0.1 RFIM - Part 2

(Lesson ? of 02/12/19)
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We were trying to compute the *mean free energy* $\bar{F}$ of a Random Field Ising Model (RFIM). We noted that $\bar{F}$ depends on $\overline{\ln Z}$, which can be computed more easily by first evaluating $\overline{Z^m}$, for which we found the following expression:

$$\overline{Z^n} = \sum_{\{S^a\}} \exp\left[\frac{\beta J}{N}\sum_a \left(\sum_i S_i^a\right)^2 + \frac{\beta^2\delta^2}{2}\sum_i\left(\sum_a S_i^a\right)^2\right] \tag{1}$$

where we are using the *replica trick*, averaging over n replicas of a system with N spins. The notation $\{S^a\}$ denotes a sum over every spin of every replica.
To remove the squares, we use the Hubbard-Stratonovich transformation. Let $b > 0$:

$$\exp\left(\frac{b}{2}z^2\right) = \frac{1}{\sqrt{2\pi b}}\int \mathrm{d}x \exp\left(-\frac{x^2}{2b} \pm zx\right)$$

Otherwise, if the exponential argument is negative:

$$-\exp\left(-\frac{b}{2}z^2\right) = \frac{1}{\sqrt{2\pi b}}\int \mathrm{d}x \exp\left(-\frac{x^2}{2b} \pm izx\right)$$

These are just kinds of multivariate Gaussian integrals.

In our case, we choose:

$$z_a = \sqrt{2J\beta}\sum_i S_i^a$$

(the 2 factor is necessary to have a $b/2$ in the exponential) and $b = 1/N$, leading to:

$$\exp\left(\frac{b}{2}z_a^2\right) = \frac{1}{\sqrt{2\pi b}}\int \mathrm{d}x_a \exp\left(-\frac{x_a^2}{2b} + z_a x_a\right) \qquad \forall a$$

Substituting back in (1) we get:

$$\overline{Z^n} = \left(\frac{N}{2\pi}\right)^{n/2}\sum_{\{S^a\}}\int\prod_a \mathrm{d}x_a \exp\left[-\frac{N}{2}\sum_a x_a^2 + \sqrt{2J\beta}\sum_i\sum_a S_i^a x_a + \frac{\beta^2\delta^2}{2}\sum_i\left(\sum_a S_i^a\right)^2\right]$$

Note that now S_i^a appear *by itself* (there are no j (?)).

$$\overline{Z^n} = \left(\frac{N}{2\pi}\right)^{n/2}\prod_a \mathrm{d}x_a \exp\left[N\left(-\frac{1}{2}\sum_a x_a^2 + \log Z_1\right)\right]$$

$$Z_1(x_a) = \sum_{\{S^a=\pm 1\}} \exp\left[\sqrt{2\beta J}\sum_a x_a S^a + \frac{\beta^2\delta^2}{2}\left(\sum_a S^a\right)^2\right]$$

We now use compute the integrals with the *saddle point approximation* for $N \to \infty$. Also, we assume that $x_a = x \quad \forall a = 1, \ldots, n$, which works for this specific system. This means that we can simplify sums:

$$\sum_a x_a^2 = nx^2$$

So, we proceed to find the exponential maximum by differentiating:

$$\frac{\partial}{\partial x}\left[-\frac{1}{2}nx^2 + \log Z_1(x)\right] \stackrel{!}{=} 0 \Rightarrow nx = \frac{\partial}{\partial x}\log Z_1(x)$$

Denote the solution as x_m. Then we have:

$$nx_m = \frac{\sqrt{2\beta J}\sum\limits_{S^a=\pm 1}\left(\sum\limits_a S^a\right)e^{A[S,x_m]}}{\sum\limits_{S^a=\pm 1} e^{A[S,x_m]}}$$

where:

$$A[S,x] = \sqrt{2\beta J}x\sum_a S^a + \frac{\beta^2\delta^2}{2}\left(\sum_a S^a\right)^2$$

Note that nx_m looks like the *average* over a certain ensemble:

$$\langle y\rangle = \frac{\sum\limits_S ye^{-\beta H}}{\sum\limits_S e^{-\beta H}}$$

Rearranging:

$$\frac{x_m}{\sqrt{2\beta J}} = \frac{\sum\limits_{\{S^a=\pm 1\}}\frac{1}{n}\sum\limits_a S^a e^A}{\sum\limits_{\{S^a=\pm 1\}} e^A} \equiv \langle S\rangle_A = m$$

where m is the **magnetization** of the system.
Note that:

$$A = -\beta\tilde{H} = -\beta\left(\frac{1}{\sqrt{\beta}}(\ldots) + \beta(\ldots)\right)$$

so generally only one of the two terms will be dominant at a given temperature.
Recalling that $x_m = \sqrt{2\beta J}m$ we have, in summary:

$$\overline{Z^n} \approx \exp\left(N[-n\beta Jm^2 + \log Z_1(m)]\right)$$

$$Z(m) = \sum_{\{S^a=\pm 1\}} e^{A[S,m]}$$

$$A[S,m] = 2\beta Jm\sum_a S^a + \frac{\beta^2\delta^2}{2}\left(\sum_a S^a\right)^2$$

$$m = \frac{1}{Z_1(m)}\sum_{S^a=\pm 1}\left(\frac{1}{n}\sum_a S^a\right)e^{A[S,m]}$$

