0.1 Continuity of Brownian Path

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Consider a particle starting in $x = 0$ at $t = 0$, and traversing N points $\{x_i\}_{i=1,\dots,N}$ such that all increments $\Delta x_i = x_i - x_{i-1}$ are *independent* and described by a *gaussian pdf*. The density function for such a trajectory $\{x_i\}$ is the usual product of transition probabilities:

$$d\mathbb{P}_{t_1,\dots,t_N}(x_1,\dots,x_N) = \left(\prod_{i=1}^N \frac{dx_i}{\sqrt{4\pi\Delta t_i D}}\right)\exp\left(-\sum_{i=1}^N \frac{(\Delta x_i)^2}{4D\Delta t_i}\right) \quad {\scriptstyle \begin{array}{c}\Delta t_i = t_i - t_{i-1}\\ \Delta x_i = x_i - x_{i-1}\end{array}} \tag{1}$$

We now show that, taking the continuum limit $\max_i \Delta t_i \to 0$ leads to paths $\{x(\tau)\}$ that are *almost surely continuous*. In other words, for any interval $T \subseteq \mathbb{R}$, the subset $N \subset \mathbb{R}^T$ of functions that are discontinuous has 0 Wiener measure.
Mathematically, we want to show that, as $\Delta t_i \to 0$, the probability that Δx_i is close to 0 approaches certainty:

$$\lim_{\Delta t_i \to 0} \mathbb{P}(|\Delta x_i| < \epsilon) = 1 \quad \forall \epsilon > 0$$

This is just the probability that, during time Δt_i, the particle makes a jump of size lower than ϵ:

$$\begin{aligned}\mathbb{P}(|\Delta x_i| < \epsilon) &= \mathbb{P}(x_{i-1} - \epsilon < x_i < x_{i-1} + \epsilon | x(t_{i-1}) = x_i) = \\ &= \int_{x_{i-1}-\epsilon}^{x_{i-1}+\epsilon} \frac{dx_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{(x_i - x_{i-1})^2}{4D\Delta t_i}\right) = \\ &\underset{(a)}{=} \int_{-\epsilon}^{+\epsilon} \frac{d\Delta x_i}{\sqrt{4\pi D\Delta t_i}} \exp\left(-\frac{(\Delta x_i)^2}{4D\Delta t_i}\right)\end{aligned}$$

where in (a) we translated the variable of integration $\Delta x_i = x_i - x_{i-1}$.
With another change of variables:

$$\frac{(\Delta x_i)^2}{\Delta t_i} = z^2 \Rightarrow z = \frac{\Delta x_i}{\sqrt{\Delta t_i}} \Rightarrow d\Delta x_i = dz\,\sqrt{\Delta t_i}$$

we get:

$$\mathbb{P}(|\Delta x_i| < \epsilon) = \int_{|z|<\epsilon/\sqrt{\Delta t_i}} \frac{dz\,\cancel{\sqrt{\Delta t_i}}}{\sqrt{4\pi D\cancel{\Delta t_i}}} \exp\left(-\frac{z^2}{4D}\right)$$

And taking the continuum limit leads to:

$$\lim_{\Delta t_i \to 0} \mathbb{P}(|\Delta x_i| < \epsilon) = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{4\pi D}} \exp\left(-\frac{z^2}{4D}\right) = 1$$

0.2 Differentiability of Brownian Path

With a very similar calculation (here omitted) we can also show that:

$$\lim_{\Delta t_i \downarrow 0} \left(\left|\frac{\Delta x_i}{\Delta t_i}\right| > k\right) = 1 \quad \forall k > 0$$

meaning that Brownian paths are *almost surely everywhere non-differentiable*.

