0.1 Introduction

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One of the cardinal principle in cosmology is the **Copernican Principle**, which states that we do not occupy a preferred position in the universe. In other words, the Earth is not special.

This is generalized and formalized by the **cosmological principle**:

Every comoving observer sees the Universe around her/him, at a **fixed time**, as being **homogeneous** and **isotropic**.

This is a guiding principle - that is something intuitively true, that is used to construct and simplify theories (such as the *principle of inertia* in classical mechanics). Let us explain it in more detail:

- First of all, the cosmological principle holds only when seeing the universe as a whole, and so only at the **largest scales**. In fact, a proposition so general can only be considered if we forget all about details, such as the local structure of planets, stars and galaxies. More precisely, it holds in the scale of hundreds of Mpc, where a parsec (pc) is defined as the distance where one astronomical unit subtends an angle of one arcsecond ($1\,\mathrm{pc} = 3.26\,\mathrm{ly}$).

- **Comoving**. The Cosmic Radiowave Background is a signal in the infrared, at temperature of about $3\,\mathrm{K}$, that permeates the universe. In particular, there is a *dipole effect* - that is the CMB, as observed by Earth, appears hotter in a certain direction, and colder in the opposite one, by about $1\,\mathrm{mK}$. This can be explained by the Doppler effect, by assuming that our frame of reference is moving with respect to the CMB with a huge velocity of $\sim 630\,\mathrm{km\,s^{-1}}$. This motion cannot be accounted by the revolution of Earth around the Sun, or the Sun around the center of the galaxy - but it's generated by the attraction between *clusters of galaxies*. The presence of this dipole component means that Earth is not a *comoving observer*, that is it's not stationary with respect to the CMB.
  We then define a **comoving observer** as one that observes the CMB dipole moment as zero - that is one for which the CMB appears completely uniform within a $\mu\mathrm{K}$.
  Note that the existence of preferred reference frames is not allowed in special relativity, but it is in *general relativity*, that is necessary to describe our universe.

- For **fixed time** we define an instant in the proper time of a comoving observer.

- **Isotropy** means that the Universe "looks the same in every direction" (i.e., it's rotationally invariant): this is something that can, and in fact is, observed. On the other hand, **homogeneity** is the phylosophical assumption - as it is not verifiable - that the same results about isotropy could be inferred if we started from any other position in the universe (i.e. the universe is translationally invariant). This is indeed true if Earth is really a typical planet - which holds only in approximation, as the existence of life requires a specific subset of all the conditions possible in the universe (leading to the necessity of discussing also the *anthropic principle*).

0.1.1 Metric

Let's start our discussion by describing the *geometry* of the universe, which, as we will see, it's tied to its contents, and characteristics (as suggested by General Relativity).

In special relativity, spacetime is described by the Minkowski metric, for which the line element ds^2 has the form:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

A general line element, that is the four-distance between two infinitely close events, is given by:

$$ds^2 = g_{ab}dx^a dx^b \tag{2}$$

where g_{ab} is the metric tensor. By comparing (1) and (2) we note that the Minkowski metric is given by:

$$g_{ab} = \eta_{ab} = \mathrm{diag}([1, -1, -1, -1])$$

Let $dl^2 = dx^2 + dy^2 + dz^2$, so that we can write $ds^2 = c^2dt^2 - dl^2$. Then, we can transform it to spherical coordinates:

$$dl^2 = d\rho^2 + \rho^2 d\Omega^2 \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2; \quad \rho \in \mathbb{R}_+, \theta \in [0, \pi), \varphi \in [0, 2\pi)$$

which leads to the spherical version of the Minkowski metric:

$$ds^2 = c^2dt^2 - d\rho^2 - \rho^2 d\Omega^2 \tag{3}$$

This is one of the building blocks of the Roberston-Walker metric, which is the preferred metric for studying spacetime from the reference frame of a comoving observer - something really useful in cosmology. In fact, it can be shown that the R-W metric is the only metric compatible with the cosmological principle which is maximally symmetric:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right]$$

where $a(t)$ is the *scale factor*, it has units of length ($[a(t)] = \mathscr{L}$), and it is a function of the proper time in a comoving frame of reference. $a(t)$ represents, in a certain sense, the *expansion* or *contraction* of the universe.
On the other hand, $r = \rho/a(t)$ is an adimensional variable, and k, the *curvature parameter*, has values in $[-1, 1]$. In particular:

- $k = 1$ leads to *closed* universes (with a boundary, but can be limitless, like the surface of a ball)
- $k = -1$ leads to *open* universes (boundless and limitless)
- $k = 0$ leads to *flat* universes

Note that as $\{0\}$ is only a point of null Lebesgue-measure in the set $[-1, 1]$, the probability that a universe will have $k = 0$ is negligible.

If $k = 0$, the R-W metric leads simply to the Minkowski metric:

$$k = 0 \Rightarrow c^2 dt^2 - a^2(t)[dr^2 + r^2 d\Omega^2]$$

which is equal to (3) with the substitution $r = \rho/a(t)$.

