0.1

(Lesson ? of 05/12/19) Compiled: December 5, 2019

We introduced the notion of 4-spin S^α, which, in the gyro rest frame is $S^\alpha = (0, \boldsymbol{S})$. We showed that $\boldsymbol{u} \cdot \boldsymbol{S} = 0$ always, and along a geodesic $\boldsymbol{S} \cdot \boldsymbol{S} = \text{Constant} = S_*^2$. The spin satisfies the following equation:

$$\frac{\mathrm{d}S^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta S^\gamma = 0$$

We consider now a *circular orbit* around a central mass M. Consider the gyroscope moving in the plane $\theta = \pi/2$, initially pointing in the radial direction. What will be the final direction after *one full rotation*?
Thanks to the problem's symmetry, we noted that:

$$S^\theta = 0 \Rightarrow S^\alpha = (S^t, S^r, 0, S^\varphi)$$

From $\boldsymbol{S} \cdot \boldsymbol{u} = 0$ we have that:

$$S^t = R^2 \Omega \left(1 - \frac{2GM}{R}\right)^{-1} S^\varphi$$

where:

$$u^\alpha = u^t(1, 0, 0, \Omega) \qquad \Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

as:

$$u^3 = u^t \Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$$

Also:

$$\Omega^2 = \frac{GM}{R^3}$$

Initially, only $S^r \neq 0$, and so $S^\varphi = S^t = 0$.
During last lecture we wrote the equation for S^r:

$$\frac{\mathrm{d}S^r}{\mathrm{d}\tau} + (3GM - R)\, \Omega u^t S^\varphi = 0$$

We now compute the equation for S^φ. The geodesics equation reads:

$$\frac{\mathrm{d}S^\varphi}{\mathrm{d}\tau} + \Gamma^3_{\beta\gamma} u^\beta S^\varphi = 0$$

The only non-zero components can be for $\beta = \{0, 3\}$ and $\gamma = \{0, 1, 3\}$, so all the possible combinations are:

$$\Gamma^3_{00}, \Gamma^3_{01}, \Gamma^3_{03}$$
$$\Gamma^3_{30}, \Gamma^3_{31}, \Gamma^3_{33}$$

As the Schwarzschild metric is diagonal, a non-zero Christoffel symbol must have always two equal indices. This leaves us with:

$$\begin{gathered}\Gamma^3_{00}, \cancel{\Gamma^3_{01}}, \Gamma^3_{03} \\ \Gamma^3_{30}, \Gamma^3_{31}, \Gamma^3_{33}\end{gathered}$$

When two indices are the same, the third one denotes the derivative. As φ derivatives are null:

$$\begin{gathered}\cancel{\Gamma^3_{00}}, \cancel{\Gamma^3_{01}}, \Gamma^3_{03} \\ \Gamma^3_{30}, \Gamma^3_{31}, \cancel{\Gamma^3_{33}}\end{gathered}$$

and also the time derivative of g_{33} are null:

$$\begin{gathered}\cancel{\Gamma^3_{00}}, \cancel{\Gamma^3_{01}}, \cancel{\Gamma^3_{03}} \\ \cancel{\Gamma^3_{30}}, \Gamma^3_{31}, \cancel{\Gamma^3_{33}}\end{gathered}$$

So we just have to compute Γ^1_{31}:

$$\Gamma^3_{31} = \frac{1}{2}g^{33}(\cancel{g_{31,3}} + g_{33,1} - \cancel{g_{31,3}}) = \frac{1}{2r^2}\frac{\partial}{\partial r}r^2 = \frac{1}{r}$$

Substituting back:

$$\frac{\mathrm{d}S^\varphi}{\mathrm{d}\tau} + \frac{1}{r}\underbrace{u^t\Omega}_{u^3} S^t\bigg|_{r=R} = 0$$

