0.1 A first functional

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Consider a Brownian trajectory $x(\tau)$ (from now on, we will assume that all trajectories start in $x = 0$ at $t = 0$), and a functional that *weights* every traversed point $x(\tau)$ with a function $a\colon \mathbb{R} \to \mathbb{R}$, and then applies another function $F\colon \mathbb{R} \to \mathbb{R}$ to the total integral:

$$F[x(\tau)] = F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)$$

For simplicity, we set $D = 1/4$, so that:

$$\mathrm{d}\mathbb{P}_{t_1,\dots,t_n}(x_1,\dots,x_n|0,0) = \exp\left(-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{\pi\Delta t_i}\right)\prod_{i=1}^n \frac{\mathrm{d}x_i}{\sqrt{\pi\Delta t_i}}$$

This is equivalent to a time rescaling $t \to \tau = 4Dt$.
We want now to compute $\langle F\rangle$:

$$I_3 \equiv \langle F[x(\tau)]\rangle_w = \int_{\mathcal{C}\{0,0;t\}} \mathrm{d}_W x(\tau)\, F[x(\tau)]$$

Note: the next computations will follow the book. Prof. Maritan's method for evaluating I_3 is quicker, but more advanced, and will be presented at the end.

Then we start by discretizing, by choosing a time grid $0 = t_0 < t_1 < \cdots < t_N = t$:

$$I_3 = \lim_{N\to\infty} I_3^{(N)}$$

$$I_3^{(N)} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}x_1}{\sqrt{\pi\Delta t_1}}\cdots\int_{-\infty}^{+\infty}\frac{\mathrm{d}x_N}{\sqrt{\pi\Delta t_N}} F\left(\sum_{i=1}^N a_i x_i \Delta t_i\right)\exp\left(-\sum_{i=1}^N \frac{(x_i - x_{i-1})^2}{\Delta t_i}\right)$$

$$a_i \equiv a(t_i)$$
$$x_i \equiv x(t_i)$$

This integral can be evaluated by transforming it to a *gaussian integral* that we already know. So we start by changing variables:

$$x_i - x_{i-1} = y_i \qquad i = 1,\dots,N \tag{1}$$

Note that:

$$\sum_{j=1}^i y_j = \cancel{x_1} - \underbrace{x_0}_{=0} + x_2 - \cancel{x_1} + \cdots + x_i - \cancel{x_{i-1}} = x_i \qquad 1 \le i \le N$$

So, when we compute the transformation of the volume element:

$$\det\left|\frac{\partial\{x_i\}}{\partial\{y_j\}}\right| = \det\begin{vmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ \vdots & \vdots & \ddots & 0\\ 1 & 1 & \cdots & 1\end{vmatrix}_{N\times N} = 1$$

as the determinant of a lower triangular matrix is equal to the product of the diagonal entries.
All that's left is to transform the argument of F. Let's start by writing the first terms of the sum and apply the change of variables:

$$\begin{aligned}\sum_{i=1}^{N} a_i x_i \Delta t_i &= a_1 x_1 \Delta t_1 + a_2 x_2 \Delta t_2 + \cdots = \\ &= a_1(y_1)\Delta t_1 + a_2(y_1 + y_2)\Delta t_2 + \cdots = \\ &= y_1 \left(\sum_{j=1}^{N} a_j \Delta t_j \right) + y_2 \left(\sum_{j=2}^{N} a_j \Delta t_j \right) + \cdots + y_N a_N \Delta t_N = \\ &= \sum_{i=1}^{N} y_i \underbrace{\left(\sum_{j=i}^{N} a_j \Delta t_j \right)}_{A_i} \equiv \sum_{i=1}^{N} A_i y_i \qquad (2)\end{aligned}$$

Substituting everything back:

$$I_3^{(N)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}y_1}{\sqrt{\pi \Delta t_1}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}y_N}{\sqrt{\pi \Delta t_N}} F\left(\sum_{i=1}^{N} A_i y_i \right) \exp\left(-\sum_{i=1}^{N} \frac{y_i^2}{\Delta t_i} \right) \qquad A_i = \sum_{j=i}^{N} a_j \Delta t_j$$

We can simplify this integral a bit more by rescaling the y_i:

$$z_i = A_i y_i \qquad \mathrm{d}y_i = \frac{\mathrm{d}z_i}{A_i}$$

As each y_i is transformed independently, the jacobian is diagonal.

$$I_3^{(N)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_1}{\sqrt{\pi A_1^2 \Delta t_1}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_N}{\sqrt{\pi A_N^2 \Delta t_N}} F(z_1 + \cdots + z_N) \exp\left(-\sum_{i=1}^{N} \frac{z_i^2}{A_i^2 \Delta t_i} \right)$$

This is the expected value of a function of the *sum* of N normally distributed random variables $\{z_i\}$. The idea is now to *isolate* one of them from the argument of F, integrate over it, and reiterate. This is done by changing variables yet again:

$$\begin{cases} \eta = z_1 + z_2 \\ \xi = z_2 \end{cases} \Rightarrow \begin{cases} z_1 = \eta - \xi \\ z_2 = \xi \end{cases} \Rightarrow \det \left| \frac{\partial\{z_1, z_2\}}{\partial\{\eta, \xi\}} \right| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

leading to:

$$\begin{aligned} I_3^{(N)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\eta}{\sqrt{\pi A_1^2 \Delta t_1}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\sqrt{\pi A_2^2 \Delta t_2}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_3}{\sqrt{\pi A_3^2 \Delta t_3}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_N}{\sqrt{\pi A_N^2 \Delta t_N}} \cdot \\ \cdot F(\eta + z_3 + \cdots + z_N) \exp\left(-\frac{(\eta - \xi)^2}{A_1^2 \Delta t_1} - \frac{\xi^2}{A_2^2 \Delta t_2} - \sum_{i=3}^{N} \frac{z_i^2}{A_i^2 \Delta t_i} \right) \end{aligned}$$

