0.0.1 Correlation function in the Mean Field

(Lesson 27 of 11/05/20) Compiled: June 2, 2020

We want to verify (**??**) in the case of the mean field approximation.

So, we start with a *separable* variational ansatz (**??**, pag. **??**):

$$\rho_0(\boldsymbol{\sigma}) = \prod_x \rho_x(\sigma_x) \qquad \rho_x(\sigma_x) = \frac{1 + m_x\sigma_x}{2} \quad m_x \in [-1, 1] \tag{1}$$

In this case, spins are independent, and so:

$$\langle \sigma_x \sigma_y \rangle_0 = \sum_{\{\boldsymbol{\sigma}\}} \rho_0(\boldsymbol{\sigma}) \sigma_x \sigma_y = \langle \sigma_x \rangle_0 \langle \sigma_y \rangle_0 \underset{(??)}{=} m_x m_y$$

More in general, the n-point correlator between m **distinct** $x_1, \dots, x_m$ spins is the product of the *local magnetizations* m_{x_i}:

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_0 = \prod_{i=1}^{n} m_{x_i}$$

This means that the correlation function G_{xy} is trivial:

$$G_{xy} \equiv \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle = m_x m_y - m_x m_y = 0$$

A more interesting (and accurate) result can be obtained if we start from the **exact** partition function:

$$Z(\boldsymbol{h}) = \sum_{\{\boldsymbol{\sigma}\}} \exp\left(-\beta H(\boldsymbol{\sigma}) + \sum_x h_x \sigma_x \right)$$

The magnetization m_{x_i} at position x_1 can be obtained by deriving $\ln Z(\boldsymbol{h})$ with respect to the local field h_{x_1} at that position (**??**):

$$\frac{\partial}{\partial h_{x_1}} \ln Z(\boldsymbol{h}) = \frac{\sum_{\{\boldsymbol{\sigma}\}} \exp\left(-\beta H(\boldsymbol{\sigma}) + \sum_x h_x \sigma_x \right) \sigma_{x_1}}{Z(\boldsymbol{h})} = \langle \sigma_{x_1} \rangle = m_{x_1}$$

And if we differentiate once more, with respect to h_{x_2} with $x_2 \neq x_1$ (see the steps preceding (**??**) at pag. **??**):

$$\begin{aligned} \frac{\partial^2}{\partial h_{x_1} \partial h_{x_2}} \ln Z(\boldsymbol{h}) = \frac{\partial}{\partial h_{x_1}} \langle \sigma_{x_1} \rangle = \frac{\sum_{\{\boldsymbol{\sigma}\}} \exp\left(-\beta H(\boldsymbol{\sigma}) + \sum_x h_x \sigma_x \right) \sigma_{x_1} \sigma_{x_2}}{Z} + \\ - \frac{\sum_{\{\boldsymbol{\sigma}\}} \exp\left(-\beta H(\boldsymbol{\sigma}) + \sum_x h_x \sigma_x \right) \sigma_{x_1}}{Z} \cdot \frac{\sum_{\{\boldsymbol{\sigma}\}} \exp\left(-\beta H(\boldsymbol{\sigma}) + \sum_x h_x \sigma_x \right) \sigma_{x_2}}{Z} = \\ = \langle \sigma_{x_1} \sigma_{x_2} \rangle - \langle \sigma_{x_1} \rangle \langle \sigma_{x_2} \rangle \end{aligned}$$

And so we get an exact result for the two-point correlation function:

$$G_{x_1 x_2} \equiv \frac{\partial m_{x_1}}{\partial h_{x_2}} = \langle \sigma_{x_1} \sigma_{x_2} \rangle - \langle \sigma_{x_1} \rangle \langle \sigma_{x_2} \rangle \tag{2}$$ eqn:twop

In the mean field, local magnetizations obey equation (**??**):

$$m_x(\boldsymbol{h}, K) = \tanh\left[K \sum_{y \in \langle y,x \rangle} m_y + h_x\right] \qquad \texttt{eqn:var-sol2} \tag{3}$$

where the sum is over all nodes y that are neighbours of x. Solving for h_x leads to:

$$h_x = \tanh^{-1} m_x - \sum_y K_{xy} m_y \qquad K_{xy} \underset{(a)}{\equiv} K\delta_{|\boldsymbol{r}_x - \boldsymbol{r}_y|,a} = \begin{cases} K & |\boldsymbol{r}_x - \boldsymbol{r}_y| = a \\ 0 & \text{otherwise} \end{cases} \qquad \texttt{eqn:hx-step} \tag{4}$$

Here we denote with $\boldsymbol{r}_x$ the position of node x in the lattice, so that $|\boldsymbol{r}_x - \boldsymbol{r}_y|$ is the distance between spins x and y. Neighbouring cells are separated only by the grid step a, and we use this fact to rewrite in (a) the sum over neighbours of x as a sum over *all nodes* by adding an appropriate Kronecker delta.

Differentiating both sides of (**??**) with respect to h_z leads to:

$$\delta_{xz} = \frac{\partial h_x}{\partial h_z} = \frac{1}{1 - m_x^2} \underbrace{\frac{\partial m_x}{\partial h_z}}_{G_{xz}\,(\texttt{??})} - \sum_y K_{xy} \underbrace{\frac{\partial m_y}{\partial h_z}}_{G_{yz}\,(\texttt{??})} =$$

$$= \sum_y \underbrace{\left[\frac{\delta_{xy}}{1 - m_x^2} - K_{xy}\right]}_{A_{xy}} G_{yz}$$

This can be rewritten in matrix form as follows:

$$\mathbb{1} = \mathbf{AG} \qquad \texttt{eqn:AG-mtx} \tag{5}$$

where the entries of A are:

$$\mathbf{A} = \begin{pmatrix} \frac{1}{1-m_1^2} - \cancel{K_{11}} & -K_{12} & -K_{13} & \cdots \\ -K_{21} & \frac{1}{1-m_2^2} - \cancel{K_{22}} & -K_{23} & \cdots \\ -K_{31} & -K_{32} & \frac{1}{1-m_3^2} - \cancel{K_{33}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad A_{xy} = \frac{\delta_{xy}}{1 - m_x^2} - K_{xy}$$

In the case of a **cubic lattice** with **periodic boundary conditions** the system is translationally invariant, and so it is reasonable to assume a **uniform magnetization**, i.e. $h_x \equiv h$ and $m_x \equiv m(h, T)$. This leads to:

$$A_{xy} = \frac{\delta_{xy}}{1 - m^2} - K_{xy} = \begin{cases} \frac{1}{1-m^2} & x = y \\ -K & |\boldsymbol{r}_x - \boldsymbol{r}_y| = a \end{cases}$$

And from (**??**) we obtain:

$$\mathbf{G}^{-1} = \mathbf{A} \Leftrightarrow (\mathbf{G}^{-1})_{xy} = A_{xy} = \frac{\delta_{xy}}{1 - m^2} - K_{xy}$$

Note that the entries of $\mathbf{G}^{-1}$ depend only on *differences of positions* (in K_{xy}), meaning that $\mathbf{G}^{-1}$ is translationally invariant. Explicitly, the (reciprocal of the) correlation between spins that are *the same distance apart* is the same, and so:

$$(\mathbf{G}^{-1})_{xy} \equiv G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) = G^{-1}(\boldsymbol{r}_x + \boldsymbol{n}a, \boldsymbol{r}_y + \boldsymbol{n}a) \qquad \boldsymbol{n} \in \mathbb{Z}^d$$

Choosing $\boldsymbol{n}$ so that $\boldsymbol{r}_y = -\boldsymbol{n}a$ we get:

$$G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) = G^{-1}(\boldsymbol{r}_x - \boldsymbol{r}_y, 0) \equiv G^{-1}(\boldsymbol{r}_x - \boldsymbol{r}_y) \tag{6}$$ eqn:corr_translational_invariance

Translational invariance implies that $\mathbf{G}^{-1}$ is diagonalized by the Fourier basis.