We can rewrite it as:

$$\frac{\mathrm{d}S^\varphi}{\mathrm{d}\tau} + \frac{\Omega}{R}\frac{\mathrm{d}t}{\mathrm{d}\tau}S^r = 0$$

And changing variables (*or multiplying by* $\mathrm{d}\tau\,/\,\mathrm{d}t$...):

$$\frac{\mathrm{d}S^\varphi}{\mathrm{d}t} + \frac{\Omega}{R}S^r = 0$$

If we do the same with the first equation we arrive at:

$$\begin{cases}\frac{\mathrm{d}S^r}{\mathrm{d}t} + (3GM - R)\Omega S^\varphi = 0 \\ \frac{\mathrm{d}S^\varphi}{\mathrm{d}t} + \frac{\Omega}{R}S^r = 0\end{cases}$$

If we differentiate the first one with respect to t, we can then substitute inside the second one, and obtain a second order differential equation with only one variable (r):

$$\begin{aligned}\frac{\mathrm{d}^2S^r}{\mathrm{d}t^2} + (3GM - R)\Omega\frac{\mathrm{d}S^\varphi}{\mathrm{d}t} &= 0 \\ \Rightarrow \frac{\mathrm{d}^2S^r}{\mathrm{d}t^2} + (3GM - R)\Omega\frac{-\Omega}{R}S^r &= 0\end{aligned}$$

After some algebra:

$$\frac{\mathrm{d}S^r}{\mathrm{d}t} + \underbrace{\left(1 - \frac{3GM}{R}\right)\Omega^2}_{\omega^2} S^r = 0$$

which is the equation of the simple harmonic oscillator. Note that:

$$1 - \frac{3GM}{R} > 0$$

as for a *stable* circular orbit we need $R > 6GM$. Then the solution is:

$$S^r = A\cos\left(\bar{\Omega}t\right) \qquad \bar{\Omega} = \left(1 - \frac{3GM}{R}\right)^{1/2}\Omega$$

We choose the cosine because S^r is maximum *at the start*, i.e. for $t = 0$.
Substituting this solution in the first equation (as we know S^r, it is easy to compute $\mathrm{d}S^r \,/\, \mathrm{d}t$):

$$\begin{aligned}
-A\bar{\Omega}\sin\left(\bar{\Omega}t\right) &= (R - 3GM)\Omega S^\varphi \\
-A\bar{\Omega}\sin\left(\bar{\Omega}t\right) &= R - \Omega\left(1 - \frac{3GM}{R}\right)S^\varphi \\
-A\bar{\Omega}\sin\left(\bar{\Omega}t\right) &= R\Omega\frac{\bar{\Omega}^2}{\Omega^2}S^\varphi
\end{aligned}$$

leading to:

$$S^\varphi = -\frac{A}{R}\frac{\Omega}{\bar{\Omega}}\sin\left(\bar{\Omega}t\right)$$

To compute the integration constant A we use $\boldsymbol{S}\cdot\boldsymbol{S} \equiv S_*^2 = \text{Constant}$. Evaluating it at the starting time:

$$\boldsymbol{S}\cdot\boldsymbol{S} = S^\mu S^\nu g_{\mu\nu} = (S^r)^2 g_{11} = A^2\left(1 - \frac{2GM}{R}\right)^{-1} \Rightarrow A = S_*\sqrt{1 - \frac{2GM}{R}}$$

Summarizing:

$$S^\mu(t) = (S^t, S^r, 0, S^\varphi); \qquad S^\mu(0) = (S_r(0), 0, 0, 0)$$

$$\begin{cases} S^t = R^2\Omega\left(1 - \frac{2GM}{R}\right)^{-1} S^\varphi \\ S^r = A\cos\left(\bar{\Omega}t\right) \\ S^\varphi = -\frac{A}{R}\frac{\Omega}{\bar{\Omega}}\sin\left(\bar{\Omega}t\right) \end{cases} \qquad \bar{\Omega} = \sqrt{1 - \frac{3GM}{R}}\Omega;\ A = S_*\sqrt{1 - \frac{2GM}{R}};\ \boldsymbol{S}\cdot\boldsymbol{S} = S_*^2$$