Note how ξ does not enter in the F argument, and so we can integrate over it:

$$
\begin{aligned}
I_\xi &= \int_{-\infty}^{+\infty} \mathrm{d}\xi \, \frac{1}{\sqrt{\pi A_1^2 \Delta t_1}\sqrt{\pi A_2^2 \Delta t_1}} \exp\left(-\frac{(\eta-\xi)^2}{A_1^2\Delta t_1} - \frac{\xi^2}{A_2^2 \Delta t_2}\right) = \\
&= \int_{-\infty}^{+\infty} \mathrm{d}\xi \, (\cdots) \exp\left(-\frac{\xi^2(A_2^2\Delta t_1 + A_2^2 \Delta t_2) - (2\eta A_2^2 \Delta t_2) - (-\eta^2 A_2^2 \Delta t_2)}{A_1^2 A_2^2 \Delta t_1 \Delta t_2}\right)
\end{aligned}
$$

Recall the gaussian integral formula:

$$
\int_{-\infty}^{+\infty} \exp(-a x^2 + b x + c) \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right) \tag{3}
$$

which evaluates to:

$$
I_\xi = \frac{1}{\sqrt{\pi(A_1^2 \Delta t_1 + A_2^2 \Delta t_2)}} \exp\left(-\frac{\eta^2}{A_1^2 \Delta t_1 + A_2^2 \Delta t_2}\right)
$$

and substituting back in $I_3^{(N)}$:

$$
\begin{aligned}
I_3^{(N)} = \int_{-\infty}^{+\infty} & \frac{\mathrm{d}\eta}{\sqrt{\pi A_1^2 \Delta t_1 + \pi A_2^2 \Delta t_2}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_3}{\sqrt{\pi A_3^2 \Delta t_3}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_N}{\sqrt{\pi A_N^2 \Delta t_N}} \cdot \\
& \cdot F(\eta + z_3 + \cdots + z_N) \exp\left(-\frac{\eta^2}{A_1^2 \Delta t_1 + A_2^2 \Delta t_2} - \sum_{i=3}^{N} \frac{z_i^2}{A_1^2 \Delta t_i}\right)
\end{aligned}
$$

We can now reiterate this procedure until only one integration is left:

$$
I_3^{(N)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\sqrt{\pi \sum_{i=1}^N A_i^2 \Delta t_i}} F(z) \exp\left(-\frac{z^2}{\sum_{i=1}^N A_i^2 \Delta t_i}\right)
$$

We are now finally ready to take the continuum limit $\Delta t_i \to 0$, $N \to \infty$. Note that:

$$
\lim_{\Delta t_i \to 0} A_i = \int_\tau^t a(s) \, \mathrm{d}s = A(\tau) \tag{4}
$$

as the discrete sum goes from $t_i = \tau$ to $t_N = t$. Then:

$$
R \equiv \lim_{\Delta t \to 0} \sum_{i=1}^{N} A_i^2 \Delta t_i = \int_0^t \mathrm{d}\tau \left(\int_\tau^t \mathrm{d}s \, a(s)\right)^2
$$

and so:

$$
I_3 = \lim_{N\to\infty} I_3^{(N)} = \int_{-\infty}^{+\infty} \mathrm{d}z \, \frac{F(z)}{\sqrt{\pi R}} \exp\left(-\frac{z^2}{R}\right)
$$

And to recover D we can just substitute $R \to 4DR$.

0.1.1 Alternative method

We consider now a different (quicker) technique to compute I_3. So we start again from:

$$I_3 \equiv \langle F[x(\tau)]\rangle_w = \int_{C\{0,0;t\}} \mathrm{d}_W x(\tau)\, F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)$$

It is convenient to apply the change of variables we did in (2). We can do *before* discretizing, by defining $A(\tau)$ as in (4):

$$A(\tau) \equiv \int_\tau^t a(s)\,\mathrm{d}s \tag{5}$$

Note that $\dot{A}(\tau) = -a(\tau)$, and so the argument of F becomes:

$$\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau = -\int_0^t \partial_\tau A(\tau)x(\tau)\,\mathrm{d}\tau$$

Integrating by parts:

$$= -\cancel{[x(\tau)A(\tau)]_{\tau=0}^{\tau=t}} + \int_0^t A(\tau)\dot{x}(\tau)\,\mathrm{d}\tau$$

And now we discretize the path over the instants $0 = t_0 < t_1 < \cdots < t_N$, so that:

$$\begin{aligned}\int_0^t A(\tau)\dot{x}(\tau)\,\mathrm{d}\tau &= \lim_{\Delta t_i \to 0} \sum_{i=1}^N A(t_i)\frac{x(t_i) - x(t_{i-1})}{\Delta t_i}\Delta t_i = \\ &= \lim_{N\to\infty} \sum_{i=1}^N A_i(x_i - x_{i-1}) = \lim_{N\to\infty}\sum_{i=1}^N A_i \Delta x_i \qquad \begin{array}{c} x_i \equiv x(t_i) \\ A_i \equiv A(t_i)\end{array}\end{aligned}$$