Nonetheless, it is sometimes useful to consider "formal derivatives" of a Brownian path, that acquire a definite meaning only when considering a *finite discretization*. For example, we can start from (1) and rewrite it as:

$$d\mathbb{P}_{t_1,\ldots,t_N}(x_1,\ldots,x_N) = \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i}{\sqrt{4\pi\Delta t_i}}\right) \exp\left(-\frac{1}{4D}\sum_{i=1}^{N} \Delta t_i \left(\frac{\Delta x_i}{\Delta t_i}\right)^2\right)$$

Then, in the continuum limit $\Delta t_i \to 0$, the sum in the exponential argument becomes a Riemann integral:

$$\sum_{i=1}^{N} \Delta t_i \left(\frac{\Delta x_i}{\Delta t_i}\right)^2 \xrightarrow[\Delta t \to 0]{} \int_0^t \mathrm{d}\tau \underbrace{\left(\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\right)^2}_{\dot{x}^2(\tau)} \qquad t = t_N$$

where $t = t_N$. Substituting it back leads to:

$$\mathrm{d}x_w(\tau) = \prod_{\tau=0^+}^{t} \frac{\mathrm{d}x(\tau)}{\sqrt{4\pi D\,\mathrm{d}\tau}} \exp\left(-\frac{1}{4D}\int_0^t \dot{x}^2(\tau)\,\mathrm{d}\tau\right)$$

This expression has no rigorous meaning in this form ($\dot{x}(\tau)$ *does not* exists!) but can be *formally manipulated* into other expressions that *have* a definite meaning, thus proving useful for the discussion.

0.3 Forces on the particle

We want now to generalize the framework we obtained to the case of a diffusing particle subject to *external forces*, e.g. a drop of ink diffusing through a water medium in the presence of gravity.
To do this, we first return to the beginning, deduce a Master Equation for a more general *evolution*, and then choose the right probability distribution reproducing the behaviour in presence of forces.
So, let's start by considering a particle moving on a *uniform* one-dimensional lattice ($x_i = i \cdot l$, $t_n = n \cdot \epsilon$), and satisfying the Markovian property, meaning that the probability $W_i(t_{n+1})$ of being at the position labelled by i at the *next time-step* t_{n+1} depends only on the current state t_n, that is on the current probabilities $W_j(t_n)\,\forall j$) and on the current transition probabilities $W_{ij}(t_n)$ from j to i:

$$W_i(t_{n+1}) = \sum_{j=-\infty}^{+\infty} W_{ij}(t_n) W_j(t_n) \tag{2}$$

Previously, we assumed that:

$$W_{ij}(t_n) = \delta_{j,i-1}P_+ + \delta_{j,i+1}P_-$$

Which means that the particle only jumps from adjacent positions, one step at a time, and cannot remain at the same place. This Master Equation leads, in $d = 3$ and in the continuum limit, to the usual Diffusion Equation:

$$\frac{\partial}{\partial t} W(\boldsymbol{x}, t|\boldsymbol{x_0}, t_0) = \nabla^2 W(\boldsymbol{x}, t|\boldsymbol{x_0}, t_0)$$

We now consider a more general case, where we drop the discretization of the space domain, allowing *jumps* of *any size* in $\mathbb{R}$. Then (2) becomes:

$$W(x, t_{n+1})\,\mathrm{d}x = \int_{-\infty}^{+\infty} \mathrm{d}z\, W(x, t_{n+1}|x - z, t_n) W(x - z, t_n) \tag{3}$$

The integrand is the probability of the particle being in $[x - z, x - z + \mathrm{d}x]$ at time t_n and making a jump of size z to reach $[x, x + \mathrm{d}x]$ at time t_{n+1}. By summing over *all possible jump sizes* we compute the total probability of the particle being *near* the arrival position.

If we require *jumps* to be **independent** of each other[1], as it is physically evident by the problem's symmetry, then the *jump* probabilities $W(x, t_{n+1}|x - z, t_n)$ depend only on the *jump size* z.

Assuming a *isolate system*, as the particle cannot *escape*, **probability is conserved**:

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}x\, W(x, t_{n+1}) &\overset{!}{=} \int_{\mathbb{R}} \mathrm{d}y\, W(y, t_n) \\
&\underset{(3)}{=} \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}x W(x, t_{n+1}|x - z, t_n)\, W(x - z, t_n) = \\
&\underset{(a)}{=} \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}y\, W(y + z, t_{n+1}|y, t_n) W(y, t_n) = \\
&\underset{(b)}{=} \left(\int_{\mathbb{R}} \mathrm{d}z\, W(\bar{y} + z, t_{n+1}|\bar{y}, t_n) \right) \left(\int_{\mathbb{R}} \mathrm{d}y\, W(y, t_n) \right) \qquad \forall \bar{y} \in \mathbb{R}
\end{aligned}$$

where in (a) we changed variables $x \mapsto y = x - z$, with $\mathrm{d}y = \mathrm{d}x$, and in (b) we used the *independent increments* property ($\bar{y}$ is a arbitrary constant). Comparing the first and last lines leads to:

$$\int_{\mathbb{R}} \mathrm{d}z\, W(y + z, t_{n+1}|y, t_n) = 1$$

Intuitively, if the particle *cannot disappear*, it *must make a jump*.

