(Lesson 4b of 21/10/19) Compiled: December 2, 2019

0.1 Wiener's integral

Consider an *unconstrained* Brownian particle, moving on the real line, starting in x_0 at t_0. By solving the diffusion equation we found that the probability of finding the particle in $[x, x+\mathrm{d}x]$ at time $t > t_0$ is given by the **propagator**:

$$\mathbb{P}\{x(t) \in [x, x+\mathrm{d}x] | x(t_0) = x_0\} = W(x,t|x_0,t_0)\,\mathrm{d}x = \\ = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp\left(-\frac{(x-x_0)^2}{4D(t-t_0)}\right)\mathrm{d}x \qquad (1)$$

By integrating (1) we can then find the probability of finding the particle inside an interval $[A, B]$ at time t:

$$\mathbb{P}\{x(t) \in [A,B] | x(t_0) = x_0\} = \int_A^B \mathrm{d}x\, W(x,t|x_0,t_0) \qquad t > t_0$$

We are now interested in computing the expected value $\langle f \rangle$ of **functionals** f of the trajectory, i.e. of quantities depending on several (or all) points of the trajectory $x(\tau)$ of a Brownian particle.

- The simplest example is the **correlation function**, which is defined as the product of the particle's position at two different times $t_1 < t_2$:

$$f(\{x(t_1), x(t_2)\}) = x(t_1)x(t_2) \quad t_1 < t_2$$

- A more general (and difficult) case is given by a function of the *entire* trajectory, such as:

$$f(\{x(\tau)\colon 0 < \tau \leq \tau\}) = g\left(\int_0^t x(\tau)a(\tau)\,\mathrm{d}\tau\right) \qquad a,\ g\colon \mathbb{R} \to \mathbb{R}$$

0.1.1 Functions of a discrete number of points

Let's start from the simplest case, and consider the **correlation function**:

$$f(\{x(t_1), x(t_2)\}) = x(t_1)x(t_2) \qquad t_1 < t_2$$

To compute $\langle f \rangle$ we will need the *joint probability distribution* $g(x_1, x_2)$ that gives the probability of $x(t_1)$ being "close to" x_1 **and** $x(t_2)$ "close to" x_2 *for the same trajectory*. Let us denote the three events of interest:

A: Particle starts in x_0 at t_0

B: Particle is close to x_1 at t_1 $(x(t_1) \in [x_1, x_1 + \mathrm{d}x_1])$

C: Particle is close to x_2 at t_2 $(x(t_2) \in [x_2, x_2 + \mathrm{d}x_2])$

We are interested in the joint probability $\mathbb{P}(C,B|A)$ (the order is defined by $t_2 > t_1 > t_0$). From probability theory:

$$\mathbb{P}(C,B|A) = \mathbb{P}(C|B,A)\mathbb{P}(B|A)$$

We already now how to compute probabilities like $\mathbb{P}(B|A)$, but not like $\mathbb{P}(C|B,A)$. Fortunately, that is not needed.
Recall, in fact, that Brownian motion is a *Markovian process*, meaning that the *future* depends only on the *present state*, i.e. the particle *has no memory*. So, **subsequent displacements are independent**: the probability of the particle going from x_1 to x_2 is the same whether it has started at x_0 or at any other point $\tilde{x}_0$. In other words, if we take the *present state* as the particle being in x_1 at t_1, the future (position at $t_2 > t_1$) depends only on that, and not on the past (position at t_0). So:

$$\mathbb{P}(C|B,A) = \mathbb{P}(C|B)$$

leading to:

$$\mathbb{P}(C,B|A) = \mathbb{P}(C|B)\mathbb{P}(B|A)$$

Inserting the *propagators* (1):

$$\mathrm{d}\mathbb{P}_{t_1,t_2}(x_1,x_2) \equiv W(x_2,t_2|x_1,t_1)W(x_1,t_1|x_0,t_0)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

This is the joint probability we need to compute $\langle f \rangle$. Of course, nothing stops us from considering N "jumps" instead of only 2:

$$\begin{aligned}\mathrm{d}\mathbb{P}_{t_1,\dots,t_n}(x_1,\dots,x_n|x_0,t_0) &\equiv W(x_n,t_n|x_{n-1},t_{n-1})\dots W(x_1,t_1|x_0,t_0)\,\mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_n = \\ &= \exp\left(-\sum_{i=1}^n \frac{(x_i-x_{i-1})^2}{4\pi D\Delta t_i}\right)\prod_{i=1}^n \frac{\mathrm{d}x_i}{\sqrt{4\pi D\Delta t_i}} \qquad (2)\end{aligned}$$