Substituting back (here $D = 1/4$ for simplicity):

$$\begin{aligned} I_3 &= \lim_{N\to\infty} I_3^{(N)} \\ I_3^{(N)} &= \int_{\mathbb{R}^N}\left(\prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{\pi \Delta t_i}}\right)\exp\left(-\sum_{i=1}^N \frac{(\Delta x_i)^2}{\Delta t_i}\right) F\left(\sum_{i=1}^N A_i \Delta x_i\right)\end{aligned}$$

The idea is now to *apply* a change of random variable, rewriting the average $\langle F[x(\tau)]\rangle_w$ (according to the distribution of *paths*) as the average $\langle F(z)\rangle_{p(z)}$, where $p(z)$ is the distribution followed by the argument of F:

$$\sum_{i=1}^N A_i \Delta x_i$$

So, we begin by inserting the appropriate δ:

$$I_3^{(N)} = \int_{\mathbb{R}^N}\left(\prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{\pi\Delta t_i}}\right)\exp\left(-\sum_{i=1}^N\frac{(\Delta x_i)^2}{\Delta t_i}\right) F\left(\sum_{i=1}^N A_i\Delta x_i\right) \underbrace{\int_{\mathbb{R}} \mathrm{d}z\,\delta\left(z - \sum_{i=1}^N A_i \Delta x_i\right)}_{=1}$$

Exchanging the integrals leads to:

$$\langle F\left(\sum_{i=1}^{N} A_i \Delta x_i\right)\rangle_w = \langle F(z)\rangle_{p(z)} =$$

$$= \int_{\mathbb{R}} \mathrm{d}z\, F(z) \underbrace{\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i}{\sqrt{\pi \Delta t_i}}\right) F\left(\sum_{i=1}^{N} A_i \Delta x_i\right) \delta\left(z - \sum_{i=1}^{N} A_i \Delta x_i\right) \exp\left(-\sum_{i=1}^{N} \frac{(\Delta x_i)^2}{\Delta t_i}\right)}_{p(z)}$$

We can evaluate $I_3^{(N)}$ by transforming it to a *gaussian integral*. First, we remove the δ with a Fourier transform:

$$2\pi\delta(x) = \int_{\mathbb{R}} e^{i\alpha x}\, \mathrm{d}\alpha$$

which, in this case, leads to:

$$\delta\left(z - \sum_{i=1}^{N} A_i \Delta x_i\right) = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \exp\left(i\alpha \left(z - \sum_{i=1}^{N} A_i \Delta x_i\right)\right)$$

Substituting back:

$$I_3^{(N)} = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \mathrm{d}z\, F(z) e^{i\alpha z} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i}{\sqrt{\pi \Delta t_i}}\right) \exp\left(-\sum_{i=1}^{N} \frac{\Delta x_i^2}{\Delta t_i} - i\alpha \sum_{i=1}^{N} A_i \Delta x_i\right)$$

We see that the last term is similar to a multivariate gaussian with a imaginary term, that we know how to integrate. We just need to remove the *differences* in the exponential with a change of variables (as in (1)):

$$\begin{aligned} y_1 &= \Delta x_1 = x_1 - \overbrace{x_0}^{=0} = x_1 \\ y_2 &= \Delta x_2 = x_2 - x_1 \\ &\vdots \\ y_N &= \Delta x_N = x_N - x_{N-1} \end{aligned}$$

The volume element will be transformed by the determinant of the Jacobian:

$$J = \det \frac{\partial(x_1 \dots x_N)}{\partial(y_1 \dots y_N)} = \left[\det \frac{\partial(y_1 \dots y_N)}{\partial(x_1 \dots x_N)}\right]^{-1} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}^{-1} = 1$$

where we used the fact that $\det A^{-1} = (\det A)^{-1}$, and that the determinant of a lower triangular matrix is just the product of the diagonal entries.

The integral then becomes:

$$
\begin{aligned}
I_3^{(N)} &= \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \mathrm{d}z\, F(z) e^{i\alpha z} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \frac{\mathrm{d}y_i}{\sqrt{\pi \Delta t_i}} \right) \exp\left(-\sum_{i=1}^N \frac{y_i^2}{\Delta t_i} - i\alpha \sum_{i=1}^N A_i y_i \right) = \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \mathrm{d}z\, F(z) e^{i\alpha z} \left[\prod_{i=1}^N \int_{\mathbb{R}} \frac{\mathrm{d}y_i}{\sqrt{\pi \Delta t_i}} \exp\left(-\frac{y_i^2}{\Delta t_i} - i\alpha A_i y_i \right) \right]
\end{aligned}
$$

The terms in the product are all independent gaussian integrals. Recall that:

$$
\int_{\mathbb{R}} \mathrm{d}k\, e^{-iak^2 - ibk} = \sqrt{\frac{\pi}{ia}} \exp\left(\frac{ib^2}{4a} \right) \tag{6}
$$

So, with $ia = 1/\Delta t_i$ and $b = \alpha A_i$ we get:

$$
\int_{\mathbb{R}} \frac{\mathrm{d}y_i}{\sqrt{\pi \Delta t_i}} \exp\left(-\frac{y_i^2}{\Delta t_i} - i\alpha A_i y_i \right) = \exp\left(-\frac{\alpha^2 A_i^2 \Delta t_i}{4} \right)
$$

and substituting back in the integral leads to:

$$
\begin{aligned}
I_3^{(N)} &= \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \mathrm{d}z\, F(z) e^{i\alpha z} \left[\prod_{i=1}^N \exp\left(-\frac{\alpha^2 A_i^2 \Delta t_i}{4} \right) \right] = \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \mathrm{d}z\, F(z) e^{i\alpha z} \exp\left(-\frac{\alpha^2}{4} \sum_{i=1}^N A_i^2 \Delta t_i \right)
\end{aligned}
$$