Translational invariance and Fourier basis

This can be quickly shown in the *continuum limit*, i.e. if we treat $\boldsymbol{r}_x$ and $\boldsymbol{r}_y$ as continuous variables. Then, ignoring the normalization constants:

$$\begin{aligned}
\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q}) &\propto \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{r}_x \, e^{i\boldsymbol{p}\cdot\boldsymbol{r}_x} \int_{\mathbb{R}^d} e^{i\boldsymbol{q}\cdot\boldsymbol{r}_y} \, G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) \overset{\texttt{eqn:fourier-basis1}}{=} \qquad (7)\\
&\underset{(??)}{=} \int_{\mathbb{R}^{2d}} \mathrm{d}^d \boldsymbol{r}_x \, \mathrm{d}^d \boldsymbol{r}_y \, e^{i\boldsymbol{p}\cdot\boldsymbol{r}_x} e^{i\boldsymbol{q}\cdot\boldsymbol{r}_y} \, G^{-1}(\boldsymbol{r}_x - \boldsymbol{r}_y) = \\
&\underset{\boldsymbol{t}=\boldsymbol{r}_x-\boldsymbol{r}_y}{=} \int_{\mathbb{R}^{2d}} \mathrm{d}^d \boldsymbol{t} \, \mathrm{d}^d \boldsymbol{r}_y \, e^{i\boldsymbol{p}\cdot\boldsymbol{t}} e^{i\boldsymbol{r}_y\cdot(\boldsymbol{p}+\boldsymbol{q})} G^{-1}(\boldsymbol{t}) = \\
&= \underbrace{\int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{t} \, e^{i\boldsymbol{p}\cdot\boldsymbol{t}} G^{-1}(\boldsymbol{t})}_{\mathcal{F}[G^{-1}](\boldsymbol{p})} \underbrace{\int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{r}_y \, e^{i\boldsymbol{r}_y\cdot(\boldsymbol{p}+\boldsymbol{q})}}_{\delta^d(\boldsymbol{p}+\boldsymbol{q})} = \tilde{G}^{-1}(\boldsymbol{p}) \delta^d(\boldsymbol{p}+\boldsymbol{q})
\end{aligned}$$

And so $\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q})$ can be nonzero only if $\boldsymbol{p} = -\boldsymbol{q}$, meaning that can be seen as a (infinite) diagonal matrix[1]. Then, the inverse of a diagonal matrix is obtained by replacing each element in the diagonal with its reciprocal: bhfn:1

$$\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{\mathcal{F}[G^{-1}](\boldsymbol{p})} \delta^d(\boldsymbol{p} + \boldsymbol{q})$$

However, $\boldsymbol{r}_x$ and $\boldsymbol{r}_y$ are constrained to **discrete** positions in the cubic lattice - in other words $G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y)$ should really be a function $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$. By adding some Dirac deltas, we can extend the domain to $\mathbb{R}^d \times \mathbb{R}^d$, essentially making the function vanish for all non-integers arguments:

$$G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) = \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d} G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) \delta^d(\boldsymbol{r}_x - \boldsymbol{n}a) \delta^d(\boldsymbol{r}_y - \boldsymbol{m}a) \qquad \boldsymbol{r}_x, \boldsymbol{r}_y \in \mathbb{R}^d$$

G^{-1} has a period of a for all its arguments, and so its Fourier transform will have[2] a period of $2\pi/a$. bhfn:2

With a **symmetric** choice for the normalization, the Fourier transform becomes:

$$\begin{aligned}
\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q}) = \left(\frac{a}{2\pi}\right)^d \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{p} \, e^{-i\boldsymbol{p}\cdot\boldsymbol{r}_x} \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{q} \, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_y} \cdot \\
\cdot \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d} G^{-1}(\boldsymbol{r}_x, \boldsymbol{r}_y) \delta^d(\boldsymbol{r}_x - \boldsymbol{n}a) \delta^d(\boldsymbol{r}_y - \boldsymbol{m}a) =
\end{aligned}$$

[1] ∧A discrete matrix $\mathcal{M}_{N\times N}(\mathbb{R})$ is diagonal if it has entries $A_{xy} = A_x \delta_{xy}$. Here we have some sort of "continuous matrix" - which should really be intended as the matrix representation in the Fourier basis of a linear operator on $L^2(\mathbb{R}^d)$. So, in a sense, it is a $\infty \times \infty$ matrix "centred in $\mathbf{0}$". In the $d = 1$ case, it can be thought as of a *plane*, with entries at every point (p, q), and the diagonal being the second-fourth quadrant bisector $p = -q$.

[2] ∧See `https://www.gnu.org/software/gnuastro/manual/html_node/Dirac-delta-and-comb.html` for the proof.

$$\underset{(a)}{=} \left(\frac{a}{2\pi}\right)^d \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \sum_{\boldsymbol{m}\in\mathbb{Z}^d} e^{-ia\boldsymbol{p}\cdot\boldsymbol{n}} e^{-ia\boldsymbol{q}\cdot\boldsymbol{m}} \underbrace{G^{-1}(\boldsymbol{n}a, \boldsymbol{m}a)}_{G^{-1}([\boldsymbol{n}-\boldsymbol{m}]a)} =$$

$$\underset{\boldsymbol{t}=\boldsymbol{n}-\boldsymbol{m}}{=} \left(\frac{a}{2\pi}\right)^d \sum_{\boldsymbol{t}\in\mathbb{Z}^d} \sum_{\boldsymbol{m}\in\mathbb{Z}^d} e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}} e^{-ia\boldsymbol{m}\cdot(\boldsymbol{p}+\boldsymbol{q})} G^{-1}(\boldsymbol{t}a) =$$

$$= \left(\frac{a}{2\pi}\right)^d \underbrace{\sum_{\boldsymbol{t}\in\mathbb{Z}^d} e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}} G^{-1}(\boldsymbol{t}a)}_{\text{1st}} \underbrace{\sum_{\boldsymbol{m}\in\mathbb{Z}^d} e^{-ia\boldsymbol{m}\cdot(\boldsymbol{p}+\boldsymbol{q})}}_{\text{2nd}}$$

where in (a) we used the deltas to collapse the integrals.
For the second sum, recall that[3]: bhfn:3

$$(\text{2nd}) = \frac{1}{F} \sum_{k=-\infty}^{+\infty} \exp\left(\pm i \frac{2k\pi f}{F}\right) = \sum_{n=-\infty}^{+\infty} \delta(f - nF)$$

In our case:

$$\left(\frac{2\pi}{a}\frac{a}{2\pi}\right)^d \sum_{\boldsymbol{m}\in\mathbb{Z}^d} \exp\left(-i\frac{2\pi\boldsymbol{m}\cdot(\boldsymbol{p}+\boldsymbol{q})}{2\pi/a}\right) = \left(\frac{2\pi}{a}\right)^d \sum_{\boldsymbol{m}\in\mathbb{Z}^d} \delta\left(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{m}\frac{2\pi}{a}\right)$$

Regarding the remaining sum, recall that $\boldsymbol{t}a = \boldsymbol{r}_x - \boldsymbol{r}_y$:

$$(\text{1st}) = G^{-1}(\boldsymbol{t}a) \equiv G^{-1}(\boldsymbol{r}_x - \boldsymbol{r}_y) = A_{xy} = \frac{\delta_{xy}}{1-m^2} - K\delta_{|\boldsymbol{r}_x-\boldsymbol{r}_y|,a}$$

