0.1 The 3rd integral

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We considered the following integral:

$$\langle F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)\rangle$$

To solve this, we need to discretize the path, and then consider the continuum limit.
We start by applying the transformation:

$$A(\tau) = \int_\tau^t a(\tau')\,\mathrm{d}\tau'$$

Note that $\partial_\tau A(\tau) = -a(\tau)$, and that $A(t) = 0$. The integral becomes:

$$\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau = -\int_0^t \partial_\tau A(\tau)x(\tau)\,\mathrm{d}\tau$$

Integrating by parts we get:

$$= -A(\tau)x(\tau)\Big|_0^t + \int_0^t A(\tau)\dot{x}(\tau)\,\mathrm{d}\tau$$

The first term is 0, as $A(\tau) = 0$, and $x(0) = 0$ (paths, for simplicity, always start from 0 - as any generic path can be shifted to satisfy this condition).
We then discretize the path, starting from $t_0 = 0$ and arriving to t_N. Let $\dot{x}(\tau)\,\mathrm{d}\tau \equiv \mathrm{d}x = x_i - x_{i-1}$, where $x_i \equiv x(t_i)$. Then, the integral is just the limit of the Riemann sum:

$$= \lim_{N\to\infty} \sum_{i=1}^N A(t_i)(x_i - x_{i-1})$$

For simplicity, denote $A_i \equiv A(t_i)$ and $(x_i - x_{i-1}) \equiv \Delta x_i$. Then:

$$F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right) = F\left(\int_0^t A(\tau)\dot{x}(\tau)\,\mathrm{d}\tau\right) = \lim_{N\to\infty} F\left(\sum_{i=1}^N A_i \Delta x_i\right)$$

We consider now the expectation value. If we are allowed to bring the average inside the integral, we get:

$$\langle F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)\rangle = \lim_{N\to\infty} \int \prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{4\pi D\Delta t_i}} \exp\left(-\sum_{i=1}^N \frac{\Delta x_i^2}{\Delta t_i}\right) \cdot F\left(\sum_{i=1}^N \Delta x_i A_i\right) \int \mathrm{d}z \cdot\delta\left(z - \sum_{i=1}^N A_i \Delta x_i\right)$$

where $D = 1/4$ and $t = t/(4D)$. The last term is equal to 1, and it is introduced just to exchange the integrals and lead to:

$$\int \mathrm{d}z\, F(z) \int \prod_i \frac{\mathrm{d}x_i}{\sqrt{\pi \Delta t_i}} F\left(\sum_i A_i \Delta x_i\right) \delta\left(z \sum_i A_i \Delta x_i\right) \cdot \exp\left(-\sum_i \frac{\Delta x_i^2}{\Delta t_i}\right)$$

Then we write the δ in terms of its Fourier transform:

$$\delta(z\sum_i A_i\Delta x_i) = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \exp\Big(i\alpha(z - \sum_i A_i\Delta x_i)\Big)$$

leading to:

$$= \int \frac{\mathrm{d}\alpha}{2\pi} \int \mathrm{d}z\, F(z) e^{i\alpha z} \int \prod_i \frac{\mathrm{d}x_i}{\sqrt{\pi\Delta t_i}} \exp\left(-\sum_{i=1}^N \frac{\Delta x_i^2}{\Delta t_i} - i\sum_{i=1}^N A_i\Delta x_i\right)$$

Then consider the change of variables: $y_i = \Delta x_i = x_1 - x_0 = x_1,\ y_2 = \Delta x_2 = x_2 - x_1; \dots, y_N = \Delta x_N = x_N - x_{N-1}$. The volume element will be transformed by the determinant of the Jacobian:

$$J = \det \frac{\partial(x_1 \dots x_N)}{\partial(y_1 \dots y_N)} = \left[\det \frac{\partial(y_1 \dots y_N)}{\partial(x_1 \dots x_N)}\right]^{-1} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}^{-1} = 1$$

