0.1 Concepts

(Lesson ? of 13/12/19) Compiled: December 13, 2019

0.1.1 Planck's scale

The Compton wavelength for an energetic particle is defined as:

$$\lambda = \frac{\hbar c}{\mathcal{E}}$$

Recall the expression for the Schwarzschild radius:

$$r_s = \frac{GM}{c^2} = \frac{G\mathcal{E}}{c^4}$$

If $\lambda < r_S$ we are in **quantum gravity** regime. This happens when:

$$\frac{\hbar c}{\mathcal{E}} = \frac{G\mathcal{E}}{c^4} \Rightarrow \mathcal{E}_p = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \times 10^9 \text{ J}$$

This is called **Planck energy**. Beyond this energy scale, quantum field theory breaks down, as the particles *collapse* into black holes.
Recall that $1\,\text{eV} = 1.6 \times 10^{-19}$ J, and so:

$$\mathcal{E}_p = 1.22 \times 10^{19}\,\text{GeV}$$

In $c = 1$ units, this energy corresponds also to a mass, denoted as **Planck mass**. We also introduce the *reduced planck mass* defined as:

$$M_p = \frac{\mathcal{E}_p}{\sqrt{8\pi}} = 2.43 \times 10^{19}\,\text{GeV}$$

0.1.2 Natural units

We start by defining $c = 1$ as a *dimensionless quantity*, meaning that *lengths* and *times* have the same dimension.
Also setting $\hbar = 1$ means that *angular momentum* is *dimensionless*:

$$1 = [\text{Angular Momentum}] = [\text{Length}][\text{Velocity}][\text{Mass}]$$

This means that:

$$[\text{Length}] = [\text{Time}] = \frac{1}{[\text{Mass}]} = \frac{1}{[\text{Energy}]}$$

We can now rewrite Einstein's equations in natural units. Recall that:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Then from:

$$M_p = \frac{1}{\sqrt{8\pi}} \underbrace{\frac{1}{\sqrt{G}}}_{\mathcal{E}_p} \Rightarrow 8\pi G = \frac{1}{M_p^2}$$

and so:

$$G_{\mu\nu} = \frac{T_{\mu\nu}}{M_p^2}$$

0.2 Cosmology

Consider a *homogeneous* and *isotropic* universe, i.e. the Friedman Lemaitre Robertson Walker solution (FLRW).

- **Homogeneous**: any point is "like" every other point
- **Isotropic**: any direction is "like" every other direction

For example, a system that is *homogeneous but not isotropic* is the volume inside a charged capacitor (the preferred direction is given by the electric field). An example for a system that is *isotropic but not homogeneous* is the field generated by a point-charge, as it is spherically symmetric.

There are three possible cases of curvature:

- **Flat**: 0 curvature, meaning that the angles of any triangle add up to 180°.
- **Closed**: Positive curvature, where the angles of a triangle add up to > 180°
- **Open**: negative curvature (like the surface of a saddle), where the angles of a triangle add up to < 180°

Experimentally, our universe is very close to being *flat*. However, as very large radii R of curvature generate spaces that are *very close to flat*, we cannot know for sure.

The FLRW solution supposes a *flat* universe:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2]$$

The only possible non-zero Christoffel's symbols are:

$$\Gamma^0_{ij}, \qquad \Gamma^i_{0j}$$

Expanding:

$$\Gamma^0_{ij} = \frac{1}{2} g^{00}(\cancel{g_{0j,i}} + \cancel{g_{i0,j}} - g_{ij,0}) = \frac{1}{2}(-1)(-1)\frac{\partial}{\partial t}(a^2\delta_{ij}) = a\dot{a}\delta_{ij}$$

as $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij} = a^2\delta_{ij}$.

