(Lesson ? of 12/12/19) Compiled: December 12, 2019

0.1 Sherrington KirkPatrick - part 2

We want to compute the *free energy*:

$$
\begin{aligned}
f &= \lim_{N\to\infty} -\frac{1}{N\beta}\overline{\log(Z_J)} \\
Z_J &= \sum_{\{S_1,\dots,S_N\}} \exp(-\beta H_J[S_1,\dots,S_N]) \\
H_J &= -\sum_{i<j} J_{ij} S_i S_j \\
P(J_i) &= \frac{1}{\sigma}\sqrt{\frac{N}{2\pi}}\exp\left(-\frac{NJ_i^2}{2\sigma^2}\right)
\end{aligned}
$$

Introducing the *replica trick* we compute $\overline{\log(Z_J)}$ in terms of $\overline{Z^n}$:

$$
\overline{Z^n} = \exp\left(\frac{N\beta^2\sigma^2 n}{4}\right)\sum_{\substack{\{S_1,\dots,S_n\}\\ \alpha=1,\dots,n}} \exp\left(\frac{\beta^2\sigma^2}{4N}\sum_{\alpha,\beta}\left(\sum_{i=1}^N S_i^\alpha S_i^\beta\right)^2\right)
$$

To *unfold* the square we do an Hubbard-Stratonovich transformation:

$$
\overline{Z^n} = \sum_{\substack{\{S_1^\alpha,\dots,S_N^\alpha\}\\ \alpha=1,\dots,n}} \int_{-\infty}^{+\infty}\left(\prod_{\alpha<\beta}^n \mathrm{d}q_{\alpha\beta}\right)\exp\left(-\frac{N\beta^2\sigma^2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2\right)\exp\left(\beta^2\sigma^2\sum_{\alpha<\beta} q_{\alpha\beta}\sum_{i=1}^N S_i^\alpha S_i^\beta\right) \tag{1}
$$

Note that now the spin products are *not squared*, and we introduced variables $q_{\alpha\beta}$ that, as we will show later on, represent *overlaps*.
We can then bring the summation *inside* the integral, and, noting that it only acts on the highlighted term, we only need to compute the following:

$$
\begin{aligned}
\sum_{\substack{\{S_1^\alpha,\dots,S_N^\alpha\}\\ \alpha=1,\dots,n}} \exp\left(\beta^2\sigma^2\sum_{\alpha<\beta} a_{\alpha\beta}\sum_{i=1}^N S_i^\alpha S_i^\beta\right) &= \prod_{i=1}^N\left[\sum_{\substack{\{S_i^\alpha\}\\ \alpha=1,\dots,n}}\exp\left(\beta^2\sigma^2\sum_{\alpha<\beta} q_{\alpha\beta} S_i^\alpha S_i^\beta\right)\right] = \\
&= \left[\sum_{\substack{\{S^\alpha\}\\ \alpha=1,\dots,n}}\exp\left(\beta^2\sigma^2\sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta\right)\right]^N
\end{aligned} \tag{2}
$$

We then define:

$$
L(q_{\alpha\beta}) = \beta^2\sigma^2\sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta
$$

So that:

$$(2) = [\mathrm{Tr}\, e^{L}]^{N} = \exp\left(N \log[\mathrm{Tr}\, e^{L(q_{\alpha\beta})}]\right)$$

This explicit computation is very technical, especially in the limit $n \to 0$ we are interested on, and so we will not see it in full detail.
Substituting back in (1) leads to:

$$\begin{aligned}
\overline{Z^n} &= \exp\left(\frac{n\beta^2\sigma^2 N}{4}\right) \int_{-\infty}^{+\infty} \prod_{\alpha<\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-\frac{N\beta^2\sigma^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 + N \log[\mathrm{Tr}\, e^{L(q_{\alpha\beta})}]\right) = \\
&= \exp\left(\frac{n\beta^2\sigma^2 N}{4}\right) \int_{-\infty}^{+\infty} \prod_{\alpha<\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-nNA[q_{\alpha\beta}]\right)
\end{aligned}$$

$$A(q_{\alpha\beta}) = \left[\frac{1}{n}\frac{\beta^2\sigma^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{1}{n}\log[\mathrm{Tr}\, e^{L(q_{\alpha\beta})}]\right]$$

We compute the integral with a saddle point approximation. So we start by minimizing the argument:

$$\left.\frac{\partial A}{\partial q_{\alpha\beta}}(q_{\alpha\beta})\right|_{q^*_{\alpha\beta}} \stackrel{!}{=} 0$$

leading to:

$$\overline{Z^n} = \exp\left(-nNA[q^*_{\alpha\beta}]\right) \tag{3}$$

And now we can return to the free energy:

$$f = \lim_{\substack{N\to\infty \\ n\to 0}} -\frac{1}{nN\beta}(\overline{Z^n} - 1) \tag{4}$$

Expanding the exponential in (1) to first order and inserting in (4):

$$f = \frac{1 - nNA[q^*_{\alpha\beta}] - 1}{Nn}(-\frac{1}{\beta}) = \frac{1}{\beta}A[q^*_{\alpha\beta}]$$

(where we are ignoring the prefactor $\exp\bigl(n\beta^2\sigma^2 N/4\bigr)$).