Then, the average of a generic function $f(x(t_1),\dots,x(t_n))$ of the positions of the particle at times $t_1 < t_2 < \dots < t_n$ is defined as:

$$\langle f(x(t_1),\dots,x(t_n))\rangle_w = \int_{\mathbb{R}^n} f(x_1,\dots,x_n)\,\mathrm{d}\mathbb{P}_{t_1,\dots,t_n}(x_1,\dots,x_n|x_0,t_0)$$

0.1.2 Functionals of the whole trajectory

Note that (**??**) can be interpreted as the *infinitesimal volume element* spanned by all the *trajectories* passing through *tiny gates*,
As always, we start from the simplest case, and then work our way up to the most complex one. So, let's start with the two points case: $f(x(t_1),x(t_2))$. For the average $\langle f \rangle$ we need to *weight* every possible value of f with the probability that f assumes that value, which will depend on the likelihood of the inputs $x(t_1)$ and $x(t_2)$. Thus, this becomes a problem of computing *probabilities of compound events* - that is, of the particle passing through a specific sets of points at certain times. As $x(\tau) \in \mathbb{R}$, any kind of $\mathbb{P}\{x(t_1) = x_1, x(t_2) \in x_2\}$ for certain t_1, t_2 and x_1, x_2 will

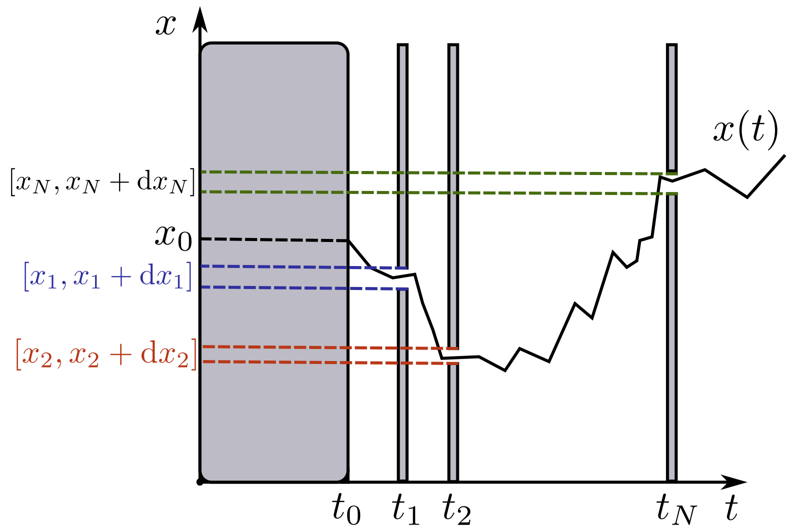

Figure (1)

be 0. We need, in general, to consider instead a *range* of possibilities, i.e. that the particle passes through a set of **gates**, so that $x(t_1) \in [A_1, B_1]$ and $x(t_2) \in [A_2, B_2]$.

In general, if we consider N gates $[A_i, B_i]_{i=1,\dots,N}$ the probability of a particle passing through all of them will be:

$$\begin{aligned}
&\mathbb{P}\{x(t_1) \in [A_1, B_1], x(t_2) \in [A_2, B_2], \dots, x(t_N) \in [A_N, B_N]\} = \\
&= \int_{A_1}^{B_1} \mathrm{d}x_1\, W(x_1, t_1 | x_0, t_0) \int_{A_2}^{B_2} \mathrm{d}x_2\, W(x_2, t_2 | x_1, t_1) \cdots \int_{A_N}^{B_N} \mathrm{d}x_N\, W(x_N, t_N | x_{N-1}, t_{N-1}) = \\
&= \int_{A_1}^{B_1} \frac{\mathrm{d}x_1}{\sqrt{4\pi D(t_1 - t_0)}} \exp\left(-\frac{(x_1 - x_0)^2}{4D(t_1 - t_0)}\right) \cdots \int_{A_N}^{B_N} \frac{\mathrm{d}x_N}{\sqrt{4\pi D(t_N - t_{N-1})}} \exp\left(-\frac{(x_N - x_{N-1})^2}{4D(t_N - t_{N-1})}\right)
\end{aligned}$$

This is because the events of passing through two different gates are always *independent*: the transition probability between two gates depends only on their distance, and not on the *history* of the particle[1].

