

0.1 Continuity of Brownian Path

Consider a particle starting in $x = 0$ at $t = 0$, and traversing N points $\{x_i\}_{i=1,\dots,N}$ such that all increments $\Delta x_i = x_i - x_{i-1}$ are *independent* and described by a *gaussian pdf*. The density function for such a trajectory $\{x_i\}$ is the usual product of transition probabilities:

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$$d\mathbb{P}_{t_1,\dots,t_N}(x_1,\dots,x_N) = \left(\prod_{i=1}^N \frac{dx_i}{\sqrt{4\pi\Delta t_i D}} \right) \exp \left(- \sum_{i=1}^N \frac{(\Delta x_i)^2}{4D\Delta t_i} \right) \quad \begin{matrix} \Delta t_i = t_i - t_{i-1} \\ \Delta x_i = x_i - x_{i-1} \end{matrix} \quad (1)$$

We now show that, taking the continuum limit $\max_i \Delta t_i \rightarrow 0$ leads to paths $\{x(\tau)\}$ that are *almost surely continuous*. In other words, for any interval $T \subseteq \mathbb{R}$, the subset $N \subset \mathbb{R}^T$ of functions that are discontinuous has 0 Wiener measure.

Mathematically, we want to show that, as $\Delta t_i \rightarrow 0$, the probability that Δx_i is close to 0 approaches certainty:

$$\lim_{\Delta t_i \rightarrow 0} \mathbb{P}(|\Delta x_i| < \epsilon) = 1 \quad \forall \epsilon > 0$$

This is just the probability that, during time Δt_i , the particle makes a jump of size lower than ϵ :

$$\begin{aligned} \mathbb{P}(|\Delta x_i| < \epsilon) &= \mathbb{P}(x_{i-1} - \epsilon < x_i < x_{i-1} + \epsilon | x(t_{i-1}) = x_i) = \\ &= \int_{x_{i-1}-\epsilon}^{x_{i-1}+\epsilon} \frac{dx_i}{\sqrt{4\pi D \Delta t_i}} \exp \left(- \frac{(x_i - x_{i-1})^2}{4D\Delta t_i} \right) = \\ &\stackrel{(a)}{=} \int_{-\epsilon}^{+\epsilon} \frac{d\Delta x_i}{\sqrt{4\pi D \Delta t_i}} \exp \left(- \frac{(\Delta x_i)^2}{4D\Delta t_i} \right) \end{aligned}$$

where in (a) we translated the variable of integration $\Delta x_i = x_i - x_{i-1}$.

With another change of variables:

$$\frac{(\Delta x_i)^2}{\Delta t_i} = z^2 \Rightarrow z = \frac{\Delta x_i}{\sqrt{\Delta t_i}} \Rightarrow d\Delta x_i = dz \sqrt{\Delta t_i}$$

we get:

$$\mathbb{P}(|\Delta x_i| < \epsilon) = \int_{|z| < \epsilon/\sqrt{\Delta t_i}} \frac{dz \sqrt{\Delta t_i}}{\sqrt{4\pi D \Delta t_i}} \exp \left(- \frac{z^2}{4D} \right)$$

And taking the continuum limit leads to:

$$\lim_{\Delta t_i \rightarrow 0} \mathbb{P}(|\Delta x_i| < \epsilon) = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{4\pi D}} \exp \left(- \frac{z^2}{4D} \right) = 1$$

0.2 Differentiability of Brownian Path

With a very similar calculation (here omitted) we can also show that:

$$\lim_{\Delta t_i \downarrow 0} \left(\left| \frac{\Delta x_i}{\Delta t_i} \right| > k \right) = 1 \quad \forall k > 0$$

meaning that Brownian paths are *almost surely everywhere non-differentiable*.