Here on, for notation simplicity, we denote:

$$W(y + z, t_{n+1}|y, t_n) \equiv W(+z|y, t_n)$$

Starting from (3) and taking the continuum limit in time we can write a *more*

[1] ∧This is a stronger requirement than the Markovian property. In fact, *independent increments* imply a *Markov process*, but the converse is not true. See `http://statweb.stanford.edu/~adembo/math-136/Markov_note.pdf`

general diffusion equation. We start by constructing the difference quotient:

$$
\begin{aligned}
W(x,t_{n+1}) - W(x,t_n) &= \int_{\mathbb{R}} \mathrm{d}z\, W(+z|x-z,t_n)W(x-z,t_n) - W(x,t_n) = \\
&= \int_{\mathbb{R}} \mathrm{d}z\, W(+z|x-z,t_n)W(x-z,t_n) - \underbrace{\int \mathrm{d}z\, W(+z|x,t_n)}_{=1} W(x,t_n) = \\
&= \int_{\mathbb{R}} \mathrm{d}z \Big[\underbrace{W(+z|x-z,t_n)W(x-z,t_n)}_{F_z(x-z)} - \underbrace{W(+z|x,t_n)W(x,t_n)}_{F_z(x)}\Big] = \\
&= \int_{\mathbb{R}} \mathrm{d}z \left[F_z(x-z) - F_z(x)\right] = \\
&\underset{(a)}{=} \int_{\mathbb{R}} \mathrm{d}z \left[\cancel{F_z(x)} - z\frac{\partial}{\partial x}F_z(x) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}[F_z(x)] + \dots - \cancel{F_z(x)}\right] = \\
&= -\int_{\mathbb{R}} \mathrm{d}z\, z\frac{\partial}{\partial x}[F_z(x)] + \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}z\, z^2\frac{\partial^2}{\partial x^2}[F_z(x)] + \dots = \\
&\underset{(b)}{=} -\frac{\partial}{\partial x}\Bigg[\underbrace{\left(\int_{\mathbb{R}} \mathrm{d}z\, z\, W(+z|x,t_n)\right)}_{\mu_1(x,t_n)} W(x,t_n)\Bigg] + \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigg[\underbrace{\left(\int_{\mathbb{R}} \mathrm{d}z\, z\, W(+z|x,t_n)\right)}_{\mu_2(x,t_n)} W(x,t_n)\Bigg] + \dots
\end{aligned}
$$

where $F_z(x)$ is the probability of a *jump* of size z from the position x. In (a) we expanded F_z about x, and in (b) we exchanged the order of integrals and derivatives. Then we define the k-th moment of the *jump* pdf as follows:

$$
\mu_k(x,t) = \int_{\mathbb{R}} \mathrm{d}z\, z^k\, W(+z|x,t)
$$

This allows us to rewrite the above difference in a more compact form:

$$
W(x,t_{n+1}) - W(x,t_n) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!}\frac{\partial^k}{\partial x^k}\left(\mu_k(x,t_n)W(x,t_n)\right)
$$

Physically, as probability is conserved, by the continuity equation, the *change* in probability density equals the divergence of a *flux*, which is just the x derivative in this one-dimensional case. So, if we *extract* a derivative, we can write the flux explicitly:

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(\sum_{k=1}^{+\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1}}{\partial x^{k-1}}(\mu_k(x,t_n)W(x,t_n))\right) \\
&\equiv -\frac{\partial}{\partial x}J(x,t_n)
\end{aligned}
$$

where $J(x,t_n)$ is the *outward flux* at x, meaning that if $J > 0$, then $W(x,t_{n+1}) < W(x,t_n)$ (the particle *escapes* from x to another place), and otherwise if $J < 0$ we have $W(x,t_{n+1}) > W(x,t_n)$ (the particle is *sucked in* x).

