0.1 Gravitational Waves - part 2

(Lesson ? of 09/01/20)
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Recall that we used a *perturbed* metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad h_{\mu\nu} \ll 1$$

We are looking for a *vacuum solution*, $R_{\mu\nu} = 0$, and so we are dealing only with the *propagation* of a gravitational wave, and not its production. Expanding $R_{\mu\nu}$ to $O(h)$:

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\partial_\mu(\partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h) + (\mu \leftrightarrow \nu)$$

Making a *infinitesimal* change of coordinates:

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

we have:

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$$

$h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ are both describing the same wave, but with different expressions. To make them equal, we need to fix the gauge. Usually, this is done in such a way to simplify the equation. For example, consider the *harmonic gauge*:

$$\partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h = 0$$

leading to:

$$\Box h_{\mu\nu} = 0$$

One solution is the plane wave:

$$h_{\mu\nu} = C_{\mu\nu} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

with $\boldsymbol{k}\cdot\boldsymbol{k} = 0$, k^μ being the 4-wave vector, and $P^\mu = \hbar k^\mu \Rightarrow P^2 = 0 \equiv m^2$. Note that gravitational waves propagate along *null geodesics* (as does light).
The harmonic gauge *does not completely fix* the gauge, as there is still some freedom left. In fact, suppose that $h_{\mu\nu}$ satisfies the harmonic gauge, and ϵ_μ satisfies $\Box \epsilon_\mu = 0$, then:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$$

also satisfies the harmonic gauge.
Let's prove it.

$$0 \stackrel{?}{=} \partial_\lambda \tilde{h}^\lambda_\mu - \frac{1}{2}\partial_\mu \tilde{h} = \partial_\lambda[\cancel{h^\lambda_\mu} - \partial^\lambda \epsilon_\mu - \partial_\mu \epsilon^\lambda] - \frac{1}{2}\partial_\mu[\cancel{h} - 2\partial\cdot\epsilon]$$

with $h = h^\mu_\mu$, $\partial = \partial^\mu_\mu$, and where we applied the harmonic gauge condition to cancel two terms. Then:

$$= -\Box\epsilon_\mu - \partial_\mu\partial\cdot\epsilon + \partial_\mu\partial\cdot\epsilon$$

We want not to impose 4 additional conditions that remove the residual freedom $x^\mu \to x^\mu + \epsilon^\mu$ with $\Box\epsilon^\mu = 0$

$$\Box h_\mu\nu = 0 \to h_{\mu\nu} = C_{\mu\nu}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \qquad k^2 = 0, \quad C_{\mu\nu} = \text{Const.}$$
$$\Box\epsilon_{\mu\nu} = 0 \to \epsilon_\mu = \gamma_\mu e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad k^2 = 0 \quad \gamma_\mu = \text{const}$$

Applying the coordinate change to the plane wave $h_{\mu\nu} \to \tilde{h}_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu$ we get:

$$\tilde{C}_{\mu\nu}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = C_{\mu\nu}e^{ikx} - \partial_\mu[\gamma_\nu e^{ikx}] - \partial_\nu[\gamma_\mu e^{ikx}]$$

Computing the derivatives:

$$\tilde{C}_{\mu\nu}e^{ikx} = C_{\mu\nu}e^{ikx} - ik_\mu\gamma_\nu e^{ikx} - ik_\nu\gamma_\mu e^{ikx}$$

Removing the phases e^{ikx} we have found that, under a change of coordinates $x^\mu \to x^\mu + \epsilon^\mu$ the constants transform as:

$$C_{\mu\nu} \to \tilde{C}_{\mu\nu} - ik_\mu\gamma_\nu - ik_\nu\gamma_\mu$$

So the plane wave remains a plane wave:

$$h_{\mu\nu} = C_{\mu\nu}e^{ikx} \to \tilde{h}_{\mu\nu} = \tilde{C}_{\mu\nu}e^{ikx}$$

but the coefficients are now different. However, $C_{\mu\nu}$ and $\tilde{C}_{\mu\nu}$ both describe the same physical situations, as $x^\mu \to x^\mu + \epsilon^\mu$ is just a change of coordinates.
This is because the gauge is not completely fixed - we need additional conditions. As before, we choose them so that the problem become simpler:

$$\tilde{C}_{00} = \tilde{C}_{0i} = 0$$

These fix completely the residual gauge freedom.