Note that $x = y$ if and only if $\boldsymbol{r}_x - \boldsymbol{r}_y = 0$, i.e. $\boldsymbol{t} = \boldsymbol{0}$, and so $\delta_{xy} = \delta^d_{\boldsymbol{t},\boldsymbol{0}}$, which denotes a d-dimensional Kronecker delta:

$$\delta^d_{\boldsymbol{t},\boldsymbol{0}} = \begin{cases} 1 & \boldsymbol{t} = \boldsymbol{0} \Leftrightarrow t_1 = \cdots = t_d = 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly:

$$\delta_{|\boldsymbol{r}_x-\boldsymbol{r}_y|,a} = \delta_{\|\boldsymbol{t}\|a,a} = \delta_{\|\boldsymbol{t}\|,1}$$

Substituting in the sum:

$$\sum_{\boldsymbol{t}\in\mathbb{Z}^d} e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}} \left(\frac{\delta^d_{\boldsymbol{t},\boldsymbol{0}}}{1-m^2} - K\delta_{\|\boldsymbol{t}\|,1}\right) = \underbrace{e^{-ia\boldsymbol{p}\cdot\boldsymbol{0}}}_{1} \frac{1}{1-m^2} - K \sum_{\substack{\boldsymbol{t}\in\mathbb{Z}^d \\ \|\boldsymbol{t}\|=1}} e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}}$$

For a integer valued vector $\boldsymbol{t} \in \mathbb{Z}^d$, a unitary norm can be obtained if and only if *exactly one* of its components is ± 1. Thus:

$$\sum_{\substack{\boldsymbol{t}\in\mathbb{Z}^d \\ \|\boldsymbol{t}\|=1}} e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}} = \sum_{\mu=1}^{d} \left[e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}^\mu_+} + e^{-ia\boldsymbol{p}\cdot\boldsymbol{t}^\mu_-}\right] = \qquad \left(\boldsymbol{t}^\mu_\pm = (\underset{1}{0}, \ldots, \underset{\mu}{\pm 1}, \ldots, \underset{d}{0})\right)$$

[3] ∧See equation 4 in http://fourier.eng.hmc.edu/e102/lectures/ExponentialDelta.pdf for the proof.

$$= \sum_{\mu=1}^{d} \left[e^{-iap_\mu} + e^{+iap_\mu} \right] = \sum_{\mu=1}^{d} 2\cos(ap_\mu)$$

And so at the end we get:

$$\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q}) = \left(\frac{1}{1-m^2} - 2K \sum_{\mu=1}^{d} \cos(ap_\mu) \right) \sum_{\boldsymbol{m} \in \mathbb{Z}^d} \delta^d \left(\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{m} \frac{2\pi}{a} \right)$$

which is periodic with period $2\pi/a$ in each component of both arguments - and so $\boldsymbol{p}$ and $\boldsymbol{q}$ vary within[4] $(-\pi/a, +\pi/a)^d$. This means that:

$$p_\mu + q_\mu \in \left(-\frac{2\pi}{a}, +\frac{2\pi}{a} \right) \qquad \forall \mu = 1, \ldots, d$$

And so:

$$\sum_{\boldsymbol{m} \in \mathbb{Z}^d} \delta^d(\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{m} \frac{2\pi}{a}) = \delta^d(\boldsymbol{p} + \boldsymbol{q})$$

because all other δs with some $m_\mu \neq 0$ vanish. Thus $\mathcal{F}[G^{-1}](\boldsymbol{p}, \boldsymbol{q})$ is **diagonal**, and its matrix inverse is:

$$\tilde{G}(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{(1-m^2)^{-1} - 2K \sum_{\mu=1}^{d} \cos(ap_\mu)} \delta^d(\boldsymbol{p} + \boldsymbol{q})$$

Symmetric normalization. We are using a *symmetric* normalization for the direct and inverse Fourier transforms (here for $d = 1$ for simplicity):

$$\mathcal{F}[f(x)](p) \equiv \tilde{f}(p) = \sqrt{\frac{a}{2\pi}} \sum_{x=-\infty}^{+\infty} e^{ixp} f(x)$$

$$\mathcal{F}^{-1}[\tilde{f}(p)](x) = \sqrt{\frac{a}{2\pi}} \int_{-\pi/a}^{+\pi/a} \tilde{f}(p) e^{ixp} \, \mathrm{d}p$$

Where the factor is exactly $1/\sqrt{T}$, with T being the period of $\tilde{f}(p)$, which is $2\pi/a$ in our case.
This is done so that $\mathcal{F}$ is a **unitary** linear operator mapping elements (i.e. functions) of $L_2(\mathbb{R})$ to elements of $L_2([-\pi/a, \pi/a])$. This is necessary to avoid altering the determinant of the matrix during transformation.
Physically, this is the only way to make both G^{-1} and $\tilde{G}^{-1}$ adimensional - as they should be.

To find G_{xy} we anti-transform. Applying the definition of the inverse transform of the DTFT (with a symmetric choice for normalization) leads to:

$$G(\boldsymbol{r}_x, \boldsymbol{r}_y) = \left(\frac{a}{2\pi} \right)^d \int_{\left(-\frac{\pi}{a}, +\frac{\pi}{a}\right)^{2d}} \mathrm{d}^d\boldsymbol{q} \, \mathrm{d}^d\boldsymbol{p} \, e^{i\boldsymbol{p}\cdot\boldsymbol{r}_x} e^{i\boldsymbol{q}\cdot\boldsymbol{r}_y} \frac{\delta^d(\boldsymbol{p} + \boldsymbol{q})}{(1-m^2)^{-1} - 2K \sum_{\mu=1}^{d} \cos(ap_\mu)} =$$

[4] ∧The extrema are **not** contained. This can be proved by considering the *discrete finite case*, i.e. a lattice with a finite number N of spins, computing the DFT and taking the thermodynamic limit $N \to \infty$.

$$\underset{\boldsymbol{q}=-\boldsymbol{p}}{=}\left(\frac{a}{2\pi}\right)^d \int_{\left(-\frac{\pi}{a},+\frac{\pi}{a}\right)^d} \mathrm{d}^d\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{r}_x-\boldsymbol{r}_y)} \underbrace{\frac{1}{(1-m^2)^{-1}-2K\sum_{\mu=1}^{d}\cos(ap_\mu)}}_{\tilde{G}(\boldsymbol{p})} \tag{8}$$

eqn:G-eigenvalue

The integral's cubic domain is also called the **first Brillouin zone**[5] of the cubic lattice with grid step a. $\tilde{G}(\boldsymbol{p})$ are then the **eigenvalues** of the matrix G_{xy} (when $N \to \infty$), and $(a/2\pi)^{d/2}e^{i\boldsymbol{p}\cdot\boldsymbol{r}_x}$ are the (orthonormal) eigenvectors.

bhfn:5

Note that for $|\boldsymbol{x}-\boldsymbol{y}| \gg a$, i.e. spins that are very far apart, the oscillation of the complex exponential $e^{i\boldsymbol{p}\cdot(\boldsymbol{r}_x-\boldsymbol{r}_y)}$ is very rapid, i.e. a tiny difference in $\boldsymbol{p}$ amounts to a high change in phase. Then the integrand will *oscillate* rapidly about 0, making the integral vanish. Thus, the only significant contributions are when $\boldsymbol{p} \approx \boldsymbol{0}$, i.e. the ones with a "stationary phase"[6]. So, we can expand $\tilde{G}(\boldsymbol{p})$ in series:

bhfn:6

$$\begin{aligned}\tilde{G}(\boldsymbol{p}) &= \left[\frac{1}{1-m^2}-2K\sum_{\mu=1}^{d}\cos(ap_\mu)\right] = \left[\frac{1}{1-m^2}-2K\sum_{\mu=1}^{d}\left(1-\frac{(ap_\mu)^2}{2}+O(p_\mu^4)\right)\right] = \\ &= \Big[\underbrace{\frac{1}{1-m^2}-2dK}_{A}+K\|\boldsymbol{p}\|^2a^2+O(\|\boldsymbol{p}\|^4)\Big]^{-1} \approx \Big[Ka^2\Big(\|\boldsymbol{p}\|^2+\frac{A}{Ka^2}\Big)\Big]^{-1} = \\ &= \frac{1}{Ka^2}\frac{1}{\|\boldsymbol{p}\|^2+Ka^2/A}\end{aligned}$$