If we integrate both sides over x and apply the probability conservation we get the boundary conditions for the flux:

$$\begin{aligned}\int_{\mathbb{R}} (W(x,t_{n+1}) - W(x,t_n))\,\mathrm{d}x &= \int_{\mathbb{R}} \mathrm{d}x \left(-\frac{\partial}{\partial x} J(x,t_n)\right) \\ 1 - 1 = -J(x,t_n)\Big|_{-\infty}^{+\infty} &= J(-\infty,t_n) - J(+\infty,t_n)\end{aligned}$$

This means that, in a *isolate system*, the *flux* at $\pm\infty$ must be the same.
Finally, normalizing by the time interval we get the complete difference quotient, which will become a time derivative in the continuum limit.

$$\frac{W(x,t_{n+1}) - W(x,t_n)}{t_{n+1}-t_n} = \frac{\partial}{\partial x}\left\{\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial x^{k-1}} \frac{\mu_k(x,t_n) W(x,t_n)}{t_{n+1}-t_n}\right\} \tag{4}$$

Letting $t_{n+1} - t_n = \epsilon$, in the limit $\epsilon \to 0$ the left side will be $\dot{W}(x,t)$.

All that's left is to find an explicit definition for the *jump pdf* $W(+z|x,t)$. Previously, we assumed a **gaussian** pdf for the displacements:

$$z \sim \frac{1}{\sqrt{4\pi D\epsilon}} \exp\left(-\frac{(\Delta x)^2}{4D\epsilon}\right)$$

With this choice, the first two moments become:

$$\mu_1 = 0 \qquad \mu_2 = 2D\epsilon$$

And the variance:

$$\mathrm{Var}(z) = \mu_2 - \mu_1^2 = 2D\epsilon \propto \epsilon$$

However, for a particle subject to a force we would expect to have a *preferred jump direction*, leading to a *constant velocity motion* in the direction of the force. So we require a different μ_1:

$$\langle z \rangle = \mu_1 = \int_{\mathbb{R}} z W(+z|x,t) \propto \epsilon f(x)$$

We still want to fix the variance to be proportional to ϵ, as it is expected in a diffusion process.
An appropriate choice for such a distribution is given by:

$$W(+z|x,t) = F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}} \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}}\right) \tag{5}$$

with $F, \hat{D} \colon \mathbb{R} \to \mathbb{R}$ functions, satisfying certain conditions, and with a physical meaning that we will now see.
First of all, we check the normalization:

$$1 \stackrel{!}{=} \int_{\mathbb{R}} \mathrm{d}z\, W(+z|x,t) = \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} \int_{\mathbb{R}} \mathrm{d}z\, F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) \underset{(a)}{=} \int_{\mathbb{R}} \mathrm{d}y\, F(y)$$

where in (a) we changed variables:

$$y = \frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}} \qquad \mathrm{d}z = \sqrt{\epsilon \hat{D}(x,t)}\, \mathrm{d}y \tag{6}$$

Then we compute the first moment:

$$\begin{aligned}
\langle z \rangle = \mu_1(x,t) &= \int_{\mathbb{R}} \mathrm{d}z\, z\, F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} = \\
&\underset{(6)}{=} \int_{\mathbb{R}} \mathrm{d}y \left(\epsilon f(x,t) + y\sqrt{\epsilon \hat{D}(x,t)}\right) F(y) = \\
&= \epsilon f(x,t) \underbrace{\int_{\mathbb{R}} F(y)\, \mathrm{d}y}_{=1} + \sqrt{\epsilon \hat{D}(x,t)} \int_{\mathbb{R}} yF(y) \stackrel{!}{=} \epsilon f(x,t)
\end{aligned}$$

So, in order to have the right normalization and the desired $\langle z \rangle$ we need:

$$\begin{cases} \int_{\mathbb{R}} \mathrm{d}y\, F(y) = 1 \\ \int_{\mathbb{R}} \mathrm{d}y\, yF(y) = 0 \end{cases}$$