Initially $\boldsymbol{S}$ is in the radial component, meaning that it's equal to the unit vector $\hat{e}^r$ in the radial direction for that observer (O).
At a later t, suppose that the gyroscope points at a different angle $\Delta\varphi$. The $\cos(D\varphi)$ is the **ratio** between the *radial component of S^α measured by O at time*

t, and the *radial component of S^α measured by O at time* 0. This is really what needs to be done when *we make the measurement in practice*. So:

$$\cos(\Delta\varphi) = \frac{\hat{\boldsymbol{e}}^r \cdot \boldsymbol{S}(t)}{\hat{\boldsymbol{e}}^r \cdot \boldsymbol{S}(t=0)} = \frac{(\hat{\boldsymbol{e}}^r)^\alpha g_{\alpha\beta} S^\beta(t)}{(\hat{\boldsymbol{e}}^r)^\alpha g_{\alpha\beta} S^\beta(t=0)}$$

where:

$$\hat{\boldsymbol{e}}^r = \left(0, \frac{1}{\sqrt{g_{11}}}, 0, 0\right)$$

So:

$$\cos(\Delta\varphi) = \frac{(\hat{\boldsymbol{e}}^r)^1 g_{11} S^1(t)}{(\hat{\boldsymbol{e}}^r)^1 g_{11} S^1(t=0)} = \frac{S^r(t)}{S^r(t=0)} = \cos\left(\bar{\Omega}t\right)$$

We know that when an observer is moving, due to Lorentz transformation, both the *time* component and the *spatial component in that direction* will change. So, to compute S^α in a *rest frame*, it is necessary to make a boost in the φ direction, which is the only direction of motion in that circular orbit. This, however, will make calculations much harder, and we are interested only in the r component - which does not change in the boost, as the observer does not move in that direction. So, even if we don't do the boost - and so we are not really in a rest frame (note, in fact, that S^t generally is $\neq 0$ during that motion), the calculation for $\cos(\Delta\varphi)$, as it only involves r components, provides the right result.
(If we were instead to compute $\tan\Delta\varphi$, we would need also the φ component, and so we would need to do the boost).

$$\cos(\Delta\varphi) = \cos\left(\bar{\Omega}t\right) \Rightarrow \Delta\varphi = \pm\bar{\Omega}t$$

Which one should we choose? We now what happens when $M = 0$, and so we choose the one that makes sense in that limit. In this case, the gyroscope always points in the *same direction*. If the orbit is counter-clockwise, this means that the gyroscope, with respect to the *local rotating basis* is rotating *clockwise*, meaning that:

$$\Delta\varphi = -\Omega t$$

[Insert fig. 1]
When $M = 0$, however, the coordinate system is still rotating at Ω, but the vector just at $\bar{\Omega} < \Omega$. So, from the point of view of someone at infinity, the gyroscope is rotating at $\Gamma - \bar{\Omega}$, meaning that it's *bending towards* the direction of motion:

$$\Omega_{\substack{\text{wrt}\\\text{infinity}}} = \Omega_{\substack{\text{of } \hat{\boldsymbol{e}}^r,\ \hat{\boldsymbol{e}}^\varphi\\\text{coordinate}\\\text{system}}} - \Omega_{\substack{\text{of spin}\\\text{in that}\\\text{coordinate}\\\text{system}}} = \Omega - \bar{\Omega}$$

How much $\Delta\varphi_\infty$ it is accumulated in 1 orbit?

$$\Delta\varphi_{\substack{\text{wrt}\\\text{infinity}}} = (\Omega - \bar{\Omega})t = \frac{2\pi}{\Omega}(\Omega - \bar{\Omega}) = 2\pi\left(1 - \frac{\bar{\Omega}}{\Omega}\right) = 2\pi\left[1 - \sqrt{1 - \frac{3GM}{R}}\right]$$

$$\underset{GM \ll R}{\approx} 2\pi\left[1 - \left(1 - \frac{3GM}{2R}\right)\right] = \frac{3\pi GM}{R}$$

where we used $\Omega = \mathrm{d}\varphi \,/\, \mathrm{d}t \Rightarrow t = (2\pi)/\Omega$.
This is the effect of **geodetic precession**. Note that angular momentum in GR is conserved as a 4-vector.