A general 4D spacetime has 10 degrees of freedom (time, 3 for position, 3 for velocity, 3 for Lorentz boosts). The Minkowski metric is maximally symmetric, that is it preserves all 10 symmetries. Other metrics with this same property are the DeSitter and Anti-DeSitter spacetimes.
However, the observed universe does not exhibit time translation invariance, that is it evolves in a irreversible future which is different from the past. Then, the CMB offers a preferred reference frame, that of a *comoving observer*. So, 4 symmetries are broken (time and Lorentz boosts).
Under these assumptions, simply searching for a metric that preserves the remaining 6 symmetries leads to the R-W metric previously discussed.

Let's now compare the R-W metric to metric on different manifolds, to better understand the role of k. Consider the following 2D manifolds (for simplicity):

- A flat **plane** has the metric:

  $$dl^2 = a^2(dr^2 + r^2 d\theta^2) \tag{4}$$

  where a is included to make r adimensional.

- The surface of a **sphere** has the line-element:

  $$dl^2 = a^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{5}$$

- The surface of a **hyperboloid** has the line element:

  $$dl^2 = a^2(d\theta^2 + \sinh^2\theta d\varphi^2) \tag{6}$$

We now show that, with a change of variables, each of these cases leads to a version of the R-W metric with a different value for k.

- Starting from the line element on the *sphere* (5), and substituting $r = \sin\theta$ (and $dr = \cos\theta\, d\theta \Rightarrow d\theta = dr/\cos\theta$) leads to:

$$dl^2 = a^2\left(\frac{dr^2}{\cos^2\theta} + r^2 d\varphi^2\right) = a^2\left(\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right)$$

which is the space term ($dt = 0$) of R-W with $k = +1$:

$$ds^2 \underset{k=1}{=} c^2dt^2 - a^2\left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right]$$

- Following the same procedure, starting with the hyperboloid line element and substituting $r = \sinh\theta$ leads to R-W with $k = -1$

Applying the reverse substitutions to R-W leads to:

$$ds^2 = cdt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\varphi^2\right) =$$

$$= c^2dt^2 - a^2\begin{cases} d\chi^2 + \sin^2\chi d\Omega^2 & k = +1, r = \sin\chi \\ d\chi^2 + \chi^2 d\Omega^2 & k = 0, r = \chi \\ d\chi^2 + \sinh^2\chi d\Omega^2 & k = -1, r = \sinh\chi \end{cases}$$

Using cartesian coordinates, the R-W line element is given by:

$$ds^2 = c^2dt^2 - a^2(t)\left(1 + \frac{k|x|^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2)$$

It is possible to simplify the R-W metric by a substitution, passing to **conformal time** $dt \equiv a(\eta)\eta$, $a(\eta) \equiv a(t(\eta))$, $\int d\eta = \int dt/a(t)$ leading to:

Not really sure about this part

$$ds^2 = a^2(\eta)\left(c^2 d\eta^2 - \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)\right)$$

0.1.2 Friedman equations

The evolution of a universe for which the cosmological principle holds is described by the Friedman equations, where a time derivative (denoted with a dot - like in $\dot{a}$) is interpreted with respect to the proper time of a comoving observer. These are the following:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$
$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right)$$
$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)$$

where $\rho = \rho(t)$ is the *energy density*, and $P = P(t)$ is the *isotropic pressure*.
We define the **Hubble parameter** as:

$$H(t) \equiv \frac{\dot{a}}{a}$$

Its evolution is given by:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

If the universe where flat, that is with $k = 0$, the previous relation becomes:

$$H^2(t) = \frac{8\pi G}{3}\rho_c(t) \Rightarrow \rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

ρ_c is called the **critical density**. We then define the ratio $\Omega(t)$ (**density parameter**):

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}$$

Depending on the value of Ω, k can be inferred:

- $\Omega > 1 \Leftrightarrow k = 1$
- $\Omega = 1 \Leftrightarrow k = 0$
- $\Omega < 1 \Leftrightarrow k = -1$

So, by measuring the density parameter, we can infer properties of the geometry of spacetime.

$H_0 = H(t_0)$, with t_0 meaning the current proper time, is the Hubble constant (current spacetime expansion rate), and has been measured:

$$H_0 = 100h\frac{\text{km/s}}{\text{Mpc}} = 70\frac{\text{km/s}}{\text{Mpc}}$$

where $h = 0.7$.
All current measurements suggest that $\Omega_0 \approx 1$, which means a *flat universe*. This is due to the presence of three main components: visible matter, dark matter and dark energy. Unfortunately, 95% of of the universe is made of the unobservable ("dark") parts.

Suppose that we want to measure the current energy density in galaxies ρ_{0g}. This is related to the mean intrinsic luminosity of galaxies per unit volume $\mathcal{L}_g$ by:

$$\rho_{0g} = \mathcal{L}_g\langle\frac{M}{L}\rangle$$

where $\langle M/L \rangle$ is the mean mass to light ratio of galaxies (usually expressed in solar units, where $M_\odot = 1.99 \times 10^{33}$ g is the mass of the sun, and $L_\odot = 3.9 \times 10^{33}$ erg/s $= 3.9 \times 10^{26}$ W.
Measuring $\mathcal{L}_g$ is not easy, as even all the weakest sources need to be considered. One trick to help with that is given by the definition of expected value of a continuous variable:

$$\mathcal{L}_g = \int_0^{+\infty} \mathrm{dL}\, L\phi(L)$$

where is the luminosity function, that is the number of objects per unit volume with luminosity between L and $L + dL$ (in analogy with a probability density).
$\phi(L)$ is somewhat easier to determine, and can be extrapolated from observations.