Recall that a has dimension of length, ($[a]=\mathsf{L}$) and that $\boldsymbol{p}$ of the reciprocal of length ($[\|\boldsymbol{p}\|]=\mathsf{L}^{-1}$), while the constant A defined in the above expression is a pure number. Then $[A/Ka^2]=\mathsf{L}^{-2}$ and so we denote it $\xi^{-2}=A/ka^2$, where now $[\xi]=\mathrm{L}$ is the **correlation** length:

$$\tilde{G}(\boldsymbol{p}) \approx \frac{1}{Ka^2}\frac{1}{\|\boldsymbol{p}\|^2+\xi^{-2}} \qquad \xi^{-2} \equiv \frac{A}{Ka^2} = \frac{a^{-2}}{K}\Big[\frac{1}{1-m^2}-2dK\Big] \tag{9}$$

eqn:G-corr

In the mean field approximation, m is given by the solution of (**??**), which near criticality is given by (**??**, pag. **??**):

$$m^2(K) = \begin{cases} 3\dfrac{K-K_c}{K_c} & K > K_c \\ 0 & K < K_c \end{cases}$$

with $K_c = 1/2d$.

[5] ∧Lattices are at the foundation of solid state physics. In particular, the Fourier transform of a lattice - represented as a "grid of δs" - is still a lattice in the space of *frequencies*, and it's called the **reciprocal lattice**. The first Brillouin zone is just how the first unit cell of the lattice appears after the Fourier transform. In the case of a cubic lattice, it is still cubic, but with a different length.

[6] ∧This is in fact a generalization of the *saddle-point approximation* to complex integrals with an oscillating term

Then, substituting in the definition of ξ, for $K < K_c$ we get:

$$\xi^{-2} = \frac{a^{-2}}{K^2}\left[\frac{1}{1-m^2} - \frac{K}{K_c}\right] \underset{K<K_c}{=} \frac{a^{-2}}{K}\left[1 - \frac{K}{K_c}\right] = \frac{a^{-2}}{K}\left(\frac{K_c - K}{K_c}\right) = \frac{a^{-2}}{K}\frac{|K - K_c|}{K_c}$$

And for $K > K_c$, expanding around $m \approx 0$:

$$\xi^{-2} = \frac{a^{-2}}{K^2}\left[1 + m^2 + O(m^4) - \frac{K}{K_c}\right] \underset{K>K_c}{\approx} \frac{a^{-2}}{K}\left[1 + 3\frac{K - K_c}{K_c} - \frac{K}{K_c}\right] = 2\frac{a^{-2}}{K}\frac{|K - K_c|}{K_c}$$

So, in both cases, we have:

$$\xi^{-2} \propto \left|\frac{K - K_c}{K_c}\right| \qquad K \approx K_c$$

Meaning that the correlation length diverges near criticality:

$$\xi \propto \left|\frac{K_c}{K - K_c}\right|^{1/2} \qquad K \approx K_c$$

This is consistent with (**??**, pag. **??**):

$$\xi \propto |t|^{-\nu}$$

with $\nu = 1/2$ in the mean field approximation.

Substituting back in (**??**) and collecting a $\|\boldsymbol{p}\|^2$:

$$\tilde{G}(\boldsymbol{p}) \approx \frac{1}{ka^2\|\boldsymbol{p}\|^2}\Big(1 + \frac{1}{(\xi\|\boldsymbol{p}\|)^2}\Big)^{-1} = \|\boldsymbol{p}\|^{-2}\tilde{g}(\|\boldsymbol{p}\|\xi) \tag{10}$$ eqn:corr-transform

All that's left is to verify that (**??**) is indeed compatible with a correlation function given by the ansatz (**??**). So, let $h = 0$ for simplicity, and assume that:

$$G_{xy} = \|\boldsymbol{r}_x - \boldsymbol{r}_y\|^{-(d-2+\eta)}\hat{g}\left(\frac{\|\boldsymbol{r}_x - \boldsymbol{r}_y\|}{\xi}\right) \tag{11}$$ eqn:assume-ansatz

The Fourier transform of (**??**), up to normalization, is given by:

$$\tilde{G}(\boldsymbol{p}) \propto \int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{r}\, \|\boldsymbol{r}\|^{-(d-2+\eta)}\hat{g}\left(\frac{\|\boldsymbol{r}\|}{\xi}\right) e^{i\boldsymbol{r}\cdot\boldsymbol{p}}$$

By rescaling $\boldsymbol{r} = \boldsymbol{v}/\|\boldsymbol{p}\|$ we can make $\hat{g}$ a function of $\|\boldsymbol{p}\|\xi$ as in (**??**):

$$\begin{aligned}
\tilde{G}(\boldsymbol{p}) &\propto \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\boldsymbol{v}}{\|\boldsymbol{p}\|^d}\left\|\frac{\boldsymbol{r}}{\boldsymbol{p}}\right\|^{-(d-2+\eta)}\hat{g}\left(\frac{\|\boldsymbol{v}\|}{\|\boldsymbol{p}\|\xi}\right)\exp\Big(i\boldsymbol{v}\cdot\underbrace{\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}}_{\hat{\boldsymbol{p}}}\Big) = \\
&= \|\boldsymbol{p}\|^{-(2-\eta)}\underbrace{\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{v}\, \|\boldsymbol{v}\|^{-(d-2+\eta)}\hat{g}\left(\frac{\|\boldsymbol{v}\|}{\|\boldsymbol{p}\|\xi}\right)e^{i\boldsymbol{v}\cdot\hat{\boldsymbol{p}}}}_{\tilde{g}(\|\boldsymbol{p}\|\xi)} = \\
&= \|\boldsymbol{p}\|^{-2+\eta}\tilde{g}(\|\boldsymbol{p}\|\xi)
\end{aligned}$$

And so (**??**) is compatible with (**??**). Moreover, we see that $\eta = 0$ in the mean field approximation (but in general it can be $\neq 0$).

As a bonus, we see that the scaling relation (**??**, pag. **??**) is respected in the mean field:

$$\gamma = \nu(2 - \eta)$$

As we have found $\gamma = 1$, $\nu = 1/2$ and $\eta = 0$.

Finally, notice that:

$$\sum_y G_{xy} = \langle \sigma_x \sum \sigma_y \rangle - \langle \sigma_x \rangle \langle \sum_y \sigma_y \rangle \underset{(a)}{=} \frac{\partial}{\partial h} \langle \sigma_x \rangle = \chi$$

where in (a) we used the fluctuation-dissipation theorem (**??**, pag. **??**). Then, the sum over all y is equal to the Fourier transform evaluated at $\boldsymbol{p} = \mathbf{0}$. First we rewrite it as:

$$\sum_y G_{xy} \equiv \sum_{\boldsymbol{r}_y \in \mathbb{Z}^d} G(\boldsymbol{r}_x, \boldsymbol{r}_y) = \sum_{\boldsymbol{t} \in \mathbb{Z}^d} G(\boldsymbol{r}_x - \boldsymbol{r}_y, 0) = \sum_{\boldsymbol{t} \in \mathbb{Z}^d} G(\boldsymbol{t})$$

Then, evaluating $\tilde{G}(\mathbf{0})$ gives the desired sum:

$$\tilde{G}(\mathbf{0}) = \sum_{\boldsymbol{t} \in \mathbb{Z}^d} G(\boldsymbol{t}) e^{-i\boldsymbol{p}\cdot\boldsymbol{t}} \Big|_{\boldsymbol{p}=\mathbf{0}} = \sum_{\boldsymbol{t} \in \mathbb{Z}^d} G(\boldsymbol{t}) \underset{(??)}{=} \left(\frac{1}{1 - m^2} - 2Kd \right)$$

0.0.2 Example in a real system

An example of the emergence of scaling laws in real complex systems can be seen by examining some features of a **forest**, in which many trees *compete* for a set of resources (elements in soil, light, etc.).