0.2 Slowly rotating geometry

Let's examine what happens to a gyroscope moving in a *slowly rotating geometry*. We start from the line element:

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\text{Schwarzschild}} - \frac{4GJ}{r}\sin^2\theta \,\mathrm{d}t\,\mathrm{d}\varphi + \text{Terms of order } J^2 \text{ or higher}$$

(see homework to check that this metric satisfies Einstein's equation up to linear order in J).

Now we have *non-diagonal* elements:

$$g_{03} = g_{30} = -\frac{2GJ}{r}\sin^2\theta$$

If we insert $c \neq 1$:

$$\underbrace{\mathrm{d}s^2}_{[\mathcal{L}^2]} = \mathrm{d}s^2_{\text{Schwarz.}} - \cdot\frac{4GJ}{c^3r^2}\sin^2\theta\,\underbrace{(r\,\mathrm{d}\varphi)}_{[\mathcal{L}]}\,\underbrace{(c\,\mathrm{d}t)}_{[\mathcal{L}]}$$

This means:

$$\left[\frac{GJ}{c^3r^2}\right] = \mathcal{N}$$

is a pure number, and so:

$$[J] = \left[\frac{c^3r^2}{G}\right] = \frac{\mathrm{m}^2\,\mathrm{s}^{-3}\mathrm{m}^2}{\mathrm{N}\,\mathrm{m}^2\,\mathrm{kg}^{-2}} = \frac{\mathrm{m}^3}{\mathrm{s}^3}\frac{\mathrm{kg}^2}{\mathrm{N}} = \frac{\mathrm{m}^3}{\mathrm{s}^3}\frac{\mathrm{kg}^2}{\mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2}} = \mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}$$

This means that J has the dimensions of an angular momentum.
To simplify calculations, we consider the motion of a gyroscope moving in free fall *along the rotation axis*, and initially aligned along the $\hat{x}$ direction.
If $J = 0$, obviously the gyroscope will remain aligned along $\hat{x}$. We expect a change of $\vec{S}$ in the order of $O\left(GJ/(c^3r^2)\right)$. This is quantity that is *dimensionless* and *very small* (an analogue of GM/R in a rotating geometry). Note, in fact, that in this case the gyroscope is not orbiting M, and so any change of its spin is due to

J (the mass, by itself, *cannot* produce a rotation of this gyroscope - so terms of $O(GM/R^2)$ alone are automatically null).
Then, for M small, we will also have terms of order:

$$O\left(\frac{GJ}{c^3 r^2} \times \frac{GM}{rc^2}\right)$$

where the GM/r^2 we had before reappears. Note that this terms are a *correction of a correction*, and so we can ignore them, and set $M = 0$. This means that we can use the following metric:

$$\mathrm{d}s^2 = \underbrace{\eta_{\mu\nu}}_{M=0} \mathrm{d}x^\mu \, \mathrm{d}x^\nu - \frac{4GJ}{r} \sin^2\theta \, \mathrm{d}t \, \mathrm{d}\varphi$$

Summarizing, we are consider a rotating mass M that produces a rotating geometry. We mathematically *split* the influence of J and M on the gyroscope rotation. We note that M alone *cannot* produce any effect, while J alone can. Also, terms with both M and J are of higher order. So, for simplicity, we can ignore M, and mathematically set it to 0.