We need to verify that we *can* impose these conditions. **Goal**: show that, starting from a generic $C_{\mu\nu}$, we can always find γ_μ (i.e. find ϵ_μ, that is *an appropriate change of variables*) such that $\tilde{C}_{00} = \tilde{C}_{0i} = 0$ $(i = 1, 2, 3)$. We start from the transformation rule we found earlier:

$$C_{\mu\nu} \to \tilde{C}_{\mu\nu} - ik_\mu\gamma_\nu - ik_\nu\gamma_\mu$$

Setting $\mu = \nu = 0$:

$$\tilde{C}_{00} = C_{00} - 2ik_0\gamma_0 \Rightarrow \gamma_0 = \frac{C_{00}}{2ik_0} \Rightarrow \tilde{C}_{00} = 0$$

Note that there is *only one possible choice* for the change of coordinates, i.e. only one value for γ_0.
If $\mu = 0$, $\nu = i$:

$$\tilde{C}_{0i} = C_{0i} - ik_0\gamma_i - ik_i\gamma_0 = C_{0i} - ik_0\gamma_i - \cancel{i}k_i \frac{C_{00}}{2\cancel{i}k_0} \stackrel{!}{=} 0$$

leading to:

$$ik_0\gamma_i = C_{0i} - \frac{k_i}{2k_0}C_{00} \Rightarrow \gamma_i = \frac{1}{ik_0}\left(C_{0i} - \frac{k_i}{2k_0}C_{00}\right)$$

Note γ_μ is completely fixed by the need to get $\tilde{C}_{00} = 0$, $\tilde{C}_{0i} = 0$. Here we assume $k_\mu \neq 0$. As we will show, this amount on assuming that the gravitational wave has a non-zero frequency. The degenerate case of a 0 frequency wave - which means to "have nothing" - generates diverging terms in this gauge. This is *a gauge artefact*, that can be fixed by just making a different gauge choice.

Summarizing, the gravitational wave equation in the harmonic gauge with the additional constraints to fix completely the gauge is:

$$\begin{cases} \Box h_{\mu\nu} = 0 \\ \partial_\lambda h^\lambda_\mu - \frac{1}{2}\partial_\mu h = 0 \\ h_{00} = h_{0i} = 0 \end{cases}$$

$h_{\mu\nu}$ has 10 degrees of freedom (because of symmetry), these ones:

$$h_{\mu\nu} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}$$

The harmonic gauge is a set of 4 differential equations (4 constraints), and then we have another 4 constraints from the residual gauge fixing. So a gravitational wave will have $10 - 4 - 4 = 2$ degrees of freedom (physically, they are 2 polarizations).

Let's see this concretely (assuring that all 8 constraints are independent) for the plane-wave solution:

$$h_{\mu\nu} = C_{\mu\nu}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

We start from the harmonic gauge (H.G.) condition for $\mu = 0$:

$$\partial_\lambda h^\lambda_0 - \frac{1}{2}\partial_0 h = 0$$

Recall that $h^\alpha_\beta = \eta^{\alpha\gamma}h_{\gamma\beta}$, and so:

$$\partial_0 h^0_0 + \partial_i h^i_0 - \frac{1}{2}\partial_0 h = 0$$

As $\eta^{00} = -1$ and $\eta^{ij} = \delta_{ij}$ we have, because $h_{00} = h_{0i} = 0$:

$$-\cancel{\partial_0 h_{00}} + \cancel{\partial_i h_{i0}} - \frac{1}{2}\partial_0 h = 0 \Rightarrow \partial_0 h = 0 \Rightarrow \partial_0 (Ce^{ikx}) = 0 \Rightarrow ik_0 Ce^{ikx} = 0$$

This means that:

$$C \equiv C^\mu_\mu = 0 = C^0_0 + C^i_i = -C_{00} + C_{ii} = 0$$

So: $h = 0$ and $h_{ii} = 0$. This means that the gravitational wave is *traceless* (this can be proved in the general case of a *non-planar* wave).

