0.1 Integrals in $\mathbb{C}$

0.1.1 Fourier Integral

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Recall the **Fourier transform** of a function $f(x)$:

$$\mathcal{F}(f(x)) \equiv \tilde{f}(k) \equiv \int_{-\infty}^{+\infty} e^{-ikx} f(x) \, \mathrm{d}x$$

The *inverse* Fourier transform is then:

$$\mathbb{F}^{-1}(\tilde{f}(k)) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} \tilde{f}(k) \, \mathrm{d}k$$

Note that, by our convention, we will put the $(2\pi)^{-1}$ normalization factor only in the inverse, and not in the direct Fourier transform.

- One simple Fourier transform is that of the $\delta(x)$ distribution:

  $$\mathbb{F}(\delta(x)) = \int_{-\infty}^{+\infty} e^{-ikx} \delta(x) \, \mathrm{d}x = 1$$

  In fact, we could use this relation to *define* $\delta(x)$:

  $$\delta(x) \equiv \mathbb{F}^{-1}(1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} \, \mathrm{d}k$$

- The **Heaviside function** is defined as:

  $$\theta_0(x) = \begin{cases} -1 & x < 0 \\ \frac{1}{2} & x = 0 \\ +1 & x > 0 \end{cases}$$

  If we *remove* the middle value, we can write the resulting function $\theta(x)$ in terms of the *sign function*:

  $$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & < 0 \end{cases}$$

  noting that:

  $$\theta(x) = \frac{1}{2} + \frac{\mathrm{sign}(x)}{2}$$

  To compute the Fourier transform of $\theta(x)$ we start by differentiating:

  $$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{sign}(x)}{2} = \delta(x)$$

and then make use of a property of Fourier transforms:

$$\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right) = ik\tilde{f}(k)$$

(Prove this as **exercise**) leading to:

$$\begin{aligned}\tilde{\theta}(k) &= \frac{1}{ik}\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\right) + \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{ik}\cdot 1 + \frac{1}{2}\mathcal{F}(1) = \\ &\underset{(a)}{=} \frac{1}{ik} + \pi\delta(x)\end{aligned}$$

(As **exercise**, prove that $\mathcal{F}(1) = 2\pi\delta(x)$, the result we used in (a)).

0.1.2 Fresnel integral

We now introduce a new kind of integral, which will be useful to solve the Schrödinger equation after applying a Fourier transform:

$$i\mathcal{G}(\vec{r}, t; \vec{r}\,', t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\exp\left(-\frac{i\hbar}{2m}k^2(t-t') - i\vec{k}(\vec{r}-\vec{r}\,')\right)$$

where $\mathcal{G}$ is the *propagator* (a measure of probability of transition between two positions in a quantum system - will be explained in detail later in the course).

Example 1:

Consider the following integral:

$$I(a,b) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}e^{-iak^2-ibk} = \frac{1}{(4\pi ai)^{1/2}}\exp\left(\frac{ib^2}{4a}\right) \qquad a, b \in \mathbb{R}$$

Note that $I(a,b)$ is similar to $Z(A,b)$, but it lies in the complex plane.

1. Suppose $a > 0$. We consider a slight "nudge" of the imaginary axis by a a small clockwise angle $\epsilon > 0$, that is:

$$i_\epsilon = \exp\left(i\left(\frac{\pi}{2}-\epsilon\right)\right) = e^{i2\Phi_\epsilon} \qquad \Phi_\epsilon = \frac{\pi}{4} - \frac{\epsilon}{2}$$

We replace i with i_ϵ in the first exponential term:

$$I_\epsilon(a,b) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\exp\bigl(-ak^2e^{2i\Phi_\epsilon} - ibk\bigr)$$

Note that $I_\epsilon \to I$ as $\epsilon \to 0$. Also, as:

$$\mathrm{Re}(e^{2i\Phi_\epsilon}) = \cos\left(\frac{\pi}{2}-\epsilon\right) > 0$$

I_ϵ is well defined.
Now, consider the following change of variables:

$$ke^{i\Phi_\epsilon} = z \Rightarrow \mathrm{d}k = \mathrm{d}z\, e^{-i\Phi_\epsilon}$$

and let $b' = e^{-i\Phi_\epsilon}b$ for simplicity. We arrive at:

$$I_\epsilon(a,b) = \frac{e^{-i\Phi_\epsilon}}{2\pi}\int_\gamma e^{-az^2-izb'}\,\mathrm{d}z$$

where γ is the new path of integration, i.e. a line passing through the origin of the complex plane, forming an angle of Φ_ϵ with the real axis.
To solve the complex integral we invoke, as usual, the Cauchy Residual Theorem. For that we need a close path Γ:

