0.1 Critical Behaviours and Scaling Laws

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Power laws

In the last section, we were able to finally describe a **phase-transition** , by analysing the Ising Model in the mean field approximation in $d > 1$. Mathematically, we observed how the **spontaneous magnetization** $M(K, h)$, when K is chosen in the proximity of the *critical parameter* K_c needed for the phase-transition, is described by a **power law** (**??**).

This happens to be a very general kind of behaviour, proper of **not only mean field** models. Scaling laws such as (**??**) were originally formulated from empirical evidence, and then given a theoretical foundation in the 1960s by Widom, Kadanoff and Kenneth Wilson, leading to the field of **renormalization group theory**. In this framework, all critical phenomena can be treated on equal ground, and general results can be mathematically proven.

The importance of scaling laws, and especially the values of their *critical exponents* (such as β for the IM) resides in their **universality**, i.e. in the fact that they are *largely independent* on the "model's details". In other words, the very same scaling law can describe two systems that - from the outside - seem completely different - but that share some fundamental characteristic (e.g. symmetry).

So, let's continue using the Ising Model in the mean field approximation as a *concrete* example, and let's focus on deriving and understanding scaling laws.

Recall the expression for the variational free energy (**??**):

$$\beta \frac{F_V(m, K, h)}{N} = -Kdm^2 + \frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} - hm \qquad (1)$$

F_V is closest to the *true* "unapproximated" free energy F when it's minimum:

$$\frac{\partial}{\partial m} F_V(m, K, h) \overset{!}{=} 0 \underset{(??)}{\Rightarrow} m(h, K) = \tanh(2dKm + h) \qquad \texttt{eqn:minimizing-eq} \; (2)$$

Let's invert (2) and