Both conditions are satisfied, for example, by all even normalized functions.
For the second moment:

$$\begin{aligned}
\mu_2(x,t) &= \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} \int_{\mathbb{R}} \mathrm{d}z\, z^2 F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) = \\
&\underset{(6)}{=} \int_{\mathbb{R}} \mathrm{d}y\, (\epsilon f(x,t) + y\sqrt{\epsilon \hat{D}(x,t)})^2 F(y) = \\
&= \int_{\mathbb{R}} \mathrm{d}y\, F(y)[\epsilon^2 f^2 + y^2 \hat{D}\epsilon + \cancel{2\epsilon\sqrt{\epsilon \hat{D}} f y}] = \\
&= \epsilon^2 f^2 + \hat{D}\epsilon \int_{\mathbb{R}} \mathrm{d}y\, y^2 F(y) = \epsilon^2 f^2 + \hat{D}\epsilon \langle y^2 \rangle_{F(y)}
\end{aligned}$$

And so the variance becomes:

$$\mathrm{Var}(z) = \mu_2 - \mu_1^2 = \epsilon \hat{D} \langle y^2 \rangle_{F(y)} \propto \epsilon$$

which is proportional to ϵ as desired. For notational simplicity, we introduce a new function $D\colon \mathbb{R} \to \mathbb{R}$ such that:

$$\mathrm{Var}(z) = \epsilon \hat{D} \langle y^2 \rangle_{F(y)} \equiv 2D(x,t)\epsilon \Rightarrow \mu_2(x,t) = \epsilon^2 f^2 + 2D(x,t)$$

We note that higher order moments are all of order $O(\epsilon^{3/2})$. For example, the third moment is:

$$\begin{aligned}
\mu_3(x,t) &= \frac{1}{\sqrt{\epsilon\hat{D}(x,t)}\int_{\mathbb{R}}} \mathrm{d}z\, z^2 F\left(\frac{z-\epsilon f(x,t)}{\sqrt{\epsilon\hat{D}(x,t)}}\right) = \\
&\underset{(6)}{=} \int_{\mathbb{R}} \mathrm{d}y\, (\epsilon f(x,t) + y\sqrt{\epsilon\hat{D}(x,t)})^3 F(y) = \\
&= \int_{\mathbb{R}} \mathrm{d}y \left(\epsilon^3 f^3 + y^3(\epsilon\hat{D})^{3/2} + \cancel{3\epsilon^2 f^2 y\sqrt{\epsilon\hat{D}}} + 3\epsilon^2 f\hat{D}y^2\right) F(y) = \\
&= \epsilon^3 f^3 + (\epsilon\hat{D})^{3/2} + 3\epsilon^2 f\hat{D}\langle y^2\rangle_{F(y)} = O(\epsilon^{3/2})
\end{aligned}$$

Substituting back (5) in (4) we arrive to:

$$\begin{aligned}
\frac{W(x,t_{n+1}) - W(x,t_n)}{\epsilon} = -\frac{\partial}{\partial x}\left[W(x,t_n)\underbrace{\frac{\mu_1(x,t_n)}{\epsilon}}_{f(x,t)}\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\underbrace{\frac{\mu_2(x,t_n)}{\epsilon}}_{\epsilon f^2 + 2D(x,t)} W(x,t_n)\right] + \\
\underbrace{+\frac{1}{3!}\frac{\partial^3}{\partial x^3}\left[W(x,t_n)\frac{\mu_3(x,t_n)}{\epsilon}\right] + \dots}_{O(\epsilon^{1/2})}
\end{aligned}$$

Taking the limit $\epsilon \to 0$, we are left with:

$$\begin{aligned}
\frac{\partial W(x,t)}{\partial t} &= -\frac{\partial}{\partial x}[f(x,t)W(x,t)] + \frac{1}{\cancel{2}}\frac{\partial^2}{\partial x^2}[\cancel{2}D(x,t)W(x,t)] = \\
&= -\frac{\partial}{\partial x}\left[f(x,t)W(x,t) - \frac{\partial}{\partial x}\big(D(x,t)W(x,t)\big)\right]
\end{aligned}$$