However, for computing expected values of functions we are interested in *tiny gates*, so that the value of f at a gate is well defined (otherwise we would not know which value of f we are *weighting* with the trajectories probability). So, we diminish the size of gates, and instead of integrating the transition probabilities over sets $[A_i, B_i]$, we consider just their differentials:

$$W(x_t, t | x_0, t_0)\, \mathrm{d}x_t \equiv \mathbb{P}\{x(t) \in [x_t, x_t + \mathrm{d}x_t\,, x(t_0) = x_0]\}$$

So, we can now compute the (infinitesimal) probability that a Brownian particle will be very close to x_1 at $t = t_1$, and to x_2 at $t = t_2$:

$$\begin{aligned}
&\mathbb{P}\{x(t_1) \in [x_1, x_1 + \mathrm{d}x_1], x(t_2) \in [x_2, x_2 + \mathrm{d}x_2]\} = \\
&= W(x_2, t_2 | x_1, t_1) W(x_1, t_1 | x_0, t_0)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&\equiv dP_{t_2, t_1}(x_2, x_1 | x_0, t_0)
\end{aligned}$$

[1] ∧For example, the fact that a particle has travelled to the right for $0 < t < t_1$ tells nothing on the motion after t_1.

And then we can compute the expected value of f:

$$\langle f(x(t_1), x(t_2))\rangle = \iint_{\mathbb{R}^2} x_1 x_2 f(x_1, x_2) dP_{t_2,t_1}(x_2, x_1|x_0, t_0)$$

$$\begin{aligned}\langle x(t_1)x(t_2)\rangle &= \iint_{\mathbb{R}^2} x_1 x_2 dP_{t_2,t_1}(x_2, x_1|x_0, t_0) = \\ &= \int_{\mathcal{C}\{x_0,t_0;t\}} x(t_1)x(t_2)\, \mathrm{d}_W x(\tau) = \\ &= \iint_{\mathbb{R}^2} \mathrm{d}x_1\, \mathrm{d}x_2\, x_1 x_2 \int_{\mathcal{C}\{x_0,t_0;x_1,t_1\}} \mathrm{d}_W x(\tau) \int_{\mathcal{C}\{x_1,t_1;x_2,t_2\}} \mathrm{d}_W x(\tau) \int_{\mathcal{C}\{x_2,t_2;t\}} \mathrm{d}_W x(\tau)\end{aligned}$$

In a certain (probabilistic) sense, dP_{t_2,t_1} measures the *volume* of all trajectories passing "really close" to x_1 at t_1 and x_2 at t_2. The power of this idea becomes clear when we extend the number of gates N to infinity, while decreasing the interval $\Delta t_i = t_i - t_{i-1}$ between them:

$$\begin{aligned}\lim_{\substack{\Delta t_i \to 0\\ N\to\infty}} \mathbb{P}\{x(t_1) \in \mathrm{d}x_1\,, \dots, x(t_N) \in \mathrm{d}x_N\} &= \lim_{\substack{\Delta t_i \to 0\\ N\to\infty}} \exp\left(-\sum_{i=1}^N \frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})}\right) \prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{4\pi D(t_i - t_{i-1})}} = \\ &= \lim_{\substack{\Delta t_i \to 0\\ N\to\infty}} \exp\left(-\frac{1}{4D} \sum_{i=1}^N \frac{(x_i - x_{i-1})^2}{(t_i - t_{i-1})^{2}} \Delta t_i\right) \prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{4\pi D \Delta t_i}} = \\ &\underset{(a)}{=} \exp\left(-\frac{1}{4D}\int_0^t \mathrm{d}\tau\, \dot{x}^2(\tau)\right) \prod_{\tau=0}^t \frac{\mathrm{d}x\,(\tau)}{\sqrt{4\pi D\,\mathrm{d}\tau}}\end{aligned}$$

where in (a) we replaced the infinite "dense" sum with a formal integral (Riemann sum) of $(\mathrm{d}x\,/\,\mathrm{d}t)^2 = \dot{x}^2(\tau)$.

