0.1 Schwarzschild Black Hole

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The 3D Plot. Starting from the $\{U, V\}$ plot, consider the universe as seen at a constant time $V = A < 0$ (obviously this is only a *mathematical* view, as no physical observer can ever see an extended region at *the same exact instant*). Here we can use V as a *time coordinate*, as in the metric $\mathrm{d}V^2$ has a *minus* sign, just like $\mathrm{d}t^2$. Now, $V = A$ is a horizontal line, that intercepts two separate region of spacetime (one with $U > 0$ and $U < 0$). If we add another coordinate θ, these two regions becomes separate *planes* (they are geometrically *different spaces*, as they are *not connected*). We can plot them by embedding in a fictitious 3D space. Also, to aid visualization, we can *deform* the two planes inside their horizons, so that the two points at $r = 0$ lie *closer* together (in the abstract 3D space) than all the other points. Then, if we consider other pictures at different $V = A$, with A closer and closer to 0, we can see a *bridge* forming between the two spaces, which however exists for not enough time to be physically traversable.

0.2 Complement on geodesics

(see additional material at the end of the lecture notes on geodesics).
We already noted that an observer experiencing geodesic motion does not *feel* any acceleration at all. We defined the 4-acceleration as:

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = u^\nu \nabla_\nu u^\mu = \frac{D}{\mathrm{d}\tau} u^\mu$$

But what is the acceleration $|\boldsymbol{a}|$ *felt* by the observer?
We start from:

$$0 = u^\nu \nabla_\nu (-1) = u^\nu \nabla_\nu (u^\mu u_\mu)$$

Applying Leibniz rule:

$$= u^\mu u^\nu \nabla_\nu u_\mu + u_\mu u^\nu \nabla_\nu u^\mu$$

These two terms are actually the same, as the metric is *covariantly constant*:

$$= u^\mu u^\nu \nabla_\nu u_\mu + u_\alpha g^{\alpha\mu} u^\nu \nabla_\nu g_{\mu\beta} u^\beta = u^\mu u^\nu \nabla_\nu u_\mu + u_\alpha \underbrace{g^{\alpha\mu} g_{\mu\beta}}_{\delta^\alpha_\beta} u^\nu \nabla_\nu u^\beta =$$

$$= 2 u_\mu u^\nu \nabla_\nu u^\mu = 2 \boldsymbol{u} \cdot \boldsymbol{a}$$

and so:

$$\boldsymbol{u} \cdot \boldsymbol{a} = 0$$

Now, the acceleration *felt* by an observer A is the same acceleration of A with respect to an observer B who is in a LIF (free fall) and who has the same velocity of A at the instant of the measurement.
[Insert figure (1)]

So, let's compute the acceleration of A in the frame of B:

$$a^\mu = u^\nu \nabla_\nu u^\mu = \underbrace{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}}_{\mathrm{d}\tau=\mathrm{d}t \text{ in LIF}} + \underbrace{\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta}_{=0 \text{ in LIF}} = \frac{\mathrm{d}u^\mu}{\mathrm{d}t}$$

At that instant A is *at rest* in this frame, meaning that $u^\mu = (1, \mathbf{0})$. Also, as $\boldsymbol{a} \cdot \boldsymbol{u} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$ in a LIF, we have:

$$a^\mu = (0, \boldsymbol{a})$$

Then:

$$a^\mu a_\mu = |\boldsymbol{a}|^2 \qquad \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = |\boldsymbol{a}_{\text{felt}}|$$

Summarizing:

1. Go in any frame
2. Compute $A^\mu = u^\nu \nabla_\nu u^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$ in that frame
3. Compute $\sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ (the scalar product will be the same in every frame)

Example 1 (Uniformly accelerated observer in Minkowski Spacetime)**:**

Consider an uniformly accelerated observer in flat spacetime:

$$x(t) = \frac{\sqrt{1 + k^2 t^2}}{k}$$

Recall that, using the proper time τ as the parameterization variable, we get:

$$\begin{cases} t = \frac{1}{k} \sinh(k\tau) \\ x = \frac{1}{k} \cosh(k\tau) \end{cases}$$

We already now that this observer *feels* a constant acceleration k. We want now to check that:

1. $\boldsymbol{u} \cdot \boldsymbol{a} = 0$
2. $\sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = k$

The 4-position is:

$$x^\mu = \left(\frac{1}{k} \sinh(k\tau), \frac{1}{k} \cosh(k\tau), 0, 0 \right)$$

We can immediately compute the 4-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = (\cosh(k\tau), \sinh(k\tau), 0, 0)$$