This is the **Fokker-Planck equation**, describing the diffusion process in the presence of a *force* $f(x,t)$, and a diffusion parameter $D(x,t)$.
Note that, in absence of forces $f(x,t) \equiv 0$ and with a constant diffusion $D(x,t) \equiv D$ we retrieve the usual *diffusion equation*:

$$\frac{\partial}{\partial t}W(x,t) = D\frac{\partial^2}{\partial x^2}W(x,t)$$

0.4 Langevin equation

The Fokker-Planck equation involves *probability distributions*, meaning that it describes the behaviour of *ensembles of trajectories* at once. However, we can find an equivalent description by focusing on a *single path*.
We start with a Wiener process, that is a stochastic process with *independent* and *gaussian* increments and *continuous* paths. Considering a time discretization $\{t_i\}$, the evolution of a single trajectory is described by:

$$x(t_{i+1}) = x(t_i) + \Delta x(t_i) \tag{7}$$

where each increment $\Delta x(t_i)$ is sampled from a gaussian pdf:

$$\Delta x_i(t_i) \sim \frac{1}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{(\Delta x)^2}{4D\Delta t_i}\right)$$

To simplify notation, we change variables, so that:

$$\frac{\Delta B^2}{2} = \frac{\Delta x^2}{4D} \Rightarrow \Delta B = \frac{\Delta x}{\sqrt{2D}}$$

If $x \sim p(x)$, and $y = y(x) \sim g(y)$, then by the rule for a change of random variables we have:

$$g(y) = p(x(y))\frac{\mathrm{d}x(y)}{\mathrm{d}y}$$

In this case:

$$\Delta B \sim \frac{1}{\sqrt{4\pi D\Delta t_i}} \exp\left(-\frac{(\Delta B)^2}{2\Delta t_i}\right) \underbrace{\frac{\mathrm{d}\Delta x}{\mathrm{d}\Delta B}}_{\sqrt{2D}} = \frac{1}{\sqrt{2\pi\Delta t_i}} \exp\left(-\frac{(\Delta B)^2}{2\Delta t_i}\right)$$

Note that now $\langle \Delta B^2(t_i)\rangle = \Delta t_i$, leaving out the D.
Substituting in (7) and rearranging we get:

$$x(t_{i+1}) - x(t_i) = \sqrt{2D}\Delta B(t_i) \tag{8}$$

We want now to form a time derivative in the left side, in order to arrive a (stochastic) differential equation for paths. To do this, we first extract a Δt_i factor from $\Delta B(t_i)$ by performing another change of variables:

$$\Delta B(t_i) \equiv \Delta t_i \xi(t_i) \tag{9}$$

so that $\Delta x_i = \sqrt{2D}\Delta t_i \xi_i$, and all the *randomness* is now contained in the random variable ξ, which is distributed according to:

$$\xi(t_i) \sim \frac{1}{\sqrt{2\pi\Delta t_i}} \exp\left(-\frac{\Delta t_i^2 \xi_i^2}{2\Delta t_i}\right) \underbrace{\frac{\mathrm{d}\Delta B_i}{\mathrm{d}\xi(t_i)}}_{\Delta t_i} = \sqrt{\frac{\Delta t_i}{2\pi}} \exp\left(-\frac{\Delta t_i}{2}\xi_i^2\right) \qquad \xi_i \equiv \xi(t_i)$$

Substituting back in (8) and dividing by Δt_i leads to:

$$\frac{x(t_{i+1}) - x(t_i)}{\Delta t_i} = \sqrt{2D}\xi(t_i)$$

And by taking the continuum limit $\Delta t_i \to 0$ we get the **Langevin equation** for a Brownian particle:

$$\dot{x}(t) = \sqrt{2D}\xi(t) \tag{10}$$

We can see $\xi(t)$ as a *highly irregular, quickly varying function*, which, in a certain sense, expresses the result of Brownian collisions at a certain instant. In particular, the following holds:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(t') \rangle = \delta(t-t')$$

meaning that the values of $\xi(t)$ at different instants are completely *independent*.