- Start following γ
- Follow an arc of circumference γ_+ to a point $R \gg 0$ on the real axis.
- Follow the real axis $\bar{\gamma}_R$ from right to left, up to $-R$
- Follow an arc of circumference γ_- leading back to γ

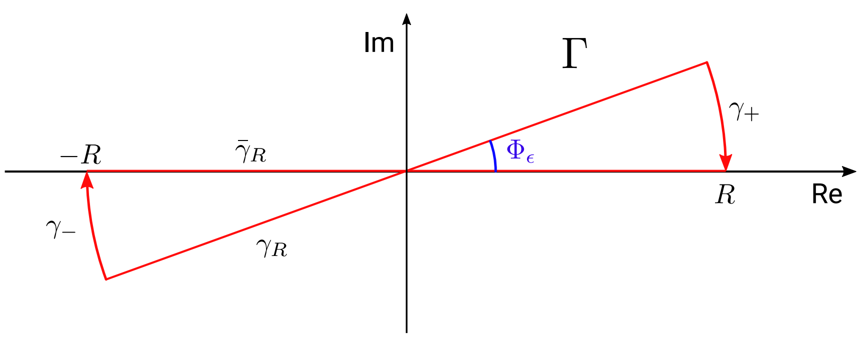

Figure (1) – Path of the integral

So that:

$$I_{R,\epsilon}(a,b) = \int_\Gamma \mathrm{d}z\, e^{-az^2-izb'} = 0$$

as there are no singularities inside $\Gamma = \gamma \cup \gamma_+ \cup \bar{\gamma}_R \cup \gamma_-$, where:

$$\begin{aligned}
\gamma_+ &= \{z = Re^{i\theta}, \theta \in [0, \Phi_\epsilon]\} \\
\gamma_- &= \{z = Re^{i\theta}, \theta \in [\pi, \pi + \Phi_\epsilon]\} \\
\gamma_R &= \{z \in \gamma, |z| \leq R\} \\
\bar{\gamma}_R &= [-R, R]
\end{aligned}$$

Note that:

$$\begin{aligned}
\left|I^{\gamma_+}_{R,\epsilon}\right| &\xrightarrow[R\to\infty]{} 0 \\
\left|I^{\gamma_-}_{R,\epsilon}\right| &\xrightarrow[R\to\infty]{} 0
\end{aligned}$$

In fact:

$$\left|\int_{\gamma_+} e^{-az^2-ib'z}\,\mathrm{d}z\right| = \left|-Ri\int_{\theta=0}^{\theta=\Phi_\epsilon}\mathrm{d}\theta\exp\left\{i\theta - aR^2e^{2i\theta} - iRb'e^{i\theta}\right\}\right| \leq$$
$$\leq R\int_0^{\Phi_\epsilon}\mathrm{d}\theta\exp\left\{-aR^2\cos(2\theta)\right\}\left|\exp\left\{iRbe^{-i\Phi_\epsilon+i\theta}\right\}\right| \tag{1}$$

Note that $\cos(2\theta) > 0$, as $2\theta < \pi/2$. Then:

$$(1) \leq R\int_0^{\Phi_\epsilon}\mathrm{d}\theta\exp\left\{-aR^2\cos(2\theta) + Rb\sin(-\Phi_\epsilon+\theta)\right\} \leq Re^{-aR^2\sin\Phi_\epsilon}\xrightarrow[R\to\infty]{} 0$$

And the same steps can be repeated for γ_-.

Then:

$$I_{\gamma_R} = -I_{\bar{\gamma}_R} = I_{\bar{\gamma}_R^{-1}}$$

And so:

$$I_\epsilon(a,b) = \frac{1}{2\pi}e^{-i\Phi_\epsilon}\int_{-\infty}^{+\infty}e^{-az^2-ib'z}\,\mathrm{d}z \underset{(a)}{=} \frac{1}{2\pi}e^{-i\Phi_\epsilon}\sqrt{\frac{\pi}{a}}\exp\left(-\frac{(b')^2}{4a}\right) =$$
$$= (4\pi a)^{-\frac{1}{2}}e^{-i\Phi_\epsilon}\exp\left(-\frac{(b')^2}{4a}\right)$$

where in (a) we used the Gaussian integral (prove as **exercise**).
Substituting back:

$$I(a,b) = \lim_{\epsilon\to0} I_\epsilon(a,b) = (4\pi ai)^{-\frac{1}{2}}\exp\left(\frac{ib^2}{4a}\right)$$

where we used:

$$e^{i\Phi_\epsilon} \to \exp\left(i\frac{\pi}{4}\right) = \sqrt{i}$$
$$(b')^2 = (be^{-i\Phi_\epsilon})^2 \to -b^2$$

2. Otherwise, if $a < 0$, note that $I(a,b) = I(-a,-b)^*$. Since $ia = (-ia)^*$ and $b^2 = (b^2)^*$ we arrive at:

$$I(a,b) = \lim_{\epsilon\to0} I_\epsilon(a,b) = (4\pi ai)^{-\frac{1}{2}}\exp\left(\frac{ib^2}{4a}\right)$$

as before.