Nonetheless, it is sometimes useful to consider “formal derivatives” of a Brownian path, that acquire a definite meaning only when considering a *finite discretization*. For example, we can start from (1) and rewrite it as:

$$d\mathbb{P}_{t_1, \dots, t_N}(x_1, \dots, x_N) = \left(\prod_{i=1}^N \frac{dx_i}{\sqrt{4\pi\Delta t_i}} \right) \exp \left(-\frac{1}{4D} \sum_{i=1}^N \Delta t_i \left(\frac{\Delta x_i}{\Delta t_i} \right)^2 \right)$$

Then, in the continuum limit $\Delta t_i \rightarrow 0$, the sum in the exponential argument becomes a Riemann integral:

$$\sum_{i=1}^N \Delta t_i \left(\frac{\Delta x_i}{\Delta t_i} \right)^2 \xrightarrow{\Delta t \rightarrow 0} \int_0^t d\tau \underbrace{\left(\frac{dx_i}{d\tau} \right)^2}_{\dot{x}^2(\tau)} \quad t = t_N$$

where $t = t_N$. Substituting it back leads to:

$$dx_w(\tau) = \prod_{\tau=0^+}^t \frac{dx(\tau)}{\sqrt{4\pi D d\tau}} \exp \left(-\frac{1}{4D} \int_0^t \dot{x}^2(\tau) d\tau \right)$$

This expression has no rigorous meaning in this form ($\dot{x}(\tau)$ *does not* exists!) but can be *formally manipulated* into other expressions that *have* a definite meaning, thus proving useful for the discussion.

0.3 Forces on the particle

We want now to generalize the framework we obtained to the case of a diffusing particle subject to *external forces*, e.g. a drop of ink diffusing through a water medium in the presence of gravity.

To do this, we first return to the beginning, deduce a Master Equation for a more general *evolution*, and then choose the right probability distribution reproducing the behaviour in presence of forces.

So, let's start by considering a particle moving on a *uniform* one-dimensional lattice ($x_i = i \cdot l$, $t_n = n \cdot \epsilon$), and satisfying the Markovian property, meaning that the probability $W_i(t_{n+1})$ of being at the position labelled by i at the *next time-step* t_{n+1} depends only on the current state t_n , that is on the current probabilities $W_j(t_n) \forall j$ and on the current transition probabilities $W_{ij}(t_n)$ from j to i :

$$W_i(t_{n+1}) = \sum_{j=-\infty}^{+\infty} W_{ij}(t_n) W_j(t_n) \quad (2)$$

Previously, we assumed that:

$$W_{ij}(t_n) = \delta_{j,i-1} P_+ + \delta_{j,i+1} P_-$$

Which means that the particle only jumps from adjacent positions, one step at a time, and cannot remain at the same place. This Master Equation leads, in $d = 3$ and in the continuum limit, to the usual Diffusion Equation:

$$\frac{\partial}{\partial t} W(\mathbf{x}, t | \mathbf{x}_0, t_0) = \nabla^2 W(\mathbf{x}, t | \mathbf{x}_0, t_0)$$

We now consider a more general case, where we drop the discretization of the space domain, allowing *jumps* of *any size* in \mathbb{R} . Then (2) becomes:

$$W(x, t_{n+1}) dx = \int_{-\infty}^{+\infty} dz W(x, t_{n+1} | x - z, t_n) W(x - z, t_n) \quad (3)$$

The integrand is the probability of the particle being in $[x - z, x - z + dx]$ at time t_n and making a jump of size z to reach $[x, x + dx]$ at time t_{n+1} . By summing over *all possible jump sizes* we compute the total probability of the particle being *near* the arrival position.

If we require *jumps* to be **independent** of each other¹, as it is physically evident by the problem's symmetry, then the *jump* probabilities $W(x, t_{n+1} | x - z, t_n)$ depend only on the *jump size* z .

Assuming a *isolate system*, as the particle cannot *escape*, **probability is conserved**:

$$\begin{aligned} \int_{\mathbb{R}} dx W(x, t_{n+1}) &\stackrel{!}{=} \int_{\mathbb{R}} dy W(y, t_n) \\ &\stackrel{(3)}{=} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dx W(x, t_{n+1} | x - z, t_n) W(x - z, t_n) = \\ &\stackrel{(a)}{=} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dy W(y + z, t_{n+1} | y, t_n) W(y, t_n) = \\ &\stackrel{(b)}{=} \left(\int_{\mathbb{R}} dz W(\bar{y} + z, t_{n+1} | \bar{y}, t_n) \right) \left(\int_{\mathbb{R}} dy W(y, t_n) \right) \quad \forall \bar{y} \in \mathbb{R} \end{aligned}$$