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik} g_{jk,0} = \frac{1}{2} a^{-2}\delta_{ik}\frac{\partial}{\partial t}(a^2\delta_{kj}) = \frac{\dot{a}}{a}\delta_{ij}$$

Then $\Gamma^0_{ii} = 3a\dot{a}$ (repeated indices denote a sum), and $\Gamma^i_{0i} = 3\dot{a}/a$.
We can now compute the Ricci tensors. R_{0i} is immediately 0, as there are no non-vanishing g_{0i}, and also no non-zero spatial derivatives - so there isn't anything that can contribute to indices $0i$. We also know that R_{ij} must be proportional to δ_{ij} - as all spatial derivatives vanish.
Then, recall the full formula:

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\alpha} - \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\lambda}$$

and then we can compute:

$$R_{00} = \cancel{\partial_\alpha \Gamma^\alpha_{00}} - \partial_0 \Gamma^\alpha_{0\alpha} + \cancel{\Gamma^\lambda_{00}\Gamma^\alpha_{\lambda\alpha}} - \Gamma^\lambda_{0\alpha}\Gamma^\alpha_{0\lambda}$$

as $\Gamma^\alpha_{00} = 0$. Then, for the second term, α can only range over the spatial indices, and the same for the last term:

$$\begin{aligned} R_{00} &= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i} = -\partial_0 \left(\frac{3\dot{a}}{a}\right) - \frac{\dot{a}}{a}\delta_{ij}\frac{\dot{a}}{a}\delta_{ji} = \\ &= -\frac{3\ddot{a}}{a} + \cancel{\frac{3\dot{a}^2}{a^2}} - \cancel{3\frac{\dot{a}^2}{a^2}} = -3\frac{\ddot{a}}{a} \end{aligned}$$

as $\delta_{ij}\delta_{ji} = (\mathbb{I}_3)_{ii} = 3$.
Then:

$$R_{ij} = \partial_\alpha \Gamma^\alpha_{ij} - \cancel{\partial_j \Gamma^\alpha_{i\alpha}} + \Gamma^\alpha_{ij}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\lambda_{i\alpha}\Gamma^\alpha_{j\lambda}$$

where the second term vanishes are there are no non-zero spatial derivatives. Then $\Gamma^\alpha_{ij} \neq 0$ only for $\alpha = 0$, and the same for the second term. For the last, if $\alpha = 0$, $\lambda = k$, or if $\alpha = k$ then $\lambda = 0$:

$$\begin{aligned} R_{ij} &= \partial_0 \Gamma^0_{ij} + \Gamma^0_{ij}\Gamma^k_{0k} - \Gamma^0_{ik}\Gamma^k_{j0} - \Gamma^k_{i0}\Gamma^0_{jk} = \\ &= \partial_0[a\dot{a}\delta_{ij}] + a\dot{a}\delta_{ij}3\frac{\dot{a}}{a} - a\dot{a}\delta_{ik}\frac{\dot{a}}{a}\delta_{kj} - \frac{\dot{a}}{a}\delta_{ki}a\dot{a}\delta_{jk} = \\ &= (a\ddot{a} + \dot{a}^2)\delta_{ij} + 3\dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} = (a\ddot{a} + 2\dot{a}^2)\delta_{ij} = \\ &= \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)a^2\delta_{ij} \end{aligned}$$

Summarizing, we have:

$$\begin{aligned} R_{00} &= -\frac{3\ddot{a}}{a} \\ R_{ij} &= a^2\delta_{ij}\left[\frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right] \end{aligned}$$

The scalar curvature R is then:

$$R = g^{00}R_{00} + g^{ij}R_{ij} = \frac{3\ddot{a}}{a} + \frac{\delta_{ij}}{a^2}a^2\delta_{ij}\left[\frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right] = \frac{6\dot{a}^2}{a^2} + \frac{6\ddot{a}}{a}$$

We then compute the Einstein tensor:

$$\begin{aligned} G_{00} &= R_{00} - \frac{R}{2}g_{00} = -\cancel{\frac{3\ddot{a}}{a}} + \frac{3\dot{a}^2}{a^2} + \cancel{\frac{3\ddot{a}}{a}} = \frac{3\dot{a}^2}{a^2} \\ G_{ij} &= R_{ij} - \frac{R}{2}g_{ij} = a^2\delta_{ij}\left[\frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right] - a^2\delta_{ij}\left[\frac{3\dot{a}^2}{a^2} + \frac{3\ddot{a}}{a}\right] = a^2\delta_{ij}\left[-\frac{\dot{a}^2}{a^2} - \frac{2\ddot{a}}{a}\right] \end{aligned}$$