Applying the continuum limit ($N \to \infty$, $\Delta t_i \to 0$), the exponential argument becomes the limit of a Riemann sum, i.e. a integral:

$$
\sum_{i=1}^N A(t_i)^2 \Delta t_i \xrightarrow[N\to\infty]{} \int_0^t A^2(\tau)\,\mathrm{d}\tau \underset{(5)}{=} \int_0^t \mathrm{d}\tau \left(\int_\tau^t \mathrm{d}s\, a(s) \right)^2 \equiv R(t)
$$

Substituting back:

$$
I_3 \equiv \langle F\left(\int_0^t a(\tau) x(\tau) \right) \rangle = \lim_{N\to\infty} I_3^{(N)} = \int_{\mathbb{R}} \mathrm{d}z \int_R \frac{\mathrm{d}\alpha}{2\pi} \exp\left(-\frac{\alpha^2}{4} R(t) + i\alpha z \right)
$$

All that's left is to evaluate the last gaussian integral thanks to (6) with $ia = R(t)/4$ and $b = -z$, leading to:

$$
I_3 = \int_{\mathbb{R}} \mathrm{d}z\, F(z) \frac{1}{2\pi} \sqrt{\frac{4\pi}{R(t)}} \exp\left(-\frac{z^2}{R(t)} \right) = \frac{1}{\sqrt{\pi R(t)}} \int_{\mathbb{R}} \mathrm{d}z\, F(z) \exp\left(-\frac{z^2}{R(t)} \right)
$$

So, we showed that:

$$
\langle F\left(\int_0^t a(\tau) x(\tau)\,\mathrm{d}\tau \right) \rangle_w = \sqrt{\frac{1}{\pi R(t)}} \int_{\mathbb{R}} \mathrm{d}z\, F(z) \exp\left(-\frac{z^2}{R(t)} \right); \qquad R(t) \equiv \int_0^t \mathrm{d}\tau \left(\int_\tau^t a(s)\,\mathrm{d}s \right)^2 \tag{7}
$$

Example 1 (Generating function)**:**

Let $F(z) = e^{hz}$. Inserting in (7) results in:

$$\langle \exp\left(h\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)\rangle_w = \frac{1}{\sqrt{\pi R}}\int_{\mathbb{R}} \mathrm{d}z \exp\left(-\frac{z^2}{R}+hz\right) \underset{(a)}{=} \exp\left(\frac{h^2 R}{4}\right) \equiv G(h) \tag{8}$$

where in (a) we used formula (3) with $a = 1/R$ and $b = h$.

Note that $G(h)$ is the **generating function** (CFR def. in 10/10 notes) of the integral:

$$I = \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau$$

We can then retrieve the n-th moment of I by computing the n-th derivative of $G(h)$:

$$\frac{\mathrm{d}^n}{\mathrm{d}h^n} G(h)\Big|_{h=0} = \langle I^n\rangle_w$$

We can see this by differentiating the left side of (8):

$$G'(h) = \langle \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau \exp\left(h\int_0^\tau ax\,\mathrm{d}\tau\right)\rangle_w$$

and then setting $h = 0$:

$$G'(0) = \langle \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\rangle_w = \langle I\rangle_w$$

Then, differentiating the right side of (8) we have immediately the result:

$$\langle I\rangle_w = G'(h)\Big|_{h=0} = \frac{h}{2} R \exp\left(\frac{h^2 R}{4}\right)\Big|_{h=0} = 0$$

If we differentiate again we get the second moment:

$$G''(h) = \frac{R}{2}\exp\left(\frac{h^2R}{4}\right) + \frac{h^2}{4}R^2\exp\left(\frac{h^2R}{4}\right) \Rightarrow G''(0) = \langle I^2\rangle_w = \frac{R}{2}$$

Consider now a generic odd moment:

$$\langle \left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)^{2k+1}\rangle_w = 0 \qquad \forall k \in \mathbb{N}$$

In fact, if we expand $G(h)$, we get:

$$G(h) = \sum_{n=0}^{\infty}\left(\frac{R}{4}\right)^n \frac{1}{n!}h^{2n}$$

Since all the powers are even, if we differentiate an odd number of times and set $h = 0$ we are "selecting" an odd power - which just is not there - and so the result will be 0.

On the other hand, an even moment leads to:

$$\langle \left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)^{2k}\rangle_w = \left(\frac{R}{2}\right)^2 \frac{(2k)!}{2^k k!}$$

(computations omitted).

0.2 Exponential functional

We consider now the following functional:

$$F[x(\tau)] = \exp\left(-\int_0^t \mathrm{d}\tau\, P(\tau)x^2(\tau)\right)$$

As before, we wish to compute $\langle F\rangle_w$. We start by discretizing the path over a **uniform**[1] grid $0 = t_0 < t_1 < \cdots < t_N = t$ so that $\Delta t_i = t_i - t_{i-1} \equiv \epsilon = t/N$.

$$\begin{aligned} I_4 &\equiv \int_{\mathcal{C}\{0,0;t\}} \mathrm{d}_W x(\tau) \exp\left(-\int_0^t \mathrm{d}\tau\, P(\tau)x^2(\tau)\right) = \lim_{N\to\infty} I_4^{(N)} \\ I_4^{(N)} &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_1}{\sqrt{\pi\epsilon}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_N}{\sqrt{\pi\epsilon}} \exp\left(-\sum_{i=1}^N P_i x_i^2 \epsilon - \sum_{i=1}^N \frac{(x_i - x_{i-1})^2}{\epsilon}\right) \qquad \begin{array}{l} x_i \equiv x(ti) \\ P_i \equiv P(t_i) \end{array} \end{aligned} \tag{9}$$

