrm{d}\tau \ \text{ is minimized} \Rightarrow \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0$$

0.1.2 Spacelike geodesics

Equivalently, we can consider the *shortest spatial paths*, that is the ones that minimize *spatial distance*:

$$\int_A^B \mathrm{d}s \ \text{ is minimum}$$

Of course, these trajectories are not followed by free particles (they are space-like). If we repeat the same calculations we made for the timelike geodesics, we arrive at the following differential equation:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \frac{\mathrm{d}x^\gamma}{\mathrm{d}s} = 0$$

0.1.3 Null geodesics

Massless particles, like photons, move along different trajectories, that satisfy:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0$$

(This also needs to be experimentally verified).

0.2 Solution of the Geodesic Equation

Example 1 (Geodesics on 2D Euclidean Plane)**:**

Obviously, the geodesics on the 2D plane are just *straight lines*. We will prove it in a more complex (and instructive) manner, that is by using *polar coordinates* (why not?).

Recall that a *spacelike geodesic* is the trajectory that *minimizes* the *length* to go from a point A to a point B.
In polar coordinate $x^\mu = (r, \theta)$ the Euclidean metric is:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

The Christoffel symbol is defined as:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} (g_{\lambda\gamma,\beta} + g_{\beta\lambda,\gamma} - g_{\beta\gamma,\lambda})$$

For the r-coordinate:

$$\begin{aligned} \Gamma^{r}_{\beta\gamma} &= \frac{1}{2} (\cancel{g_{r\gamma,\beta}} + \cancel{g_{\beta r,\gamma}} - g_{\beta\gamma,r}) = \frac{1}{2}(-2r) = -r \\ \Gamma^{\theta}_{\beta\gamma} &= \frac{1}{2} (g_{\theta\gamma,\beta} + g_{\beta\theta,\gamma} - g_{\beta\gamma,\theta}) = \frac{1}{r} \end{aligned}$$

The only non-zero symbols are:

$$\Gamma^{r}_{\theta\theta} = -r \qquad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}$$

Inserting in the geodesics equation:

$$\frac{\mathrm{d}^2 x^{\alpha}}{\mathrm{d}s^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}s} \frac{\mathrm{d}x^{\gamma}}{\mathrm{d}s} = 0$$

leads to:

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - r \left(\frac{\mathrm{d}\theta}{\mathrm{d}s} \right)^2 &= 0 \\ \frac{\mathrm{d}^2 \theta}{\mathrm{d}s^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\theta}{\mathrm{d}s} &= 0 \end{aligned}$$

To know the solution I need to know the coordinates $x^1, x^2, x^3 \dots$ of all points in the curve, as functions of the distance s from A: $x^1(s), x^2(s), \dots$. So we need to know N functions (one per dimension). Also $2N$ initial conditions are needed (this is a second order differential equation).

When solving a differential equation it is useful to find *first integrals*, that is quantities that are constant along the geodesic. First, note that we can rewrite the second equation as:

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}s} \left(r^2 \frac{\mathrm{d}\theta}{\mathrm{d}s} \right) = 0 \Rightarrow \frac{1}{r^2} \left(2r \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\theta}{\mathrm{d}s} + r^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} \right) = \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = 0$$

(In the general case, try with r^{α} for a generic constant α).

Then, we can solve this as a *first order differential equation* (much simpler):

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}s} = A = \text{constant} \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{A}{r^2}$$

By using the definition of $\mathrm{d}s$:

$$\mathrm{d}s^2 + \mathrm{d}r^2 + r^2 \,\mathrm{d}\theta^2$$

we can derive another relation:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \frac{A^2}{r^4}\,\mathrm{d}s^2 \Rightarrow \mathrm{d}s^2\left(1 - \frac{A^2}{r^2}\right) = \mathrm{d}r^2 \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 - \frac{A}{r^2}}$$

where we omitted the $\pm$ as they will lead the same result at the end in this case. So, we obtained another *first order differential equation*.
We are interested in the trajectory, not in a parametrization, so we search $r(\theta)$ or $\theta(r)$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}\theta\,/\,\mathrm{d}s}{\mathrm{d}r\,/\,\mathrm{d}s} = \frac{A/r^2}{\sqrt{1 - A^2 r^{-2}}} \Rightarrow \mathrm{d}\theta = \frac{A}{r^2}\left(1 - \frac{A^2}{r^2}\right)^{-1/2}\mathrm{d}r$$

Then we integrate:

$$\theta - \theta_0 = \int_{\theta_0}^{\theta} \mathrm{d}\theta \int_{r_0}^{r} \frac{A}{r^2}\left(1 - \frac{A^2}{r^2}\right)^{-1/2}\mathrm{d}r$$

With the change of variables $\xi = A/r$, $-Ar^{-2}\,\mathrm{d}r = \mathrm{d}\xi$ we arrive at:

$$\theta - \theta_0 = -\int \frac{\mathrm{d}\xi}{\sqrt{1 - \xi^2}} = \arccos(\xi) = \arccos\left(\frac{A}{r}\right)$$

To see that these are indeed straight lines, we write:

$$r\cos(\theta - \theta_0) = A \Rightarrow r\cos\theta\cos\theta_0 + r\sin\theta\sin\theta_0 = A \Rightarrow x\cos\theta_0 + y\sin\theta_0 = A$$

and by solving it:

$$y = \underbrace{-\frac{\cos\theta_0}{\sin\theta_0}}_{\alpha}\,x + \underbrace{\frac{A}{\sin\theta_0}}_{\beta} = \alpha x + \beta$$

0.3 Euler-Lagrange Equations

If we compute the proper time interval between two points:

$$\tau_{AB} = \int_A^B \mathrm{d}\tau = \int \sqrt{-g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta} = \int \mathrm{d}\sigma\,\sqrt{-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}} \equiv \int \mathrm{d}\sigma\, L\left[x^\alpha, \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\right]$$

where L is called a **Lagrangian**.