(It is a tri-diagonal matrix).
Then we arrive at:

$$\int \frac{\mathrm{d}\alpha}{2\pi} \int \mathrm{d}z\, F(z) e^{i\alpha z} \int \prod_{i=1}^N \frac{\mathrm{d}y_i}{\sqrt{\pi\Delta t_i}} \exp\left(-\sum_{i=1}^N \frac{y_i^2}{\Delta t_i} - i\sum_{i=1}^N A_i y_i\right) \prod_{i=1}^N \int \frac{\mathrm{d}y_i}{\sqrt{\pi\Delta t_i}} \exp\left(-\frac{y_i^2}{\sqrt{\pi\Delta t_i}} - i\alpha A_i y_i\right)$$

which is a gaussian integral, with solution:

$$= \int \frac{\mathrm{d}\alpha}{2\pi} \left[\int \mathrm{d}z\, F(z) \prod_i \exp\left(-\frac{\alpha^2}{4} A_i^2 \Delta t_i\right) e^{i\alpha z}\right]$$

Note that:

$$\prod_i \exp\left(-\frac{\alpha^2}{4} A_i^2 \Delta t_i\right) = \exp\left(-\frac{\alpha^2}{4} \sum_{i=1}^N A_i^2 \Delta t_i\right)$$

and then, in the limit the Riemann summation becomes an integral:

$$\sum_{i=1}^N A^2(t_i)\Delta t_i \xrightarrow[N\to\infty]{} \int_0^t A^2(\tau)\,\mathrm{d}\tau$$

so that:

$$\int_0^t \mathrm{d}\tau\, A^2(\tau) = \int_0^t \mathrm{d}\tau \left(\int_\tau^t \mathrm{d}\tau\, a(\tau)\right)^2 \equiv R(t)$$

Finally, we arrive at:

$$\langle F\left(\int_0^t a(\tau)x(\tau)\right)\rangle = \int \mathrm{d}z \int \frac{\mathrm{d}\alpha}{2\pi} \exp\left(-\frac{\alpha^2}{4} R(t) + i\alpha z\right)$$

which is also a gaussian integral!

$$= \int \mathrm{d}z\, F(z) \frac{1}{2\pi}\sqrt{\frac{4\pi}{R(t)}} \exp\left(-\frac{z^2}{R(t)}\right) = \int \mathrm{d}z\, F(z) \sqrt{\frac{\pi}{R(t)}} \exp\left(-\frac{z^2}{R(t)}\right)$$

So, we showed that:

$$\langle F\left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)\rangle = \sqrt{\frac{\pi}{R(t)}} \int \exp\left(-\frac{z^2}{R(t)} F(z)\,\mathrm{d}z\right); \qquad R(t) \equiv \int_0^t \left(\int_\tau^t a(\tau')\,\mathrm{d}\tau'\right)^2$$

0.1.1 Example 1

Choose:

$$F(z) = e^{hz}$$

Then:

$$\langle \exp\left(h\int_0^\tau a(\tau)x(\tau)\,\mathrm{d}\tau\right)\rangle = \sqrt{\frac{\pi}{R}} \int \mathrm{d}z \exp\left(-\frac{z^2}{R} + hz\right) = \exp\left(\frac{h^2 R}{4}\right) \equiv G(h)$$

This is just the *generating function* of $h(z)$. In fact:

$$G'(h) = \langle \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau \exp\left(h\int_0^\tau ax\,\mathrm{d}\tau\right)\rangle$$

and setting $h = 0$ leads to the first moment of $h(z)$:

$$G'(0) = \langle \int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\rangle$$

As we know $G(z)$, we can differentiate the result, obtaining:

$$G'(h) = \frac{h}{2} R \exp\left(\frac{h^2 R}{4}\right)$$

and then $G'(0) = 0$.
Then, for the second moment:

$$G''(h) = \frac{R}{2}\exp\left(\frac{h^2 R}{4}\right) + \frac{h^2}{4}R^2 \exp\left(\frac{h^2 R}{4}\right) \Rightarrow G''(0) = \frac{R}{2}$$