We start from the simplifying assumption that each tree of a certain "size" r (which can be measured as its height, or the diameter of its trunk) mainly *competes* with trees of similar or bigger size - as everything much smaller will have a negligible effect on it.
Then we measure the distance r_i between the tree of size r and the closest *bigger* tree, which will follow some distribution $\mathbb{P}(r_i|r)$. In particular, we consider the *accumulated* distribution given by:

$$\mathbb{P}^{>}(r_i|r) \equiv \int_{r_i}^{\infty} P(r_i'|r) \, \mathrm{d}r_i'$$

After some sophisticated analysis, a reasonable *ansatz* for $\mathbb{P}^{>}$ is found:

$$\mathbb{P}^{>}(r_i|r) = F\left(\frac{r_i}{r^{2/3}} \right) \qquad \texttt{eqn:tree-ansatz} \quad (12)$$

For each r, $\mathbb{P}^{>}(r_i|r)$ is plotted in fig. **??**.

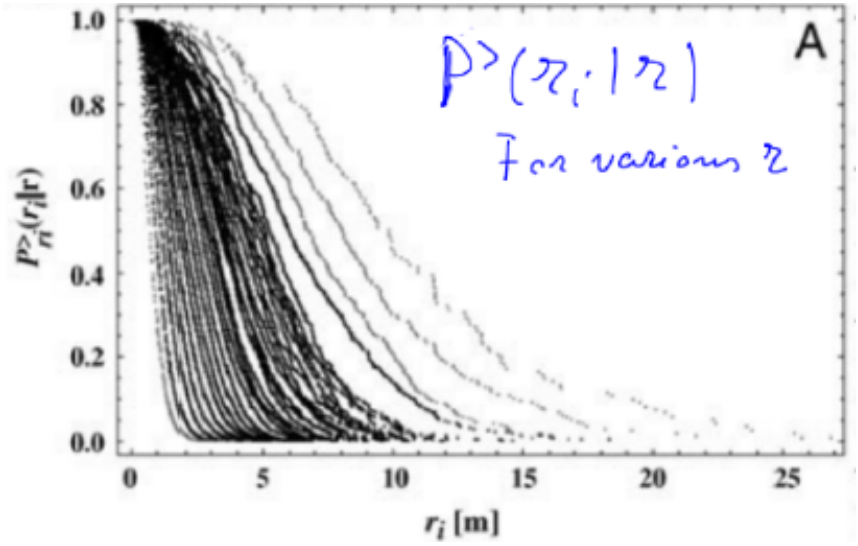

Figure (1) – Plot of the *accumulated* probabilities for r_i for various tree sizes r. `fig:tree_size`

If the ansatz in (**??**) is correct, then by rescaling the x axis to $r_i/r^{2/3}$ we should see all curves "collapsing" into one. This indeed happens in fig. **??**.

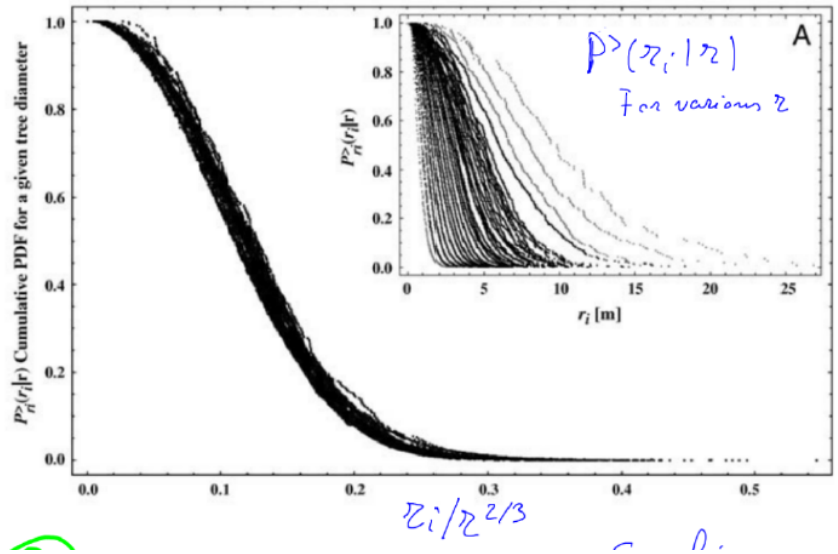

Figure (2) – All curves from fig. **??** collapse (approximately) into one when rescaling the x axis according to ansatz (**??**) `fig:tree-collapse`

This means that there is some kind of *emergent behaviour* in the forest: the distribution of tree sizes is not completely random, but exhibits a *scaling behaviour*, which is similar to the one we studied in the Ising model. In a sense, the forest has "self-tuned" to a state "near criticality". Yet, it is not clear *why* this is the case - for example what is the evolutionary advantage in this kind of "self tuning".

Chapter 1

Beyond Mean Field and Universality

Many years of study of phase transitions have shown that there are few *types* of critical behaviours, meaning that many systems exhibit the same kind of behaviour near criticality.

A summary of the critical exponents for several *classes* of models is shown in fig. ??.

Table 1-1 Values of critical exponents

Exponent	Mean field theory	Ising model $d = 2$	Ising model $d = 3$†	Heisenberg model $d = 3$†	Spherical model $\epsilon > 0$
α	0	0	0.12	−0.06	$-\frac{\epsilon}{2-\epsilon}$
α'		0			
β	$\frac{1}{2}$	$\frac{1}{8}$	0.31		$\frac{1}{2}$
γ	1	$\frac{7}{4}$	1.25	1.38	$\frac{2}{2-\epsilon}$
γ'				not defined	
ν	$\frac{1}{2}$	1	0.64	0.7	$\frac{1}{2-\epsilon}$
ν'					
δ	3	15	5.0		$\frac{6-\epsilon}{2-\epsilon}$
η	0	$\frac{1}{4}$	0.04	0.05	0

†Approximate results from numerical extrapolations of high-temperature and low-temperature series. The uncertainties are higher for $n = 3$ than $n = 1$ and for the primed indices.

Figure (1.1) – Values of critical exponents for several models. As these exponents do not depend on all the details of each model, there is no need to fully specify each Hamiltonian. For example, the IM in $d = 2$ could have more complex couplings between spins (other than the nearest neighbour ones we considered until now). `fig:table-criticality`

Here there is no need to **fully specify** the **Hamiltonian** of each model: for instance, when talking about a Ising Model in $d = 2$, the exponents are the same no matter how complex the spin couplings may be (for example, not only nearest neighbours may be interacting).

Moreover, if we consider higher and **higher dimensions** , the exponents **tend** to the values obtained in the **mean field** approximation. For example, δ is 15 in $d = 2$, and 5 in $d = 3$, which is closer to the MF value of 3.
There is also a dependence on the **symmetries** of each model. As an example we may consider the Heisenberg model, a generalization of the IM in which each spin is a 3-vector. Now the system is globally *rotationally symmetric*, meaning that the symmetry group is O(3). In the Ising Model with *binary* spins, the spin-flip symmetry is instead described by the $\mathbb{Z}_2$ group. So, even if the two models are studied in the same number of dimension (e.g. $d = 3$), the scaling exponents will be different, as can be seen in fig. **??**.