Now, if $\mu = i$ the harmonic-gauge condition, as $h = 0$, reduces to:

$$\partial_\lambda h^\lambda_i = 0 \Rightarrow -\partial_0 h_{0j} + \partial_i h_{ij} = 0 \Rightarrow \partial_i h_{ij} = 0$$

This means, as we will see, that the G.W. is *transverse* (oscillates in the direction perpendicular to its propagation).

So, we are left to solve:

$$\begin{cases} \Box h_{\mu\nu} = 0 \Rightarrow h_{\mu\nu} = C_{\mu\nu} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \; \boldsymbol{k}^2 = 0 \\ h_{00} = h_{0i} = 0 \\ h_{ii} = 0 \\ \partial_i h_{ij} = 0 \end{cases}$$

Applying these conditions, we get new expressions for the matrix $C_{\mu\nu}$:

$$\begin{cases} C_{00} = C_{0i} = 0 \\ C_{ii} = 0 \\ ik_i C_{ij} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0 \Rightarrow k_i C_{ij} = 0 \end{cases}$$

Let's fix $\boldsymbol{k}$ along the $\hat{z}$ axis: $\boldsymbol{k} = (0, 0, k)$. Then the third condition becomes:

$$C_{3j} = 0 \Rightarrow C_{31} = C_{32} = C_{33} = 0$$

As $C_{\mu\nu}$ is symmetric, we can rewrite this as:

$$C_{13} = C_{23} = C_{33} = 0$$

As $C_{ii} = 0$:

$$C_{11} + C_{22} + \cancel{C_{33}} = 0$$

Summarizing:

$$\begin{cases} C_{00} = C_{01} = C_{02} = C_{03} = 0 \\ C_{11} + C_{22} = 0 \\ C_{13} = C_{23} = C_{33} = 0 \end{cases}$$

We have two independent solutions:

$$C_{12} = C_{21} \neq 0 \vee C_{22} = -C_{11} \neq 0$$

As the wave equation is linear, also any linear combination of these solutions will be a solution. In particular, it will be a matrix $C_{\mu\nu}$ that is a linear combination of the following two basis elements:

$$e_{ij,+} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_{ij,\times} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We need to normalize them:

$$e_{ij,+}e_{ij,+} = e_{ij,+}e_{ji,+} = \text{trace}(e_+e_+) = \text{Tr}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] =$$

$$= \text{Tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2$$

$$e_{ij,+}e_{ij,\times} = \text{Tr}(e_+e_\times) = \text{Tr}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{Tr}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$e_{ij,\times}e_{ij,\times} = \text{Tr}(e_\times e_\times) = \text{Tr}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{Tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2$$

And so:

$$\text{Tr}(e_r e_s) = 2\delta_{rs} \qquad \{r, s\} \in \{+, x\}$$

A G.W. can be decomposed as:

$$h_{ij}(x) = (h_+ e_{ij,+} + h_\times e_{ij,\times})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad \boldsymbol{k}^2 = 0$$

where h_+ and h_x are the *amplitudes* of the two polarizations.