Consider now a generic odd moment:

$$\langle \left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)^{2k+1}\rangle = 0 \qquad \forall k \in \mathbb{N}$$

In fact, if we expand $G(h)$, we get:

$$G(h) = \sum_{n=0}^{\infty} \left(\frac{R}{4}\right)^n \frac{1}{n!} h^{2n}$$

Since all the powers are even, if we differentiate an odd number of times and set $h = 0$ we are "selecting" an odd power - which just is not there - and so the result will be 0.
On the other hand, an even moment leads to:

$$\langle \left(\int_0^t a(\tau)x(\tau)\,\mathrm{d}\tau\right)^{2k}\rangle = R^k \frac{(2k)!}{2^k k!}$$

0.2 The 4th integral

Consider now:

$$\langle \exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)\rangle$$

To solve this, we will use a method introduced by Gelfand-Yaglom.
Expanding the average:

$$\int \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{\sqrt{\pi \Delta t_i}} \exp\left(-\sum_i \frac{\Delta x_i^2}{\Delta t_i} - \sum_{i=1}^{N} P_i x_i^2 \Delta t_i\right)$$

with $x_i \equiv x(t_i), P_i \equiv P(t_i)$, and we consider the limit for $N \to \infty$ in the Riemann summation. Recall that $\Delta x_i = x_i - x_{i-1}$, and denote $\Delta t_i = \epsilon = t/N$ (regular discretization in time - in fact the result is independent on the chosen mesh, but proving involves a much more heavy notation, which is omitted for simplicity). This leads to:

$$\int \prod_i \frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}} e^{-\boldsymbol{x}^T \boldsymbol{a} \boldsymbol{x}}$$

where a is a $N \times N$ tri-diagonal matrix:

$$a = \begin{pmatrix} a_1 & -1/\epsilon & 0 & \dots & 0 \\ -\epsilon^{-1} & a_2 & -\epsilon^{-1} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\epsilon^{-1} & a_{N-1} & -\epsilon^{-1} \\ 0 & 0 & 0 & -\epsilon^{-1} & a_N \end{pmatrix}$$

so that the integral becomes:

$$= \frac{1}{\epsilon^{N/2}(\det a)^{1/2}} = \frac{1}{(\det(\epsilon a))^{1/2}}$$

Consider the determinant:

$$\epsilon a = \begin{vmatrix} \epsilon a_1 & -1 & 0 & \dots & 0 \\ -1 & \epsilon a_2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & \epsilon a_{N-1} & -1 \\ 0 & \dots & 0 & -1 & \epsilon a_N \end{vmatrix} \equiv D_1^{(N)}$$

If we consider the block starting from a_k:

$$D_k^{(N)} \equiv= \begin{vmatrix} \epsilon a_k & -1 & 0 & \dots & 0 \\ -1 & \epsilon a_{k+1} & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & a_{N-1} & -1 \\ 0 & \dots & 0 & -1 & \epsilon a_N \end{vmatrix}$$

then we note that $a_1 = \epsilon P_i + 2/\epsilon$ for $0 < i < N$ and $a_N = \epsilon P_n + 1/\epsilon$, leading to:

$$D_k^{(N)} = \epsilon a_k D_{k+1}^{(N)} + 1 \begin{vmatrix} -1 & -1 & 0 & \dots & 0 \\ 0 & \epsilon a_{k+2} & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & \epsilon a_N \end{vmatrix} = \epsilon a_k D_{k+1}^{(N)} - 1 D_{k+2}^{(N)}$$

If we now consider:

$$D_k^{(N)} - 2D_{k+1}^{(N)} + D_{k+2}^{(N)} = \epsilon a_k D_{k+1}^{(N)} - 2D_{k+1}^{(N)} = D_{k+1}^{(N)}(\epsilon a_k - 2)$$