From this analysis, we find that the critical exponents do not depend on the specificity of the Hamiltonian, but only on two *general* characteristics:

- The **dimensionality** of the model
- The **symmetries** of the Hamiltonian

1.1 From discrete to continuous variables

Consider a Ising Model, with a general Hamiltonian including also more complex spin-spin interactions:

$$-\beta H(\boldsymbol{\sigma}) = \sum_{x,y} K_{xy}\sigma_x\sigma_y + \sum_{x,y,z,t} K_{xyzt}\sigma_x\sigma_y\sigma_z\sigma_t + \dots \tag{1.1}$$ eqn:H-complex

The term K_{xy} (and similarly the others) could include not only neighbouring spins, but also next-to-neighbouring spins and so on. One important constraint is to allow only interaction between an **even** number of spins, so that the Hamiltonian has still a *spin-flip* symmetry:

1. Symmetry

$$H(\boldsymbol{\sigma}) = H(-\boldsymbol{\sigma})$$

which is described by the cyclic group $\mathbb{Z}_2$.

The partition function is given by:

$$Z = \sum_{\{\boldsymbol{\sigma}\}} e^{-\beta H(\boldsymbol{\sigma})}$$

This is a function of **discrete** binary variables $\sigma_x = \pm 1$. We can write it as a function of *continuous* variables φ_x by *zeroing* all values where $\varphi_x \neq \pm 1$ with a Dirac delta:

$$Z = \int_{\mathbb{R}^N} \Big[\prod_x \mathrm{d}\varphi_x \, \delta(\varphi_x^2 - 1) \Big] e^{-\beta H(\boldsymbol{\varphi})} \tag{1.2}$$ eqn:Z-delta

The Dirac delta can then be written as the limit in which a smooth function (e.g. a gaussian) becomes more and more "peaked" (fig. **??**):

$$\delta(\varphi_x^2 - 1) = \lim_{\lambda \to +\infty} e^{-\lambda(\varphi_x^2-1)^2} \mathcal{N}(\lambda) \tag{1.3}$$ eqn:delta-limit

where $\mathcal{N}(\lambda)$ is a normalization constant. In fact, the integral of the δ is always fixed:

$$\int_{-\infty}^{+\infty} \mathrm{d}\varphi\, \delta(\varphi^2 - 1) \underset{(a)}{=} \int_{-\infty}^{+\infty} \mathrm{d}\varphi\, \frac{\delta(\varphi - 1) + \delta(\varphi + 1)}{2|\varphi|} = \frac{1}{2|1|} + \frac{1}{2|-1|} = 1 \tag{1.4}$$

eqn:delta-integral

where in (a) we used the composition formula for the δ:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

with the sum over all (simple) roots x_i of $g(x)$.

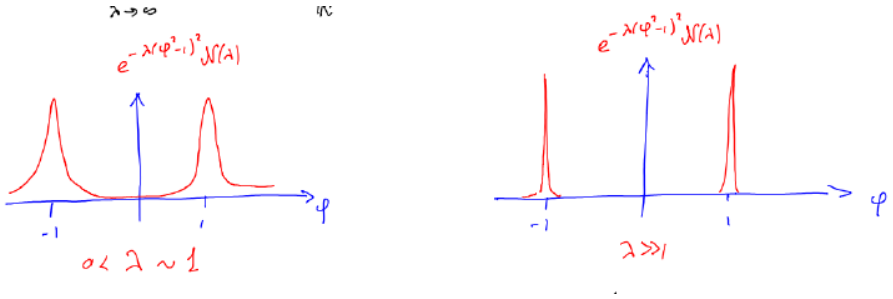

Figure (1.2) – The $\delta(\varphi_x^2 - 1)$ can be seen as the limit of a smooth function with two peaks in ± 1, which become more and more sharp as $\lambda \to +\infty$, while maintaining the area under the curve fixed to 1.

fig:delta-peaks

Imposing (**??**) in (**??**) leads to the following expression for the normalization $\mathcal{N}(\lambda)$:

$$\mathcal{N}(\lambda) = \Big[\int_{-\infty}^{+\infty} \mathrm{d}\varphi\, e^{-\lambda(\varphi^2-1)^2} \Big]^{-1}$$

Then, substituting (**??**) into the partition function (**??**) we get:

$$Z = \lim_{\lambda\to\infty} \mathcal{N}(\lambda)^N \int_{\mathbb{R}^N} \Big[\prod_x \mathrm{d}\varphi_x \Big] \exp\left(-\beta H(\boldsymbol{\varphi}) - \lambda \sum_x (\varphi_x^2 - 1)^2 \right) \tag{1.5}$$

eqn:Z-limit

Experimentally, we observe that near criticality the details of the system do not matter for describing its behaviour - and so we expect that the system with finite λ (e.g. $\lambda \sim 1$) will behave *similarly* to the one with $\lambda \to +\infty$. Mathematically, studying the first case allows us to deal with *smooth* functions.

So, removing the limit from (**??**) is equivalent to study the following Hamiltonian:

$$-\beta H_{\mathrm{tot}}(\boldsymbol{\varphi}) = -\beta H(\boldsymbol{\varphi}) - \lambda \sum_x (\varphi_x^2 - 1)^2$$

2. Translational invariance

Let's now assume that all spin-spin interactions are **translationally invariant**, meaning that the interaction terms K_{xy}, K_{xyzt} and so on are all functions of *distances*.

Explicitly, let's focus for instance on K_{xy}. Translational invariance means that we can write it as:

$$K_{xy} = K_2(\boldsymbol{r}_x - \boldsymbol{r}_y)$$

for some function K_2. Then, due to the $\mathbb{Z}_2$ symmetry, $K_2(\boldsymbol{r}_x - \boldsymbol{r}_y) = K_2(\boldsymbol{r}_y - \boldsymbol{r}_x)$. In fact, if the two spins are the same (both $+1$ or -1), then exchanging them will not make any difference. If they are different, i.e. one $+1$ and the other -1, exchanging them is equivalent to a *spin-flip*, and so the result will still be the same.
Thus, if we rewrite $\boldsymbol{r} = \boldsymbol{r}_x - \boldsymbol{r}_y = \boldsymbol{n}a$, with $\boldsymbol{n} \in \mathbb{Z}^d$, then $K_2(\boldsymbol{r})$ is an **even** function, and depends only on $\|\boldsymbol{r}\|$ due to translational invariance.
This means that all averages of only one component of $\boldsymbol{r}$ are zero:

$$\sum_{\boldsymbol{r}} K_2(\|\boldsymbol{r}\|) r_\alpha = 0 \qquad \alpha = 1, \dots, d \tag{1.6}$$ eqn:r-average

because K_2 is even, while r_α is odd.

3. Short range interactions

We also assume interactions to be **short range**. This means that the average of two components of $\boldsymbol{r}$ is proportional to a^2:

$$\sum_{\boldsymbol{r}} K_2(\|\boldsymbol{r}\|) r_\alpha r_\beta = \frac{\delta_{\alpha\beta}}{d} \sum_{\boldsymbol{r}} K_2(\|\boldsymbol{r}\|) \|\boldsymbol{r}\|^2 \propto a^2 < \infty \tag{1.7}$$ eqn:rr-average

The $\delta_{\alpha\beta}$ comes from the fact that we expect different directions to be independent (**isotropy**), and the d is just a normalization for the Kronecker delta.

Similar relations are expected to hold for all higher order interaction terms (K_{xyzt} and so on).

With all these assumptions, we can simplify the Hamiltonian (**??**). For instance, the K_{xy} term becomes:

$$\sum_{xy} K_2(\boldsymbol{r}_x - \boldsymbol{r}_y) \varphi(\boldsymbol{r}_x) \varphi(\boldsymbol{r}_y)$$

with $\varphi(\boldsymbol{r}_x) \equiv \varphi_x$ and $\varphi(\boldsymbol{r}_y) \equiv \varphi_y$. In this notation, x and y are *numeric* indices for the spins, and $\boldsymbol{r}_x$, $\boldsymbol{r}_y \in \mathbb{Z}^d a$ are their positions in the lattice. Then $\varphi(\boldsymbol{r}_x) \in \mathbb{R}$ refers to the *spin* of the cell x at position $\boldsymbol{r}_x$.

