0.1 Relativity of simultaneity

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Consider two events A and B that happen at the same time as measured by the inertial frame of reference of observer O': $t'_A = t'_B \equiv t'_{AB}$.
Suppose that O' is moving at a constant velocity v relative to another observer O.
How to represent these events in a spacetime diagram?
The idea is to simply use the usual:

$$\begin{cases} ct' = \gamma ct - \gamma \frac{v}{c} x \\ x' = \gamma x - \gamma \frac{v}{c} ct \end{cases} \quad \text{or} \quad \begin{cases} ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \\ x' = \frac{x - \frac{v}{c}ct}{\sqrt{1-\frac{v^2}{c^2}}} \end{cases}$$

So starting from:

$$ct'_{AB} = \frac{ct - \frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow ct = \frac{v}{c}x + \sqrt{1-\frac{v^2}{c^2}}\, ct'_{AB}$$

which is a line parallel to the x' axis, which is different from the x axis. So, the observer O will measure a non-zero time difference between the two events A and B.

0.2 Length contraction

The **length** of an object is defined as the spatial distance between two simultaneous events situated at both ends:

- A occurs at $t_A = 0$, $x_A = 0$
- B occurs at $t_B = 0$, $x_B = L$

So, by looking at the ends of an object *at the same time* (relative to the O observer) one can compute the object's length ($L = x_B - x_A$).
What is the length as measured by a different observer O', in relative motion at velocity v wrt O?
By using:

$$x'_A = \frac{x_A - \frac{v}{c}ct_A}{\sqrt{1-\frac{v^2}{c^2}}}; \quad x'_B = \frac{x_B - \frac{v}{c}ct_B}{\sqrt{1-\frac{v^2}{c^2}}}$$

and taking the difference, recalling that $t_A = t_B$:

$$L' = x'_B - x'_A = \frac{x_B - x_A}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow L = \sqrt{1-\frac{v^2}{c^2}}\, L$$

This is the phenomenon of length contraction.

0.3 Velocity addition

Consider two observers O and O' in relative motion at velocity v, and a point P with velocity V as measured by O, or V' as seen by O'. What is the relation between V and V'?

From the point of view of O', the measured velocity is defined as:

$$V' = \frac{dx'}{dt'} \underset{(a)}{=} \frac{\cancel{\gamma}[dx - vdt]}{\cancel{\gamma}\left[dt - \dfrac{v\,dx}{c^2}\right]} = \frac{\dfrac{dx}{dt} - v\dfrac{dt}{dt}}{\dfrac{dt}{dt} - \dfrac{v}{c^2}\dfrac{dx}{dt}} = \frac{V - v}{1 - \dfrac{v}{c^2}V}$$

when in (a) we used a differential of the Lorentz transformations.
Some considerations:

- **Non relativistic limit** ($v \ll c$, or informally $c \to \infty$): $V' = V - v$
- $V = c \Rightarrow V' = \dfrac{c - v}{1 - \frac{vc}{c^2}} = \dfrac{c - v}{\frac{c-v}{c}} = c$, so *light has the same speed for all observers*. This proves that Lorentz transformations are the correct ones in a universe where light always move at speed c in every frame.

0.4 Four-vectors

Let's review all the previous effects and concepts by building a general and useful mathematical framework.

4-vectors are "objects that transform as (ct, x, y, z) under a Lorentz Boost". We will denote this object with:

$$x^\mu = (\underbrace{ct}_{\mu=0}, x, y, \underbrace{z}_{\mu=3}) \qquad \mu = \{0, 1, 2, 3\}$$

0.4.1 Distance and Metric

In physics, space is endowed with the notion of *scalar product*, a mathematical structure that gives meaning to the concept of **distance**. For example, for a vector $\vec{d}$ in two dimensions, it is defined its length as $\sqrt{\vec{d} \cdot \vec{d}} = \sqrt{dx^2 + dy^2}$.

More generally, one can define a scalar product by defining its action on two vectors:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = \begin{pmatrix} A_x, A_y \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} A_x, A_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

The identity matrix in this relation *defines* the scalar product. If we used a different matrix, we would have obtained a different scalar product, and thus a different

distance.
That "matrix" is called the **metric** of space. In general relativity, we will see that gravitational sources can *influence* the way in which distance as measured - that is they alter the geometry of spacetime.