Exercise 1:

Compute the Gaussian integral:

$$I = \int_{-\infty}^{+\infty} e^{-az^2+bz}\,\mathrm{d}z = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right) \qquad a \in \mathbb{R}_+, b \in \mathbb{C}$$

Hint : let $b = \beta + i\nu$, with $\beta, \nu \in \mathbb{R}$, and perform a shift of the integration path $z = x + iq$ $(x, q \in \mathbb{R})$ to remove all complex terms from the exponential.

0.1.3 Indented integrals in $\mathbb{C}$

We start from:

$$\lim_{\epsilon\to 0} \frac{1}{x - x_0 - i\epsilon} = \mathcal{P}\left(\frac{1}{x - x_0}\right) + i\pi\delta(x - x_0) \tag{2}$$

where $\mathcal{P}$ denotes the *principal component*. This is nothing but a crude abbreviation for the following formula:

$$\lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0 - i\epsilon}\,\mathrm{d}x = \mathcal{P}\int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0} + i\pi f(x_0) \tag{3}$$

where $f(x)$ is an analytical function.
Graphically, the limit in the left side is “pulling” a pole at $i\epsilon$ towards the point x_0 on the real axis. Then, the principal component on the other side represents a small deformation (an “indent”) of the real axis that “accomodates” the incoming pole. Recall that the principal value is defined as:

$$\mathcal{P} = \lim_{\delta\to 0}\left[\int_{-\infty}^{x_0-\delta} \frac{f(x)}{x - x_0}\,\mathrm{d}x + \int_{x_0+\delta}^{+\infty} \frac{f(x)}{x - x_0}\,\mathrm{d}x\right]$$

that is an integral that stops “just before” a pole x_0 and restarts right after, somewhat “taming” certain kinds of singular integrals that would otherwise be undefined. For example:

$$\mathcal{P}\int_{-\infty}^{+\infty} \frac{1}{x}\,\mathrm{d}x = \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta} \frac{1}{x}\,\mathrm{d}x + \int_{\delta}^{+\infty} \frac{1}{x}\,\mathrm{d}x\right] = \lim_{\delta\to 0} 0$$

To prove (3) we make use again of the Cauchy Residual Theorem. Consider a path Γ around the singularity $x_0 + i\epsilon$:

$$\int_\Gamma \frac{f(x)}{x - (x_0 + i\epsilon)} = 2\pi i f(x_0 + i\epsilon)$$

Now consider the integral on the right side of (3), and consider a Γ along the real axis, that makes a semicircle Γ_{int} with $\operatorname{Im} z < 0$ around x_0, accommodating the singularity that is shifted from the limit, and then a large semicircle Γ_{ext} with $\operatorname{Im} z > 0$.
Assume that the integrand function $f(z)$ is analytic for $\operatorname{Im} z \geq 0$, and that $f(z) \to 0$ as $|z| \to \infty$, so that:

$$\int_{\Gamma_{\text{ext}}} \cdots \xrightarrow[R\to\infty]{} 0$$

Then:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{f(x)}{x-x_0-i\epsilon} = \mathcal{P}\left(\frac{f(x)}{x-x_0-i\epsilon}\right)+\int_{\Gamma_{\text{int}}}\frac{f(z)}{z-x_0}\,\mathrm{d}z =$$
$$= \mathcal{P}\left(\frac{f(x)}{x-x_0-i\epsilon}\right)+i\pi f(x_0)$$

Exercise 2:

Prove that:

$$\lim_{\epsilon\to 0}\frac{1}{x-x_0+i\epsilon} = \mathcal{P}\left(\frac{1}{x-x_0}\right) = i\pi\delta(x-x_0)$$

Example 2 (Laplace's formula)**:**

Consider the following integral:

$$I(n) = \int_{-1}^{1}(\cos x)^n\,\mathrm{d}x$$

in the limit $n\to\infty$.
Note that, as n increases, the only relevant part of the integral is that around the maximum, meaning around $x=0$. Thus we can expand around it:

$$\cos x \approx 1-\frac{x^2}{2}$$

And then approximate with a gaussian:

$$I(n)\approx\int_{-1}^{+1}\exp\left(-\frac{n}{2}x^2\right)\mathrm{d}x = \cdots = \sqrt{\frac{2}{n}\pi}$$

[Laplace's formula]