Changing variables to $\boldsymbol{r} = \boldsymbol{r}_x - \boldsymbol{r}_y$ leads to:

$$\sum_{y} \varphi(\boldsymbol{r}_y) \sum_{\boldsymbol{r}} K_2(\boldsymbol{r}) \varphi(\boldsymbol{r}_y + \boldsymbol{r})$$

4. Criticality

Now we use the fact that K_2 is **short range**, and so the dominant contributions to the sum are the ones with small $\boldsymbol{r}$. Near **criticality**, we expect the correlation length to diverge, meaning that spins that are far apart can be highly correlated. Qualitatively, this leads to spin configurations that are "smooth", in the sense that neighbouring spins are similarly aligned. Mathematically, this allows us to treat $\varphi(\boldsymbol{r})$ as a smooth function of the position $\boldsymbol{r}$, and in particular to expand it in series around $\boldsymbol{r} = \boldsymbol{0}$:

$$\sum_{y} \varphi(\boldsymbol{r}_y) \sum_{\boldsymbol{r}} K_2(\boldsymbol{r}) \left[\varphi(\boldsymbol{r}_y) + \sum_{\alpha=1}^{d} r_\alpha \frac{\partial}{\partial r_\alpha} \varphi(\boldsymbol{r}) \Big|_{\boldsymbol{r}=\boldsymbol{r}_y} + \frac{1}{2} \sum_{\alpha,\beta=1}^{d} r_\alpha r_\beta \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \varphi(\boldsymbol{r}) \Big|_{\boldsymbol{r}=\boldsymbol{r}_y} + \dots \right] =$$

We can then exchange some of the sums to use (**??**) and (**??**):

$$= \sum_{y} \varphi(\boldsymbol{r}_y) \underbrace{\sum_{\boldsymbol{r}} K_2(\boldsymbol{r})}_{c_2} \varphi(\boldsymbol{r}_y) + \sum_{y} \varphi(\boldsymbol{r}_y) \sum_{\alpha=1}^{d} \underbrace{\Big(\sum_{\boldsymbol{r}} K_2(\boldsymbol{r}) r_\alpha \Big)}_{0\,(??)} \frac{\partial \varphi}{\partial r_\alpha}(\boldsymbol{r}_y) +$$

$$+\sum_y \varphi(\boldsymbol{r}_y) \sum_{\alpha,\beta=1}^{d} \underbrace{\Big(\sum_{\boldsymbol{r}} \frac{r_\alpha r_\beta}{2} K_2(\boldsymbol{r})\Big)}_{\propto \delta_{\alpha\beta} a^2 \text{ (??)}} \frac{\partial^2 \varphi}{\partial r_\alpha \partial r_\beta}(\boldsymbol{r}_y) + \dots =$$

And inserting a coefficient c_3 to account for the proportionality:

$$= c_2 \sum_y \varphi(\boldsymbol{r}_y)^2 + c_3 a^2 \sum_y \varphi(\boldsymbol{r}_y) \underbrace{\sum_{\alpha=1}^{d} \frac{\partial^2 \varphi}{\partial r_\alpha^2}(\boldsymbol{r}_y)}_{\nabla^2 \varphi(\boldsymbol{r}_y)} =$$

$$= c_2 \sum_y \varphi(\boldsymbol{r}_y)^2 + c_3 a^2 \sum_y \varphi(\boldsymbol{r}_y) \nabla^2(\boldsymbol{r}_y)$$

Taking the **continuum limit** $a \to 0$ (an "infinitely dense" lattice) the summation becomes an integral. We then *rescale* the integration variable from $\boldsymbol{r}_y = \boldsymbol{y} a$, resulting in an additional a^{-d} factor, so that the integrand is adimensional:

$$\underset{a\to 0}{=} \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{y}\, a^{-d} \Big[c_2 \varphi(a\boldsymbol{y})^2 + c_3 a^2 \varphi(a\boldsymbol{y}) \nabla^2 \varphi(a\boldsymbol{y}) + \dots \Big] =$$

Finally, integrating by parts two times the highlighted term, and ignoring the resulting surface terms, leads to:

$$= \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{y}\, a^{-d} \left[c_2 \varphi^2(a\boldsymbol{y}) - c_3 a^2 (\nabla \varphi(a\boldsymbol{y}))^2 + \dots \right]$$

A similar procedure can be done also for the other interaction terms. For instance, consider the *quartic* term:

$$\sum_{x,y,z,t} K_{xyzt} \varphi(\boldsymbol{r}_x) \varphi(\boldsymbol{r}_y) \varphi(\boldsymbol{r}_z) \varphi(\boldsymbol{r}_t)$$

Translational invariance implies that:

$$K_{xyzt} = K_4(\boldsymbol{r}_x - \boldsymbol{r}_y, \boldsymbol{r}_z - \boldsymbol{r}_y, \boldsymbol{r}_t - \boldsymbol{r}_y)$$

And so we can rewrite the term in the Hamiltonian as follows:

$$\sum_{xyzt} K_{xyzt} \varphi(\boldsymbol{r}_x) \varphi(\boldsymbol{r}_y) \varphi(\boldsymbol{r}_z) \varphi(\boldsymbol{r}_t) =$$

$$\underset{\substack{\boldsymbol{r}_1 \equiv \boldsymbol{r}_x - \boldsymbol{r}_y \\ \boldsymbol{r}_2 \equiv \boldsymbol{r}_z - \boldsymbol{r}_y \\ \boldsymbol{r}_3 \equiv \boldsymbol{r}_t - \boldsymbol{r}_y}}{=} \sum_y \varphi(\boldsymbol{r}_y) \sum_{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3} K_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) \varphi(\boldsymbol{r}_y + \boldsymbol{r}_1) \varphi(\boldsymbol{r}_y + \boldsymbol{r}_2) \varphi(\boldsymbol{r}_y + \boldsymbol{r}_3)$$

And then expand around $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3 = \boldsymbol{0}$.

After all these manipulations, the Hamiltonian will look like:

$$-\beta H_{\text{tot}}(\boldsymbol{\varphi}) = \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{y}\, a^{-d} \Big[- c_3 a^2 (\nabla \varphi)^2 + \hat{c}_2 \varphi^2 + \hat{c}_4 \varphi^4 + \dots + d_2 a^2 (\nabla \varphi)^2 + \dots \Big]$$

The yellow term comes from the binary interactions, the blue ones from the quartic interactions, and the green includes contributions from both of them.