where in (a) we changed variables $x \mapsto y = x - z$, with $dy = dx$, and in (b) we used the *independent increments* property (\bar{y} is a arbitrary constant). Comparing the first and last lines leads to:

$$\int_{\mathbb{R}} dz W(y + z, t_{n+1} | y, t_n) = 1$$

Intuitively, if the particle *cannot disappear*, it *must make a jump*. Here on, for notation simplicity, we denote:

$$W(y + z, t_{n+1} | y, t_n) \equiv W(+z | y, t_n)$$

Starting from (3) and taking the continuum limit in time we can write a *more*

¹^This is a stronger requirement than the Markovian property. In fact, *independent increments* imply a *Markov process*, but the converse is not true. See http://statweb.stanford.edu/~adembo/math-136/Markov_note.pdf

general diffusion equation. We start by constructing the difference quotient:

$$\begin{aligned}
W(x, t_{n+1}) - W(x, t_n) &= \int_{\mathbb{R}} dz W(+z|x - z, t_n) W(x - z, t_n) - W(x, t_n) = \\
&= \int_{\mathbb{R}} dz W(+z|x - z, t_n) W(x - z, t_n) - \underbrace{\int_{\mathbb{R}} dz W(+z|x, t_n) W(x, t_n)}_{=1} = \\
&= \int_{\mathbb{R}} dz \left[\underbrace{W(+z|x - z, t_n) W(x - z, t_n)}_{F_z(x-z)} - \underbrace{W(+z|x, t_n) W(x, t_n)}_{F_z(x)} \right] = \\
&= \int_{\mathbb{R}} dz [F_z(x - z) - F_z(x)] = \\
&\stackrel{(a)}{=} \int_{\mathbb{R}} dz \left[\cancel{F_z(x)} - z \frac{\partial}{\partial x} F_z(x) + \frac{z^2}{2} \frac{\partial^2}{\partial x^2} [F_z(x)] + \dots - \cancel{F_z(x)} \right] = \\
&= - \int_{\mathbb{R}} dz z \frac{\partial}{\partial x} [F_z(x)] + \frac{1}{2} \int_{\mathbb{R}} dz z^2 \frac{\partial^2}{\partial x^2} [F_z(x)] + \dots = \\
&\stackrel{(b)}{=} - \frac{\partial}{\partial x} \left[\underbrace{\left(\int_{\mathbb{R}} dz z W(+z|x, t_n) \right)}_{\mu_1(x, t_n)} W(x, t_n) \right] + \\
&\quad + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\underbrace{\left(\int_{\mathbb{R}} dz z W(+z|x, t_n) \right)}_{\mu_2(x, t_n)} W(x, t_n) \right] + \dots
\end{aligned}$$

where $F_z(x)$ is the probability of a *jump* of size z from the position x . In (a) we expanded F_z about x , and in (b) we exchanged the order of integrals and derivatives. Then we define the k -th moment of the *jump* pdf as follows:

$$\mu_k(x, t) = \int_{\mathbb{R}} dz z^k W(+z|x, t)$$

This allows us to rewrite the above difference in a more compact form:

$$W(x, t_{n+1}) - W(x, t_n) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} (\mu_k(x, t_n) W(x, t_n))$$

Physically, as probability is conserved, by the continuity equation, the *change* in probability density equals the divergence of a *flux*, which is just the x derivative in this one-dimensional case. So, if we *extract* a derivative, we can write the flux explicitly:

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left(\sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial x^{k-1}} (\mu_k(x, t_n) W(x, t_n)) \right) \\
&\equiv - \frac{\partial}{\partial x} J(x, t_n)
\end{aligned}$$

where $J(x, t_n)$ is the *outward flux* at x , meaning that if $J > 0$, then $W(x, t_{n+1}) < W(x, t_n)$ (the particle *escapes* from x to another place), and otherwise if $J < 0$ we have $W(x, t_{n+1}) > W(x, t_n)$ (the particle is *sucked in* x).

