In this lesson we will look at two proofs:

1. Continuity of Brownian Paths
2. Differentiability of Brownian Paths

Both proofs are available on the textbook, but they are not rigorous (don't look at them!).

0.1 Continuity of Brownian Path

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Consider a Brownian path $x(t)$, i.e. a set of contiguous segments connecting discrete points. We can write the probability density for a path as:

$$\mathrm{d}P_{t_1,\dots,t_N}(x_1,\dots,x_N) = \prod_{i=1}^N \left[\frac{\mathrm{d}x_i}{\sqrt{4\pi\Delta t_i D}} \exp\left(-\sum_{i=1}^N (x_i - x_{i-1})^2 \frac{1}{4D\Delta t_i}\right)\right]$$

with $\Delta t_i = t_i - t_{i-1}$, $\Delta x_i = x_i - x_{i-1}$.
Now, if we consider the continuum limit ($\max_i \Delta t_i \downarrow 0$), is the path $x(t)$ a continuous function? Mathematically, we want to show that, as $\Delta t_i \to 0$, also $\Delta x_i \to 0$. More precisely, as this process is stochastic, we need to use a limit "in the probabilistic sense", that is:

$$\lim_{\Delta t_i \to 0} P(|\Delta x_i| < \epsilon) = 1 \quad \forall \epsilon > 0$$

Note that this expression alone is just a necessary condition for continuity, and a full proof would require a more advanced math, so we will limit ourselves to this simple case.
By simply plugging in the probability distribution:

$$P(|\Delta x_i| < \epsilon) = \int_{|\Delta x_i| < \epsilon} \frac{\mathrm{d}\Delta x_i}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{(\Delta x_i)^2}{4D\Delta t_i}\right)$$

where we applied a translation to the integration variable ($x_i \to \Delta x_i$), which does not modify the volume element.
With another change of variables ($\Delta x_i/\sqrt{\Delta t_i} = z$) we arrive at:

$$= \int_{|z| < \epsilon/\sqrt{\Delta t_i}} \mathrm{d}z \exp\left(-\frac{z^2}{4D}\right)\frac{1}{\sqrt{4\pi D}} \xrightarrow{\Delta t_i \downarrow 0} \int \mathrm{d}z \frac{1}{\sqrt{4\pi D}} \exp\left(-\frac{z^2}{4D}\right) = 1$$

0.2 Differentiability of Brownian Path

With a very similar calculation we can also show that:

$$\lim_{\Delta t_i \downarrow 0} \left(\left|\frac{\Delta x_i}{\Delta t_i}\right| > k\right) = 1 \quad \forall k > 0$$

that is the Brownian motion is "fractal", in the sense that it is continuous everywhere, but nowhere differentiable.
If we now go back to the initial probability distribution, and rewrite it as:

$$dP_{t_1,\ldots,t_N}(x_1,\ldots,x_N) = \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{\sqrt{4\pi\Delta t_i}} \exp\left(-\frac{1}{4D}\sum_{i=1}^{N}\Delta t_i\left(\frac{\Delta x_i}{\Delta t_i}\right)^2\right)$$

Note that if $\Delta t \downarrow 0$, the summation in the exponential can be regarded as a Riemann sum:

$$\sum_{i=1}^{N}\Delta t_i\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 \xrightarrow[\Delta t\to 0]{} \int \mathrm{d}\tau\left(\frac{\Delta x_i}{\tau}\right)^2$$

leading to:

$$\mathrm{d}x_w(\tau) = \prod_{\tau=0^+}^{t} \frac{\mathrm{d}x(\tau)}{\sqrt{4\pi D\,\mathrm{d}\tau}} \exp\left(-\frac{1}{4D}\int_0^t \dot{x}^2(\tau)\,\mathrm{d}\tau\right)$$

with $\dot{x} = \mathrm{d}x \,/\, \mathrm{d}\tau$. Note that this is just a formal expression, with no defined meaning (it is valid only introducing a discretization) - which however will be useful as can be manipulated (formally) into expressions with a definite meaning.