Let's now try to get a better understanding of this solution. First of all, note that the metric is real $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$, but the solution we have is complex. However, as the differential equation has real coefficients, then if $h_{\mu\nu}$ solves it, also its complex conjugate $h^*_{\mu\nu}$ solves it. Then also their linear combinations will be solutions, and so we can combine them to write a solution that is manifestly real:

$$\begin{aligned} C_{ij}e^{ikx} &= C_{ij}[\cos(kx) + i\sin(kx)] \text{ is a solution} \\ C_{ij}e^{-ikx} &= C_{ij}[\cos(kx) - i\sin(kx)] \text{ is a solution} \\ \Rightarrow &\; C_{ij}[C_1\cos(kx) + C_2\sin(kx)] \text{ is a solution} \end{aligned}$$

Absorbing the constants:

$$C_{ij}\cos(kx+\varphi) \text{ is a solution}$$

where φ is an arbitrary phase.
So, we can rewrite a generic plane wave solution as:

$$h_{ij}(z) = \sum_{r=+,\times} h_r e_{ij,r} \cos(kt - kz + \varphi_r)$$

where we used $\boldsymbol{k}\cdot\boldsymbol{x}+\varphi = -kt+kz+\varphi$ and changed the overall sign and renamed $-\varphi \to \varphi$. Note that it is periodic in time with period $T = 2\pi/k$. We then define the frequency as:

$$f = \frac{1}{T} = \frac{k}{2\pi}$$

Let's now focus on getting a geometric interpretation for the polarizations.

$$e_{ij,+} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Recall that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ tells us how to measure *distances*. Suppose we have two objects with a fixed comoving distance:

$$\int \mathrm{d}x \sqrt{g_{xx}}$$

In presence of a gravitational wave, the non-zero $h_{\mu\nu}$ will *vary* their distance. For the + polarization, at $t = 0$, distances along $\hat{x}$ *rise* and along $\hat{y}$ lower. These two directions exchange after half a period (as the wave will have travelled $\lambda/2$). Note that changes happen only in directions $\perp$ to that of motion (transverse wave).
Now, consider a 45° rotation:a

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cos(45^\circ) & \sin(45^\circ) & 0 \\ -\sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For a vector along the diagonal, such as $(1, 1, 0)$, we get:

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

We can write this relation as $\tilde{x}^i = R^i_j x^j$, or more compactly as $\tilde{\boldsymbol{x}} = R\boldsymbol{x}$. Recall that $R^T R = \mathbb{I}$ for rotation matrices (as they are orthogonal). The line element does not change:

$$\mathrm{d}l^2 = \mathrm{d}x^i\, g_{ij}\, \mathrm{d}x^j = \mathrm{d}\tilde{x}^i\, \tilde{g}_{ij}\, \mathrm{d}\tilde{x}^j = \mathrm{d}x^T\, g\, \mathrm{d}x = \mathrm{d}\tilde{x}^T\, \tilde{g}\, \mathrm{d}x = \mathrm{d}x^T\, R^T \tilde{g} R\, \mathrm{d}x$$

as $\mathrm{d}\tilde{x} = R\,\mathrm{d}x$. This means that $R^T \tilde{g} R = g \Rightarrow \tilde{g} = R g R^T$, which is the formal way for writing:

$$\tilde{g}_{ij} = \frac{\partial x^m}{\partial \tilde{x}^i} g_{mn} \frac{\partial x^n}{\partial \tilde{x}^j}$$

If we focus on the space coordinates $g_{ij} = \delta_{ij} + h_{ij}$:

$$\mathbb{I} + \tilde{h} = R(\mathbb{I} + h)R^T = R\mathbb{I}R^T + RhR^T = \mathbb{I} + RhR^T$$

and so: $\tilde{h} = RhR^T$. We can now tackle the $\times$ polarization.

$$h = e_\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then:

$$\tilde{h} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = e_+$$

So e_+ is just $e_\times$ rotated by $45°$.
A generic case will have the effect of e_+ *superimposed* to that of $e_\times$, with possibly different amplitudes:

$$h_{ij} = \sum_{r=+,\times} h_r e_{ij,r} \cos(kt - kz + \varphi_r)$$