Now, we minimize τ_{AB}:

$$0 = \delta\tau = \int \mathrm{d}\sigma \left[\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right)} \frac{\mathrm{d}\delta x^\alpha}{\mathrm{d}\sigma} \right] =$$
$$= \int \mathrm{d}\sigma \left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial L}{\partial \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right)} \right) \right] \delta x^\alpha(\sigma) = 0$$

This holds for *every possible variation*, meaning that also the integrand must vanish:

$$\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial L}{\partial \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right)} \right) = 0$$

These are the **Euler-Lagrange equations**, with:

$$L \left[x^\alpha, \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right] \equiv \sqrt{-g_{\alpha\beta}(x) \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}$$

0.4 Killing vectors

For every *symmetry of the metric* (i.e. the metric does not depend on a certain coordinate) there is a *conserved quantity* (a constant of motion). We will now show why.

First, if a metric is x-independent, we define the Killing vector $\xi^\alpha = (0, 1, 0, 0)$, i.e. a vector that goes in the direction where the metric does not change:

$$\frac{\partial g_{\alpha\beta}}{\partial x^1} = 0$$

Therefore, also L does not depend on that coordinate:

$$\frac{\partial L}{\partial x^1} = 0$$

And so, substituting in the Euler-Lagrange equations:

$$\cancel{\frac{\partial L}{\partial x^1}} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial L}{\partial \left(\frac{\partial x^1}{\partial \sigma} \right)} \right) = 0$$

and so:

$$\frac{\partial L}{\partial \left(\frac{\partial x^1}{\partial \sigma} \right)} = \text{constant}$$

Explicitly:

$$\text{constant} = \frac{\partial L}{\partial\left(\frac{\mathrm{d}x^1}{\mathrm{d}\sigma}\right)} = \frac{1}{2\sqrt{\cdots}}\left[-g_{\mu\nu}\underbrace{\frac{\partial\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right)}{\partial\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\right)}}_{\delta^\mu_\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} - g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\underbrace{\frac{\partial\left(\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right)}{\partial\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\right)}}_{\delta^\nu_\alpha}\right]\Bigg|_{\alpha=1} =$$

$$= \frac{1}{2L}\left[-g_{\alpha\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} - g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right]\Bigg|_{\alpha=1} \underset{(a)}{=} \frac{-2g_{\alpha\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}{2L}\Bigg|_{\alpha=1} = \frac{-2g_{1\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}{2L}$$

where L comes from the derivative of the square root, and in (a) we renamed $\nu \to \mu$ and then used the symmetry $g_{\alpha\mu} = g_{\mu\alpha}$ to collect the metric. Recall that:

$$\frac{1}{L}\frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\mathrm{d}}{\mathrm{d}\tau}$$

and so:

$$= -g_{1\beta}\underbrace{\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}}_{u^\beta} = -\overbrace{\xi^\alpha}^{(0,1,0,0)} g_{\alpha\beta}u^\beta = -\boldsymbol{\xi}\cdot\boldsymbol{u}$$

So, if $g_{\mu\nu}$ does not epend on the direction ξ^α, the quantity:

$$\boldsymbol{\xi}\cdot\boldsymbol{u} = \text{constant}$$

that is, $\boldsymbol{\xi}\cdot\boldsymbol{p} = \text{constnat}$, where $p^\mu = mu^\mu$ is the 4-momentum.

Example 2 (Conserved quantity in polar coordinates)**:**

Consider the Euclidean 2D metric in polar coordinates:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Note that this metric *does not depend* on θ, so $\xi = (0,1)$ is a Killing vector (choosing the (r,θ) basis). Then:

$$\boldsymbol{u} = \left(\frac{\mathrm{d}r}{\mathrm{d}s}, \frac{\mathrm{d}\theta}{\mathrm{d}s}\right)$$

and, as demonstrated, $\boldsymbol{\xi}\cdot\boldsymbol{u}$ is constant, that is:

$$\xi^\alpha g_{\alpha\beta}u^\beta = \xi^\theta g_{\theta\beta}u^\beta = g_{\theta\theta}u^\theta = r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}$$

which is the same result implied by the second geodesic equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0 \Rightarrow r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = \text{constant}$$

Note that the choice of coordinates for writing the metric will make some Killing vector easier to see. In fact, the independence of the metric on a coordinate is just a sufficient condition to find a Killing vector (a necessary one involves Lie derivatives, and we will not examine it in this course)