Then:

$$\boldsymbol{u}\cdot\boldsymbol{u} = u^\mu \eta_{\mu\nu} u^\nu = -(u^0)^2 + (u^1)^2 = -\cosh^2(k\tau) + \sinh^2(k\tau) = -1$$

The 4-acceleration:

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$

but in Minkowski spacetime all the Christoffel symbols are 0 (flat spacetime). So:

$$a^\mu = (k\sinh(k\tau), k\cosh(k\tau), 0, 0)$$

And we can finally check:

$$\boldsymbol{a}\cdot\boldsymbol{u} = -a^0u^0 + a^1u^1 = -\cosh(k\tau)k\sinh(k\tau) + \sinh(k\tau)k\cosh(k\tau) = 0$$

And also:

$$\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} = \sqrt{-a_0^2 + a_1^2} = \sqrt{-k^2\sinh^2(k\tau) + k^2\cosh^2(k\tau)} = k$$

Example 2 (Observer at rest in Schwarzschild)**:**

For an observer at rest:

$$x^\mu = (t(\tau), r, \theta, \varphi)$$

with r, θ, φ are all **constants**. So:

$$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0, 0\right)$$

We can find the missing component by using the normalization:

$$-1 = \boldsymbol{u}\cdot\boldsymbol{u} = u^\mu g_{\mu\nu} u^\nu = g_{00}(u^0)^2 = -\left(1 - \frac{2GM}{r}\right)(u^0)^2$$

leading to:

$$u^0 = \frac{1}{\sqrt{-g_{00}}} = \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

Substituting back:

$$u^\mu = \left(\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right)$$

The 4-acceleration:

$$a^\mu = \cancel{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = \Gamma^\mu_{00}(u^0)^2$$

as only $u^0 \neq 0$, and u^μ is constant. Then:

$$G^\mu_{00} = \frac{1}{2} g^{\mu\lambda} (\cancel{g_{\lambda 0,0}} + \cancel{g_{\lambda 0,0}} - g_{00,\lambda})$$

and so the only non-zero symbol is:

$$\Gamma^1_{00} = \frac{1}{2} \left(1 - \frac{2GM}{r}\right) - \frac{\partial}{\partial r} \left(-1 + \frac{2GM}{r}\right) = \frac{1}{2} \left(1 - \frac{2GM}{r}\right) \frac{2GM}{r^2}$$

Substituting back:

$$a^1 = \left(1 - \frac{2GM}{r}\right) \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} = \frac{GM}{r^2}$$

and so:

$$a^\mu = \left(0, \frac{GM}{r^2}, 0, 0\right)$$

Then:

$$|\boldsymbol{a}_{\text{felt}}| = \sqrt{a_\mu a^\mu} = \sqrt{g_{11} a^1 a^1} = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

Note that when $r \gg 2GM$:

$$|\boldsymbol{a}_{\text{felt}}| = \frac{GM}{r^2}$$

which is just the Newtonian gravitational acceleration.
Otherwise, when $r \to 2GM$, $|\boldsymbol{a}_{\text{felt}}| \to \infty$, meaning that it is not possible to remain stationary at the Schwarzschild horizon. Note that this result is *physical*, and not due to a bad choice of coordinates.

0.3 Spin

In geodetic motion:

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = u^\nu \nabla_\nu u^\mu = \frac{D}{\mathrm{d}\tau} u^\mu$$

where the capital D denotes a *total derivative*.
We define a **gyroscope** to be an object with *angular momentum*. In the rest frame of the object we define it to be:

$$S^\mu = (0, \boldsymbol{S})$$

Immediately, in the rest frame:

$$\boldsymbol{u} \cdot \boldsymbol{S} = 0$$

As the result is a scalar, this relation will be true in all frames.
A free object in Minkowski spacetime in his own rest frame has a constant $\boldsymbol{S}$:

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}t} = 0$$

In a LIF, for a moving object, we expect:

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = u^\nu \frac{\partial S^\mu}{\partial x^\nu} = 0$$

Generically:

$$u^\nu \nabla_\nu S^\mu = 0$$

This is the same relation we had for a^μ, meaning that S^μ is *constant* along the trajectory:

$$\frac{\mathrm{D}S^\mu}{\mathrm{d}\tau} = 0$$

Also:

$$u^\nu \nabla_\nu (S^\mu S_\mu) = 2S_\mu u^\nu \nabla_\nu S^\mu = 0$$

meaning that $\boldsymbol{S} \cdot \boldsymbol{S}$ is conserved during motion. By the same argument, also the *product* of two different spins is conserved: $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 = \text{Constant}$.