0.3 Forces on the particle

Let's return to the beginning. We started with a particle capable of moving in discrete steps (with probabilities P_+ and P_- of going to the right or to the left), leading to the Master Equation, then to the Diffusion Equation and finally to the Path Integral.
In particular, we arrived at:

$$\frac{\partial}{\partial t}w(\boldsymbol{x},t|\boldsymbol{x_0},t_0) = \nabla^2 w(\bar{\boldsymbol{x}},t|\bar{\boldsymbol{x}}_{\boldsymbol{0}},t_0)$$

We want now to generalize this expression to the presence of *forces* acting on the particle. We recall the usual discretization $x_i = i\cdot l$, $t_n = n\cdot\epsilon$. In general, the probability that the particle will be at position i at time t_{n+1} depends only on the states (probabilities) of the previous time step t_n:

$$w_i(t_{n+1}) = \sum_j W_{ij}(t_n)w_j(t_n)$$

where $W_j(t_n)$ is the probability of the particle being at j at t_n, and $W_{ij}(t_n)$ is the probability of jumping from j to i at t_n (transition probability).
In the first lecture we assumed that:

$$w_{ij}(t_n) = \delta_{j,i-1}P_+ + \delta_{j,i+1}P_-$$

i.e. the particle only jumps from adjacent positions, one step at a time, and cannot remain at the same place.
Now we drop that assumption, leading to:

$$w(x,t_{n+1})\,\mathrm{d}x = \int \mathrm{d}z\, W(z|x-z,t_n)w(x-z,t_n)\,\mathrm{d}x$$

i.e. the particle can make jumps of *any* size, and is not restricted to the discretization. The integrand is just the probability for a particle starting from $[x, x+\mathrm{d}x]$ to make a jump of size z and arriving at $[x-z, x+\mathrm{d}x-z]$.
In the discrete case we previously considered, the condition $\sum_i W_{ij}(t_n) = 1$ (along with the master equation $w_i(t_{n+1}) = \sum_j W_{ij}(t_n)w_j(t_n)$) implied the conservation of probability $\sum_i w_i(t_{n+1}) = \sum_i w_i(t_n) = \cdots = \sum_i w_i(0) = 1$.
Now, in the more general case, we have:

$$\begin{aligned} w(x,t_{n+1}) &= \int \mathrm{d}z\, W(z|x-z,t_n)w(x-z,t_n) = \\ &= \int \mathrm{d}y \int \mathrm{d}z\, W(z|y,t_n)w(y,t_n) = \int \mathrm{d}y\, w(y,t_n) \end{aligned}$$

with $y = x - z$ only if:

$$\int \mathrm{d}z\, W(z|y,t) = 1 \quad \forall y, \forall t \geq 0$$

We now consider the continuum limit in time:

$$w(x,t_{n+1}) - w(x,t_n) = \int \mathrm{d}z\, W(z|x-z,t_n)w(x-z,t_n) - w(x,t_n)$$

Multiplying by 1:

$$\begin{aligned} w(x,t_{n+1}) - w(x,t_n) &= \int \mathrm{d}z\, W(z|x-z,t_n)w(x-z,t_n) - \int \mathrm{d}z\, W(z|x,t_n) w(x,t_n) = \\ &= \int \mathrm{d}z \left[\underbrace{W(z|x-z,t_n)w(x-z,t_n)}_{F_z(x-z)} - \underbrace{W(z|x,t_n)w(x,t_n)}_{F_z(x)} \right] = \\ &= F_z(x) - z\frac{\partial}{\partial x}F_z(x) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}F_z(x) + \cdots = \\ &= -\int \mathrm{d}z\, z \frac{\partial}{\partial x}\left[W(z|x,t_n)w(x,t_n)\right] + \frac{1}{2}\int \mathrm{d}z\, z^2 \frac{\partial^2}{\partial x^2}\left[W(z|x,t_n)w(x,t_n)\right] + \dots \end{aligned}$$

We then consider a sort of k-th moment of the distribution:

$$\mu_k(x,t) = \int \mathrm{d}z\, z^k W(z|x,t)$$

so that $\mu_0(x,t) = 1$ due to the normalization. Returning to the previous expression:

$$\begin{aligned} w(x,t_{n+1}) - w(x,t_n) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial}{\partial k}[\mu_k(x,t)w(x,t_n)] = \\ &= \frac{\partial}{\partial x}\left\{ \sum_{k=1}^{\infty} (-1)^k \frac{\partial^{k-1}}{\partial x^{k-1}}[\mu_k(x,t)w(x,t_n)] \right\} = -\frac{\partial}{\partial x}J(x,t_n) \end{aligned}$$

where the right side can be interpreted as the flux J.
If we integrate over x, the left side is equal to 0 (due to the normalization):

$$0 = \sum_{k\geq 1} (-1)^k \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial x}\right)^k [\mu_k(x,t) w(x,t_n)]$$

So, if the flux at the boundaries is 0, then the probability is conserved (e.g. if the system is closed).
If we now divide by the time interval (to get the derivative):

$$\frac{w(x,t_{n+1}) - w(x,t_n)}{t_{n+1} - t_n} = \frac{\partial}{\partial x} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial x^{k-1}} \frac{\mu_k(x,t_n) w(x,t_n)}{t_{n+1} - t_n} \right\}$$