0.2 Notes 1

Recall that $W(x,t)\,\mathrm{d}x$ is the probability of finding the Brownian particle in the interval $[x, x + \mathrm{d}x]$ at time t.
Then, letting the initial condition be $W(x, t_0|x_0, t_0) = \delta(x - x_0)$ (particle located in x_0 at t_0), the following holds (prove it explicitly as exercise):

$$\begin{aligned}&\int \mathrm{d}x'\, W(x,t|x',t')W(x',t'|x_0,t_0) = W(x,t|x_0,t_0) = \\ &= \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp\left(-\frac{(x-x_0)^2}{4D(t-t_0)}\right)\end{aligned}$$

Define:

$$dP_{t,t'}(x, x'|x_0, t_0) = W(x,t|x',t')W(x',t'|x_0,t_0)\,\mathrm{d}x\,\mathrm{d}x'$$

with $t_0 < t' < t$ as the probability of finding a particle in $[x, x+\mathrm{d}x]$ at time t, and then in $[x', x' + \mathrm{d}x']$ at time t'. Then:

$$\langle x'(t)x(t)\rangle = \int dP_{t,t'}(x, x'|x_0, t_0) x_0 x'$$

Consider a function g of n points of the trajectory, sampled at times $t_1, t_2, \ldots, t_n$:

$$g(x(t_1), x(t_2), \ldots, x(t_n))$$

To compute $\langle g \rangle$, we need to extend the joint pdf:

$$dP_{t_n, t_{n-1}, \ldots, t_1}(x_n, \ldots, x_1, x_0, t_0) \equiv W(x_n, t_n | x_{n-1} t_{n-1}) \cdots W(x_1, t_1 | x_0, t_0) \prod_{i=1}^{n} \mathrm{d}x_i \tag{3}$$

leading to:

$$\langle g(x(t_1), x(t_2), \ldots, x(t_n)) \rangle = \int dP_{t_n, t_{n-1}, \ldots, t_2, t_1}(x_n, \ldots, x_1 | x_0, t_0) g(x_1, \ldots, x_n)$$

Expanding (2):

$$dP_{t_n, t_{n-1}, \ldots, t_2, t_1} = \int \prod_{i=1}^{n} \frac{\mathrm{d}x_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(- \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{4 D \Delta t_i} \right)$$

Consider now a function of the *whole* trajectory:

$$F(\{x(\tau) \colon 0 < \tau \leq t\})$$

For example:

$$F = f\left(\int_0^t x(\tau) a(\tau) \, \mathrm{d}\tau \right)$$

with a given function $a(\tau)$, such as $a(\tau) = 1$ or $a(\tau) = e^{-\tau/\tau_0}$. To compute the average of F we introduce the Wiener measure $d_W x$, i.e. a generalization of (2) to the continuum, so that:

$$\langle F \rangle = \int d_W x F(\{x(\tau) \colon 0 < \tau \leq \tau\})$$

and the integral is over a *space of trajectories* $x \colon T \to R$, with $R \subseteq \mathbb{R}$, denoted with $\mathbb{R}^T$ (generalizing the common notation). For example: $T = [0, \infty]$.

We have to define a sigma algebra in this space $\mathbb{R}^T$ in order to define a measure $d_W x$, i.e. a domain of measurable sets for which a probability measure makes sense.

We start by defining a set of intervals $H_i \subset \mathbb{R}$ (for example $H_i = (x_i, x_i + \Delta x_i)$). We then consider the set of functions having values inside these H_i: $\mathbb{R}^T \colon \{x(t_i) \in H_i\}_{i=1,\ldots,n}$. In other words, this is the set of trajectories that pass through each H_i at instant t_i. Then we define the measure:

$$\begin{aligned} &\mu_W(\{x(t_1) \in H_1, x(t_2) \in H_2, \ldots, x(t_n) \in H_n\}) = \\ &= \int dP_{t_n, \ldots, t_1}(x_n, \ldots, x_1 | x_0, t_0) \mathbb{I}_{H_1}(x_1) \cdots \mathbb{I}_{H_n}(x_n) \end{aligned}$$

where $\mathbb{I}_{H_i}$ are *characteristic sets*:

$$\mathbb{I}_{H_i}(x) = \begin{cases} 1 & x \in H_i \\ 0 & x \notin H_i \end{cases}$$

