0.1 Orbits in GR

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During last lecture, we derived:

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) = \underbrace{\mathcal{E}}_{(e^2-1)/2} \qquad V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$$

With two constants of motion:

$$\xi^\mu = (1,0,0,0) \Rightarrow e = \left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{Constant}$$
$$\xi^\mu = (0,0,0,1) \Rightarrow l = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \text{Constant}$$

This describes the motion of a planet constrained to a plane ($\theta = \pi/2$) in the Schwarzschild metric.
In the case of *circular orbits* we have:

$$\begin{cases} -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = \frac{e^2-1}{2} \qquad (V = e) \\ \frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4} = 0 \qquad (\frac{\partial v}{\partial r} = 0) \end{cases}$$

From the second equation we can arrive to:

$$r = \frac{L^2}{2GM}\left[1 + \sqrt{1 - 12\left(\frac{GM}{l}\right)^2}\right]$$

Now, we consider the following:

$$(\text{Eq.}1) + r\left(1 - \frac{r}{2GM}\right)(\text{E}q.2) \Rightarrow \frac{l}{e} = \sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1} \tag{1}$$

We are interested in computing the **angular velocity** Ω, defined with respect to *coordinate time*:

$$\Omega \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

This leads to an interesting relation:

$$\Omega = \frac{\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}}{\frac{\mathrm{d}t}{\mathrm{d}\tau}} = \frac{l/r^2}{e\left(1 - \frac{2GM}{r}\right)^{-1}} \underset{(1)}{=} \frac{\sqrt{GMr}\cancel{\left(1 - \frac{2GM}{r}\right)^{-1}}\frac{1}{r^2}}{\cancel{\left(1 - \frac{2GM}{r}\right)^{-1}}} \Rightarrow \Omega^2 = \frac{GMr}{r^4} = \frac{GM}{r^3}$$

which is exactly the same relation that holds in Newtonian mechanics (due to sheer coincidence).

0.1.1 A general orbit

Let's go back to the general equation:

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}(r) = \mathcal{E} = \frac{e^2-1}{2}$$

Following the same steps we did in Newtonian mechanics, we simplified the problem introducing $u \equiv r^{-1}$, leading to:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{l^2} + 3GMu^2$$

We know already one solution: the circular orbit, with $u \equiv u_c$. We can then consider a *perturbation*:

$$u(\varphi) = u_c[1 + w(\varphi)] \equiv w \ll 1$$

and then derive the equation (neglecting the $O(w^2)$ terms):

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} + (1 - 6GMu_c)w \approx 0$$

which is just the harmonic oscillator equation. One solution is:

$$w = A\cos\left(\sqrt{1 - 6GMu_c}\,\varphi\right) \qquad A \ll 1$$

Going back to r:

$$r(\varphi) = \frac{r_c}{1 + A\cos\left(\sqrt{1 - 6GM/r_c}\,\varphi\right)} \qquad r_c = \frac{1}{u_c}$$

If $A > 0$ (and $A \ll 1$), when the argument of cosine is 0 then:

$$r = \frac{r_c}{1+A}$$

This is the *perihelion*, as r is smallest when the denominator is greatest. On the other hand, if the argument is π, then we have the aphelion:

$$r = \frac{r_c}{1-A}$$

With 2π we are back to $r_c/(1+A)$ - the next perihelion. Note that one orbit happens whenever the *argument* of the cosine changes by 2π. This is equal to a change of 2π of the coordinate φ only if we neglect the term $6GM/r_c$ which comes from GR. So, in Newtonian mechanics, one orbit equals a 2π rotation in φ, but not in GR - here, to have a 2π change of the argument of the cosine, φ must change a bit more: $2\pi + \delta\varphi_{\text{precession}}$. Then:

$$\Delta\varphi_{1\text{ orbit}} = \frac{2\pi}{\sqrt{1 - \dfrac{6GM}{r_c}}} \approx 2\pi\left(1 + \frac{3GM}{r_c}\right)$$

and so:

$$\delta\varphi_{\text{precession}} = \Delta\varphi_{\text{1orbit}} - 2\pi = 6\pi \frac{GM}{r_c}$$

Plugging in the Newtonian result for $r_c = l^2/(GM)$ (as the eventual corrections would lead to terms of higher order), we arrive finally at:

$$\delta\varphi_{\text{precession}} = 6\pi \left(\frac{GM}{l}\right)^2$$

We can know evaluate this quantity with the real case of the planet Mercury. First, we need to put back the powers of c. We can do this by *dimensional analysis*. From $F = GMm/r^2$, we note that $[G] = \mathrm{N\,m^2\,kg^{-2}} = \mathrm{kg\,m\,s^{-2}\,m^2\,kg^{-2}} = \mathrm{m^3\,kg^{-1}\,s^{-2}}$. $[M] = \mathrm{kg}$, $[l] = [rv] = \mathrm{m^2\,s^{-1}}$ and so:

$$\left[\frac{GM}{l}\right] = \frac{\mathrm{m^3\,kg^{-1}\,s^{-2}\,kg}}{\mathrm{m^2\,s^{-1}}} = \mathrm{m\,s^{-1}}$$

