

Consider a potential  $U(x)$  with a local minimum at  $x = c$ , and a local maximum at  $x = d$ , with  $c < d$ . Consider a particle starting at  $x = c$ . We wish to compute the average first visit time of  $d$ , denoted with  $\langle T(c \rightarrow d) \rangle$ . This can be done by redefining the system as the half-line  $[-\infty, d]$ , with  $x = -\infty$  being a *reflective* boundary, and  $x = d$  an *absorbing* one. We can do this because we are not interested in the behaviour *after* passing  $d$ , but just in the mean arrival times. So  $A(x) = -\partial_x U(x)/\gamma$ . Supposing to be at equilibrium,  $D(x) \equiv D = 1/(\gamma B)$ . Letting  $a = -\infty$  and  $b = d$  leads to:

$$\begin{aligned} T_d(x) &= \int_x^d dy \int_{-\infty}^y \beta \gamma dz \exp \left( - \int_z^y -dv \frac{\partial_v U(v)}{\gamma} \gamma \beta \right) = \\ &= \beta \gamma \int_x^d dy \int_{-\infty}^y dz \exp(\beta[U(y) - U(z)]) = \\ &= \beta \gamma \int_x^d dy e^{\beta U(y)} \underbrace{\int_{-\infty}^y dz e^{-\beta U(z)}}_{e^{F(y)}} = \beta \gamma \int_x^d dy e^{\beta U(y) + F(y)} \end{aligned}$$

It is not possible to evaluate this integral in the general case. However, in the limit  $\beta \rightarrow \infty$  ( $T \rightarrow 0$ ) we can use the saddle-point approximation. Recall Laplace's formula:

$$\int_a^b e^{Mf(x)} dx \underset{M \rightarrow +\infty}{\approx} \sqrt{\frac{2\pi}{M|f''(x_0)|}} e^{Mf(x_0)}$$

where  $f'(x_0) = 0$  and  $f''(x_0) < 0$ .

For the integral in  $dz$ ,  $f(z) = -U(z)$ . We search for a maximum of  $f(z)$ , i.e. a minimum of  $U(z)$ , which is  $z = c$ . So:

$$\int_{-\infty}^y e^{-\beta U(z)} dz = \sqrt{\frac{2\pi}{\beta U''(c)}} e^{-\beta U(c)}$$

This is a constant, and can be brought outside the integral over  $dy$ . Then, by applying Laplace's formula once again:

$$\int_c^d dy e^{\beta U(y)} = \sqrt{\frac{2\pi}{\beta |U''(d)|}} e^{\beta U(d)}$$

as now  $f(y) = U(y)$ , and  $U$  has a local maximum in  $y = d$ . Finally, this leads to:

$$T_d(c) \underset{T \rightarrow 0}{\approx} \frac{2\pi\gamma}{\sqrt{U''(c)|U''(d)|}} \exp(\beta[U(d) - U(c)])$$

Note that the mean transition time from  $c$  to  $d$  diverges exponentially as the barrier's height  $U(d) - U(c)$  rises. Equivalently, the *escape transition rate*  $1/T_d(c) \rightarrow 0$ .