Note that, as we saw previously, Brownian paths are not differentiable - and so $\dot{x}(t)$ does not exist, and this is just a *formal* equation, with a definite meaning only in a given *discretization*. Also, note that $\xi(t)$ is a random variable, and so this is an example of a **stochastic differential equation**. It is not clear how to find a solution to such an equation, or even how to *define* what a solution should be - and this will be the main topic of the next section.
We can rewrite (10) in a more *rigorous* form by "multiplying by dt", i.e. performing the change of variables (9), which - in the continuum limit - is $\mathrm{d}B = \xi\,\mathrm{d}t$, leading to:

$$\mathrm{d}x(t) = \sqrt{2D}\,\mathrm{d}B \qquad \mathrm{d}B \sim \frac{1}{\sqrt{2\pi\,\mathrm{d}t}} \exp\left(-\frac{\mathrm{d}B^2}{2\,\mathrm{d}t}\right)$$

Before moving on, we want to generalize this equation to the presence of *external forces*. As we saw previously, this just results in adding a *constant velocity motion* to the particle, leading to the full **Langevin equation**:

$$\begin{aligned} \dot{x}(t) &= f(x,t) + \sqrt{2D(x,t)}\xi(t) \\ \mathrm{d}x(t) &= f(x,t)\,\mathrm{d}t + \sqrt{2D(x,t)}\,\mathrm{d}B \qquad \mathrm{d}B \sim \frac{1}{\sqrt{2\pi\,\mathrm{d}t}} \exp\left(-\frac{\mathrm{d}B^2}{2\,\mathrm{d}t}\right) \end{aligned} \tag{11}$$

The *physical* meaning of $f(x,t)$ and $D(x,t)$ can be more clearly seen by comparing (11) to the equation of motion of the Brownian particle.
Consider a particle of mass m immersed in a fluid, with a radius a that is much larger than the surrounding molecules (typically $\sim 10^{-9}$ to 10^{-7} m). The forces acting on it will be that of *viscous friction* $-\gamma\dot{\boldsymbol{r}}$, eventual *external forces* $\boldsymbol{F}_{\text{ext}}$ (e.g. gravity), and a rapidly varying and *random* term $\boldsymbol{F}_{\text{noise}}$, encompassing the effect of the large number of collisions ($\sim 10^{12}$/s) with the smaller fluid particles:

$$m\ddot{\boldsymbol{r}}(t) = -\gamma\dot{\boldsymbol{r}} + \boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{noise}}(t)$$

Dividing both sides by γ:

$$\frac{m}{\gamma}\ddot{\boldsymbol{r}}(t) = -\dot{\boldsymbol{r}} + \frac{\boldsymbol{F}_{\text{ext}}(\boldsymbol{r},t)}{\gamma} + \frac{\boldsymbol{F}_{\text{noise}}(t)}{\gamma} \tag{12}$$

Assuming a spherical particle, γ is given by Stokes law to be $6\pi a\eta$, where η is the viscosity of the surrounding fluid.
Note that, if we ignore the external force and the random term, the equation becomes:

$$\frac{\mathrm{d}\dot{\boldsymbol{r}}(t)}{\mathrm{d}t} = -\frac{\gamma}{m}v(t)$$

which has solution:

$$\dot{\boldsymbol{r}}(t) = \exp\left(-\frac{t}{\tau_B}\right)\dot{\boldsymbol{r}}(0) \qquad \tau_B = \frac{m}{\gamma}$$

τ_B is in the scale of 10^{-3} s, and represents the timescale of reaching equilibrium, i.e. 0 velocity. So, for Brownian motion to happen, $\boldsymbol{F}_{\text{noise}}$ is necessary. Also, if we are interested in the motion on the scale of seconds, we can neglect the acceleration term. This is the **overdamped limit** (in analogy to a damped oscillator with high loss of energy due to attrition, so that it quickly reaches equilibrium without ever "overshooting"). Given that assumption, (12) becomes:

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{F}_{\text{ext}}}{\gamma} + \frac{\boldsymbol{F}_{\text{noise}}}{\gamma}$$

Which, for a particle moving in one dimension, reduces to:

$$x(t) = \underbrace{\frac{F_{\text{ext}}}{\gamma}}_{f(x,t)} + \underbrace{\frac{F_{\text{noise}}}{\gamma}}_{\sqrt{2D(x,t)}\xi(t)}$$

Comparing with (11) gives the physical meaning of $f(x,t)$ and $D(x,t)$.