Thanks to the **Kolomogorov theorem** we can *extend* this measure, defined in the "tube that passes through all gates" $\{x(t_i) \in H_i\}$ to the entire $\mathbb{R}^T$.
Knowing that this measure exists in the *continuous case*, we can give meaning to a *continuum limit* of the *discrete case*. More precisely, in order to compute a function of the entire trajectory:

$$F(\{x(\tau)\colon 0 < \tau < t\})$$

we start with a discretization $t_1 < t_2 < \cdots < t_N < t \equiv t_{N+1}$, evaluate a *discretized* function $F_N(x(t_1), \ldots, x(t_N))$ and then consider the limit $N \to \infty$:

$$\lim_{N\to\infty} \langle F_N(x(t_1), \ldots, x(t_N)) \rangle$$

meaning that $\Delta t_i \to 0$, where $\Delta t_i = t_i - t_{i-1}$. We know how to compute the average of a function that depends on a *finite* set of trajectory points:

$$\lim_{N\to\infty} \int \prod_{i=1}^{N+1} \frac{\mathrm{d}x_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\sum_{i=1}^{N+1} \frac{(x_i - x_{i-1})^2}{4D\Delta t_i}\right) F_N(x_1, \ldots, x_N)$$

The normalization condition:

$$\begin{aligned} 1 &= \int dP_{t_1,\ldots,t_N}(x_1, \ldots, x_N | x_0, t_0) = \\ &= \int \prod_{i=1}^{N+1} \frac{\mathrm{d}x_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D\Delta t_i}\right) = \\ &= \int \frac{\mathrm{d}x_N}{(4\pi D \Delta t_N)} \exp\left(-\frac{(x_N - x_{N-1})^2}{4D\Delta t_N}\right) \end{aligned}$$

Example 1 (Correlator function)**:**

$$\langle x(t_1')x(t_2')\rangle = \int \mathrm{d}x_1'\,\mathrm{d}x_2'\, W(x_2', t_2' | x_1', t_1') W(x_1', t_1' | 0, 0) x_1' x_2'$$

The same result can be obtained by using Wiener's measure:

$$\begin{aligned} \langle x(t_1')x(t_2')\rangle &= \int \mathrm{d}_W x\, x_1(t_1')x_2(t_2') = \int d\mathbb{P}_{t_1,\ldots,t_N}(x_1, \ldots, x_N | 0, 0) x(t_k) x(t_n) = \\ &= \int \mathrm{d}x_1 \ldots \mathrm{d}x_N\, W(x_N, t_N | x_{N-1}, t_{N-1}) \ldots W(x_1, t_1 | 0, 0) x_k x_n = \\ &= \int \mathrm{d}x_k\, \mathrm{d}x_n\, W(x_n, t_n | x_k | t_k) W(x_k, t_k | 0, 0) x_k x_n \end{aligned}$$

Note that:

$$W(x, t|0, 0) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right) = \int \mathrm{d}_W x\, \delta(x(t) - x)$$

0.3 Change of random variables

Consider a random variable $X \sim q(x)$, with $q(x)$ being a generic distribution (e.g. $q(x) = \mu e^{-\mu x}$). Now consider a function $y(x)$, e.g. $y(x) = x^2$. Y is then a new random variable, with a certain distribution $p(y)$. We now want to compute $p(y)$ starting from $q(x)$ and $y(x)$.
Suppose that $y(x)$ is invertible. Then, if we extract a value from X, it will be inside $[x, x + \mathrm{d}x]$ with a probability $q(x)\,\mathrm{d}x$. Knowing X, we can use $y(x)$ to uniquely determine Y, that will be in $[y, y + \mathrm{d}y]$ with the same probability. So, the following holds:

$$q(x)\,\mathrm{d}x = p(y)\,\mathrm{d}y \tag{4}$$

We can compute $\mathrm{d}y$ by *nudging* $y(x)$, and expanding in Taylor series:

$$y(x + \mathrm{d}x) \equiv y + \mathrm{d}y + O(\mathrm{d}y^2) = y(x) + \underbrace{\mathrm{d}x\, y'(x)}_{\mathrm{d}y} + O(\mathrm{d}x^2)$$

and so $\mathrm{d}y = \mathrm{d}x\, y'(x)$. Substituting in (3) we get:

$$q(x)\,\mathrm{d}x = p(y)\,\mathrm{d}y = p(y(x))y'(x)\,\mathrm{d}x \Rightarrow p(y) = q(x(y))\frac{\mathrm{d}x}{\mathrm{d}y}(x(y))$$