0.4.2 The Minkowski metric

Let's introduce the **Minkowski** metric as:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's find the *distance* associated to that metric. We start by taking two events very close to each other, that is separated by an infinitesimal distance, given by their *separation vector*:

$$dx^\mu = (cdt, dx, dy, dz)$$

The distance squared between the two events is then:

$$\begin{aligned} \text{Distance}^2 &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} dx^\mu \eta_{\mu\nu} dx^\nu = \\ &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2 = ds^2 \end{aligned}$$

We obtain the invariant four-distance previously defined: this proves that the Minkowski metric is the one best suited for special relativity.

0.4.3 Einstein's notation

Repeated indices are implicitly summed over. So, instead of writing:

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} dx^\mu \eta_{\mu\nu} dx^\nu$$

we will just write:

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$$

Indices that appear once are *free indices*, and they will appear also in the result. Indices that appear twice are summed over, and are not free indices, and they disappear in the result.
Be careful not to use the same index more than twice!

0.4.4 Inverse Minkowski metric

Every matrix A with $|det\rangle\langle det|(A) \neq 0$ can be inverted. That means there is a (unique) matrix A^{-1} such that $AA^{-1} = A^{-1}A = \mathbb{I}$. In the case of the Minkowski metrix we get:

$$|det\rangle\langle det|(\eta) = -1$$

We denote its inverse with $\eta^{\mu\nu}$ (indices up), so that:

$$\eta^{-1}\eta = \mathbb{I} \Leftrightarrow \eta^{\overbrace{\mu}^{column}\overbrace{\nu}^{row}} \eta_{\underbrace{\nu}_{col}\underbrace{\alpha}_{row}} = \delta^{\mu}_{\diamond\alpha}$$

($\diamond$ is a spacer for indices).

$$\delta^{\mu}_{\diamond\alpha} = \begin{cases} 1 & \text{if } \mu = \alpha \\ 0 & \text{if } \mu \neq \alpha \end{cases}$$

In the particular case of the Minkowski metric $\eta^{\mu\nu} = \eta_{\mu\nu}$, in fact:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is usually not true for a general metric $g^{\mu\nu} \neq g_{\mu\nu}$.

0.4.5 Lorentz Boosts

How are Lorentz boosts expressed in this formalism?
Recall the transformation relations:

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}; \qquad x' = \frac{x - vt}{1 - \frac{v^2}{c^2}}$$

We define:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

So we can write the transformations in a more compact way:

$$\begin{cases} ct' = \gamma ct - \beta\gamma x \\ x' = \gamma x - \beta\gamma ct \end{cases}$$

In matrix notation this becomes much simpler:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^{\mu}_{\diamond\nu}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow x^{\mu\prime} = \Lambda^{\mu}_{\diamond\nu} x^{\nu}$$

(and this is the best way to remember them).

The infinitesimal invariant four-distance is:

$$ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

In a different frame of referente:

$$ds' = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} \Lambda^{\mu}_{\diamond\alpha} dx^{\alpha} \Lambda^{\nu}_{\diamond\beta} dx^{\beta}$$

Of course they are the same, meaning that the Λ matrix has a peculiar property:

$$\text{Invariance of 4-distance} \Leftrightarrow \eta_{\mu\nu} \Lambda^{\mu}_{\diamond\alpha} \Lambda^{\nu}_{\diamond\beta} = \eta_{\alpha\beta}$$

Let's multiply both parts by the inverse matrix $\eta^{\beta\sigma}$:

$$\eta_{\mu\nu} \boxed{\Lambda^{\mu}_{\diamond\alpha} \Lambda^{\nu}_{\diamond\beta} \eta^{\beta\sigma}} = \Lambda^{\mu}_{\diamond\alpha} \boxed{\eta_{\mu\nu} \Lambda^{\nu}_{\diamond\beta} \eta^{\beta\sigma}} = \eta_{\alpha\beta} \eta^{\beta\sigma} = \delta^{\diamond\sigma}_{\alpha} \tag{1}$$

so the yellow part is equal to the inverse metric, and the light blue part is the inverse of $\Lambda^{\mu}_{\diamond\alpha}$.