Denote $t = N\epsilon$, $\tau = (k-1)\epsilon$, so that:

$$D_k^{(N)} \xrightarrow[N\to\infty]{\epsilon\downarrow 0, \tau \text{ const}} D(\tau)$$

and so the difference considered before becomes the second derivative:

$$D(\tau) - 2D(\tau+\epsilon) + D(\tau+2\epsilon) = \epsilon^2 P(\tau+\epsilon) D(\tau+\epsilon)$$

as:

$$\epsilon a_k - 2 = \epsilon^2 P_k + 2 - 2 = \epsilon^2 P_k \to \epsilon^2 P(\tau+2) = \epsilon^2 P(\tau) + O(\tau^3)$$

Dividing by ϵ^2:

$$\frac{D(\tau+\epsilon) + D(\tau) - 2D(\tau+\epsilon)}{\epsilon^2} = P(\tau+\epsilon)D(\tau+\epsilon)$$

Expanding $D(\tau)$ around $\tau + \epsilon$:

$$D(\tau) = D(\tau+\epsilon) - \epsilon D'(\tau+\epsilon) + \frac{\epsilon^2}{2} D''(\tau+\epsilon)$$

and also for $D(\tau + 2\epsilon)$:

$$D(\tau+2\epsilon) = D(\tau+\epsilon) + \epsilon D'(\tau+\epsilon) + \frac{\epsilon^2}{2} D''(\tau+\epsilon) + O(\epsilon^3)$$

Subtracting term by term and dividing by ϵ^2:

$$D''(\tau+\epsilon) + O(\epsilon^2) = P(\tau+\epsilon)D(\tau+\epsilon)$$

and for $\epsilon \to 0$ we are left with:

$$D''(\tau) = P(\tau)D(\tau)$$

This is the Gelfand method, to represent an infinite dimensional matrix in a recursive way and arrive at a differential equation.
Returning to the initial integral:

$$\langle \exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)\rangle = \lim_{N\uparrow\infty} \frac{1}{\sqrt{D_1^{(N)}}} = \frac{1}{\sqrt{D(0)}} \equiv I_0$$

However, to be able to integrate this differential equation, we need two boundary conditions.
We start from:

$$D_N^{(N)} = \epsilon a_N = \epsilon^2 P_N + 1 \xrightarrow[\epsilon\to 0]{} 1 = D(t)$$

as:

$$\tau = (k-1)\epsilon \underset{k=N}{=} (N-1)\epsilon = t - \tau = \ldots$$

For the first derivative, we need:

$$D_{N-1}^{(N)} = \begin{vmatrix} \epsilon a_{N-1} & -1 \\ -1 & \epsilon a_N \end{vmatrix} = 1 + \epsilon^2 P_{N-1} + 2\epsilon^2 P_N + P_N P_{N-1}\epsilon^4$$

so that:

$$D'(t) = \frac{D_N^{(N)} - D_{N-1}^{(N)}}{\epsilon} = -(P_N P_{N-1}\epsilon^4 + \epsilon^2 P_N + \epsilon^2 P_{N-1})\frac{1}{\epsilon} \to 0$$

So we get the two boundary conditions:

$$D(t) = 1 \qquad D'(t) = 0$$

Before proceeding with the solution, we need another result:

$$\langle \exp\left(-\int_0^t P(\tau)x^2(\tau)\,\mathrm{d}\tau\right)\delta(x(t)-x)\rangle \equiv I$$

which is computed by resorting to the discretization trick:

$$I_N = \int \prod_{i=1}^N \frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}} \exp\left(-\sum_{i=1}^N \frac{\Delta x_i^2}{\Delta t_i} - \sum_i \epsilon P_i x_i^2\right)\delta(x_N - x)$$

using the Fourier transform for the δ:

$$\delta(x_N - x) = \int \frac{\mathrm{d}\alpha}{2\pi} e^{i\alpha(x_N - x)}$$

and exchanging the integrals:

$$= \int \mathrm{d}\alpha\, e^{-i\alpha x} \int \prod_i \frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}} \exp\left(\underbrace{-\sum_i \left(\frac{\Delta x_i^2}{\epsilon} + P_i \epsilon x_i^2 \right)}_{\boldsymbol{x}^T a \boldsymbol{x}} + i\alpha x_N \right)$$