For a more general $y(x)$, $x \sim q(x)$ and $y = y(x)$, the expected value of a function f is:

$$\begin{aligned}
\langle f(y)\rangle &= \int \mathrm{d}x\, f(y(x))q(x) = \int \mathrm{d}y\, f(y)p(y) = \\
&= \int \mathrm{d}x\, f(y(x))q(x) \underbrace{\int \mathrm{d}z\, \delta(z - y(x))}_{=1} = \\
&= \int \mathrm{d}z\, f(z) \underbrace{\int \mathrm{d}x\, q(x)\delta(z - y(x))}_{\langle \delta(z - y(x))\rangle}
\end{aligned}$$

and so:

$$p(z) = \int \mathrm{d}x\, q(x)\delta(z - y(x)) = \langle \delta(z - y(x))\rangle$$

which, in general, is not:

$$p(z) \neq q(x(z))\frac{\mathrm{d}x(z)}{\mathrm{d}z}$$

However, if $y(x)$ is invertible, i.e. $\mathrm{sgn} y'(x) = \text{constant}$ ($y'(x) \neq 0$ always), then:

$$\delta(z - y(x)) = \frac{\delta(x - x(z))}{|y'(x)|}$$

leading to:

$$p(z) = \langle \frac{\delta(x - x(z))}{|y'(x)|}\rangle_q = \int \mathrm{d}x\, q(x)\frac{\delta(x - x(z))}{|y'(x)|} = q(x(z))|y'(x(z))|^{-1} = q(x(z))\frac{\mathrm{d}x}{\mathrm{d}y}$$

and we recover the previous formula.

0.4 The 1st integral

So we can now write:

$$W(x,t|0,0) = \int \mathrm{d}_W x\, \delta(x(t)-x) =$$
$$= \text{”} \lim_{N\to\infty} \text{”} \int \prod_{i=1}^{N+1} \frac{\mathrm{d}x_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(- \sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D\Delta t_i} \right) \delta(x_{N+1}-x)$$

where $t_n = t$, $x(t_n) = x_{N+1}$.

Recall that:

$$W(x_t,t|0,0) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x_t^2}{4Dt} \right)$$

If we set $x_t = 0$ (for simplicity), we get:

$$W(0,t|0,0) = \frac{1}{\sqrt{4\pi D t}} \tag{5}$$

As an exercise to get some familiarity with Wiener integrals, we want now to *re-derive* this result, by instead solving a *path integral*:

$$W(0,t|0,0) = \int_{\mathcal{C}\{0,0;0,t\}} \mathrm{d}_W x(\tau) = \langle \delta(0-x) \rangle = \langle \delta(x) \rangle \equiv I_1???$$

Here we are computing the *fraction* of Brownian paths that start in $x=0$ at $t=0$ and return in $x=0$ after a fixed interval t.
The standard way to compute a Wiener integral is to *discretize* it, and then take a *continuum* limit. So, consider a uniform time discretization $\{t_i\}_{i=1,\dots,N+1}$, ϵ apart from each other, so that:

$$t_i - t_{i-1} = \epsilon = \frac{t}{N+1} \quad \forall i = 1,\dots,N+1$$

Note that the end-points are $x_0 = x_{N+1} = 0$.
Then:

$$I_1 \equiv \lim_{\substack{\epsilon\to 0\\ N\to\infty}} I_1^{(N)} \tag{6}$$

$$I_1^{(N)} \equiv \frac{1}{(\sqrt{4\pi D\epsilon})^{N+1}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_{-\infty}^{+\infty} \mathrm{d}x_2 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_N \exp\left(-\frac{1}{4D\epsilon} \sum_{i=0}^{N} (x_{i+1}-x_i)^2 \right) \tag{7}$$

Let's focus on the summation in the exponential:

$$\sum_{i=0}^{N}(x_{i+1}-x_i)^2 = x_1^2 + \cancel{x_0}^2 - 2x_0x_1 + x_2^2 + x_1^2 - 2x_1x_2 + \cdots + \cancel{x_{N+1}}^2 + x_N^2 - 2x_N x_{N+1} =$$
$$= 2(x_1^2 + \cdots + x_N^2) - 2(x_1x_2 + x_2x_3 + \cdots + x_{N-1}x_N) =$$
$$= 2\left(\sum_{i=1}^{N} x_i^2 \right) - 2\left(\sum_{i=1}^{N-1} x_i x_{i+1} \right)$$