But $\delta\varphi_{\text{precession}}$ must be a pure number, and so we have to divide by c^2:

$$\delta\varphi_{\text{precession}} = 6\pi \left(\frac{GM}{cl}\right)^2$$

Then, inserting all the numbers (with at least 3 significant digits each):

$$\begin{aligned} G &= 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \\ M &= 1.99 \times 10^{30}\,\mathrm{kg} \\ l &= rv\Big|_{\text{perihelion Mercury}} = 4.60 \times 10^7\,\mathrm{km} \cdot 590\,\mathrm{km\,s^{-1}} \\ c &= 3.00 \times 10^8\,\mathrm{m\,s^{-1}} \end{aligned}$$

(note that r and v must be measured at the same *point*, e.g. the perihelion) leads to:

$$\delta\varphi_{\text{precession}} = 5.02 \times 10^{-7}\,\mathrm{rad}$$

This is the precession that *accumulates* at *every* orbit. To compute the total drift in a year, we need the orbital period of Mercury, which is $T = 88.0\,\text{days} = 2.41 \times 10^{-3} \cdot 100\,\text{years}$. We then find:

$$\frac{\delta\varphi_{\text{precession}}}{T} = \frac{43''}{100\,\text{years}}$$

Which is exactly compatible to the measured result!

0.2 Radial orbit - Dive into Black Hole

Consider an object with all the mass concentrated at the origin (black hole), and we study the motion with $l = 0$, that *falls straight* to the origin. Suppose we start at rest at infinity.
Recalling the previous equation:

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} = \frac{e^2-1}{2}$$

and:

$$e = \left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1$$

In fact, as e is a constant of motion, we can evaluate it at infinity, where GM/r is negligible, and as the object is at rest we have $\mathrm{d}t = \mathrm{d}\tau$. Then the 4-velocity components are:

$$u^0 \equiv u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{1-2GM/r}; \quad u^r = \frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}}; \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0; \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = 0$$

and so:

$$u^\alpha = \left(\left(1 - \frac{2GM}{r}\right)^{-1}, -\sqrt{\frac{2GM}{r}}, 0, 0\right)$$

Let's check if the norm is equal to -1:

$$\boldsymbol{u}\cdot\boldsymbol{u} = g_{00}u^0u^0 + g_{11}u^1u^1 = -\left(1-\frac{2GM}{r}\right)(1-\frac{2GM}{r})^{-2} + \left(1-\frac{2GM}{r}\right)^{-1}\frac{2GM}{r} =$$
$$= \frac{-1+2GM/r}{1-2GM/r} = -1$$

Then rearranging the conservation of energy:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{1-2GM/r}$$

We now rearrange the differential equation:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}}$$

(the - sign comes from the fact that we are *approaching* the black hole). Integrating:

$$\int_0^r r^{1/2}\,\mathrm{d}r = -\sqrt{2GM}\int_0^\tau \mathrm{d}\tau$$

so that $\tau = 0$ at $r = 0$ at the end of the motion, so that $r(\tau)$ occurs for negative τ. This leads to:

$$\frac{2}{3}r^{3/2} = \sqrt{2GM}(-\tau) \Rightarrow r(\tau) = \left(\frac{3}{2}\right)^{2/3}(2GM)^{1/3}(-\tau)^{1/3}$$

Recall that τ is the time measured by the infalling particle. The Schwarzschild metric is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$

We would suspect that something bad happens at $r = 2GM$. However, from the point of view of the particle, the solution $r(\tau)$ has no singularity at $r = 2GM$. In fact, the particle crosses this *horizon* in a finite time (as locally measured). However, at $r = 2GM$, a finite $\Delta\tau$ corresponds to an *infinite* Δt ($\mathrm{d}t\,/\,\mathrm{d}\tau$ diverges at $r = 2GM$). So it takes an infinite coordinate time for the particle to cross the horizon.
This likely means that the coordinates $\{t, r\}$ are a bad choice at the horizon.