We fix the energy-momentum tensor, considering a universe filled by a *perfect fluid*:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$$

where ρ is the energy density, p is the pressure and u^ν the 4-velocity. Recall that:

$$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} \right)$$

We know that $u^0 = \mathrm{d}t \,/\, \mathrm{d}\tau > 0$, and also that:

$$0 = G_{0i} = \frac{T_{0i}}{M_p^2} \qquad T_{0i} = (\rho + p)u^0 u^i + \cancel{pg^{0i}}$$

and so $u^i = 0$, meaning that the *cosmic fluid* is *at rest*. Obviously, this can't happen in *every frame*, meaning that there is a **special frame of reference**: that where the cosmic fluid is at rest. This means that, while the theory is Lorentz-invariant, the universe *isn't*, because there is a uniquely identifiable special frame of reference (that of an observer stationary with respect to the Cosmic Microwave Background, meaning that he does not observe any dipole effect).
As $u^i = 0$, from $\boldsymbol{u} \cdot \boldsymbol{u} = g_{00} = -1$ we have $u^0 = 1$. This leads to:

$$\begin{aligned}
&T^{00} = \rho + p - p = \rho \Rightarrow T_{00} = \rho \\
&T_{ij} = pg_{ij} = a^2 p \delta_{ij} \\
&T_{0i} = 0
\end{aligned}$$

We can finally write the Einstein's equations:

$$\begin{aligned}
\frac{3\dot{a}^2}{a^2} &= \frac{\rho}{M_p^2} \\
a^2 \delta_{ij} \left[-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] &= a^2 \delta_{ij} \frac{p}{M_p^2}
\end{aligned}$$

leading to:

$$\begin{cases}
\dfrac{3\dot{a}^2}{a^2} = \dfrac{\rho}{M_p^2} \\
-\dfrac{2\ddot{a}}{a} - \dfrac{\dot{a}^2}{a^2} = \dfrac{\rho}{M_p^2}
\end{cases}$$

Let's now verify Bianchi identity:

$$\nabla_\mu G^{\mu\nu} = 0$$

For $\nu = 0$:

$$\begin{aligned}
\partial_\mu G^{\mu 0} + \Gamma^\mu_{\mu\lambda} G^{\lambda 0} + \Gamma^0_{\mu\lambda} G^{\mu\lambda} &= \partial_0 G^{00} + \Gamma^i_{i0} G^{00} + \Gamma^0_{ij} G^{ij} = \\
&= \partial_0 \left(\frac{3\dot{a}^2}{a^2} \right) + \frac{3\dot{a}}{a} \frac{3\dot{a}^2}{a^2} + a\dot{a}\delta_{ij} \frac{1}{a^2} \delta_{ij} \left[-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] = \\
&= 6\frac{\dot{a}}{a} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] + \frac{9\dot{a}^3}{a^3} - 6\frac{\dot{a}}{a}\frac{\ddot{a}}{a} - \frac{3\dot{a}^3}{a^3} = 0
\end{aligned}$$

And for $\nu = 1$:

$$\cancel{\partial_\mu G^{\mu i}} + \cancel{\Gamma^\mu_{\mu\lambda} G^{\lambda i}} + \cancel{\Gamma^i_{\mu\lambda} G^{\mu\lambda}} = 0$$

In the first term, a non-vanishing derivative implies $\mu = 0$, but $G^{\mu i} \neq 0$ for $\mu = i$, and so the term vanishes. A similar reasoning applies to the other two terms.
So, the Bianchi identity is satisfied.