Letting $t_{n+1} - t_n = \epsilon$, in the limit $\epsilon \to 0$ the left side will be $\dot{w}(x,t)$. Now, recall that for the pdf:

$$\frac{1}{\sqrt{4\pi D\epsilon}} \exp\left(-\frac{(\Delta x)^2}{4D\epsilon}\right)$$

we have $\langle \Delta x \rangle = 0$ and $\langle (\Delta x)^2 \rangle = 2D\epsilon$, with $|\Delta x| \approx \sqrt{\epsilon}$.
If the particle is subject to a force, then it will move in a *preferred direction*. So we want:

$$\langle z \rangle = \int z W(z|x,t) \propto \epsilon f(x)$$

where $f(x)$ is the force (e.g. $f(x) = mg$ for gravity). In fact, in a constant regime, the velocity of a particle is constant (when friction and the force balance out).
And then, the variance of z should be:

$$\langle (z - \langle z \rangle)^2 \rangle = \mathrm{Var}(z) \propto \epsilon$$

So we make an *ansatz*:

$$W(z|x,t) = F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}}$$

Let's see why this guess is good. First, we integrate over z to check the normalization:

$$\begin{aligned} 1 &\overset{!}{=} \int_{-\infty}^{+\infty} \mathrm{d}z\, W(z|x,t) = \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} \int_{-\infty}^{+\infty} \mathrm{d}z\, F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) = \\ &= \int_{-\infty}^{+\infty} \mathrm{d}y\, F(y) \end{aligned}$$

Then:

$$\langle z \rangle = \int \mathrm{d}z\, z F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} = \int \mathrm{d}y\, (\epsilon f(x,t) + y\sqrt{\epsilon \hat{D}(x,t)}) F(y) \overset{!}{=} \epsilon f(x,t)$$

So, summarizing, we require two conditions for F:

$$1 = \int \mathrm{d}y \, F(y)$$
$$0 = \int \mathrm{d}y \, yF(y)$$

Note that all normalized even functions (where $F(y) = F(-y)$) satisfy both of them (but they are not the only ones).
Consider now the higher moments:

$$\begin{aligned}
\mu_2(x,t) &= \frac{1}{\sqrt{\epsilon \hat{D}(x,t)}} \int \mathrm{d}z \, z^2 F\left(\frac{z - \epsilon f(x,t)}{\sqrt{\epsilon \hat{D}(x,t)}}\right) = \int \mathrm{d}y \, (\epsilon f(x,t) + y\sqrt{\epsilon \hat{D}(x,t)})^2 F(y) = \\
&= \int \mathrm{d}y \, F(y) \left[\epsilon^2 f^2 + 2\epsilon\sqrt{\epsilon \hat{D}} f y + y^2 \hat{D}\epsilon\right] = \\
&= \epsilon^2 f^2 + 0 + \epsilon \hat{D} \int \mathrm{d}y \, y^2 F(y) = \\
&= \epsilon^2 f^2 + \epsilon \hat{D} \langle y^2 \rangle_F
\end{aligned}$$

and so:

$$\mathrm{Var}(z) = \mu_2 - \mu_1^2 = \epsilon \hat{D} \langle y^2 \rangle_F$$

which is consistent with the random walk properties.
Summarizing:

$$\mu_1(x,t) = \epsilon f(x,t) \qquad \mu_2(x,t) = \epsilon \hat{D}(x,t) \langle y^2 \rangle_F = 2D(x,t)$$

Now, for the third moment:

$$\langle z^3 \rangle = \cdots = \int \mathrm{d}y \, F(y) (\epsilon f + y\sqrt{\epsilon \hat{D}})^3$$

Recall that:

$$\frac{w(x,t_{n+1}) - w(x,t_n)}{\epsilon} = -\frac{\partial}{\partial x}\left[w(x,t) \underbrace{\frac{\mu_1(x,t)}{\epsilon}}_{f(x,t)}\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[w(x,t) \underbrace{\frac{\mu_2(x,t)}{\epsilon}}_{2D(x,t)}\right] + \frac{1}{3!}\frac{\partial^3}{\partial x^3}\left[w\frac{\mu_3}{\epsilon}\right] + \dots$$

The higher terms are at least of $O(\sqrt{\epsilon})$, and thus vanish in the limit.

$$\dot{w}(x,t) = -\frac{\partial}{\partial x} \underbrace{\left[f(x,t) w(x,t) - \frac{\partial}{\partial x}(D(x,t) w(x,t))\right]}_{J(x,t)}$$

Then, if $f(x,t) = 0$ (free particle) and $D(x,t) = D$ constant, then we get the diffusion equation:

$$\dot{w}(x,t) = D \frac{\partial^2}{\partial x^2} w(x,t)$$

However, this equation is much more general, and it is called the **Fokker-Plank** equation.