We now need to get rid of the remaining $(\sum_a S^a)^2$ by applying a second Hubbard-Stratonovich transformation. So we start from:

$$e^A = \exp\left(2\beta J m \sum_a S_a\right) \exp\left(\frac{\beta^2\delta^2}{2}\left[\sum_a S^a\right]^2\right)$$

Applying H-S with $b = 1$, $z = \beta\delta\sum_a S^a$ and $x = \nu$ we get:

$$\exp\left(\frac{\beta^2\delta^2}{2}\left[\sum_a S^a\right]^2\right) = \int \frac{\mathrm{d}\nu}{2\pi} \exp\left(-\frac{1}{2}\nu^2 + \nu\left[\beta\delta\sum_a S_a\right]\right)$$

And so:

$$\begin{aligned}
e^{A[S,m]} &= \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + \underbrace{(2\beta J m + \beta\delta\nu)}_{\eta}\sum_a S^a\right) \\
Z_1(m) &= \sum_{\{S^a=\pm 1\}} e^{A[S,m]} = \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2\right) \prod_a \sum_{S^a=\pm 1} e^{\nu S^a} = \\
&= \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2\right)[2\cosh\nu]^n = \\
&= \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh\nu]\right)
\end{aligned}$$

and so $Z_1(m) \xrightarrow[n\to 0]{} 1$.

$\boxed{\textbf{Exercise 0.1.1 } \text{(Magnetization)}\textbf{:}}$

> Prove that:
>
> $$m = \frac{1}{Z_1(m)} \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh(2\beta J m + \beta\delta\nu)]\right)\tanh[2\beta J m + \beta\delta\nu]$$

Note that as $n \to 0$ the magnetization becomes:

$$m = \int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2\right)\tanh(2\beta J m + \beta\delta\nu)$$

We can finally go back, recalling that $h = \delta\nu$ (gaussian noise), and so:

$$m = \int \underbrace{\frac{\mathrm{d}h}{\sqrt{2\pi\delta^2}} \exp\left(-\frac{h^2}{2\delta^2}\right)}_{p(h)} \tanh(\beta(2Jm + h)) = \overline{\tanh(\beta(2Jm + h))}$$

If $\beta \to \infty$ ($T \to 0$), the tangent becomes like a *periodic step function*, which is averaged with gaussian weights.
To find the *critical line* separating the *paramagnetic* and *ferromagnetic* behaviours, we need to evaluate:

$$\frac{\partial}{\partial m} m_{\mathrm{sc}}(m)\bigg|_{m=0} = 1 \tag{2}$$

Exercise 0.1.2 (Critical line)**:**

> Prove that the critical line satisfies the condition:
>
> $$2\beta J \int \mathrm{d}h\, p(h) \frac{1}{[\cosh(\beta h)]^2} = 1$$
>
> (just differentiate (2))

This can be written in a different manner. First, let:

$$J' = \frac{J}{\delta} \qquad \beta' = \beta\delta \qquad \tilde{h} = \beta h$$

so that:

$$2\beta' J' \int \frac{\mathrm{d}\tilde{h}}{\sqrt{2\pi}} \exp\left(-\frac{\tilde{h}^2}{2\beta'^2}\right) \frac{1}{(\cosh \tilde{h})^2} = 1$$

Exercise 0.1.3 (Critical ratio at $\boldsymbol{T = 0}$)**:**

> Show that the *para-ferro* transition at $T = 0$ takes place when:
>
> $$\frac{2J}{\delta} = \sqrt{\frac{\pi}{2}}$$

Finally, we can compute the **free energy** (let $k_B = 1$):

$$\begin{aligned}
\bar{F} &= -T\overline{\ln Z} = -T \frac{\partial}{\partial n} \overline{Z^n}\Big|_{n=0} = \\
&\underset{\substack{\text{Saddle}\\\text{point}}}{\approx} -T \frac{\partial}{\partial n} \left[\exp\left(N(-n\beta J n^2 + \log Z_1)\right)\right]_{n=0} = \\
&= -TN \left[-\beta J m^2 + \frac{\partial}{\partial n} \ln Z_1\right]_{n=0} = \\
&= N\left[Jm^2 - \frac{T}{Z_1}\frac{\partial}{\partial n} Z_1\right] = \\
&\underset{Z_1 \to 1}{=} N\left[Jm^2 - T\int \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2\right) \ln(2\cosh(2\beta J m + \beta\delta\nu))\right] = \\
&= N\left[Jm^2 - T\int \frac{\mathrm{d}h}{\sqrt{2\pi\delta^2}} \exp\left(-\frac{h^2}{2\nu^2}\right) \log\left[2\cosh\left(\beta(2Jm + h)\right)\right]\right]
\end{aligned}$$

which is the *free energy* averaged over the disorder. Note that the magnetization and the energy are interdependent.