The exponential argument is a **quadratic form**:

$$\begin{aligned} &-\epsilon(P_1 x_1^2 + \cdots + P_N x_N^2) - \frac{1}{\epsilon}[\cancel{x_0^2} + x_1^2 - \cancel{2x_0}x_1 + x_1^2 + x_2^2 - 2x_1x_2 + \cdots + x_{N-1}^2 + x_N^2 - 2x_{N-1}x_N] = \\ &= -\epsilon \sum_{i=1}^N P_i x_i^2 - \frac{1}{\epsilon}\left[2\sum_{i=1}^{N-1} x_i^2 + x_N^2 - 2\sum_{i=1}^N x_{i-1}x_i\right] = \\ &= -\left[x_1^2\left(\epsilon P_1 + \frac{2}{\epsilon}\right) + \cdots + x_{N-1}^2\left(\epsilon P_{N-1} + \frac{2}{\epsilon}\right) + x_N^2\left(\epsilon P_N + \frac{1}{\epsilon}\right) + \frac{2}{\epsilon}\sum_{i=1}^N x_i x_{i-1}\right] = \\ &= -\sum_{i,j=1}^N A_{ij} x_i x_j \end{aligned}$$

where A_{ij} are matrix elements of a matrix A_N:

$$A_{ij} = \delta_{ij} a_i - \frac{1}{\epsilon}(\delta_{i,j-1} + \delta_{i-1,j}) \qquad a_i = P_i\epsilon + \frac{1}{\epsilon}(2 - \delta_{iN})$$

$$A_N = \begin{pmatrix} a_1 & -1/\epsilon & 0 & \dots & 0 \\ -\epsilon^{-1} & a_2 & -\epsilon^{-1} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\epsilon^{-1} & a_{N-1} & -\epsilon^{-1} \\ 0 & 0 & 0 & -\epsilon^{-1} & a_N \end{pmatrix}$$

We can now rewrite $I_4^{(N)}$ as:

$$I_4^{(N)} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}}\right) e^{-\boldsymbol{x}^T A_N \boldsymbol{x}} \qquad \boldsymbol{x}^T = (x_1, \dots, x_N)$$

[1] ∧The same result can be proved without this assumption, but which a much more heavy notation.

This is the integral of a multivariate gaussian, and evaluates to:

$$I_4^{(N)} = \frac{1}{\epsilon^{N/2}(\det A_N)^{1/2}} = \frac{1}{(\det(\epsilon A_N))^{1/2}}$$

as for a $N \times N$ matrix we have $\det(\epsilon A_N) = \epsilon^N \det A_N$. This has the advantage of removing all denominators in A_N.
To compute this determinant we use a method suggested by Gelfand and Yaglom (1960). We start by denoting with $D_k^{(N)}$ the determinant of the matrix obtained by removing the first $k-1$ rows and columns from ϵA_N:

$$D_k^{(N)} \equiv \begin{vmatrix} \epsilon a_k & -1 & 0 & \dots & 0 \\ -1 & \epsilon a_{k+1} & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & a_{N-1} & -1 \\ 0 & \dots & 0 & -1 & \epsilon a_N \end{vmatrix}$$

So that $D_1^{(N)} = \det \epsilon A_N$ is the determinant we want to compute.
Expanding $D_k^{(N)}$ from the first row we get:

$$D_k^{(N)} = \epsilon a_k D_{k+1} - (-1) \begin{vmatrix} -1 & -1 & 0 & \dots & 0 \\ 0 & \epsilon a_{k+2} & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & a_{N-1} & -1 \\ 0 & 0 & 0 & -1 & \epsilon a_N \end{vmatrix} =$$

$$\underset{(a)}{=} \epsilon a_k D_{k+1}^{(N)} + (-1) D_{k+2}^{(N)} = \epsilon \left(\epsilon P_k + \frac{2}{\epsilon} \right) D_{k+1}^{(N)} - D_{k+2}^{(N)} =$$
$$= (\epsilon^2 P_k + 2) D_{k+1}^{(N)} - D_{k+2}^{(N)}$$

where in (a) we expanded the last determinant following the first column.
Rearranging:

$$\frac{D_k^{(N)} - 2D_{k+1}^{(N)} + D_{k+2}^{(N)}}{\epsilon^2} = P_k D_{k+1}^{(N)} \tag{10}$$

We introduce now the variable $\tau = (k-1)t/N$, representing the *fraction* of removed rows/columns in each determinant, rescaled to the final time t. Performing a continuum limit $N \to \infty$ we can then map $D_k^{(N)} \xrightarrow[N \to \infty]{} D(s)$. Then, the relation (10) becomes a *differential equation*:

$$\frac{\mathrm{d}^2 D(\tau)}{\mathrm{d}\tau^2} = P(\tau) D(\tau) \tag{11}$$

In fact, note that the first term of (10) is a second derivative in the *finite difference* approximation. This can be shown by Taylor expanding a generic function $f(x)$ to get the points immediately before and after:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O((\Delta x)^3)$$
$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O((\Delta x)^3)$$