0.4.6 Rising and lowering of the indices

Some notation rules:

- When contract (= sum over) with η, lower the index
- When contract with η^{-1}, rise the index

In this notation, than, by observing (**??**) we can define:

$$\Lambda^{\mu}_{\diamond\alpha} \Lambda^{\diamond\sigma}_{\mu} = \delta^{\diamond\sigma}_{\alpha}$$

We can also define a 4-vector with *lower* index:

$$x_{\mu} \equiv \eta_{\mu\nu} x^{\nu}$$

How does this vector transform under a Lorentz Boost?
Recall that:

$$x'^{\mu} = \Lambda^{\mu}_{\diamond\nu} x^{\nu}$$

Multiplying both sides by $\eta_{\alpha\mu}$:

$$\eta_{\alpha\mu} x'^{\mu} = \eta_{\alpha\mu} \Lambda^{\mu}_{\diamond\nu} x^{\nu}$$

we can lower the index of $\Lambda^{\mu}_{\diamond\nu}$:

$$= \Lambda_{\alpha\nu} x^{\nu}$$

and we insert an identity $\delta^{\sigma}_{\diamond\nu}$:

$$= \Lambda_{\alpha\sigma} \delta^{\sigma}_{\diamond\nu} x^{\nu} = \Lambda_{\alpha\sigma} \eta^{\sigma\beta} \eta_{\beta\nu} x^{\nu} = \Lambda_{\alpha}^{\diamond\beta} x_{\beta}$$

We have thus found the transformation relation for this kind of vector.

Summarizing:

- We call a **contravariant vector** a vector with "upper indices":

$$x'^{\mu} = \Lambda^{\mu}_{\diamond\nu} x^{\nu}$$

- A **covariant vector** is then the *lower indices version*:

$$x'_{\alpha} = \Lambda_{\alpha}^{\diamond\beta} x_{\beta}$$

  Both kind of vectors are *defined* by their transformation properties.

In fact, vector are defined in physics as entities that transform in a certain manner under rotation. For example, $\vec{F} = m\vec{a}$ implicitly contains an important statement: this law *does not* depend on the specific direction, it is *invariant* (or *covariant*) under rotation.
In analogy, relativity stands from the principle that *laws of physics are the same in every inertial frame of reference*. So it is useful to write physical laws in a *manifestally covariant form*, that is are immediately recognizable as something that transforms in a nice way, respecting the relativity principle.

Note that the *contraction* between a contravariant vector and a covariant vector is a **scalar**, i.e. an object that is **invariant** under a boost. For example, if we contract two boosted vectors we get the same result as if we contracted the same two vectors before boosting:

$$A'_{\mu} B'^{\mu} = \Lambda_{\mu}^{\diamond\alpha} A_{\alpha} \Lambda^{\mu}_{\diamond\beta} B^{\beta} = \delta^{\alpha}_{\diamond\beta} A_{\alpha} B^{\beta} = A_{\alpha} B^{\alpha}$$

In particular, an important scalar is the infinitesimal four-distance. Its invariance can now be seen immediately by simply applying the rule on lowering indices:

$$ds^2 = dx^{\mu} \eta_{\mu\nu} dx^{\nu} = dx^{\mu} dx_{\mu}$$

Note that when we have a contraction it does not matter which index is up and which is down:

$$A_{\mu} B^{\mu} = A_{\mu} \eta^{\mu\nu} B_{\nu} \underset{(a)}{=} A_{\mu} \eta^{\nu\mu} B_{\nu} \underset{(b)}{=} \eta^{\nu\mu} A_{\mu} B_{\nu} = A^{\nu} B_{\nu}$$

where in (a) we used the symmetry of $\eta^{\mu\nu}$, allowing the exchange $\nu \leftrightarrow \nu$. By rearranging (b) one can then use $\eta^{\nu\mu}$ to rise the index of A_{μ}.

0.4.7 Tensors

Tensors are objects *with many high/low indices*, where each index transforms independently under a boost. For example:

$$A'^{\alpha} = \Lambda^{\alpha}_{\diamond\mu} A^{\mu}; \qquad A'_{\mu\nu} = \Lambda_{\mu}^{\diamond\alpha}\Lambda_{\nu}^{\diamond\beta} A'_{\alpha\beta}; \qquad A'^{\mu\nu}_{\diamond\diamond l} = \Lambda^{\mu}_{\diamond\alpha}\Lambda^{\nu}_{\diamond\beta}\Lambda_{l}^{\diamond\gamma} A^{\alpha\beta}_{\diamond\diamond\gamma}$$

(this transformations are *by definition*).

Contracting indices between two tensors or within the same tensor reduces their rank (the number of matrices needed for a transformation), because only free indices can transform:

$$A'_{\alpha\beta}B'^{\beta} = \Lambda_{\alpha}^{\diamond\mu}\,\Lambda_{\beta}^{\diamond\nu}\,A_{\mu\nu}\,\Lambda^{\beta}_{\diamond l}\,B^{l} = \Lambda_{\alpha}^{\diamond\mu}\,\delta^{\nu}_{\diamond l}\,A_{\mu\nu}B^{l} = \Lambda_{\alpha}^{\diamond\mu}A_{\mu\nu}B^{\nu}$$