We want now to verify $\nabla_\mu T^{\mu\nu} = 0$, which directly follows from Einstein's equation. This is a **local conservation law**.
For $\nu = 0$:

$$\begin{aligned} \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} &= \partial_0 T^{00} + \Gamma^i_{i0} T^{i0} + \Gamma^0_{ij} T^{ij} = \\ &= \partial_0 \rho + 3\frac{\dot{a}}{a}\rho + \dot{a}a\delta_{ij}\frac{1}{a^2}\delta_{ij} p = \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}p \underset{!}{=} 0 \end{aligned}$$

So we get another equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

As $\nabla_\mu T^{\mu\nu} = 0$ is a consequence of Einstein's equation, this relation we just found can be retrieved by manipulating the two equations we previously got.
First, let's examine quickly the remaining case for $\nu = i$:

$$\cancel{\partial_\mu T^{\mu i}} + \cancel{\Gamma^\mu_{\mu\lambda} T^{\lambda i}} + \cancel{\Gamma^i_{\mu\lambda} T^{\mu\lambda}} = 0$$

which is trivially satisfied.
Then, taking the ∂_0 of the first Einstein's equation we get:

$$6\frac{\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] = \frac{\dot{\rho}}{M_p^2}$$

And multiplying the second one by $3\dot{a}/a$:

$$-6\frac{\dot{a}}{a}\frac{\ddot{a}}{a} - 3\frac{\dot{a}^3}{a^3} = \frac{3\dot{a}}{a}\frac{p}{M_p^2}$$

If we now add them, we can remove the $\ddot{a}$ term:

$$-9\frac{\dot{a}^3}{a^3} = \frac{\dot{\rho} + \frac{3\dot{a}}{a}p}{M_p^2}$$

Taking again the first equation and multiplying it by $3\dot{a}/a$ leads to:

$$\frac{9\dot{a}^3}{a^3} = \frac{3\dot{a}}{a}\frac{\rho}{M_p^2}$$

And adding these two equations makes $\dot{a}^3$ vanish:

$$0 = \frac{\dot{\rho} + \frac{3\dot{a}}{a}p + \frac{3\dot{a}}{a}\rho}{M_p^2} \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

and this is the equation we got from $\nabla_\mu T^{\mu\nu} = 0$, proving that indeed it follows only from the other two.
To solve these equations, as one of them is redundant, we can *consider only two of them at a time*. The easy choice is the first and the third, as they're both first order:

$$\begin{cases} \dot{\rho} + 3\dfrac{\dot{a}}{a}(\rho + p) = 0 \\ \dfrac{\dot{a}^2}{a^2} = \dfrac{\rho}{3M_p^2} \end{cases}$$

0.2.1 Sources

The **equation of state** reads:

$$w \equiv \frac{p}{\rho}$$

If $w = -1$, then $p = -\rho$ and so $\rho =$ constant. This means that, for an expanding universe, the energy density *does not drop* - meaning that *space itself* has energy, called **vacuum energy**.
This is coherent with particle physics, as *vacuum* just denotes the *lowest energy state* (e.g. the Higgs potential). However, this creates problems with quantum mechanics.

0.2.2 Einstein cosmological constant

Einstein's equation can, in principle, be modified in the following way:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu}$$

which *satisfies* Bianchi identity if Λ is a constant:

$$\nabla_\mu (\Lambda g^{\mu\nu}) = 0$$

as the metric is covariantly constant.
Rearranging:

$$G_{\mu\nu} = \frac{1}{M_p^2} \left(T_{\mu\nu} - \Lambda M_p^2 g_{\mu\nu} \right)$$

and so we can interpret the role of Λ (cosmological constant) as a *source* of energy.
The components of the Einstein tensor become:

$$\begin{aligned} G_{00} &= \frac{1}{M_p^2} \left(\rho + \Lambda M_p^2 \right) \\ G_{ij} &= \frac{1}{M_p^2} \left(a^2 \delta_{ij} p - \Lambda M_p^2 a^2 \delta_{ij} \right) \end{aligned}$$

And defining:

$$\rho_{\text{tot}} = \rho + \Lambda M_p^2$$
$$p_{\text{tot}} = p - \Lambda M_p^2$$

and so:

$$\frac{p_\Lambda}{\rho_\Lambda} = -1$$

So the cosmological constant has the same effect of a *vacuum energy* (that we saw in the previous paragraph). The idea is that, if Λ is very small, it will not be measurable inside the solar system, but it will have a significant effect on the evolution of the entire universe.