Summing side by side, and denoting $f(x) \equiv f_i$, $f(x-\Delta x) \equiv f_{i-1}$ and $f(x+\Delta x) = f_{i+1}$:

$$f_{i+1} + f_{i-1} = 2f_i + f_i''(\Delta x)^2 + O((\Delta x)^3)$$

Rearranging, shifting $i \to i+1$ and ignoring the higher order terms leads to:

$$\frac{f_{i+2} - 2f_{i+1} + f_i}{(\Delta x)^2} = f_{i+1}''$$

Analogously, this can be seen by computing the second derivative as *the derivative of the first derivative* in terms of incremental ratios:

$$f''(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x} \right) =$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

Returning to the problem, we note that the determinant of the full matrix, in the continuum limit, is given by:

$$\det(\epsilon A_N) = D_1^{(N)} \xrightarrow[N\to\infty]{} D(0)$$

(as $s = (k-1)t/N\Big|_{k=1} \equiv 0$). So, we just need to solve (11) and evaluate it at $\tau = 0$.
To do this, we first need *two* boundary conditions, as (11) is a second order differential equation.
Noting that $D_N^{(N)}$ is just the last diagonal entry, we have:

$$D_N^{(N)} = \epsilon a_N = \epsilon^2 p_N + 1 \xrightarrow[\substack{N\to\infty \\ \epsilon\to 0}]{} 1$$

As $s = (k-1)t/N\Big|_{k=N} = t$ for $N \to \infty$, this means that:

$$D(t) = 1$$

For the second boundary condition, we search a relation for the first derivative at $\tau = t$:

$$\frac{\mathrm{d}D(\tau)}{\mathrm{d}\tau}\Big|_{\tau=t} = \lim_{N\to\infty} \frac{D_N^{(N)} - D_{N-1}^{(N)}}{\epsilon}$$

$D_{N-1}^{(N)}$ can be computed directly:

$$D_{N-1}^{(N)} = \begin{vmatrix} P_{N-1}\epsilon^2 + 2 & -1 \\ -1 & P_N\epsilon^2 + 1 \end{vmatrix} = P_{N-1}P_N\epsilon^4 + \epsilon^2(P_{N-1} + 2P_N) + 1$$

leading to:

$$\frac{\mathrm{d}D(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=t} = \lim_{\epsilon\to 0} \frac{\epsilon^2 P_N + 1 - P_{N-1}P_N\epsilon^4 - \epsilon^2(P_{N-1} + 2P_N) - 1}{\epsilon} = 0$$

Summarizing, we found that:

$$I_4 \equiv \langle \exp\left(-\int_0^t \mathrm{d}\tau\, P(\tau)x^2(\tau)\right)\rangle_w = \frac{1}{\sqrt{D(0)}}$$

where $D(\tau)$ is the solution of the differential equation:

$$\frac{\mathrm{d}^2 D(\tau)}{\mathrm{d}\tau^2} = P(\tau)D(\tau)$$

with the following boundary conditions:

$$\begin{cases} D(t) = 1 \\ \dot{D}(t) = \frac{\mathrm{d}D(\tau)}{\mathrm{d}\tau}\Big|_{\tau=t} = 0 \end{cases}$$

$\boxed{\textbf{Example 2 } (\boldsymbol{P(\tau) = k^2}\text{, free end-point})\textbf{:}}$

> Let's compute I_4 with the choice of $P(\tau) = k^2$. The differential equation becomes:
>
> $$\frac{\mathrm{d}^2 D(\tau)}{\mathrm{d}\tau^2} = k^2 D(\tau)$$
>
> which is that of a *harmonic repulsor*. The solution is a linear combination of exponentials:
>
> $$D(\tau) = Ae^{k\tau} + Be^{-k\tau} \tag{12}$$
>
> Differentiating:
>
> $$\dot{D}(\tau) = k(Ae^{k\tau} - Be^{-k\tau})$$
>
> We can now impose the boundary conditions:
>
> $$\begin{cases} D(t) \overset{!}{=} 1 = Ae^{kt} + Be^{-kt} & (a) \\ \dot{D}(t) \overset{!}{=} 0 = \cancel{k}(Ae^{kt} - Be^{-kt}) & (b) \end{cases}$$

leading to:

$$\begin{aligned}(a)+(b)\colon\, 2Ae^{kt}=1 &\Rightarrow A=\frac{1}{2}e^{-kt}\\ (a)-(b)\colon\, 2Be^{-kt}=1 &\Rightarrow B=\frac{1}{2}e^{kt}\end{aligned}$$

So the solution is:

$$D(\tau)=\frac{1}{2}\left[e^{k(t-\tau)}+e^{-k(t-\tau)}\right]=\cosh(k(t-\tau)) \tag{13}$$

from which:

$$I_4=\overset{(N)}{\lim_{N\to\infty}}=\frac{1}{\sqrt{D(0)}}=\frac{1}{\sqrt{\cosh(kt)}}$$

0.2.1 Fixed endpoint

We consider now a small variation of I_4, where we integrated on paths with a fixed *end-point* $x(t)\equiv x_t$:

$$\hat{I}_4=\langle\exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)\delta(x\ -x(t)\ )\rangle_w=\int_{\mathcal{C}\{0,0;x_t,t\}}\exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)$$

First, we rewrite the δ in terms of a Fourier transform:

$$I_4'=\int_{-\infty}^{+\infty}\frac{\mathrm{d}\alpha}{2\pi}e^{i\alpha x}\langle\exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)e^{-i\alpha x(t)}\rangle_w$$