0.2.1 Escape velocity

We can now reverse our reasoning to compute the *escape velocity.* In practice, we just reverse the sign of the velocity:

$$u^\alpha = \left(\left(1 - \frac{2GM}{r}\right)^{-1}, +\sqrt{\frac{2GM}{r}}, 0, 0\right)$$

which is a radial motion with $e = 1$ that reaches $r = \infty$ at rest.
Consider an observer at rest at distance r from the black hole, who launches radially a projectile. What is the $\boldsymbol{v}_{\text{escape}}$ needed for that object to *escape* the gravitational pull of the black hole?
Note that:

$$u^\alpha_{\text{observer}} = \left(\frac{1}{\sqrt{-g_{00}}}, \boldsymbol{0}\right) = \left(\left(1 - \frac{2GM}{r}\right)^{-1}, \boldsymbol{0}\right) \Rightarrow \boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{u}_{\text{obs}} = -1$$

The energy of the projectile as measured by the observer is:

$$\mathcal{E} = -\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{p}_{escape}$$

where $\boldsymbol{p}_{\text{escale}} = m\boldsymbol{u}_{\text{escape}}$. Expanding:

$$\mathcal{E} = -g_{00}(\boldsymbol{u}_{\text{obs}})^0(\boldsymbol{p}_{\text{esc}})^0 = \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1/2} m\left(1 - \frac{2GM}{r}\right)^{-1} = \frac{m}{\sqrt{1 - \dfrac{2GM}{r}}}$$

Recall that:

$$\mathcal{E} = \frac{m}{\sqrt{1 - v^2}}$$

so that:

$$|\boldsymbol{v}_{\text{escape}}| = \sqrt{\frac{2GM}{r}}$$

Again, by coincidence, this is the same result obtained in Newtonian mechanics.
So, for an observer *on the horizon*, the escape velocity is c, and *inside the horizon*, it exceeds c (it is not possible to escape anymore).

0.3 Motion of light

Light must experience gravity - this can be seen by using the *equivalence principle*. Mathematically, light follows *geodesics*, and so we can study it by solving the *geodesics equation*.
$\xi^\mu = (1, 0, 0, 0)$ is a Killing vector, meaning that $e \equiv -\boldsymbol{\xi} \cdot \boldsymbol{u} = (1 - 2GM/r)\,\mathrm{d}t \,/\, \mathrm{d}\lambda$ is a conserved quantity. Also $\xi^\mu = (0, 0, 0, 1)$ is a Killing vector, and then $l \equiv \boldsymbol{\xi} \cdot \boldsymbol{u} = r^2\,\mathrm{d}\varphi \,/\, \mathrm{d}\lambda$ is constant (here we consider the $\theta = \pi/2$ plane). For light:

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \qquad \boldsymbol{u} \cdot \boldsymbol{u} = 0 \Rightarrow 0 = g_{00}u^0u^0 + g_{11}u^1u^1 + g_{33}u^3u^3$$

(the g_{22} term vanishes, as $u^2 = \mathrm{d}\theta \,/\, \mathrm{d}\lambda = 0$). Checking the normalization:

$$0 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{e}{1 - 2GM/r}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{l}{r^2}\right)^2$$

Multiplying everything by $1 - 2GM/r$ and dividing by l^2 leads to:

$$0 = -\frac{e^2}{r^2} + \frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

Rearranging:

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2}$$

where we introduce:

$$b^2 = \frac{l^2}{e^2} \qquad W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

Note that λ is just a parameter introduced "by convenience" to characterize motion. If we reparametrize $\lambda \to k\lambda$ nothing should change in the motion. This leads to $e \to e/k$ and $l \to l/k$, so that $b^2 = l^2/e^2 \to b^2$ and also $\frac{1}{l^2}(\frac{\mathrm{d}}{\mathrm{d}\lambda})^2$ stays the same. So the equation we found is *invariant* under the reparameterization $\lambda \to k\lambda$. This means that a physical parameter must only involve *ratios* of l and e.
Note also that the equation is *even* in l. This means that motion does not depend on the *orientation* of the viewer (the sign of $\mathrm{d}\varphi \,/\, \mathrm{d}\lambda$ depends on the observer's position). So we can assume $l > 0$ without loss of generality.
One last thing to do before solving the equation is to *plot* $W_{\text{eff}}(r)$, or better, $G^2M^2W_{\text{eff}}$ over r/GM. For $r \to 0$ the potential diverges to $-\infty$, and then goes to 0 (in the first quadrant) as $r \to \infty$. This means that W_{eff} has a maximum, which is effectively found at $r = 3GM$, where $W_{\text{eff}} = 1/(27G^2M^2)$ (by taking the derivative and setting it to 0).
So, mathematically there are *circular orbits* (with $r = 3GM$), but they are not stable - and so physically they don't happen (a photon *cannot* orbit a black hole forever).

Also, a photon with "energy" (in the sense of effective potential W_{eff} plot) less than $\max W_{\text{eff}}$, coming from infinity, "bounces back", i.e. escapes away after a certain amount of time. This happens when:

$$\frac{1}{b^2} < W_{\text{eff},max} = \frac{1}{27G^2M^2} \Rightarrow l^2 > 27G^2M^2e^2 \Rightarrow l > \sqrt{27}GMe$$

meaning that the photon has *enough angular momentum*.