Example 3 (Geodetic Precession)**:**

> We consider a gyroscope going around a Schwarzschild geometry (non-rotating mass). We will see that a different observer will see the gyroscope *precess* during that motion.
> The 4-velocity of the gyroscope is:
>
> $$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \right)$$
>
> as $r \equiv R$ and $\theta = \pi/2$ are both constants. Then $u^\varphi = u^t \Omega$ as:
>
> $$u^\mu = \underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}}_{u^t} \left(1, 0, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}t} \right)$$
>
> So:
>
> $$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} \right)^2 = \Omega^2 = \frac{GM}{R^3}$$
>
> If we now use the normalization:
>
> $$\begin{aligned} -1 = u^\mu u_\mu &= g_{00}(u^t)^2 + g_{33}(u^\varphi)^2 = \\ &= -(u^t)^2 \left(1 - \frac{2GM}{r} - \frac{GM}{r} \right) \end{aligned}$$

$$u^t = \frac{1}{\sqrt{1 - \frac{3GM}{R}}}$$

If we now write the spin:

$$S^\mu = (S^t, S^r, S^\theta, S^\varphi)$$

$S^\theta = 0$ at the start, and will remain 0 for all motion due to the system's symmetry (there is no reason for such a rotation). Then:

$$\begin{aligned} 0 = \boldsymbol{S} \cdot \boldsymbol{u} = g_{\mu\nu} S^\mu u^\nu &= g_{00} S^t u^t + g_{33} S^\varphi u^\varphi = \\ &= -\left(1 - \frac{2GM}{R}\right) S^t \cancel{u^t} + R^2 S^\varphi \cancel{u^t} \Omega = \\ &= \left(1 - \frac{2GM}{R}\right)^{-1} R^2 \Omega S^\varphi \end{aligned}$$

We now only need to compute the evolution of S^r and S^φ, and then we can compute S^t with the relation just found. From the equation of the spin, we now that S^μ is constant along the geodesic:

$$\frac{\mathrm{d}S^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta S^\gamma = 0$$

We start with:

$$\frac{\mathrm{d}S^1}{\mathrm{d}\tau} + \Gamma^1_{\beta\gamma} u^\beta S^\gamma = 0$$

What are the non-zero components? We see that $\beta = 0, 3$ and $\gamma = 0, 1, 3$. All possible symbols are then:

$$\Gamma^1_{00}, \Gamma^1_{01}, \Gamma^1_{03}, \Gamma^1_{30}, \Gamma^1_{31}, \Gamma^1_{33}$$

Recall the definition of $\Gamma^\alpha_{\beta\gamma}$:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} (g_{\lambda\gamma,\beta} + g_{\beta\lambda,\gamma} - g_{\beta\gamma,\lambda})$$

If the metric is diagonal (as in this case), then $\lambda = \alpha$, and we get:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha} (g_{\alpha\gamma,\beta} + g_{\beta\alpha,\gamma} - g_{\beta\gamma,\alpha})$$

and to get a non-zero result, at least two indices (between α, β, γ) must be the same. So:

$$\Gamma^1_{00}, \Gamma^1_{01}, \cancel{\Gamma^1_{03}}, \cancel{\Gamma^1_{30}}, \Gamma^1_{31}, \Gamma^1_{33}$$

When two indices are the same, the third one denotes the *derivative* (look at the expression). As the metric is stationary (time derivatives are null) and does not depend on φ, also $\Gamma^1_{01}, \Gamma^1_{31} = 0$. So only Γ^1_{00} and Γ^1_{33} are left to compute.

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(-1)\frac{\partial}{\partial r} g_{00} = \left(1 - \frac{2GM}{R}\right)\frac{GM}{R^2}$$
$$\Gamma^1_{33} = \frac{1}{2} g^{11}(-1)\frac{\partial}{\partial r} g_{33} = -\left(1 - \frac{2GM}{R}\right) R$$

We can now write the equations:

$$\frac{\mathrm{d}S^1}{\mathrm{d}\tau} + \Gamma^1_{00}\underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}}_{u^t} S^t + \Gamma^1_{33}\underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}\Omega}_{u^3} S^\varphi = 0$$

leading to:

$$\frac{\mathrm{d}S^1}{\mathrm{d}t} + \Gamma^1_{00} S^t + \Gamma^1_{33}\Omega S^\varphi = 0$$

where we used $\frac{\mathrm{d}S^1}{\mathrm{d}\tau} = u^t \frac{\mathrm{d}S^1}{\mathrm{d}t}$ to simplify away the u^t. Inserting the Christoffel symbols we arrive at:

$$\frac{\mathrm{d}S^r}{\mathrm{d}t} + (3GM - R)\Omega S^\varphi = 0$$