We want now to understand what is the physical meaning of $q_{\alpha\beta}$. Stepping back, we start from the expression from $\overline{Z^n}$ right after the replica trick:

$$\begin{aligned}
\overline{Z^n} &= \sum_{\substack{\{S_1^\alpha,\dots,S_N^\alpha\} \\ \alpha=1,\dots,n}} \int_{-\infty}^{+\infty} \prod_{\alpha<\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-\frac{N\beta^2\sigma^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 + \frac{N}{N}\beta^2\sigma^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_{i=1}^{N} S_i^\alpha S_i^\beta\right) = \\
&= \sum_{\substack{\{S_1^\alpha,\dots,S_N^\alpha\} \\ \alpha=1,\dots,n}} \int_{-\infty}^{+\infty} \prod_{\alpha<\beta} \mathrm{d}q_{\alpha\beta} \exp\left(-Nu(q_{\alpha\beta}, S_i^\alpha, \dots, S_N^\alpha)\right)
\end{aligned}$$

$$u = \frac{\beta^2\sigma^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \beta^2\sigma^2 \sum_{\alpha<\beta} q_{\alpha\beta} \frac{1}{N} \sum_{i=1}^{N} S_i^\alpha S_i^\beta$$

And then we compute the integral by saddle point approximation:

$$\frac{\partial u}{\partial q_{\alpha\beta}}(q_{\alpha\beta})\bigg|_{q^*_{\alpha\beta}} \stackrel{!}{=} 0 \Rightarrow q^*_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} S_i^\alpha S_i^\beta$$

So each term $q_{\alpha\beta}$, in the saddle point approximation, represents the normalized scalar product of spins of the replicas α and β. So $q_{\alpha\beta}$ is a real symmetric matrix. We also have a second order phase transition at *critical temperature* T_c. For $T > T_c$ (1) the system is ergodic, and for $T < T_c$ (2) is non ergodic, and the phase space splits in *disjoint ergodic components*, where a system starting in one of them *cannot evolve* to one of the others.
The two possibilities correspond to two different ansatz for $q_{\alpha\beta}$:

1. $$\begin{pmatrix} 1 & q_0 & \dots & q_0 \\ q_0 & \ddots & q_0 & \vdots \\ \vdots & q_0 & \ddots & q_0 \\ q_0 & \dots & q_0 & 1 \end{pmatrix}_{n\times n} = q_{\alpha\beta}$$
   This ansatz leads to the minimum of the free energy.

2. It is easier to understand the ansatz for $n \gg 1$. Here we have a *hierarchical ansatz*. We start by constructing:
   $$q_{\alpha\beta} = \begin{pmatrix} M_1 & Q_0 & \dots & Q_0 \\ Q_0 & M_1 & Q_0 & \vdots \\ \vdots & Q_0 & \ddots & Q_0 \\ Q_0 & \dots & Q_0 & M_1 \end{pmatrix}$$
   where Q_0 are matrices $m_1 \times m_1$ with entries all equal to q_0. Every M_1 block is a $m_1 \times m_1$ matrix with the *same structure*:
   $$M_1 = \begin{pmatrix} M_2 & Q_1 & \dots & Q_1 \\ Q_1 & \ddots & Q_1 & \vdots \\ \vdots & \ddots & M_2 & Q_1 \\ Q_1 & \dots & Q_1 & M_2 \end{pmatrix}$$
   with Q_1 blocks of size $m_2 \times m_2$ of all entries equal to q_1. We can reiterate this structure to find M_2, and then (for $n \to \infty$) take this process to M_∞, so that $n > m_1 > m_2 > m_3 > \cdots > m_\infty$, and also $q_0 < q_1 < \cdots < q_\infty$. Then, in the continuum limit we have a function $q(x)\colon [0,1] \mapsto [0,1]$, which is *monotonic*, and so we can invert it:
   $$x[q] = \int_0^q \mathrm{d}q\, p(q)$$
   which is the *cumulative probability* of getting an overlap $\leq q$ between two replicas sampled at random from a Boltzmann distribution. Only then we can take the limit for $n \to 0$.