If we integrate both sides over x and apply the probability conservation we get the boundary conditions for the flux:

$$\int_{\mathbb{R}} (W(x, t_{n+1}) - W(x, t_n)) dx = \int_{\mathbb{R}} dx \left(-\frac{\partial}{\partial x} J(x, t_n) \right)$$

$$1 - 1 = -J(x, t_n) \Big|_{-\infty}^{+\infty} = J(-\infty, t_n) - J(+\infty, t_n)$$

This means that, in a *isolate system*, the *flux* at $\pm\infty$ must be the same.

Finally, normalizing by the time interval we get the complete difference quotient, which will become a time derivative in the continuum limit.

$$\frac{W(x, t_{n+1}) - W(x, t_n)}{t_{n+1} - t_n} = \frac{\partial}{\partial x} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial x^{k-1}} \frac{\mu_k(x, t_n) W(x, t_n)}{t_{n+1} - t_n} \right\} \quad (4)$$

Letting $t_{n+1} - t_n = \epsilon$, in the limit $\epsilon \rightarrow 0$ the left side will be $\dot{W}(x, t)$.

All that's left is to find an explicit definition for the *jump pdf* $W(+z|x, t)$. Previously, we assumed a **gaussian** pdf for the displacements:

$$z \sim \frac{1}{\sqrt{4\pi D\epsilon}} \exp\left(-\frac{(\Delta x)^2}{4D\epsilon}\right)$$

With this choice, the first two moments become:

$$\mu_1 = 0 \quad \mu_2 = 2D\epsilon$$

And the variance:

$$\text{Var}(z) = \mu_2 - \mu_1^2 = 2D\epsilon \propto \epsilon$$

However, for a particle subject to a force we would expect to have a *preferred jump direction*, leading to a *constant velocity motion* in the direction of the force. So we require a different μ_1 :

$$\langle z \rangle = \mu_1 = \int_{\mathbb{R}} z W(+z|x, t) \propto \epsilon f(x)$$

We still want to fix the variance to be proportional to ϵ , as it is expected in a diffusion process.

An appropriate choice for such a distribution is given by:

$$W(+z|x, t) = F\left(\frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}} \frac{1}{\sqrt{\epsilon \hat{D}(x, t)}}\right) \quad (5)$$

with $F, \hat{D}: \mathbb{R} \rightarrow \mathbb{R}$ functions, satisfying certain conditions, and with a physical meaning that we will now see.

First of all, we check the normalization:

$$1 \stackrel{!}{=} \int_{\mathbb{R}} dz W(+z|x, t) = \frac{1}{\sqrt{\epsilon \hat{D}(x, t)}} \int_{\mathbb{R}} dz F\left(\frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}}\right) \stackrel{(a)}{=} \int_{\mathbb{R}} dy F(y)$$

where in (a) we changed variables:

$$y = \frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}} \quad dz = \sqrt{\epsilon \hat{D}(x, t)} dy \quad (6)$$

Then we compute the first moment:

$$\begin{aligned} \langle z \rangle &= \mu_1(x, t) = \int_{\mathbb{R}} dz z F \left(\frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}} \right) \frac{1}{\sqrt{\epsilon \hat{D}(x, t)}} = \\ &\stackrel{(6)}{=} \int_{\mathbb{R}} dy \left(\epsilon f(x, t) + y \sqrt{\epsilon \hat{D}(x, t)} \right) F(y) = \\ &= \epsilon f(x, t) \underbrace{\int_{\mathbb{R}} F(y) dy}_{=1} + \sqrt{\epsilon \hat{D}(x, t)} \int_{\mathbb{R}} y F(y) dy \stackrel{!}{=} \epsilon f(x, t) \end{aligned}$$

So, in order to have the right normalization and the desired $\langle z \rangle$ we need:

$$\begin{cases} \int_{\mathbb{R}} dy F(y) = 1 \\ \int_{\mathbb{R}} dy y F(y) = 0 \end{cases}$$