0.4 Langenvin equation

Following the approach of the first lessons, we now search for a general solution for the motion.
In general, the position at the next time step for a free particle is given by:

$$x(t_{i+1}) = x(t_i) + \Delta x(t_i)$$

where $\Delta x(t_i)$ is sampled from a gaussian distribution:

$$\Delta x(t_i) \sim \frac{1}{\sqrt{4\pi D \Delta t_i}} \exp\left(-\frac{\Delta x^2}{4D\Delta t_i}\right)$$

We can rewrite it as:

$$x(t_{i+1}) = x(t_i) + \sqrt{2D}\Delta B(t_i)$$

with:

$$\Delta x_i = \sqrt{2D}\Delta B(t_i); \quad \Delta B(t_i) \sim \frac{1}{\sqrt{2\pi\Delta t_i}} \exp\left(-\frac{\Delta B_i^2}{2\Delta t_i}\right)$$

and $\langle \Delta B^2(t_i)\rangle = \Delta t_i$.
So, if we define $\Delta B(t_i) = \Delta t_i \xi(t_i)$, dividing both sides by Δt_i we get:

$$\dot{x}(t) = \sqrt{2D}\xi(t)$$

which is the **Langenvin** equation for a Brownian particle.
However, recall that the spatial derivative for a brownian trajectory does not exist - so this is a *quasi-equation*, without a rigorous meaning.

$$\xi(t_i) \sim \sqrt{\frac{\Delta t_i}{2\pi}} \exp\left(-\frac{\Delta t_i \xi_i^2}{2}\right)$$

and then:

$$P(\xi \dots) \propto \exp\left(-\frac{1}{2}\int \xi^2(\tau)\,\mathrm{d}\tau\right)$$

and $\langle \xi(\tau)\rangle = 0$, $\langle \xi(\tau)\xi(\tau')\rangle = \delta(\tau - \tau')$.

$$\mathrm{d}x\,(t) = \sqrt{2D}\,\mathrm{d}B\,; \qquad \mathrm{d}B \sim \frac{1}{\sqrt{2\pi\,\mathrm{d}t}} \exp\left(-\frac{\mathrm{d}B^2}{2\,\mathrm{d}t}\right)$$

Now consider a particle moving in 3D:

$$m\ddot{\boldsymbol{r}}(t) = -\gamma\dot{\boldsymbol{r}} + \boldsymbol{F}_{\mathrm{ext}} + \boldsymbol{F}_{\mathrm{noise}}(t)$$

e.g. $\boldsymbol{F}_{\text{ext}} = -\hat{\boldsymbol{z}}gm$ for gravity, and $\boldsymbol{F}_{\text{noise}}$ is the *random force* due to collisions with other particles.
If $\boldsymbol{F}_{\text{ext}}$ is conservative, we can write:

$$\boldsymbol{F}_{\text{ext}}(\boldsymbol{r},t) = -\boldsymbol{\nabla} V(\boldsymbol{r},t)$$

with a certain potential $V(\boldsymbol{r},t)$.
Dividing both sides by γ:

$$\frac{m}{\gamma}\ddot{\boldsymbol{r}}(t) = -\dot{\boldsymbol{r}} + \frac{\boldsymbol{F}_{\text{ext}}(\boldsymbol{r},t)}{\gamma} + \frac{\boldsymbol{F}_{\text{noise}}(t)}{\gamma}$$

where, by Stokes law, $\gamma = 6\pi a\eta$ and η is the viscosity of the surrounding fluid. Note that $[m\gamma^{-1}] = [t]$ - so this ratio sets a *timescale* τ, i.e. the characteristic time $\bar{v} - v(t) = e^{-t/\tau}$, where $\bar{v}$ is the final (constant) velocity.
If m/γ is much smaller than the timescale of observation, then we can neglect the acceleration term (i.e. the final constant state is reached almost immediately).
We can then write:

$$\dot{\boldsymbol{r}} = \underbrace{\frac{\boldsymbol{F}_{\text{ext}}}{\gamma}}_{f} + \frac{\boldsymbol{F}_{\text{noise}}}{\gamma} \underset{d=1}{\Rightarrow} \dot{x}(t) = f(x,t) + \sqrt{2D(x,t)}$$

This has no meaning unless we define a discretization:

$$\mathrm{d}x\,(t) = f(x,t)\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}B \qquad \mathrm{d}B \sim \frac{1}{\sqrt{2\pi\,\mathrm{d}t}} \exp\left(-\frac{(\mathrm{d}B)^2}{2\,\mathrm{d}t}\right)$$

which is a *stochastic differential equation.* But how to integrate it?