Then we discretize the path as before, with $0=t_0<t_1<\cdots<t_N=t$ uniformly distributed ($\Delta t_i=t_i-t_{i-1}\equiv\epsilon=t/N$):

$$\begin{aligned}\hat{I}_4&=\lim_{N\to\infty}\hat{I}_4^{(N)}\\ \hat{I}_4^{(N)}&=\int_{\mathbb{R}}\frac{\mathrm{d}\alpha}{2\pi}e^{i\alpha x}\int_{\mathbb{R}^N}\left(\prod_{i=1}^N\frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}}\right)\exp\left(-\sum_{i=1}^N P_i x_i^2\epsilon-\sum_{i=1}^N\frac{(x_i-x_{i-1})^2}{\epsilon}-i\alpha x_N\right)\end{aligned}$$

where the red terms are the only differences from (9). We can rewrite the quadratic form with the matrix A_N as before:

$$\hat{I}_4^{(N)}=\int_{\mathbb{R}}\frac{\mathrm{d}\alpha}{2\pi}e^{i\alpha x}\int_{\mathbb{R}^N}\left(\prod_{i=1}^N\frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}}\right)\exp(-\boldsymbol{x}^TA_N\boldsymbol{x}-i\alpha x_N)$$

Also, we can express $i\alpha x_N$ as a scalar product:

$$i\alpha x_N=\boldsymbol{h}^T\boldsymbol{x}\qquad h_l=\delta_{lN}(-i\alpha)$$

So that we can now use the gaussian integral:

$$\int_{\mathbb{R}^N}\mathrm{d}^N\boldsymbol{x}\exp\left(-\frac{1}{2}\boldsymbol{x}^TA\boldsymbol{x}+\boldsymbol{b}\cdot\boldsymbol{x}\right)=\exp\left(\frac{1}{2}\boldsymbol{b}\cdot A^{-1}\boldsymbol{b}\right)(2\pi)^{N/2}(\det A)^{-1/2}$$

with $A = 2A_N$ and $\boldsymbol{b} = \boldsymbol{h}$:

$$I' \equiv \frac{1}{\sqrt{(\pi\epsilon)^N}} \int_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{x} \exp\bigl(-\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{h}^T \boldsymbol{x}\bigr) = \frac{1}{\sqrt{(\pi\epsilon)^N}} \exp\left(\frac{1}{4} \boldsymbol{h}^T A^{-1} \boldsymbol{h}\right) (\cancel{2}\pi)^{N/2} (\cancel{2^N} \det A_N)^{-1/2} =$$

$$= \frac{1}{\sqrt{(\cancel{\pi}\epsilon)^N}} \sqrt{\frac{\cancel{\pi^N}}{\det A_N}} \exp\left(\frac{1}{4}(-i\alpha)^2 (A_N^{-1})_{NN}\right) = \underbrace{\sqrt{\frac{1}{\epsilon^N \det A_N}}}_{I_0} \exp\left(-\frac{1}{4}\alpha^2 (A_N^{-1})_{NN}\right)$$

where $(A_N^{-1})_{NN}$ is the last diagonal element of the inverse matrix of A_N. Substituting back:

$$\hat{I}_4^{(N)} = I_0 \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \exp\left(i\alpha x - \frac{1}{4}\alpha^2 (A_N^{-1})_{NN}\right)$$

which is again a gaussian integral, and following formula (6) with $a = (A_N^{-1})_{NN}/4$ and $b = -x$ leads to:

$$\hat{I}_4^{(N)} = \frac{I_0}{\cancel{2}\pi} \sqrt{\frac{\cancel{4}\pi}{(A_N^{-1})_{NN}}} \exp\left(-\frac{x^2}{(A_N^{-1})_{NN}}\right) = \frac{I_0}{\sqrt{\pi (A_N^{-1})_{NN}}} \exp\left(-\frac{x^2}{(A_N^{-1})_{NN}}\right) \quad (14)$$

All that's left is to compute $(A_N^{-1})_{NN}$ and take the continuum limit. Recall from linear algebra that:

$$A^{-1} = \frac{1}{\det A} C_{ji}$$

where C_{ij} are the *cofactors* of A, i.e. the determinants of the matrices obtained from A by removing the i-th row and j-th column. In our case:

$$(A_N^{-1})_{NN} = \frac{C_{NN}}{\det A_N}$$

Before, we obtained $\det A_N$ by means of $D_k^{(N)}$, i.e. the determinants of the matrices obtained by removing the first $k-1$ rows and columns, so that $D_1^{(N)} = \epsilon^N \det A_N$. This leads to:

$$(A_N^{-1})_{NN} = \frac{\epsilon^N}{D_1^{(N)}} C_{NN}$$

For C_{NN} we have to compute the determinant of the $(N-1) \times (N-1)$ matrix $A_*^{(N-1)}$, obtained by removing the last row and column from A_N. Note that $A_*^{(N-1)} \neq A^{(N-1)}$, as they differ for the *last diagonal element* which is:

$$(A_*^{(N-1)})_{N-1,N-1} = P_{N-1}\epsilon + \frac{2}{\epsilon} \neq (A_{N-1,N-1}^{(N-1)}) = P_{N-1}\epsilon + \frac{1}{\epsilon} \quad (15)$$

We proceed in a similar manner, defining $\hat{D}_k^{(N-1)}$ to be the determinant of the matrix obtained by removing the first $k-1$ rows and columns from $\epsilon A_*^{(N-1)}$ (again, we multiply by ϵ to remove denominators):

$$\hat{D}_k^{(N-1)} = \begin{vmatrix} \epsilon a_k & -1 & 0 & \dots & 0 \\ -1 & \epsilon a_{k-1} & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & \epsilon a_{N-2} & -1 \\ 0 & \dots & \dots & -1 & \epsilon a_{N-1} \end{vmatrix}$$