[Insert figure 1]

0.2 p-spin model

For $p = 3$ the p-spin model involves an energy defined as follows:

$$H_J = -\sum_{i<j<k} J_{ijk}\sigma_i\sigma_j\sigma_k \qquad \sigma_i \in \mathbb{R} \tag{5}$$

Note that now spins $\boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_N\}$ *are not discrete*, but are real numbers: $\sigma_i \in (-\infty, +\infty)$.
To keep the energy bounded we introduce a *spherical constraint*, i.e. we fix the norm of the spin vector $\boldsymbol{\sigma}$:

$$N \overset{!}{=} \sum_{i=1}^{N} \sigma_i^2$$

We can generalize (5) to a generic value of $p \in \mathbb{N}$:

$$H_J = -\sum_{i_1<\dots<i_p} J_{i_1,\dots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}$$

We will make explicit computations in the case of $p = 3$, but all the conclusions will be general for systems with $p > 2$. $p = 2$ is a special case, and must be analysed separately.
We choose the J_{ijk} with a gaussian pdf:

$$P(J_{ijk}) = \frac{N^{\frac{p-1}{2}}}{\sqrt{p!\pi}} \exp\left(-\frac{N^{p-1}}{p!} J_{ijk}^2\right)$$

The spins are sampled with a Boltzmann pdf:

$$P_J(\sigma_1, \dots, \sigma_N) = \exp(-\beta H_J[\sigma_1, \dots, \sigma_N])\delta\left(N - \sum_{i=1}^{N} \sigma_i^2\right)$$

And then the partition function is defined as:

$$Z_J = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \mathrm{d}\sigma_i \, P_J(\sigma_1, \dots, \sigma_N)$$

As before, we are interested in computing the **free energy**:

$$f = \lim_{N\to\infty} -\frac{1}{N^\beta}\overline{\log(Z_J)}$$

Note that, in this model, we have *non-linear interactions*, that is interactions of more than two spins at once (terms of order > 2).

Again we employ the replica trick:

$$f = \lim_{\substack{N\to\infty \\ n\to 0}} -\frac{1}{N\beta}\frac{\overline{Z^n}-1}{n}$$

$$\overline{Z^n} = \int_{-\infty}^{+\infty} \prod_{i<j<k} \mathrm{d}J_{ijk}\, P(J_{ijk}) \left[\int_{-\infty}^{+\infty} \prod_{i=1}^{N} \mathrm{d}\sigma_i \exp(-\beta H_J[\sigma_1,\dots,\sigma_N])\delta\left(N - \sum_{i=1}^{N}\sigma_i^2\right)\right]^n =$$

$$= \int_{-\infty}^{+\infty} \prod_{i<j} \mathrm{d}J_{ijk}\, P(J_{ijk}) \int_{-\infty}^{+\infty} \underbrace{\prod_{\alpha=1}^{n}\prod_{i=1}^{N} \mathrm{d}\sigma_i{}^\alpha}_{\mathcal{D}\sigma} \exp\left(\beta \sum_{\alpha=1}^{n} \underbrace{\sum_{i<j<k}}_{\sim N^3} J_{ijk}\sigma_i^\alpha\sigma_j^\alpha\sigma_k^\alpha\right) \prod_{\alpha=1}^{n} \delta\left(N - \sum_{i=1}^{N}[\sigma_i^\alpha]^2\right)$$

For the central sum we have to compute $\sim N^3$ gaussian integrals, of the form:

$$\int_{-\infty}^{+\infty} \mathrm{d}J_{ijk} \exp\left(-\frac{N^{p-1}}{p!}J_{ijk}^2 + \beta J_{ijk}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha\sigma_k^\alpha\right) = \exp\left(\frac{\beta^2 p!}{4N^{p-1}}\left(\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha\sigma_k^\alpha\right)^2\right) =$$

$$= \exp\left(\frac{\beta^2 p!}{4N^{p-1}}\sum_{\alpha<\beta}^{n}(\sigma_i^\alpha\sigma_i^\beta)(\sigma_j^\alpha\sigma_j^\beta)(\sigma_k^\alpha\sigma_k^\beta)\right)$$

Taking into account the N^3 terms:

$$\exp\left(\frac{\beta^2 p!}{4N^{p-1}}\sum_{\alpha<\beta}\sum_{i<j<k}(\sigma_i^\alpha\sigma_i^\beta)(\sigma_j^\alpha\sigma_j^\beta)(\sigma_k^\alpha\sigma_k^\beta)\right)$$

Note that:

$$p!\sum_{i_1<i_2<\dots i_p} \equiv \sum_{i_1,i_2,\dots,i_p}$$

and so:

$$\exp\left(\frac{\beta^2 \cancel{p!}}{4N^{p-1}}\sum_{\alpha<\beta}\sum_{ijk}(\sigma_i^\alpha\sigma_i^\beta)(\sigma_j^\alpha\sigma_j^\beta)(\sigma_k^\alpha\sigma_k^\beta)\right)$$

Also, note that:

$$\exp\left(\frac{\beta^2}{4}N\sum_{\alpha<\beta}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^\alpha\sigma_i^\beta\right)^p\right)$$