As $\varphi \in \mathbb{R}$, through a change of variables we can *fix* one of the coefficients - for example c_3 to $1/2$. So we consider the transformation $\varphi = \zeta\phi$, with $\zeta \in \mathbb{R}$ constant such that:

$$c_3 a^{2-d}(\nabla\varphi)^2 = c_3 a^{2-d}\zeta^2(\nabla\phi)^2 \stackrel{!}{=} \frac{1}{2}(\nabla\phi)^2 \Rightarrow \zeta^2 = \frac{a^{d-2}}{c_3}$$

After this, the Hamiltonian becomes:

$$-\beta H_{\text{tot}}(\boldsymbol{\varphi}) \equiv -\beta\mathcal{H}(\boldsymbol{\phi}) = -\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{y}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{\mu}{2}\phi^2 + g_4\phi^4 + g_6\phi^6 + \cdots + \right. \tag{1.8}$$

eqn:continuum-limit

$$\left. + f_2(\nabla\phi)^2\phi^2 + \ldots\right] \tag{1.9}$$

It is not important to specify exactly the dependence of these new coefficients $\mu, g_4, g_6, f_2, \ldots$ on the old ones $c_3, \hat{c}_2, \hat{c}_4, d_2, \ldots$. For now, let's just observe their order on a:

$$\mu \propto a^{-2};\ g_4 \propto a^{d-4};\ g_6 \propto a^{2(d-3)};\ g_8 \propto a^{3d-8};\ f_2 \propto a^{d-2}$$

When taking the continuum limit $a \to 0$, μ always diverges, while the other coefficients either vanish or diverge depending on d. For instance:

- $d > 4$: only μ is diverging
- $3 < d < 4$: μ and g_4 diverge
- $8/3 < d < 3$: μ, g_4, g_6 diverge

In particular, this means that for $d > 4$, a purely gaussian model can be used to describe the system, and it's able to capture all the behaviour of the system near criticality:

$$-\beta H_G(\boldsymbol{\phi}) = -\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{x}\, h_g(\boldsymbol{\phi}, \partial_\alpha\boldsymbol{\phi}) \qquad h_G(\boldsymbol{\phi}) = \frac{1}{2}\left[(\nabla\phi)^2 + \boldsymbol{\phi}^2\right]$$

This is the essence of **universality**: a critical system can be described with few parameters, which depend only on the symmetry and the dimensionality (assuming short-range interactions).

From a physical point of view, we are interested only in the $d = 3$ (general systems) and $d = 2$ cases (interfaces/surfaces of systems). However, all other possibilities are still relevant theoretically. In particular, the concept of *fractional dimensions* enables *perturbative expansions*. The idea is that for $d \lesssim 4$ we can write $d = 4 - \epsilon$, with $\epsilon \approx 0$. This means that g_4 is "less important" than μ, and can be treated perturbatively starting from a Gaussian model, leading to results that agree well with experiments. In particular, this has lead to very powerful *renormalization group techniques*, which are able to shed light onto the so-called "universality classes", i.e. very general "types" of models with similar critical behaviour.

1.2 Back to the discrete world

In summary, the continuum limit $a \to 0$ has shown to us that only a **finite number of terms** in the **Hamiltonian** are **important**.
Thus, a **discrete model** with just the "right terms" so that its continuum limit matches just the first terms of (**??**) is enough to describe *all the systems* in the same *universality class* near criticality!

One such example is the Nearest Neighbour Ising Model that we have already studied. In the same universality class we find the "Next neighbour" Ising Model, in which a spin may interact with a *neighbour* of its *neighbours* (fig. **??**). Explicitly, the Hamiltonian is given by:

$$-\beta H(\boldsymbol{\sigma}) = K \sum_{\langle x,y \rangle} \sigma_x \sigma_y + L \sum_{\langle\langle x,y \rangle\rangle} \sigma_x \sigma_y$$

where the second sum is over all x and y that *share* a neighbour z (i.e. that are "neighbours of neighbours").

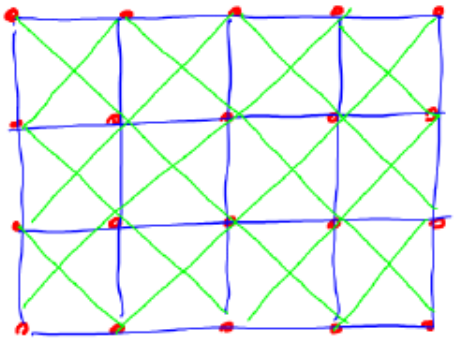

Figure (1.3) – Diagram of the interactions in a Next Neighbour Ising Model. Spins in the lattice are represented as red dots, the usual *nearest neighbour* interactions are in blue, and the *next neighbour* interactions are in green.
`fig:n-nIM`

Results from renormalization group theory show that the phase diagram of this kind of model is that of fig. **??**.

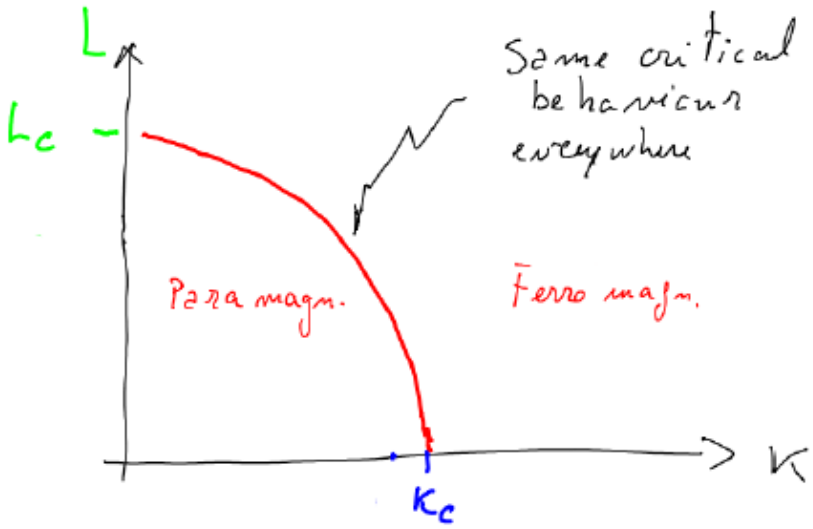

Figure (1.4) – Phase diagram of the Next Neighbour Ising Model. `fig:critical-n-nIM`

When the system is near criticality (i.e. along the red line of fig. **??**) its behaviour is described by the **same set of critical exponents** that appear in the Nearest Neighbour Ising Model we previously examined.

Let's add *another term* to the Hamiltonian, describing *quartic* interactions:

$$-\beta H(\boldsymbol{\sigma}) = K \sum_{\langle x,y \rangle} \sigma_x \sigma_y + L \sum_{\langle\langle x,y \rangle\rangle} \sigma_x \sigma_y + M \sum_{[xyzt]} \sigma_x \sigma_y \sigma_z \sigma_t \qquad (1.10)$$ `eqn:H-complicated`

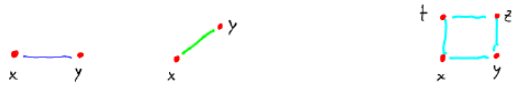

Figure (1.5) – Types of interactions in (**??**) `fig:interactions`

In this case, the phase diagram is shown in fig. **??**. Near the *red surface*, at which a second order transition happens, the system's behaviour is again described by the same critical exponents that appear in the Nearest Neighbour Ising Model! They are only different when crossing the *green surface* (corresponding to a first order transition, for which there is no universality in principle) and the boundary between the two surfaces, called the *tri-critical line* (which belongs to a *different* universality class).

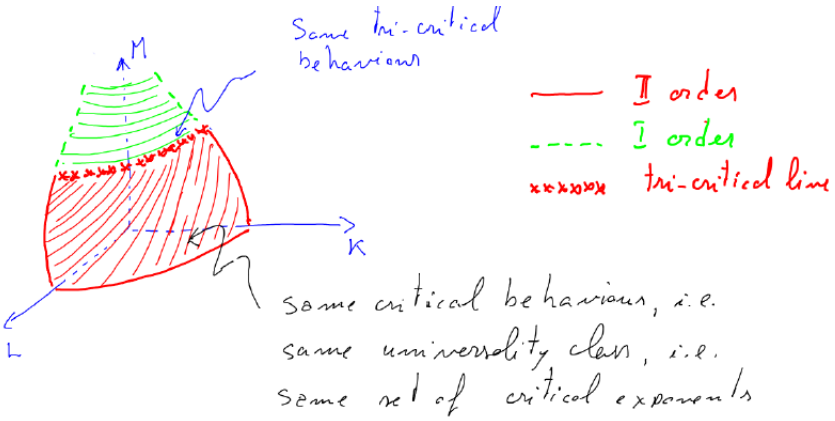

Figure (1.6) – Phase diagram for the model (??). fig:complicated-phases