Both conditions are satisfied, for example, by all even normalized functions. For the second moment:

$$\begin{aligned} \mu_2(x, t) &= \frac{1}{\sqrt{\epsilon \hat{D}(x, t)}} \int_{\mathbb{R}} dz z^2 F \left(\frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}} \right) = \\ &\stackrel{(6)}{=} \int_{\mathbb{R}} dy (\epsilon f(x, t) + y \sqrt{\epsilon \hat{D}(x, t)})^2 F(y) = \\ &= \int_{\mathbb{R}} dy F(y) [\epsilon^2 f^2 + y^2 \hat{D} \epsilon + 2\epsilon \sqrt{\epsilon \hat{D}} f y] = \\ &= \epsilon^2 f^2 + \hat{D} \epsilon \int_{\mathbb{R}} dy y^2 F(y) = \epsilon^2 f^2 + \hat{D} \epsilon \langle y^2 \rangle_{F(y)} \end{aligned}$$

And so the variance becomes:

$$\text{Var}(z) = \mu_2 - \mu_1^2 = \epsilon \hat{D} \langle y^2 \rangle_{F(y)} \propto \epsilon$$

which is proportional to ϵ as desired. For notational simplicity, we introduce a new function $D: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$\text{Var}(z) = \epsilon \hat{D} \langle y^2 \rangle_{F(y)} \equiv 2D(x, t) \epsilon \Rightarrow \mu_2(x, t) = \epsilon^2 f^2 + 2D(x, t)$$

We note that higher order moments are all of order $O(\epsilon^{3/2})$. For example, the

third moment is:

$$\begin{aligned}
\mu_3(x, t) &= \frac{1}{\sqrt{\epsilon \hat{D}(x, t)} \int_{\mathbb{R}} dz z^2 F\left(\frac{z - \epsilon f(x, t)}{\sqrt{\epsilon \hat{D}(x, t)}}\right) = \\
&\stackrel{(6)}{=} \int_{\mathbb{R}} dy (\epsilon f(x, t) + y \sqrt{\epsilon \hat{D}(x, t)})^3 F(y) = \\
&= \int_{\mathbb{R}} dy \left(\epsilon^3 f^3 + y^3 (\epsilon \hat{D})^{3/2} + 3\epsilon^2 f^2 y \sqrt{\epsilon \hat{D}} + 3\epsilon^2 f \hat{D} y^2 \right) F(y) = \\
&= \epsilon^3 f^3 + (\epsilon \hat{D})^{3/2} + 3\epsilon^2 f \hat{D} \langle y^2 \rangle_{F(y)} = O(\epsilon^{3/2})
\end{aligned}$$

Substituting back (5) in (4) we arrive to:

$$\begin{aligned}
\frac{W(x, t_{n+1}) - W(x, t_n)}{\epsilon} &= -\frac{\partial}{\partial x} \left[W(x, t_n) \underbrace{\frac{\mu_1(x, t_n)}{\epsilon}}_{f(x, t)} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\underbrace{\frac{\mu_2(x, t_n)}{\epsilon}}_{\epsilon f^2 + 2D(x, t)} W(x, t_n) \right] + \\
&\quad + \underbrace{\frac{1}{3!} \frac{\partial^3}{\partial x^3} \left[W(x, t_n) \frac{\mu_3(x, t_n)}{\epsilon} \right]}_{O(\epsilon^{1/2})} + \dots
\end{aligned}$$

Taking the limit $\epsilon \rightarrow 0$, we are left with:

$$\begin{aligned}
\frac{\partial W(x, t)}{\partial t} &= -\frac{\partial}{\partial x} [f(x, t) W(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [2D(x, t) W(x, t)] = \\
&= -\frac{\partial}{\partial x} \left[f(x, t) W(x, t) - \frac{\partial}{\partial x} (D(x, t) W(x, t)) \right]
\end{aligned}$$

This is the **Fokker-Planck equation**, describing the diffusion process in the presence of a *force* $f(x, t)$, and a diffusion parameter $D(x, t)$.

Note that, in absence of forces $f(x, t) \equiv 0$ and with a constant diffusion $D(x, t) \equiv D$ we retrieve the usual *diffusion equation*:

$$\frac{\partial}{\partial t} W(x, t) = D \frac{\partial^2}{\partial x^2} W(x, t)$$

0.4 Langevin equation

The Fokker-Planck equation involves *probability distributions*, meaning that it describes the behaviour of *ensembles of trajectories* at once. However, we can find an equivalent description by focusing on a *single path*.