This is a *quadratic form*, i.e. a polynomial with all terms of order 2. So, it can be written in *matrix form*:

$$= \sum_{k,l=1}^{N} x_k A_{kl} x_l = \boldsymbol{x}^T A_N \boldsymbol{x}$$

for an appropriate choice of entries A_{kl} of the $N \times N$ matrix A_N:

$$A_{kk} = 2;\; A_{kl} = -(\delta k, l+1 + \delta_{k+1,l}) \Rightarrow A_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Substituting back in (6):

$$I_1^{(N)} = \frac{1}{(\sqrt{4\pi D\epsilon})^{N+1}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_N \exp\left(-\frac{\boldsymbol{x}^T A_N \boldsymbol{x}}{4D\epsilon}\right)$$

Recall the multivariate Gaussian integral:

$$\int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_N \exp\left(-\sum_{ij}^{N} B_{ij} x_i x_j\right) = \frac{(\sqrt{\pi})^N}{\sqrt{\det B}}$$

leading to:

$$I_1^{(N)} = \frac{1}{(\sqrt{4\pi D\epsilon})^{N+1}} \sqrt{\frac{\pi^N}{\det\left(A_N \left[\frac{1}{4D\epsilon}\right]^N\right)}} \underset{(a)}{=} \frac{1}{(\sqrt{4\pi D\epsilon})^{N+1}} \frac{\sqrt{4\pi D\epsilon}^N}{\sqrt{\det A_N}} = \frac{1}{\sqrt{4\pi D\epsilon}} \frac{1}{\sqrt{\det A_N}} \tag{8}$$

where in (a) we used the property of the determinant $\det(cA) = c^n \det(A)\, \forall c \in \mathbb{R}$. Now, all that's left is to compute the determinant of A_N. Fortunately, as A_N is a *tri-diagonal* matrix, there is a recurrence relation in terms of the leading principal minors of A_N, which turns out to be multiples of the determinants of A_{N-1} and A_{N-2}.
Explicitly, consider A_N:

$$\det A_N \equiv \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & \color{purple}{-1} \\ 0 & \cdots & 0 & -1 & \color{red}{2} \end{vmatrix}_{N\times N}$$

and start computing the determinant following the last column. The only non-zero contributions are:

$$\det A_N = \underbrace{(-1)^{(N-1)+N}(-1)}_{+1} \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 \end{vmatrix}_{(N-1)\times(N-1)} + (-1)^{2N}(2) \det A_{N-1} =$$

$$= (-1)^{2(N-1)}(-1) \det A_{N-2} + 2 \det A_{N-1} = 2 \det A_{N-1} - \det A_{N-2} \tag{9}$$

where the terms in blue are just the alternating signs from the determinant expansion, and the other colours identify the matrix entries that are being used. Then, it is just a matter of computing the first two terms of the succession ($|A_N| = \det A_N$ for brevity):

$$|A_1| = 2 \qquad |A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

And now we can use (8) to iteratively compute all $|A_N|$, e.g. $|A_3| = 2 \cdot 3 - 2 = 4$. To find $|A_N|$ for a *generic* N, we need to make an hypothesis, and then verify that it is compatible with (8). In this case, note that $|A_N| = N + 1$ $(*)$ for all the examples we explicitly computed. Then, by induction:

$$|A_{N+1}| \underset{(8)}{=} 2 \cdot |A_N| - |A_{N-1}| \underset{(*)}{=} 2 \cdot (N+1) - (N - 1 + 1) = 2N + 2 - N = (N+1) + 1$$

which is indeed compatible with $(*)$. So, substituting back in (7) we get:

$$I_1^{(N)} = \frac{1}{\sqrt{4\pi D\epsilon}} \frac{1}{\sqrt{N+1}} \underset{(a)}{=} \frac{1}{\sqrt{4\pi Dt}}$$

where in (a) we used $\epsilon = t/(N+1) \Rightarrow N + 1 = t/\epsilon$ from the discretization. Note that this result is *constant* with respect to ϵ or N (recall that t is fixed beforehand) and so taking the *continuum* limit leads immediately to I_1 (5):

$$I_1 \equiv \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \frac{1}{\sqrt{4\pi Dt}} = \frac{1}{\sqrt{4\pi Dt}}$$

which is coherent with the result we previously computed (4).