So $\hat{D}_1^{(N-1)} = \epsilon^{N-1} \det A_*^{(N-1)} = \epsilon^{N-1} C_{NN}$ leading to:

$$(A_N^{-1})_{NN} = \frac{\epsilon^N}{D_1^{(N)}} \frac{1}{\epsilon^{N-1}} \hat{D}_1^{(N-1)} = \epsilon \frac{\hat{D}_1^{(N-1)}}{D_1^{(N)}}$$

For simplicity, it is convenient to define $\tilde{D}_1^{(N-1)} \equiv \epsilon \hat{D}_1^{(N-1)}$, so that:

$$(A_N^{-1})_{NN} = \frac{\tilde{D}_1^{(N-1)}}{D_1^{(N)}} \tag{16}$$

Repeating the steps for the continuum limit, we get the same differential equation for $\tilde{D}(\tau)$:

$$\partial_\tau^2 \tilde{D}(\tau) = P(\tau) D(\tau)$$

However, due to (15), the boundary conditions are now different:

$$\tilde{D}_{N-1}^{(N-1)} = \epsilon(\epsilon^2 P_{N-1} + 2) = P_{N-1}\epsilon^3 + 2\epsilon \xrightarrow[\epsilon \to 0]{} 0 = \tilde{D}(t)$$

$$\tilde{D}_{N-2}^{(N-1)} = \epsilon \begin{vmatrix} P_{N-2}\epsilon^2 + 2 & -1 \\ -1 & P_{N-1}\epsilon^2 + 2 \end{vmatrix} = \epsilon(P_{N-1}P_{N-2}\epsilon^4 + 2(p_{N-1} + P_{N-2})\epsilon^2 + 3)$$

$$\frac{\tilde{D}_{N-1}^{(N-1)} - \tilde{D}_{N-2}^{(N-1)}}{\epsilon} = -1 + O(\epsilon^2) \xrightarrow[\epsilon \to 0]{} -1 = \left.\frac{\mathrm{d}\tilde{D}(\tau)}{\mathrm{d}\tau}\right|_{\tau=t}$$

Then, substituting (16) in (14) we get:

$$\hat{I}_4^{(N)} = \frac{I_0}{\sqrt{\pi (A_N^{-1})_{NN}}} \exp\left(-x^2 \frac{D_1^{(N)}}{\tilde{D}_1^{(N-1)}}\right) \qquad I_0 = \frac{1}{\sqrt{\epsilon^N \det A_N}} = \frac{1}{\sqrt{D_1^{(N)}}}$$

$$I_4 = \lim_{N \to \infty} \hat{I}_4^{(N)} = \frac{1}{\sqrt{\pi \tilde{D}(0)}} \exp\left(-x^2 \frac{D(0)}{\tilde{D}(0)}\right) \tag{17}$$

Where $D(\tau)$ and $\tilde{D}(\tau)$ are solutions of the following differential equations with the following boundary conditions:

$$\tilde{D}''(\tau) = P(\tau)\tilde{D}(\tau) \qquad \begin{cases} \tilde{D}(t) = 0 \\ \left.\frac{\mathrm{d}\tilde{D}(\tau)}{\mathrm{d}\tau}\right|_{\tau=t} = -1 \end{cases}$$

$$D''(\tau) = P(\tau)D(\tau) \qquad \begin{cases} D(t) = 1 \\ \left.\frac{\mathrm{d}D(\tau)}{\mathrm{d}\tau}\right|_{\tau=t} = 0 \end{cases}$$

Example 3 ($\boldsymbol{P(\tau) = k^2}$ with fixed end-point):

Let $P(\tau) = k^2$, with $k \in \mathbb{R}$ constant. We already solved the equation for $D(\tau)$ with the right boundary conditions in (13):

$$D(\tau) = \cosh(k(t-\tau))$$

For $\tilde{D}(\tau)$ we start from the general integral (12) and impose the appropriate boundary conditions:

$$\begin{cases} \tilde{D}(t) = \tilde{A}e^{kt} + \tilde{B}e^{-kt} = 0 & (a) \\ \frac{\mathrm{d}\tilde{D}(\tau)}{\mathrm{d}\tau}\Big|_{\tau=t} = k(\tilde{A}e^{kt} - \tilde{B}e^{-kt}) = -1 & (b) \end{cases}$$

so that:

$$\begin{aligned} k(a) + (b) &: 2\tilde{A}ke^{kt} = -1 \Rightarrow \tilde{A} = -\frac{1}{2k}e^{-kt} \\ k(a) - (b) &: 2Bke^{-kt} = 1 \Rightarrow \tilde{B} = \frac{1}{2k}e^{kt} \end{aligned}$$

leading to the solution:

$$\tilde{D}(\tau) = \frac{1}{2k}(e^{k(t-\tau)} - e^{-k(t-\tau)}) = \frac{1}{k}\sinh(k(t-\tau))$$

Finally, using the result we found in (17):

$$\begin{aligned} \langle \exp\left(-k^2 \int_0^t x^2(\tau)\,\mathrm{d}\tau\,\delta(x - x(t))\right)\rangle_w &= \int_{\mathcal{C}\{0,0;x_t,t\}} \exp\left(-k^2 \int_0^t x^2(\tau)\,\mathrm{d}\tau\right) \mathrm{d}_W x(\tau) = \\ &= \sqrt{\frac{k}{\sinh(kt)}} \exp\bigl(-kx_t^2 \coth(kt)\bigr) \end{aligned}$$