We start with a Wiener process, that is a stochastic process with *independent* and *gaussian* increments and *continuous* paths. Considering a time discretization $\{t_i\}$, the evolution of a single trajectory is described by:

$$x(t_{i+1}) = x(t_i) + \Delta x(t_i) \tag{7}$$

where the increment $\Delta x(t_i)$ is sampled from a gaussian pdf:

$$\Delta x_i(t_i) \sim \frac{1}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{(\Delta x)^2}{4D \Delta t_i}\right)$$

To simplify notation, we change variables, so that:

$$\frac{\Delta B^2}{2} = \frac{\Delta x^2}{4D} \Rightarrow \Delta B = \frac{\Delta x}{\sqrt{2D}}$$

If $x \sim p(x)$, and $y = y(x) \sim g(y)$, then by the rule for a change of random variables we have:

$$g(y) = p(x(y)) \frac{dx(y)}{dy}$$

In this case:

$$\Delta B \sim \frac{1}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{(\Delta B)^2}{2\Delta t_i}\right) \underbrace{\frac{d\Delta x}{d\Delta B}}_{\sqrt{2D}} = \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left(-\frac{(\Delta B)^2}{2\Delta t_i}\right)$$

Note that now $\langle \Delta B^2(t_i) \rangle = \Delta t_i$, leaving out the D .

Substituting in (7) and rearranging we get:

$$x(t_{i+1}) - x(t_i) = \sqrt{2D} \Delta B(t_i) \quad (8)$$

We want now to form a time derivative in the left side, in order to arrive a (stochastic) differential equation for paths. To do this, we first extract a Δt_i factor from $\Delta B(t_i)$ by performing another change of variables:

$$\Delta B(t_i) \equiv \Delta t_i \xi(t_i) \quad (9)$$

so that $\Delta x_i = \sqrt{2D} \Delta t_i \xi$, and all the *randomness* is now contained in the random variable ξ , which is distributed according to:

$$\xi(t_i) \sim \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left(-\frac{\Delta t_i^2 \xi_i^2}{2\Delta t_i}\right) \underbrace{\frac{d\Delta B_i}{d\xi(t_i)}}_{\Delta t_i} = \sqrt{\frac{\Delta t_i}{2\pi}} \exp\left(-\frac{\Delta t_i}{2} \xi_i^2\right) \quad \xi_i \equiv \xi(t_i)$$

Substituting back in (8) and dividing by Δt_i leads to:

$$\frac{x(t_{i+1}) - x(t_i)}{\Delta t_i} = \sqrt{2D} \xi(t_i)$$

And by taking the continuum limit $\Delta t_i \rightarrow 0$ we get the **Langevin equation** for a Brownian particle:

$$\dot{x}(t) = \sqrt{2D} \xi(t) \quad (10)$$

Note that, as we saw previously, Brownian paths are not differentiable - and so this is just a *formal* equation, with a definite meaning only in a given *discretization*. Also, note that $\xi(t)$ is a random variable, and so this is an example of a **stochastic differential equation**. It is not clear how to find a solution to such an equation, or even how to *define* what a solution should be - and this will be the main topic of the next section.

We can rewrite (10) in a more *rigorous* form by “multiplying by dt ”, i.e. performing the change of variables (9), which - in the continuum limit - is $dB = \xi dt$, leading to:

$$dx(t) = \sqrt{2D} dB \quad dB \sim \frac{1}{\sqrt{2\pi dt}} \exp\left(-\frac{dB^2}{2 dt}\right)$$

Before moving on, we want to generalize this equation to the presence of *external forces*. As we saw previously, this just results in adding a *constant velocity motion* to the particle, leading to the full **Langevin equation**:

$$\begin{aligned} \dot{x}(t) &= f(x, t) + \sqrt{2D}\xi(t) \\ dx(t) &= f(x, t) dt + \sqrt{2D} dB \quad dB \sim \frac{1}{\sqrt{2\pi dt}} \exp\left(-\frac{dB^2}{2 dt}\right) \end{aligned}$$