Leading to:

$$I_N = \int \frac{\mathrm{d}\alpha}{2\pi} e^{i\alpha x} \int \prod_i \frac{\mathrm{d}x_i}{\sqrt{\pi\epsilon}} \exp\left(-\boldsymbol{x}^T a \boldsymbol{x} + i\alpha x_N\right)$$

If we denote $i\alpha x_N = \boldsymbol{h}^T \boldsymbol{x}$, with $h_l = \delta_{lN}(-i\alpha)$, then:

$$\int \exp\left(-\boldsymbol{x}^T a \boldsymbol{x} + \boldsymbol{h}^T \boldsymbol{x}\right) = \frac{\pi^{N/2}}{|a|^{1/2}} \exp\left(\frac{1}{4} \boldsymbol{h}^T a^{-1} \boldsymbol{h}\right) = \frac{\pi^{N/2}}{|a|^{1/2}} \exp\left(-\frac{1}{4} \alpha^2 (a^{-1})_{NN}\right)$$

Substituting back:

$$I_N = I_0 \int \frac{\mathrm{d}\alpha}{2\pi} \exp\left(i\alpha x - \frac{1}{4}\alpha^2 (a^{-1})_{NN}\right) = I_0 \frac{1}{2\pi} \sqrt{4\pi} \left(\frac{1}{(a^{-1})_{NN}}\right)^{1/2} \exp\left(-\frac{x^2}{(a^{-1})_{NN}}\right)$$

Recalling the form of a, we can compute $(a^{-1})_{NN}$:

$$(a^{-1})_{NN} = \cdots = \text{See the notes}$$

And through the magical power of friendship, we finally arrive to:

$$I = \lim_{N\to\infty} I_N = \frac{1}{\sqrt{\pi \widetilde{D}(0)}} \exp\left(-x^2 \frac{D(0)}{\widetilde{D}(0)}\right)$$

where:

$$\widetilde{D}''(\tau) = P(\tau)\widetilde{D}(\tau); \qquad \widetilde{D}(t) = 0 \quad \widetilde{D}(t) = -1$$

which are different initial conditions than that of D:

$$D''(\tau) = P(\tau)D(\tau) \qquad D(t) = 1; \quad D'(t) = 0$$

As $P(\tau) = k^2$ independent of τ, we arrive at:

$$I_0 = \langle \exp\left(-k^2 \int_0^t x^2(\tau)\, \mathrm{d}\tau\right) \rangle$$
$$I = \langle \exp\left(-k^2 \int_0^t x^2(\tau)\, \mathrm{d}\tau\, \delta(x - x(t))\right) \rangle$$

Recall that:

$$D''(\tau) = k^2 D(\tau); \quad D(\tau) = Ae^{k\tau} + Be^{-k\tau}$$

Imposing the initial conditions:

$$\begin{aligned} D(t) &= Ae^{kt} + Be^{-kt} = 1 \\ D'(t) &= kAe^{kt} - kBe^{-kt} = 0 \end{aligned}$$

Solving for A and B we get:

$$D(\tau) = \frac{1}{2}(e^{k(t-\tau)} + e^{-k(t-\tau)}) = \cosh(k(t-\tau))$$

If we repeat the same steps for $\widetilde{D}$, with $\widetilde{D}(t) = 0$ and $\widetilde{D}'(t) = -1$, the solution will be:

$$\widetilde{D}(\tau) = \widetilde{A}e^{k\tau} + \widetilde{B}e^{-k\tau}$$

and solving for $\widetilde{A}$ and $\widetilde{B}$ leads to:

$$D(\tau) = \frac{1}{2k}(e^{k(t-\tau)} + e^{-k(t-\tau)}) = \frac{1}{k}\sinh(k(t-\tau))$$

Ans substituting back:

$$I = \sqrt{\frac{k}{\pi \sinh(kt)}} \exp\left(-x^2 k \coth(kt)\right)$$